Molecules and Molecular Spectroscopy:

Learning about molecules from Quantum theory

and

Learning about Quantum theory from molecules

William G. Harter Research Sketch 11. 17 . 17

A sketch of modern molecular spectroscopy

The molecular frequency hierarchy

Units of frequency (Hz), wavelength (m), energy (eV), and wavenumber (cm^{-1})

Spectral windows in atmosphere due to molecules

Example of~16μm (670cm^{-1}) spectral hierarchy of CO_2 (simple*)*

Example of~16μm (631cm^{-1}) spectral hierarchy of CF_4 (complicated*)*

Example of ~16μm (615cm^{-1}) spectral hierarchy of SF_6 (really complicated*)*

Rotational Energy Surface (RES) analysis, J-vector geometry, and tunneling

Nuclear spin hyperfine effects rule mol-spec.

Quantum "revivals" of gently localized rotor waves:

Bohr-rotor wave dynamics gives lessons for quantum number theory

Gaussian wave-packet bandwidth and uncertainty

Gaussian Bohr-rotor revivals and quantum fractals

Understanding fractals using geometry of fractions (Rationalizing rationals)

Farey-Sums and Ford-products

Ford Circles and Farey-Trees

The simplest molecule: A pair of head-on lasers gives lessons for relativistic quantum theory

Light wave zeros draw Minkowski coordinate grid

***Relawavity** geometry of waves defines space-time warp*

...and per-space-time quantum mechanics

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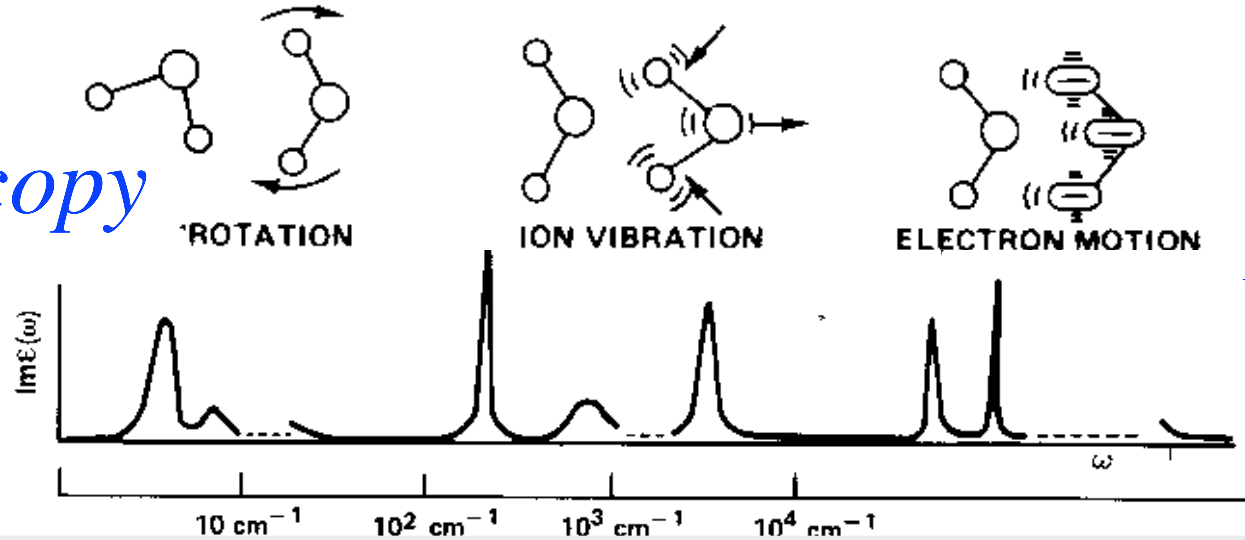
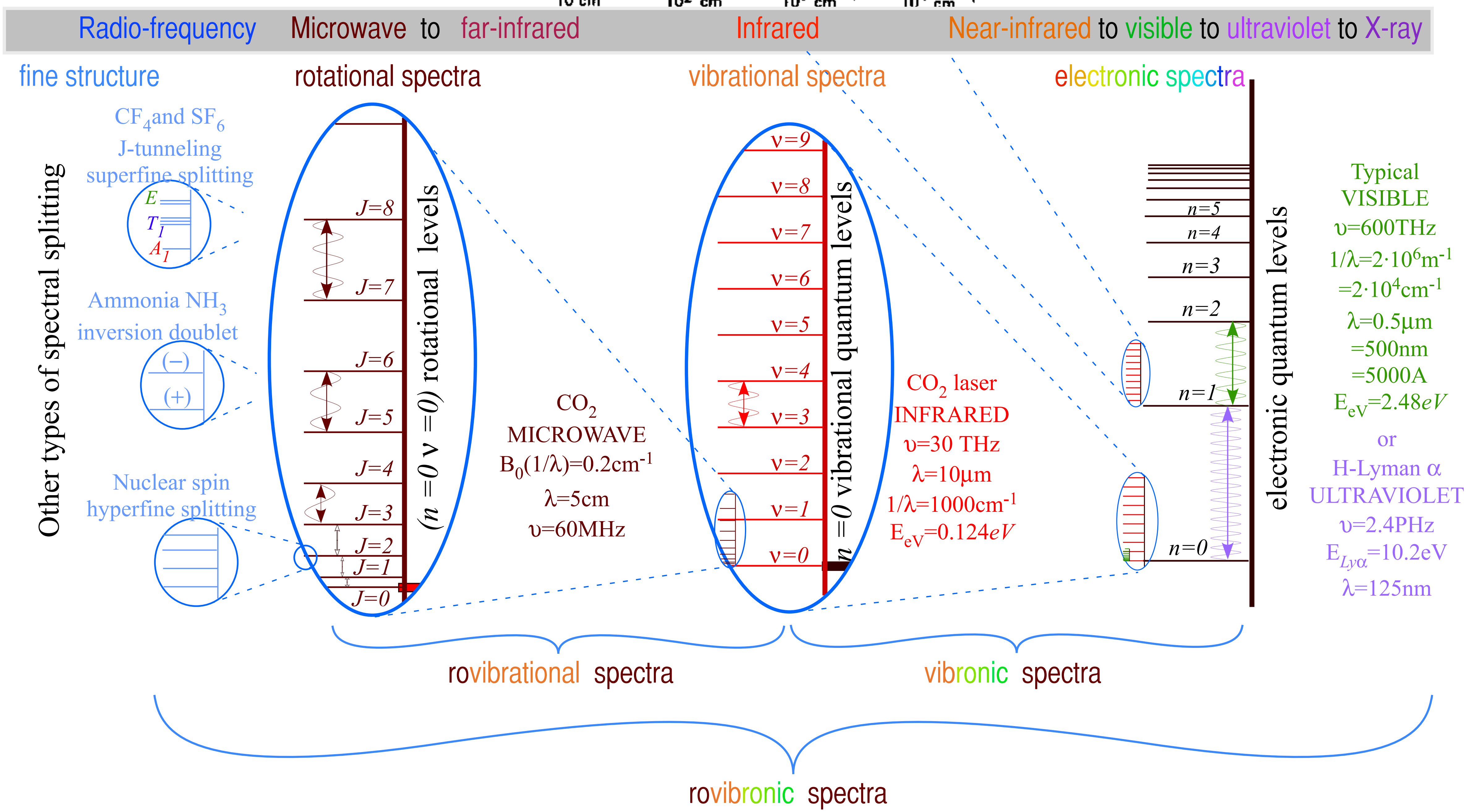


From Fig. 6.5.5.
Principles of Symmetry, Dynamics, and Spectroscopy
W. G. Harter, Wiley Interscience, NY (1993)

The frequency hierarchy



Spectral Quantities

Frequency υ
Hertz(*sec*$^{-1}$)
THz $10^{12}s^{-1}$
GHz $10^{9}s^{-1}$
MHz $10^{6}s^{-1}$
kHz $10^{3}s^{-1}$

Wavelength λ
meters(*m*)
fm $10^{-15}m$
pm $10^{-12}m$
nm $10^{-9}m$
μm $10^{-6}m$
mm $10^{-3}m$
cm $10^{-2}m$
km $10^{3}m$

Wavenumber
per meter(m^{-1})
cm^{-1} $10^{2}m^{-1}$

Energy $eh\upsilon$
electonVolts(*eV*)

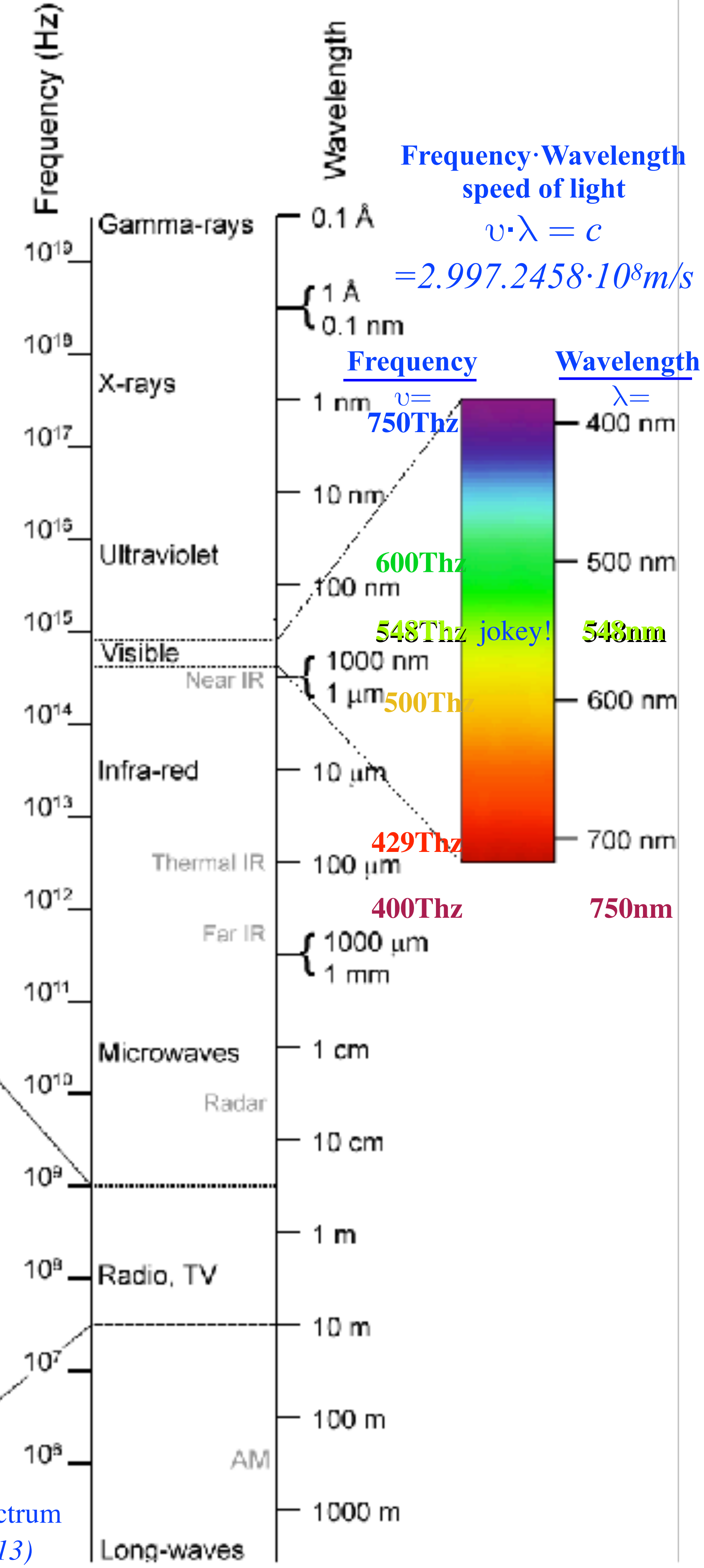
Units of frequency (Hz), wavelength (m), and energy (eV)

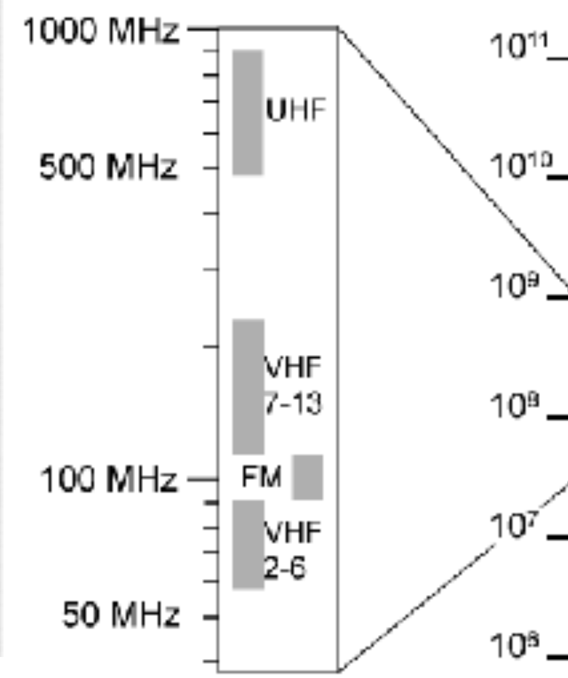
CLASS	FREQUENCY	WAVELENGTH	ENERGY
γ	300 EHz	1 pm	1.24 MeV
HX	30 EHz	10 pm	124 keV
	3 EHz	100 pm	12.4 keV
SX	300 PHz	1 nm	1.24 keV
EUV	30 PHz	10 nm	124 eV
NUV	3 PHz	100 nm	12.4 eV
NIR	300 THz	1 μm	1.24 eV
MIR	30 THz	10 μm	124 meV
FIR	3 THz	100 μm	12.4 meV
EHF	300 GHz	1 mm	1.24 meV
SHF	30 GHz	1 cm	124 μeV
UHF	3 GHz	1 dm	12.4 μeV
VHF	300 MHz	1 m	1.24 μeV
HF	30 MHz	10 m	124 neV
MF	3 MHz	100 m	12.4 neV
LF	300 kHz	1 km	1.24 neV
VLF	30 kHz	10 km	124 peV
VF/ULF	3 kHz	100 km	12.4 peV
SLF	300 Hz	1 Mm	1.24 peV
ELF	30 Hz	10 Mm	124 feV
	3 Hz	100 Mm	12.4 feV

From: Electromagnetic Spectrum
Wikepedia Commons (2013)

Exa: 10^{18}
Peta: 10^{15}
Tera: 10^{12}
Giga: 10^{9}
Mega: 10^{6}
kilo: 10^{3}

milli: 10^{-3}
micro: 10^{-6}
nano: 10^{-9}
pico: 10^{-12}
femto: 10^{-15}
atto: 10^{-18}





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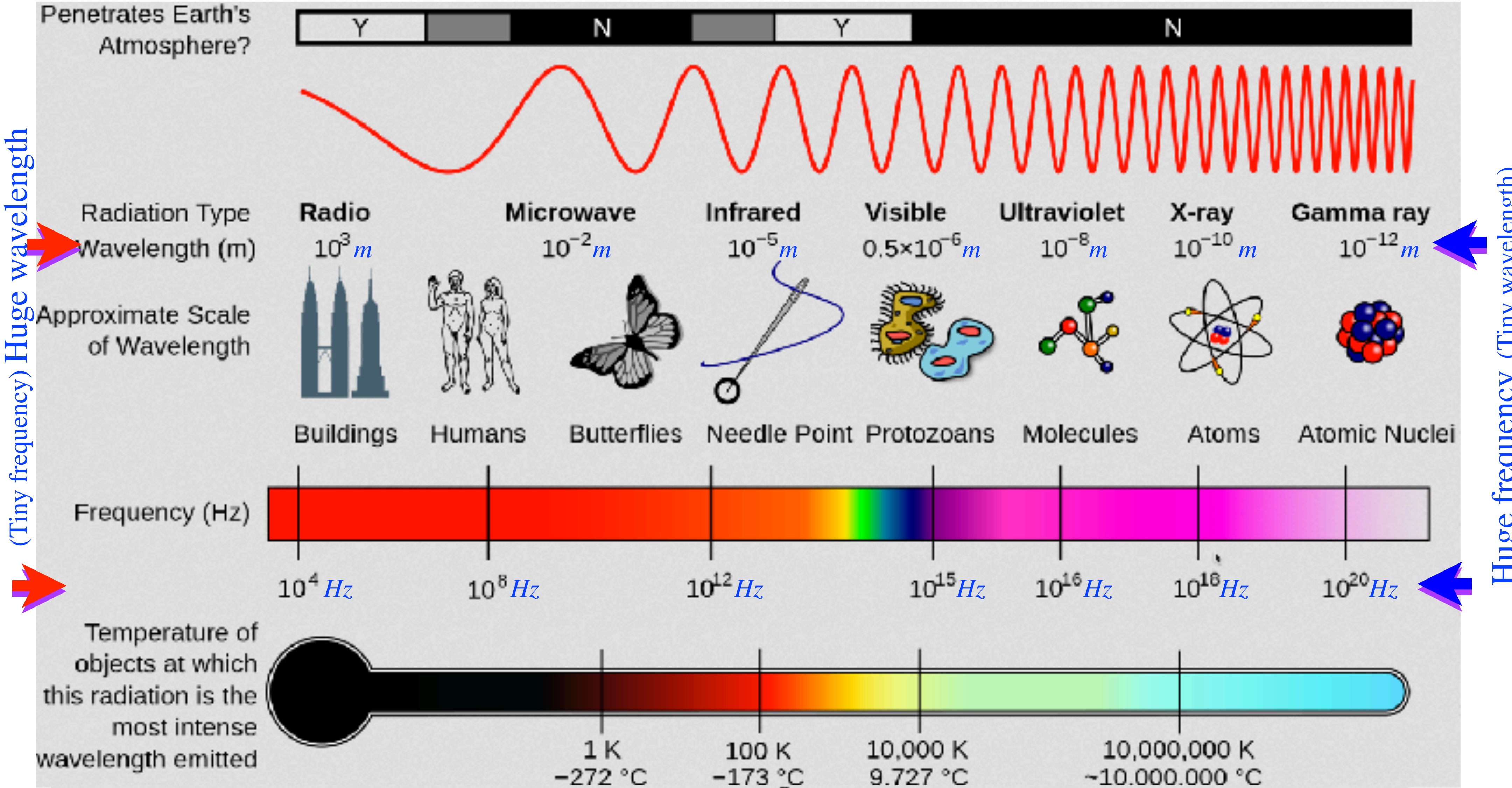

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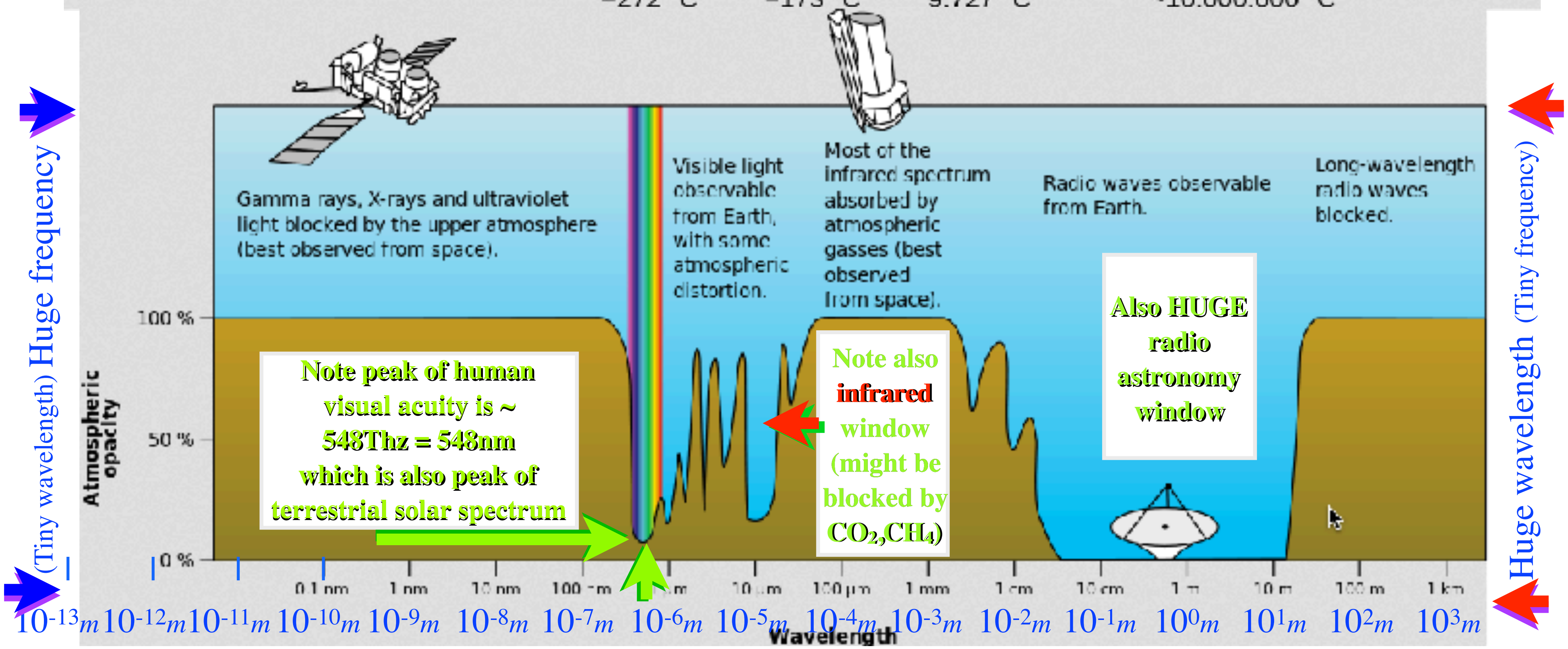

Spectral windows in Earth atmosphere

From: Electromagnetic Spectrum
Wikepedia Commons (2013)

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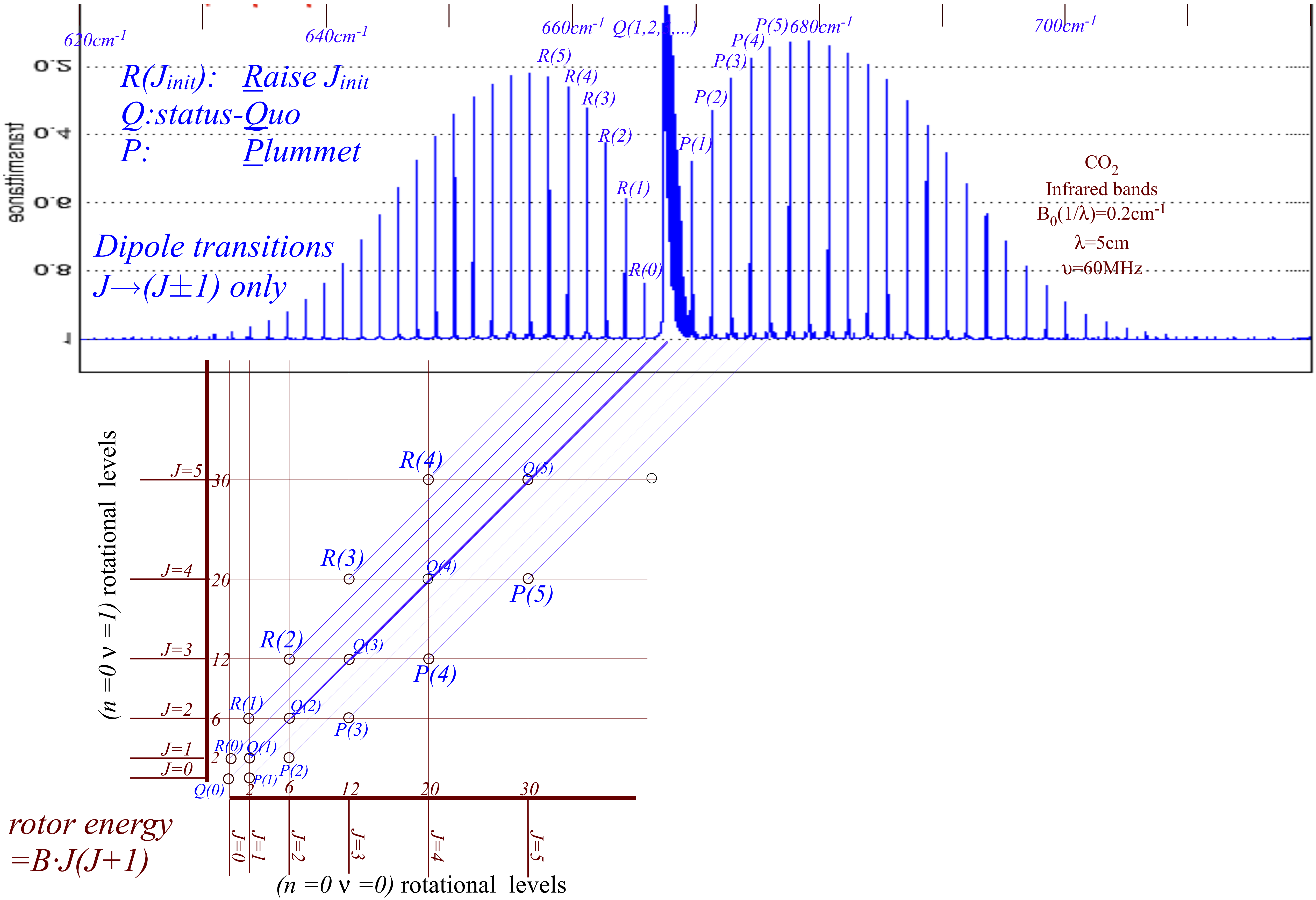
Example of CO_2 rotational $(\upsilon=0)\Leftrightarrow(\upsilon=1)$ bands



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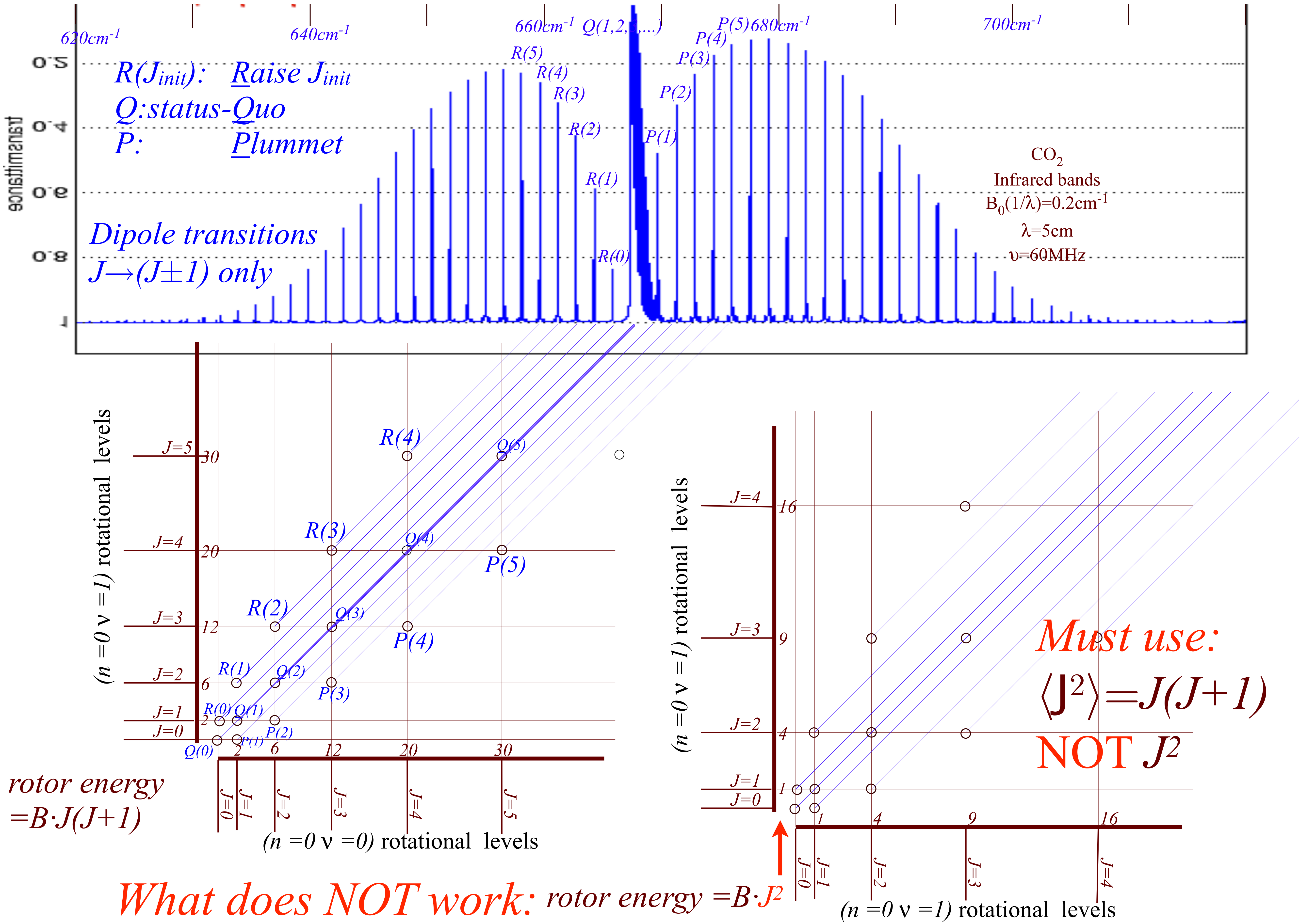
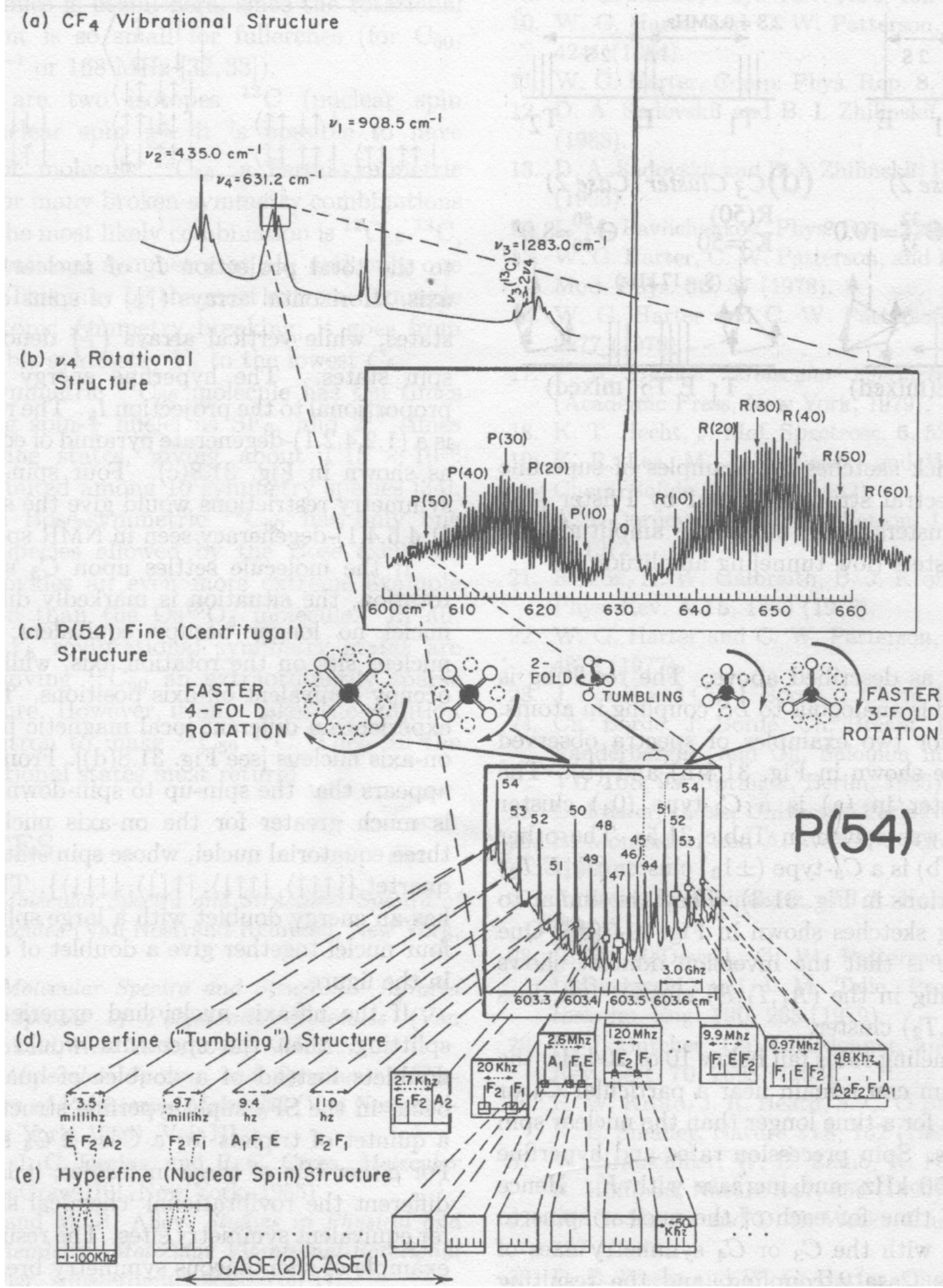


Example of frequency hierarchy for 16μm spectra of CF_4 (Freon-14)

W.G.Harter
Ch. 31
Atomic, Molecular, & Optical Physics Handbook
Am. Int. of Physics
Gordon Drake Editor
(1996)



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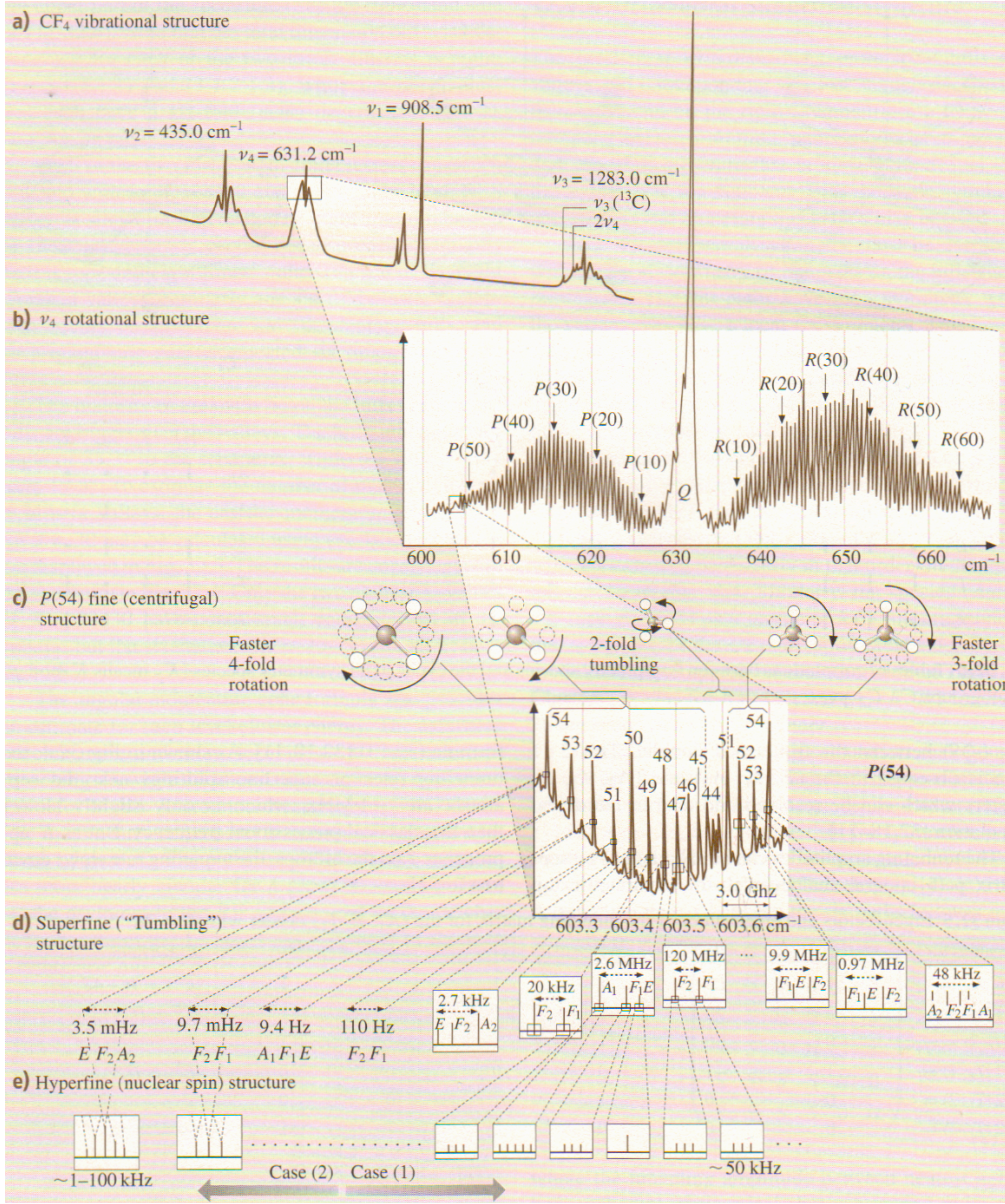
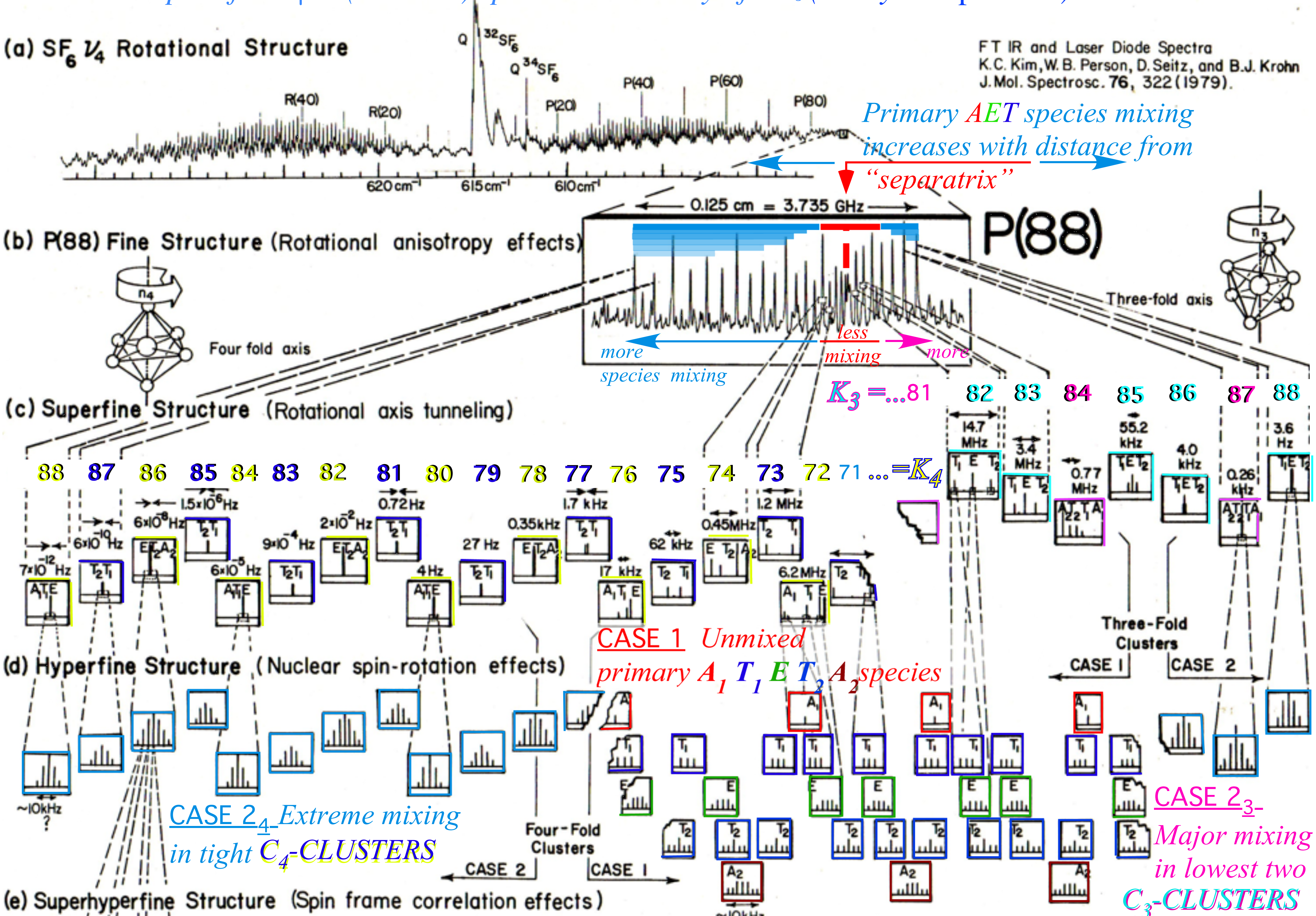
W.G.Harter Fig. 32.7 Springer Handbook of Atomic, Molecular, & Optical Physics Gordon Drake Editor (2005)



Example of ~16μm (615cm^{-1}) spectral hierarchy of SF_6 (really complicated)



(Next page: approximate theory)

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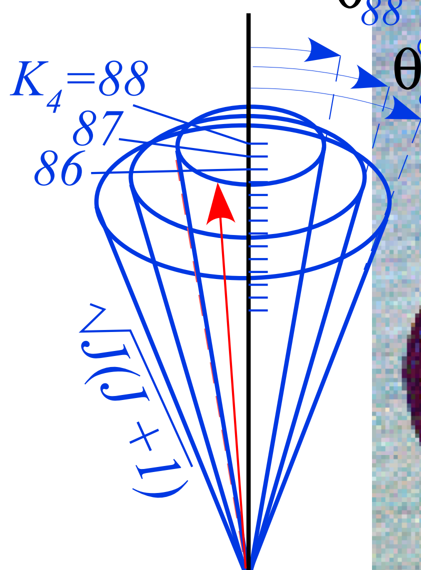
$$\langle H\rangle \sim \nu_{vib} + BJ(J+1) + \langle H^{Scalar\ Coriolis}\rangle + \langle H^{Tensor\ Centrifugal}\rangle + \langle H^{Tensor\ Coriolis}\rangle + \langle H^{Nuclear\ Spin}\rangle + \ldots$$

O_h or T_d Spherical Top: *(Hecht CH_4 Hamiltonian 1960)*

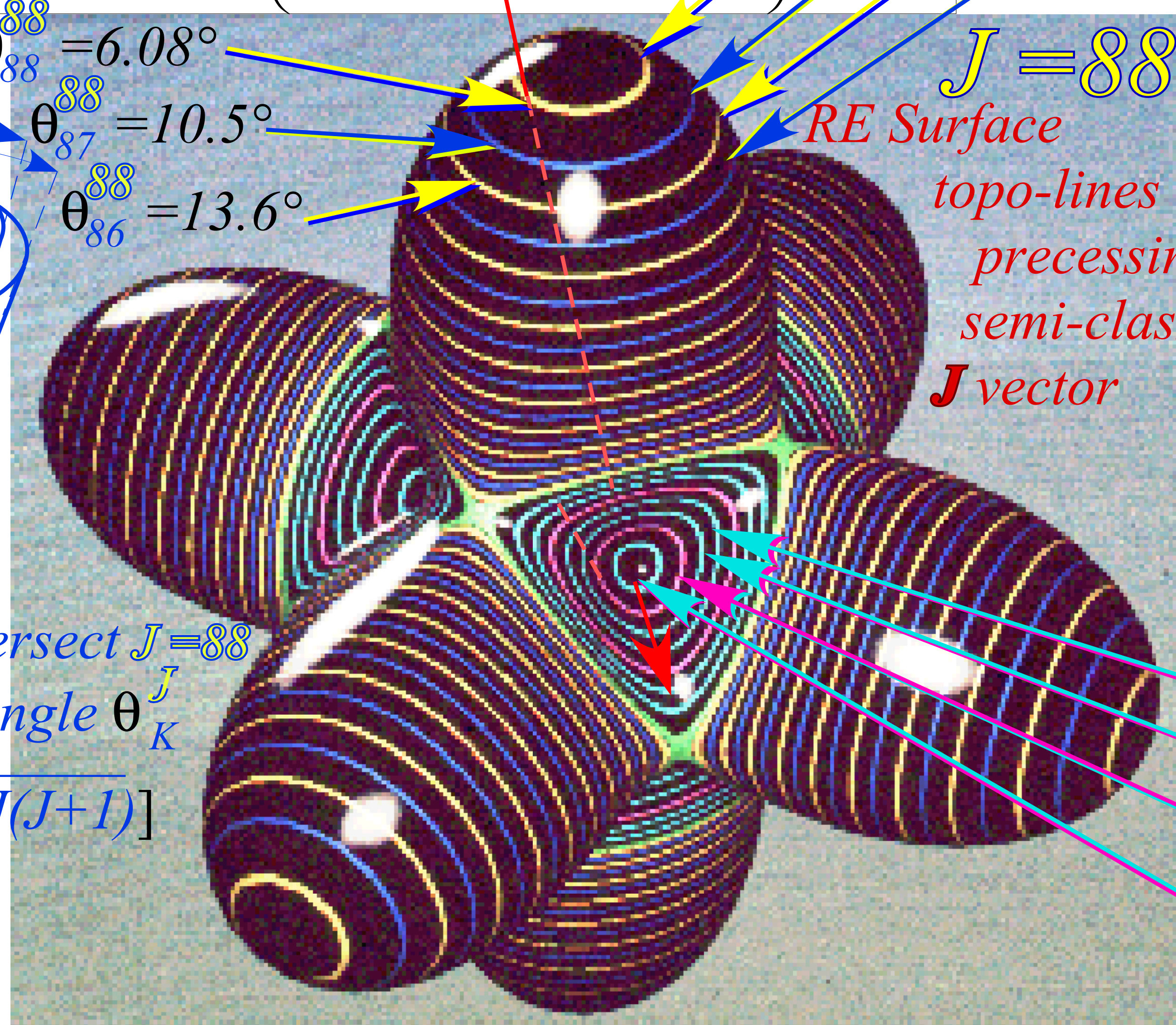
$$\mathbf{H} = B\left(\mathbf{J}_x^2 + \mathbf{J}_y^2 + \mathbf{J}_z^2\right) + t_{440}\left(\mathbf{J}_x^4 + \mathbf{J}_y^4 + \mathbf{J}_z^4 - \frac{3}{5}J^4\right) + \cdots$$

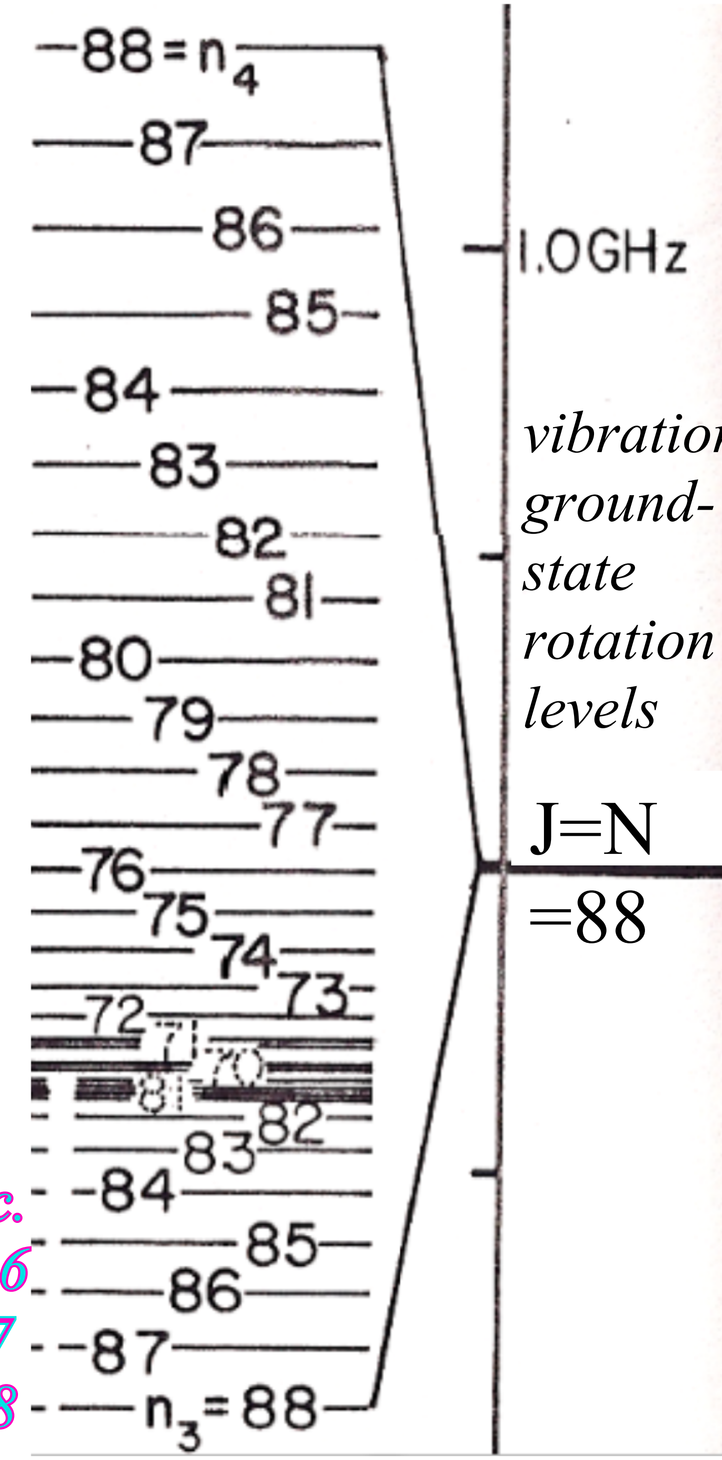
$$= BJ^2 + t_{440}\left(\mathbf{T}_0^4 + \sqrt{\frac{5}{14}}\left[\mathbf{T}_4^4 + \mathbf{T}_{-4}^4\right]\right) + \cdots$$



(J,K) cones intersect J=88 RE surface at angle θ_K^J

$\theta_K^J = \text{acos}[K/\sqrt{J(J+1)}]$

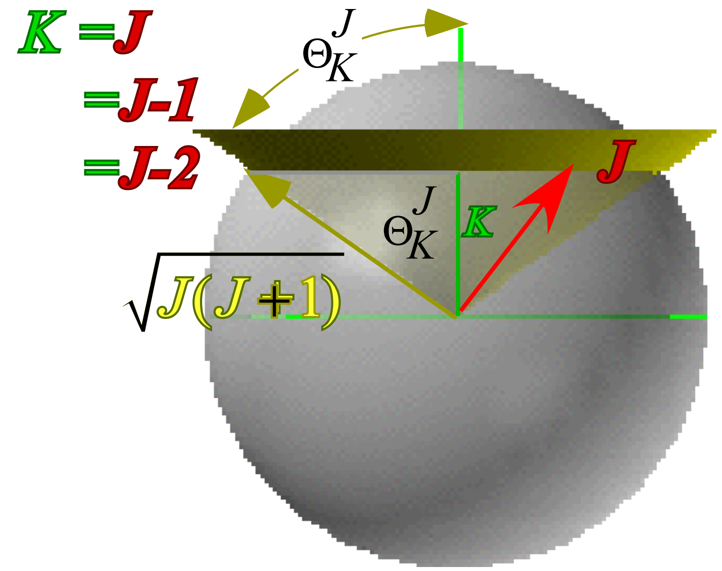




(next page shows slice)

Finding Hamiltonian Eigensolutions by Geometry using Uncertainty Cone Angles

$$\cos\Theta_K^J = \frac{K}{\sqrt{J(J+1)}}$$

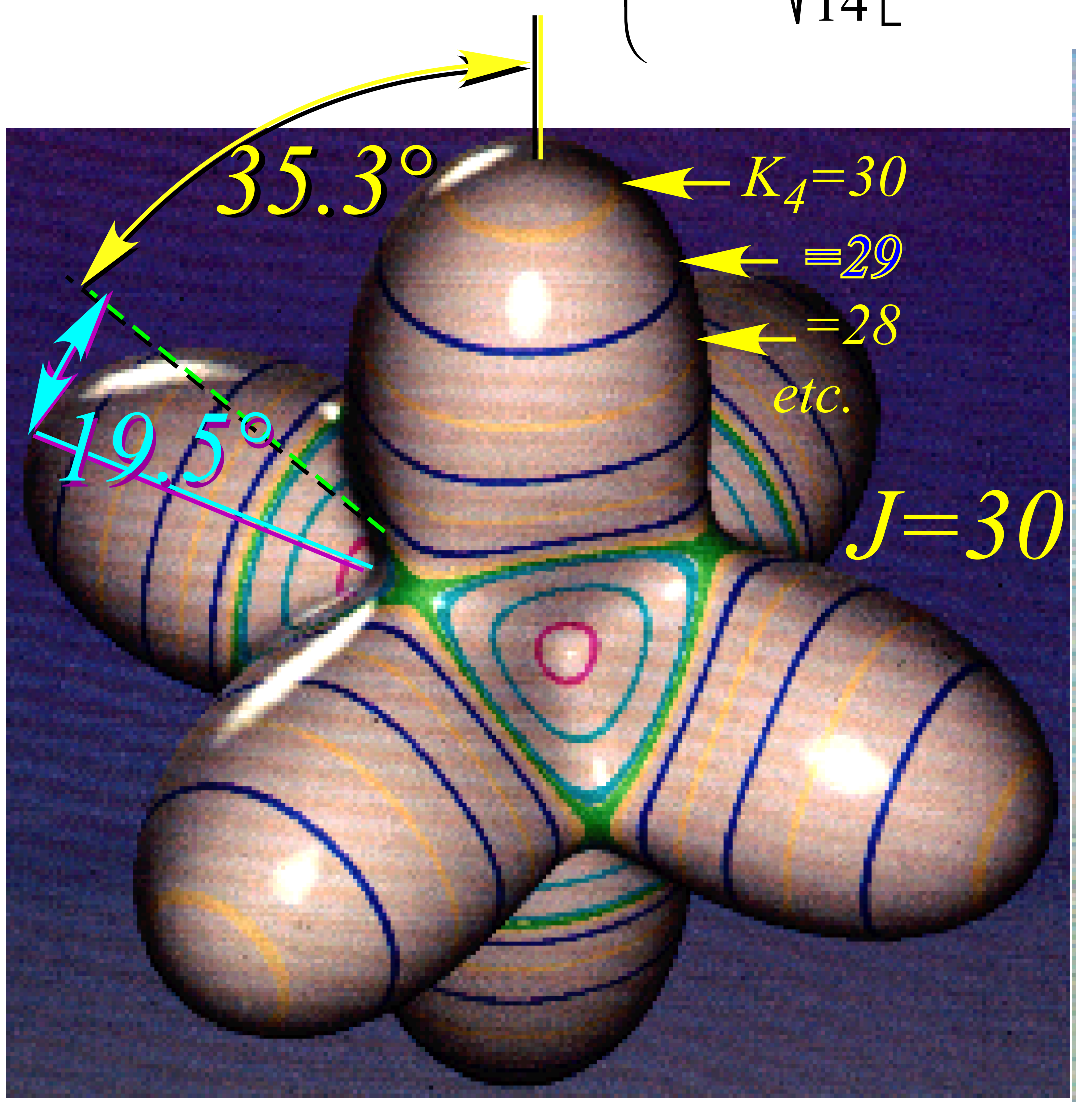


J cone intersects J=88 RE surface to give approx. K=J, J-1, J-2... energy levels

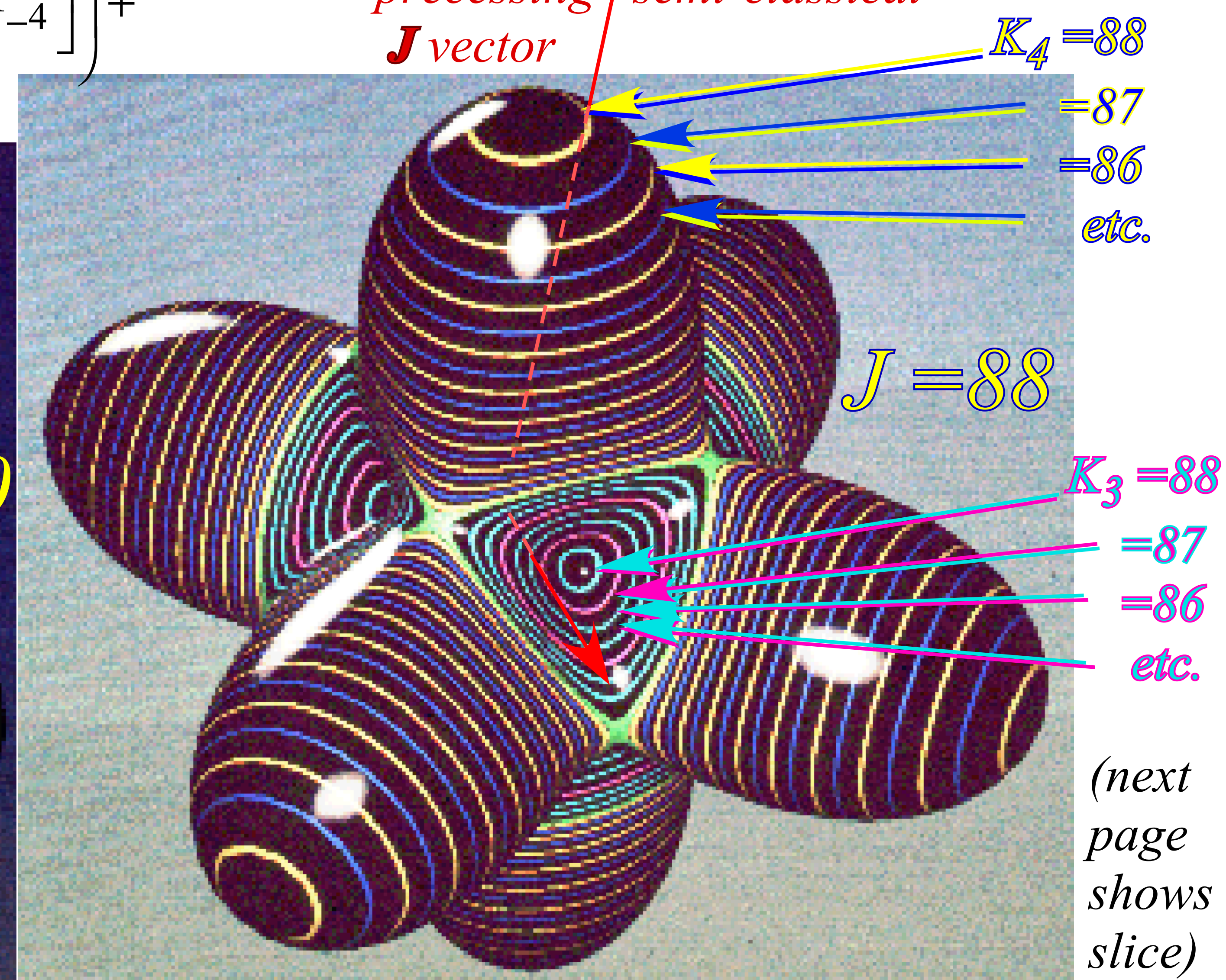
O_h or T_d Spherical Top: *(Hecht Ro-vib Hamiltonian 1960)*

$$\mathbf{H} = B\left(\mathbf{J}_x^2 + \mathbf{J}_y^2 + \mathbf{J}_z^2\right) + t_{440}\left(\mathbf{J}_x^4 + \mathbf{J}_y^4 + \mathbf{J}_z^4 - \frac{3}{5}J^4\right) + \cdots$$

$$= BJ^2 + t_{440}\left(\mathbf{T}_0^4 + \sqrt{\frac{5}{14}}\left[\mathbf{T}_4^4 + \mathbf{T}_{-4}^4\right]\right) + \cdots$$



RE Surface topo-lines track precessing semi-classical J vector

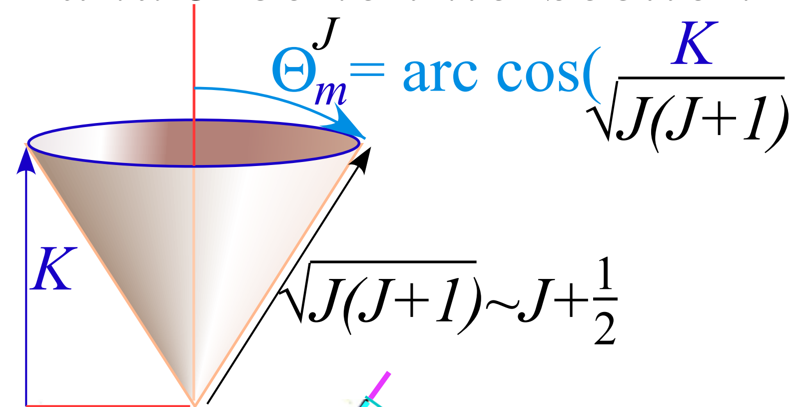


(next page shows slice)

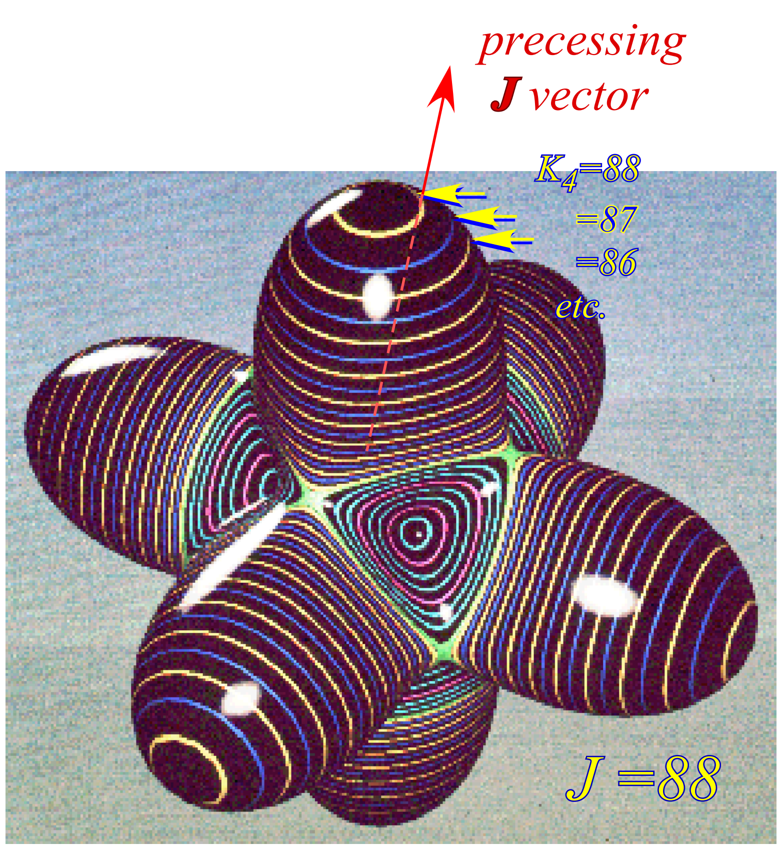
SF_6 Spectra of O_h Ro-vibronic Hamiltonian described by RE Tensor Topography and J-cone intersection

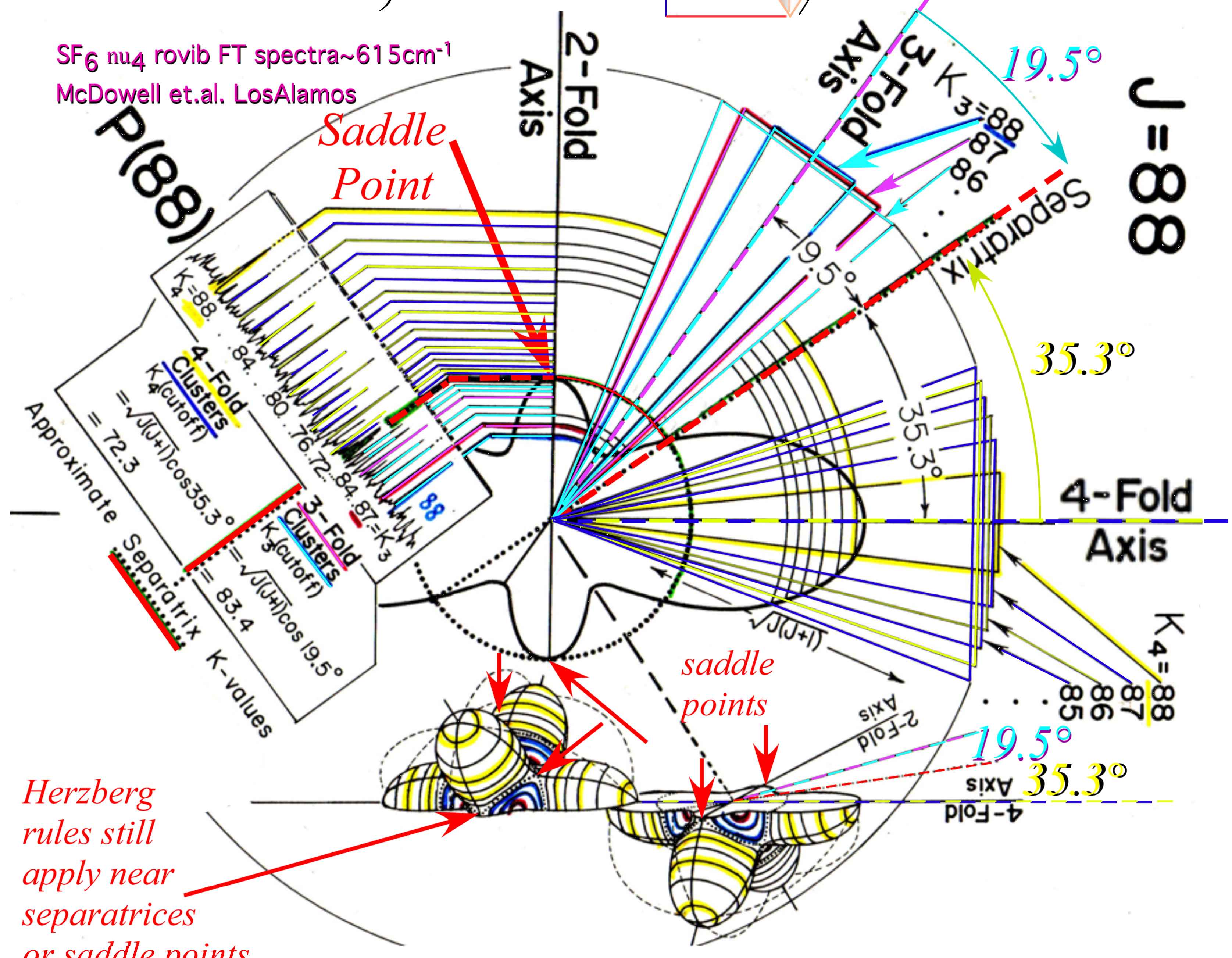
$$\mathbf{H} = B\left(\mathbf{J}_x^2 + \mathbf{J}_y^2 + \mathbf{J}_z^2\right) + t_{440}\left(\mathbf{J}_x^4 + \mathbf{J}_y^4 + \mathbf{J}_z^4 - \frac{3}{5}J^4\right) + \cdots$$

$$= \quad B\mathbf{J}^2 \quad + t_{440}\left(\mathbf{T}_0^4 + \sqrt{\frac{5}{14}}\left[\mathbf{T}_4^4 + \mathbf{T}_{-4}^4\right]\right) + \cdots$$

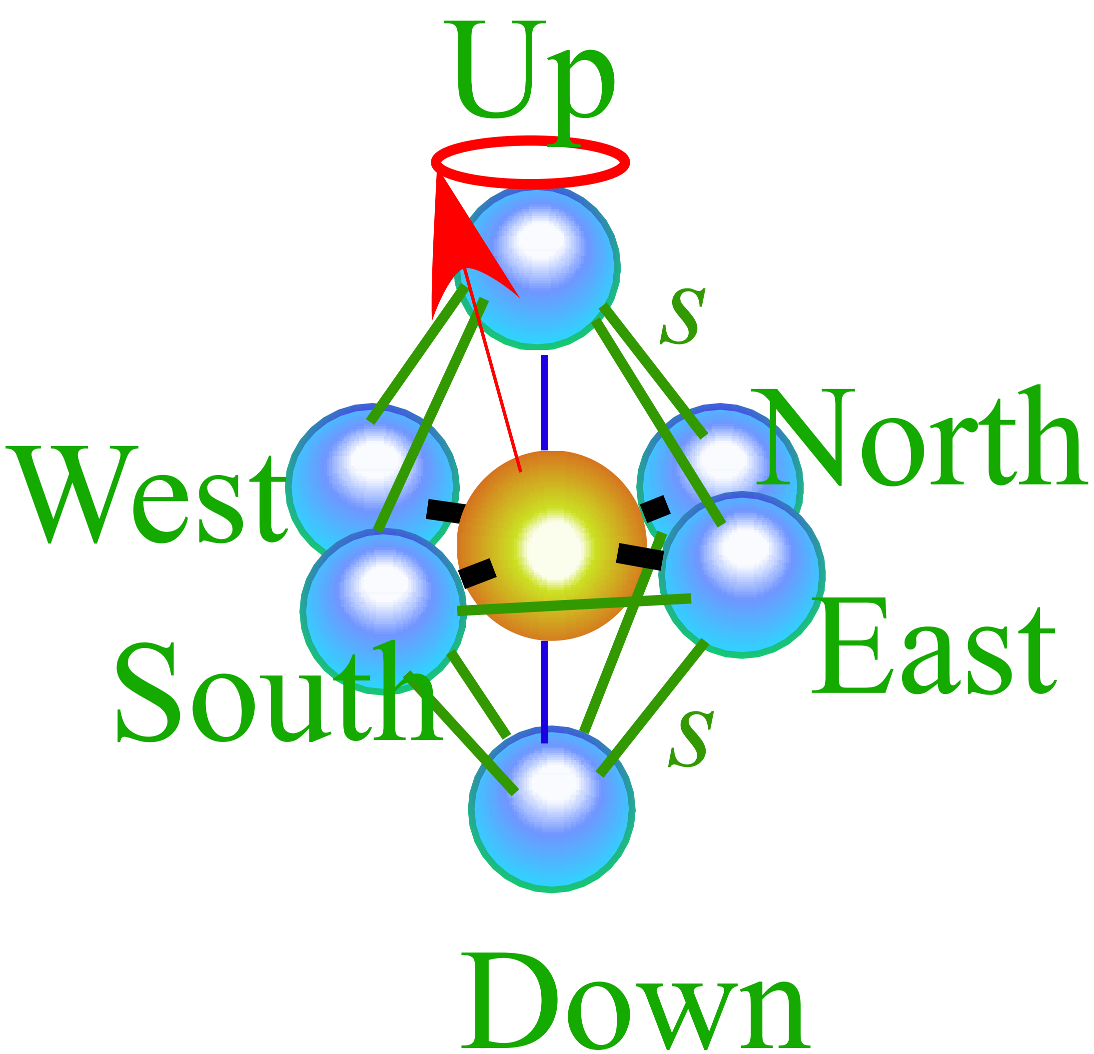


Rovibronic Energy (RE) Tensor Surface





*Internal **J** gets "stuck" on RES axes*
Must "tunnel" axis-to-axis at rate s



|U>|D>|E>|W>|N>|S>

H	*0*	*s*	*s*	*s*	*s*
0	*H*	*s*	*s*	*s*	*s*
s	*s*	*H*	*0*	*s*	*s*
s	*s*	*0*	*H*	*s*	*s*
s	*s*	*s*	*s*	*H*	*0*
s	*s*	*s*	*s*	*0*	*H*

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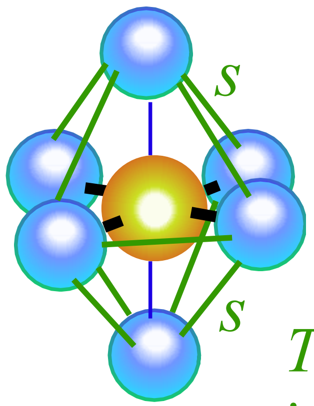
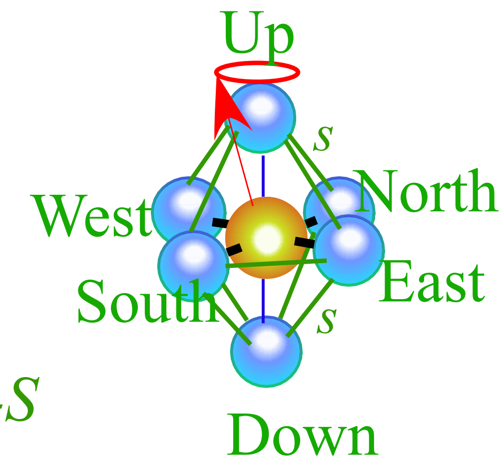


Tunneling s=-S
is negative here



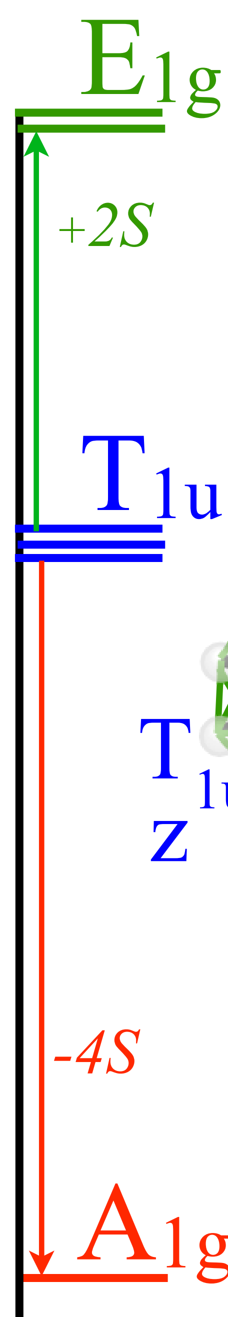
|U>|D>|E>|W>|N>|S>

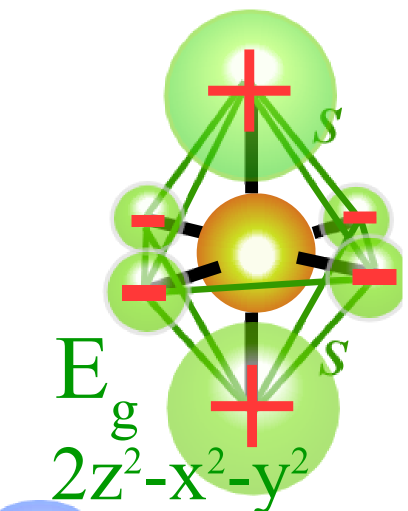
$$\begin{pmatrix} H & 0 & s & s & s & s \\ 0 & H & s & s & s & s \\ s & s & H & 0 & s & s \\ s & s & 0 & H & s & s \\ s & s & s & s & H & 0 \\ s & s & s & s & 0 & H \end{pmatrix}$$

$$\begin{vmatrix} H & 0 & s & s & s & s \\ 0 & H & s & s & s & s \\ s & s & H & 0 & s & s \\ s & s & 0 & H & s & s \\ s & s & s & s & H & 0 \\ s & s & s & s & 0 & H \end{vmatrix} \begin{vmatrix} +2 \\ +2 \\ -1 \\ -1 \\ -1 \\ -1 \end{vmatrix} \frac{1}{\sqrt{12}} = (H-2s) \begin{vmatrix} +2 \\ +2 \\ -1 \\ -1 \\ -1 \\ -1 \end{vmatrix} \frac{1}{\sqrt{12}}$$

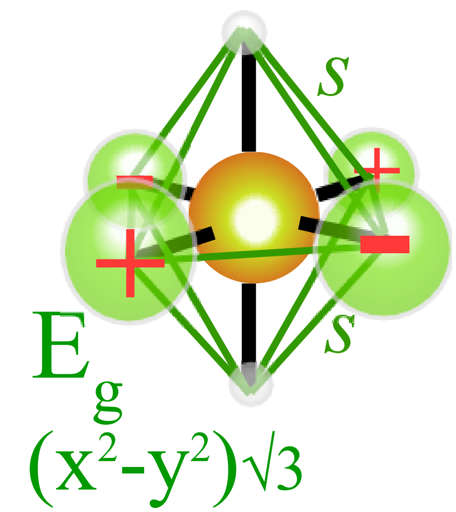
$$\begin{vmatrix} H & 0 & s & s & s & s \\ 0 & H & s & s & s & s \\ s & s & H & 0 & s & s \\ s & s & 0 & H & s & s \\ s & s & s & s & H & 0 \\ s & s & s & s & 0 & H \end{vmatrix} \begin{vmatrix} +1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \frac{1}{\sqrt{2}} = (H+0) \begin{vmatrix} +1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \frac{1}{\sqrt{2}}$$

$$\begin{vmatrix} H & 0 & s & s & s & s \\ 0 & H & s & s & s & s \\ s & s & H & 0 & s & s \\ s & s & 0 & H & s & s \\ s & s & s & s & H & 0 \\ s & s & s & s & 0 & H \end{vmatrix} \begin{vmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{vmatrix} \frac{1}{\sqrt{6}} = (H+4s) \begin{vmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{vmatrix} \frac{1}{\sqrt{6}}$$

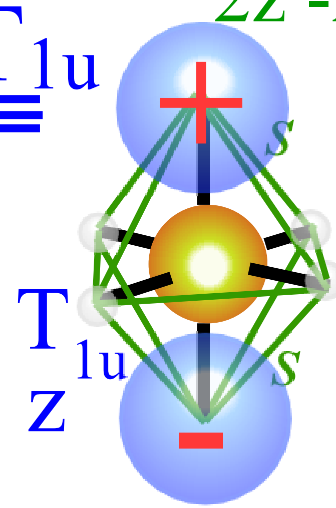




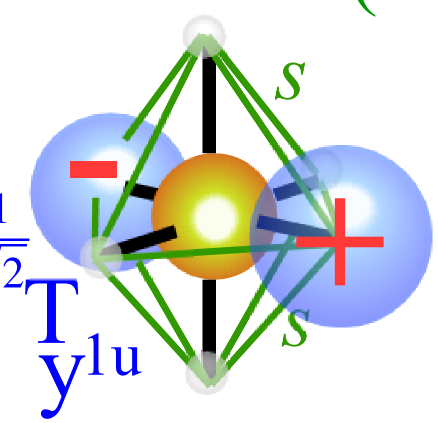
$$\begin{vmatrix} +2 \\ +2 \\ -1 \\ -1 \\ -1 \\ -1 \end{vmatrix} \frac{1}{\sqrt{12}}$$

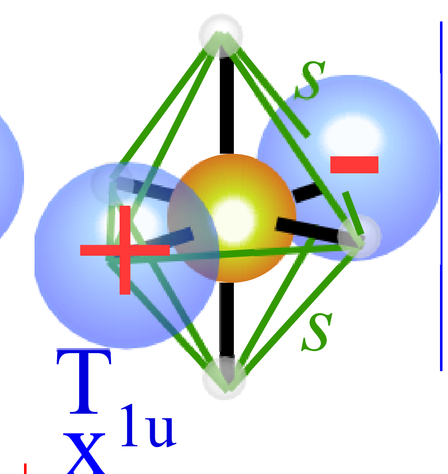


$$\begin{vmatrix} 0 \\ 0 \\ -1 \\ -1 \\ +1 \\ +1 \end{vmatrix} \frac{1}{2}$$

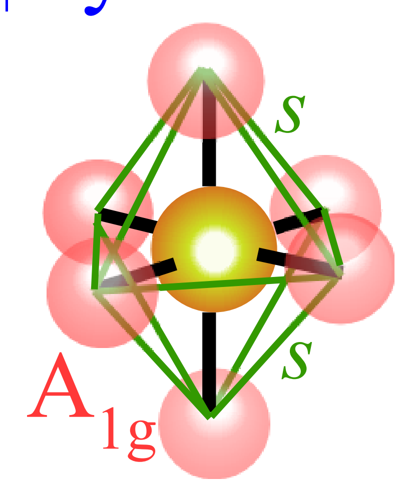


$$\begin{vmatrix} 0 \\ 0 \\ +1 \\ -1 \\ 0 \\ 0 \end{vmatrix} \frac{1}{\sqrt{2}}$$





$$\begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ +1 \end{vmatrix} \frac{1}{\sqrt{2}}$$



$$\begin{vmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{vmatrix} \frac{1}{\sqrt{6}}$$

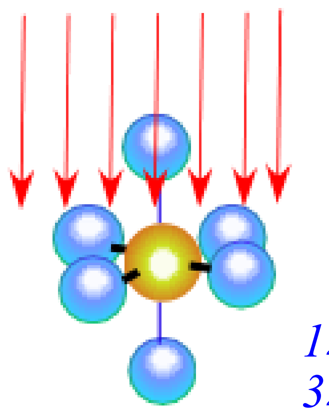
Duality: The "Flip Side" of Symmetry Analysis.

LAB versus BODY, STATE versus PARTICLE,

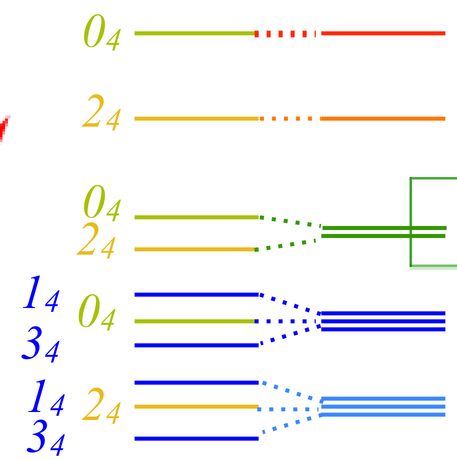
boils down to :

OUTSIDE versus INSIDE

OUTSIDE or LAB
Symmetry reduction results in Level or Spectral SPLITTING
External B-field does Zeeman splitting



Example:
Cubic-Octahedral O reduced to Tetragonal C_4



C_4	0_4	1_4	2_4	3_4
A_1	1	.	.	.
A_2	.	.	1	.
E	1.	.	1	.
T_1	1	1	.	1
T_2	.	1	1	1

INSIDE or BODY
Symmetry reduction results in Level or Spectral UN-SPLITTING ("clustering")

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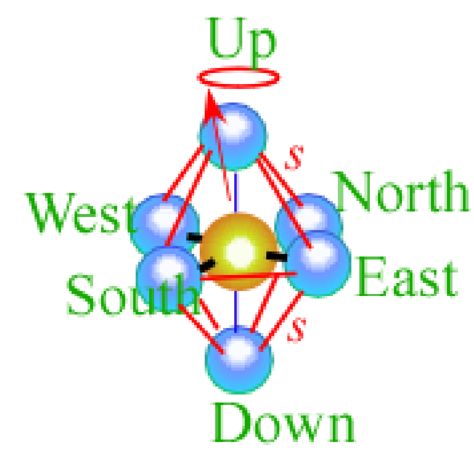
Must "tunnel" axis-to-axis at rate s



|U>|D>|E>|W>|N>|S>

H	0	s	s	s	s
0	H	s	s	s	s
s	s	H	0	s	s
s	s	0	H	s	s
s	s	s	s	H	0
s	s	s	s	0	H

Tunneling (s) between axes splits the 0_4 cluster as shown on following pages

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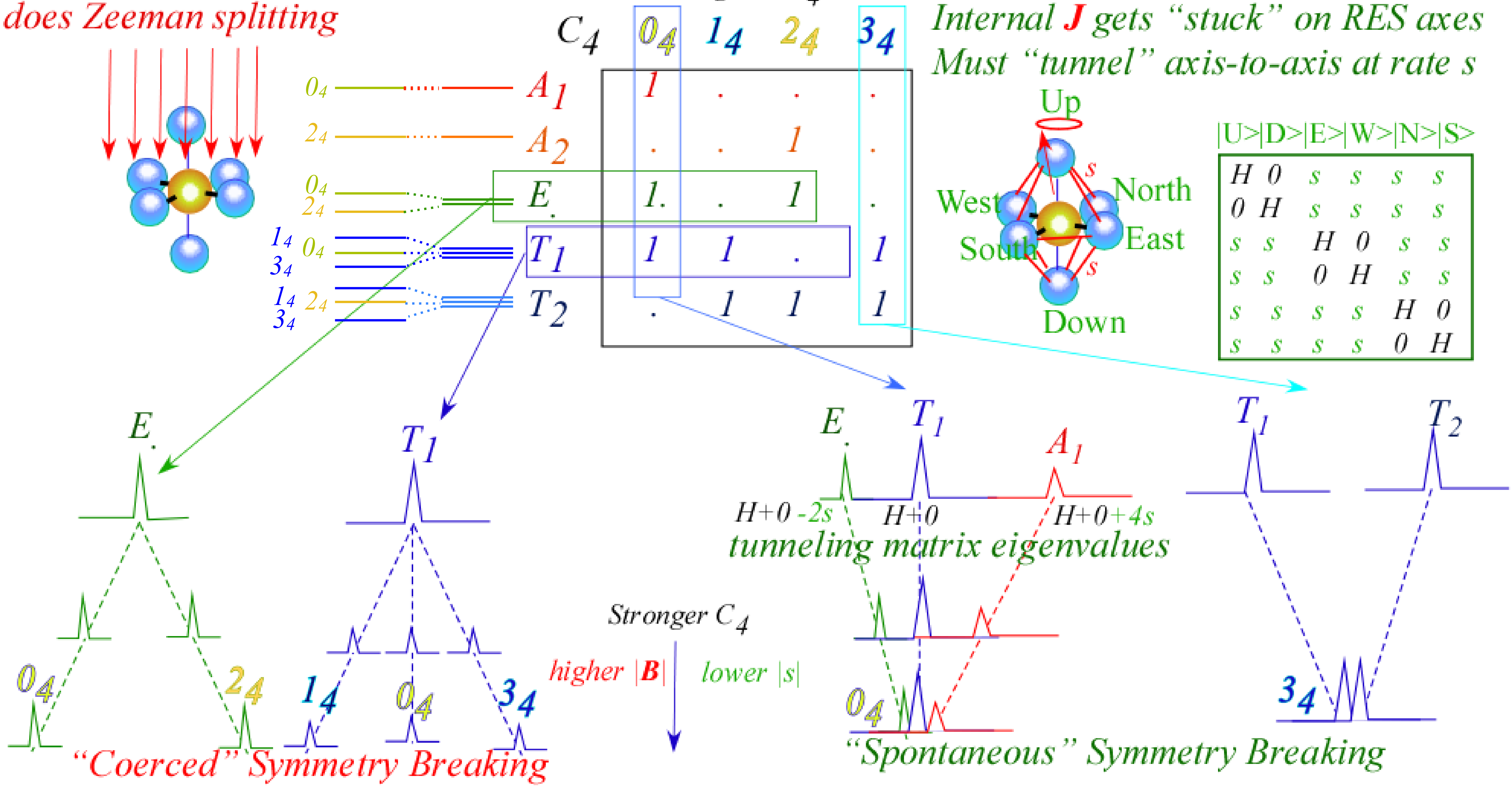

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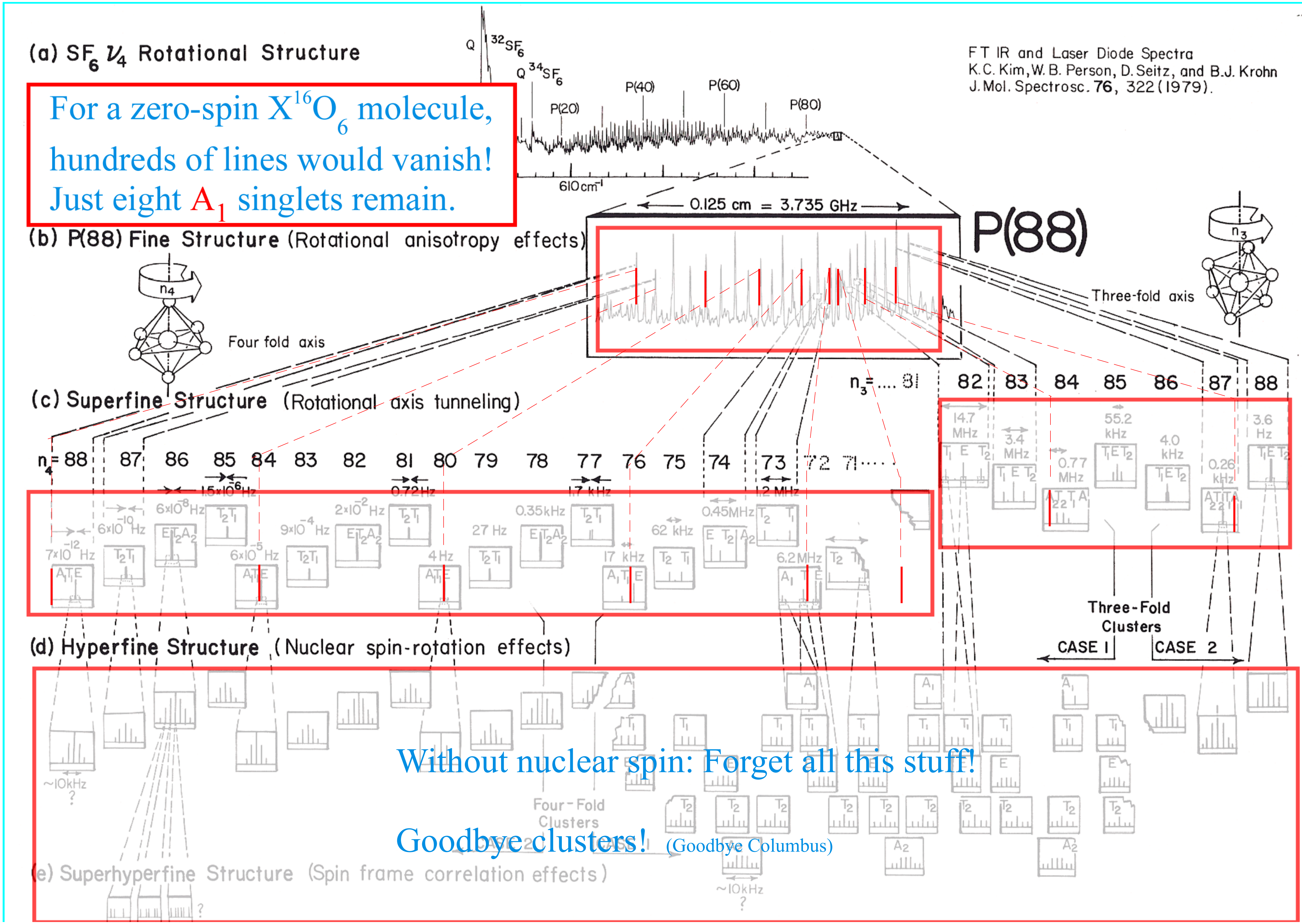
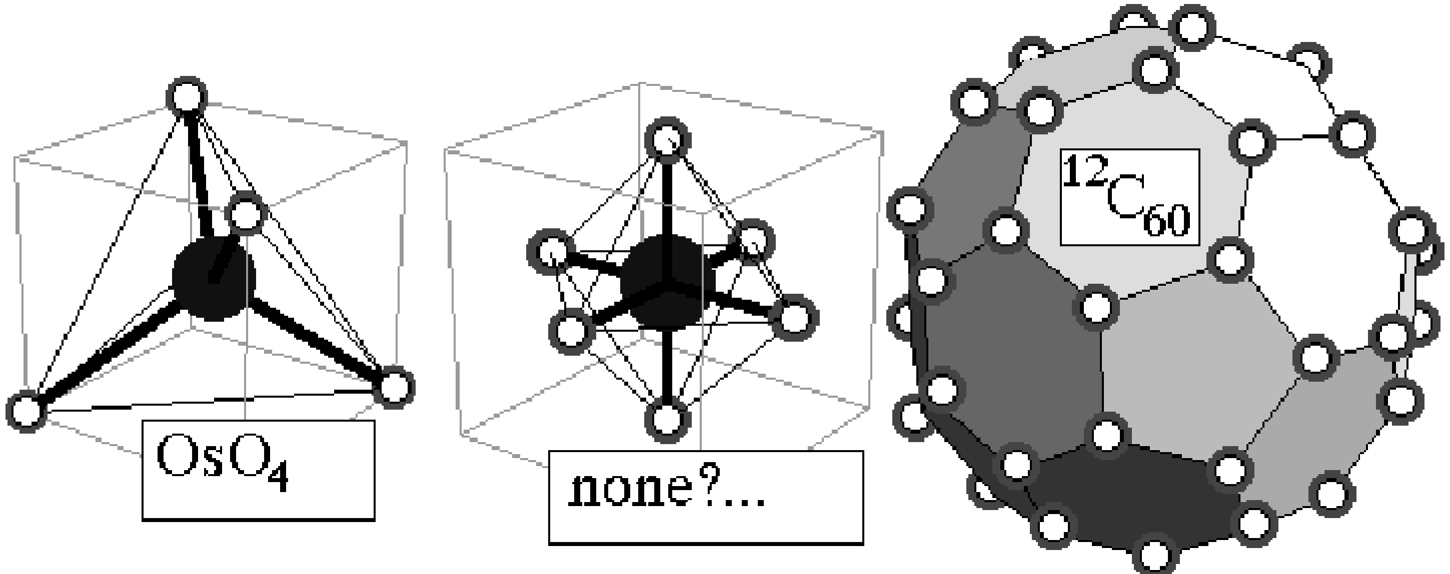


(a) SF6 ν4 Rotational Structure
For a zero-spin X16O6 molecule, hundreds of lines would vanish! Just eight A1 singlets remain.
FT IR and Laser Diode Spectra
K.C. Kim, W.B. Person, D. Seitz, and B.J. Krohn
J. Mol. Spectrosc. 76, 322 (1979).
Q 32SF6
Q 34SF6
P(20)
P(40)
P(60)
P(80)
610 cm-1
0.125 cm = 3.735 GHz
P(88)
(b) P(88) Fine Structure (Rotational anisotropy effects)
Four fold axis
Three-fold axis
(c) Superfine Structure (Rotational axis tunneling)
n3 = ... 81 82 83 84 85 86 87 88
n4 = 88 87 86 85 84 83 82 81 80 79 78 77 76 75 74 73 72 71 ....
Three-Fold Clusters
CASE 1
CASE 2
(d) Hyperfine Structure (Nuclear spin-rotation effects)
Without nuclear spin: Forget all this stuff!
Four-Fold Clusters
Goodbye clusters! (Goodbye Columbus)
(e) Superhyperfine Structure (Spin frame correlation effects)

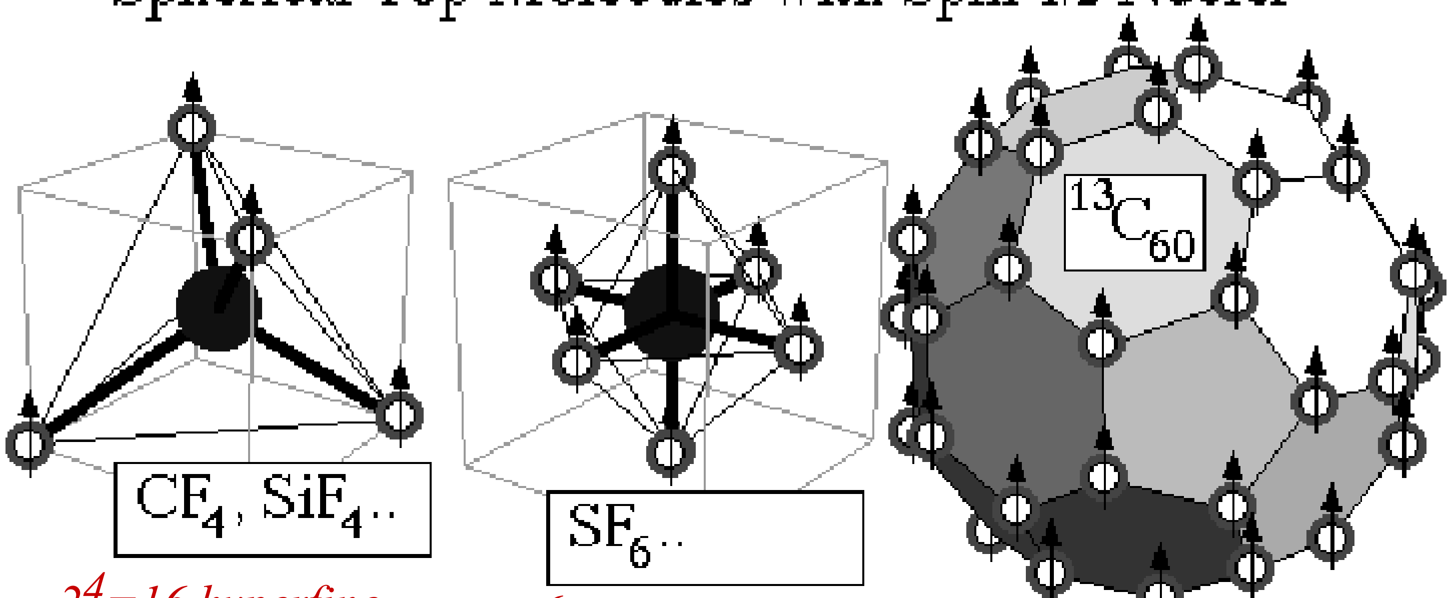
Some examples of Bose Exclusion

Spherical Top Molecules with Spin-0 Nuclei



Only 1 hyperfine state: I=0

Spherical Top Molecules with Spin-1/2 Nuclei



2^4=16 hyperfine states: I=0-2

2^6=64 hyperfine states: I=0-3

2^{60}=1.15x10^{18} hyperfine states: I=0-30

Some examples of Fermi (non) Exclusion

Some examples of Bose Exclusion

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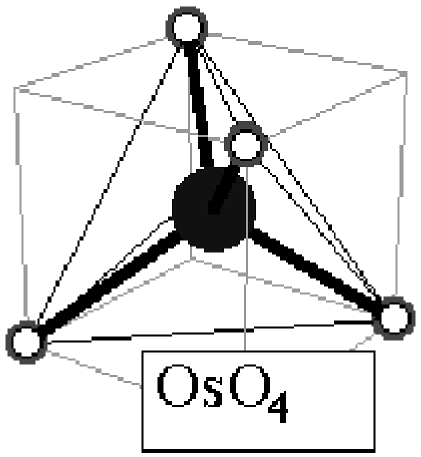

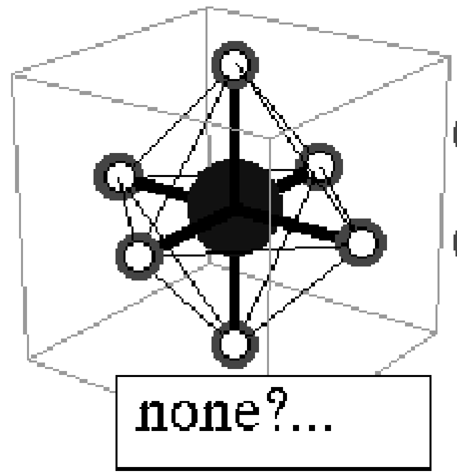

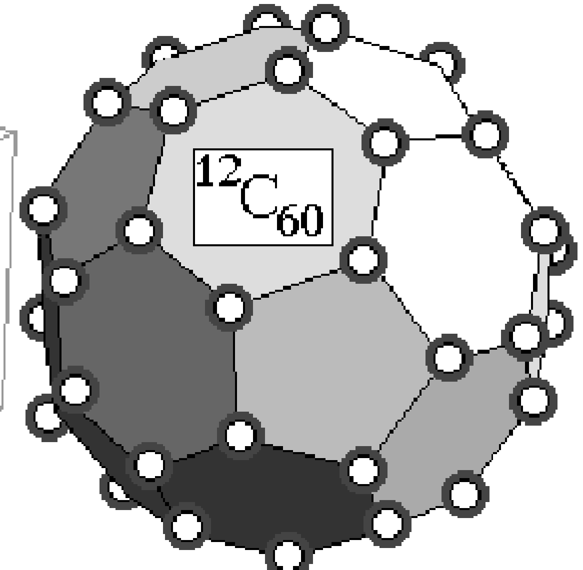

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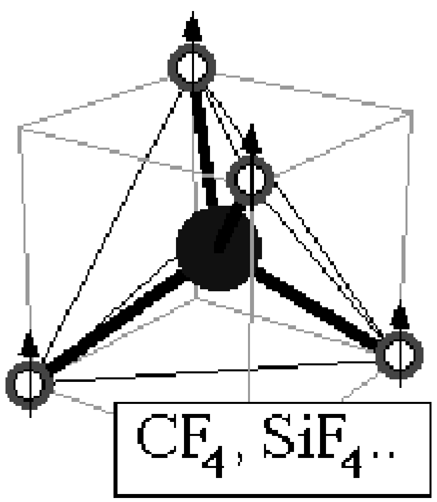

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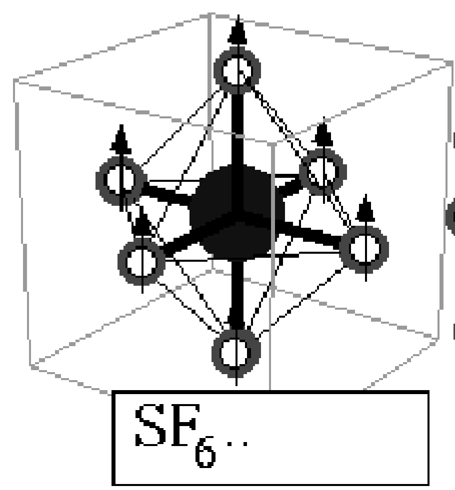

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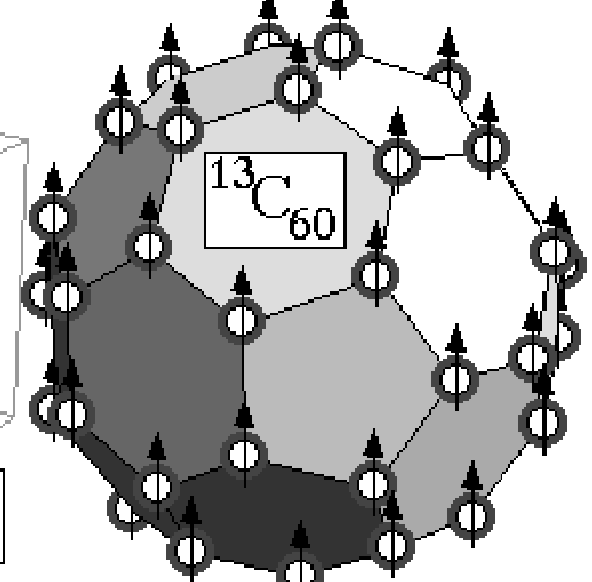


2^{60}=1.15x10^{18} hyperfine states: I=0-30

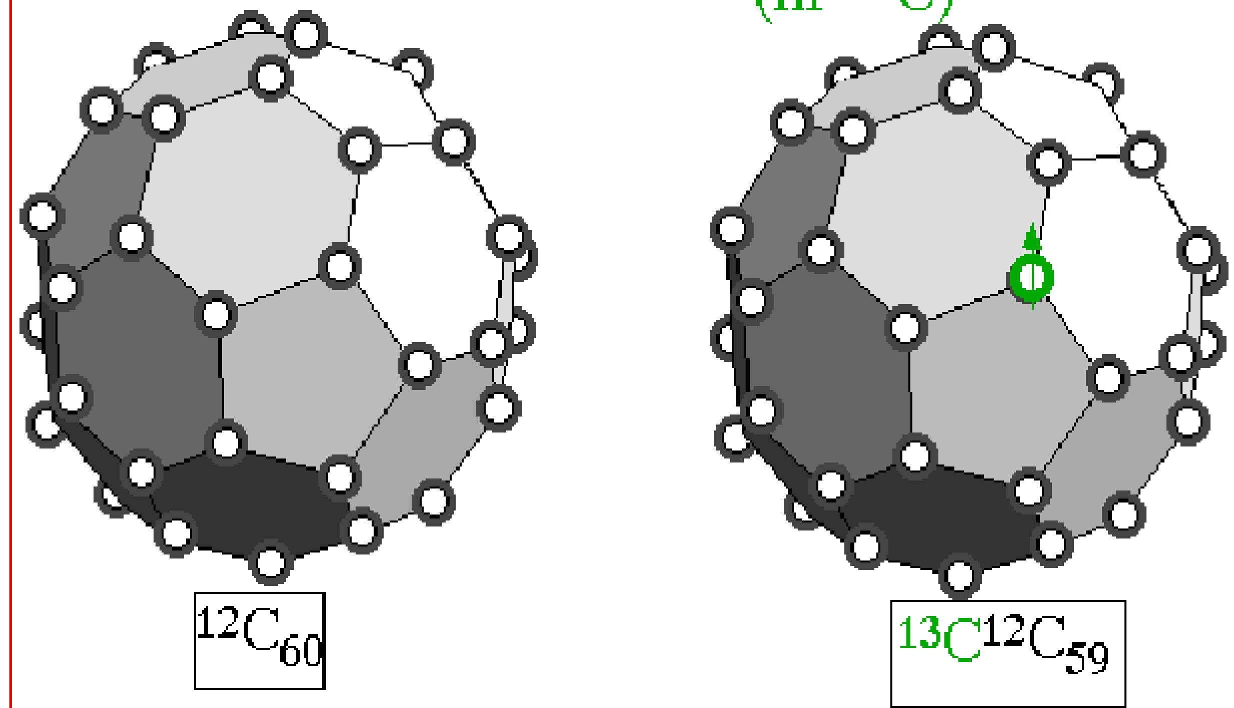
Some examples of Fermi (non) Exclusion

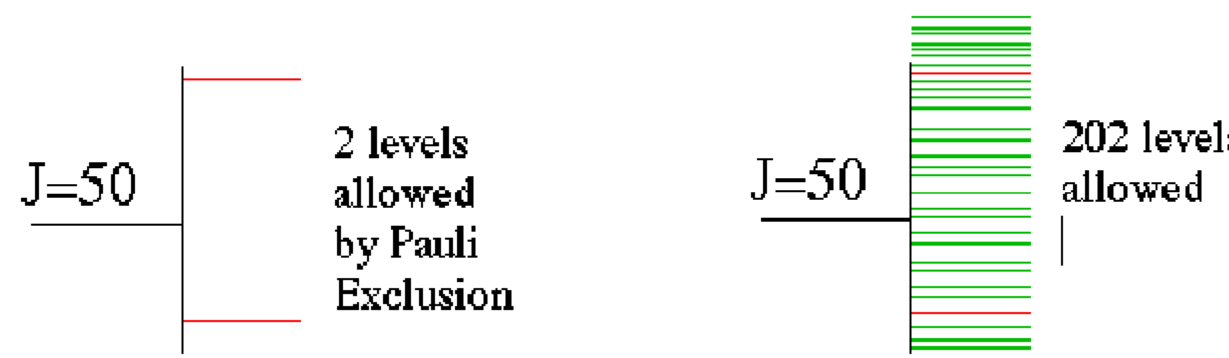
$^{12}C_{60}$ is the "Achilles Molecule" (Felled by one neutron)

Example of extreme symmetry exclusion

... (and partial recovery)

Y_h Symmetry reduced to C_v by a single neutron (in ^{13}C)





Question: Where did those 200 levels go?

Better Question: Where did those 1.15 octillion levels go?

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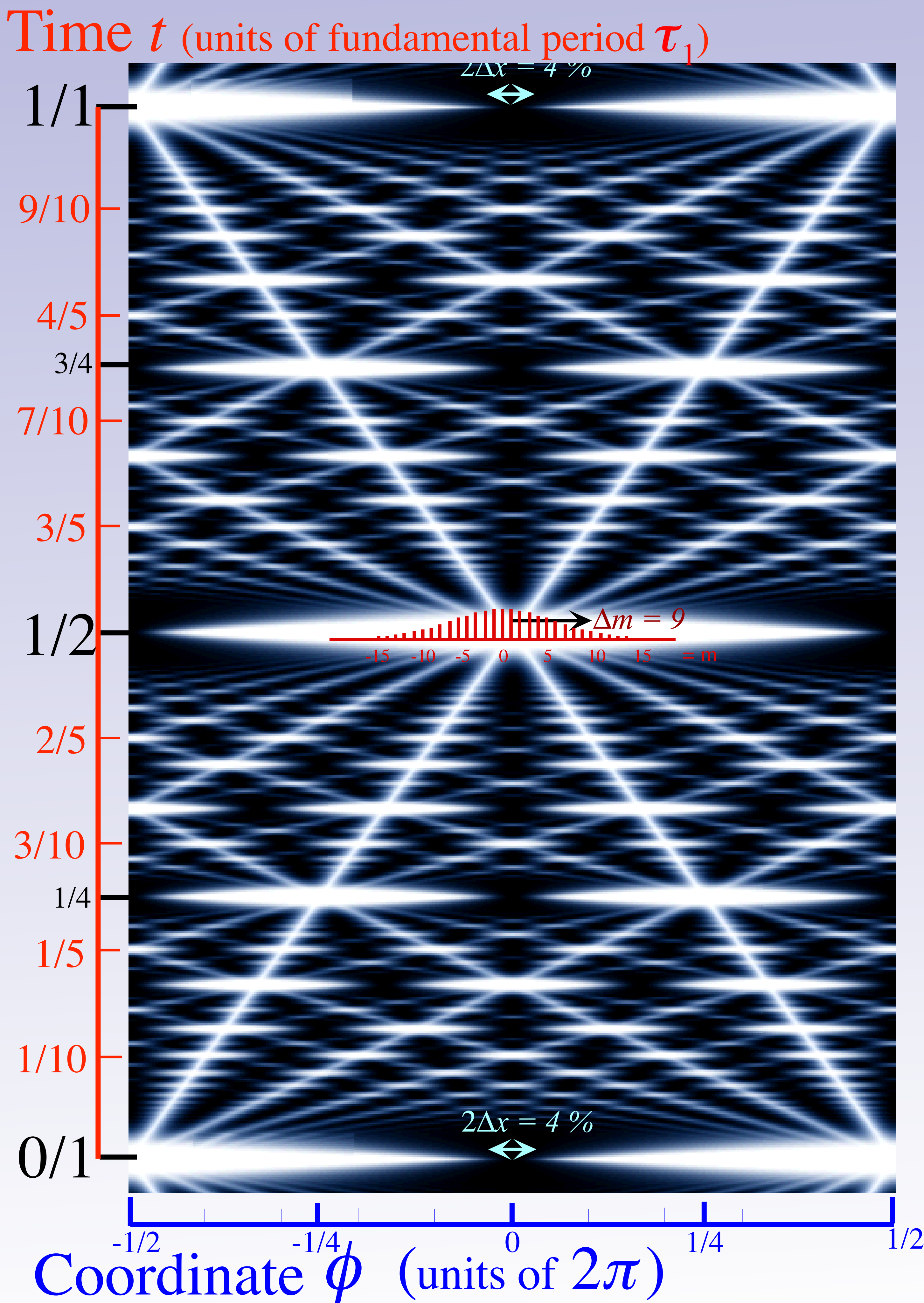
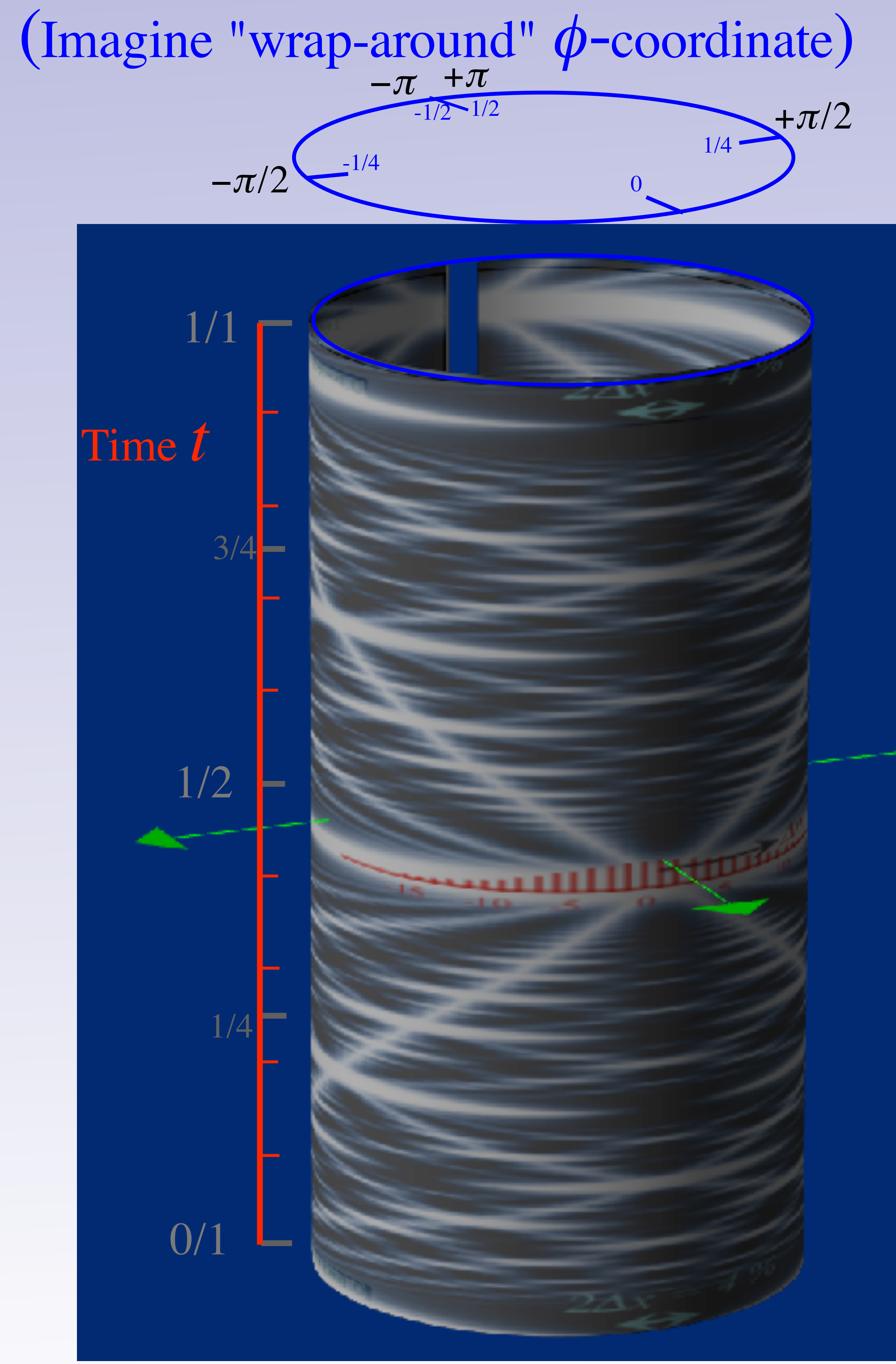




[*Harter, J. Mol. Spec. 210, 166-182 (2001)*]

Web simulation *or:* http://www.uark.edu/ua/modphys/markup/WaveItWeb.html

http://www.uark.edu/ua/modphys/markup/WaveItWeb.html?scenario=Quantum%20Carpet

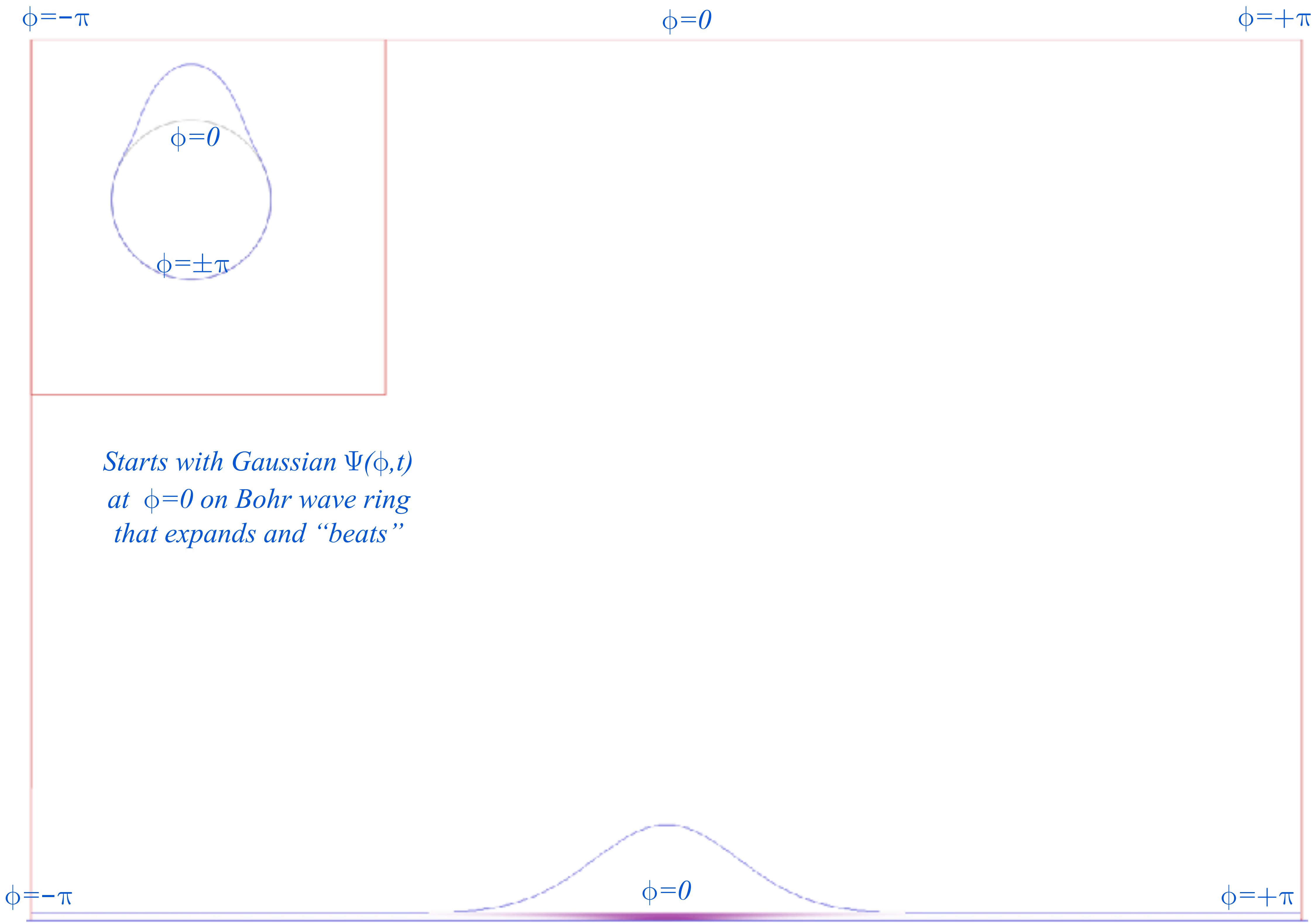
Also, try testing *or else* markup

Click here....

Launch | Fourier Control | Scenarios | Pause | Set T=0 | Zero Amps | T-Scale= 1

..then here....

Twelve (n=12) oscillator
Twelve (n=12) oscillator
Twelve (n=12) oscillator
C(n) Character Table
Quantum Carpet



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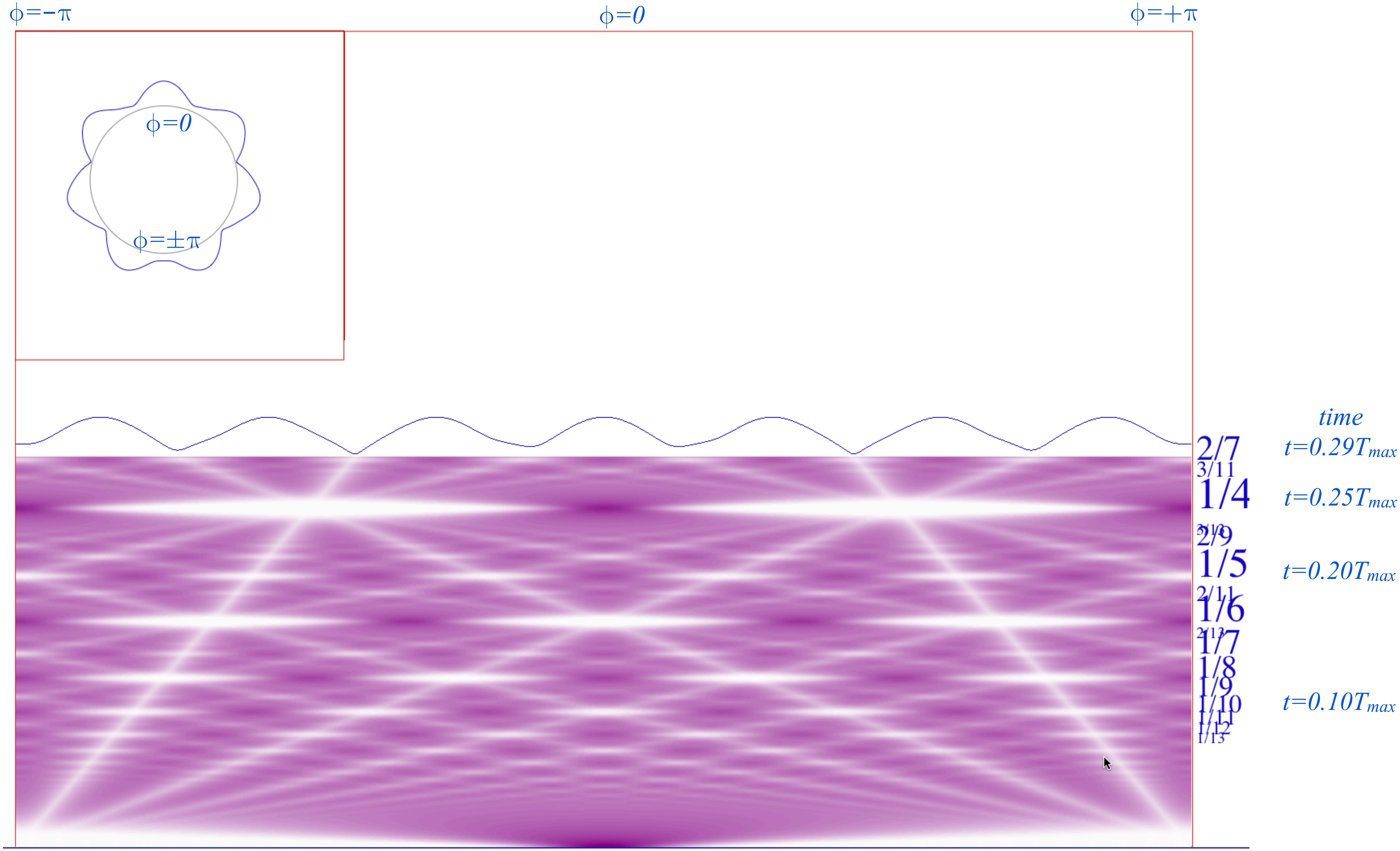
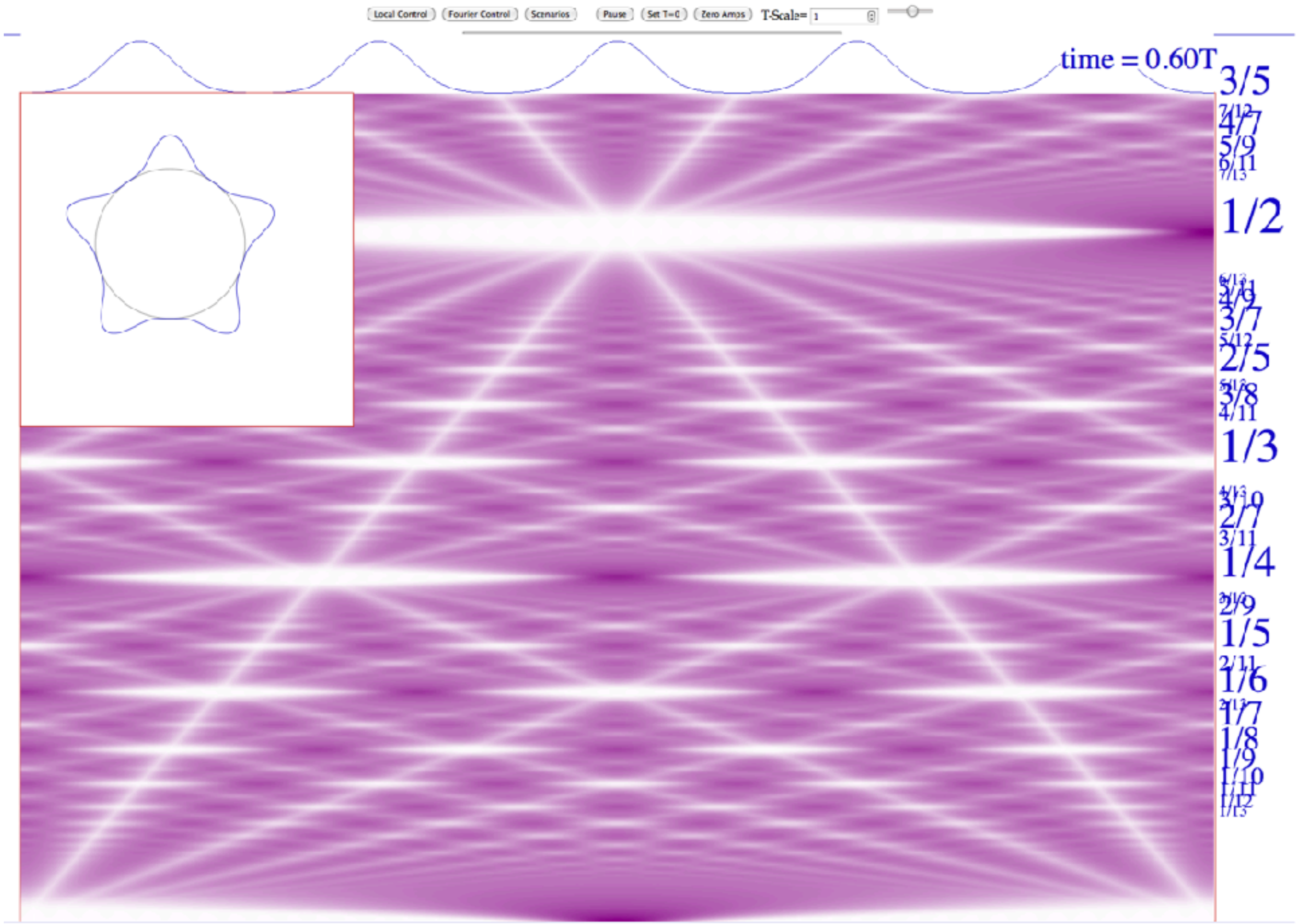
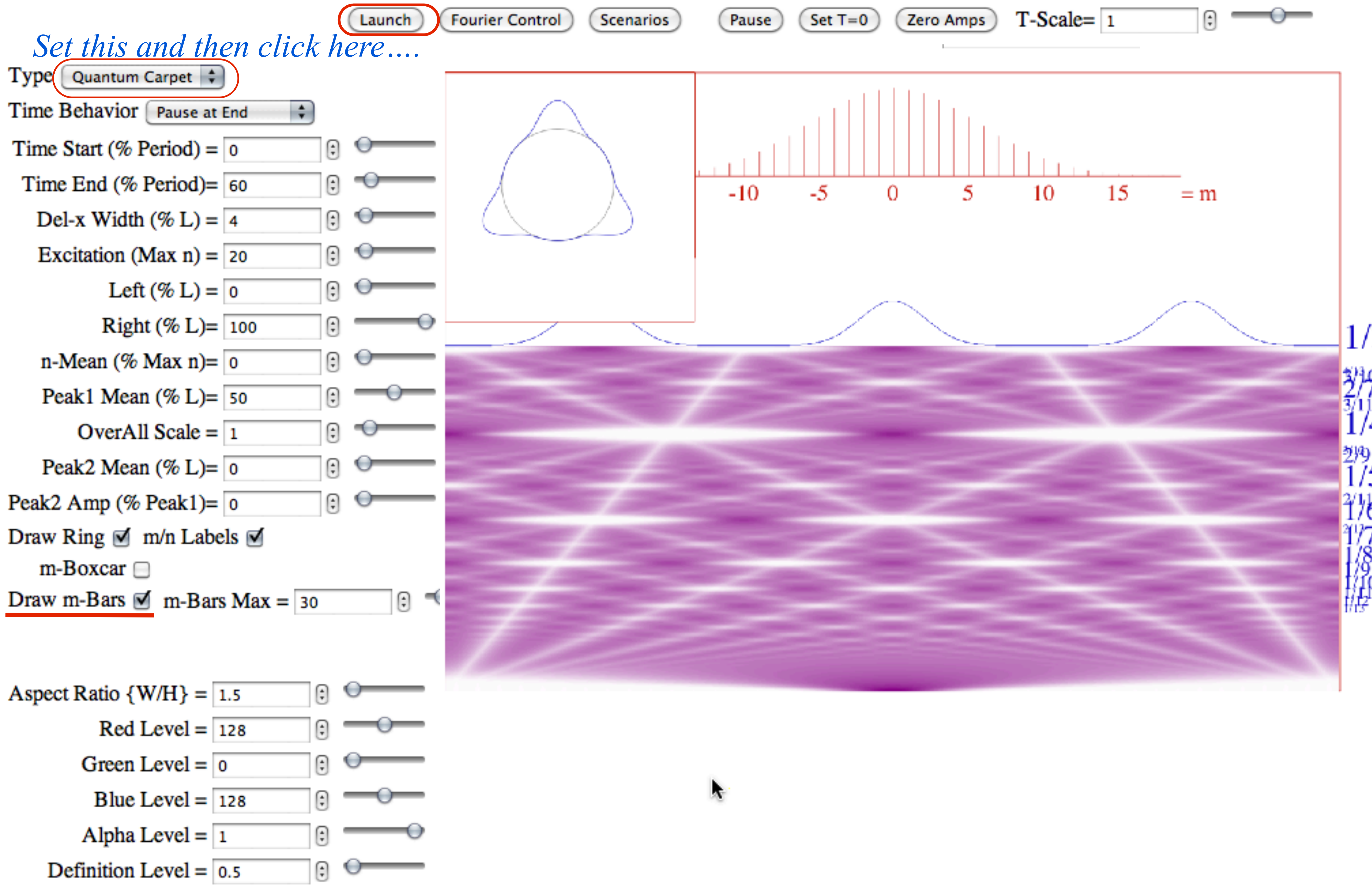
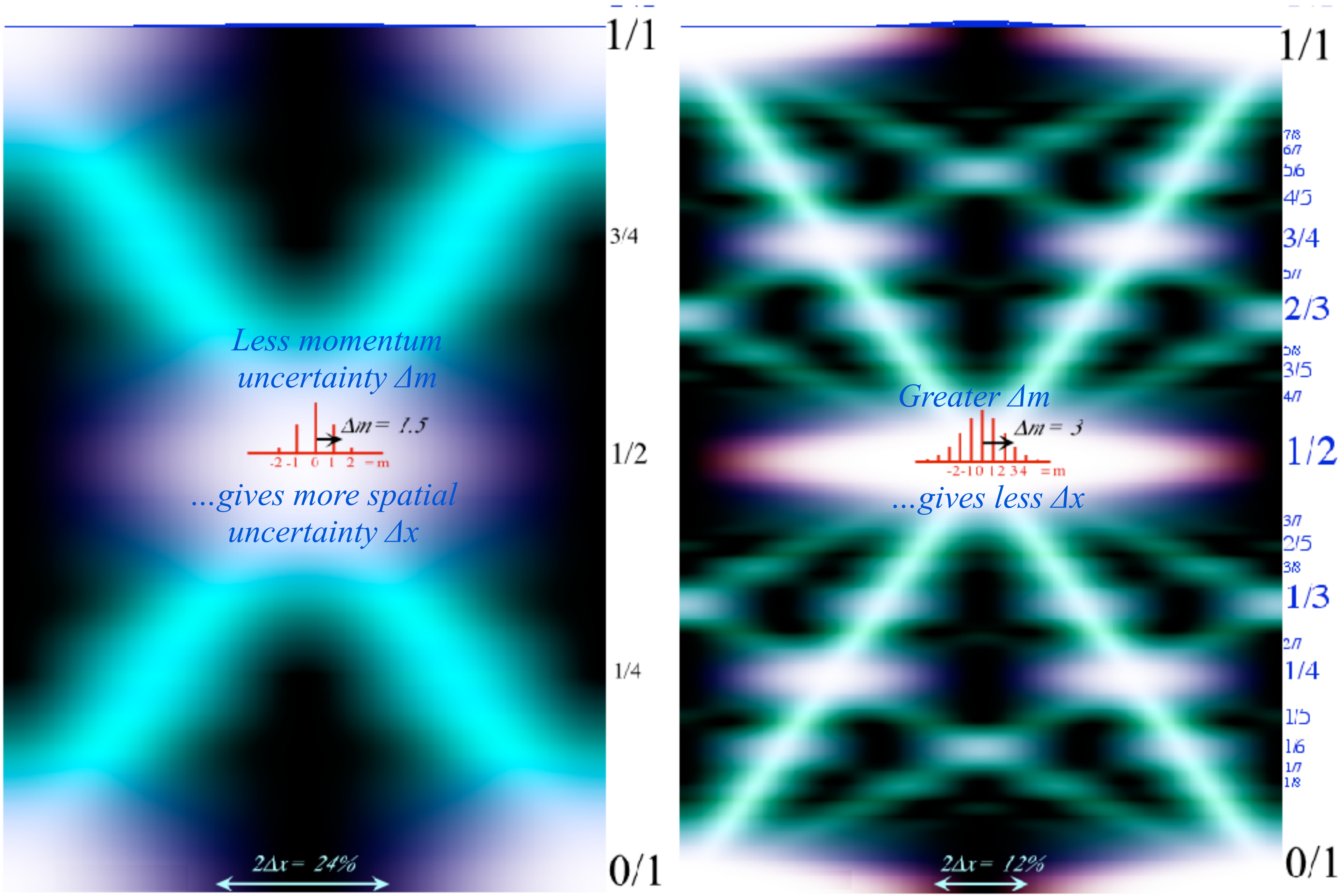
..then here.... Quantum Carpet

simulation




Local Control
Fourier Control
Scenarios
Pause
Set T=0
Zero Amps
T-Scale=
time = 0.60T
3/5
4/7
5/9
1/2
3/7
2/5
3/8
4/11
1/3
2/7
3/11
1/4
2/9
1/5
2/11
1/6
1/7
1/8
1/9
1/10


Launch
Fourier Control
Scenarios
Pause
Set T=0
Zero Amps
T-Scale= 1
Set this and then click here....
Type Quantum Carpet
Time Behavior Pause at End
Time Start (% Period) = 0
Time End (% Period)= 60
Del-x Width (% L) = 4
Excitation (Max n) = 20
Left (% L) = 0
Right (% L)= 100
n-Mean (% Max n)= 0
Peak1 Mean (% L)= 50
OverAll Scale = 1
Peak2 Mean (% L)= 0
Peak2 Amp (% Peak1)= 0
Draw Ring
m/n Labels
m-Boxcar
Draw m-Bars
m-Bars Max = 30
Aspect Ratio {W/H} = 1.5
Red Level = 128
Green Level = 0
Blue Level = 128
Alpha Level = 1
Definition Level = 0.5
-10
-5
0
5
10
15
= m
1/3
2/7
1/4
1/5
1/6
1/7
1/8
1/9
1/10

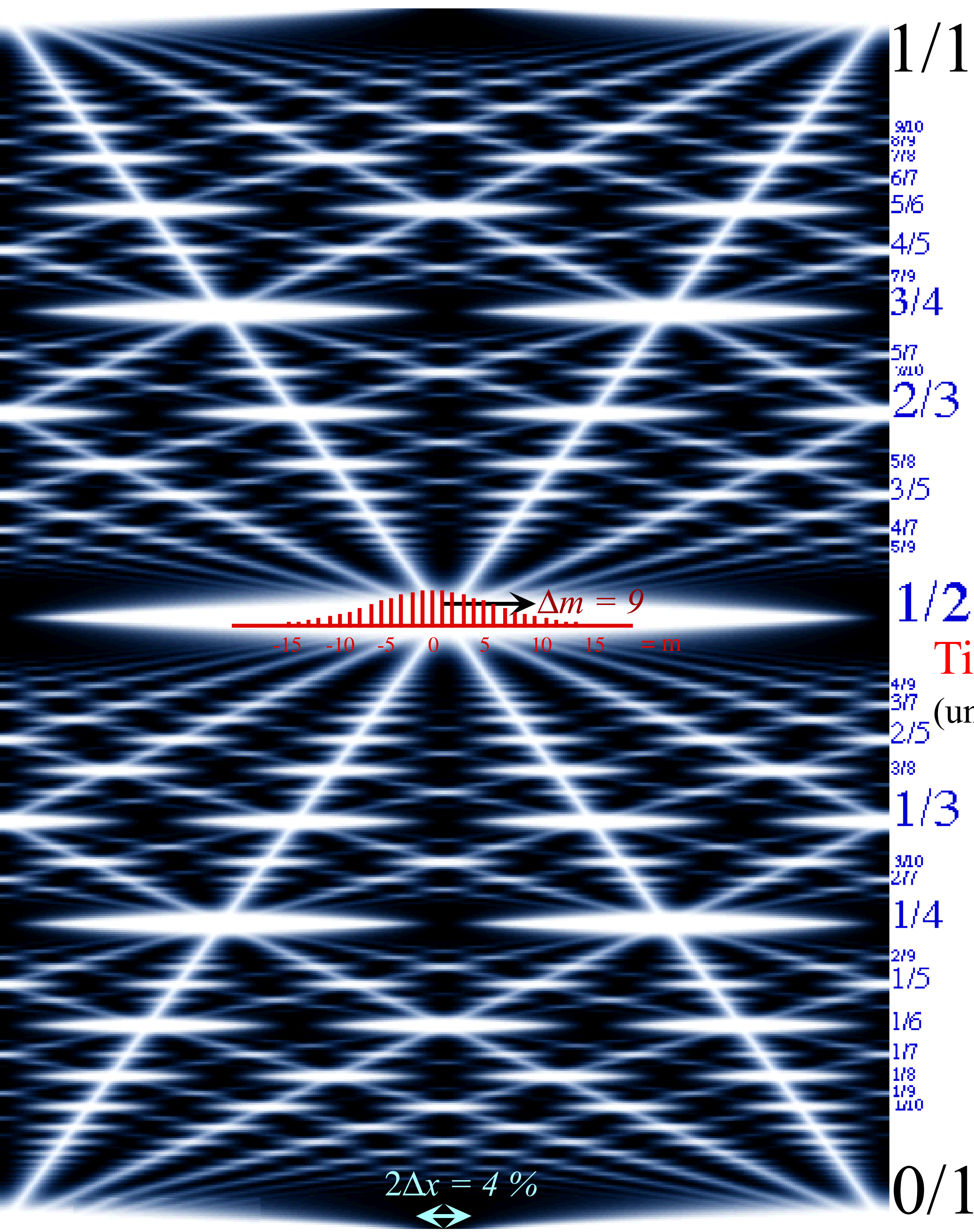


[Harter, J. Mol. Spec. 210, 166-182 (2001)]

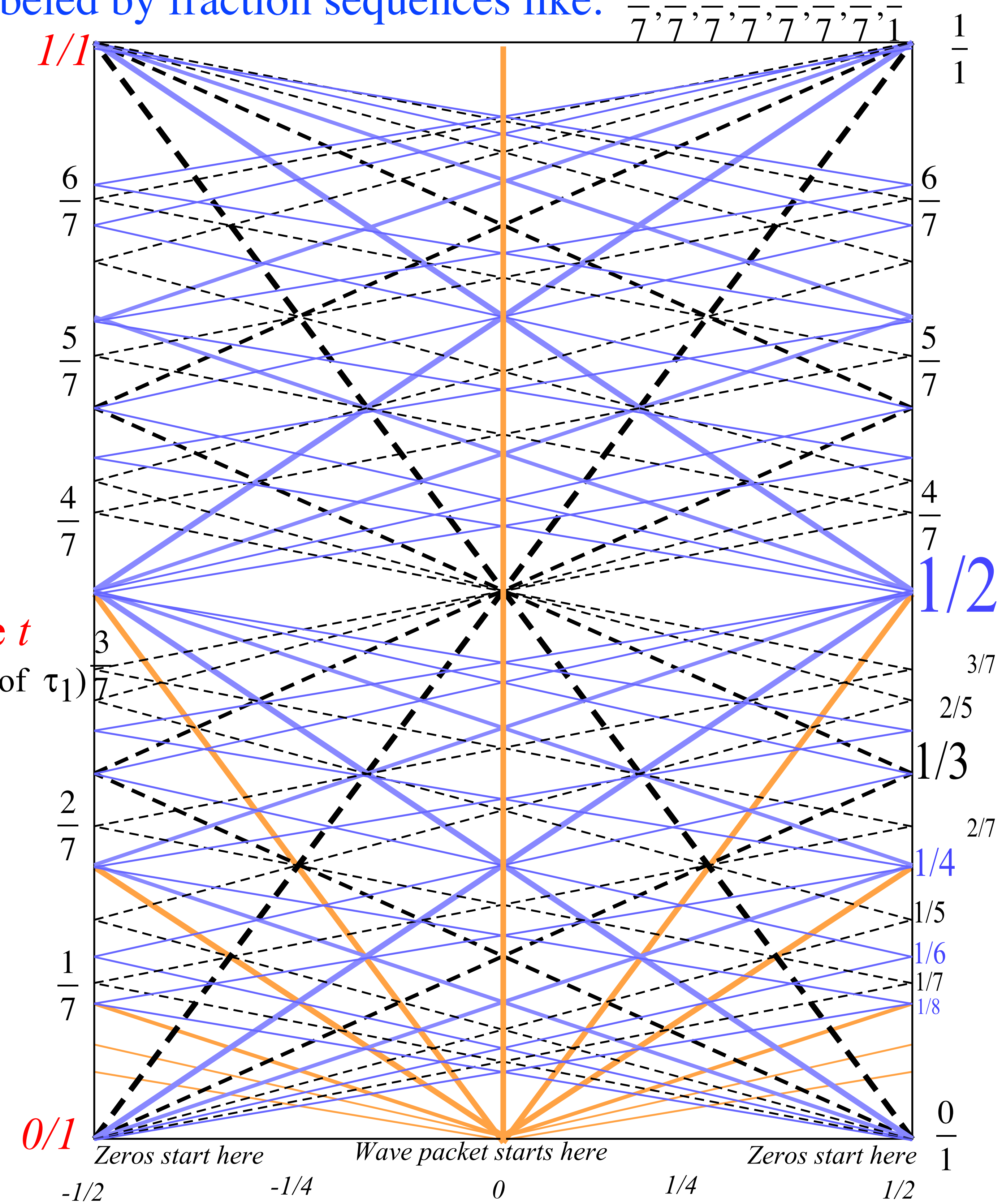
N-level-system and revival-beat wave dynamics

(9 or10-levels (0, ±1, ±2, ±3, ±4,..., ±9, ±10, ±11...) excited)

Zeros (clearly) and "particle-packets" (faintly) have paths labeled by fraction sequences like: $\frac{0}{7}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \frac{1}{1}$



Time *t* (units of τ_1)



Coordinate ϕ (units of 2π)

[*Harter, J. Mol. Spec. 210, 166-182 (2001)*]

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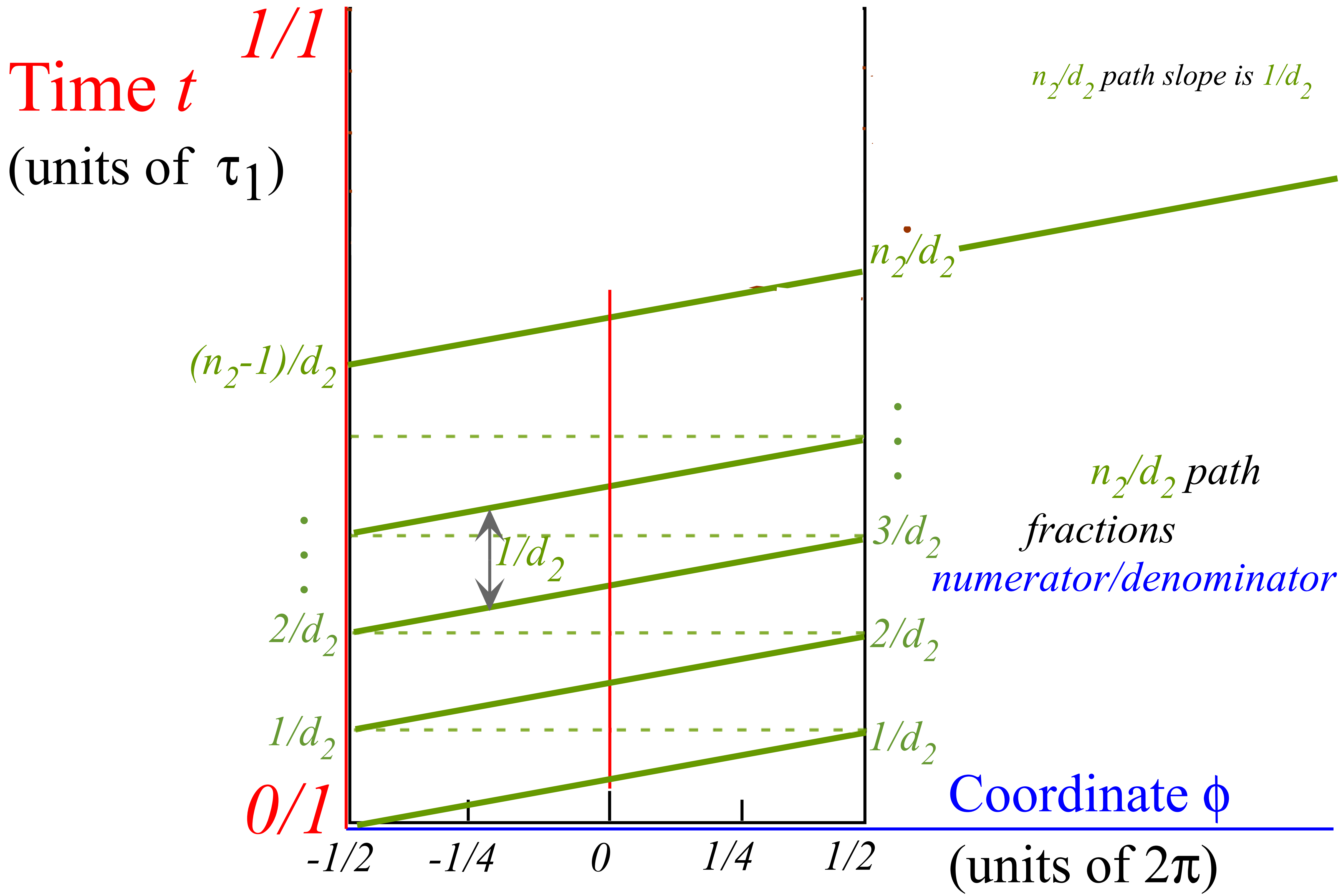
Farey Sum algebra of revival-beat wave dynamics

Label by *numerators N* and *denominators D* of rational fractions *N/D*



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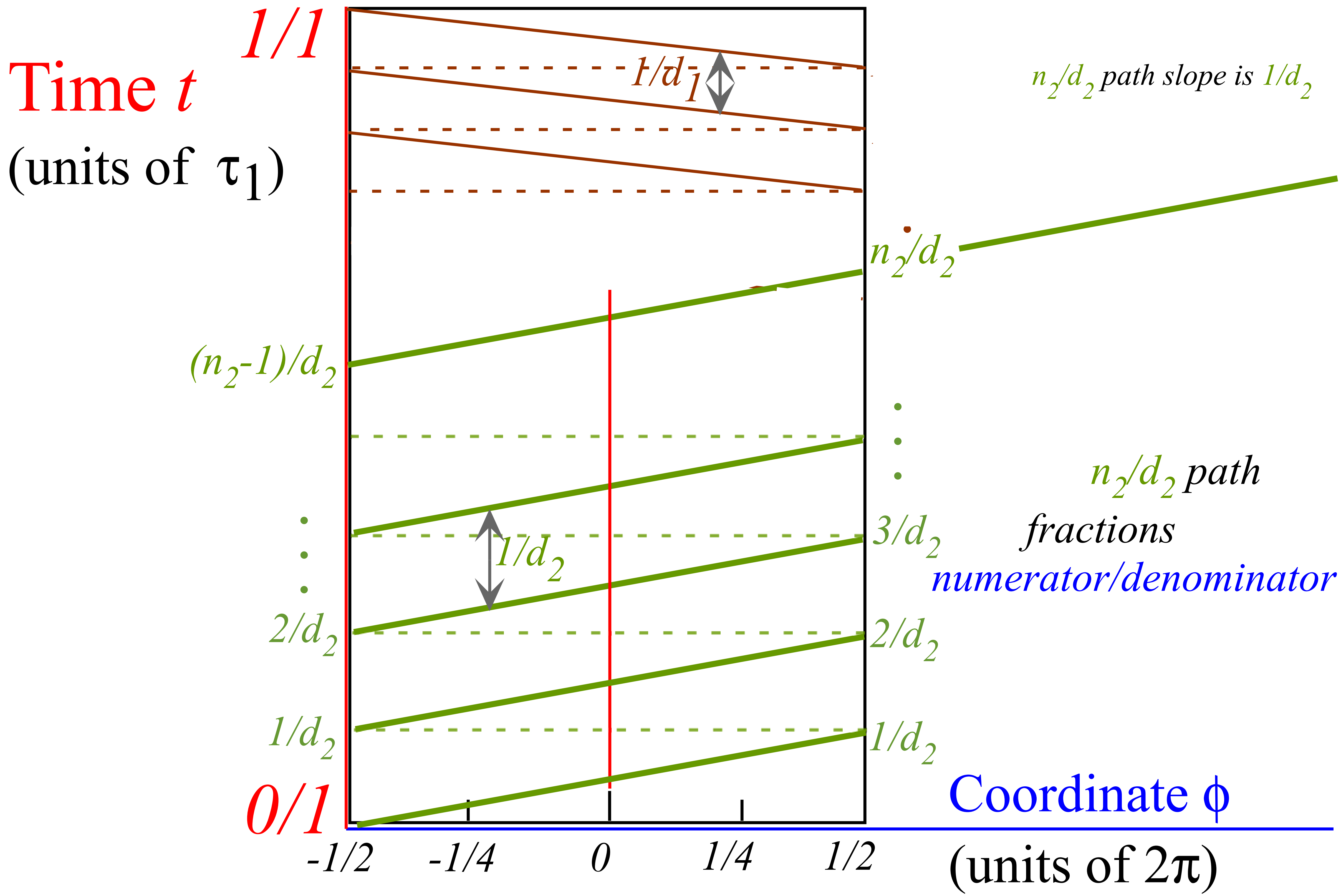

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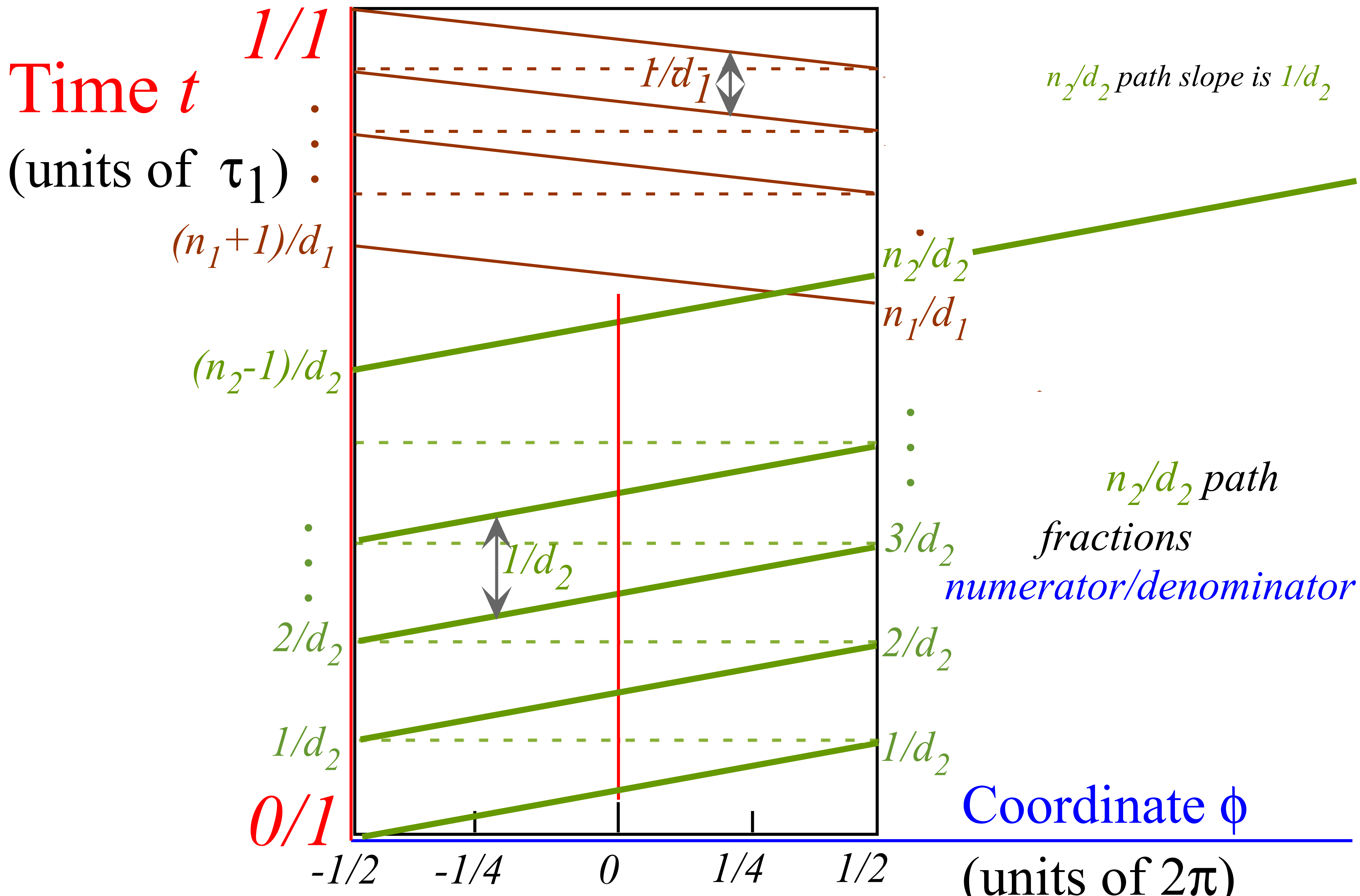

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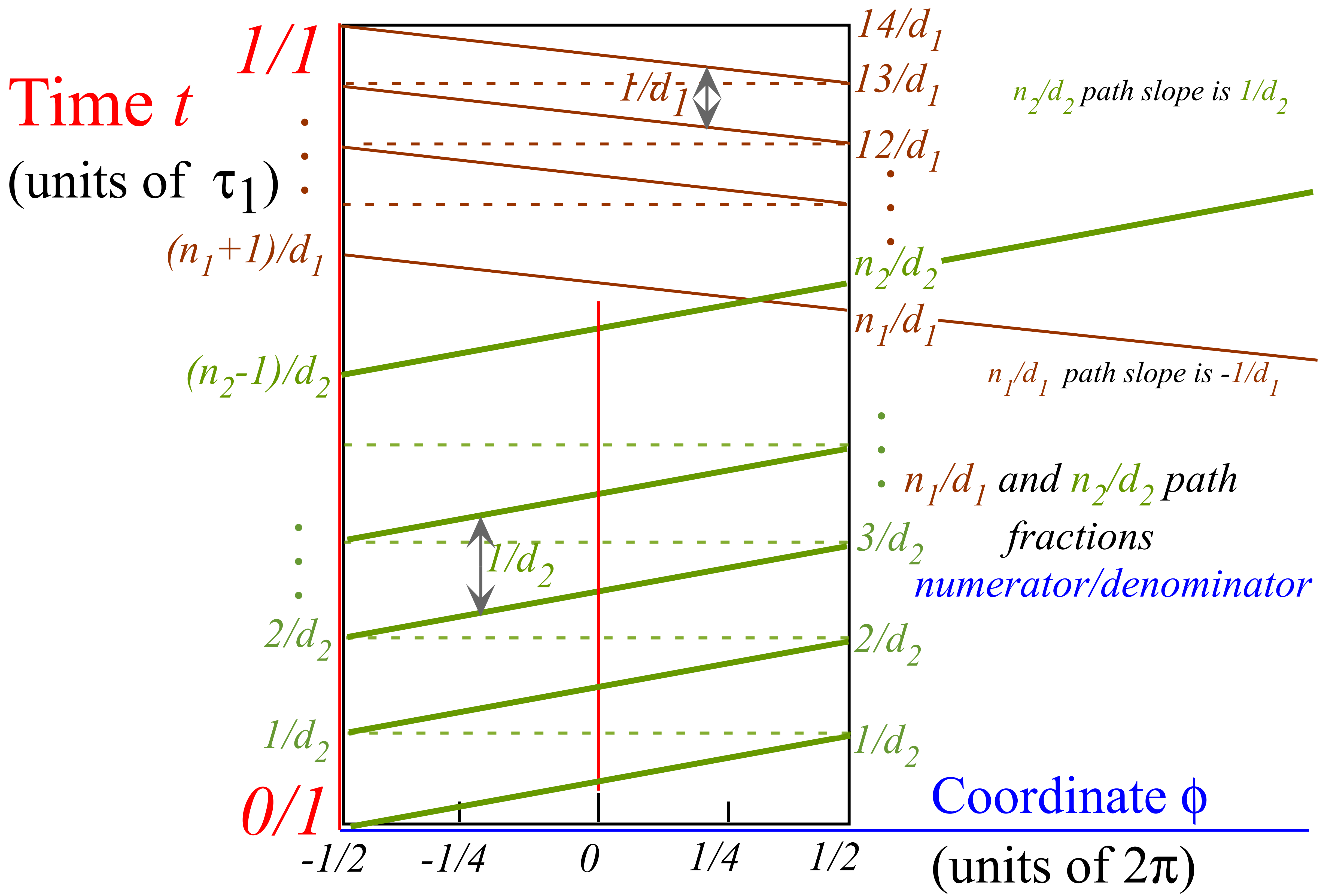

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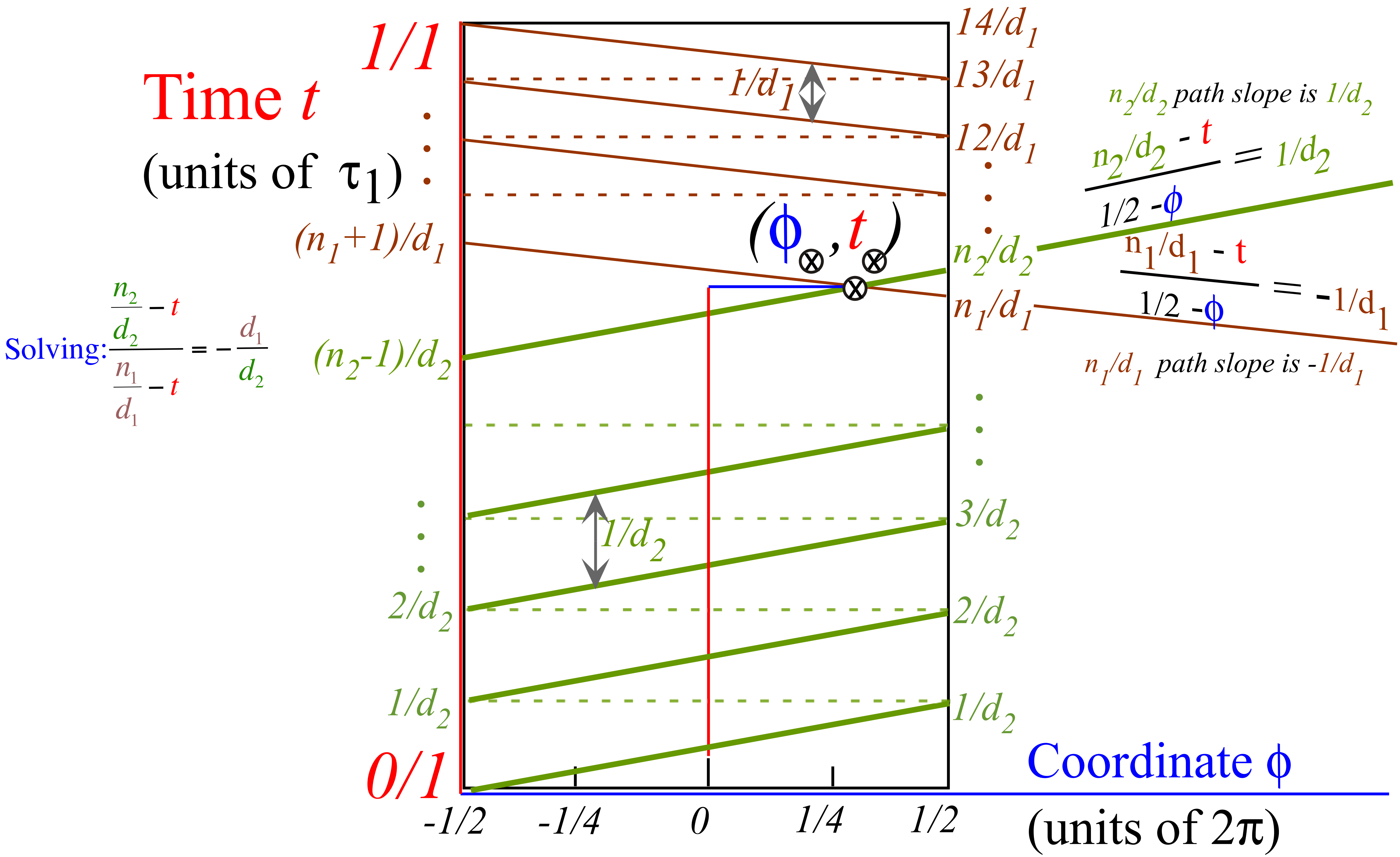

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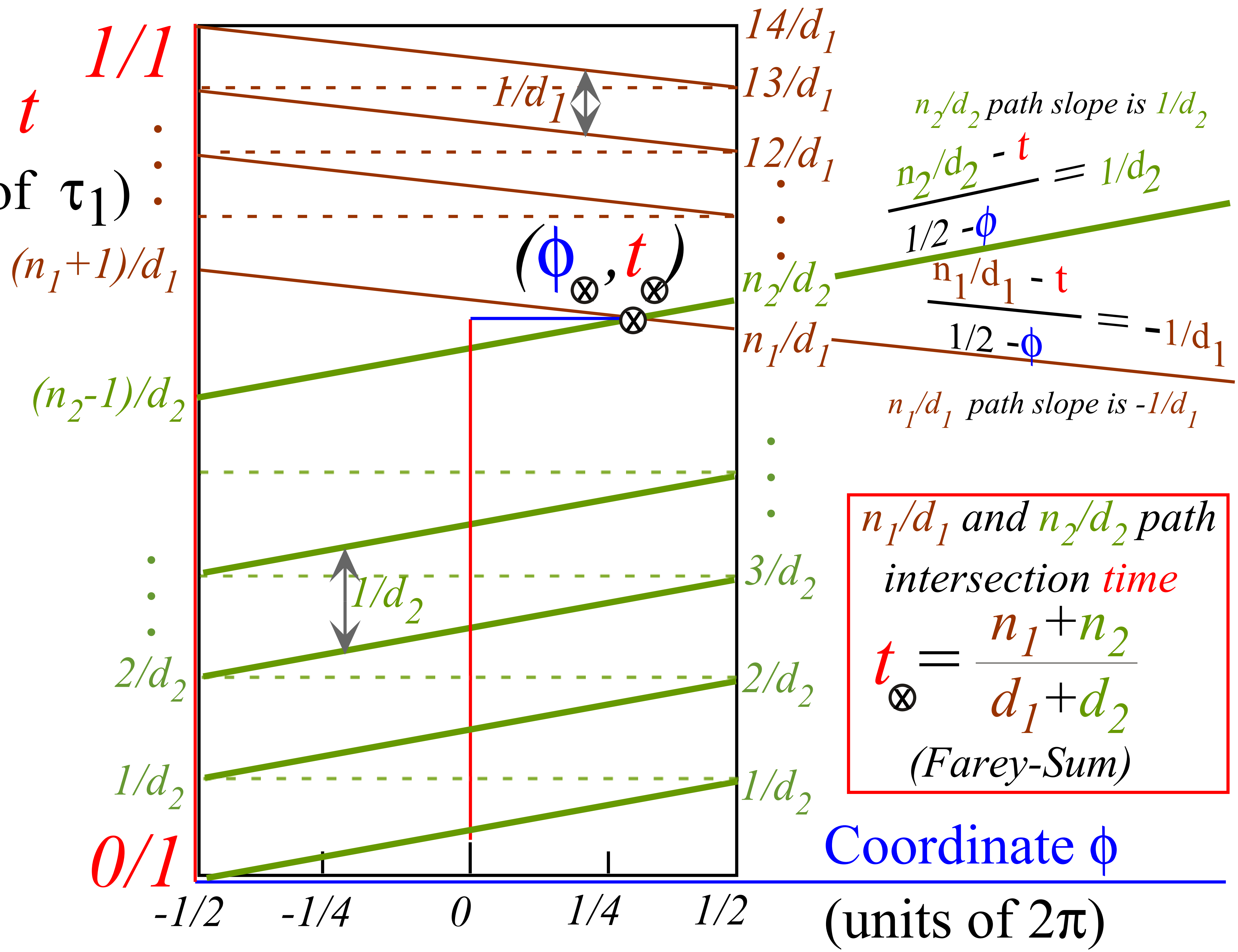
Solving: $$\frac{\frac{n_2}{d_2}-t}{\frac{n_1}{d_1}-t}=-\frac{d_1}{d_2}$$

$$\frac{n_2}{d_2}-t=-\frac{d_1}{d_2}\frac{n_1}{d_1}+\frac{d_1}{d_2}t$$

$$=-\frac{n_1}{d_2}+\frac{d_1}{d_2}t$$

$$n_2+n_1=d_2t+d_1t$$



n_1/d_1 and n_2/d_2 path intersection time

$$t_\otimes=\frac{n_1+n_2}{d_1+d_2}$$

(Farey-Sum)

[John Farey, Phil. Mag.(1816)]

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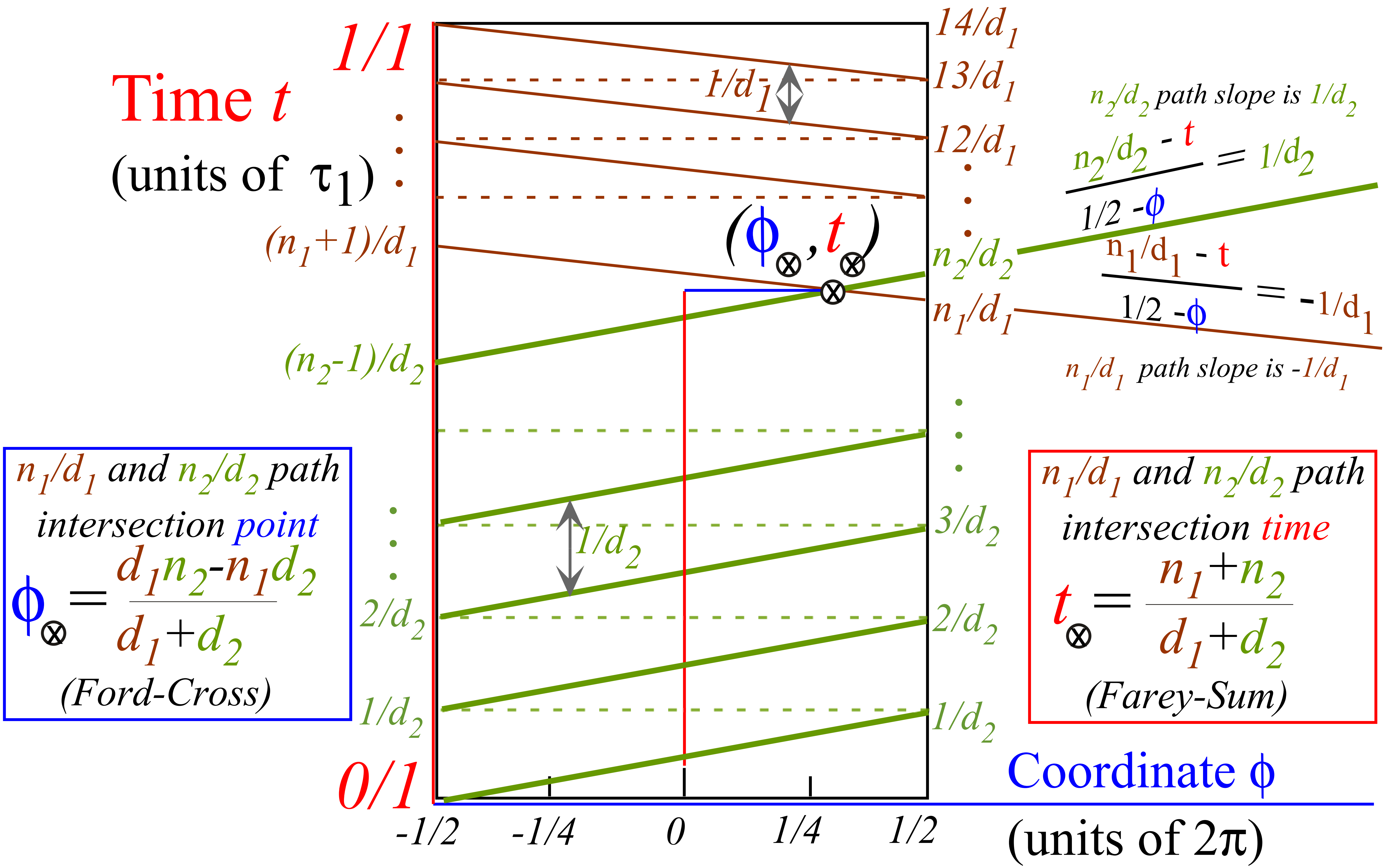

[Lester. R. Ford, Am. Math. Monthly 45,586(1938)]

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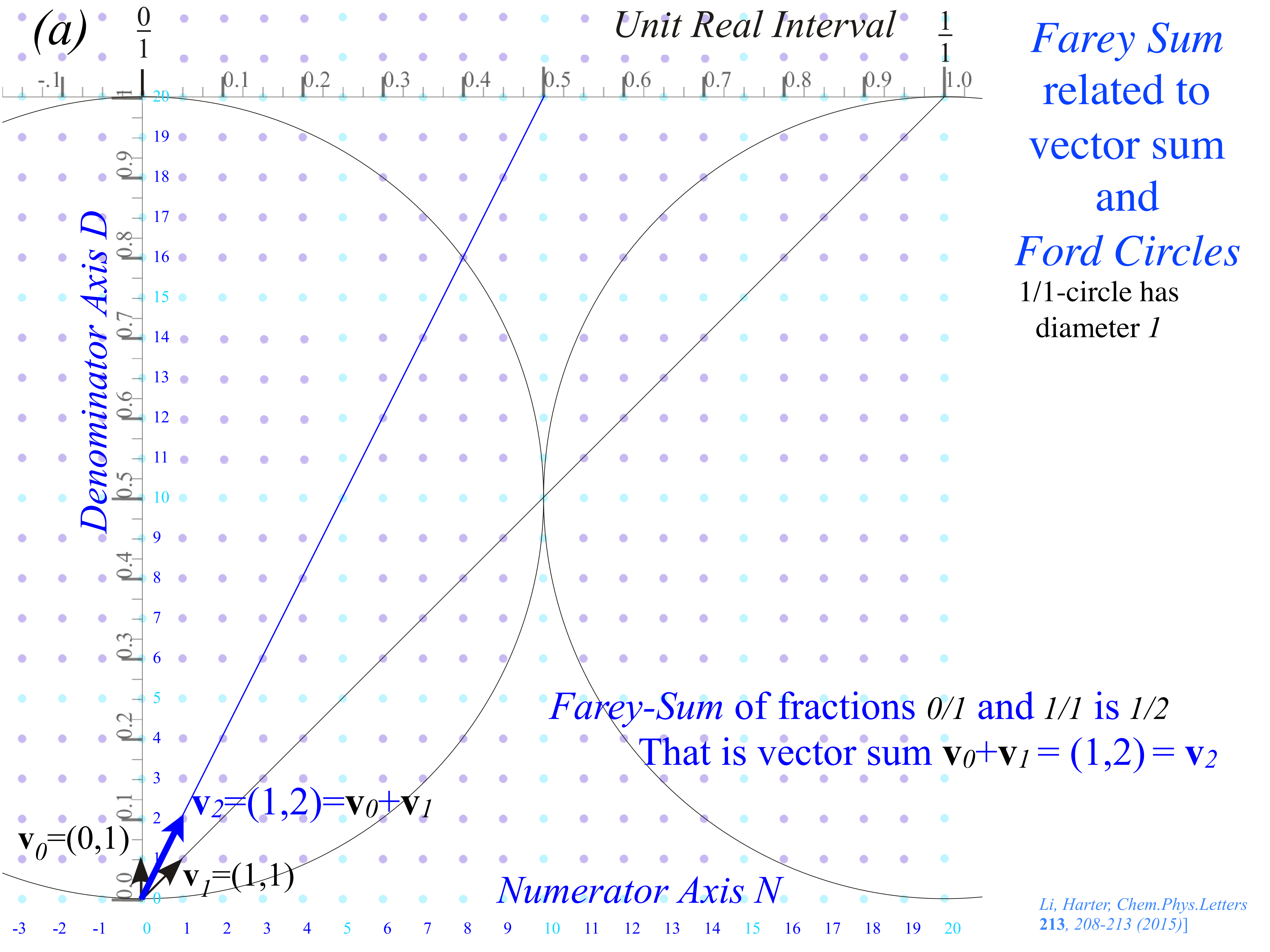
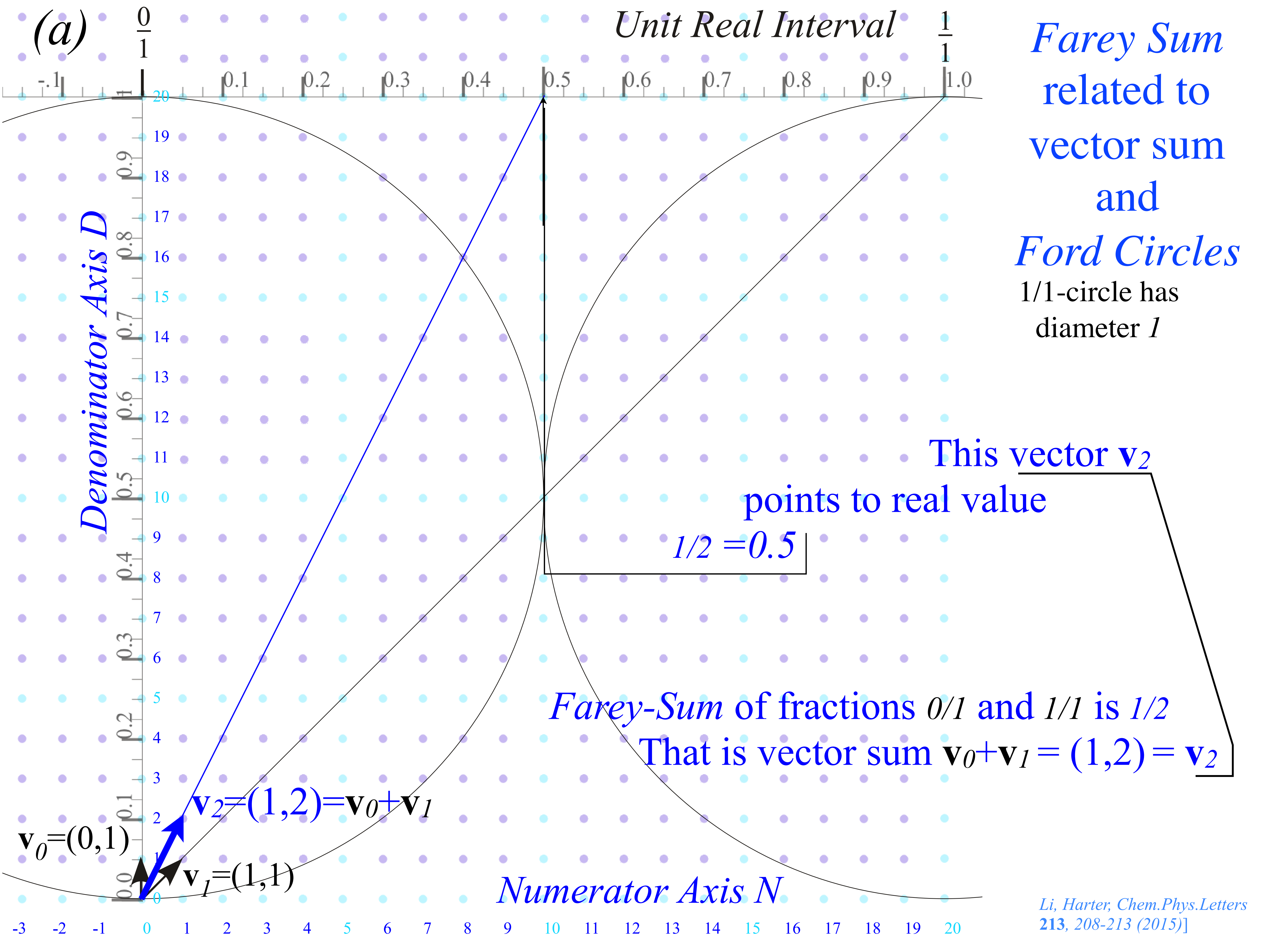


(a)
Unit Real Interval
Farey Sum related to vector sum and Ford Circles
1/1-circle has diameter 1
Denominator Axis D
Numerator Axis N
Farey-Sum of fractions 0/1 and 1/1 is 1/2
That is vector sum v0+v1 = (1,2) = v2
v2=(1,2)=v0+v1
v0=(0,1)
v1=(1,1)
Li, Harter, Chem.Phys.Letters 213, 208-213 (2015)]



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1/1
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1/1-circle has diameter 1
Denominator Axis D
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This vector v2 points to real value 1/2 =0.5
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v2=(1,2)=v0+v1
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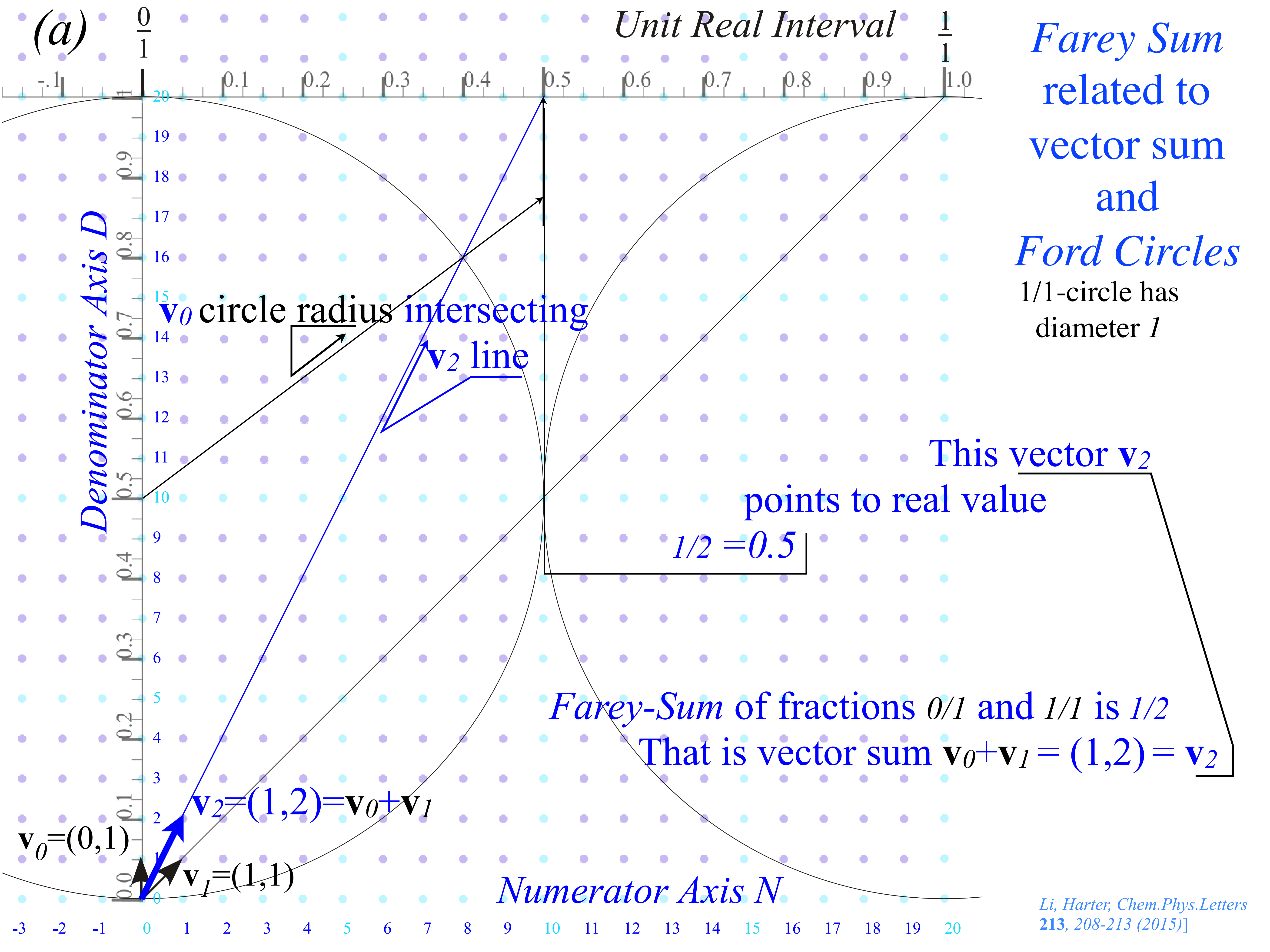

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$\mathbf{v}_0$ circle radius intersecting $\mathbf{v}_2$ line
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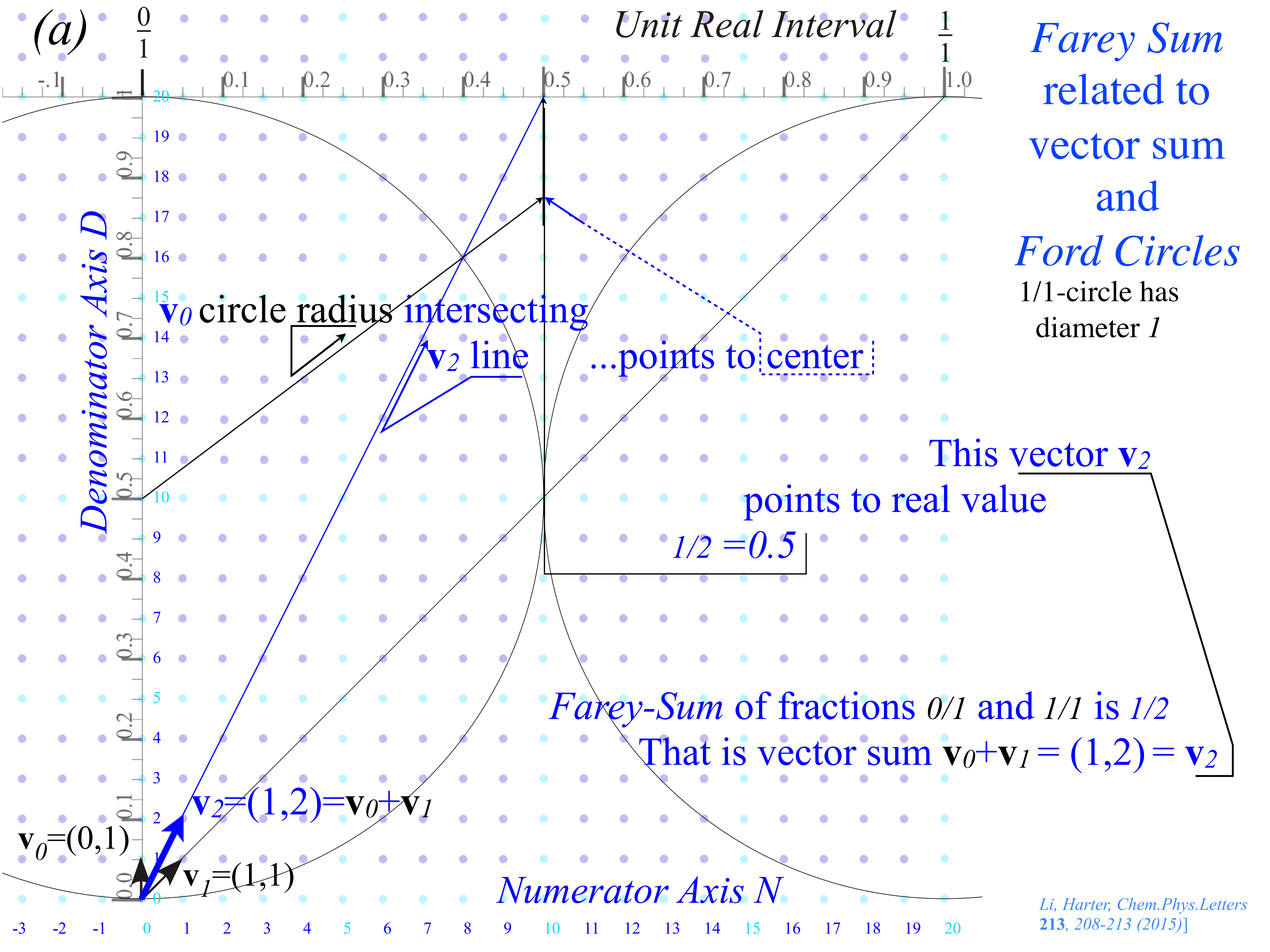

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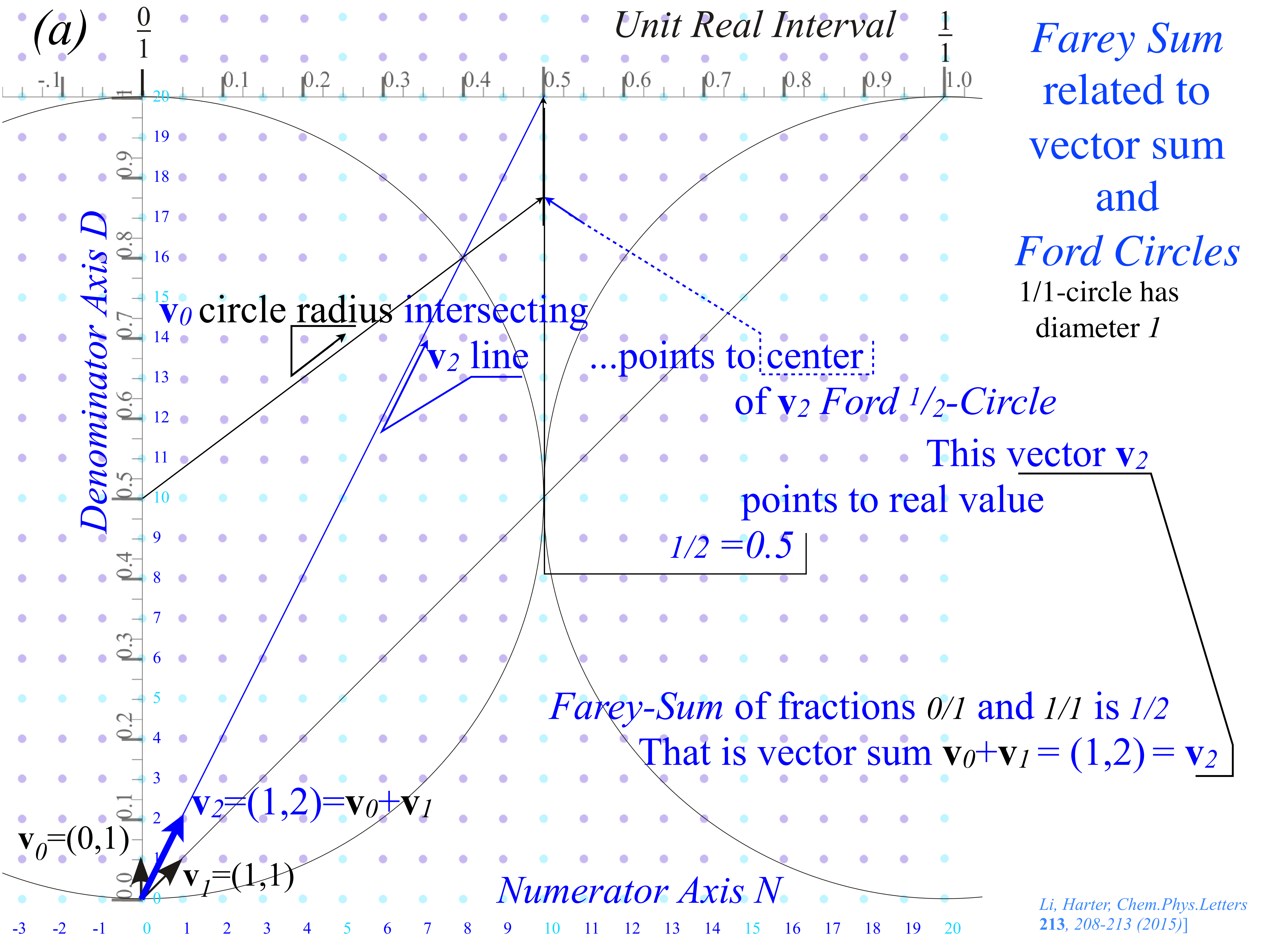

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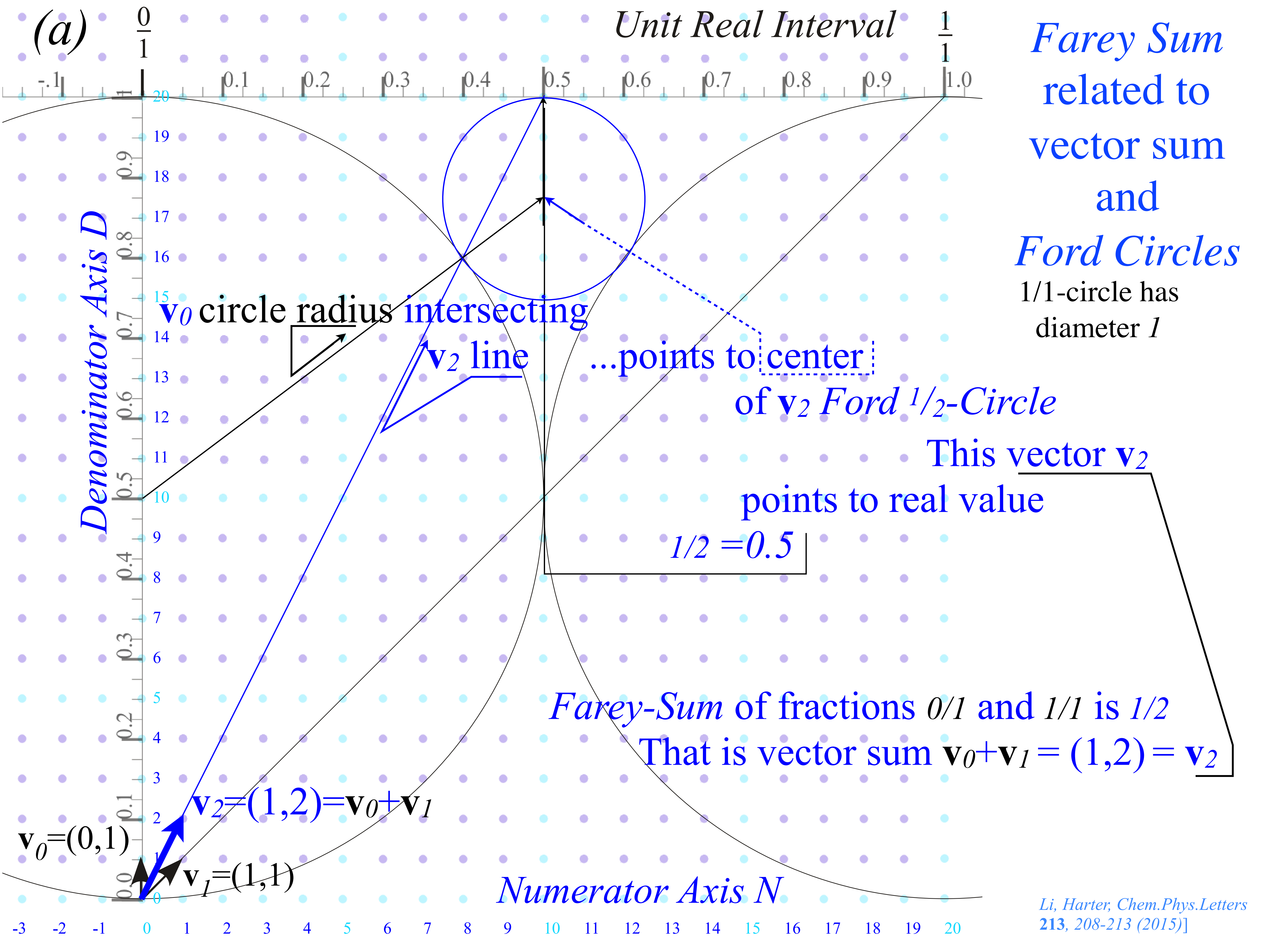

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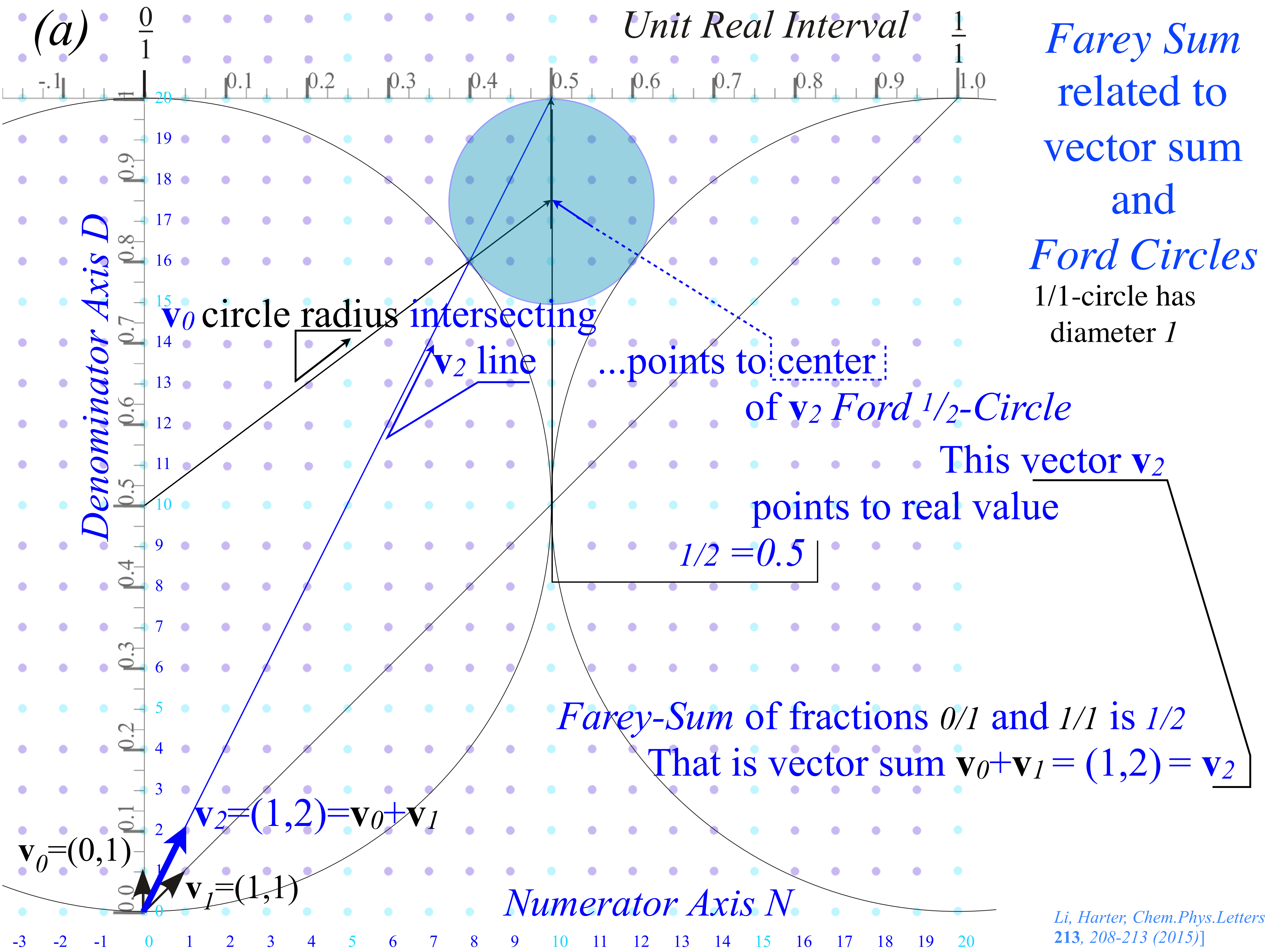
Li, Harter, Chem.Phys.Letters 213, 208-213 (2015)]



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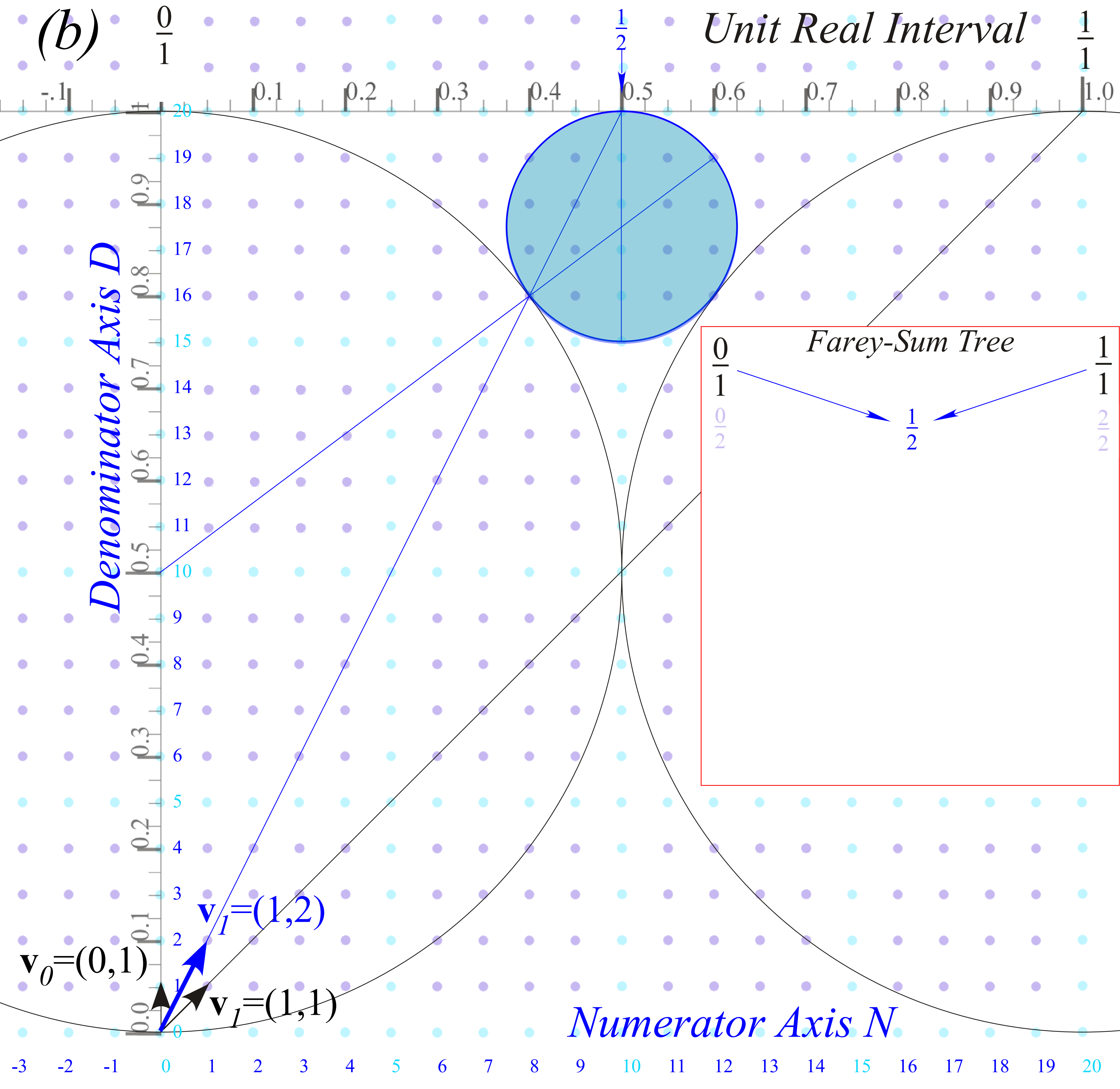
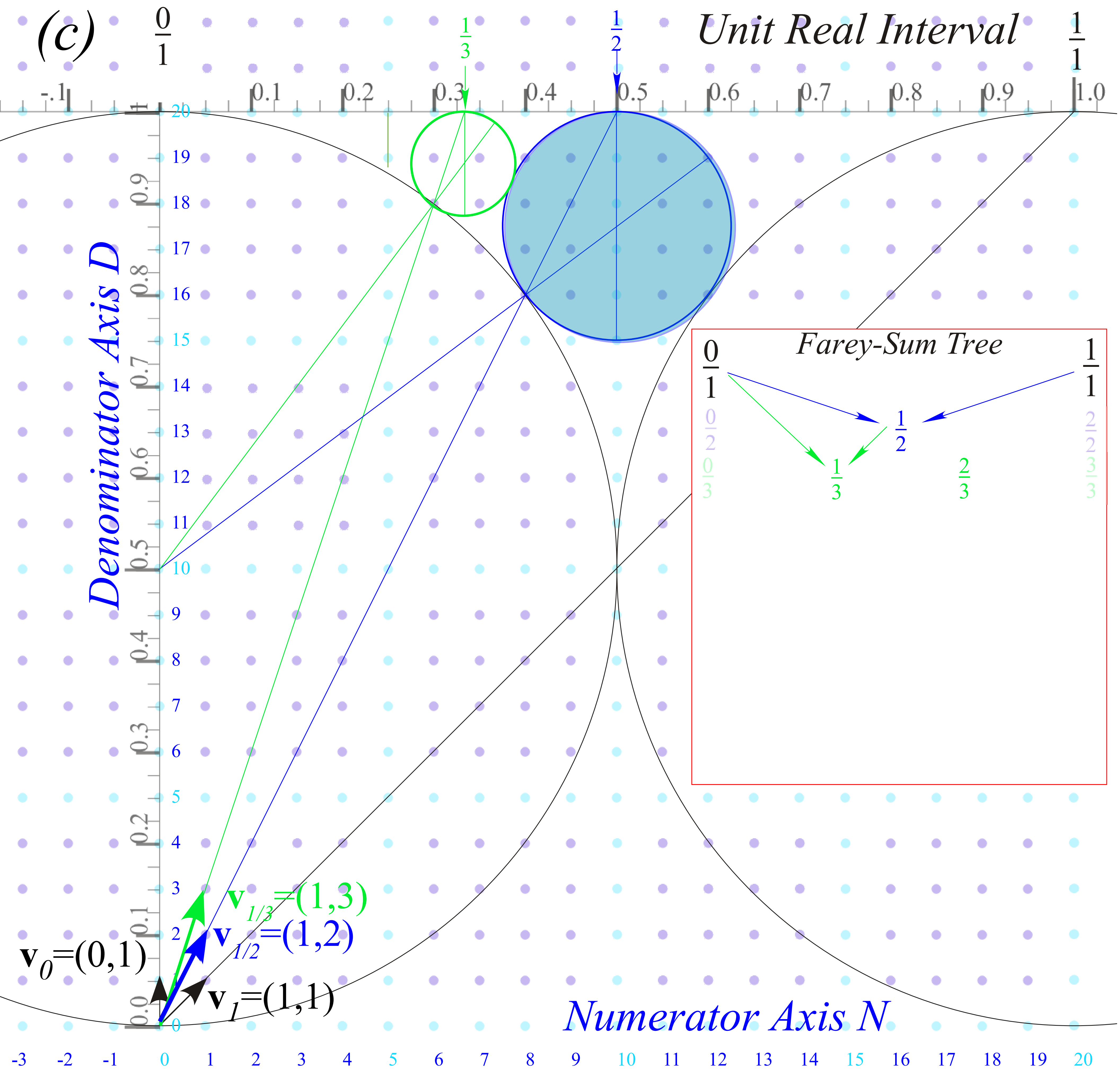


Farey Sum related to vector sum and *Ford Circles*

1/1-circle has diameter *1*

1/2-circle has diameter $1/2^2=1/4$

Li, Harter, Chem.Phys.Letters **213**, *208-213 (2015)*]

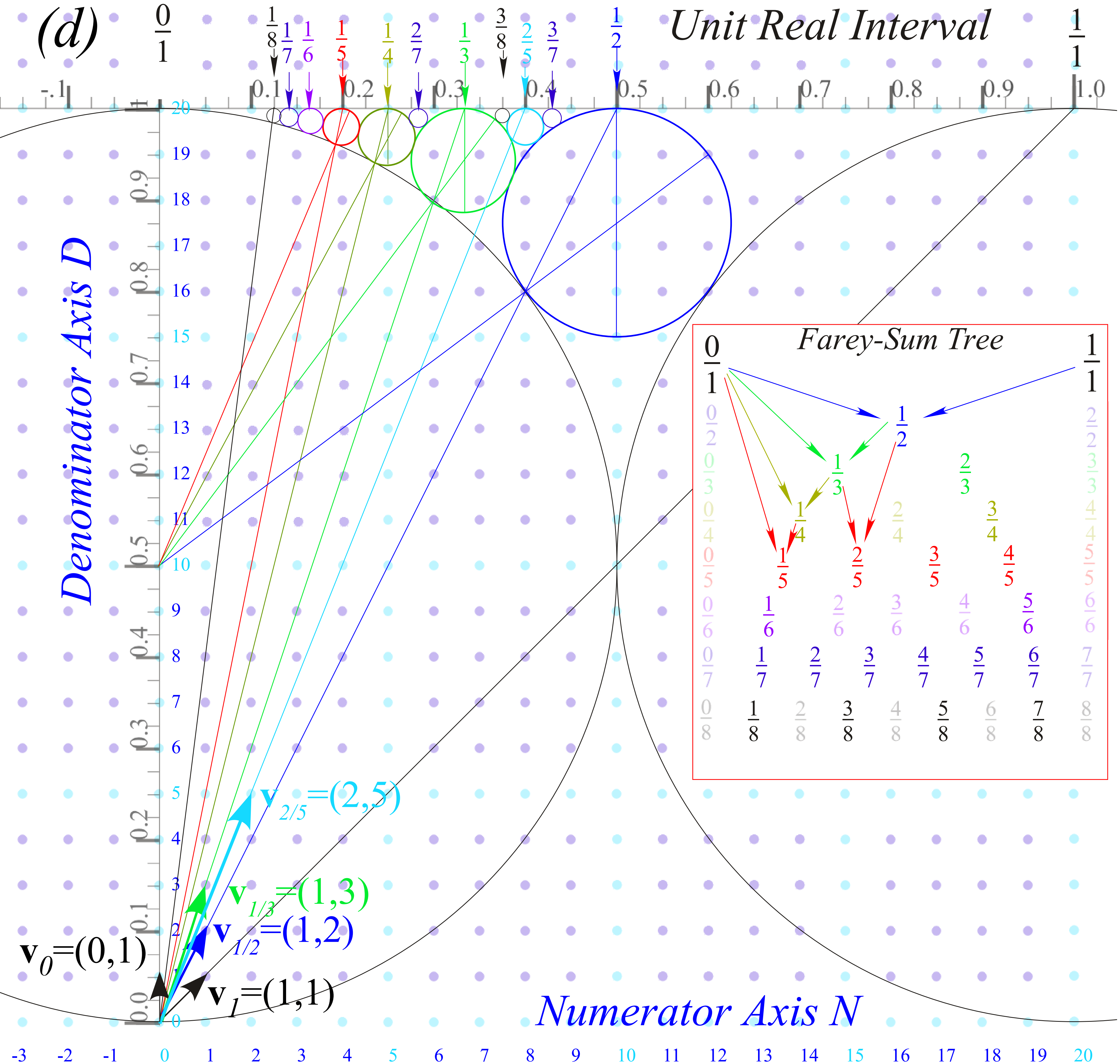


Farey Sum related to vector sum and *Ford Circles*

1/2-circle has diameter $1/2^2=1/4$

1/3-circles have diameter $1/3^2=1/9$

Li, Harter, Chem.Phys.Letters **213**, *208-213 (2015)]*



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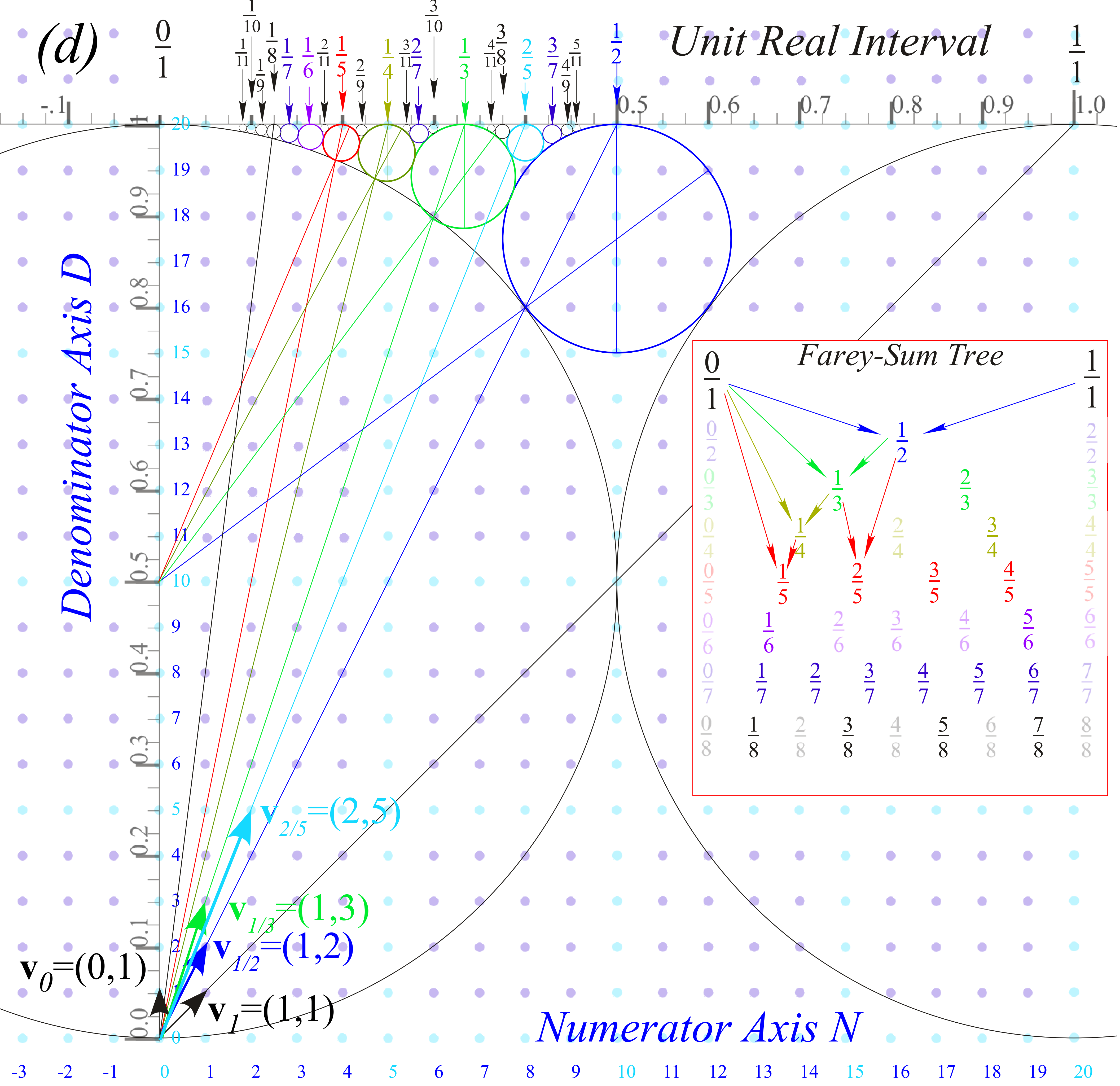

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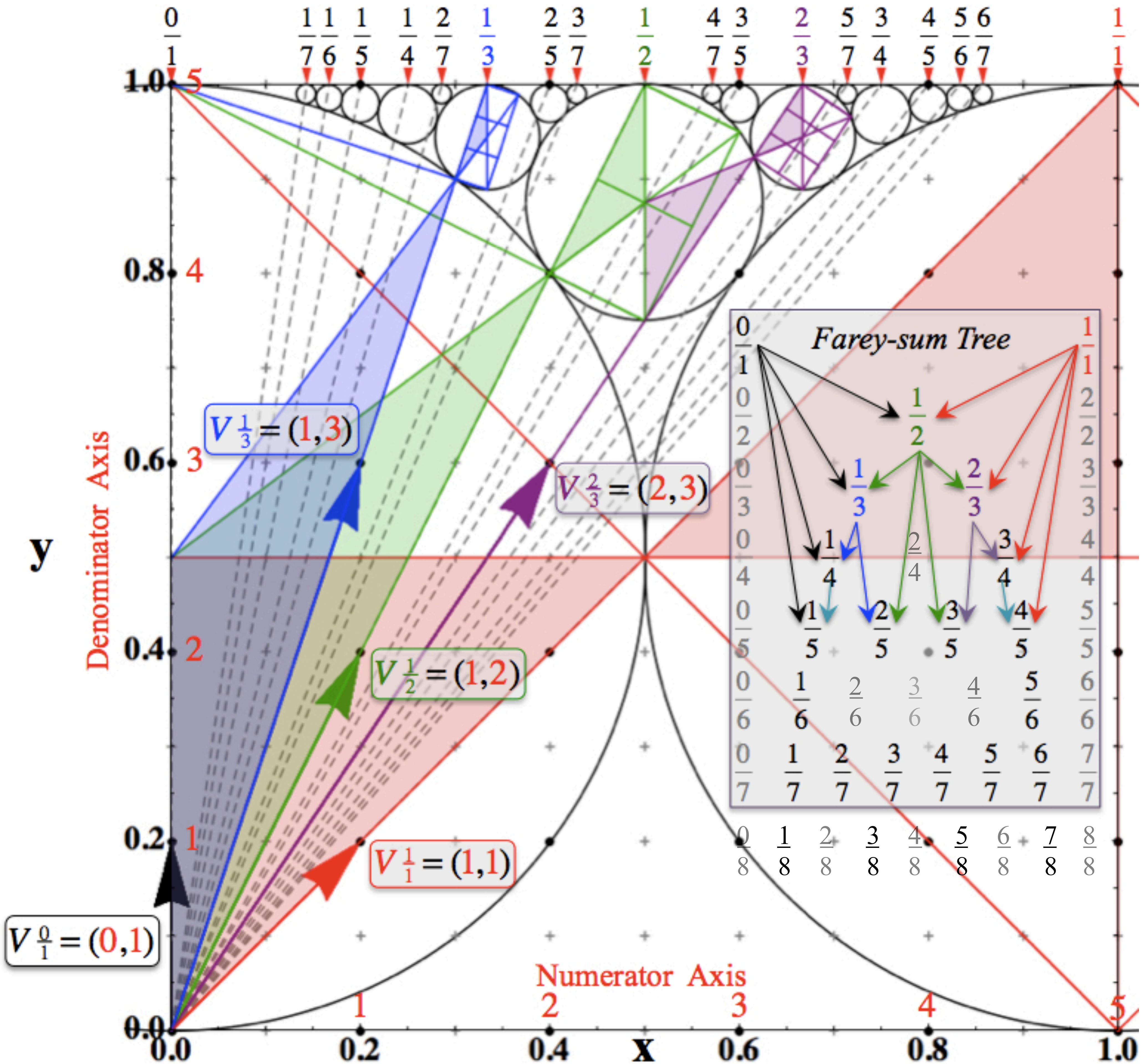
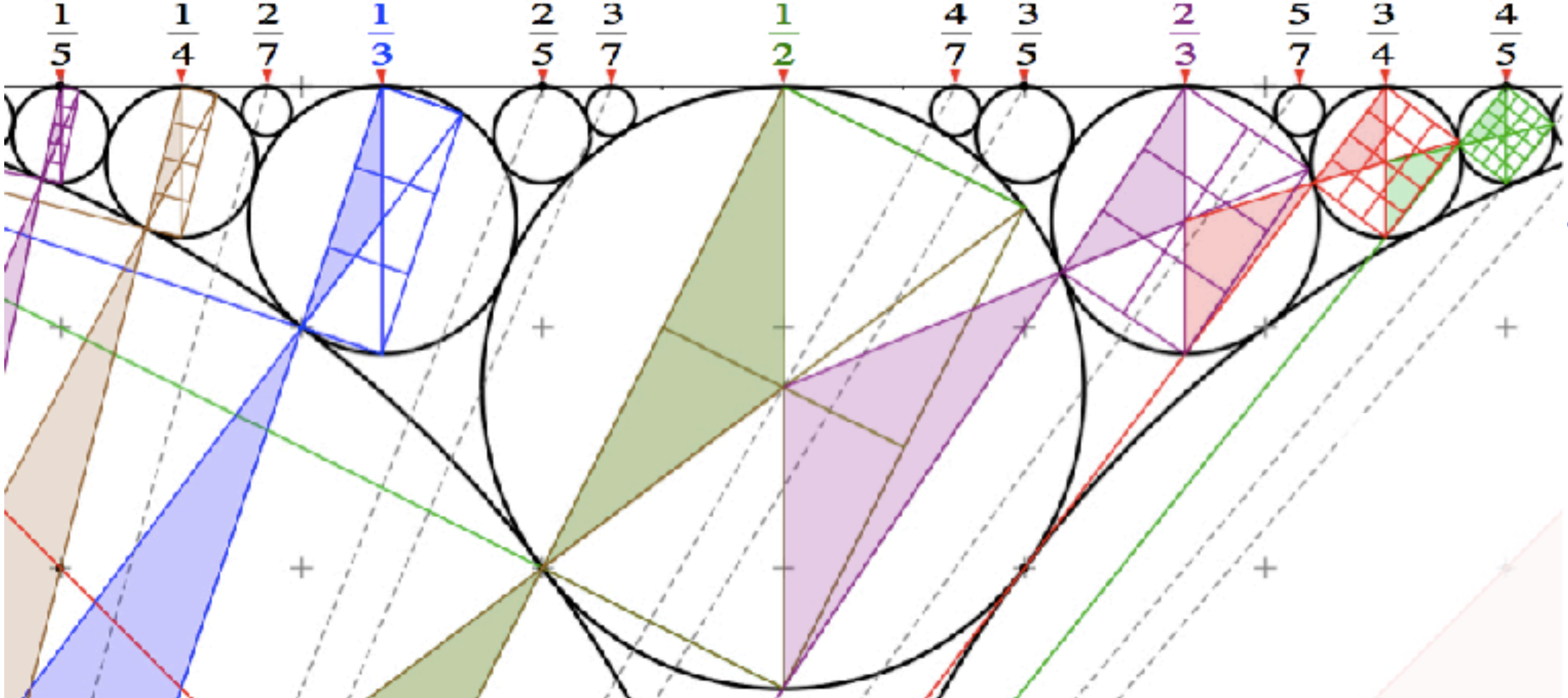


Thales Rectangles

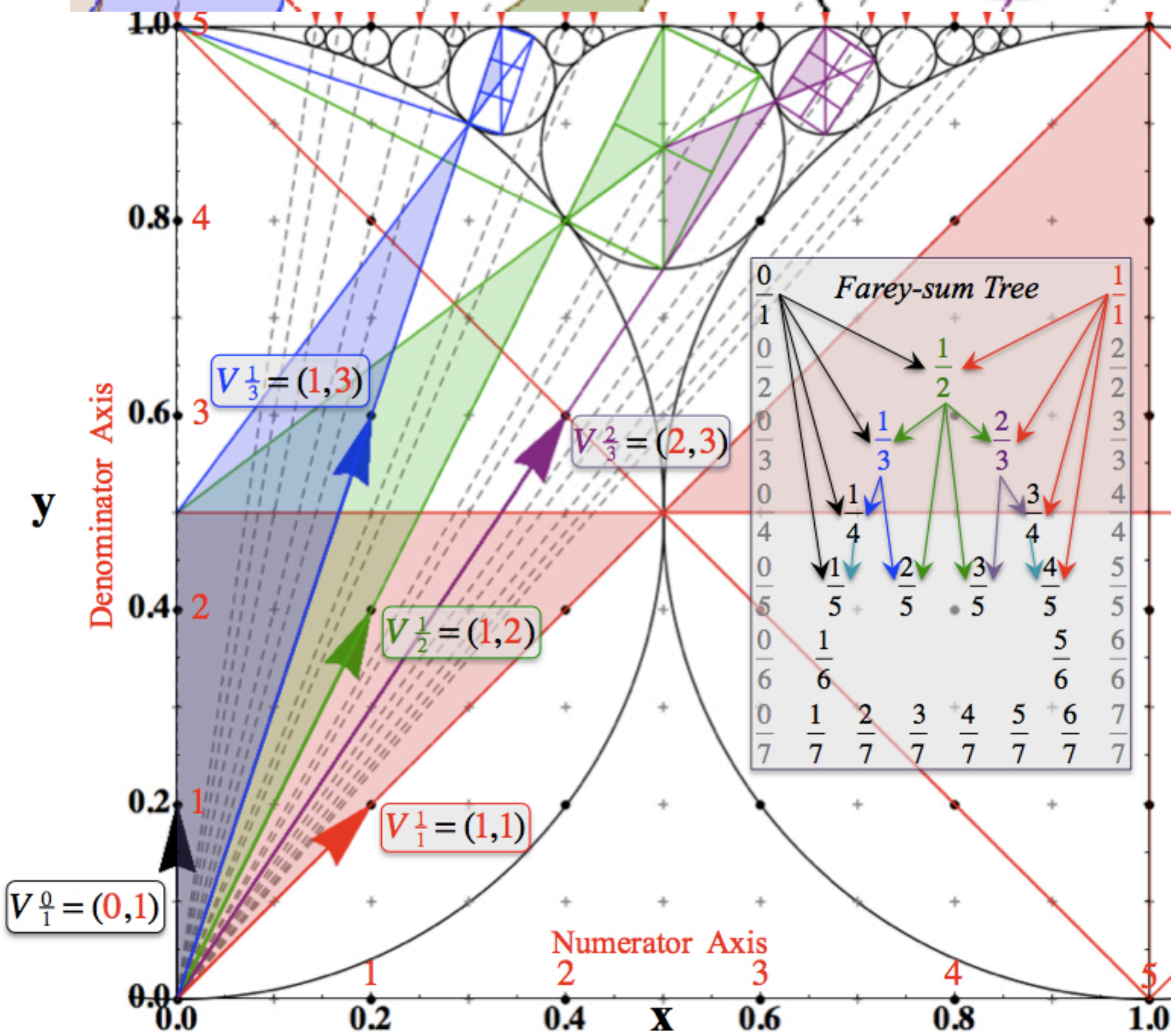
provide analytic geometry of fractal structure

Li, Harter, Chem.Phys.Letters **213**, 208-213 (2015)]



“Quantized” Thales Rectangles

provide analytic geometry of fractal structure



Li, Harter, Chem.Phys.Letters **213**, *208-213 (2015)*]

A sketch of modern molecular spectroscopy

- *The molecular frequency hierarchy*
- *Units of frequency (Hz), wavelength (m), energy (eV), and wavenumber (cm^{-1})*
- *Spectral windows in atmosphere due to molecules*
  - *Example of~16μm (670cm^{-1}) spectral hierarchy of CO_2* (simple*)*
  - *Example of~16μm (631cm^{-1}) spectral hierarchy of CF_4* (complicated*)*
  - *Example of ~16μm (615cm^{-1}) spectral hierarchy of SF_6 (really* complicated*)*
- *Rotational Energy Surface (RES) analysis, J-vector geometry, and tunneling*
  - *Nuclear spin hyperfine effects rule mol-spec.*

Quantum "revivals" of gently localized rotor waves:

Bohr-rotor wave dynamics gives lessons for quantum number theory

- *Gaussian wave-packet bandwidth and uncertainty*
  - *Gaussian Bohr-rotor revivals and quantum fractals*

Understanding fractals using geometry of fractions (Rationalizing rationals)

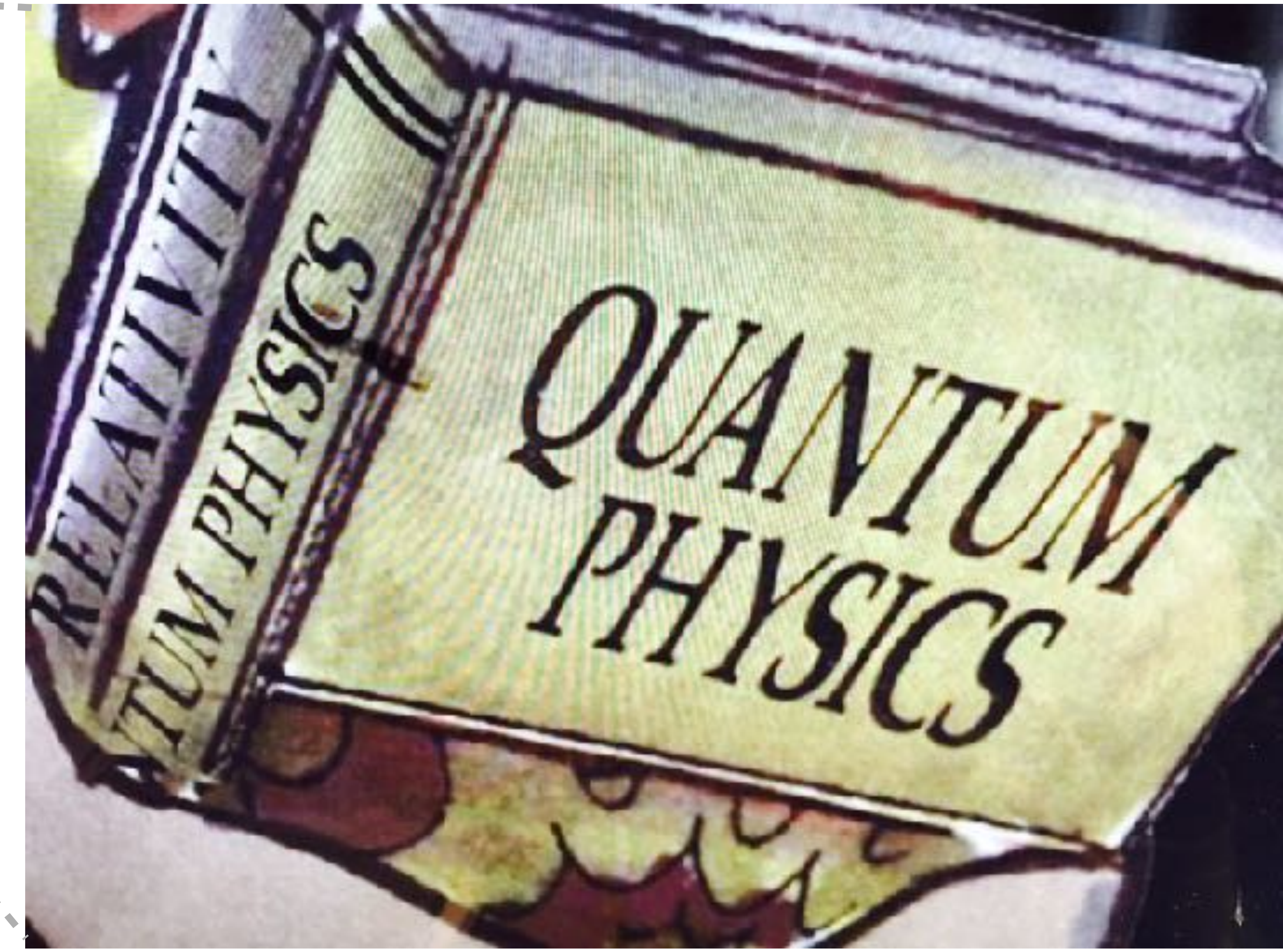
- *Farey-Sums and Ford-products*
- *Ford Circles and Farey-Trees*

Unifying Relativity with Quantum Theory (Why a ***Men In Black*** candidate shot little Suzy)

→ *The simplest molecule: A pair of head-on lasers gives lessons for relativistic quantum theory*

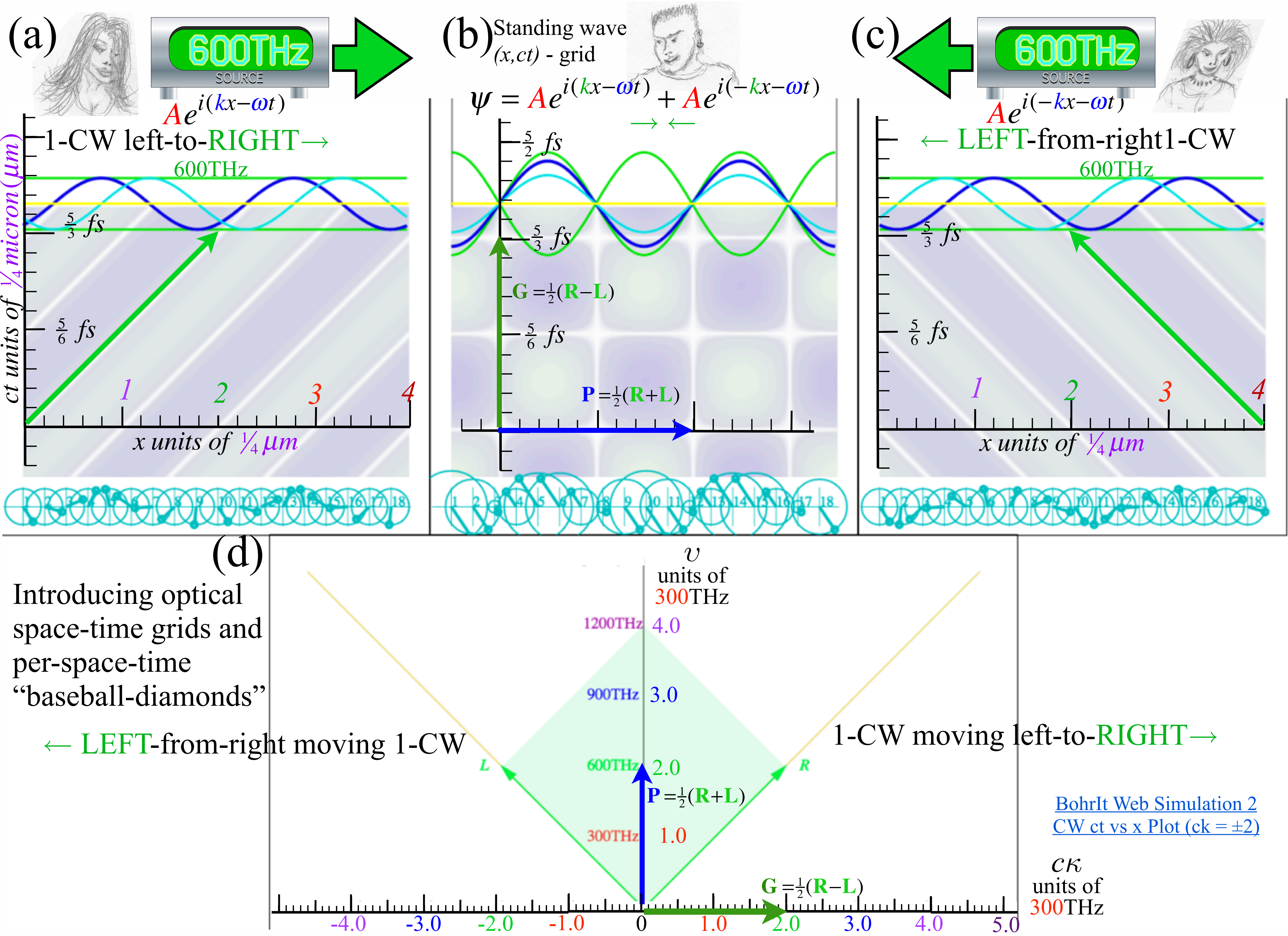
- *Light wave zeros draw Minkowski coordinate grid*
  - ***Relawavity*** *geometry of waves defines space-time warp*
    - *...and per-space-time quantum mechanics*

Men In Black candidate shot little Suzy

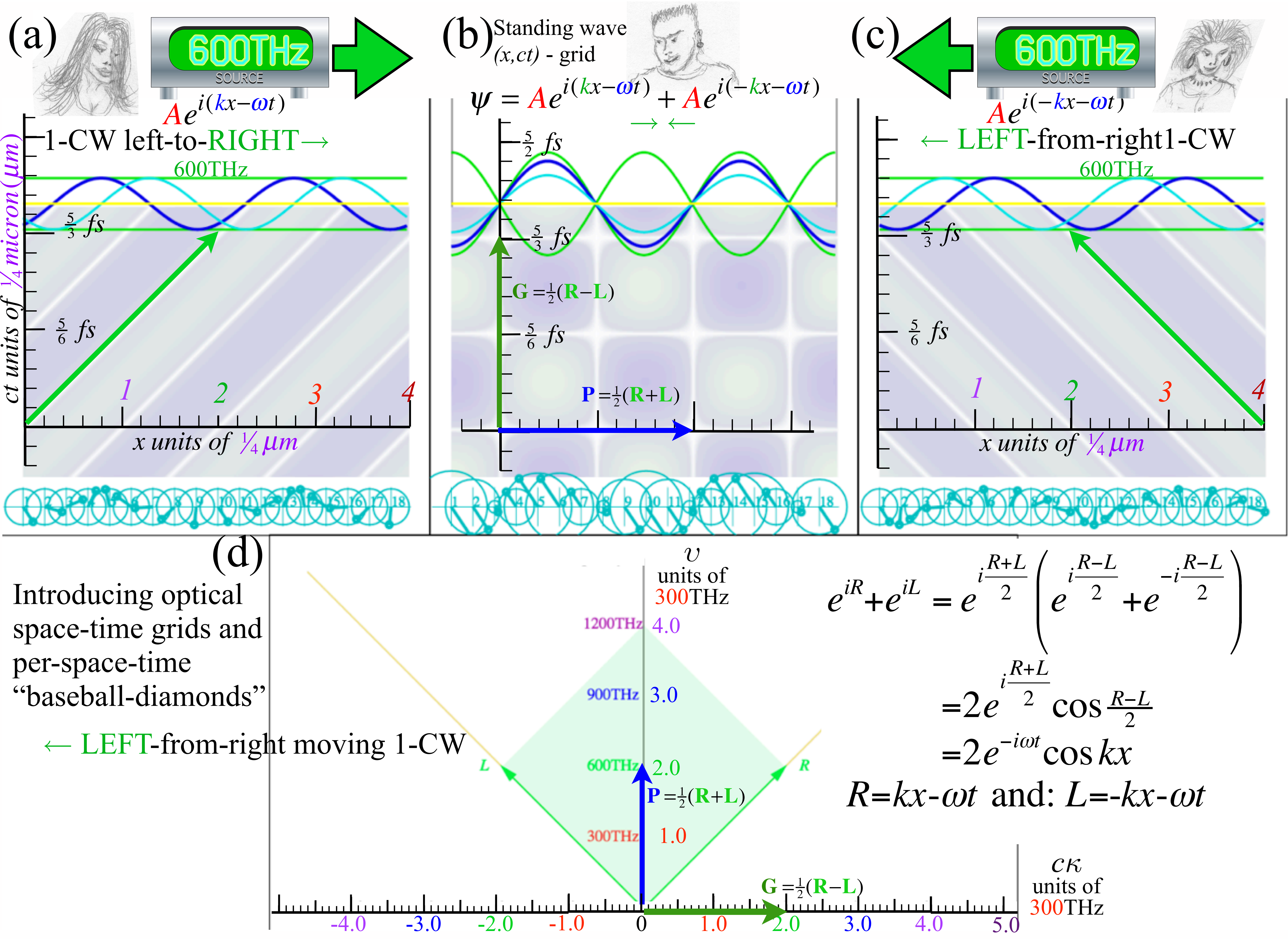


Bad Suzy!
Relativity and Quantum Theory need to be unified in *one* book *half* the size of those old tomes!

It's called ***Relawavity***.

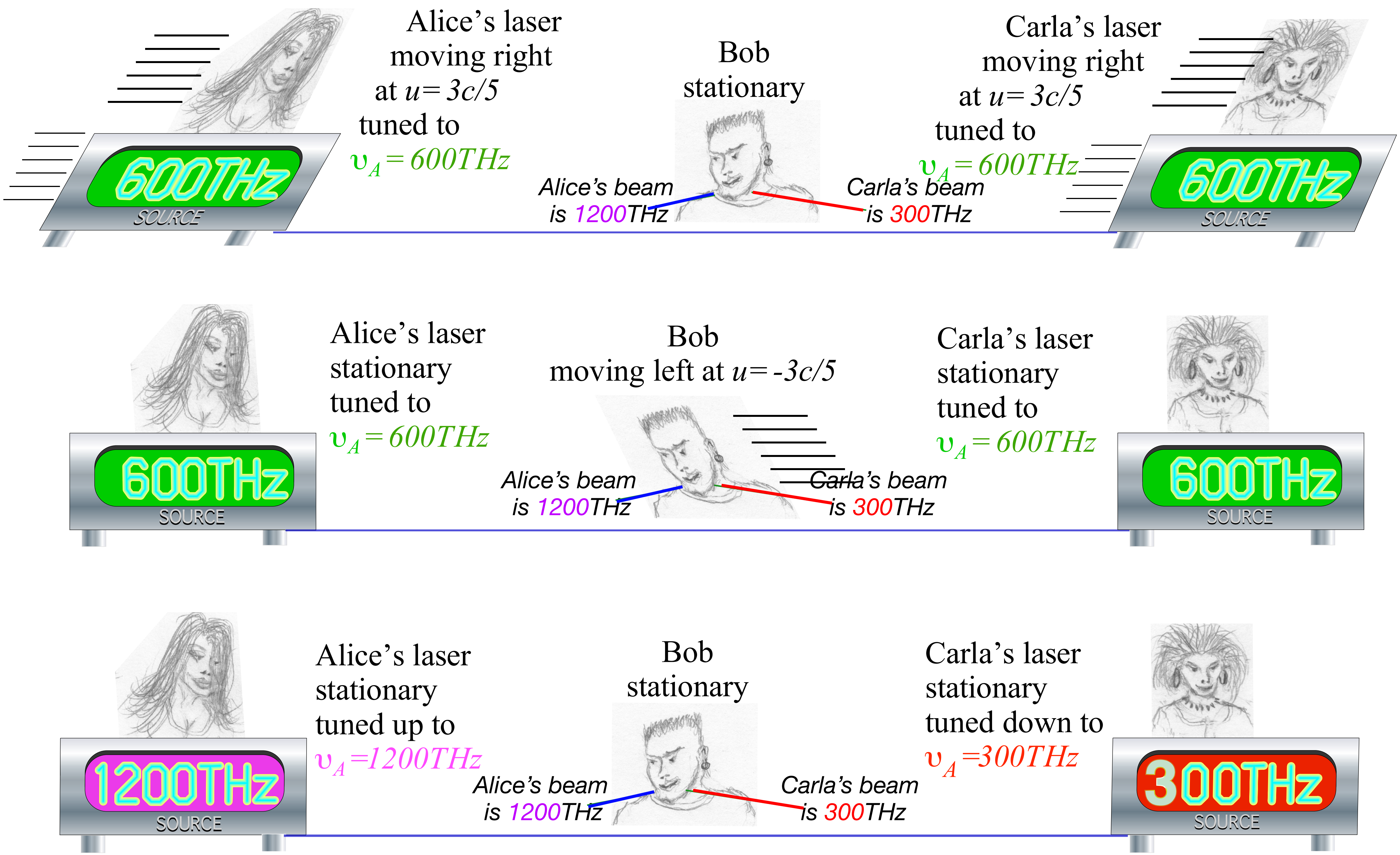


(a)
600THz
SOURCE
$Ae^{i(kx-\omega t)}$
1-CW left-to-RIGHT→
600THz
ct units of ¼ micron (μm)
$\frac{5}{3}$ fs
$\frac{5}{6}$ fs
1
2
3
4
x units of ¼ μm
(b)
Standing wave
(x,ct) - grid
$\psi = Ae^{i(kx-\omega t)} + Ae^{i(-kx-\omega t)}$
→ ←
$\frac{5}{2}$ fs
$\frac{5}{3}$ fs
$\mathbf{G} = \frac{1}{2}(\mathbf{R}-\mathbf{L})$
$\frac{5}{6}$ fs
$\mathbf{P} = \frac{1}{2}(\mathbf{R}+\mathbf{L})$
(c)
600THz
SOURCE
$Ae^{i(-kx-\omega t)}$
← LEFT-from-right1-CW
600THz
$\frac{5}{3}$ fs
$\frac{5}{6}$ fs
1
2
3
4
x units of ¼ μm
(d)
Introducing optical space-time grids and per-space-time “baseball-diamonds”
υ units of 300THz
1200THz 4.0
900THz 3.0
600THz 2.0
300THz 1.0
← LEFT-from-right moving 1-CW
1-CW moving left-to-RIGHT→
L
R
$\mathbf{P} = \frac{1}{2}(\mathbf{R}+\mathbf{L})$
$\mathbf{G} = \frac{1}{2}(\mathbf{R}-\mathbf{L})$
cκ units of 300THz
-4.0
-3.0
-2.0
-1.0
0
1.0
2.0
3.0
4.0
5.0
BohrIt Web Simulation 2
CW ct vs x Plot (ck = ±2)



(a)
600THz
SOURCE
$Ae^{i(kx-\omega t)}$
1-CW left-to-RIGHT→
600THz
ct units of ¼ micron (μm)
$\frac{5}{3}$ fs
$\frac{5}{6}$ fs
1
2
3
4
x units of ¼ μm
(b) Standing wave
(x,ct) - grid
$\psi = Ae^{i(kx-\omega t)} + Ae^{i(-kx-\omega t)}$
→ ←
$\frac{5}{2}$ fs
$\frac{5}{3}$ fs
$\mathbf{G} = \frac{1}{2}(\mathbf{R}-\mathbf{L})$
$\frac{5}{6}$ fs
$\mathbf{P} = \frac{1}{2}(\mathbf{R}+\mathbf{L})$
(c)
600THz
SOURCE
$Ae^{i(-kx-\omega t)}$
← LEFT-from-right 1-CW
600THz
$\frac{5}{3}$ fs
$\frac{5}{6}$ fs
1
2
3
4
x units of ¼ μm
(d)
Introducing optical space-time grids and per-space-time “baseball-diamonds”
← LEFT-from-right moving 1-CW
υ units of 300THz
1200THz 4.0
900THz 3.0
600THz 2.0
300THz 1.0
L
R
$\mathbf{P} = \frac{1}{2}(\mathbf{R}+\mathbf{L})$
$\mathbf{G} = \frac{1}{2}(\mathbf{R}-\mathbf{L})$
cκ units of 300THz
-4.0
-3.0
-2.0
-1.0
0
1.0
2.0
3.0
4.0
5.0
$e^{iR}+e^{iL} = e^{i\frac{R+L}{2}}\left(e^{i\frac{R-L}{2}}+e^{-i\frac{R-L}{2}}\right)$
$=2e^{i\frac{R+L}{2}}\cos\frac{R-L}{2}$
$=2e^{-i\omega t}\cos kx$
$R=kx-\omega t$ and: $L=-kx-\omega t$

Three scenarios that look the same to Bob



Much cheaper to do the 3rd scenario!$!

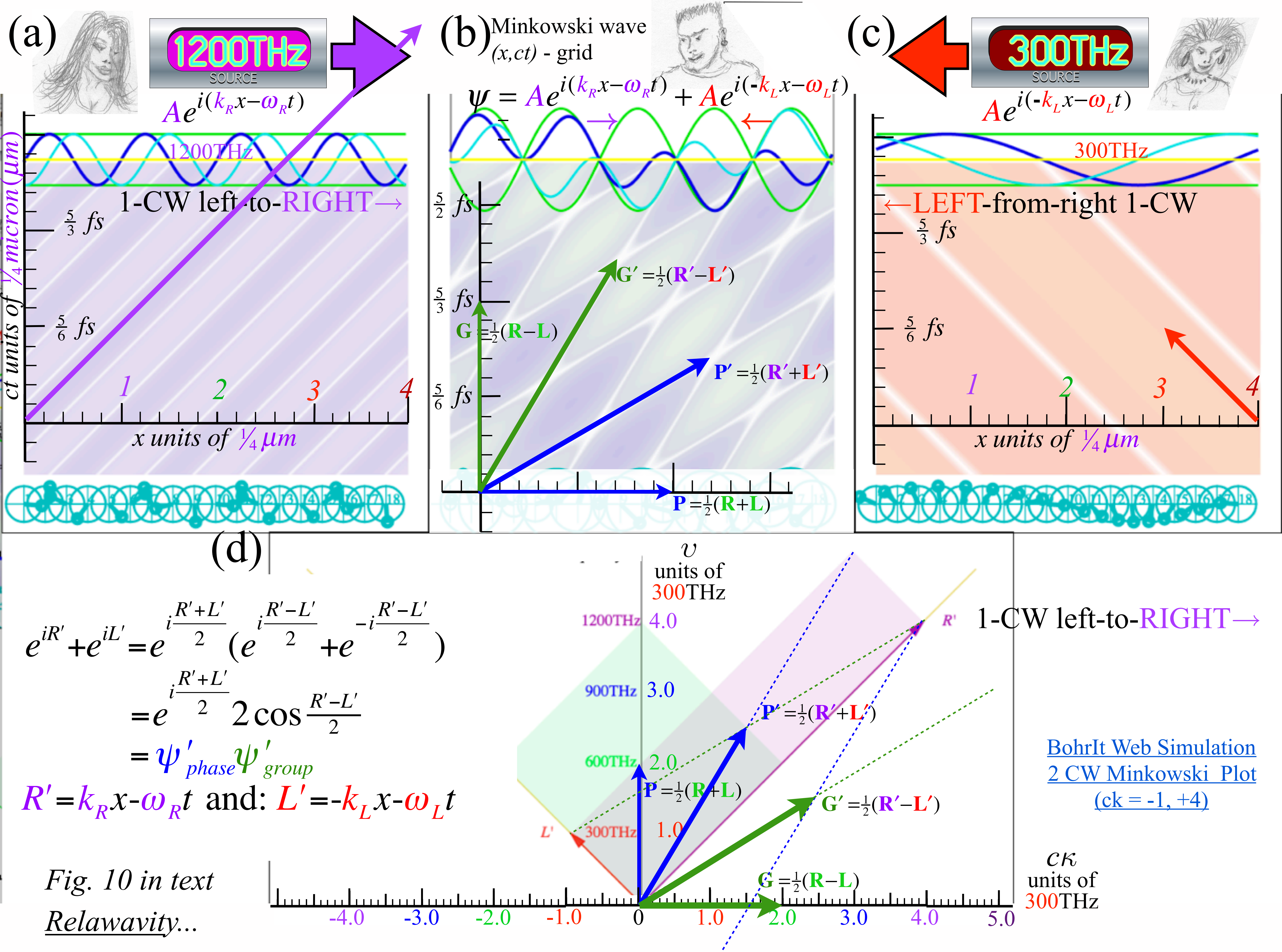


Fig. 10 in text
Relawavity...

Lorentz transformations...

write **G′** *and* **P′** *in terms of* **G** *and* **P** *using* $\cosh\rho$ *and* $\sinh\rho$

$$\mathbf{G}' = \begin{pmatrix} c\kappa'_{group} \\ \upsilon'_{group} \end{pmatrix} = \upsilon_A \begin{pmatrix} \cosh\rho \\ \sinh\rho \end{pmatrix} = \upsilon_A \begin{pmatrix} 5/4 \\ 3/4 \end{pmatrix}$$

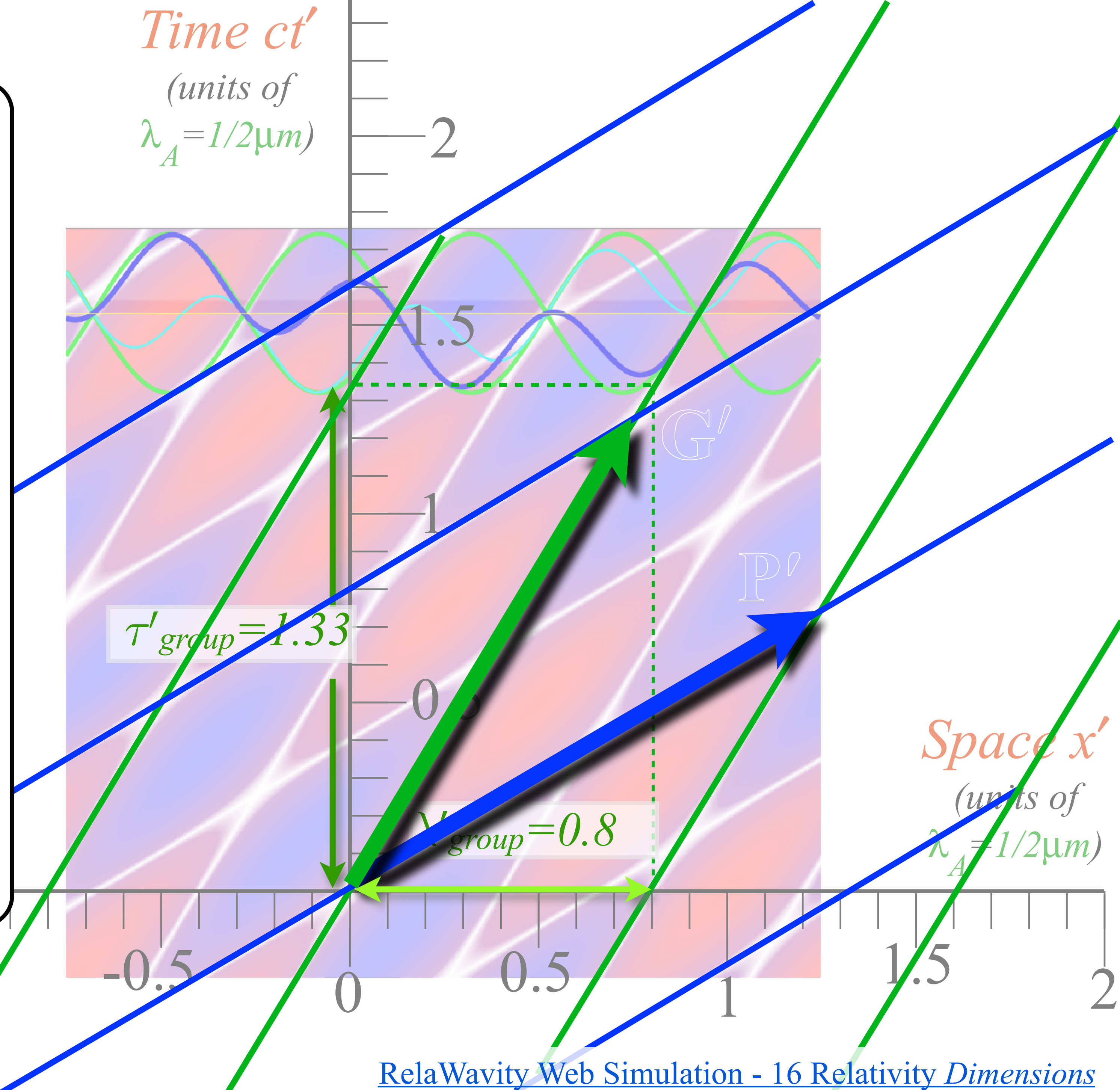
$$= \upsilon_A \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cosh\rho + \upsilon_A \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sinh\rho$$

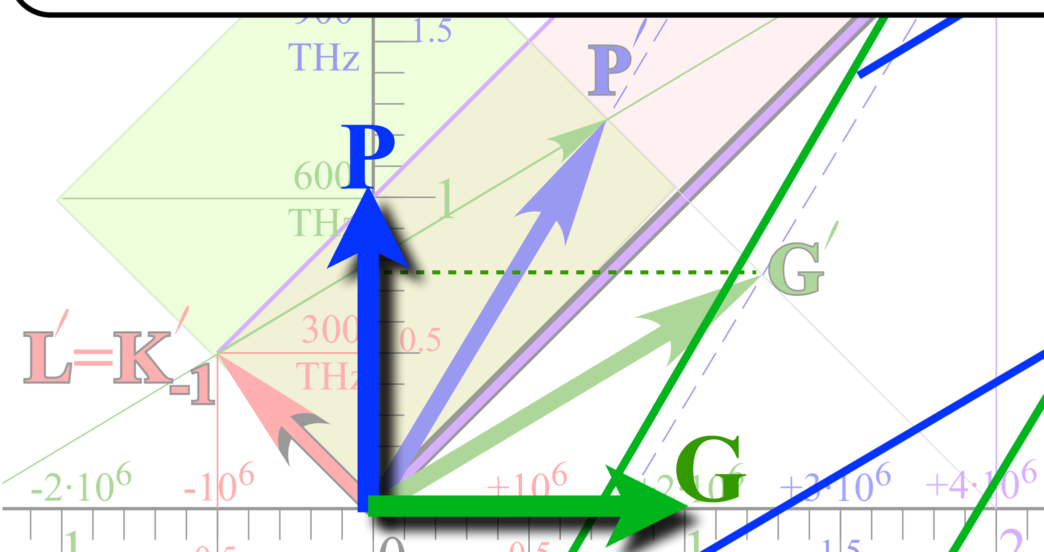
$$\mathbf{G}' = \mathbf{G} \cosh\rho + \mathbf{P} \sinh\rho$$

$$\mathbf{P}' = \begin{pmatrix} c\kappa'_{phase} \\ \upsilon'_{phase} \end{pmatrix} = \upsilon_A \begin{pmatrix} \sinh\rho \\ \cosh\rho \end{pmatrix} = \upsilon_A \begin{pmatrix} 3/4 \\ 5/4 \end{pmatrix}$$

$$= \upsilon_A \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sinh\rho + \upsilon_A \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cosh\rho$$

$$\mathbf{P}' = \mathbf{G} \sinh\rho + \mathbf{P} \cosh\rho$$





RelaWavity Web Simulation - 16 Relativity *Dimensions*

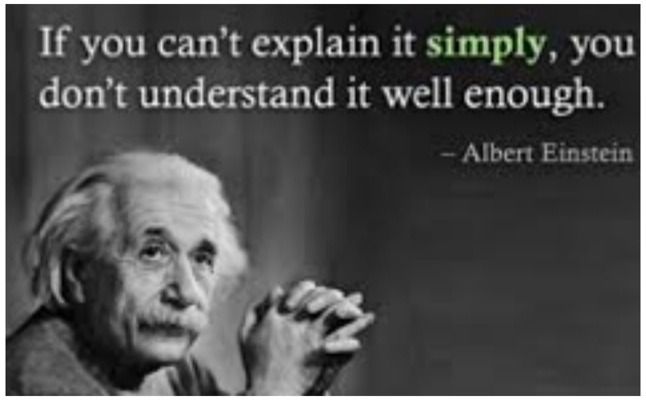
phase	$b_{RED}^{Doppler}$	$\frac{c}{V_{phase}}$	$\frac{\kappa_{phase}}{\kappa_A}$	$\frac{\tau_{phase}}{\tau_A}$	$\frac{\upsilon_{phase}}{\upsilon_A}$	$\frac{\lambda_{phase}}{\lambda_A}$	$\frac{V_{phase}}{c}$	$b_{BLUE}^{Doppler}$
group	$\frac{1}{b_{BLUE}^{Doppler}}$	$\frac{V_{group}}{c}$	$\frac{\upsilon_{group}}{\upsilon_A}$	$\frac{\lambda_{group}}{\lambda_A}$	$\frac{\kappa_{group}}{\kappa_A}$	$\frac{\tau_{group}}{\tau_A}$	$\frac{c}{V_{group}}$	$\frac{1}{b_{RED}^{Doppler}}$
rapidity ρ	$e^{-\rho}$	$\tanh\rho$	$\sinh\rho$	$\operatorname{sech}\rho$	$\cosh\rho$	$\operatorname{csch}\rho$	$\coth\rho$	$e^{+\rho}$
value for $\beta=3/5$	$\frac{1}{2}=0.5$	$\frac{3}{5}=0.6$	$\frac{3}{4}=0.75$	$\frac{4}{5}=0.80$	$\frac{5}{4}=1.25$	$\frac{4}{3}=1.33$	$\frac{5}{3}=1.67$	$\frac{2}{1}=2.0$

$$\begin{pmatrix} \cosh\rho & \sinh\rho \\ \sinh\rho & \cosh\rho \end{pmatrix}$$

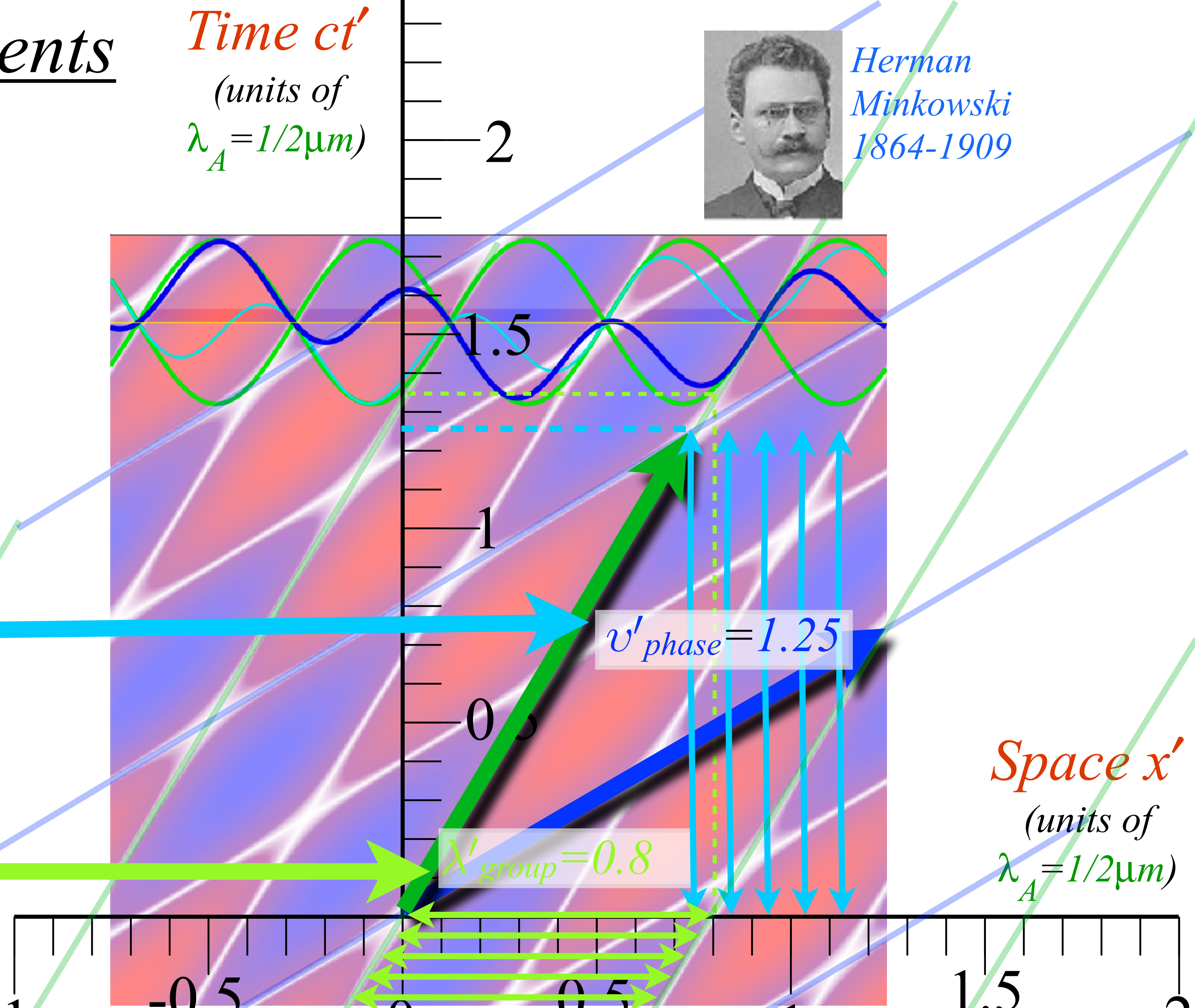
Lorentz transform matrix

Two Famous-Name Coefficients

Review of Lect. 30 p.106



Albert Einstein 1859-1955



Herman Minkowski 1864-1909

This number is called an: **Einstein time-dilation** (dilated by 25% here)

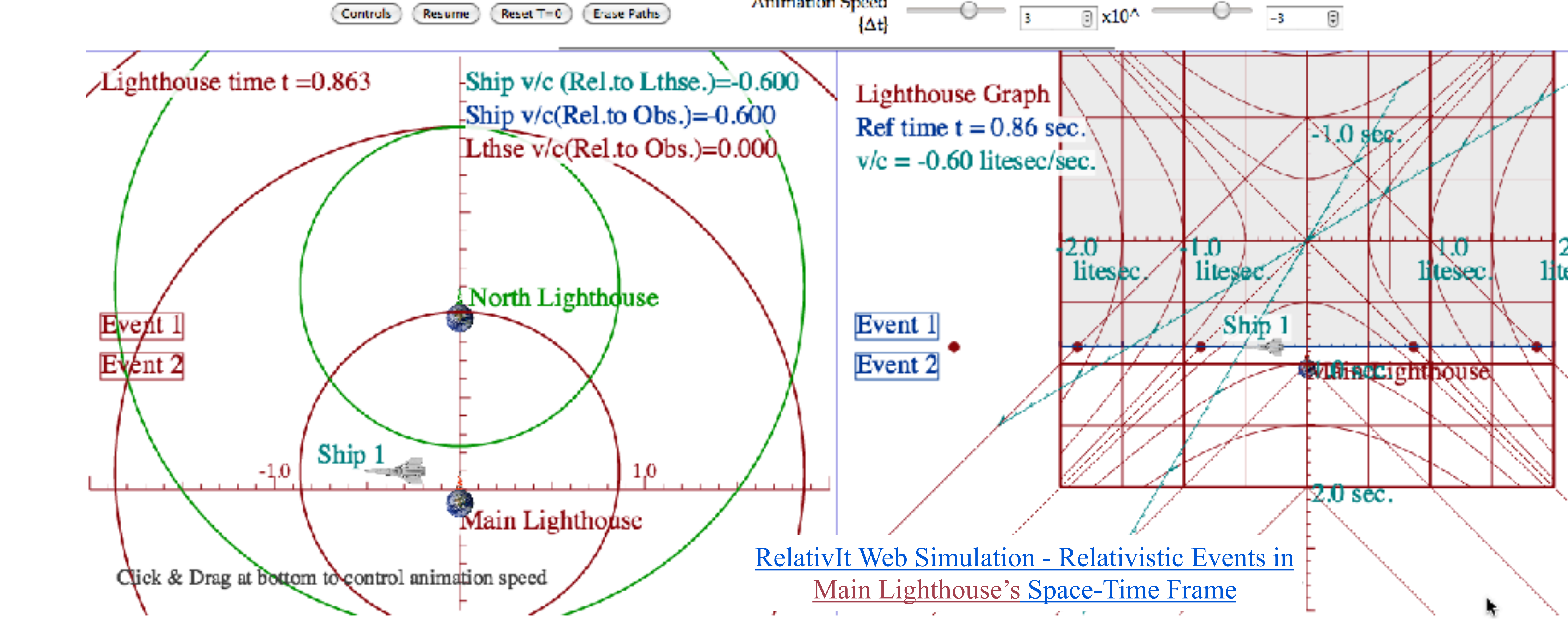
This number is called a: **Lorentz length-contraction** (contracted by 20% here)

Hendrik A. Lorentz 1853-1928

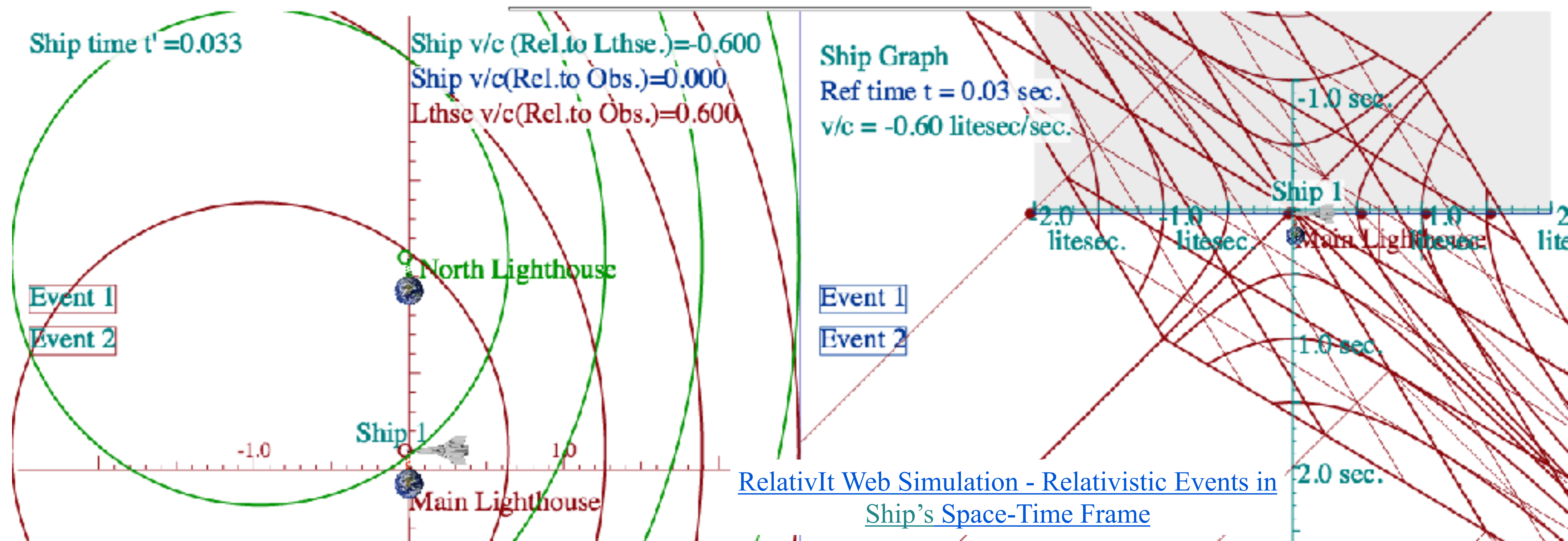
Old-Fashioned Notation

[RelaWavity Web Simulation - Relativistic *Terms* (Expanded Table)]

phase	$b_{RED}^{Doppler}$	$\frac{c}{V_{phase}}$	$\frac{\kappa_{phase}}{\kappa_A}$	$\frac{\tau_{phase}}{\tau_A}$	$\frac{\upsilon_{phase}}{\upsilon_A}$	$\frac{\lambda_{phase}}{\lambda_A}$	$\frac{V_{phase}}{c}$	$b_{BLUE}^{Doppler}$
group	$\frac{1}{b_{BLUE}^{Doppler}}$	$\frac{V_{group}}{c}$	$\frac{\upsilon_{group}}{\upsilon_A}$	$\frac{\lambda_{group}}{\lambda_A}$	$\frac{\kappa_{group}}{\kappa_A}$	$\frac{\tau_{group}}{\tau_A}$	$\frac{c}{V_{group}}$	$\frac{1}{b_{RED}^{Doppler}}$
rapidity ρ	$e^{-\rho}$	$\tanh\rho$	$\sinh\rho$	$\text{sech}\,\rho$	$\cosh\rho$	$\text{csch}\,\rho$	$\coth\rho$	$e^{+\rho}$
$\beta\equiv\frac{u}{c}$	$\sqrt{\frac{1-\beta}{1+\beta}}$	$\frac{\beta}{1}$	$\frac{1}{\sqrt{\beta^{-2}-1}}$	$\frac{\sqrt{1-\beta^2}}{1}$	$\frac{1}{\sqrt{1-\beta^2}}$	$\frac{\sqrt{\beta^{-2}-1}}{1}$	$\frac{1}{\beta}$	$\sqrt{\frac{1+\beta}{1-\beta}}$
value for β=3/5	$\frac{1}{2}$= 0.5	$\frac{3}{5}$=0.6	$\frac{3}{4}$=0.75	$\frac{4}{5}$=0.80	$\frac{5}{4}$=1.25	$\frac{4}{3}$=1.33	$\frac{5}{3}$=1.67	$\frac{2}{1}$=2.0



RelativIt Web Simulation - Relativistic Events in Main Lighthouse's Space-Time Frame



RelativIt Web Simulation - Relativistic Events in Ship's Space-Time Frame

A sketch of modern molecular spectroscopy

The molecular frequency hierarchy

Units of frequency (Hz), wavelength (m), energy (eV), and wavenumber (cm^{-1})

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Example of~16μm (670cm^{-1}) spectral hierarchy of CO_2 (simple*)*

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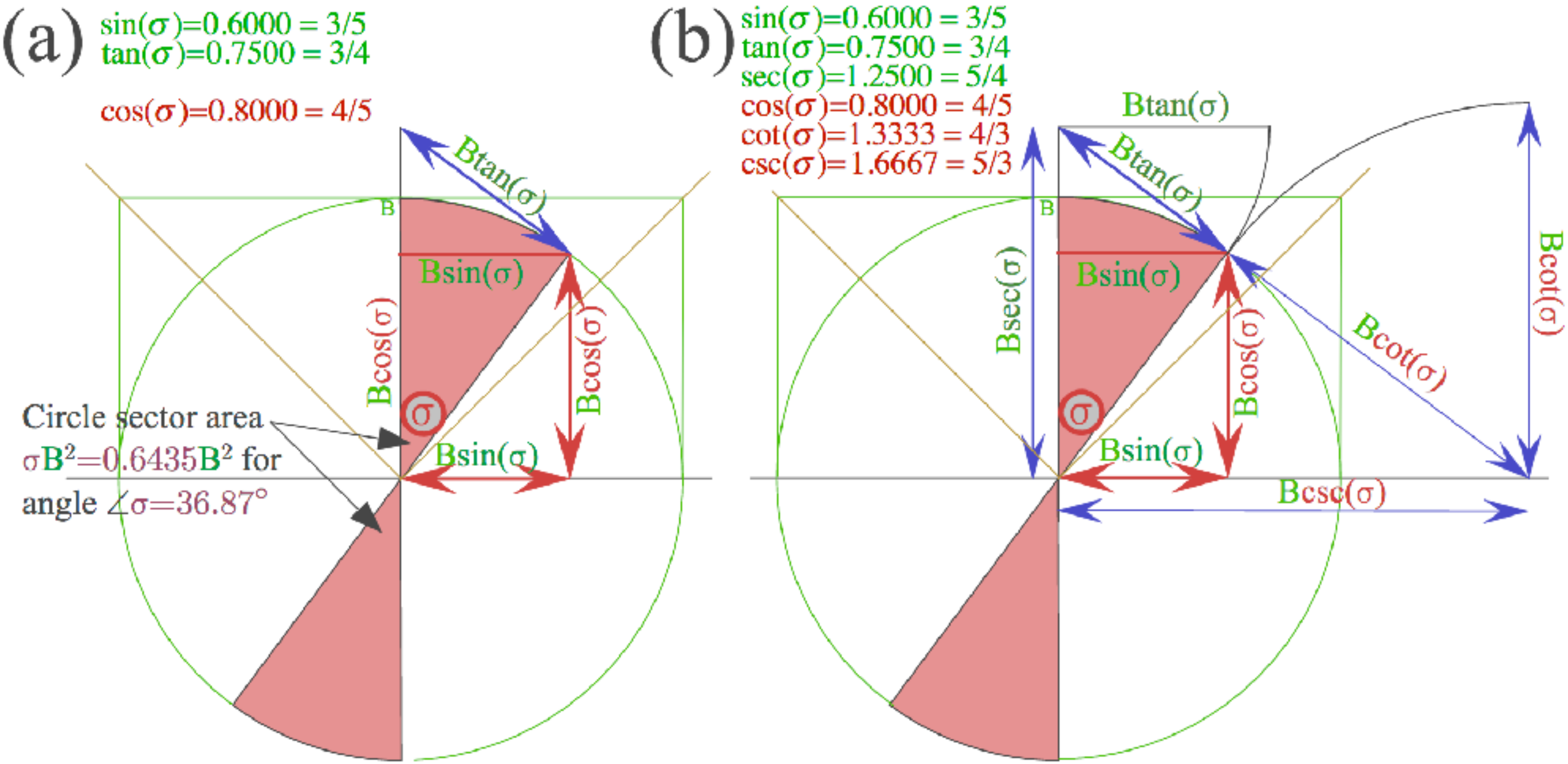
...and per-space-time quantum mechanics

➡ *Physics of relativity is mostly simple trigonometry of optical wave interference!*

And, it derives fundamentals of quantum theory, too!

Trigonometric road maps

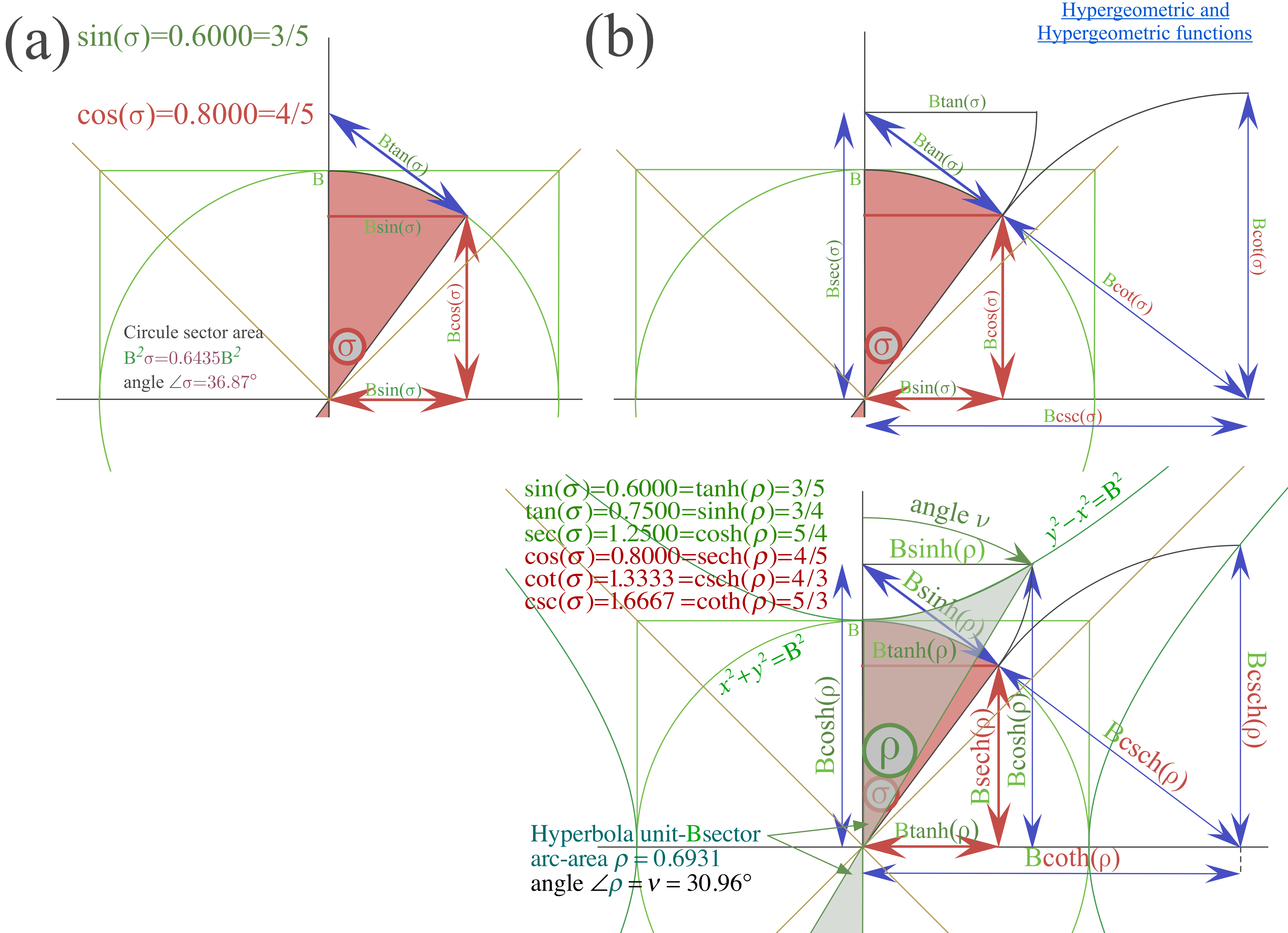


Physics of relativity is mostly simple trigonometry of optical wave interference!

RelaWavity Web Simulation
Relations between
Hypergeometric and
Hypergeometric functions

Trigonometric road maps

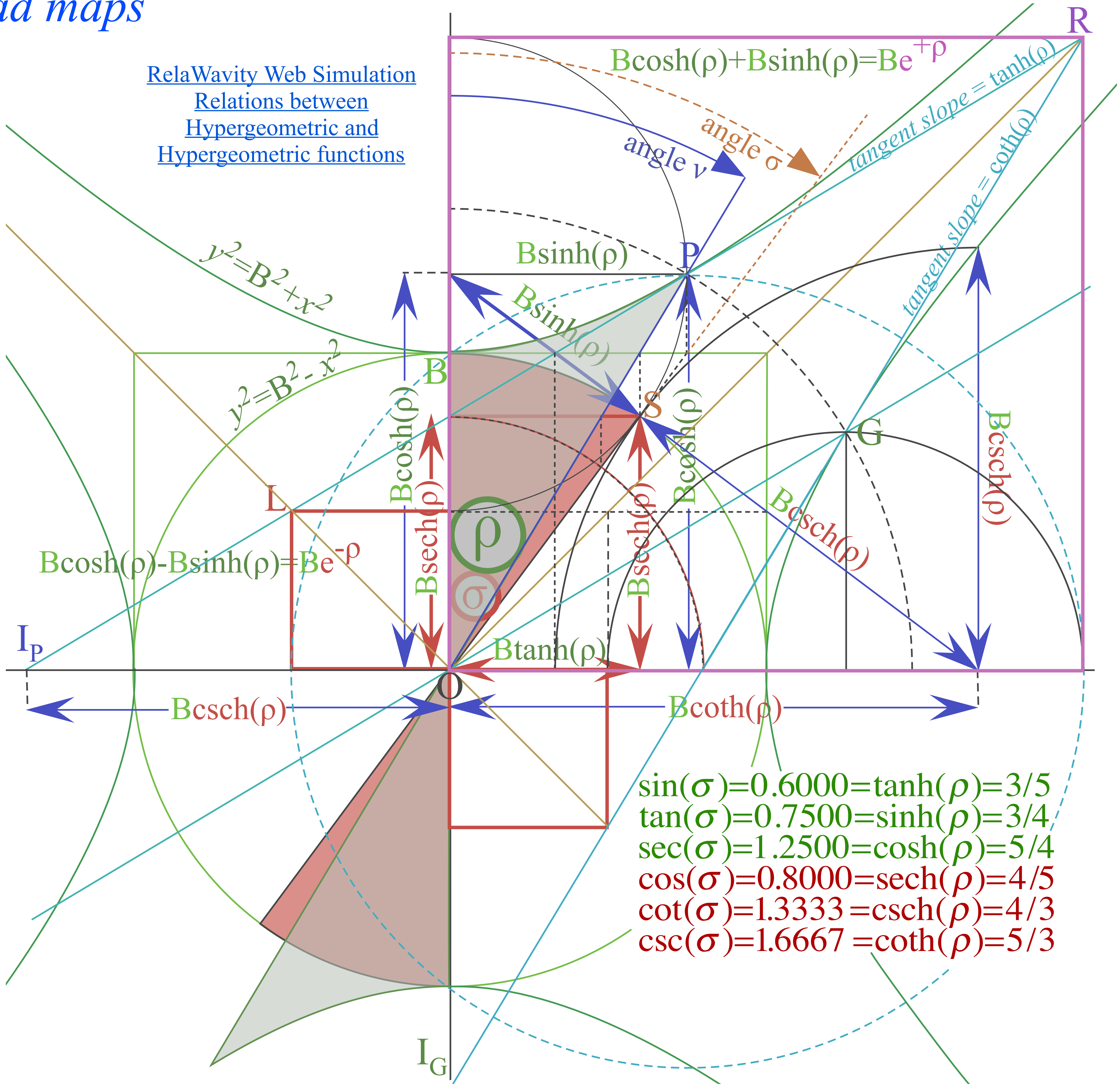
RelaWavity Web Simulation
Relations between
Hypergeometric and
Hypergeometric functions



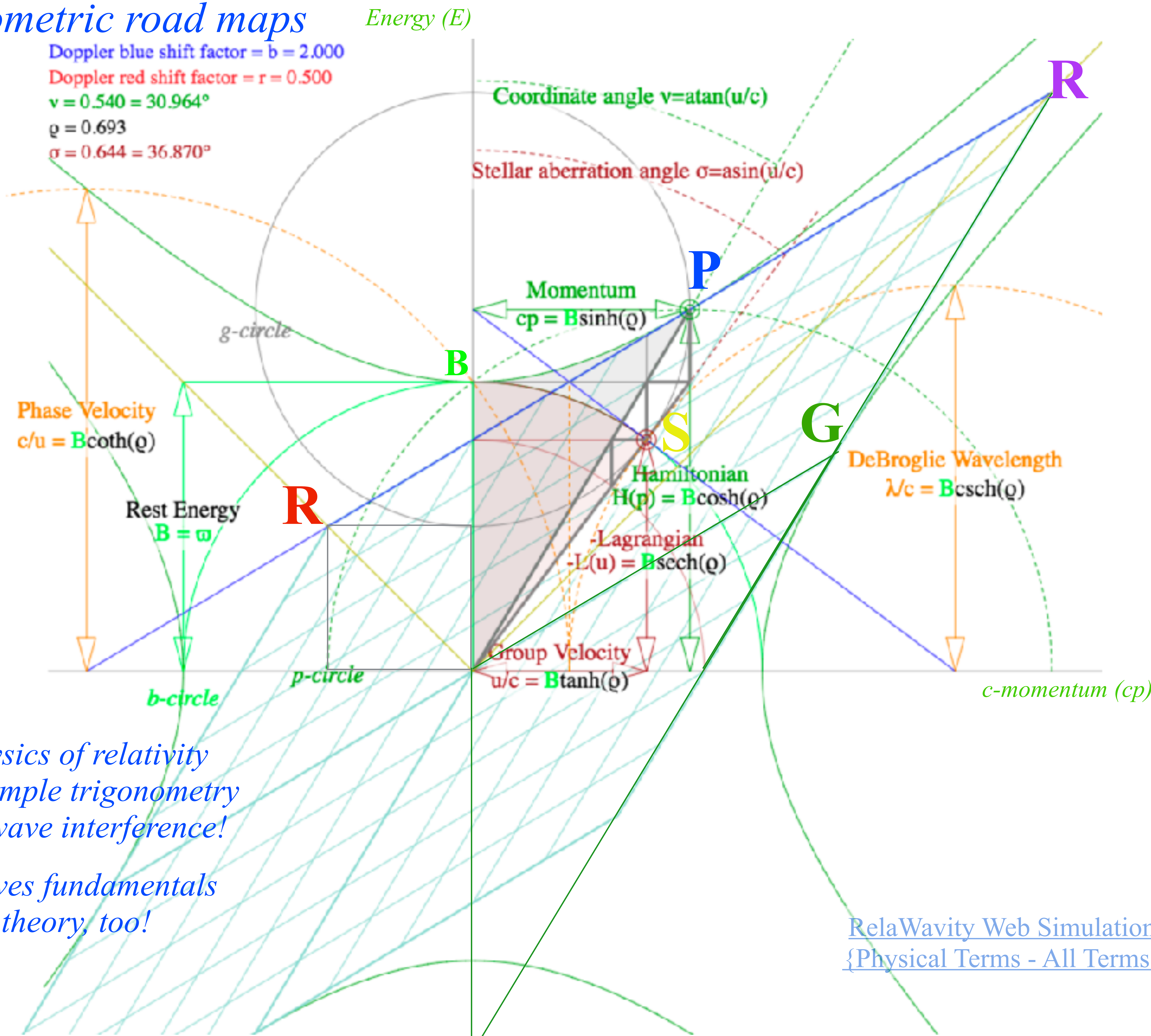
Trigonometric road maps

Need to see how trig road maps match the physical maps on next 2 pages.

[RelaWavity Web Simulation Relations between Hypergeometric and Hypergeometric functions]



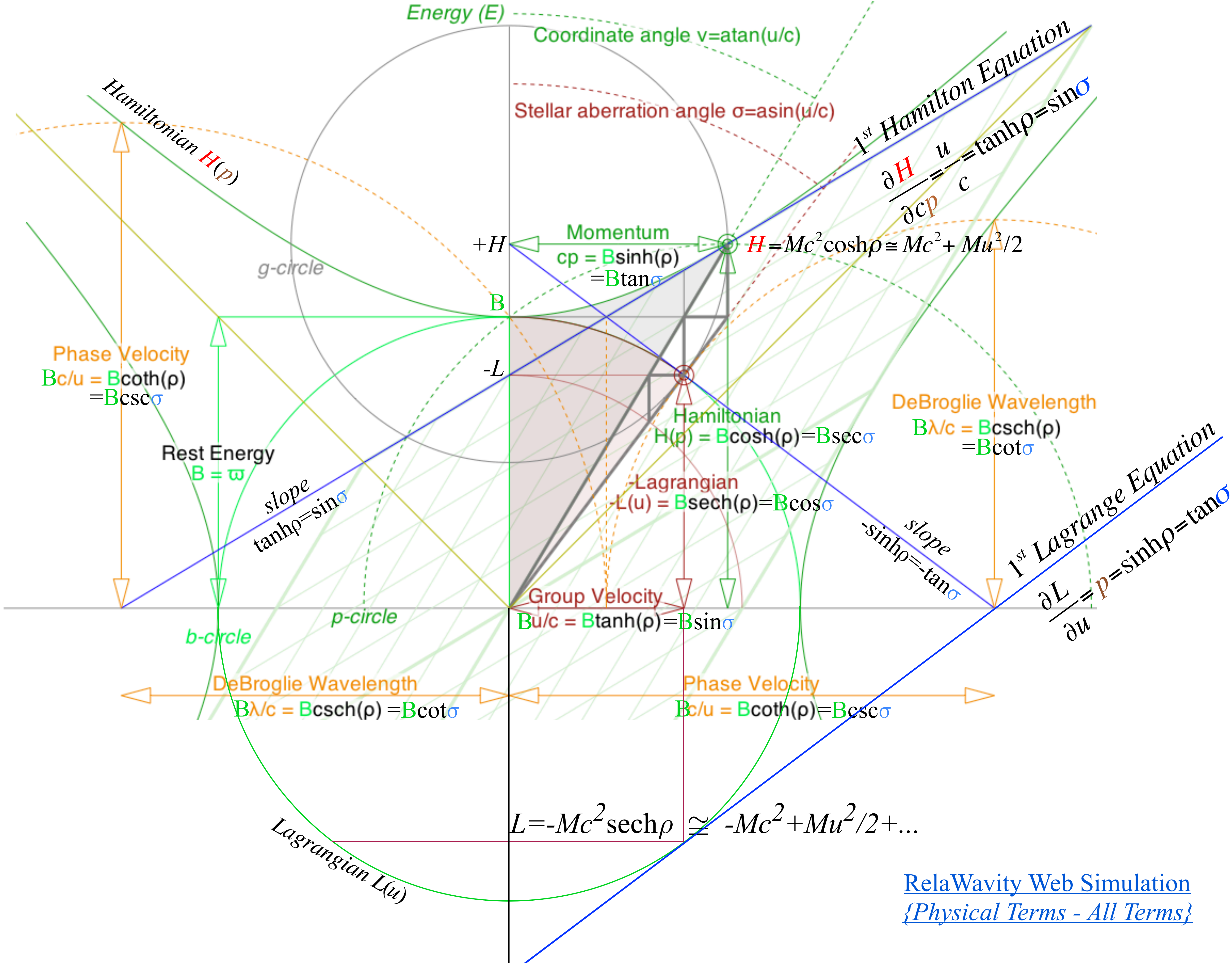
Trigonometric road maps



All this physics of relativity is mostly simple trigonometry of optical wave interference!

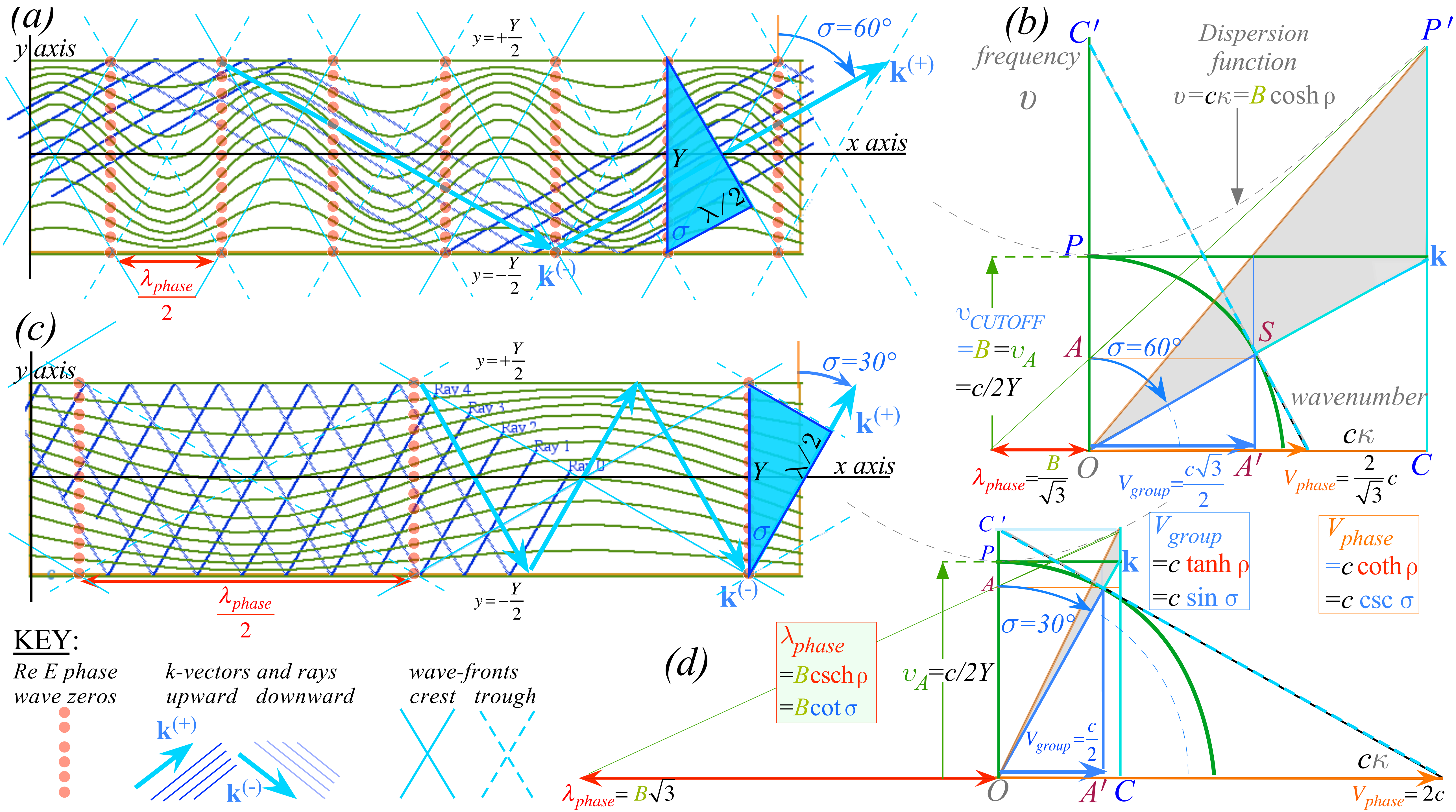
And, it derives fundamentals of quantum theory, too!

RelaWavity Web Simulation
{Physical Terms - All Terms}



[RelaWavity Web Simulation](#)
[{Physical Terms - All Terms}](#)

group	$b_{RED}^{Doppler}$	$\frac{V_{group}}{c}$	$\frac{\upsilon_{group}}{\upsilon_A}$	$\frac{\lambda_{group}}{\lambda_A}$	$\frac{\kappa_{group}}{\kappa_A}$	$\frac{\tau_{group}}{\tau_A}$	$\frac{c}{V_{group}}$	$b_{BLUE}^{Doppler}$
phase	$\frac{1}{b_{BLUE}^{Doppler}}$	$\frac{c}{V_{phase}}$	$\frac{\kappa_{phase}}{\kappa_A}$	$\frac{\tau_{phase}}{\tau_A}$	$\frac{\upsilon_{phase}}{\upsilon_A}$	$\frac{\lambda_{phase}}{\lambda_A}$	$\frac{V_{phase}}{c}$	$\frac{1}{b_{RED}^{Doppler}}$
rapidity ρ	$e^{-\rho}$	$\tanh\rho$	$\sinh\rho$	$\mathrm{sech}\,\rho$	$\cosh\rho$	$\mathrm{csch}\rho$	$\coth\rho$	$e^{+\rho}$
stellar $\forall$ *angle* σ	$1/e^{+\rho}$	$\sin\sigma$	$\tan\sigma$	$\cos\sigma$	$\sec\sigma$	$\cot\sigma$	$\csc\sigma$	$1/e^{-\rho}$
$\beta\equiv\frac{u}{c}$	$\sqrt{\frac{1-\beta}{1+\beta}}$	$\frac{\beta}{1}$	$\frac{1}{\sqrt{\beta^{-2}-1}}$	$\frac{\sqrt{1-\beta^2}}{1}$	$\frac{1}{\sqrt{1-\beta^2}}$	$\frac{\sqrt{\beta^{-2}-1}}{1}$	$\frac{1}{\beta}$	$\sqrt{\frac{1+\beta}{1-\beta}}$
value for β=3/5	$\frac{1}{2}=0.5$	$\frac{3}{5}=0.6$	$\frac{3}{4}=0.75$	$\frac{4}{5}=0.80$	$\frac{5}{4}=1.25$	$\frac{4}{3}=1.33$	$\frac{5}{3}=1.67$	$\frac{2}{1}=2.0$
		velocity	*momentum*	*Lagrangian*	*Hamiltonian*			



(a)
y axis
x axis
y=+Y/2
y=−Y/2
σ=60°
k(+)
k(-)
Y
λ/2
σ
λphase/2
(b)
C′
P′
frequency
υ
Dispersion function
υ=cκ=B cosh ρ
P
k
υCUTOFF
=B=υA
=c/2Y
A
σ=60°
S
wavenumber
cκ
λphase=B/√3
O
Vgroup=c√3/2
A′
Vphase=2/√3 c
C
(c)
y axis
x axis
y=+Y/2
y=−Y/2
σ=30°
k(+)
k(-)
Ray 4
Ray 3
Ray 2
Ray 1
Ray 0
Y
λ/2
σ
λphase/2
KEY:
Re E phase wave zeros
k-vectors and rays upward downward
wave-fronts crest trough
k(+)
k(-)
(d)
C′
P
A
k
Vgroup =c tanh ρ =c sin σ
Vphase =c coth ρ =c csc σ
σ=30°
λphase =B csch ρ =B cot σ
υA=c/2Y
Vgroup=c/2
cκ
λphase= B√3
O
A′
C
Vphase= 2c