

# QM for AMOP

## Chapter 22

# Quantum Electromagnetic Fields

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The most prevalent example of harmonic oscillator systems are electromagnetic fields, that is, light itself. Light is the genesis of Einstein-Planck quantum theory, and as shown in Unit 2, the putative source of relativity and quantum matter. A single plane em-wave mode is a 2D harmonic oscillator with two orthogonal polarization directions as described in Unit 1. So also is a single mode of a microwave cavity described by classical models in Unit 2 and by semi-classical models of light-matter interaction introduced in Unit 6. In this Unit 7 the light-matter interactions are extended to allow each protagonist, matter and light, to exhibit its quantum wave behavior in concert with the other.

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## Chapter 22 Quantum Electromagnetic Fields

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## 22.1 QUANTUM ELECTROMAGNETIC FIELDS AND TRANSITIONS

The fully quantum-mechanical treatment of electromagnetic spectral transitions will be given now. It begins by converting the classical em field equations to harmonic oscillator equations for which the quantum states are well known.

A linearly polarized plane wave may be described by the following classical vector potential derived after (17.1.12) in Unit 6.

$$\mathbf{A} = \mathbf{e}_1 2|a| \sin(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi) . \quad (22.1.1a)$$

This gives the following em fields (We add later the nonradiative or static field  $\mathbf{E} = -\nabla\Phi$ ):

$$\begin{aligned} \mathbf{E} &= -\frac{\partial \mathbf{A}}{\partial t} & \mathbf{B} &= \nabla \times \mathbf{A} \\ &= \mathbf{e}_1 E_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi) & &= (\mathbf{k} \times \mathbf{e}_1) B_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi). \end{aligned} \quad (22.1.1b)$$

The electric  $E$ -polarization vector at zero phase is along unit vector  $\mathbf{e}_1$ :

$$E_0 \mathbf{e}_1 = 2|a| \omega \mathbf{e}_1 . \quad (22.1.1c)$$

At the same time the magnetic  $B$ -polarization vector is along a unit vector  $\mathbf{b}_1 = \mathbf{e}_2$ , which is orthogonal to  $\mathbf{e}_1$  and wave vector  $\mathbf{k}$ :

$$B_0 \mathbf{b}_1 = B_0 (\mathbf{k} \times \mathbf{e}_1) = \mathbf{e}_2 2|a| \omega / c \quad (\text{Let: } k = \omega / c) \quad (22.1.1d)$$

In preparation for a quantum-mechanical theory we shall rewrite the vector potential  $\mathbf{A}$  as follows:

$$\mathbf{A} = a_{k,1} \mathbf{e}_1 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} + a_{k,1}^* \mathbf{e}_1 e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)} , \quad (22.1.2a)$$

where the complex phasor amplitude  $a = a_{k,1}$  is given by

$$a_{k,1} = -i |a_{k,1}| e^{i\phi_{k,1}} . \quad (22.1.2b)$$

This sets us up to make classical canonical phase-space coordinates  $(a, a^*)$  for the field and, eventually, the quantum field operators  $(\mathbf{a}, \mathbf{a}^\dagger)$ .

It is instructive to calculate the magnitude of the phasor for one quantum of em action. In other words, we need the magnitude of the vector potential for a wave which contains one "photon" in a cavity of volume  $V$ . The time averaged em field energy  $\langle U \rangle V$  for a plane wave in volume  $V$  follows using  $\langle \cos^2 \omega t \rangle = \frac{1}{2}$ . Average wave energy of E and B fields are equal, yet we think of B as the lesser field.

$$\begin{aligned} \langle U \rangle_V &= \left\langle \frac{\epsilon_0}{2} \mathbf{E} \cdot \mathbf{E} + \frac{1}{2\mu_0} \mathbf{B} \cdot \mathbf{B} \right\rangle_V = V \left\langle \frac{\epsilon_0}{2} 4|a|^2 \omega^2 + 4 \frac{|a|^2}{2\mu_0} k^2 \right\rangle \langle \cos^2(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi) \rangle \\ &= 2\epsilon_0 \omega^2 |a|^2 V = 2 \left( k^2 / \mu_0 \right) |a|^2 V \end{aligned} \quad (22.1.3)$$

We relate this to the Einstein-Planck energy-frequency relation ( $\langle U \rangle_V = \hbar n \omega$ ) for  $n=1$  photon.

$$|a| = \sqrt{\frac{\hbar \omega}{2\epsilon_0 \omega^2 V}} = \sqrt{\frac{\hbar}{2\epsilon_0 \omega V}} = A \quad (\text{for one } (n=1) \text{ photon}). \quad (22.1.4)$$

This is the "photon unit" of quantum field  $\mathbf{A}$ . Note that it is an inverse root function of frequency, which in turn is proportional to the magnitude  $k$  of wave vector  $\mathbf{k}$  through the vacuum dispersion relation  $\omega = ck = k/\sqrt{\mu_0 \epsilon_0}$ . Again, note that E and B contribute equally in (22.1.3).

### a. Classical Electromagnetic Fields and Operators

To completely describe an electromagnetic field in a box or "cavity" we need one phasor term like (22.1.2) for every possible value of  $\mathbf{k}$  and for each choice  $\mathbf{e}_1$  or  $\mathbf{e}_2$  of polarization orthogonal to  $\mathbf{k}$ . The complete expression for the classical  $\mathbf{A}$  is a sum over the possible modes:

$$\mathbf{A} = \sum_{\mathbf{k}} \left[ (a_{\mathbf{k}1} \mathbf{e}_1 + a_{\mathbf{k}2} \mathbf{e}_2) e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} + \text{c.c.} \right] = \sum_{\mathbf{k}} \sum_{\alpha=1}^2 \left[ a_{\mathbf{k}\alpha} \mathbf{e}_\alpha e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} + a_{\mathbf{k}\alpha}^* \mathbf{e}_\alpha e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \right]. \quad (22.1.5a)$$

Here the  $\mathbf{k}$  vector satisfies box boundary conditions, that is, periodic conditions .

$$k_\beta = n_\beta \frac{2\pi}{L} \quad (n_\beta = 1, 2, \dots, j, \quad \beta = x, y, z) \quad (22.1.5b)$$

We let  $L$  be the length of each side of a cubic box. An  $\mathbf{A}$  time derivative gives electric  $\mathbf{E}$  field.

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = \sum_{\mathbf{k}} \sum_{\alpha} \left[ i a_{\mathbf{k}\alpha} \omega \mathbf{e}_\alpha e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} - i a_{\mathbf{k}\alpha}^* \omega \mathbf{e}_\alpha e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \right]. \quad (22.1.5c)$$

An  $\mathbf{A}$  curl gives electric field. magnetic  $\mathbf{B}$  field.

$$\mathbf{B} = \nabla \times \mathbf{A} = \sum_{\mathbf{k}} \sum_{\alpha} \left[ i a_{\mathbf{k}\alpha} k \mathbf{b}_\alpha e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} - i a_{\mathbf{k}\alpha}^* k \mathbf{b}_\alpha e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \right], \quad (22.1.5d)$$

Where unit vector  $\mathbf{b}_\alpha = \mathbf{k} \times \mathbf{e}_\alpha / k$  is orthogonal to  $\mathbf{k}$  and  $\mathbf{E}$ -polarization unit vector  $\mathbf{e}$ .

### Classical Phasor Energy Relations

The classical Hamiltonian is a volume  $V$  integral of energy density (22.1.3). The electric contribution is

$$U_E V = \frac{\epsilon_0}{2} \int d^3 r \mathbf{E} \cdot \mathbf{E}, \quad (22.1.6a)$$

where

$$\begin{aligned} \mathbf{E} \cdot \mathbf{E} &= \sum_{k'\alpha'} \sum_{k\alpha} \left( i a_{k'\alpha'} \omega' \mathbf{e}_{\alpha'} e^{i(k' \cdot r - \omega' t)} + \text{c.c.} \right) \cdot \left( i a_{k\alpha} \omega \mathbf{e}_{\alpha} e^{i(k \cdot r - \omega t)} + \text{c.c.} \right) \\ &= \sum_{k'\alpha'} \sum_{k\alpha} \left[ -a_{k'\alpha'} a_{k\alpha} \omega' \omega \mathbf{e}_{\alpha'} \cdot \mathbf{e}_{\alpha} e^{i(k'+k) \cdot r - i(\omega'+\omega)t} - a_{k'\alpha'}^* a_{k\alpha}^* \omega' \omega \mathbf{e}_{\alpha'} \cdot \mathbf{e}_{\alpha} e^{i(k'+k) \cdot r + i(\omega'+\omega)t} \right. \\ &\quad \left. + a_{k'\alpha'}^* a_{k\alpha} \omega' \omega \mathbf{e}_{\alpha'} \cdot \mathbf{e}_{\alpha} e^{i(k'-k) \cdot r - i(\omega'-\omega)t} + a_{k'\alpha'} a_{k\alpha}^* \omega' \omega \mathbf{e}_{\alpha'} \cdot \mathbf{e}_{\alpha} e^{i(k'-k) \cdot r + i(\omega'-\omega)t} \right] \end{aligned} \quad (22.1.6b)$$

This simplifies if we use wave and polarization normalization conditions:

$$\int d^3 r e^{i(k'+k) \cdot r} = \delta_{k',-k} V \quad \text{and} \quad \mathbf{e}_{\alpha'} \cdot \mathbf{e}_{\alpha} = \delta_{\alpha'\alpha}$$

The result is

$$U_E V = \sum_{k\alpha} \frac{\epsilon_0 V}{2} \left[ |a_{k\alpha}|^2 \omega^2 - a_{-k\alpha}^* a_{k\alpha}^* \omega^2 e^{-2i\omega t} - a_{-k\alpha} a_{k\alpha} \omega^2 e^{-2i\omega t} \right] \quad (22.1.7)$$

The magnetic energy  $U_B V = \int d^3 r \mathbf{B} \cdot \mathbf{B} / 2\mu_0$  is like (22.1.6) if we do the following substitutions.

$$\mathbf{E} \rightarrow \mathbf{B}, \quad \frac{\epsilon_0}{2} \rightarrow \frac{1}{2\mu_0}, \quad \omega \mathbf{e}_{\alpha} \rightarrow k \mathbf{b}_{\alpha} \equiv \mathbf{k} \times \mathbf{e}_{\alpha}, \quad \omega' \mathbf{e}_{\alpha'} \equiv \mathbf{k}' \times \mathbf{e}_{\alpha'}$$

After integration the cross terms have the opposite sign as they did in (22.1.6b). We get  $\delta_{k',-k} k k' = -k^2$  in the  $\mathbf{B} \cdot \mathbf{B}$  integral instead of  $\delta_{k',-k} \omega \omega' = \omega^2$  which arose in the  $\mathbf{E} \cdot \mathbf{E}$  integral. The magnetic energy is

$$\begin{aligned} U_B V &= \sum_{k\alpha} \frac{V}{2\mu_0} \left[ 2 |a_{k\alpha}|^2 k^2 + a_{-k\alpha}^* a_{k\alpha}^* k^2 e^{2i\omega t} + a_{-k\alpha} a_{k\alpha} k^2 e^{-2i\omega t} \right] \\ &= \sum_{k\alpha} \frac{\epsilon_0 V}{2} \left[ 2 |a_{k\alpha}|^2 \omega^2 + a_{-k\alpha}^* a_{k\alpha}^* \omega^2 e^{2i\omega t} + a_{-k\alpha} a_{k\alpha} \omega^2 e^{-2i\omega t} \right]. \end{aligned} \quad (22.1.8)$$

The second line uses the optical dispersion relation.

$$\omega^2 = c^2 k^2 = k^2 / (\mu_0 \epsilon_0) \quad (22.1.9)$$

A change of sign makes the electric cross-terms in (22.1.7) cancel the magnetic ones in (22.1.8). Their sum is then just a sum of elementary mode energy density values (22.1.3). That simple formula leads to the basic Hamiltonian needed to set up quantum field theory in terms of quantum oscillators.

$$UV = (U_E + U_B)V = \sum_{\mathbf{k}\alpha} 2\varepsilon_0 \omega^2 |a_{\mathbf{k}\alpha}|^2 V. \quad (22.1.10)$$

Each  $\mathbf{k}$  and polarization  $\mathbf{e}_{\mathbf{k},\alpha}$  labeled mode is described by a classical complex phasor variable  $a_{\mathbf{k},\alpha}$ . The real and imaginary parts of  $a_{\mathbf{k},\alpha}$  can be treated as classical position  $Q_{\mathbf{k},\alpha}$  and momentum  $P_{\mathbf{k},\alpha}$  of an oscillator of unit mass as described in Chapter 20 in close analogy is to the 2-dimensional *ABCD* oscillator variables  $(X_\alpha, P_\alpha)$  defined by (10.1.1).

### Classical Field Oscillator Variables

Let us factor the phasor expression for field energy as follows so it becomes 2N dimensional oscillator energy.

$$\begin{aligned} UV &= \sum_{\mathbf{k}\alpha} 2\varepsilon_0 V \omega^2 a_{\mathbf{k}\alpha}^* a_{\mathbf{k}\alpha} = \sum_{\mathbf{k}\alpha} \frac{1}{2} \left[ 2\omega\sqrt{\varepsilon_0 V} (a_{\mathbf{k}\alpha}^{\text{Re}} - ia_{\mathbf{k}\alpha}^{\text{Im}}) \right] \left[ 2\omega\sqrt{\varepsilon_0 V} (a_{\mathbf{k}\alpha}^{\text{Re}} - ia_{\mathbf{k}\alpha}^{\text{Im}}) \right] \\ &= \sum_{\mathbf{k}\alpha} \frac{1}{2} [\omega Q_{\mathbf{k}\alpha} + iP_{\mathbf{k}\alpha}] [\omega Q_{\mathbf{k}\alpha} + iP_{\mathbf{k}\alpha}] \\ &= \sum_{\mathbf{k}\alpha} \frac{1}{2} (P_{\mathbf{k}\alpha}^2 + \omega^2 Q_{\mathbf{k}\alpha}^2) \end{aligned} \quad (22.1.11)$$

Note that vacuum dispersion frequency is a linear function  $\omega = c \cdot k$  of  $k$ , but may be a non-linear function for light-matter waves (polarons) in solids. The canonical phase space variables are

$$Q_{\mathbf{k}\alpha} = 2\sqrt{\varepsilon_0 V} a_{\mathbf{k}\alpha}^{\text{Re}} = \sqrt{\varepsilon_0 V} (a_{\mathbf{k}\alpha} + a_{\mathbf{k}\alpha}^*) \quad (22.1.12a)$$

$$P_{\mathbf{k}\alpha} = 2\omega\sqrt{\varepsilon_0 V} a_{\mathbf{k}\alpha}^{\text{Im}} = \omega\sqrt{\varepsilon_0 V} (a_{\mathbf{k}\alpha} - a_{\mathbf{k}\alpha}^*)/i. \quad (22.1.12b)$$

The inverse of the foregoing gives the original phasor variables and their conjugates in terms of P's and Q's:

$$a_{\mathbf{k}\alpha} = a_{\mathbf{k}\alpha}^{\text{Re}} + ia_{\mathbf{k}\alpha}^{\text{Im}} = \frac{1}{2\sqrt{\varepsilon_0 V}} (Q_{\mathbf{k}\alpha} + iP_{\mathbf{k}\alpha}/\omega), \quad (22.1.13a)$$

$$a_{\mathbf{k}\alpha}^* = a_{\mathbf{k}\alpha}^{\text{Re}} - ia_{\mathbf{k}\alpha}^{\text{Im}} = \frac{1}{2\sqrt{\varepsilon_0 V}} (Q_{\mathbf{k}\alpha} - iP_{\mathbf{k}\alpha}/\omega). \quad (22.1.13b)$$

The cavity energy  $UV$  in (22.1.11) shall be the classical electromagnetic field Hamiltonian function  $H = H(Q,P)$ .  $H$  describes a set of independent harmonic oscillators. To obtain a quantum field theory we make these into quantum oscillators. The situation is very similar to molecular or solid state vibration problems in which classical normal modes are quantized to give "phonons" or "polarons."

## b. Quantum Field Operators

Oscillator ladder operations  $\mathbf{a}$  and  $\mathbf{a}^\dagger$  are defined by (20.2.1) in terms of coordinate and momentum operators of a mass- $M$  particle. For each  $(\mathbf{k}, \alpha)$  mode this definition, with  $M=1$ , becomes the following.

$$\mathbf{a}_{\mathbf{k}\alpha} = \sqrt{\frac{\omega}{2\hbar}} (\mathbf{Q}_{\mathbf{k}\alpha} + i\mathbf{P}_{\mathbf{k}\alpha}/\omega), \quad (22.1.14a)$$

$$\mathbf{a}_{\mathbf{k}\alpha}^\dagger = \sqrt{\frac{\omega}{2\hbar}} (\mathbf{Q}_{\mathbf{k}\alpha} - i\mathbf{P}_{\mathbf{k}\alpha}/\omega), \quad (22.1.14b)$$

Here boldface notation  $\mathbf{Q}_{\mathbf{k}}$  and  $\mathbf{P}_{\mathbf{k}}$  indicates the quantum operators that correspond to the classical phase variables  $Q_{\mathbf{k}}$  and  $P_{\mathbf{k}}$ , respectively.

By comparing (22.1.14) with (22.1.13) we note that the ladder operators are proportional to whatever operator would correspond to the classical phasor amplitude. So with correspondence  $\mathbf{Q}_{\mathbf{k}\alpha}$  with  $Q_{\mathbf{k}}$  and  $\mathbf{P}_{\mathbf{k}}$  with  $P_{\mathbf{k}}$  we have the following phasor correspondence relations:

$$\begin{aligned} a_{\mathbf{k}\alpha} &= \frac{1}{2\sqrt{\varepsilon_0 V}} (Q_{\mathbf{k}\alpha} + iP_{\mathbf{k}\alpha}/\omega) \rightarrow \frac{1}{2\sqrt{\varepsilon_0 V}} (\mathbf{Q}_{\mathbf{k}\alpha} + i\mathbf{P}_{\mathbf{k}\alpha}/\omega) = \frac{1}{2\sqrt{\varepsilon_0 V}} \sqrt{\frac{2\hbar}{\omega}} \mathbf{a}_{\mathbf{k}\alpha} = \sqrt{\frac{\hbar}{2\varepsilon_0 \omega V}} \mathbf{a}_{\mathbf{k}\alpha}, \\ a_{\mathbf{k}\alpha}^* &= \frac{1}{2\sqrt{\varepsilon_0 V}} (Q_{\mathbf{k}\alpha} - iP_{\mathbf{k}\alpha}/\omega) \rightarrow \frac{1}{2\sqrt{\varepsilon_0 V}} (\mathbf{Q}_{\mathbf{k}\alpha} - i\mathbf{P}_{\mathbf{k}\alpha}/\omega) = \frac{1}{2\sqrt{\varepsilon_0 V}} \sqrt{\frac{2\hbar}{\omega}} \mathbf{a}_{\mathbf{k}\alpha}^\dagger = \sqrt{\frac{\hbar}{2\varepsilon_0 \omega V}} \mathbf{a}_{\mathbf{k}\alpha}^\dagger, \end{aligned} \quad (22.1.15)$$

The proportionality or scale factor in (22.1.15) is just the “photon amplitude” derived in (22.1.4). Note that coordinate and momentum operators are observables and are self-conjugate ( $\mathbf{Q} = \mathbf{Q}^\dagger$  and  $\mathbf{P} = \mathbf{P}^\dagger$ ). The phasor operator  $\mathbf{a}$  is a complex combination of observables and thus not a self-conjugate observable itself.

The oscillator Hamiltonian operator for the quantum field is the same form as (20.2.4), namely,

$$H = \sum_{\mathbf{k}\alpha} \hbar\omega_{\mathbf{k}\alpha} \left( \mathbf{a}_{\mathbf{k}\alpha}^\dagger \mathbf{a}_{\mathbf{k}\alpha} + \frac{1}{2} \right). \quad (22.1.16)$$

This is the same for the classical energy (8.5.10) or (22.1.11) except for the extra  $\hbar\omega_{\mathbf{k},\alpha}/2$  terms which are each  $(\mathbf{k}, \alpha)$  mode's quantum zero-point energy. Number operator ( $\mathbf{a}_{\mathbf{k}\alpha}^\dagger \mathbf{a}_{\mathbf{k}\alpha}$ )-eigenvalues are the number  $n_{\mathbf{k}\alpha}$  of  $(\mathbf{k}, \alpha)$  “photons” or field quanta and each  $\mathbf{a}_{\mathbf{k}\alpha}^\dagger$  (or  $\mathbf{a}_{\mathbf{k}\alpha}$ ) can raise (or lower) its assigned photon number.

$$\begin{aligned} \mathbf{a}_{\mathbf{k}\alpha}^\dagger \left| \cdots n_{\mathbf{k}\alpha} \cdots n_{\mathbf{k}'\alpha'} \cdots \right\rangle &= \sqrt{n_{\mathbf{k}\alpha} + 1} \left| \cdots n_{\mathbf{k}\alpha} + 1 \cdots n_{\mathbf{k}'\alpha'} \cdots \right\rangle, \\ \mathbf{a}_{\mathbf{k}\alpha} \left| \cdots n_{\mathbf{k}\alpha} \cdots n_{\mathbf{k}'\alpha'} \cdots \right\rangle &= \sqrt{n_{\mathbf{k}\alpha}} \left| \cdots n_{\mathbf{k}\alpha} - 1 \cdots n_{\mathbf{k}'\alpha'} \cdots \right\rangle, \\ \mathbf{a}_{\mathbf{k}'\alpha'}^\dagger \left| \cdots n_{\mathbf{k}\alpha} \cdots n_{\mathbf{k}'\alpha'} \cdots \right\rangle &= \sqrt{n_{\mathbf{k}'\alpha'} + 1} \left| \cdots n_{\mathbf{k}\alpha} \cdots n_{\mathbf{k}'\alpha'} + 1 \cdots \right\rangle, \\ \mathbf{a}_{\mathbf{k}'\alpha'} \left| \cdots n_{\mathbf{k}\alpha} \cdots n_{\mathbf{k}'\alpha'} \cdots \right\rangle &= \sqrt{n_{\mathbf{k}'\alpha'}} \left| \cdots n_{\mathbf{k}\alpha} \cdots n_{\mathbf{k}'\alpha'} - 1 \cdots \right\rangle. \end{aligned} \quad (22.1.17)$$

These relations follow (20.2.13). Here each additional quanta contributes an increase in  $\mathbf{A}$  amplitude equal to the scale factor  $\sqrt{\hbar/2\varepsilon_0\omega V}$  that appears in the correspondence relation (22.1.15).

The quantum  $\mathbf{A}$ -field operator corresponding to the classical field (22.1.5a) is found by replacing  $a_{\mathbf{k}\alpha}$  and  $a_{\mathbf{k}\alpha}^\dagger$  according to (22.1.15):

$$\mathbf{A} = \sum_{\mathbf{k}\alpha} \sqrt{\frac{\hbar}{2\varepsilon_0\omega V}} \left[ \mathbf{a}_{\mathbf{k}\alpha} \mathbf{e}_\alpha e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \mathbf{a}_{\mathbf{k}\alpha}^\dagger \mathbf{e}_\alpha e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \right]. \quad (22.1.18)$$

Time dependence of ladder operators is determined by Bloch equations:  $i\hbar\dot{\mathbf{O}} = [\mathbf{H}, \mathbf{O}]$  [Recall (10.5.12)]:

$$\begin{aligned} i\hbar\dot{\mathbf{a}}_{\mathbf{k}\alpha} &= [\mathbf{H}, \mathbf{a}_{\mathbf{k}\alpha}] & i\hbar\dot{\mathbf{a}}_{\mathbf{k}\alpha}^\dagger &= [\mathbf{H}, \mathbf{a}_{\mathbf{k}\alpha}^\dagger] \\ &= -\hbar\omega\mathbf{a}_{\mathbf{k}\alpha} & &= \hbar\omega\mathbf{a}_{\mathbf{k}\alpha}^\dagger \end{aligned}$$

Here we use the  $\mathbf{a}^\dagger\mathbf{a}$  form (22.1.16) of the field Hamiltonian and the N-dimensional commutation relation (21.1.5) that is repeated in the following [Recall also (20.2.3)]:

$$[\mathbf{a}_{\mathbf{k}\alpha}, \mathbf{a}_{\mathbf{k}'\alpha'}^\dagger] = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\alpha,\alpha'}1. \quad (22.1.19)$$

According to the foregoing  $\dot{\mathbf{a}}$  equations the ladder operators have the following time-dependent phases:

$$\mathbf{a}_{\mathbf{k}\alpha} = \mathbf{a}_{\mathbf{k}\alpha}(0)e^{i\omega t}, \quad \mathbf{a}_{\mathbf{k}\alpha}^\dagger = \mathbf{a}_{\mathbf{k}\alpha}^\dagger(0)e^{-i\omega t}$$

The phases cancel time factors in (22.1.18) to give a time-independent field operator.

$$\mathbf{A} = \sum_{\mathbf{k}\alpha} \sqrt{\frac{\hbar}{2\varepsilon_0\omega V}} \left[ \mathbf{a}_{\mathbf{k}\alpha}(0) e^{i\mathbf{k}\cdot\mathbf{r}} + \mathbf{a}_{\mathbf{k}\alpha}^\dagger(0) e^{-i\mathbf{k}\cdot\mathbf{r}} \right] \mathbf{e}_\alpha \quad (22.1.20a)$$

The electric and magnetic quantum field operators follow from (22.1.15).

$$\mathbf{E} = \sum_{\mathbf{k}\alpha} \sqrt{\frac{\hbar}{2\varepsilon_0\omega V}} \left[ i\omega\mathbf{a}_{\mathbf{k}\alpha}(0) e^{i\mathbf{k}\cdot\mathbf{r}} - i\omega\mathbf{a}_{\mathbf{k}\alpha}^\dagger(0) e^{-i\mathbf{k}\cdot\mathbf{r}} \right] \mathbf{e}_\alpha \quad (22.1.20b)$$

$$\mathbf{B} = \sum_{\mathbf{k}\alpha} \sqrt{\frac{\hbar}{2\varepsilon_0\omega V}} \left[ ik\mathbf{a}_{\mathbf{k}\alpha}(0) e^{i\mathbf{k}\cdot\mathbf{r}} - ik\mathbf{a}_{\mathbf{k}\alpha}^\dagger(0) e^{-i\mathbf{k}\cdot\mathbf{r}} \right] \mathbf{b}_\alpha \quad (22.1.20c)$$

When atoms are much smaller than the wavelength  $\lambda = 2\pi/k$  of the radiation, the fields can be simplified by the *dipole approximation*  $e^{i\mathbf{k}\cdot\mathbf{r}} \cong 1$ .



$$\mathbf{A} \equiv \sum_{\mathbf{k}\alpha} \sqrt{\frac{\hbar}{2\varepsilon_0\omega V}} \left[ \mathbf{a}_{\mathbf{k}\alpha} + i\omega \mathbf{a}_{\mathbf{k}\alpha}^\dagger \right] \mathbf{e}_\alpha = \sum_{\mathbf{k}\alpha} \sqrt{\frac{1}{\varepsilon_0 V}} \mathbf{Q}_{\mathbf{k}\alpha} \mathbf{e}_\alpha, \quad (22.1.20d)$$

$$\mathbf{E} \equiv \sum_{\mathbf{k}\alpha} \sqrt{\frac{\hbar}{2\varepsilon_0\omega V}} i\omega \left[ \mathbf{a}_{\mathbf{k}\alpha} - \mathbf{a}_{\mathbf{k}\alpha}^\dagger \right] \mathbf{e}_\alpha = \sum_{\mathbf{k}\alpha} \sqrt{\frac{1}{\varepsilon_0 V}} \mathbf{P}_{\mathbf{k}\alpha} \mathbf{e}_\alpha, \quad (22.1.20e)$$

$$\mathbf{B} \equiv \sum_{\mathbf{k}\alpha} \sqrt{\frac{\hbar}{2\varepsilon_0\omega V}} ik \left[ \mathbf{a}_{\mathbf{k}\alpha} - \mathbf{a}_{\mathbf{k}\alpha}^\dagger \right] \mathbf{b}_\alpha. \quad (22.1.20f)$$

Note also the simple connection between the approximate  $\mathbf{A}$  and  $\mathbf{E}$  and the canonical field coordinates  $\mathbf{Q}_{\mathbf{k}}$  and momenta  $\mathbf{P}_{\mathbf{k}}$  which follows from (22.1.14).

### c. Electromagnetic Quantum States and Atomic Transitions

Consider an atom coupled to an electromagnetic cavity. Suppose this system starts in a state in which the atomic state is  $|s\rangle$  and all the photon numbers  $n_{\mathbf{k}\alpha}^s$  are definitely known. We consider some of the possible final states and their probabilities as a function of time.

The states of the whole system at the start and finish will be labeled  $|S\rangle$  and  $|F\rangle$ , respectively.

The starting state  $|S\rangle$  is a ket-ket product of atomic  $|s\rangle$  and radiation  $|\cdots n_{\mathbf{k}\alpha}^s \cdots n_{\mathbf{k}'\alpha'}^s \cdots\rangle$  states:

$$|S\rangle = |s\rangle |\cdots n_{\mathbf{k}\alpha}^s \cdots n_{\mathbf{k}'\alpha'}^s \cdots\rangle$$

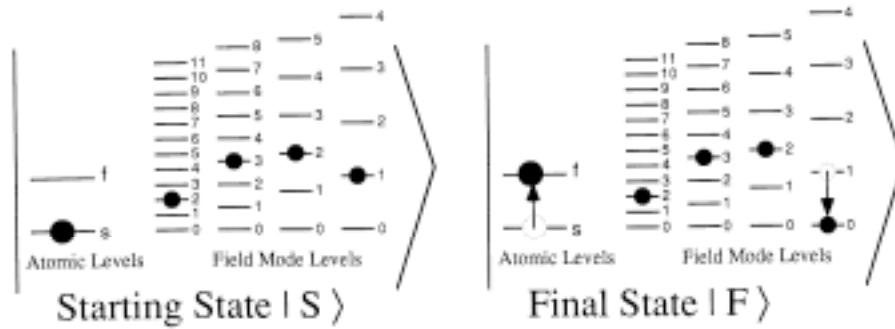
The final state is written in a similar way:

$$|F\rangle = |f\rangle |\cdots n_{\mathbf{k}\alpha}^f \cdots n_{\mathbf{k}'\alpha'}^f \cdots\rangle$$

One may picture the states by imagining atomic and electromagnetic levels as sketched in Fig.

22.1.1. A typical transition that conserves energy (more or less) can be imagined as going from the state on one side of the figure to the other. There we imagine that the atom jumps *up* from level  $|s\rangle$  to  $|f\rangle$  while simultaneously one mode number jumps *down* one level. This is an atomic *absorption* process. (The atom appears to swallow a photon.) If this is reversed, or if the atom jumps *down* from level  $|s\rangle$  to  $|f'\rangle$  while a mode number jumps *up*, the process is called an *emission*. (The atom appears to spit out a photon.) As we will see, these two processes are usually the most likely ones.

The derivation of the probabilities for quantum field atomic transitions of the type shown in Fig. 22.1.1 are given now. This derivation uses the first-order perturbation formula that only involves creating or destroying one photon. Higher  $N^{\text{th}}$ -order processes involve products of the  $1^{\text{st}}$ -order factors discussed below and are called  $N$ -photon processes for that reason.



**Fig. 22.1.1** Atom-field energy levels for initial and final states in an atomic absorption process involving a single photon from a resonant electromagnetic field mode.

The 1<sup>st</sup>-order perturbation results of (18.2.5) involve integral terms that is a *spectral functions*  $S(\Delta,t)$ .

$$c_F^{(1)}(t) = \delta_{FS} + \frac{1}{i\hbar} \int_0^t dt_1 e^{i\omega_{FW}t_1} \langle F | H_I | S \rangle = \delta_{FS} + \frac{1}{i\hbar} S(\omega_{FS}, t) \langle F | H_I | S \rangle \text{ where: } S(\Delta, t) = 2e^{it\Delta/2} \frac{\sin(t\Delta/2)}{\Delta} \tag{22.1.21}$$

The spectral function is plotted Fig. 18.2.1. Here its *detuning parameter*  $\Delta$  includes the difference of both the atomic and radiation energy.

$$\Delta = \omega_{FS} \equiv \omega_F - \omega_S = \left[ \omega_f + \sum_{\mathbf{k}\alpha} \left( n_{\mathbf{k}\alpha}^f + \frac{1}{2} \right) \omega_{\mathbf{k}} \right] - \left[ \omega_s + \sum_{\mathbf{k}\alpha} \left( n_{\mathbf{k}\alpha}^s + \frac{1}{2} \right) \omega_{\mathbf{k}} \right]. \tag{22.1.22}$$

For the absorption process depicted in Fig. 22.1.1 the mode number has gone down one step for the  $(\mathbf{k},\alpha)$  mode  $n_{\mathbf{k}\alpha}^f = n_{\mathbf{k}\alpha}^s - 1$  while the atom went up. Other quanta stay the same, so the detuning is

$$\Delta = \omega_f - \omega_s - \omega_{\mathbf{k}} = \omega_{fs} - \omega_{\mathbf{k}}.$$

Zero detuning means having a mode whose frequency  $\omega_{\mathbf{k}\alpha}$  matches the atomic transition frequency  $\omega_{fs}$ , just like the semi-classical definition (18.2.5g) for absorption resonance. (Compare it to the  $\Delta$  above.)

Now we see some important differences between quantum field theory calculations and semi-classical ones. For one thing, if you really insist on counting every photon, then the absorption and emission processes become quite distinct.

*Single-Mode Atomic Dipole Transitions*

A 1<sup>st</sup>-order  $S$  to  $F$  transition probability  $|c_F|^2$  obtained follows from (22.1.21) assuming  $S \neq F$ .

$$P_{F \leftarrow S} = |c_F|^2 = \left| \langle F | H_I | S \rangle \right|^2 \frac{\sin^2(t\Delta/2)}{\hbar^2 (\Delta/2)^2}. \tag{22.1.23}$$

We now evaluate the matrix element of the  $\mathbf{E} \cdot \mathbf{r}$  interaction operator in (17.1.20)

$$H_I = q\mathbf{E} \cdot \mathbf{r} \cong -q \sum_{\mathbf{k}\alpha} \sqrt{\frac{\hbar}{2\varepsilon_0\omega V}} \left[ i\omega \mathbf{a}_{\mathbf{k}\alpha} - i\omega \mathbf{a}_{\mathbf{k}\alpha}^\dagger \right] \mathbf{e}_\alpha \cdot \mathbf{r} \quad (22.1.24)$$

The quantized  $E$  field (22.1.20b) is used with the dipole approximation  $e^{i\mathbf{k}\cdot\mathbf{r}} \cong 1$ . The matrix element consists of a field part followed by the atomic matrix elements  $\langle f | \mathbf{r} | s \rangle = (\langle f | x | s \rangle, \langle f | y | s \rangle, \langle f | z | s \rangle)$ .

$$\langle F | H_I | S \rangle = q \sum_{\mathbf{k}\alpha} \left[ \langle \dots n_{\mathbf{k}\alpha}^f \dots | i\mathbf{a}_{\mathbf{k}\alpha}^\dagger | \dots n_{\mathbf{k}\alpha}^s \rangle - \langle \dots n_{\mathbf{k}\alpha}^f \dots | i\mathbf{a}_{\mathbf{k}\alpha} | \dots n_{\mathbf{k}\alpha}^s \rangle \right] \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \mathbf{e}_\alpha \cdot \langle f | \mathbf{r} | s \rangle.$$

The field part of this matrix element is quite selective. If more than one mode changes its photon number, the whole thing is zero. (Recall that  $\langle n | n' \rangle = \delta_{nn'}$ .) The only possible nonzero elements occur when a single mode goes up or down by exactly one photon. The two possible types of nonzero matrix elements are listed in the following:

$$\langle F | H_I | S \rangle = \left[ i\sqrt{n_{\mathbf{k}\alpha}^s + 1} + 0 \right] \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} q \mathbf{e}_\alpha \cdot \mathbf{r}_{fs} \quad \text{if all } n^f = n^s \text{ except } n_{\mathbf{k}\alpha}^f = n_{\mathbf{k}\alpha}^s + 1 \quad (1 \text{ photon emission})$$

$$\langle F | H_I | S \rangle = \left[ 0 - i\sqrt{n_{\mathbf{k}\alpha}^s} \right] \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} q \mathbf{e}_\alpha \cdot \mathbf{r}_{fs} \quad \text{if all } n^f = n^s \text{ except } n_{\mathbf{k}\alpha}^f = n_{\mathbf{k}\alpha}^s - 1 \quad (1 \text{ photon absorption})$$

$$\langle F | H_I | S \rangle = 0, \quad \text{otherwise.} \quad (22.1.25)$$

Field matrix elements follow from (22.1.17). Atomic dipole expectation value is denoted by  $\mathbf{r}_{fs} = \langle f | \mathbf{r} | s \rangle$ .

If matrix element (22.1.25) allows any transition between  $|S\rangle$  and  $|F\rangle$  it is an emission or *else* an absorption *but not both*. If it allows one, then the probability for the other is zero. This is very different from the semi-classical transition amplitude (18.2.5) or (18.3.2) in which both processes would simultaneously have non-vanishing probability. The semi-classical amplitude is a sum of a resonant and a non-resonant spectral function. The quantum amplitude (22.1.21) has only one spectral function in the  $c_F$  expression. Strict photon counting prevents absorption from interfering with emission.

#### Multimode Atomic Dipole Transitions

Suppose we accept (or are forced to accept) any of a set of possible final photon states ranging from state  $|F\rangle = |f\rangle | \dots n_{\mathbf{k}\alpha} - 1 \dots n_{\mathbf{k}'\alpha'} \dots \rangle$  in which mode  $(\mathbf{k}, \alpha)$  lost a photon to state  $|F'\rangle = |f\rangle | \dots n_{\mathbf{k}\alpha} \dots n_{\mathbf{k}'\alpha'} - 1 \dots \rangle$  in which mode  $(\mathbf{k}', \alpha')$  lost a photon. In each case the atom jumps up from state  $|s\rangle$  to state  $|f\rangle$ , but here we let it accept a photon from a range of cavity modes. It is only necessary that the donor modes have nonzero

photon number and a frequency  $\omega_{k\alpha}$  that is close enough to the atomic transition frequency  $\omega_f - \omega_s = \omega_{fs}$  so the spectral function  $|S|^2$  gives a measurable value.

The total probability for the atomic  $f \leftarrow s$  transition will be a sum of probabilities  $|c_F|^2 + \dots + |c_{F'}|^2$  as though the contribution of each mode is distinct:

$$P_{f \leftarrow s} = |c_F|^2 + \dots + |c_{F'}|^2 = \sum_{k\alpha} |c_F|^2 = \sum_{k\alpha} \frac{\sin^2(t\Delta/2)}{\hbar^2 (\Delta/2)^2} |\langle H_I \rangle|^2. \quad (22.1.26)$$

Since we are effectively counting the photons from each mode, the amplitudes  $c_F \dots c_{F'}$  have random relative phases and interference between them is washed out. Hence the total probability is the sum of their squares  $\sum |c_F|^2$  instead of the more complicated square of the sum  $|\sum c_F|^2$ . (Recall the distinction between incoherent “peeking” and coherent interference in equations (1.3.10) and (1.3.11) of Ch. 1.)

The sum over mode wave vector  $\mathbf{k}$  can be converted to an integral over  $k = [\mathbf{k}]$  or over mode frequency  $\omega = ck$ . According to (22.1.5b) the  $\mathbf{k}$  sum is a sum over integer values of photon number  $n_\alpha = k_\alpha L/2\pi = 1, 2, \dots$ , that converts to the following integral.

$$\sum_k = \sum_{n_x} \sum_{n_y} \sum_{n_z} \Delta n_x \Delta n_y \Delta n_z \equiv \int dk_x \frac{\Delta n_x}{\Delta k_x} \int dk_y \frac{\Delta n_y}{\Delta k_y} \int dk_z \frac{\Delta n_z}{\Delta k_z} = \left( \frac{L}{2\pi} \right)^3 \int dk_x \int dk_y \int dk_z.$$

Here the sum is converted to an integral over Cartesian  $k$  components using

$$\Delta n_x = 1 = \frac{L}{2\pi} \Delta k_x, \quad \Delta n_y = 1 = \frac{L}{2\pi} \Delta k_y, \quad \Delta n_z = 1 = \frac{L}{2\pi} \Delta k_z.$$

This sum can then be converted to a polar coordinate integral in  $k$  space.

$$\sum_k = \left( \frac{L}{2\pi} \right)^3 \int d^3\mathbf{k} = \frac{V}{(2\pi)^3} \int d\Omega_k \int k^2 dk = \frac{V}{(2\pi)^3} \int d\phi_k \int d\theta_k \sin\theta_k \quad (22.1.27)$$

$(\phi_k, \theta_k)$  are azimuth and polar angles of  $\mathbf{k}$ ,  $d\Omega_k$  is the incremental solid angle in  $k$ -space, and  $V = L^3$  is the cavity volume. This reduces the probability sum (22.1.26) to an integral over solid angle and  $k$  or  $\omega = ck$ .

$$P_{f \leftarrow s} = \sum_\alpha \frac{V}{(2\pi)^3} \int d\Omega_k \int_0^\infty k^2 dk |c_F|^2 = \sum_\alpha \frac{V}{(2\pi)^3} \int d\Omega_k \int_0^\infty \frac{\omega^2}{c^3} d\omega |c_F|^2 = \sum_\alpha \frac{V}{(2\pi)^3} \int d\Omega_k \int d\omega \frac{\omega^2}{c^3} \frac{\sin^2(t\Delta/2)}{\hbar^2 (\Delta/2)^2} |\langle H_I \rangle|^2 \quad (22.1.28)$$

Now we approximate the integral of the spectral function  $|S(\Delta, t)|^2$  by assuming that time  $t$  is large enough that  $|S|^2$  becomes very narrow. (Recall Fig. 18.2.2.) Then most of the probability comes from the neighborhood around zero detuning ( $\Delta = 0$ ). We may set  $\omega_{k\alpha} = \omega = \omega_{fs}$  and put all other functions of frequency outside the integral.

$$P_{f \leftarrow s} \cong \sum_{\alpha} \frac{V}{(2\pi)^3} \int d\Omega_{\mathbf{k}} \frac{\omega^2}{\hbar^2 c^3} |\langle H_I \rangle|^2 \int_0^{\infty} d\omega \frac{\sin^2(t\Delta/s)}{\hbar^2 (\Delta/2)^2}$$

The area under the spectral function is simply the elapsed time multiplied by  $2\pi$  as was noted after the Fermi Golden Rule equation (18.7.10).

$$P_{f \leftarrow s} \cong \sum_{\alpha} \int d\Omega_{\mathbf{k}} \frac{V\omega}{(2\pi)^3 \hbar^2 c^3} |\langle H_I \rangle|^2 2\pi \cdot t \quad (22.1.29)$$

The peak of the spectral function at  $\Delta = 0$  goes up quadratically with time, but the area only increases linearly. The width of the peak decreases linearly with time according to the uncertainty relation (18.2.9). As a result the time derivative or rate  $R_{f \leftarrow s} = \dot{P}_{f \leftarrow s}$  of the transition probability is a constant in this approximation, the *Fermi golden rule* of constant transition rates. (Recall (18.2.11).)

$$R_{f \leftarrow s} \cong \sum_{\alpha} \int d\Omega_{\mathbf{k}} \rho(\omega_{fs}) |\langle H_I \rangle|^2 \frac{2\pi}{\hbar^2}, \quad (22.1.30a)$$

Here the spectral density of modes  $\rho(\omega)$  is defined.

$$\rho(\omega) = \frac{V\omega^2}{(2\pi)^3 c^3}. \quad (22.1.30b)$$

The absorption dipole matrix element (22.1.25) gives the following rate if the photon number in near-resonant modes is  $n_{k\alpha}^s \equiv n$ :

$$\begin{aligned} R_{f \leftarrow s} &= n \frac{V\omega^2}{(2\pi)^3 c^3} \frac{2\pi}{\hbar^2} \frac{\hbar\omega}{2\epsilon_0 V} q^2 \sum_{\alpha} \int d\Omega_{\mathbf{k}} |\mathbf{e}_{\alpha} \cdot \mathbf{r}_{fs}|^2 \\ &= n \left( \frac{\omega^3}{\hbar c^3} \right) \left( \frac{q^2}{4\pi\epsilon_0} \right) \int d\Omega_{\mathbf{k}} \left( |\mathbf{e}_1 \cdot \mathbf{r}_{fs}|^2 + |\mathbf{e}_2 \cdot \mathbf{r}_{fs}|^2 \right). \end{aligned}$$

The polarization sum and integral is simplified by using the vector relation

$$|\mathbf{r}|^2 = |\mathbf{e}_1 \cdot \mathbf{r}|^2 + |\mathbf{e}_2 \cdot \mathbf{r}|^2 + |\hat{\mathbf{k}} \cdot \mathbf{r}|^2.$$

If we let the induced dipole  $\mathbf{r} = \mathbf{r}_{fs}$  be along the polar  $z$  axis, then  $\hat{\mathbf{k}} \cdot \mathbf{r}_{fs} = |\mathbf{r}_{fs}| \cos \theta_k$ . The sum and integral is then easily evaluated:

$$\begin{aligned} \sum_{\alpha} \int d\Omega_k |\mathbf{e}_{\alpha} \cdot \mathbf{r}_{fs}|^2 &= \int d\Omega_k \left( |\mathbf{e}_1 \cdot \mathbf{r}_{fs}|^2 + |\mathbf{e}_2 \cdot \mathbf{r}_{fs}|^2 + |\hat{\mathbf{k}} \cdot \mathbf{r}_{fs}|^2 - |\hat{\mathbf{k}} \cdot \mathbf{r}_{fs}|^2 \right) \\ &= \int d\Omega_k \left( 1 - \cos^2 \theta_k \right) |\mathbf{r}_{fs}|^2 = \int d\phi \int d\theta \sin^3 \theta |\mathbf{r}_{fs}|^2 = \frac{8\pi}{3} |\mathbf{r}_{fs}|^2. \end{aligned}$$

The resulting absorption rate is

$$R_{f \leftarrow s}(\text{absorption}) \equiv nA = n \frac{4\omega^3}{3\hbar c^3} |\mathbf{r}_{fs}|^2 = B. \quad (22.1.31)$$

The corresponding emission rate is similar except  $(n + 1)$  replaces  $n$  in the matrix element (22.1.25):

$$R_{f \leftarrow s}(\text{emission}) = (n + 1)A = A + B \quad (22.1.32a)$$

The first term is the famous *Einstein-A-coefficient* or *spontaneous decay rate* of an excited atom in a vacuum ( $n = 0$ ):

$$A = \frac{4\omega^3}{3\hbar c^3} \frac{q^2}{4\pi\epsilon_0} |\mathbf{r}_{fs}|^2. \quad (22.1.32b)$$

The second term is the *Einstein-B-coefficient*, which is the *stimulated decay rate* induced by the presence of  $n$  resonant photons:

$$B = nA = n \frac{4\omega^3}{3\hbar c^3} \frac{q^2}{4\pi\epsilon_0} |\mathbf{r}_{fs}|^2. \quad (22.1.32c)$$

$B$  is the only contribution to the absorption rate (22.1.31) since spontaneous excitation is impossible in this approximation.

### c. "Impotence" of Photon Number States

We noted that the first-order transition amplitude  $c_F^{(1)}$  in (22.1.21) could only have an absorption term or else an emission term but not both. This is because the quantum field transition matrix element (22.1.25) cannot be nonzero for both processes at once. We also noted that the semi-classical transition amplitude (18.2.5) does have terms from both processes. In fact, the derivation of resonant excitation of the oscillator expectation value  $\langle x \rangle$  depends upon the precise interference between these two terms to reproduce the classical result. [Recall comparison in (18.3.5).] At first this might seem to be a case where quantum theory is failing to approach the correct classical limit. We need to resolve this issue.

The calculation of the atomic position expectation value  $\langle \Psi | x | \Psi \rangle$  using quantum field number states is quite different and so are the results. The perturbed atom-field state is

$$|\Psi\rangle = |S\rangle + \sum_{F \neq S} e^{-i\omega_{FS}t} c_F |F\rangle,$$

where the first-order approximation to the amplitude is

$$c_F^{(1)} = \delta_{FS} + \frac{1}{i\hbar} \int_0^t dt_1 e^{i\omega_{FS}t_1} \langle F | H_I | S \rangle,$$

according to the basic time-dependent perturbation formulas (18.2.5). The matrix element is given by matrix elements (22.1.25) between the initial state  $|S\rangle = |s\rangle_{\text{atom}} |n_{k\alpha}^s \cdots n_{k'\alpha'}^s\rangle_{\text{field}}$  and final state

$|F\rangle = |f\rangle_{\text{atom}} |n_{k\alpha}^f \cdots n_{k'\alpha'}^f\rangle_{\text{field}}$ . The atom part only requires that the induced dipole moment  $q\mathbf{r}_{fs} \equiv \langle f | \mathbf{r} | s \rangle q$

be nonzero. The field part is more restrictive; it requires that exactly one mode gain or lose a single photon. The result is a perturbed state of the form

$$\begin{aligned} |\Psi\rangle = |S\rangle = & |s\rangle |n_{k\alpha}^s \cdots n_{k'\alpha'}^s\rangle + d_{k\alpha}^\alpha(f) |f\rangle |n_{k\alpha}^s - 1 \cdots n_{k'\alpha'}^s\rangle + \cdots \\ & + d_{k'\alpha'}^\alpha(f) |f\rangle |n_{k\alpha}^s \cdots n_{k'\alpha'}^s - 1\rangle + \cdots \\ & + d_{k'\alpha'}^e(f) |f\rangle |n_{k\alpha}^s + 1 \cdots n_{k'\alpha'}^s\rangle + \cdots \\ & + d_{k'\alpha'}^e(f') |f'\rangle |n_{k\alpha}^s \cdots n_{k'\alpha'}^s + 1\rangle + \cdots, \end{aligned}$$

Here the transition amplitude to a higher state  $|f\rangle$  due to absorption from mode  $(\mathbf{k}, \alpha)$  is

$$d_{k\alpha}^\alpha(f) = \frac{e^{-i\omega_f t}}{\hbar} \int_0^t dt_1 e^{i t_1 \Delta} w_k \sqrt{n_{k\alpha} + 1} \mathbf{e}_\alpha \cdot \mathbf{r}_{fs} \quad (\Delta = \omega_{fs} - \omega_k)$$

The transition amplitude to a lower state  $|f'\rangle$  due to emission from mode  $(\mathbf{k}, \alpha)$  is

$$d_{k\alpha}^e(f') = \frac{e^{-i\omega_{f'} t}}{\hbar} \int_0^t dt_1 e^{i t_1 \Delta} w_k \sqrt{n_{k\alpha} + 1} \mathbf{e}_\alpha \cdot \mathbf{r}_{f's} \quad (\Delta = \omega_{fs} + \omega_k)$$

In each case  $w_k$  is the scale factor (22.1.4) times frequency  $\omega$  and charge  $q$ .

$$w_{k\alpha} = \omega_k q \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k V}}$$

If  $|s\rangle$  is the atomic ground state, the emission terms with  $d^e$  amplitudes are nonexistent. However, there may be several atomic states  $|f\rangle, |f'\rangle, \dots$  that can be reached by near-resonant absorption. The same goes for state reached by emission if  $|s\rangle$  is an excited state.

Now the expectation value  $\langle \Psi | x | \Psi \rangle$  can be evaluated. We write the bra and ket on the left and top, respective, of the box in the following, and we collect the scalar products inside:

$$\langle \Psi | x | \Psi \rangle = \frac{\begin{array}{l} \langle n_{k\alpha}^a \dots | \langle s | x \\ + d_{k\alpha}^{\alpha*} \langle n_{k\alpha}^s - 1 \dots | \langle f | x \end{array}}{\begin{array}{l} |s\rangle | n_{k\alpha}^s \dots \rangle + d_{k\alpha}^a |f\rangle | n_{k\alpha}^s - 1 \dots \rangle + \dots \\ \langle s | x | s \rangle \langle 1 \rangle + d_{k\alpha}^a \langle s | x | f \rangle \langle 0 \rangle + \dots \\ d_{k\alpha}^{\alpha*} \langle f | x | s \rangle \langle 0 \rangle + |d_{k\alpha}^a|^2 \langle f | x | f \rangle \langle 1 \rangle + \dots \end{array}}$$

The sum includes only one field mode and one higher atomic state  $|f\rangle$ . This is enough to see that all the results must be zero if the atomic states have definite parity  $\langle s | x | s \rangle = 0 = \langle f | x | s \rangle$ . The orthogonality ( $\langle n_k | n_k - 1 \rangle = 0$ ) of the photon states kills the possibility of any contribution from the induced moment matrix elements  $\langle s | x | f \rangle$  or  $\langle f | x | s \rangle$ , however large they may be.

So photon number states are "impotent"; they cannot create a coherent excitation of an atom. This is consistent with the idea that classical phase is completely uncertain or random for a field oscillator eigenstate that has probability distributed more or less evenly over its phase space. To have a well-defined phase we need a non-stationary coherent oscillator state  $|\alpha_{k\beta}\rangle$  for a mode  $(\mathbf{k}, \beta)$  instead of a stationary eigenstate  $|n_{k\beta}\rangle$ . This will give a wave packet in  $(Q_{k\beta}, P_{k\beta})$  phase space or  $(\mathbf{A}, \mathbf{E})$  space which may have well-defined phase as shown in the Sec. 20.3.

Photon number states may be "impotent" but they are not powerless. They can create large fluctuations in expectation  $\langle \Psi | x^2 | \Psi \rangle$  even though  $\langle \Psi | x | \Psi \rangle$  is identically zero. From the foregoing calculation we get the following:

$$\langle \Psi | x^2 | \Psi \rangle = \langle s | x^2 | s \rangle + |d_{k\alpha}^a|^2 \langle f | x^2 | f \rangle + \dots \tag{22.1.33}$$

The expectation of  $x^2$  is proportional to the photon intensity  $v_{\mathbf{k}\beta}$ , the square  $|\mathbf{r}_{fs}|^2$  of the atomic induced moment, and the mean square  $x$  for the final state  $|f\rangle$ .



### d Coherent Radiation States

A much better description of a laser-cavity mode includes the nonstationary coherent or wave packet states  $|\alpha_{\mathbf{k}\beta}\rangle$ . According to (20.3.25) these may be defined as follows in terms of photon number states  $|n_{\mathbf{k}\beta}\rangle$  for a single-cavity mode  $(\mathbf{k},\beta)$ :

$$|\alpha_{\mathbf{k}\beta}\rangle = e^{-|\alpha_{\mathbf{k}\beta}|^2/2} \sum_{n_{\mathbf{k}\beta}} (\alpha_{\mathbf{k}\beta})^{n_{\mathbf{k}\beta}} |n_{\mathbf{k}\beta} 00 \dots\rangle / \sqrt{n_{\mathbf{k}\beta}!}. \quad (22.1.34)$$

The complex parameter  $\alpha_{\mathbf{k}\beta}$  is a field phasor expectation value and a quasi-eigenvalue of operator .

$$\mathbf{a}_{\mathbf{k}\beta} |\alpha_{\mathbf{k}\beta}\rangle = \alpha_{\mathbf{k}\beta} |\alpha_{\mathbf{k}\beta}\rangle, \quad \langle \alpha_{\mathbf{k}\beta} | \mathbf{a}_{\mathbf{k}\beta}^\dagger = \langle \alpha_{\mathbf{k}\beta} | \alpha_{\mathbf{k}\beta}^* . \quad (22.1.35)$$

Recall (20.3.25c). Here only the  $(\mathbf{k},\beta)$  mode is excited and all others are in their ground or vacuum states.

The  $A$ -field expectation value should equal the classical expression (22.1.2a) we began with:

$$\begin{aligned} \langle \alpha_{\mathbf{k}\beta} | \mathbf{A} | \alpha_{\mathbf{k}\beta} \rangle &= \sqrt{\frac{\hbar}{2\varepsilon_0\omega V}} \left[ \langle \alpha_{\mathbf{k}\beta} | \mathbf{a}_{\mathbf{k}\beta} | \alpha_{\mathbf{k}\beta} \rangle + \langle \alpha_{\mathbf{k}\beta} | \mathbf{a}_{\mathbf{k}\beta}^\dagger | \alpha_{\mathbf{k}\beta} \rangle \right] e_\beta \\ &= \sqrt{\frac{\hbar}{2\varepsilon_0\omega V}} \left[ \alpha_{\mathbf{k}\beta} + \alpha_{\mathbf{k}\beta}^* \right] e_\beta = \left[ \alpha_{\mathbf{k}\beta} e^{-i\omega t} + \alpha_{\mathbf{k}\beta}^* e^{i\omega t} \right] e_\beta \end{aligned}$$

In the last line is the classical value. Note that the dipole approximation  $e^{i\mathbf{k}\cdot\mathbf{r}} \cong 1$  is used here. The  $\_$  has the necessary  $e^{-i\omega t}$  time dependence noted in (20.3.29). This gives the non-stationary phase packet motion described by (20.3.34). The relation between  $\alpha_{\mathbf{k}\beta}$  and the classical amplitude  $a$  involves the phasor and the quantum scale (22.1.4):

$$\alpha_{\mathbf{k}\beta} = a_{\mathbf{k}\beta} e^{-i\omega_{\mathbf{k}} t} \sqrt{\frac{2\varepsilon_0\omega V}{\hbar}} .$$

The  $\alpha_{\mathbf{k}\beta}$  expectation value of the  $-q\mathbf{E} \cdot \mathbf{r}$  interaction in a coherent state  $|\alpha_{\mathbf{k}\beta}\rangle$  then becomes equal to the classical value quoted in (8.4.35a):

$$\langle \alpha_{\mathbf{k}\beta} | H_I | \alpha_{\mathbf{k}\beta} \rangle = -i\omega_{\mathbf{k}} q \mathbf{e}_\beta \cdot \mathbf{r} \left( \alpha_{\mathbf{k}\beta} e^{-i\omega t} - \alpha_{\mathbf{k}\beta}^* e^{i\omega t} \right).$$

It has the positive and negative frequency parts needed to coherently excite an atom.

This might lead you to believe that a coherent quantum field can reproduce the effect on an atom of a classical wave. Indeed, for very high  $\alpha_{\mathbf{k}\beta}$  and short enough time the effect of the two are nearly the same. However, eventually the coherent wave produces dephasing and rephrasing effects that are quite remarkable. These coherent decays and "revivals" are discussed in works listed at the end of this chapter. A model of coherent QED by Jaynes and Cummings is discussed next.

### Do low-quantum fields make good coordinate frames?

In the meantime, it is interesting to speculate on the ability of quantum fields, in particular, *low - quantum* fields, to make the kind of space-time coordinate grids that were used to develop the quantum theory and relativity in Unit 2.

A fundamental laser mode in a  $0.25\mu\text{m}$  cubic cavity such as the E-wave sketched in one half-wave strip of Fig. 22.1.2(c), has green light with  $\hbar\omega = 4 \cdot 10^{-19}$  Joule or  $2.5\text{eV}$  per photon. The average photon number is thus  $\bar{n} = |\alpha|^2 = 10^{10}$  for a laser with mean energy  $\bar{E} = \bar{U} \cdot V = \hbar\omega\bar{n} = 4.0\text{ nanoJ}$  in a volume  $V = (\frac{1}{4}\mu\text{m})^3$ . That amounts to a very intense micro laser! Photon number uncertainty is  $\Delta n = |\alpha| = 10^5$  in a coherent state (22.1.34) and it varies inversely to its phase uncertainty which here is a tiny value  $\Delta\Phi = \pi/\alpha \sim 3 \cdot 10^{-5}$ .

Amplitude expectation value  $\langle n|A|n\rangle$  is zero for  $|n\rangle$  states due to *incoherence* of phase, but number value  $\langle n|\mathbf{a}_k^\dagger \mathbf{a}_k|n\rangle = n$  is exact as is proper frequency  $\omega n$  due to the phase factor  $(e^{-i\omega t})^n$  of  $(\mathbf{a}_k^\dagger)^n$ . For *any* volume  $V$ , these ( $n = 10^{10}$ )-photons have total energy  $E = \hbar\omega n$  or mass  $M = E/c^2 = 10^{-25}$  kg equal to that of 59.79 H-atoms, but it's not "real" mass. (Real  $e + \bar{e}$  pair-creation means raising  $\omega$  from  $600\text{Thz}$  to  $m_e c^2/h$  or  $100\text{MegaThz}$ .) Nevertheless, "real"  $10^{-25}$  kg and an "optical"  $10^{-25}$  kg share a hyperbola  $10^{10}$  quanta above the  $n=1$  hyperbola in Fig. 5.1.1. Lorentz symmetry demands that.

A coherent-state  $|\alpha = 10^5\rangle$  also has a mass  $M = 10^{-25}$  kg but with uncertainty  $\Delta M = 10^{-30}$  kg. Its *phase* uncertainty  $3 \cdot 10^{-5}$  is low enough to plot grids like Fig. 22.1.2(c) or Fig. 22.1.3(a). A low- $\alpha$  wave state such as is used in Fig. 22.1.3(c) has too few photon counts-per-grid to plot sharply. Fig. 22.1.3(d) simulates an  $n$ -photon eigenstate  $|n\rangle$ . It is a wash even for high  $n$  since  $\Delta n = 0$  has  $\Delta\Phi = \infty$ .

However, an incoherent space-time "baseball diamond" grid like Fig. 22.1.2(d) is less sensitive to lack of phase coherence and so pure- $n$  photon number states could form diamonds since the coherent interference required for the Cartesian grid of Fig. 22.1.2(c) is not required for a pulse wave diamond grid.

So, do low-quantum fields make good coordinate frames? The answer, you see, is yes or no!

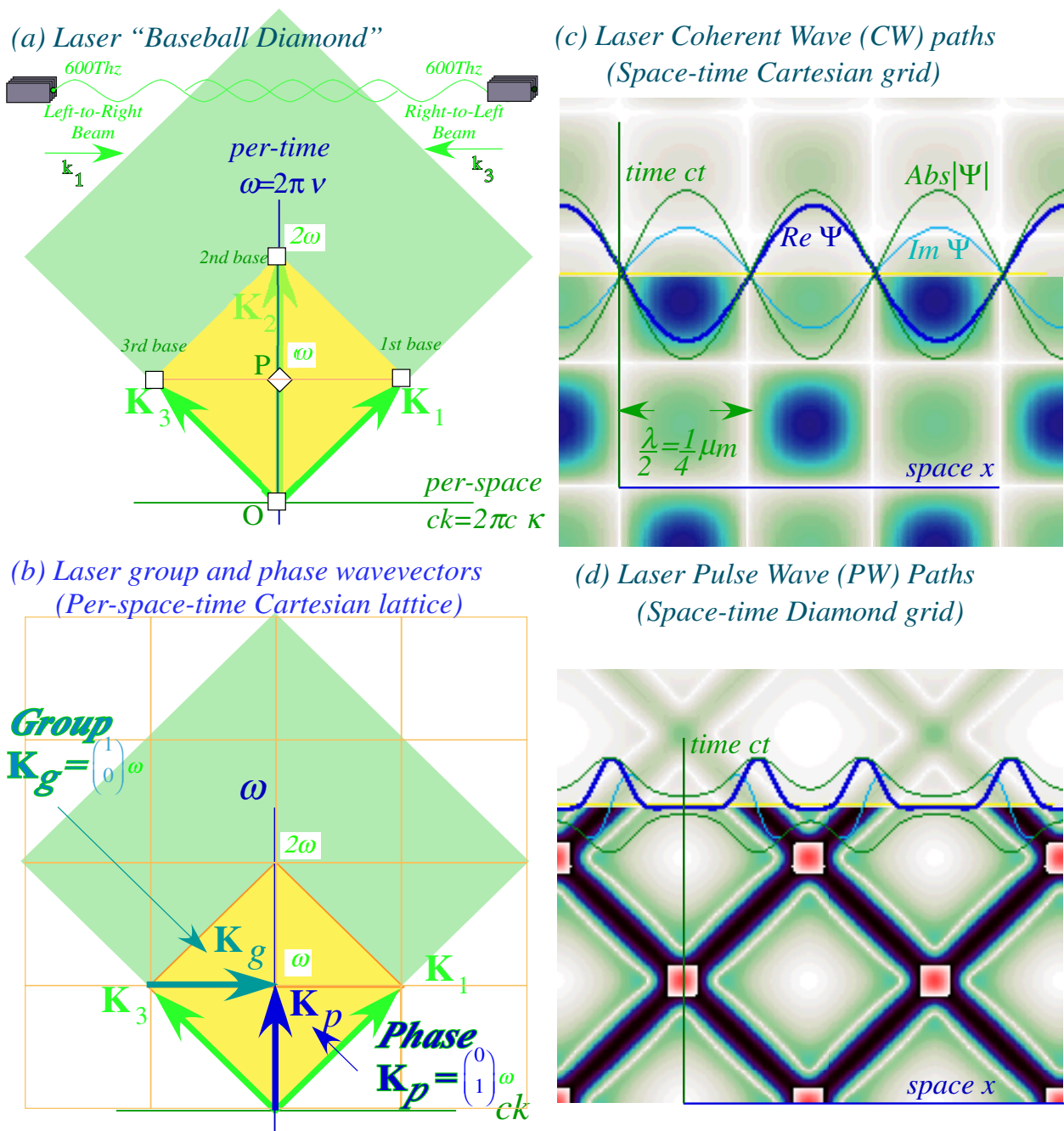


Fig. 22.1.2 Simulated space time wave coordinate grids (a-c) 600Thz Continuous Wave (CW) Cartesian per-space-time (a) and space-time. (b-d) Pulse wave (CW) light waves in per-space-time (b) and space-time (d).

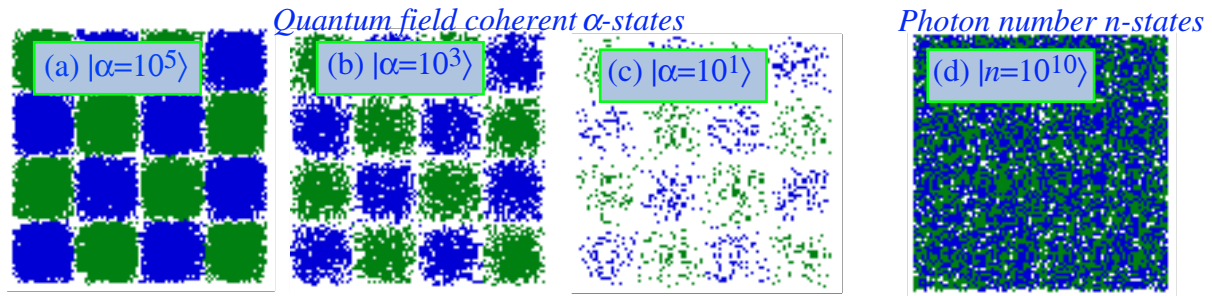


Fig. 22.1.3 Simulated spacetime photon counts for coherent (a-c) and photon-number states (d).

## 22.2 SPECTRA OF ATOMS IN LASER CAVITY

When atoms interact strongly or resonantly with an electromagnetic field, the distinction between the field and the atom is blurred. Observer and the observed become a single entity that is more than just a sum of its parts. It is as though the atom had become part of a molecule in which the excited levels involve excitation of all the constituent parts. Intra-cavity atomic theory is sometimes called *Cavity Quantum Electrodynamics* or CQED co-opting the more famous acronym QED for quantum field theory.

An atom interacting strongly with a single mode of a simple cavity is described by what is called the *Jaynes-Cummings model*. Here we will give a brief qualitative sketch of states and levels of this model. This is an important model for beginning to understand spectroscopic effects of strong laser fields. It is also a simple solvable example of an atom interacting with something that has a multitude of states. This is also the kind of problem one encounters when atomic motions go together to make a molecular rotation or vibration spectrum that are the subject of Unit 8.

### a. Jaynes-Cummings Hamiltonian

In a static electric field the two-level atomic system Hamiltonian has the following representation in the basis of atomic eigenstates given by (10.3.3) in Unit 3 or (19.1.6) in Unit 6:

$$H_{\text{atom}} = H \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + S \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - pE \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = H\mathbf{1} + S\sigma_z - pE\sigma_x. \quad (22.2.1a)$$

The electric field-dipole potential energy factor  $pE$  is folded into a Rabi coefficient in (19.1.7).

$$r = pE_z/\hbar \quad (22.2.1b)$$

An oscillating electric field Hamiltonian is transformed into rotating-wave form (19.1.10).

$$H_{\text{RW}} = \frac{\Delta}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{r}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\Delta}{2} \sigma_z + \frac{r}{2} \sigma_x. \quad (22.2.2a)$$

*Detuning factor*  $\Delta$  is the difference between the atomic transition angular frequency  $\varepsilon$  and the angular frequency  $\Omega$  of the stimulating laser.

$$\Delta = \varepsilon - \Omega \quad (22.2.2b)$$

In preparation for using a quantum field we need to separate the atomic and radiative contributions to the Hamiltonian. The terms that involve the laser frequency  $\Omega$  have been collected into the first simple term labeled  $H_{\text{field}}$ . The atom's two levels  $H_{\text{atom}}$  and their interaction  $H_{\text{interaction}}$  come next.

$$\begin{aligned} H_{\text{RW}} &= \Omega \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \frac{\varepsilon + \Omega}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varepsilon \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{r}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= H_{\text{field}} + \frac{\varepsilon}{2} (\boldsymbol{\sigma}_z + \mathbf{1}) + \frac{r}{2} \boldsymbol{\sigma}_x \\ &= H_{\text{field}} + H_{\text{atom}} + H_{\text{interaction}} \end{aligned}$$

In the Jaynes-Cummings model the classical electric field is replaced by an expression using its quantized form. However, only one mode of the field is considered and only the dipole terms (21.1.20e) are used. Also, the unit matrix term dropped. The simplified Hamiltonian is as follows.

$$\begin{aligned} H_{\text{JC}} &= H_{\text{field}} + H_{\text{atom}} + H_{\text{interaction}} \\ &= \Omega \mathbf{a}^\dagger \mathbf{a} + \frac{\varepsilon}{2} (\boldsymbol{\sigma}_z + \mathbf{1}) + i \frac{g}{2} (\mathbf{a}^\dagger - \mathbf{a}) \boldsymbol{\sigma}_x \end{aligned} \quad (22.2.3a)$$

The constant  $g$  is the Rabi factor that would correspond to a one-photon laser field. [Recall (22.1.4).]

$$g = \sqrt{\frac{\hbar \Omega}{2 \varepsilon_0 V}} \frac{q \langle 2 | z | 1 \rangle}{\hbar} \quad (22.2.3b)$$

The field amplitude for an  $N$ -photon field is proportional to  $\sqrt{N}$ , so we have the following relation between the semiclassical and quantum interaction constant.

$$r = g \sqrt{N} \quad (22.2.3c)$$

We let  $\mathbf{a}_{\mathbf{k},\alpha} \equiv \mathbf{a}$  since only one laser cavity mode ( $\mathbf{k}\alpha$ ) of frequency  $\Omega = \omega$  is being considered. By expressing  $\boldsymbol{\sigma}_x$  in terms of spinor raising and lowering operators,  $\boldsymbol{\sigma}_x = (\boldsymbol{\sigma}_+ + \boldsymbol{\sigma}_-)$ , this becomes

$$H_{\text{JC}} = \Omega \mathbf{a}^\dagger \mathbf{a} + \frac{\varepsilon}{2} (\boldsymbol{\sigma}_z + \mathbf{1}) + i \frac{g}{2} (\mathbf{a}^\dagger - \mathbf{a}) (\boldsymbol{\sigma}_+ + \boldsymbol{\sigma}_-) \quad (22.2.4)$$

A final approximation to the model keeps only the interaction term  $\mathbf{a}^\dagger \boldsymbol{\sigma}_-$  for a photon created as the atom drops from the upper-( $\uparrow$ )-level 2 to the lower-( $\downarrow$ )-level 1 and the term  $\mathbf{a} \boldsymbol{\sigma}_+$  that does the reverse.

$$H_{\text{JC}} = \Omega \mathbf{a}^\dagger \mathbf{a} + \frac{\varepsilon}{2} (\boldsymbol{\sigma}_z + \mathbf{1}) + i \frac{g}{2} (\mathbf{a}^\dagger \boldsymbol{\sigma}_- - \mathbf{a} \boldsymbol{\sigma}_+) \quad (22.2.5)$$

Let us apply the Hamiltonian (22.2.5) in turn to the radiation-atom product states with  $N = 0, 1, 2, \dots$  photons that we label  $\{|0\rangle|\downarrow\rangle, |0\rangle|\uparrow\rangle, |1\rangle|\downarrow\rangle, |1\rangle|\uparrow\rangle, |2\rangle|\downarrow\rangle, |2\rangle|\uparrow\rangle, \dots\}$ . The following state operations involve operations  $a|n\rangle = \sqrt{n}|n-1\rangle$  and  $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$  and spin operations  $\sigma_-|\uparrow\rangle = |\downarrow\rangle$ , and so on.

$$\begin{aligned}
 H_{\text{JCM}}|0\rangle|\downarrow\rangle &= (0 \cdot \Omega + 0)|0\rangle|\downarrow\rangle + i\frac{g}{2}(0 - 0), \\
 H_{\text{JCM}}|0\rangle|\uparrow\rangle &= (0 \cdot \Omega + \varepsilon)|0\rangle|\uparrow\rangle + i\frac{g}{2}(\sqrt{1}|1\rangle|\downarrow\rangle - 0), \\
 H_{\text{JCM}}|1\rangle|\downarrow\rangle &= (1 \cdot \Omega + 0)|1\rangle|\downarrow\rangle + i\frac{g}{2}(0 - \sqrt{1}|0\rangle|\uparrow\rangle), \\
 H_{\text{JCM}}|1\rangle|\uparrow\rangle &= (1 \cdot \Omega + \varepsilon)|1\rangle|\uparrow\rangle + i\frac{g}{2}(\sqrt{2}|2\rangle|\downarrow\rangle - 0), \\
 H_{\text{JCM}}|2\rangle|\downarrow\rangle &= (2 \cdot \Omega + 0)|2\rangle|\downarrow\rangle + i\frac{g}{2}(0 - \sqrt{2}|1\rangle|\uparrow\rangle), \\
 H_{\text{JCM}}|2\rangle|\uparrow\rangle &= (2 \cdot \Omega + \varepsilon)|2\rangle|\uparrow\rangle + i\frac{g}{2}(\sqrt{3}|3\rangle|\downarrow\rangle - 0).
 \end{aligned} \tag{22.2.6}$$

An infinite series of two-by-two matrices represents  $H_{\text{JCM}}$  in this basis.

|                               |                               |                                |                                |                                |                                      |          |
|-------------------------------|-------------------------------|--------------------------------|--------------------------------|--------------------------------|--------------------------------------|----------|
|                               | $ 0\rangle \downarrow\rangle$ | $ 0\rangle \uparrow\rangle$    | $ 1\rangle \downarrow\rangle$  | $ 1\rangle \uparrow\rangle$    | $ 2\rangle \downarrow\rangle, \dots$ |          |
| $ 0\rangle \downarrow\rangle$ | $0 \cdot \Omega + 0$          | $\cdot$                        | $\cdot$                        | $\cdot$                        | $\cdot$                              |          |
| $ 0\rangle \uparrow\rangle$   | $\cdot$                       | $0 \cdot \Omega + \varepsilon$ | $\frac{-ig\sqrt{1}}{2}$        | $\cdot$                        | $\cdot$                              |          |
| $ 1\rangle \downarrow\rangle$ | $\cdot$                       | $\frac{ig\sqrt{1}}{2}$         | $1 \cdot \Omega + \varepsilon$ | $\cdot$                        | $\cdot$                              | (22.2.7) |
| $ 1\rangle \uparrow\rangle$   | $\cdot$                       | $\cdot$                        | $\cdot$                        | $1 \cdot \Omega + \varepsilon$ | $\frac{-ig\sqrt{2}}{2}$              |          |
| $ 2\rangle \downarrow\rangle$ | $\cdot$                       | $\cdot$                        | $\cdot$                        | $\frac{ig\sqrt{2}}{2}$         | $2 \cdot \Omega + 0$                 |          |

The general form of each two-by-two matrix is the following:

$$\langle h_{\text{JCM}} \rangle = \langle N-1 | \langle \downarrow | \begin{array}{c|cc} & |N-1\rangle|\downarrow\rangle & |N-1\rangle|\uparrow\rangle \\ \hline |N-1\rangle\Omega + \varepsilon & & \frac{-ig\sqrt{N}}{2} \\ \langle N | \langle \uparrow | & \frac{ig\sqrt{N}}{2} & N\Omega + 0 \end{array} \tag{22.2.8}$$

In the classical limit of large  $N$  the two-by-two matrix begins to look something like the semi-classical matrix (22.2.2a). We can write the two-by-two part of the Hamiltonian as follows:

$$\begin{aligned}
\langle h_{\text{JCM}} \rangle &= (N-1)\Omega \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \Omega \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 & -i \\ i & 1 \end{pmatrix} \\
&= \left( (N-1)\Omega + \frac{\varepsilon + \Omega}{2} \right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\varepsilon - \Omega}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{g\sqrt{N}}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
&= \left( (N-1)\Omega + \frac{\varepsilon + \Omega}{2} \right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\Delta}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{r}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
&= \left( (N-1)\Omega + \frac{\varepsilon + \Omega}{2} \right) \mathbf{1} + \frac{\Delta}{2} \boldsymbol{\sigma}_z + \frac{r}{2} \boldsymbol{\sigma}_y
\end{aligned} \tag{22.2.9}$$

$N$ -photon-Rabi-factor  $r = g\sqrt{N}$  reappears on  $\boldsymbol{\sigma}_y$  due to a choice of phase for field operators. The physics is unchanged if we use a modified JCM Hamiltonian that uses  $\boldsymbol{\sigma}_x$  instead of  $\boldsymbol{\sigma}_y$ .

$$H'_{\text{JC}} = \Omega \mathbf{a}^\dagger \mathbf{a} + \frac{\varepsilon}{2} (\boldsymbol{\sigma}_z + 1) + \frac{g}{2} (\mathbf{a}^\dagger \boldsymbol{\sigma}_- + \mathbf{a} \boldsymbol{\sigma}_+)$$

This lets us use semi-classical dressed eigensolutions (19.1.20) to real two-by-two matrices.

$$\begin{aligned}
\langle h'_{\text{JCM}} \rangle &= \begin{array}{c|cc} & |N-1\rangle|\downarrow\rangle & |N\rangle|\uparrow\rangle \\ \hline \langle N-1|\langle\downarrow| & (N-1)\Omega + \varepsilon & \frac{g\sqrt{N}}{2} \\ \langle N|\langle\uparrow| & \frac{g\sqrt{N}}{2} & N\Omega \end{array} = N\Omega \mathbf{1} + \begin{pmatrix} \Delta & r \\ r & 0 \end{pmatrix} \\
&= \left( (N-1)\Omega + \frac{\varepsilon + \Omega}{2} \right) \mathbf{1} + \begin{pmatrix} \frac{\Delta}{2} & \frac{r}{2} \\ \frac{r}{2} & -\frac{\Delta}{2} \end{pmatrix} = \left( (N-1)\Omega + \frac{\varepsilon + \Omega}{2} \right) \mathbf{1} + \frac{\Delta}{2} \boldsymbol{\sigma}_z + \frac{r}{2} \boldsymbol{\sigma}_x
\end{aligned} \tag{22.2.10}$$

These matrices have the same eigenvectors as the semi-classical matrices. The only difference is that the Rabi factor  $r$  depends upon photon number  $N$  and there is a level pair for all  $N$  greater than zero. The eigenvalues are the similar, too, except for the unit matrix term that yields a ladder of doublet eigenvalues. We examine this ladder of levels now.

### b. Jaynes-Cummings Eigensolutions

An attempt to picture the dressed eigenlevels is made in Fig. 22.2.1. A column containing stacks of energy levels is shown for each of three cases of detuning: (a) laser tuned below atomic transition ( $\Delta > 0$ ); (b) at resonance ( $\Delta = 0$ ), and (c) laser tuned higher ( $\Delta < 0$ ). Recall that the detuning parameter is  $\Delta = \varepsilon - \Omega$ .

Two stacks of horizontal lines on each side of the (a), (b), and (c) columns in Fig. 22.2.1 indicate what the levels would be without any interaction between atom and field ( $r = g/\sqrt{N} = 0$ ). An  $N$ -photon level in which the atom is in the first  $|1\rangle = |\uparrow\rangle$  or second  $|2\rangle = |\downarrow\rangle$  state is labeled  $|1, N\rangle$  or  $|2, N\rangle$ ,

respectively. Each of these levels for  $N > 0$  is connected to a pair of lines in the center of the column that are shifted up and down by  $\pm\delta/2$ . The quantity  $\delta$  is the AC-Stark shift discussed in Section 19.1 (Recall (19.1.20a)). The shifted lines are the “dressed” eigenlevels with interaction turned on ( $r = g/\sqrt{N} > 0$ ). The lines that connect the shorter lines to the zero-field levels indicate the relative the greater of the two amplitudes ( $\sin \theta/2$  or  $\cos \theta/2$ ) of the zero-field states in each the dressed eigenstates corresponding to that level. Recall Eqs. (19.1.20c) and (19.1.20d).

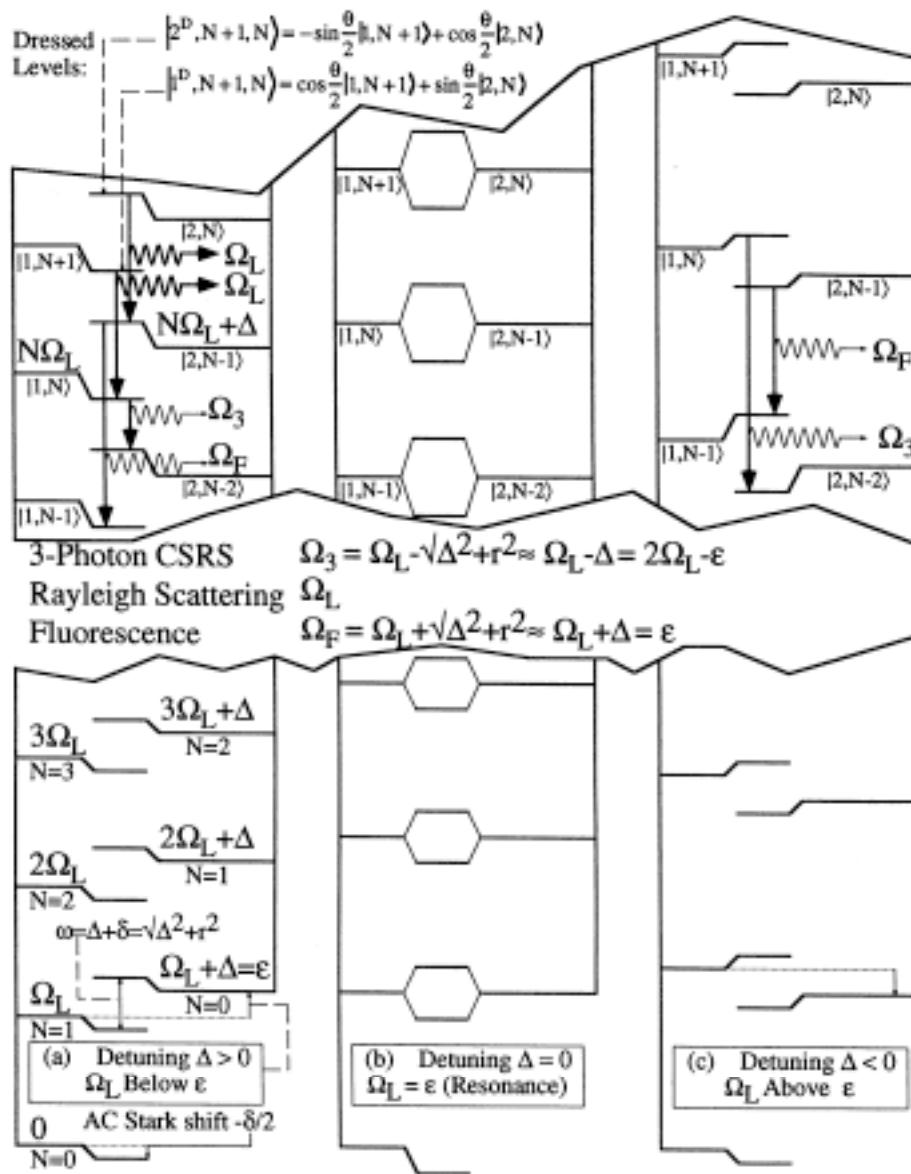
Below resonance ( $\Delta > 0$ ) the Hamiltonian rotation vector  $\omega$  makes an acute angle ( $\theta < \pi/2$ ) with the  $z$ -axis. The lower dressed eigenstate  $|1^D N + 1, N\rangle$  indicated at the top left-hand side of Fig. 22.2.1 is mostly composed of the atom-field product state  $|1, N + 1\rangle$ , while the higher-dressed state  $|2^D N + 1, N\rangle$  is mostly composed of  $|2, N\rangle$ .

As the detuning approaches resonance ( $\Delta=0$ ), the zero-field levels get lined up, the AC-shifts reach their maximum, and the rotation angle  $\theta$  approaches  $\pi/2$ . One may use the diagrams from Fig. 19.1.6 to quantify the variation. However, caution should be used since the Rabi parameter  $r = g\sqrt{N}$  is here a function of  $N$ . In other words, the Rabi parameter, which was a constant in the semi-classical theory, is now dependent upon what level you are on. It increases with the laser mode electric field amplitude, which is proportional to the root  $\sqrt{N}$  of the photon number.

At resonance ( $\Delta=0$ ) the rotation angle is  $\theta = \pi/2$ . Then the Hamiltonian rotation vector  $\Omega$  makes an angle of  $\pi/2$  with the  $z$ -axis and has its minimum magnitude of  $|\Omega| = r$ , which is the Rabi frequency. This was shown in Fig. 19.1.3c. The resonance values for the dressed eigenstate amplitudes are  $\sin \theta/2 = 1/\sqrt{2}$  and  $\cos \theta/2 = 1/\sqrt{2}$ . This corresponds to 50-50 mixtures of the atom-field product states  $|1, N + 1\rangle$  and  $|2, N\rangle$  in the dressed eigenstates  $|1^D N + 1, N\rangle$  and  $|2^D N + 1, N\rangle$ .

Above resonance ( $\Delta < 0$ ) the Hamiltonian rotation vector  $\Omega$  makes an obtuse angle ( $\theta > \pi/2$ ) with the  $z$ -axis. Now the lower-dressed eigenstate  $|1^D N + 1, N\rangle$  is mostly composed of  $|2, N\rangle$ , while the upper-dressed state  $|2^D N + 1, N\rangle$  is mostly composed of  $|1, N + 1\rangle$ , as shown in the upper right-hand side of the Fig. 22.2.1.





**Fig. 22.2.1** Level structure of 2-level atom and 1-mode cavity showing elementary transition processes of fluorescence, Rayleigh scattering, and three-photon coherent Stokes-Raman scattering (CSRS). Transitions are between levels belonging to dressed eigenstates.

**c. Transitions in the Jaynes-Cummings Model**

The diagram of levels in Fig. 22.2.1 involves just one  $\Omega_L = \Omega$  mode interacting with the two-level atom. We have ignored all the other field mode levels such as were sketched in Fig. 22.1.1. We have just concentrated on loading photons into one laser cavity mode whose frequency  $\Omega_L = \Omega$  is being tuned close to the value  $\epsilon = \omega_f - \omega_s$  of the atomic transition.

However, if these other modes are off-resonance by enough or only have one or two photons, one can treat them using perturbation theory as was discussed in Section 22.1.b. Transition rates between the dressed states of a laser-driven atom can be derived using the Fermi golden rule (22.1.30).

### *Rayleigh scattering and fluorescence*

Some of the commonly observed transitions are indicated by vertical arrows in the Fig. 22.2.1. The strongest transitions involved the so-called *Rayleigh Scattering* processes such as  $|1, N+1\rangle \rightarrow |1, N\rangle$  for  $N = 0, 1, 2, \dots$  or  $|2, N\rangle \rightarrow |2, N-1\rangle$  for  $N = 1, 2, \dots$ , where only the photon number changes and the system emits one of its laser-mode photons into an external mode of the same frequency  $\Omega_L$ . These transitions yield light with the frequency of the laser just like classical Rayleigh scattered light.

The other transitions are more complicated. One called *Fluorescence* is a transition between dressed states which involve fundamental transitions such as  $|2, N\rangle \rightarrow |1, N\rangle$  or  $|2, N-1\rangle$  and  $|1, N-1\rangle$ . The latter is the major part of the transition indicated by an  $\Omega_F$  arrow in the upper left-hand side of the figure since the initial (upper) dressed state  $|2^D N, N-1\rangle$  is mostly composed of  $|2, N-1\rangle$  and  $|1^D N, N-1\rangle$  is mostly composed of  $|1, N-1\rangle$  in the final (lower) dressed state.

The fluorescence transition angular frequency is the difference between the initial and final dressed eigenlevels connected by the  $\Omega_F$  arrow. The initial and final eigenvalues are

$$\begin{aligned}\epsilon^D(2, N, N-1) &= N\Omega_L + \Delta + \delta/2, \\ \epsilon^D(1, N-1, N-2) &= (N-1)\Omega_L - \delta/2.\end{aligned}$$

The difference is the fluorescence transition frequency,

$$\Omega_F = \Omega_L + \Delta + \delta = \Omega_L + \sqrt{\Delta^2 + r^2}. \quad (22.2.11)$$

For small Rabi factor ( $r \ll \Delta$ ) or large detuning ( $\Delta \gg r$ ) it becomes the atomic transition frequency.

$$\Omega_F \rightarrow \Omega_L + \Delta = \epsilon \quad (22.2.12)$$

This transition drops the atom from its upper state  $|2\rangle$  to its lower state  $|1\rangle$  but takes no photons out of the cavity mode since  $N$  stays constant. It emits one photon into an external mode of frequency  $\Omega_F$ .

### *Coherent Stokes Raman scattering*

Another transition called the three-photon process or *Coherent Stokes Raman Scattering* (CSRS) is a transition between dressed states, which mostly involves transitions of the type  $|1, N\rangle \rightarrow |2, N-2\rangle$ . The latter is the major part of the transition indicated by an  $\Omega_3$  arrow in the upper left-hand side of the figure

since the initial (upper) dressed state  $|1^D N, N-1\rangle$  is mostly composed of  $|1, N\rangle$ , and  $|2^D N-1, N-2\rangle$  is mostly composed of  $|2, N-2\rangle$  in the final (lower) dressed state.

The CSRS transition angular frequency is the difference between the initial and final dressed eigenlevels connected by the  $\Omega_3$  arrow. The initial and final eigenvalues are

$$\begin{aligned}\varepsilon^D(1, N, N-1) &= N\Omega_L - \delta/2, \\ \varepsilon^D(2, N-1, N-2) &= (N-1)\Omega_L + \Delta - \delta/2.\end{aligned}$$

The difference is the CSRS transition frequency.

$$\Omega_3 = \Omega_{\text{CSRS}} = \Omega_L - \Delta - \delta = \Omega_L - \sqrt{\Delta^2 + r^2} \quad (22.2.13)$$

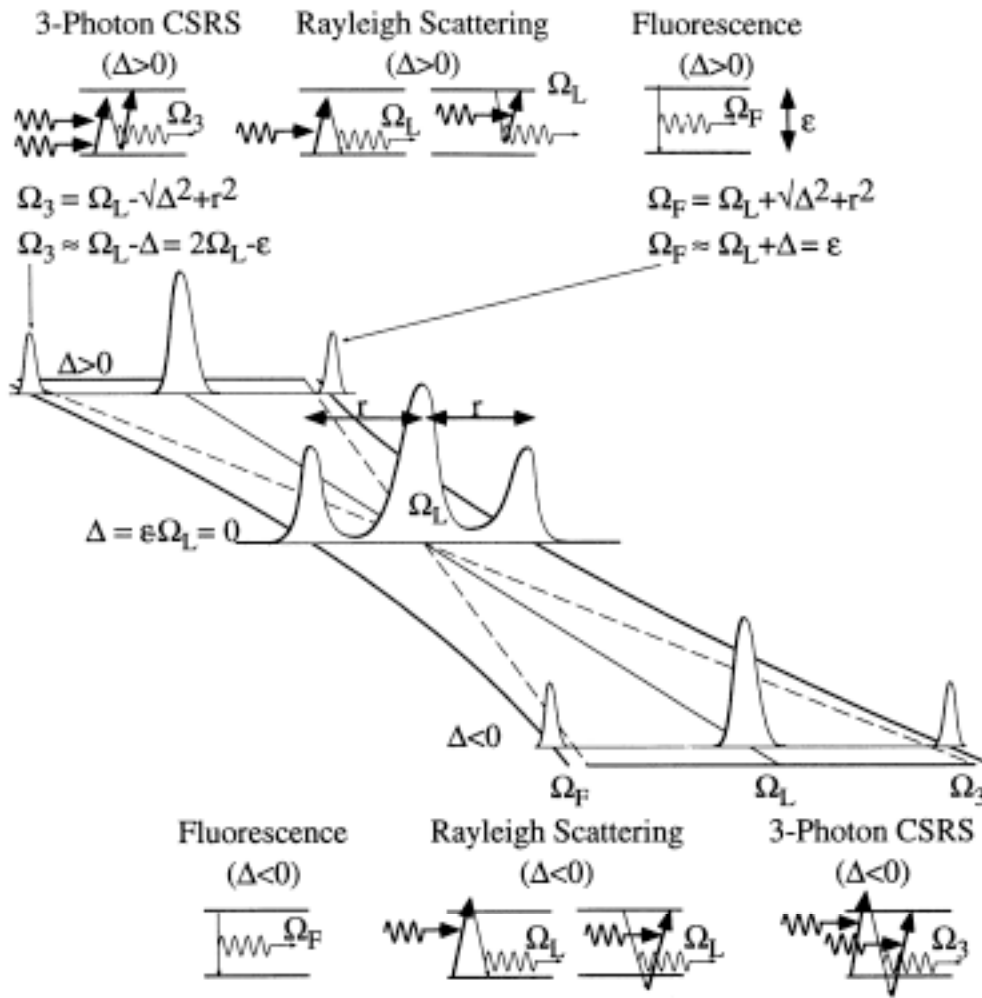
For small values of the Rabi factor ( $r \ll \Delta$ ) or large detuning ( $r \ll \Delta$ ) this becomes

$$\Omega_3 = \Omega_{\text{CSRS}} \rightarrow \Omega_L - \Delta = 2\Omega_L - \sqrt{\Delta^2 + r^2} \quad (22.2.14)$$

which is the difference between two laser photons and the atomic transition frequency. This transition raises the atom from its lower state  $|1, N\rangle$  to its upper state  $|2, N\rangle$ . It also takes two photons out of the cavity mode since  $N$  decreases by two. It emits one photon into an external mode of frequency  $\Omega_3$ , which is approximately the difference between  $2\Omega$  and the atomic transition frequency  $\varepsilon$ .

A direct transition of frequency  $\omega\sqrt{\Delta^2 + r^2} = \Delta$  between  $|1^D N, N-1\rangle$  and  $|2^D N-1, N-2\rangle$  is forbidden by  $C_2$  parity. However, in a system that does not have  $C_2$  symmetry it would be possible to have such a transition as is indicated by the small vertical arrow near the bottom of Fig. 22.2.1(a).

The next Fig. 22.2.2 shows the JTM transitions and their frequency dependence in more detail.



**Fig. 22.2.2** Structure of the Mollow spectrum and its elementary processes of fluorescence, Rayleigh scattering, and three-photon coherent Stokes-Raman scattering (CSRS).

The three allowed transitions account for the three main spectral components that may be observed coming out of the sides of a laser atom cavity. It consists of a strong Rayleigh line at  $\Omega_L = \Omega$  and two sidebands  $\Omega_F$  and  $\Omega_3$  as shown in Fig. 22.2.2. The triple-pronged spectral line is called the *Mollow Line Shape*. One sideband is centered at  $\Omega_F = \Omega + \omega$  which is approximately  $\Omega + \Delta$  far from resonance, and the other is at  $\Omega_3 = \Omega - \omega$ , which is approximately  $\Omega - \Delta$ .

Near resonance at  $\Delta=0$  the sidebands will follow AC Stark shift hyperbolic paths given by (22.2.11) and (22.2.13) rather than simply collapsing upon  $\Omega$  at  $\Delta = 0$ . The hyperbolic curves in the semiclassical level diagram in Fig. 19.1.6 are approximate traces of the spectral sidebands for variable detuning  $\Delta$  around resonance. At resonance ( $\Delta=0$ ) there will still be two sidebands but now they will be

located at  $\Omega \pm r$ , where  $r$  is the Rabi parameter. In general, the sidebands are located at  $\Omega \pm \omega$ , where  $\omega = \sqrt{\Delta^2 + r^2} \approx \Delta$  is the frequency of the Rabi precession or crank rotation shown in Fig. 19.1.3.

The sidebands correspond roughly to fluorescence and CSRS processes, respectively. With positive detuning ( $\varepsilon - \Omega = \Delta > 0$ ) the upper sideband ( $\Omega_F = \Omega + \omega \approx \Omega + \Delta$ ) is due (mostly) to fluorescence while the lower sideband ( $\Omega_3 = \Omega - \omega \approx \Omega - \Delta$ ) is due (mostly) to the three-photon CSRS process. Above resonance the detuning parameter reverses sign ( $\varepsilon - \Omega = \Delta < 0$ ) and the order is reversed. At resonance ( $\Delta = 0$ ) it is not possible to distinguish the two processes since initial states  $|1, N\rangle$  and  $|2, N-1\rangle$  and  $|2, N-2\rangle$  are mixed 50-50 as are the final states  $|1, N-1\rangle$  and  $|2, N-2\rangle$ .

Below resonance ( $\Delta > 0$ ) the CSRS  $\Omega_3$  photon has lower frequency than the  $\Omega_F$  fluorescence photon. It also must come earlier in time. The CSRS process pumps the atom from its lower state-1 into its excited state-2. Only then can it emit a fluorescence photon to put it back into its ground state. Above resonance ( $\Delta < 0$ ) the  $\Omega_3$  photon from the CSRS process has higher frequency than the fluorescence  $\Omega_F$  photon. Then the higher frequency sideband comes earlier. These time correlations have been observed.

This concludes our introduction to the recent fundamental developments in laser spectroscopy. Many details have been left out of this discussion and many new effects will soon be discovered as this new set of tools becomes more widely used. Perhaps the most important development so far lies in the way we are coming to think about the observed object (atom or molecule) and the observer's tool (radiation). In modern laser spectroscopy the distinction between the observer and the observed has practically disappeared, and the atom-radiation-cavity becomes a single quantum object.

## ADDITIONAL READING

Discussions of semiclassical quantization and wave-packet dynamics are in the following.

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An early paper on action quantization that uses color graphics to approximate quantum wave fronts is

M. J. Davis and E. J. Heller, *J. Chem. Phys.*, **75**, 3916 (1981).

The computer program *Color U(2)* mentioned at the end of Chapter 7 uses color quantization and color animation to show the dynamics of quantum wave fronts.

The idea of wave-packet coherent states can be traced back to Schrödinger.

E. Schrödinger, *Naturwissenschaften*, **14**, 664 (1926).

Their introduction in quantum optics is probably due to Glauber.

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- W. H. Miller, *J. Chem. Phys.*, **81**, 3573 (1984).

References to the original EBK quantization are as follows:

- A. Einstein, *Dent. Ges. Berlin Verh.*, **19**, 9/10 (1917).
- M. L. Brillouin, *J. Phys. Paris (Ser. 6)*, **7**, 353 (1926).
- J. B. Keller, *Ann. Phys. (N.Y.)*, **4**, 180 (1958).
- F. Reiche, *The Quantum Theory*, (Methuen, London, 1922).

A good modern reference to classical, semiclassical, and quantum theory of radiation for spectroscopy is the following:

C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Photons and Atoms* (Wiley Interscience, New York, 1989).

This discusses the  $\mathbf{A} \cdot \mathbf{P}$  versus  $\mathbf{E} \cdot \mathbf{r}$  perturbations and the Power-Zienau-Wolley transformation. A simplified discussion and other references are in the following paper:

- E. A. Power and T. Thirunamachandran, *Ann. J. Phys.*, **46**, 370 (1976).

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M. Goepfert-Mayer, *Ann. Physik (Lpzg.)*, **9**, 273 (1931).

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For an example of some of the confusion surrounding the  $\mathbf{A} \cdot \mathbf{p}$  interaction see the following paper:

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Some modern treatments of laser-atom lineshape and two-level atom models are listed below. The first papers are seminal ones by B. R. Mollow:

B. R. Mollow, *Phys. Rev.*, **188**, 1969 (1969); *Phys. Rev. A*, **2**, 76 (1969); *Phys. Rev. A*, **12**, 1919 (1969); *Phys. Rev. A*, **13**, 758 (1969).

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Applications of radiation theory to laser dynamics is the subject of the following book, which also relates the classical Lorentz model to modern theory:

P. W. Milonni and J. H. Eberly, *Lasers* (Wiley Interscience, New York, 1988).



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H. J. Yoo, J. J. Sanchez-Mondragon, and J. H. Eberly, *Phys. Rep.*, **118**, 259 (1985).

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J. Gea-Banacloche, *Phys. Rev. Lett.*, **65**, 3385 (1990); *Phys. Rev. A*, **44**, 5913 (1991); *Optical Commun.* **88**, 531 (1992).

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P. Alsing and H. J. Carmichael, *Quantum Optics*, **3**, 13 (1991).

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Many have contributed to angular momentum coupling besides the originators Wigner, Racah, Clebsch and Gordan. Some have gone on to develop similar formulas for higher unitary groups  $U(m)$ . Pioneers in this area include Schwinger, Baird, Biedenharn, Bincer, Gelfand, Louck, and Moshinsky.