# Review Topics & Formulas for Unit 3

*Fourier Series Coefficients*

$$\langle k_m|\Psi\rangle = \int_{-L/2}^{L/2} dx \langle k_m|x\rangle\langle x|\Psi\rangle$$

$$\langle k_m|x\rangle = \frac{e^{-ik_m x}}{\sqrt{L}} = \langle x|k_m\rangle^*$$

*x-Wavefunction* $\Psi(x)$=

$$\langle x|\Psi\rangle = \sum_{m=-\infty}^{m=\infty} \langle x|k_m\rangle\langle k_m|\Psi\rangle$$

*Ortho – Completeness*

$$\sum_{m=0}^{m=\infty} \langle x|k_m\rangle\langle k_m|x'\rangle = \delta(x-x')$$

$$\int_{-L/2}^{L/2} dx \langle k_m|x\rangle\langle x|k_{m'}\rangle = \delta_{m,m'}$$

*Discrete momentum m*
*Continuous position x*

*Fourier Integral Transform*

$$\langle k|\Psi\rangle = \int_{-\infty}^{\infty} dx \langle k|x\rangle\langle x|\Psi\rangle$$

$$Kernal: \langle k|x\rangle = \frac{e^{-ikx}}{\sqrt{2\pi}} = \langle x|k\rangle^*$$

*x-Wavefunction* $\Psi(x)$=

$$\langle x|\Psi\rangle = \int_{-\infty}^{\infty} dk \langle x|k\rangle\langle k|\Psi\rangle$$

*Ortho – Completeness*

$$\int_{-\infty}^{\infty} dk \langle x|k\rangle\langle k|x'\rangle = \delta(x-x')$$

$$\int_{-\infty}^{\infty} dx \langle k|x\rangle\langle x|k'\rangle = \delta(k-k')$$

*Continuous momentum k*
*Continuous position x*

*Fourier $C_N$ Transformation*

$$\langle k_m|\Psi\rangle = \sum_{p=0}^{p=N-1} \langle k_m|x_p\rangle\langle x_p|\Psi\rangle$$

$$\langle k_m|x_p\rangle = \frac{e^{-ik_m x_p}}{\sqrt{N}} = \langle x_p|k_m\rangle^*$$

*x-Wavefunction* $\Psi(x)$=

$$\langle x_p|\Psi\rangle = \sum_{m=0}^{m=N-1} \langle x_p|k_m\rangle\langle k_m|\Psi\rangle$$

*Ortho – Completeness*

$$\sum_{m=0}^{m=N-1} \langle x_p|k_m\rangle\langle k_m|x_{p'}\rangle = \delta_{p,p'}$$

$$\sum_{p=0}^{p=N-1} \langle k_m|x_p\rangle\langle x_p|k_{m'}\rangle = \delta_{m,m'}$$

*Discrete momentum m*
*Discrete position $x_p$*

---

*Time Evolution Operator* **U**

$$|\Psi(t)\rangle = \mathbf{U}(t,0)|\Psi(0)\rangle$$

*Hamiltonian Generator* **H**

$$i\hbar\frac{\partial}{\partial t}\mathbf{U}(t,0) = \mathbf{H}\,\mathbf{U}(t,0)$$

*Time Evolution Operator* **U**

$$\mathbf{U}(t,0) = e^{-i\,t\,\mathbf{H}/\hbar}$$

*Schrodinger t – Equation*

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$$

**U** *must be Unitary*

$$\mathbf{U}^\dagger(t) = \mathbf{U}^{-1}(t) = \mathbf{U}(-t)$$

$$\left(e^{-it\mathbf{H}/\hbar}\right)^\dagger = e^{it\mathbf{H}^\dagger/\hbar} = e^{it\mathbf{H}/\hbar}$$

*so* **H** *is Hermitiam* $\mathbf{H}^\dagger = \mathbf{H}$

---

*Schrodinger time-independent energy eigen equation*.

$$\mathbf{H}\,|\omega_m\rangle = \hbar\omega_m\,|\omega_m\rangle = \varepsilon_m\,|\omega_m\rangle \tag{9.3.1a}$$

**H**-eigenvalues use **r**-expansion (9.2.6) of **H** and $C_6$ symmetry $\mathbf{r}^p$-eigenvalues from (8.2.9).

$$\langle k_m|\mathbf{r}^p|k_m\rangle = e^{-ipk_m a} = e^{-ipm2\pi/N} \quad \text{where: } k_m = m(2\pi/Na)$$

$$\langle k_m|\mathbf{H}|k_m\rangle = H\langle k_m|\mathbf{1}|k_m\rangle + S\langle k_m|\mathbf{r}|k_m\rangle + T\langle k_m|\mathbf{r}^2|k_m\rangle + U\langle k_m|\mathbf{r}^3|k_m\rangle + T^*\langle k_m|\mathbf{r}^4|k_m\rangle + S^*\langle k_m|\mathbf{r}^5|k_m\rangle$$
$$= H + S\,e^{-ik_m a} + T\,e^{-i2k_m a} + U\,e^{-i3k_m a} + T^*\,e^{i2k_m a} + S^*\,e^{ik_m a} \tag{9.3.5a}$$

*Bloch dispersion relation*. And Bohr limit ($k \ll \pi/a$) approxiamtion. *Band group velocity $V_{group}$*.

$$\hbar\omega_m = E_m = H - 2|S|\cos(k_m a) = H - 2|S| + |S|(k_m a)^2 + .. \tag{9.3.8}$$

$$V_{group} = \frac{d\omega_m}{dk_m} = 2\frac{|S|}{\hbar}a\sin(k_m a) \quad \left(\cong 2\frac{|S|}{\hbar}k_m a^2 \text{ , for: } k_m \ll \pi/a\right) \tag{9.3.10}$$

*Effective mass $M_{eff}$* inversely proportional to $S$. $\quad M_{eff}(0) = \hbar^2/(2|S|a^2)$ (9.3.11a)

---

*Fourier transform of a Gaussian $e^{-(m/\Delta m)^2}$ momentum* distribution *is a Gaussian $e^{-(\phi/\Delta\phi)^2}$ in coordinate $\phi$*.

$$\langle m|\Psi\rangle = e^{-(m/\Delta m)^2} \quad \text{implies:} \quad \langle\phi|\Psi\rangle = e^{-(\phi/\Delta\phi)^2} \tag{9.3.14}$$

The relation between *momentum uncertainty* $\Delta m$ and *coordinate uncertainty* $\Delta\phi$ is a *Heisenberg relation*.

$$\Delta m/2 = 1/\Delta\phi \text{ ,or:} \quad \Delta m\,\Delta\phi = 2 \tag{9.3.15}$$

*Bohr wave quantum speed limits*

$$V_{group}^{Bohr}\left(m \leftrightarrow n\right) = \frac{\omega_m - \omega_n}{k_m - k_n} = \frac{\left(m^2 - n^2\right) h\upsilon_1}{\left(m-n\right) h / L} = (m+n)\frac{L}{\tau_1} = (m+n)V_1 \quad (9.3.16)$$

Predicting fractional revivals: *Farey Sum* $\oplus_F$ of the rational fractions $n_1/d_1$ and $n_2/d_2$

$$t_{12-intersection} = \frac{n_2 + n_1}{d_2 + d_1} = \frac{n_2}{d_2} \oplus_F \frac{n_1}{d_1} \quad (9.3.18)$$

*Appendix 9.A. Relative phase of peaks in a revival lattice*

The first derivation here of revival amplitudes at stroboscopic time fractions $t_\nu = \tau(\nu/N)$ and kaleidescopic angular positions $\phi_\rho = 2\pi(\rho/N)$ assumes $N$ is odd. At times when fraction $(\nu/N)$ is reduced, all $N$ revival peak sites hop up with identical magnitude and with particular arrangement of phases that clearly distinguishes each $\nu/N$ from all others. First we derive formulas for these phases as a function of site index $\rho$ and revival time index $\nu$. (If time fraction $\nu/N$ reduces to $\nu_R/N_R$, then use $(\nu_R, N_R)$ in place of $(\nu, N)$ to find $N_R$ peak phases of subgroup $C_{N_R}$ revivals.) The first step is to complete the square of exponent in sum.

$$\begin{aligned}\psi_0\left(\phi_\rho, t_\nu\right) &= \frac{1}{N}\Sigma_{m=0}^{N-1}\; e^{i\left(m\,\rho - m^2\,\nu\right)\frac{2\pi}{N}} = \frac{1}{N}\Sigma_{m=0}^{N-1}\; e^{-i\left(m^2\,\nu - m\,\rho + \frac{\rho^2}{4\nu}\right)\frac{2\pi}{N}}\, e^{i\frac{\rho^2}{4\nu}\frac{2\pi}{N}} \\ &= \frac{1}{N}\Sigma_{m=0}^{N-1}\; e^{-i\left(m\nu - \frac{\rho}{2}\right)\left(m - \frac{\rho}{2\nu}\right)\frac{2\pi}{N}}\, e^{i\frac{\rho^2}{4\nu}\frac{2\pi}{N}} \\ &= \frac{1}{N}\Sigma_{m=0}^{N-1}\; e^{-i(2m\nu - \rho)^2\frac{2\pi}{4\nu N}}\, e^{i\frac{\rho^2}{4\nu}\frac{2\pi}{N}}\end{aligned} \tag{A.1}$$

The integer square $(2m\nu\text{-}\rho)^2$ in the exponent is to be treated as an integer-modulo-$4\nu N$ since the phase factor repeats after that value. However, as summation index $m$ runs through the integers $m = 0, 1, 2, \ldots, N\text{-}1$ it exhausts all the possible values of $(2m\nu\text{-}p)^2$ -*mod*-$4\nu N$ for a given $\nu$ and $\rho$, and the values are the same no matter what we take for the range of $m$. For example, consider tables of phase index $(2m\nu\text{-}\rho)^2$ -*mod*-$4\nu N$ for select times of $\nu=1$ and $\nu=2$ for an $N=5$ level excitation.

$(2m\nu - \rho)^2$ **mod** $4\nu N$ for $N$=5

| $\nu$=1 | $m=0$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $\rho=0$ | $\overline{\mathbf{0}}$ | 4 | 16 | 16 | 4 | 0 | 4 |
| 1 | 1 | 1 | 9 | $\overline{\mathbf{5}}$ | 9 | 1 | 1 |
| 2 | 4 | $\overline{\mathbf{0}}$ | 4 | 16 | 16 | 4 | 0 |
| 3 | 9 | 1 | 1 | 9 | $\overline{\mathbf{5}}$ | 9 | 1 |
| 4 | 16 | 4 | $\overline{\mathbf{0}}$ | 4 | 16 | 16 | 4 |

(A.2a)

$(2m\nu - \rho)^2_{4\nu N}$ for $N$=5

| $\nu$=2 | $m=0$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10⋯ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\rho=0$ | $\overline{\mathbf{0}}$ | 16 | 24 | 24 | 16 | 0 | 16 | 24 | 24 | 16 | 0 |
| 1 | 1 | 9 | 9 | 1 | $\overline{\mathbf{25}}$ | 1 | 9 | 9 | 1 | 25 | 1 |
| 2 | 4 | 4 | 36 | $\overline{\mathbf{20}}$ | 36 | 4 | 4 | 36 | 20 | | |
| 3 | 9 | 1 | $\overline{\mathbf{25}}$ | 1 | 9 | 9 | 1 | | | | |
| 4 | 16 | $\overline{\mathbf{0}}$ | 16 | 24 | 24 | 16 | | | | | |

(A.2b)

Note that $N$ consecutive values for $m$ give the same sum no matter whether the sum starts at $m=0$ or at a *sum-shift* value $m=\mu$. The idea is to shift the summation index $m$ to $m\text{-}\mu$ so that a $(2m\nu\text{-}\rho)^2$ -*mod*-$4\nu N$ binomials in row-$\rho$ can be replaced by a simple square $(2m\nu)^2$ -*mod*-$4\nu N$ monomial found in the $\rho=0$ row. This will reduce the exponent to a term independent of site-index $\rho$ plus a $\Delta$–term independent of summation-index $m$.

It would be nice if the $\Delta$–term were also independent of $\rho$ but the tables show that is asking too much! So, $\Delta = \Delta(\rho,\nu)$ and, each of the rows $\rho = 1, .., N\text{-}1$ differ from the $\rho=0$ row by a single *modular difference* $\Delta(\rho,\nu)$ in phase index which is overlined in the table and is the *single unpaired* number in each row. For example, subtracting $\Delta(1,1)=5\text{-}mod\text{-}20 = (5)_{20}$ from the $(\rho=1)$ row of the $(\nu=1)$ table and shifting forward by $\mu_1=2$ gives the $(\rho=0)$ row $(mod\text{-}20)$ . The shifts needed to line up rows $\rho=1, 2, 3,$ and $4$ are $\mu_1=2$, $\mu_2=4$, $\mu_3=6$, and $\mu_4=8$ respectively, that is $\mu_\rho=\mu_1\rho$. These observations are summarized by a modular equation.

$$\left(2\left(m-\mu_\rho\right)\nu-\rho\right)^2 \bmod 4\nu N \equiv \left(2\left(m-\mu_\rho\right)\nu-\rho\right)^2_{4\nu N} = \left(2m\nu\right)^2_{4\nu N} - \Delta\left(\rho,\nu\right) \tag{A.3a}$$

This is supposedly valid for all values of $m$ so for $m=0$ the equation reads

$$\left(-2\mu_\rho \nu - \rho\right)^2_{4\nu N} = 0 - \Delta\left(\rho,\nu\right) \ , \tag{A.3b}$$

where $$\mu_\rho = \mu_1 \rho \, . \tag{A.3c}$$

Subtracting equation (A.3b) from (A.3a) gives the following, again valid for all $m$.

$$\left(2\left(m-\mu_\rho\right)\nu - \rho\right)^2_{4\nu N} - \left(-2\mu_\rho \nu - \rho\right)^2_{4\nu N} = \left(2m\nu\right)^2_{4\nu N}$$

$$\left(4m\nu\left(-2\mu_\rho \nu - \rho\right)\right)_{4\nu N} = \left(0\right)_{4\nu N} = \kappa 4\nu N = 0,\ 4\nu N,\ 8\nu N,\ldots,4\nu N(N-1)$$

Next, set $m=1$, and solve for the $m$-sum-shift $\mu_\rho$ of row $\rho$.

$$-8\mu_\rho \nu^2 - 4\nu\rho = -\kappa 4\nu N = 0,\ \text{-}4\nu N,\ \text{-}8\nu N,\ldots,\text{-}4\nu N(N-1)$$

$$2\mu_\rho \nu + \rho = \kappa N = 0,\ N,\ 2N,\ldots,N(N-1) \text{ or: } \mu_\rho = \frac{\kappa N - \rho}{2\nu} = (\text{integer})_N \tag{A.4a}$$

A value $\kappa=0,1,2,..,N-1$ is selected so that $m$-sum-shift $\mu_\rho$ is an integer $\mu_\rho=0,1,2,..,N-1$, too. Substituting the resulting $\mu_\rho$ value in (A.3a) gives the phase modular difference $\Delta$ first defined there and in (A.3b).

$$\Delta\left(\rho,\nu\right) = -\left(2\nu\mu_\rho + \rho\right)^2_{4\nu N} = -\left(2\nu\left(\frac{\kappa N - \rho}{2\nu}\right) + \rho\right)^2_{4\nu N} = -\left(\kappa N\right)^2_{4\nu N} \ , \tag{A.4b}$$

where

$$\kappa = \frac{2\nu\mu_\rho + \rho}{N} \, . \tag{A.4c}$$

Puttiing (A.3a) into the revival wavefunction sum (A.1) gives

$$\psi_0\left(\phi_\rho, t_\nu\right) = \frac{1}{N}\Sigma_{m=0}^{N-1}\ e^{-i\left(2m\nu-\rho\right)^2 \frac{2\pi}{4\nu N}} e^{i\frac{\rho^2}{4\nu}\frac{2\pi}{N}}$$

$$= \frac{1}{N}\Sigma_{m=0}^{N-1}\ e^{-i\left[\left(2m\nu\right)^2 - \Delta\left(\rho,\nu\right)\right]\frac{2\pi}{4\nu N}} e^{i\frac{\rho^2}{4\nu}\frac{2\pi}{N}} \qquad \left[\text{using:(A.3a)}\right]$$

$$= \frac{1}{N}\Sigma_{m=0}^{N-1}\ e^{-i\left[\left(2m\nu\right)^2 + \left(\kappa N\right)^2 - \rho^2\right]\frac{2\pi}{4\nu N}} \qquad \left[\text{using:(A.4b)}\right]$$

$$= \frac{1}{N}\Sigma_{m=0}^{N-1}\ e^{-i\left[\left(2m\nu\right)^2 + 4\mu_\rho^2\nu^2 + 4\mu_\rho \nu\rho\right]\frac{2\pi}{4\nu N}} \qquad \left[\text{using:(A.4c)}\right]$$

$$= P(\nu)e^{\frac{-i\left[\mu_\rho^2 \nu + \mu_\rho \rho\right]2\pi}{N}} = P(\nu)e^{\frac{-i\left[\mu_1^2 \nu + \mu_1\right]\rho^2\, 2\pi}{N}} \qquad \left[\text{using:(A.3c)}\right] \tag{A.5a}$$

The overall phase and amplitude prefactor $P(\nu)$ is a Gaussian sum discussed in Appendix 9B.

$$P\left(\nu\right) = \frac{1}{N}\Sigma_{m=0}^{N-1}\ e^{-i\left(2m\nu\right)^2 \frac{2\pi}{4\nu N}} = \frac{1}{N}\Sigma_{m=0}^{N-1}\ e^{-i\nu m^2 \frac{2\pi}{N}} \tag{A.5b}$$

Finally, the $(\rho=1)$ $m$-sum-shift $\mu_1$ is the first fraction $(N\text{-}1)/2\nu$, $(2N\text{-}1)/2\nu$, $(3N\text{-}1)/2\nu$, ..., or $(N^2\text{-}1)/2\nu$, to yield an integer according to (A.4a). Recall that it was assumed that $N$ and $\nu$ are relatively prime, that is, have no common factors. It seems evident that the integer arithmetic behind base-$N$ counter revivals is not trivial, even for the case of odd-$N$ .To complete this particular $N=5$ example we find the sum-shift $\mu_1$ at each revival time $\nu=1\text{-}4$.

| $\mu_1 = \dfrac{\kappa N-1}{2\nu}$ | $\kappa N-1=$ | 4 | 9 | 14 | 19 | 24 |
|---|---|---|---|---|---|---|
| $2\nu=2$ | | $\bar{\mathbf{2}}$ | . | 7 | . | 12 |
| $2\nu=4$ | | $\bar{\mathbf{1}}$ | . | . | . | 6 |
| $2\nu=6$ | | . | . | . | . | $\bar{\mathbf{4}}$ |
| $2\nu=8$ | | . | . | . | . | $\bar{\mathbf{3}}$ |

(A.6)

From the discussion of Appendix 9B come the overall prefactors $P(\nu=1)=1/\sqrt{5}$, $P(2)=-1/\sqrt{5}$, $P(3)=-1/\sqrt{5}$, and $P(\nu=1)=1/\sqrt{5}$, which are needed to complete the following $N=5$ revival table using (A.5).

| $\psi(\rho,\nu)$ | $\rho=0$ | $\rho=1$ | $\rho=2$ | $\rho=3$ | $\rho=4$ |
|---|---|---|---|---|---|
| $\nu=0$ | 1 | 0 | 0 | 0 | 0 |
| $\nu=1$ | $1/\sqrt{5}$ | $e_1^*$ | $e_1$ | $e_1$ | $e_1^*$ |
| $\nu=2$ | $-1/\sqrt{5}$ | $-e_2$ | $-e_2^*$ | $-e_2^*$ | $-e_2$ |
| $\nu=3$ | $-1/\sqrt{5}$ | $-e_2^*$ | $-e_2$ | $-e_2$ | $-e_2^*$ |
| $\nu=4$ | $1/\sqrt{5}$ | $e_1$ | $e_1^*$ | $e_1^*$ | $e_1$ |

where: $e_1 = e^{i2\pi/5}/\sqrt{5}$ $\quad e_2 = e^{2i2\pi/5}/\sqrt{5}$ (A.7)

A phasor gauge plot of the $N=5$ revivals (A.7) is shown in Fig. 9.4.3c.

The summation (A.1) for *even-N* is mostly the same as the above. Time index $\upsilon$ is replaced by $\upsilon/2$.

$$\psi_0\left(\phi_\rho,t_\nu\right)=\frac{1}{N}\Sigma_{m=0}^{N-1}\; e^{-i(m\nu-\rho)^2\frac{2\pi}{2\nu N}}e^{i\frac{\rho^2}{2\nu}\frac{2\pi}{N}}\text{, where; }\; t_\nu=\nu\frac{2\pi}{2N},\;\text{ for } N\text{-even.}$$

$$=P(\nu)e^{\frac{-i\left[\mu_\rho^2\nu+2\mu_\rho\rho\right]2\pi}{2N}}=P(\nu)e^{\frac{-i\left[\mu_1^2\nu+2\mu_1\right]\rho^2\,2\pi}{2N}} \tag{A.8a}$$

where

$$\mu_1=\frac{\kappa N-1}{\nu}=\text{first integer in }\frac{N-1}{\nu},\frac{2N-1}{\nu},\frac{3N-1}{\nu},\ldots \tag{A.8b}$$

Again the overall phase and amplitude prefactor $P(\nu)$ is a Gaussian sum discussed in Appendix B.

$$P(\nu)=\frac{1}{N}\Sigma_{m=0}^{N-1}\; e^{-i(m\nu)^2\frac{2\pi}{2\nu N}}=\frac{1}{N}\Sigma_{m=0}^{N-1}\; e^{-i\nu m^2\frac{2\pi}{2N}} \tag{A.8c}$$

This works for odd-numerator time fractions $1/2N,\ 3/2N,\ 5/2N,\ldots=\upsilon/2N$ . For the even numerator ones, we take advantage of the revival sequence $\upsilon/N = 1/N,\ 2/N,\ 3/N,\ldots.$ for $N$ cut in half and shifted by $\pi$. If $N/2$ is odd then (A.5) is used. If $N/2$ is even then (A.8) is used again, but with $N$ cut in half to $N/2$. Note that fractions with singly-even denominators have zeros at $\phi=0$ and peaks at $\phi=\pm\pi$. Fractions with odd denominators have peaks at $\phi=0$ and zeros at $\phi=\pm\pi$. Fractions with doubly-even denominators have zeros at $\phi=0$ <u>and</u> $\phi=\pm\pi$.

*Appendix 9.B. Overall phase of peaks in a revival lattice*

The evaluation of the $N$-term integral Gaussian sum

$$G(\nu) = \Sigma_{m=0}^{N-1} e^{-i\nu m^2 \frac{2\pi}{N}} = NP(\nu) \tag{B.1}$$

in the prefactor $P(\nu)=G(\nu)/N$ given by (A.5b) is, perhaps, the least trivial part of the revival formulation. The develpment involves complex Gaussian integer analysis, a subject which occupied Gauss for more than the first decade of his most productive years. Here we will be content with giving a list of the results for the first few integer combinations that would be relevant for the revivals shown previously.

| $N=$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\Sigma_{m=0}^{N-1} e^{-im^2\frac{2\pi}{N}} =$ | $0$ | $-i\sqrt{3}$ | $(1-i)\sqrt{4}$ | $\sqrt{5}$ | $0$ | $-i\sqrt{7}$ | $(1-i)\sqrt{8}$ | $\sqrt{9}$ | $0$ | $-i\sqrt{11}$ | $(1-i)\sqrt{12}$ |
| $\Sigma_{m=0}^{N-1} e^{-i2m^2\frac{2\pi}{N}} =$ | $2$ | $i\sqrt{3}$ | $0$ | $-\sqrt{5}$ | $-i\sqrt{12}$ | $-i\sqrt{7}$ | $(1-i)4$ | $\sqrt{9}$ | $\sqrt{20}$ | $i\sqrt{11}$ | $0$ |
| $\Sigma_{m=0}^{N-1} e^{-i3m^2\frac{2\pi}{N}} =$ | $0$ | $3$ | $(1+i)\sqrt{4}$ | $-\sqrt{5}$ | $0$ | $i\sqrt{7}$ | $-(1+i)\sqrt{8}$ | $-i\sqrt{27}$ | $0$ | $-i\sqrt{11}$ | $(1-i)6$ |
| $\Sigma_{m=0}^{N-1} e^{-i4m^2\frac{2\pi}{N}} =$ | $2$ | $-i\sqrt{3}$ | $4$ | $\sqrt{5}$ | $i\sqrt{12}$ | $-i\sqrt{7}$ | $0$ | $\sqrt{9}$ | $-\sqrt{20}$ | $-i\sqrt{11}$ | $-i\sqrt{48}$ |
| $\Sigma_{m=0}^{N-1} e^{-i5m^2\frac{2\pi}{N}} =$ | $0$ | $i\sqrt{3}$ | $(1-i)\sqrt{4}$ | $5$ | $0$ | $i\sqrt{7}$ | $-(1-i)\sqrt{8}$ | $\sqrt{9}$ | $0$ | $-i\sqrt{11}$ | $-(1-i)\sqrt{12}$ |
| $\Sigma_{m=0}^{N-1} e^{-i6m^2\frac{2\pi}{N}} =$ | $2$ | $3$ | $0$ | $\sqrt{5}$ | $6$ | $i\sqrt{7}$ | $(1+i)4$ | $i\sqrt{27}$ | $-\sqrt{20}$ | $i\sqrt{11}$ | $0$ |
| $\Sigma_{m=0}^{N-1} e^{-i7m^2\frac{2\pi}{N}} =$ | $0$ | $-i\sqrt{3}$ | $(1+i)\sqrt{4}$ | $-\sqrt{5}$ | $0$ | $7$ | $(1+i)\sqrt{8}$ | $\sqrt{9}$ | $0$ | $i\sqrt{11}$ | $-(1+i)\sqrt{12}$ |

(B.2)

Particuarly simple general results are had for the case of doubly-even integer.

| $N=2n$ | $4=2\cdot 2$ | $8=2\cdot 4$ | $12=2\cdot 6$ | $16=2\cdot 8$ | $20=2\cdot 10$ |
|---|---|---|---|---|---|
| $\Sigma_{m=0}^{N-1} e^{-im^2\frac{2\pi}{N}} =$ | $(1-i)$ | $(1-i)\sqrt{2}$ | $(1-i)\sqrt{3}$ | $(1-i)\sqrt{4}$ | $(1-i)\sqrt{5}$ |

(B.3)

A complex vector diagram of the first few G(u) sums is shown below in Fig. 9B.1.

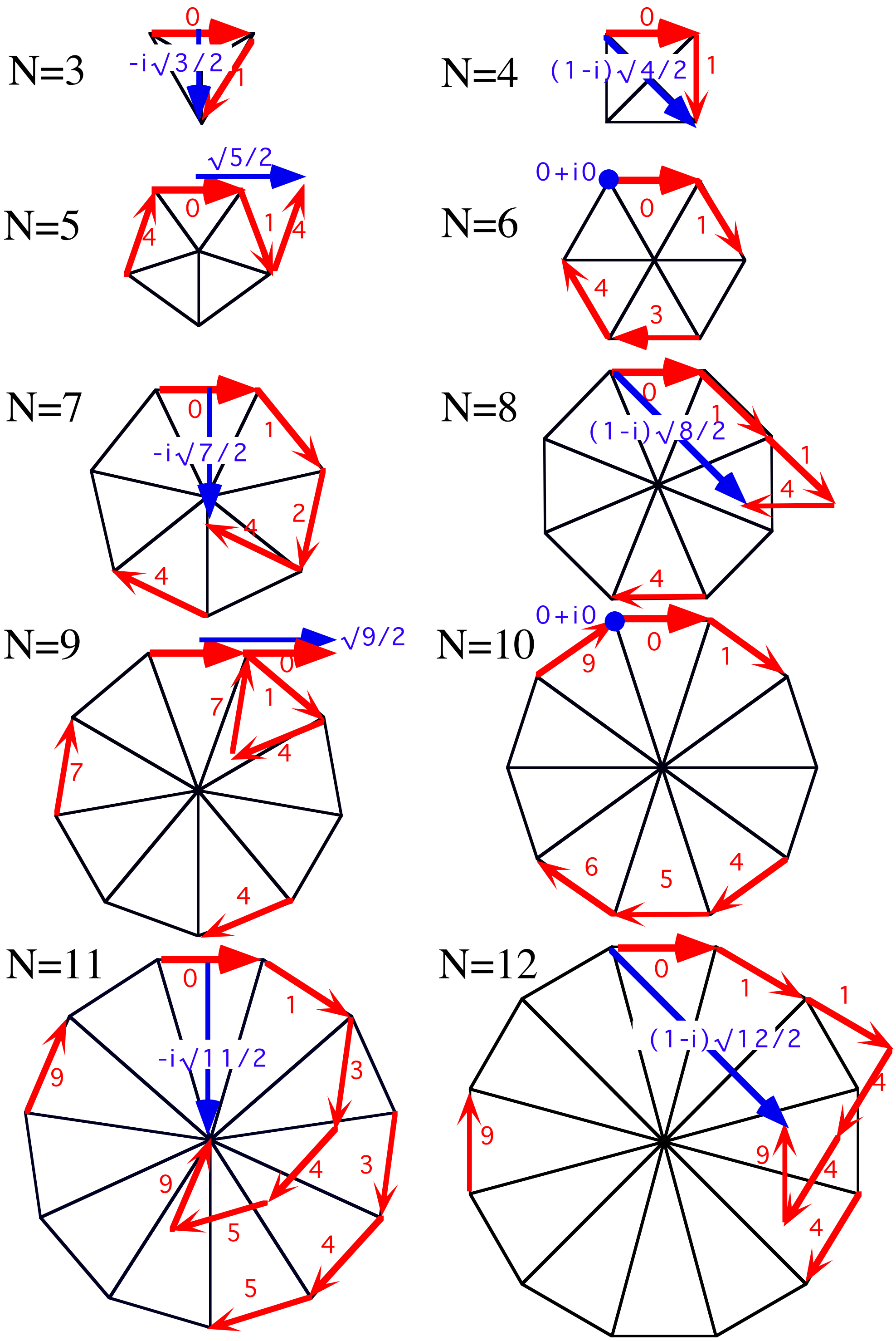


*Fig. 9B.1 Sums of modular squares $(m^2)_N = m^2 \bmod N$ (N = 3-12).*