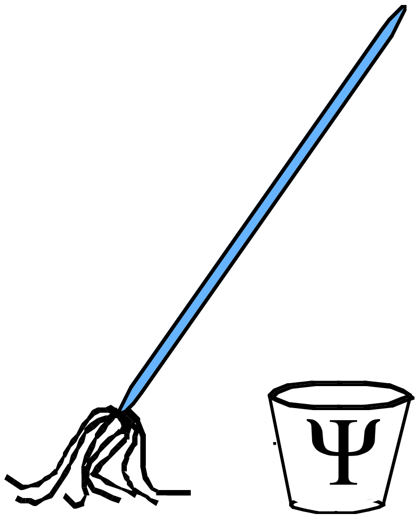

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Chapter 6
Multidimensional Waves And Modes

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Continuous wave dynamics in 3-dimensional space is determined by its wavevector components and the frequencies associated with each. This chapter introduces the geometry, relativity, and physics of wave 4-vectors that contain wavevector **k** and frequency ω. The various kinds of Doppler effects associated with optical and matter waves are derived. 3D-relativistic wave coordinates are introduced and an example involving a simple wave-guide model is worked out. Optical wave-guides have the same form of dispersion as a quantum matter wave.

Chapter 6. Multidimensional Waves and Modes

6.1. Wavevector Geometry and Transformation

Chapters 4 and 5 introduced the key ideas of wave mechanics using waves in one spatial dimension, usually labeled as the x-axis. The introduction in Ch. 4 of space-time relativity is then reduced to a study of spacetime "2-vectors" $\mathbf{X}=(x,ct)$. Per-spacetime "2-vectors" $\mathbf{K}=(ck,\omega)$ were related to $\mathbf{X}=(x,ct)$ by (4.2.11) as sketched in Fig. 4.2.11. The $\mathbf{K}$ satisfy optical dispersion $\omega=ck$ or else relativistic matter-wave dispersion $\omega^2=(\hbar Mc^2)^2-(ck)^2$ that is the key to quantum theory. Here 3D wave mechanics is discussed using 4-dimesional spacetime "4-vectors" $X^\mu=(\mathbf{r},ct)$ and per-spacetime 4-vectors $K^\mu=(c\mathbf{k},\omega)$.

(a) 3-Dimensional waves and 4-vectors

(ck,ω)-transformation formulas like (4.3.10) simplify Doppler analysis. Now we generalize the wavevector k to a three dimensional vector $\mathbf{k}$ and, for relativity, a *per-spacetime 4-vector* $k^\mu=(\mathbf{k},\omega/c)$. This is matched by a *space-time 4-vector* $x^\mu=(\mathbf{r},ct)$. The general plane wave function is the following.

$$\Psi(\mathbf{r},ct)=A\exp i(\mathbf{k}\bullet\mathbf{r}-\omega t)=Ae^{i(\mathbf{k}\bullet\mathbf{r}-\omega t)} \qquad (6.1.1)$$

Transformation equations that preserve the phase $(\mathbf{k}\bullet\mathbf{r}-\omega t)$ are obvious generalizations of (4.3.10).

$$\begin{pmatrix}x'\\y'\\z'\\ct'\end{pmatrix}=\begin{pmatrix}\cosh\rho_x&0&0&-\sinh\rho_x\\0&1&0&0\\0&0&1&0\\-\sinh\rho_x&0&0&\cosh\rho_x\end{pmatrix}\begin{pmatrix}x\\y\\z\\ct\end{pmatrix} \qquad (6.1.2a)$$

$$\begin{pmatrix}ck_x'\\ck_y'\\ck_z'\\\omega'\end{pmatrix}=\begin{pmatrix}\cosh\rho_x&0&0&-\sinh\rho_x\\0&1&0&0\\0&0&1&0\\-\sinh\rho_x&0&0&\cosh\rho_x\end{pmatrix}\begin{pmatrix}ck_x\\ck_y\\ck_z\\\omega\end{pmatrix} \qquad (6.1.2b)$$

Phase invariance axiom (4.3.6) for the 4-vectors is consistent with such a Lorentz transformation.

$$\Phi=-\mu\tau=\mathbf{k}\bullet\mathbf{r}-\omega t=\mathbf{k}\bullet\mathbf{r}-(\omega/c)(ct)=\mathbf{k}'\bullet\mathbf{r}'-(\omega'/c)(ct'). \qquad (6.1.2c)$$

(6.1.2a) preserves the individual $(\mathbf{r},ct)$-invariant of *proper time* τ. (Recall (5.1.4).)

$$(c\tau)^2=(ct)^2-r^2=(ct')^2-r'^2 \text{ where: } r=|\mathbf{r}|=\sqrt{(\mathbf{r}\bullet\mathbf{r})}=(x^2+y^2+z^2)^{1/2}. \qquad (6.1.2d)$$

(6.1.2b) preserve the individual $(\mathbf{k},\omega/c)$-invariant of *proper frequency* μ. (Recall (5.1.5).)

$$(\mu)^2=(\omega)^2-(ck)^2=(\omega')^2-(ck')^2 \text{ where: } k=|\mathbf{k}|=\sqrt{(\mathbf{k}\bullet\mathbf{k})}=(k_x^2+k_y^2+k_z^2)^{1/2}. \qquad (6.1.2e)$$

Multiplying four-vector $(\mathbf{k},\omega/c)$ by c gives a more conventional $(c\mathbf{k},\omega)$ and avoids fractions $1/c$.

There is economy in (6.1.2); one transformation matrix does both wavevector-frequency $(c\mathbf{k},\omega)$ and space-time $(\mathbf{r},ct)$. It gives the observed direction $\mathbf{k}'$ of a $(c\mathbf{k},\omega)$ light ray or any wave that was sent in a certain direction $\mathbf{k}$ from moving source. It gives the observed frequency ω', too. Both $\mathbf{k}$ and ω are linear functions of $\mathbf{k}'$ and ω'. The inverse is found by simply flipping the rapidity $+\rho_x$ to $-\rho_x$.

To visualize the $\mathbf{k}$-vector one may imagine (as in Fig. 6.1.1) parallel stacked phase planes normal to $\mathbf{k}$ all marching lockstep in the $\mathbf{k}$ direction at phase velocity $\mathbf{v}_{phase}=(\omega/k)\mathbf{e}_\mathbf{k}$, where k is the length of $\mathbf{k}$, and $\mathbf{e}_\mathbf{k}$ is a unit vector normal to the phase planes and in the direction of propagation. The distance between planes of phase Φ and $\Phi+2\pi$ is the wavelength $\lambda=2\pi/k$ as given by the usual formulas in Fig. 4.2.10. The period $\tau=2\pi/\omega$ is time between phase planes Φ and $\Phi+2\pi$ passing a point $\mathbf{r}$. Relations (6.1.2) hold whether or not V_{phase} is equal to the speed of light c and apply to all vacuum waves (6.1.1).

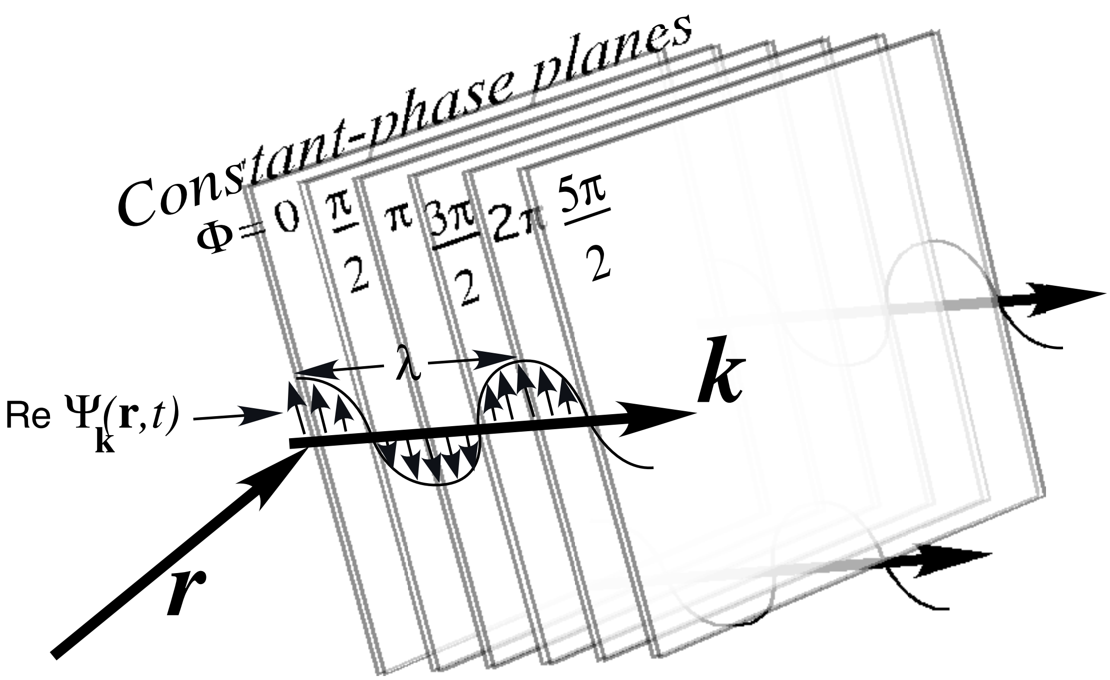

Fig. 6.1.1 Anatomy of a plane wavefunction $\Psi_k(\mathbf{r},t) = A\exp i(\mathbf{k}\bullet\mathbf{r} - \omega t)$ *with wavevector* **k**.

Use caution in interpreting the wavy arrows depicting $Re\Psi(\mathbf{r},t)$ in Fig. 6.1.1. For light waves the arrows could be actual **E**-field vectors as discussed in Ch. 1. (Recall (1.2.8-11).) Otherwise $\Psi(\mathbf{r},t)$ is a scalar wave or probability density wave that points in any (or no particular) direction. Vector waves have a polarization direction **E**. For vacuum em waves **E** is always normal to **k**, as indicated in Fig. 6.1.1.

For light waves or "photons" the *proper time* τ along its spacetime path $r=ct$ is zero by (5.1.4).

$$(c\tau)^2 = (ct)^2 - r^2 \quad (=0 \text{ for light or "photons"}) \tag{6.1.3}$$

A proper frame of a particle is one it drags with it so its r'-coordinate is origin ($r'=0$). Since light travels at c (Its position in the lab frame is $r=ct$.) it's impossible for it to "age" in its own (proper) frame. Hence its *proper frequency* μ is zero, too, by (5.1.5b). Optical phasors stop at $k=0$ since $\omega=ck$ is invariant.

$$(\mu)^2 = (\omega)^2 - (ck)^2 \quad (=0 \text{ for light or "photons"}) \tag{6.1.4}$$

The phase velocity of light is only $c=\omega/k$. Chapter 5 introduced matter waves with non-zero $\mu = Mc^2/\hbar$ that do age and only have speeds *other than* c. For real μ, V_{group} is less than c but $V_{phase} > c$. Motion of light "trapped" by waveguides and cavity walls occupies Sec. 6.3. Such imprisonment causes a non-zero "aging" rate μ for light and makes its waves "heavy" as seen already in discussion of Fig. 5.2.3.

Besides allowing a 3D range of **k**, (6.1.2) allows 3D Lorentz frame velocity **u**. Here $\mathbf{u} = u_z\mathbf{e}_z$.

$$\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\rho_z & -\sinh\rho_z \\ 0 & 0 & -\sinh\rho_z & \cosh\rho_z \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} \text{(6.1.5a)} \qquad \begin{pmatrix} ck_x' \\ ck_y' \\ ck_z' \\ \omega' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\rho_z & -\sinh\rho_z \\ 0 & 0 & -\sinh\rho_z & \cosh\rho_z \end{pmatrix} \begin{pmatrix} ck_x \\ ck_y \\ ck_z \\ \omega \end{pmatrix} \text{(6.1.5b)}$$

In this case the prime frame is moving at velocity $u_z = c\tanh\rho_z$ up the unprimed z-axis. (Set $z'=0$ in (6.1.5a). Then solve for $z = u_z t$.) Let us try to visualize $(c\mathbf{k}, \omega)$ transformations like (6.1.2b) or (6.1.5b).

(b) Transverse vs. longitudinal Doppler: Stellar aberration

A novel description of relativity by L. C. Epstein introduces a "cosmic speedometer" consisting of a telescope tube tipped to catch falling light pulses from a distant overhead star. A stationary telescope points straight up the x-axis at the apparent position S of the star. (Fig. 6.1.2a) But, for a velocity $\mathbf{u}=u_z\mathbf{e}_z$ across the star beam x-axis, the telescope must tip to catch the starlight, so the apparent position S' tips toward $\mathbf{u}$. (Fig. 6.1.2b). The telescope tips by the *stellar aberration angle* σ. The sine of the tipping angle σ is velocity ratio $\beta= u_z /c$ which is the hyper-tangent of relativistic rapidity ρ_z (or θ in (4.4.4).)

$$\beta= u_z /c =\sin \sigma = \tanh \rho_z \qquad (6.1.6)$$

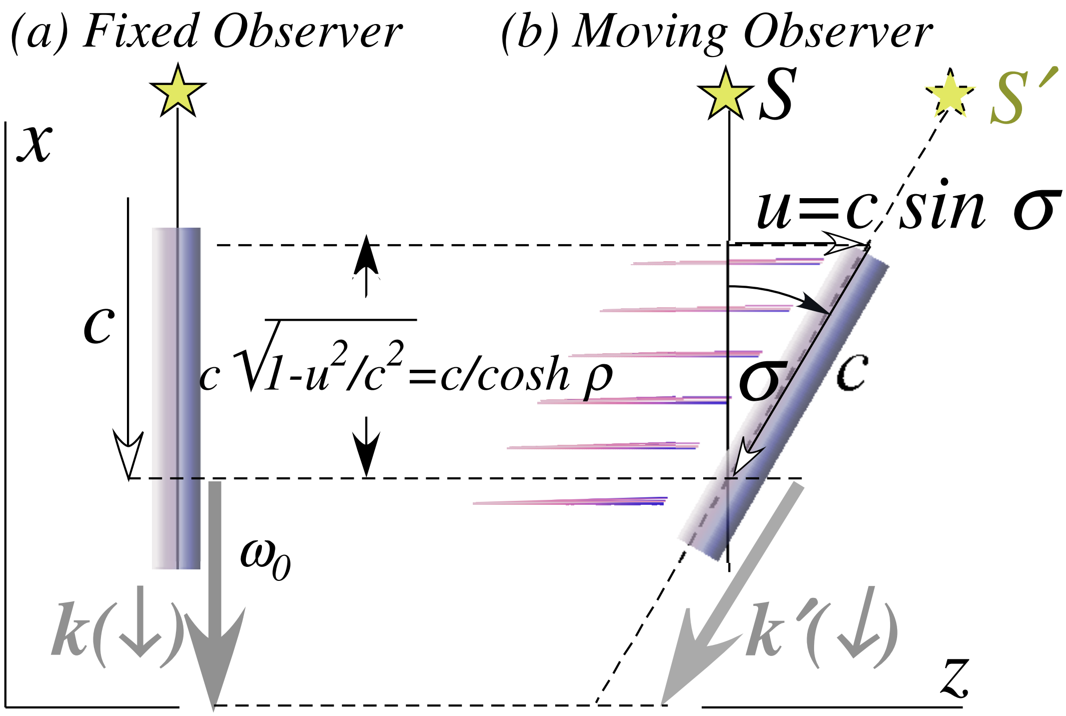

Fig. 6.1.2 Cosmic speedometer visualization of aberration angle σ and transverse Doppler shift coshρ.

Proper time τ and frequency μ invariance (6.1.2d-e) implies that 4-vector components normal to velocity $\mathbf{u}$ of a boost are unchanged. That is, a boost along z of (ct,z) to (ct',z') (or (ω,ck_z) to $(\omega',ck_{z'})$) must preserve both $(x,y)=(x',y')$ and $(ck_x,ck_y)=(ck_{x'},ck_{y'})$ just as a rotation in the xy-plane of (x,y) to (x',y') leaves unaffected the components $(ct,z)=(ct',z')$ and $(\omega,ck_z)=(\omega',ck_{z'})$ transverse to the rotation.

Invariant (6.1.3) also demands light-speed conservation as sketched in Fig. 6.1.2b. Starlight speed down the σ-tipped telescope is c, so the x-component of starlight velocity reduces from c to

$$c_{x'}=c \cos \sigma=c\sqrt{(1- u_z^2/c^2)} = c/\cosh \rho_z \,. \qquad (6.1.7)$$

Transformation (6.1.5b) assures that x-or-y-components of $\mathbf{k}\downarrow$ are unchanged by u_z-boost.

$$(ck_x,ck_y)=(ck_{x'},ck_{y'}) \qquad (6.1.8)$$

So the length of $\mathbf{k}\downarrow$ increases by a factor $\cosh \rho$ as shown in Fig. 6.1.3 as does the frequency $\omega'\downarrow$.

$$c|k'\downarrow| = c|k\downarrow| \cosh \upsilon_z = \omega_0 \cosh \upsilon_z =\omega_0/\sqrt{(1-u^2/c^2)} \qquad (6.1.9)$$

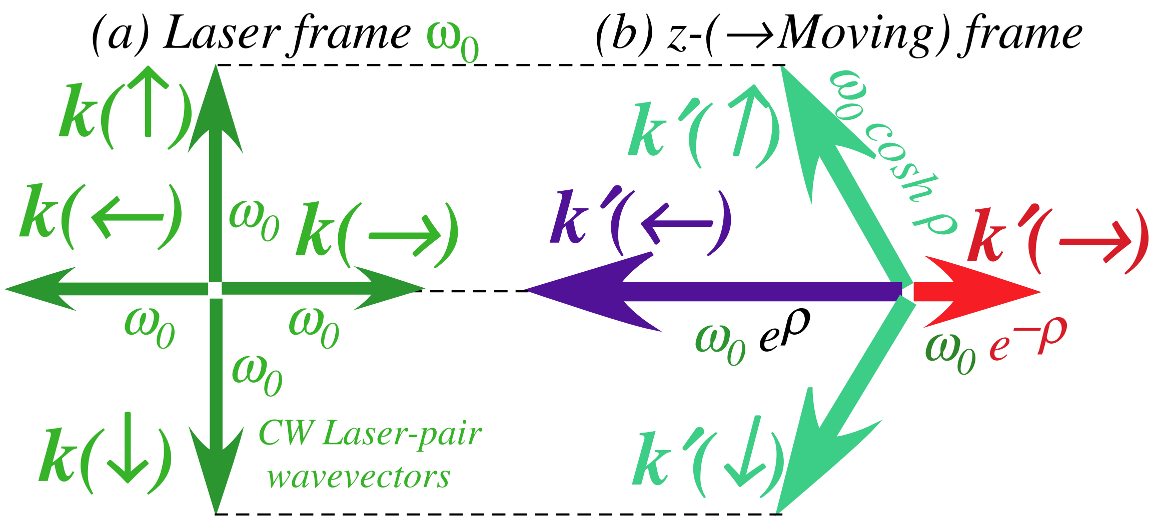

Fig. 6.1.3 CW version of cosmic speedometer showing transverse and longitudinal k-vectors.

If the observer crosses a star ray at very large velocity, that is, lets u_z approach c, then the star angle σ approaches *90°* and the frequency increases until the observer sees an X-ray or γ-ray star coming almost head on! The $\cosh\rho_z$ factor is a *transverse Doppler shift*. For large ρ_z, it approaches e^{ρ_z}, which is the ordinary *longitudinal Doppler shift* upon which the CW relativity derivations of Ch. 4 are based. Relations (6.1.6-9) are summarized in a 4-vector transformation: ω_0 has a *transverse Doppler shift* to $\omega_0\cosh\rho_z$, $ck_z=0$ becomes $ck_{z'} = -\omega_0 \sinh\rho_z$, but the x-component is unchanged: $ck_{x'} = \omega_0 = ck_x$.

$$\begin{pmatrix}\omega'_{\downarrow}\\ ck'_{x\downarrow}\\ ck'_{y\downarrow}\\ ck'_{z\downarrow}\end{pmatrix} = \begin{pmatrix}\cosh\rho_z & \cdot & \cdot & -\sinh\rho_z\\ \cdot & 1 & \cdot & \cdot\\ \cdot & \cdot & 1 & \cdot\\ -\sinh\rho_z & \cdot & \cdot & \cosh\rho_z\end{pmatrix}\begin{pmatrix}\omega_{\downarrow}\\ ck_{x\downarrow}\\ ck_{y\downarrow}\\ ck_{z\downarrow}\end{pmatrix} = \begin{pmatrix}\cosh\rho_z & \cdot & \cdot & -\sinh\rho_z\\ \cdot & 1 & \cdot & \cdot\\ \cdot & \cdot & 1 & \cdot\\ -\sinh\rho_z & \cdot & \cdot & \cosh\rho_z\end{pmatrix}\begin{pmatrix}\omega_0\\ -\omega_0\\ 0\\ 0\end{pmatrix} = \omega_0\begin{pmatrix}\cosh\rho_z\\ -1\\ 0\\ \sinh\rho_z\end{pmatrix} \tag{6.1.10a}$$

If starlight had been $\mathbf{k}_{\leftarrow}$ or $\mathbf{k}_{\rightarrow}$ waves going along $\mathbf{u}$ and z-axis, the usual longitudinal Doppler blue shifts $e^{+\rho_z}$ or red shifts $e^{-\rho_z}$ would appear on both the k-vector and the frequency, as stated by the following.

$$\begin{pmatrix}\omega'_{\rightarrow}\\ ck'_{x\rightarrow}\\ ck'_{y\rightarrow}\\ ck'_{z\rightarrow}\end{pmatrix} = \begin{pmatrix}\cosh\rho_z & \cdot & \cdot & -\sinh\rho_z\\ \cdot & 1 & \cdot & \cdot\\ \cdot & \cdot & 1 & \cdot\\ -\sinh\rho_z & \cdot & \cdot & \cosh\rho_z\end{pmatrix}\begin{pmatrix}\omega_0\\ 0\\ 0\\ \pm\omega_0\end{pmatrix} = \omega_0\begin{pmatrix}\cosh\rho_z \mp \sinh\rho_z\\ 0\\ 0\\ -\sinh\rho_z \pm \cosh\rho_z\end{pmatrix} = \omega_0\begin{pmatrix}e^{\mp\rho_z}\\ 0\\ 0\\ \pm e^{\mp\rho_z}\end{pmatrix} \tag{6.1.10b}$$

The Epstein speedometer tracks light pulses and particles in space and time. Instead of space-x and time-ct coordinates of a Minkowski graph, he plots space coordinate-x against *proper* time-$c\tau$. This view has all things, light γ and particle P included, moving at the speed of light as shown in Fig. 6.1.4. Light never ages, so its "speedometer" is tipped to the maximum along *x-axis*. (See *RelativIt* for animations, and Lewis Epstein's *Relativity Visualized*, (Insight Press San Francisco 1978) for details.)

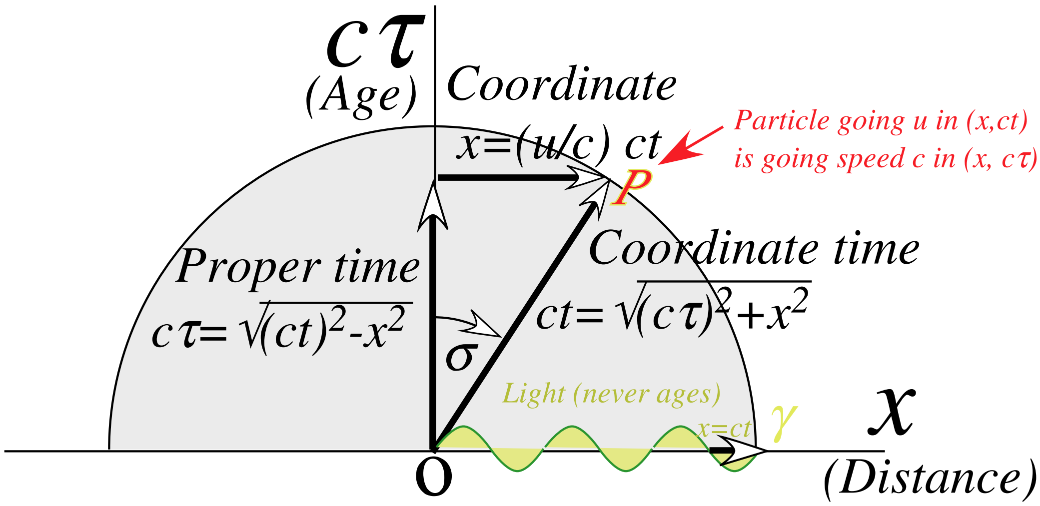

Fig. 6.1.4 Space-proper-time plot makes all objects move at speed c along their cosmic speedometer.

One nice feature of the Epstein space-proper-time view is its take of the Lorentz-Fitzgerald contraction of a proper length L to $L'=L\sqrt{1-u^2/c^2}$. (Recall discussion around (4.3.8).) As shown in Fig. 6.1.5 below, L' is simply the projection onto the x-axis of a length L tipped by σ.

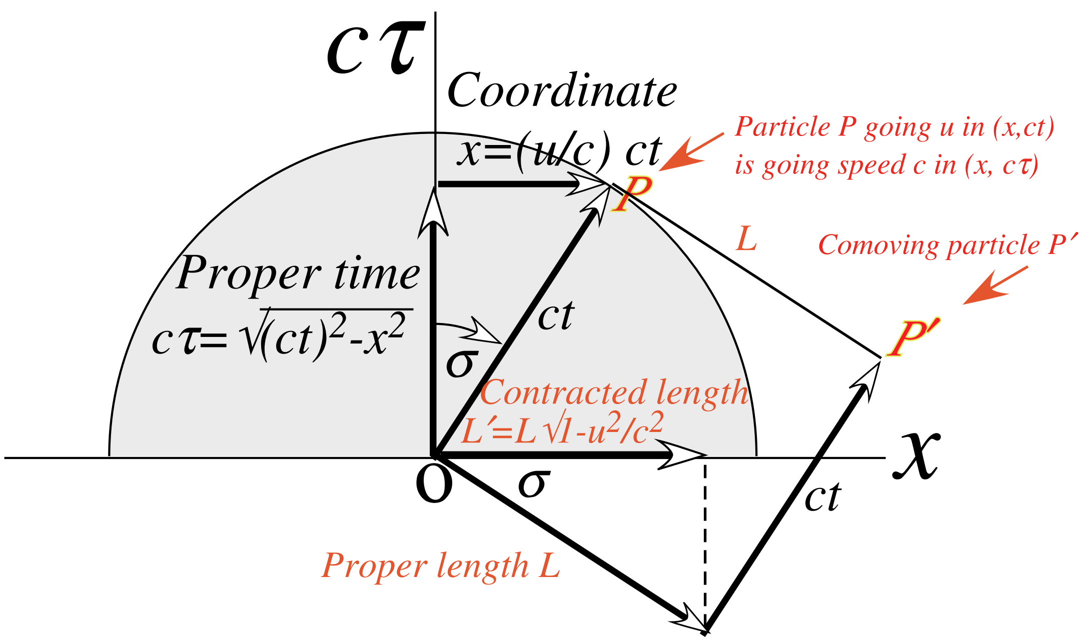

Fig. 6.1.5 Space-proper-time plot of Lorentz contraction as geometric projection of rotated line L.

The problem with the *(x,cτ)* view is that a space-time event is not plotted as a single point for all observers. Since the time parameter τ is an invariant quantity, the *(x,cτ)* graph it is not a metric space.

(c) Graphical wave 4-vector transformation

Geometric constructions combining Fig. 6.1.2 and Fig. 6.1.3 help to quantitatively visualize 4-wavevector transformations. One is shown in Fig. 6.2.1. The *c*-dial of the "speedometer" is first set to the desired **u**-speed which determines angle σ. The top of the *c*-dial (which may also represent a transverse *c***k**↑-vector in units of Lab frequency ω_0) is projected parallel to the velocity axis until it intersects the *c*-dial vertical axis. A transformed *c***k'**↑-vector of length $\omega'_\uparrow=\omega_0 \cosh\rho$ results, similar to *c***k'**↓ in (6.1.10a). Both *c***k'**↑ and *c***k'**↓ have a projection on the velocity axis of $\omega_0 \sinh\rho$ while maintaining their transverse components ω_0 and $-\omega_0$, respectively, in order to stay on the light cone. A dashed circle of radius *cosh* ρ is drawn concentric to the *c*-dial and determines the longitudinal vectors $c\mathbf{k'}_\rightarrow$ and $c\mathbf{k'}_\leftarrow$ of Doppler shifted length and frequency $\omega_0 e^{-\rho}$ and $\omega_0 e^{\rho}$, respectively, as required by transformation (6.1.10b).

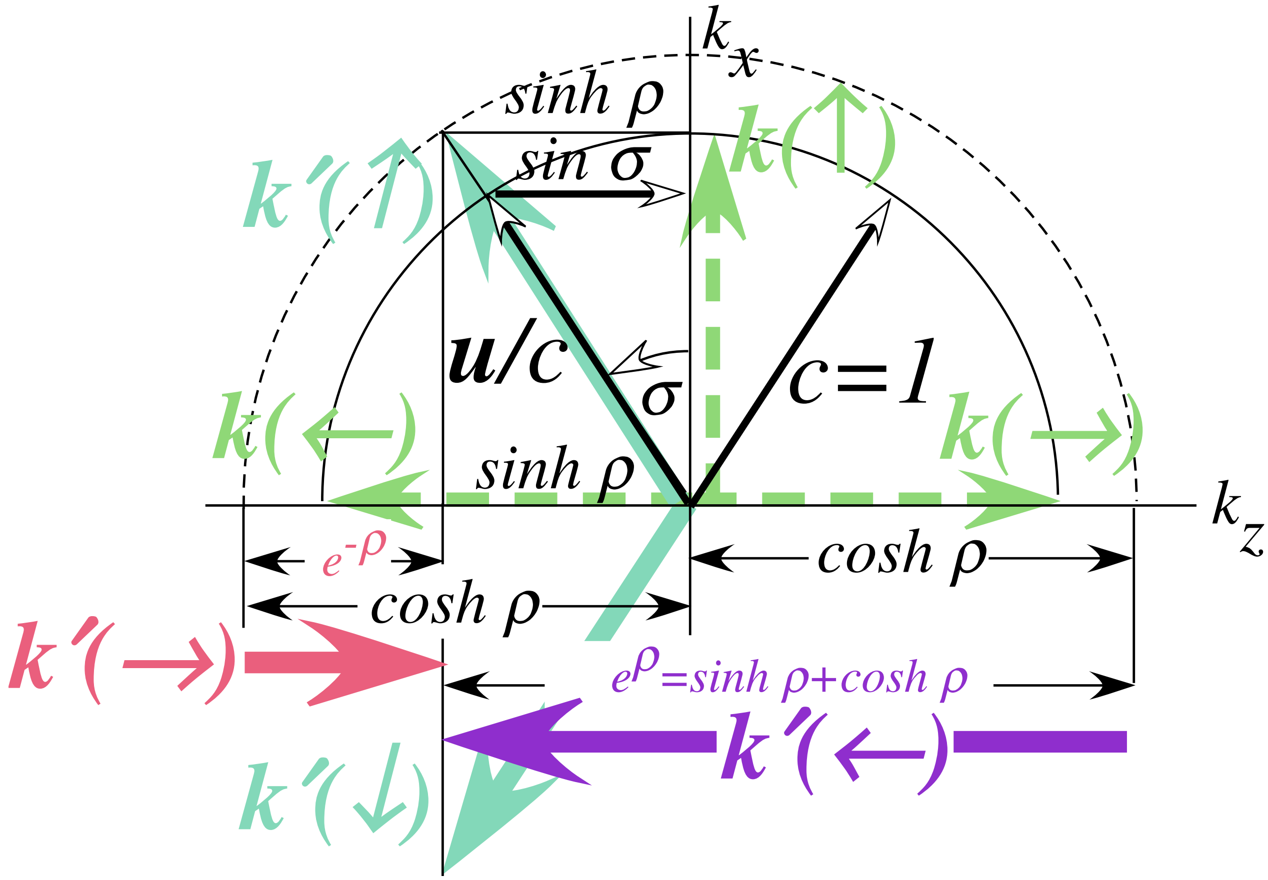

Fig. 6.1.6 CW cosmic speedometer. Geometry of Lorentz boost of counter-propagating waves.

There is similarity between the 2D per-spatial (k_x *versus* k_z) CW cosmic speedometer construction above in Fig. 6.1.6 and the one-dimensional (ω *versus* k) construction in Fig. 4.4.2(c). This is because optical invariance relation for $(\mu=0)$-γ-waves demands that the *c***k**-vector length be frequency ω.

$$\omega = ck = \sqrt{k_x^2 + k_y^2 + k_z^2}$$

That is the radius of the circle in both Fig. 4.4.2 and Fig. 6.1.6.

6.2. Laser Wave 4-Vector Coordinate Frames

Chapter 4 introduced the idea of a two-dimensional space-time *(x,ct)* coordinate system generated by a pair of continuous wave (CW) lasers. (Fig. 4.2.3) The result is the Lorentz-Einstein-Minkowski coordinates shown in Fig. 4.2.9 and labeled in Fig. 4.2.10. Now we discuss 3-dimensional problems involving 4-dimesional space-time $x^\mu = (\mathbf{r}, ct)$ coordinate systems made by counter-propagating CW lasers generating waves of 4-dimensional wavevector-frequency $k^\mu = (c\mathbf{k}, \omega)$.

(a) Counter propagating waves in space-time

One-(space) dimensional plane waves $\Psi(z,t)$ are now generalized to ones for which the word "plane" aptly describes constant phase wavefronts in Fig. 6.1.1. The general 1D wavefunction (4.2.6) is generalized to the following $\Psi_{\{\mu\}}(\mathbf{r},t)$.

$$\Psi_{A_\rightarrow,\omega_\rightarrow,\mathbf{k}_\rightarrow;A_\leftarrow,\omega_\leftarrow,\mathbf{k}_\leftarrow}(\mathbf{r},t) = A_\rightarrow e^{i(\mathbf{k}_\rightarrow\bullet\mathbf{r} - \omega_\rightarrow t)} + A_\leftarrow e^{i(\mathbf{k}_\leftarrow\bullet\mathbf{r} - \omega_\leftarrow t)} \tag{6.2.1}$$

Again, waves $\Psi_\mathbf{k}(\mathbf{r},t)$ with zero *SWR* have the simplest phase properties and transformation rules.

$$\Psi_\mathbf{k}(\mathbf{r},t) = \left(e^{i(\mathbf{k}_\rightarrow\bullet\mathbf{r} - \omega_\rightarrow t)} + e^{i(\mathbf{k}_\leftarrow\bullet\mathbf{r} - \omega_\leftarrow t)} \right)/2 \tag{6.2.2}$$

An expo-cosine identity generalizing (4.3.1) defines 3-D phase and group-envelope waves.

$$\Psi(\mathbf{r},t) = \Psi_\mathbf{k}(\mathbf{r},t) = \frac{1}{2}e^{i(\mathbf{k}_\rightarrow\bullet\mathbf{r}-\omega_\rightarrow t)} + \frac{1}{2}e^{i(\mathbf{k}_\leftarrow\bullet\mathbf{r}-\omega_\leftarrow t)}$$

$$= e^{i\frac{(\mathbf{k}_\rightarrow+\mathbf{k}_\leftarrow)\bullet\mathbf{r}-(\omega_\rightarrow+\omega_\leftarrow)t}{2}} \cos\frac{(\mathbf{k}_\rightarrow-\mathbf{k}_\leftarrow)\bullet\mathbf{r}-(\omega_\rightarrow-\omega_\leftarrow)t}{2} \tag{6.2.3a}$$

$$= e^{i(\overline{\mathbf{K}}\bullet\mathbf{r}-\overline{\Omega}t)}\cos(\overline{\mathbf{k}}\bullet\mathbf{r}-\overline{\varpi}t) \quad \text{where:} \tag{6.2.3b}$$

$$\overline{\mathbf{K}} = \frac{(\mathbf{k}_\rightarrow+\mathbf{k}_\leftarrow)}{2}, \qquad \overline{\mathbf{k}} = \frac{(\mathbf{k}_\rightarrow-\mathbf{k}_\leftarrow)}{2},$$
$$\overline{\Omega} = \frac{(\omega_\rightarrow+\omega_\leftarrow)}{2}, \qquad \overline{\varpi} = \frac{(\omega_\rightarrow-\omega_\leftarrow)}{2}. \tag{6.2.3c}$$

The Lab frame is defined by stationary phase planes separated by $\lambda_0=2\pi c/\omega_0$ normal to the beam axis-z between the two lasers of the same frequency ω_0 and equal but opposite wavevectors.

$$\Psi_{\mathbf{k}_0}(\mathbf{r},t) = e^{-i(\omega_0 t)}\cos(\mathbf{k}_0\bullet\mathbf{r}) \tag{6.2.4a}$$

where: $\overline{\mathbf{k}} = \mathbf{k}_\rightarrow = -\mathbf{k}_\leftarrow = \mathbf{k}_0$, and $\overline{\Omega} = \omega_\rightarrow = \omega_\leftarrow = \omega_0$ (6.2.4b)

The group planes of zero ReΨ are fixed normal to $\mathbf{k}_0$.

$$\mathbf{k}_0 \bullet \mathbf{r} = \pm\pi/2, \pm 3\pi/2, \ldots \tag{6.2.4c}$$

The phase zeros periodically go infinitely fast in the $\mathbf{k}_0$-direction at certain times.

$$\ldots\omega_0\, t = \pm\pi/2, \pm 3\pi/2, \ldots \tag{6.2.4d}$$

This is the same situation described before in Fig. 4.2.3a where the only boost allowed was along the beam *z*-axis as in the squared-off lasers in Fig. 4.2.3b but no 3-dimensional boosting or rotation was considered.

However, three dimensions presents a much more complicated range of possible symmetry transformations involving the six *Lorentz group* parameters for *x*, *y*, and *z* boosts and rotations or, including translations, nine parameters of the *Poincare' group*. Nevertheless, by appealing to continuous-wave optical thought experiments it is possible to simplify the derivation and visualization of this enormous symmetry of locally flat space-time for both classical and quantum theory. This, in spite of the fact that real lab experiments would be dicey at best. Squared-off laser waves of Fig. 4.2.3 would have difficulty achieving planarity over more than a few microns unless a great distance separated them.

(b) CW Cosmic positioning system

By replacing the pulsing star with orthogonal pairs of CW lasers as in Fig. 6.1.3, one might make a cosmic positioning system (CPS) similar to the Earth global positioning system (GPS) in that coherent em waves are used instead of rigid meter sticks. The wave dynamics associated with each pair automatically broadcasts a set of relativistic **k**-vectors and coordinate planes for any observer; the laser waves already "know" relativity! From the wavevectors, an observer can ascertain orientation and velocity relative to the Lab CPS system, and by coordinate plane integration, the translation position, as well. The key axiom, again, is phase invariance (6.1.2c), which for photons is (6.1.3) or for CW lasers (6.1.4).

First is individual laser phase invariance. Pairs $(ct_0,\mathbf{r}_0)$ and $(\omega_0,c\mathbf{k}_0)$ are in Lab CPS frame.

$$\Phi_{\rightarrow} = \mathbf{k}'_{\rightarrow} \bullet \mathbf{r}' - \omega'_{\rightarrow} t' = \mathbf{k}_{\rightarrow} \bullet \mathbf{r} - \omega_{\rightarrow} t = \mathbf{k}_0 \bullet \mathbf{r}_0 - \omega_0 t_0 \tag{6.2.5a}$$

$$\Phi_{\leftarrow} = \mathbf{k}'_{\leftarrow} \bullet \mathbf{r}' - \omega'_{\leftarrow} t' = \mathbf{k}_{\leftarrow} \bullet \mathbf{r} - \omega_{\leftarrow} t = -\mathbf{k}_0 \bullet \mathbf{r}_0 - \omega_0 t_0 \tag{6.2.5b}$$

Individual laser 4-vectors are, by definition, located on the light cone or null-invariant.

$$c^2\mathbf{k}'_{\rightarrow} \bullet \mathbf{k}'_{\rightarrow} - {\omega'_{\rightarrow}}^2 = c^2\mathbf{k}_{\rightarrow} \bullet \mathbf{k}_{\rightarrow} - {\omega_{\rightarrow}}^2 = c^2k_0^2 - \omega_0^2 = 0 \tag{6.2.6a}$$

$$c^2\mathbf{k}'_{\leftarrow} \bullet \mathbf{k}'_{\leftarrow} - {\omega'_{\leftarrow}}^2 = c^2\mathbf{k}_{\leftarrow} \bullet \mathbf{k}_{\leftarrow} - {\omega_{\leftarrow}}^2 = c^2k_0^2 - \omega_0^2 = 0 \tag{6.2.6b}$$

If any pair of 4-vectors $(\alpha,\mathbf{a})$ and $(\beta,\mathbf{b})$ are Lorentz covariant, then so is any linear combination of them.

$$(\gamma,\mathbf{c}) = A(\alpha,\mathbf{a}) + B(\beta,\mathbf{b}) = (A\alpha + B\beta, A\mathbf{a} + B\mathbf{b})$$

In other words, invariance of α^2-$\mathbf{a}\bullet\mathbf{a}$ = α'^2-$\mathbf{a}'\bullet\mathbf{a}'$ and β^2-$\mathbf{b}\bullet\mathbf{b}$ = β'^2-$\mathbf{b}'\bullet\mathbf{b}'$ implies invariance for the combination γ^2-$\mathbf{c}\bullet\mathbf{c}$ = γ'^2-$\mathbf{c}'\bullet\mathbf{c}'$, as well. So, phase invariance (6.1.2c) applies to sum $\overline{\mathbf{K}} = (\mathbf{k}_{\rightarrow} + \mathbf{k}_{\leftarrow})/2$ and difference $\overline{\mathbf{k}} = (\mathbf{k}_{\rightarrow} - \mathbf{k}_{\leftarrow})/2$ wavevectors attached to corresponding sum $\overline{\Omega} = (\omega_{\rightarrow} + \omega_{\leftarrow})/2$ and difference $\overline{\varpi} = (\omega_{\rightarrow} - \omega_{\leftarrow})/2$ frequencies. However, the new invariants have different values.

$$\overline{\mathbf{K}}' \bullet \mathbf{r}' - \overline{\Omega}' t' = \overline{\mathbf{K}} \bullet \mathbf{r} - \overline{\Omega} t = 0 - \omega_0 t_0 \tag{6.2.7a}$$

$$\overline{\mathbf{k}}' \bullet \mathbf{r}' - \overline{\varpi}' t' = \overline{\mathbf{k}} \bullet \mathbf{r} - \overline{\varpi} t = \mathbf{k}_0 \bullet \mathbf{r}_0 - 0 \tag{6.2.7b}$$

In fact, sum and difference vectors are not on the light cone like their laser components (6.2.6).

$$\overline{\Omega}'^2 - c^2\overline{\mathbf{K}}' \bullet \overline{\mathbf{K}}' = \overline{\Omega}^2 - c^2\overline{\mathbf{K}} \bullet \overline{\mathbf{K}} = {\omega_0}^2 - 0 = c^2k_0^2 \tag{6.2.8a}$$

$$\overline{\varpi}'^2 - c^2\overline{\mathbf{k}}' \bullet \overline{\mathbf{k}}' = \overline{\varpi}^2 - c^2\overline{\mathbf{k}} \bullet \overline{\mathbf{k}} = 0 - c^2\mathbf{k}_0 \bullet \mathbf{k}_0 = -c^2k_0^2 \tag{6.2.8b}$$

The sum vector $(\overline{\Omega}, c\overline{\mathbf{K}})$ in (6.2.8a) has a proper frequency $\mu=\omega_0=ck_0$ and behaves like a massive particle. Because of this, uniform wave guide modes have the dispersion of a massive particle which is a *mass-shell M-hyperboloid*(6.2.8a) such as was plotted in Fig. 5.2.1a. This will be shown in Sec. 6.3

The difference vector $(\overline{\varpi}, c\overline{\mathbf{k}})$ has an extraordinary imaginary proper frequency $\mu=ick_0$. Such an object is called a *Feinberg τ-Tachyon*, and is quite unlike ordinary matter. A real mass starts out at zero wavevector $(k=0)$ with real (proper) frequency $\mu=\omega_0$, zero group velocity $d\omega/dk$, and infinite phase velocity $^{\omega}/_{k}$. In contrast, tachyons start out at zero frequency $(\omega=0)$ with a real wavevector k_0 , zero phase velocity $(^{\omega}/_{k}.=0)$ and infinite group velocity $(^{d\omega}/_{dk}=\infty)$. A tachyon dispersion curve is a vertical τ-hyperbola given by (6.2.8b) and drawn below the photon asymptote in Fig. 5.2.1a. Such a τ-wave is also known as an "instanton" since it everywhere at the instant it has infinite group velocity. Our name for the τ-wave is less poetic: it is simply the group cosine envelope which is static in the Lab CPS frame. (It *defines* Lab-frame's coordinate planes.)

As the observer's rest frame changes velocity **u**, the sum vector $\left(\overline{\Omega},c\overline{\mathbf{K}}\right)$ follows an *M*-hyperbola (6.2.8a) while difference vector $\left(\overline{\omega},c\overline{\mathbf{k}}\right)$ follows a tachyon hyperbola (6.2.8b). Meanwhile, pairs of laser 4-wavevectors $(\omega_{\rightarrow},c\mathbf{k}_{\rightarrow})$ or $(\omega_{\leftarrow},c\mathbf{k}_{\leftarrow})$ (for light moving along the *x*-axis) and $(\omega_{\uparrow},c\mathbf{k}_{\uparrow})$ and $(\omega_{\downarrow},c\mathbf{k}_{\downarrow})$ (for light moving along the vertical *z*-axis) each follow null light-cone-invariants (6.2.6). The details of how $(\omega_{\rightarrow},c\mathbf{k}_{\rightarrow})$, $(\omega_{\leftarrow},c\mathbf{k}_{\leftarrow})$, $(\omega_{\uparrow},c\mathbf{k}_{\uparrow})$, and $(\omega_{\downarrow},c\mathbf{k}_{\downarrow})$ transform is sketched in Fig. 6.1.6.

(c) Wavevector defined coordinate planes

Examples of the effects of *x*-boosts, *z*-boosts and combinations of them on the CPS wavevector pairs are plotted in Fig. 6.2.1 where the relative velocity is $3c/5$. Fig. 6.2.1b is identical to the pair of wavevectors shown in the preceding Fig. 6.1.6, and Fig. 6.2.1c is the same thing rotated by $90°$.

Also shown are the CPS coordinate grid planes as seen in the observer's frame at a particular instant of the observer's time. These are planes that are the group *phase=0 mod* π planes in the CPS frame and fixed to it, so they are moving rigidly in any boosted observer's frame opposite to the boost. They are obtained from the wavevector differences such as $(\mathbf{k}_{\rightarrow} - \mathbf{k}_{\leftarrow})$ by solving (6.2.7b) as repeated below.

$$\overline{\mathbf{k}}\bullet\mathbf{r}-\overline{\omega}t=\ \mathbf{k}_0\bullet\mathbf{r}_0=0,\pm\pi,\pm2\pi,... \qquad (6.2.9)$$

At the observer instant $t=0$ one simply obtains a plane through origin and normal to $(\mathbf{k}_{\rightarrow} - \mathbf{k}_{\leftarrow})/2$.

This wave based solution is simpler than trying to use a Lorentz coordinate transformation to find observer coordinates (ct, x, y, z) in terms of CPS coordinates (ct_0, x_0, y_0, z_0). The latter would be easy if one desired observer values of CPS planes at a fixed CPS time t_0, but it is quite inconvenient if, as is the case, we desire their location as the observer sees them at his instant *t*.

The effect of doing $u_x/c=3/5$ and $u_z/c=3/5$ boosts singly and in sequence of different orders is shown in Fig. 6.2.1. A single boost induces an 80% Lorentz contraction $(1/\cosh\upsilon=0.8)$ in the direction of the boost. Two Lorentz boosts also induce a rotation, in fact, rotation group R(3) is a subgroup of the Lorentz group. The spectroscopic equivalent of the rotation is called *Thomas precession*. This important effect will be discussed later.

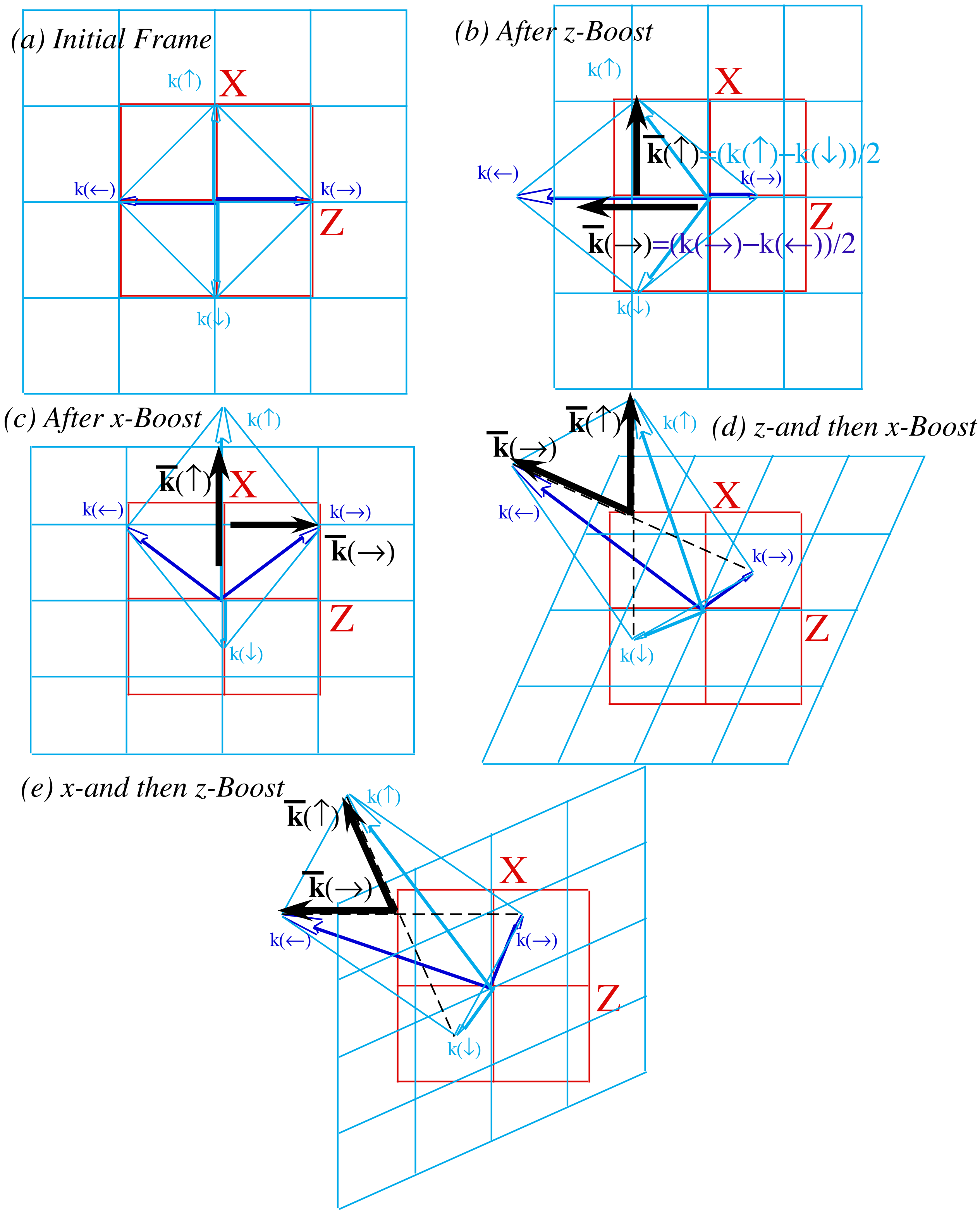

Fig. 6.2.1 Examples of sequential relativistic transformations of a tetrad of light wavevectors.

6.3. Wave Guide Dispersion and Cavity Eigenfrequencies

A wave guide confines 3-dimensional $(c\mathbf{k}, \omega)$ light waves to propagate in one dimension. The result is a dispersion function that is of the same form as (5.2.8) for quantum matter waves.

$$\omega^2=\mu^2-(ck)^2 \qquad (6.3.1)$$

Putting end plates on a guide further confines the wave to a cavity mode and restricts its frequency dispersion to discrete or "quantized" frequency eigenvalues ω_m. This is discussed below.

(a) 2-Dimensional wave mechanics: guided waves and dispersion

A two or three-dimensional wave will be seen to exceed the *c*-limit when it approaches an axis obliquely. It happens for plane waves. The phase velocities along coordinate axes are given by

$$v_x = \omega /k_x\,, \qquad v_y = \omega /k_y\,, \qquad v_z = \omega /k_z\,. \qquad (6.3.2)$$

Each of the components (k_x , k_y , k_z) must be less than or equal to magnitude k. Thus, all the component phase velocities equal or exceed the phase velocity ω /k which is c for light! In fact, water waves can exceed c ; if a wave breaks parallel to shore the 'break-line" moves infinitely fast since k_x is zero.

This has application to the basic wave mechanics of a wave guide consisting of a 'Hall of Mirrors" along the *x*-axis shown in Fig. 6.3.1. Let two parallel mirrors on either side of the *x*-axis be separated by a distance $y=W$. The South wall will be at $y=-W/2$ and the North wall at $y=W/2$. (*z*-axis or "up" is into the page of Fig. 6.3.1.) The Hall should have a floor and ceiling at $z=\pm H/2$, but its position doesn't matter as long as we consider only waves moving in the *xy*-plane direction. The effect of *H* is discussed later.

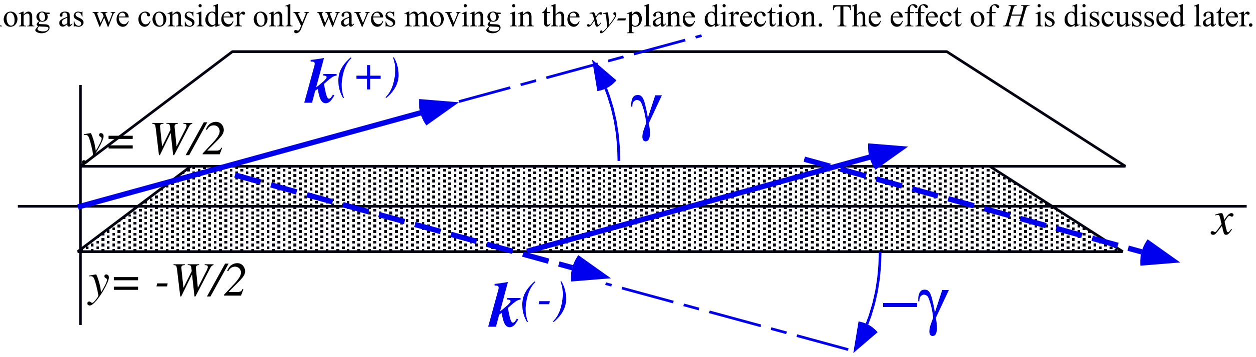

Fig. 6.3.1 A "hall of mirrors" model for an optical wave guide of width W.

Now consider what would happen if you shine a laser or maser down this hall. (In quantum jargon, "we propagate a photon beam.") Let the beam be at an angle γ to the *x*-axis in the plane of the Fig. 6.3.1. Two waves will result as indicated in Fig. 6.3.1; one you sent in with its **k**-vector $\mathbf{k}^{(+)}$ pointing at angle +γ

$$\mathbf{k}^{(+)} = (k^{(+)}{}_x,\, k^{(+)}{}_y,\, 0) = (k \cos\gamma\,,\, k \sin\gamma,\, 0\,)$$

and its *y*-reflected mirror image with its **k**-vector $\mathbf{k}^{(-)}$ pointing at angle -γ.

$$\mathbf{k}^{(-)} = (k^{(-)}{}_x,\, k^{(-)}{}_y,\, 0) = (k \cos\gamma\,,\, -k \sin\gamma,\, 0\,).$$

By adding the two waves with $\mathbf{k}^{(+)}$ and $\mathbf{k}^{(-)}$ you can make a wave function inside the Hall of Mirrors that vanishes at the mirror surface boundaries.

$$\begin{aligned}\Psi(\mathbf{r},t) &= \exp i(\,\mathbf{k}^{(+)}\bullet\mathbf{r}\; - \omega\, t) + \exp i(\,\mathbf{k}^{(-)}\bullet\mathbf{r}\; - \omega\, t)\\ &= \exp i(k\,x \cos\gamma\; + k\,y \sin\gamma - \omega\, t) + \exp i(k\,x \cos\gamma\; - k\,y \sin\gamma - \omega\, t)\\ &= \exp i(k\,x \cos\gamma\; - \omega\, t)\,[\,\exp i(\,k\,y \sin\gamma\,) + \exp i(\,-\,k\,y \sin\gamma\,)]\end{aligned}$$

$$= e^{i(k\, x \cos\gamma \;-\; \omega\, t)} \left[\, 2 \cos(\, k\, y \sin\gamma\,)\, \right] \tag{6.3.3}$$

The only requirement is that the wave function vanish on the mirror surfaces *(y=±W/2)* since we're assuming E-fields cannot penetrate these walls. This boundary condition is easily solved for the angle γ.

$$0=2\cos(\, k\, (W/2) \sin\gamma\,)\,, \quad \text{or: } k\,(W/2)\sin\gamma = \pi / 2\,, \text{ or: } \sin\gamma = \pi / (\, k\, W) \tag{6.3.4a}$$

The wavevector magnitude is related to angular frequency by the usual $k = \omega / c$. Stated another way we fix the y-component of the $\mathbf{k}^{(+)}$ or $\mathbf{k}^{(-)}$ vectors to just fit a half-wave in the width W of the Hall of Mirrors.

$$k^{(+)}{}_y = k \sin\gamma = \pi / W \tag{6.3.4b}$$

These conditions lead to what is called a *dispersion function* $\omega(k_x)$ or ω *vs.* k_x *relation*.

$$\omega = kc = c(k_x{}^2 + k_y{}^2 + k_z{}^2\,)^{1/2} = c(k_x{}^2 + \pi^2 / W^2\,)^{1/2} \tag{6.3.5a}$$

$$\omega = \sqrt{(c^2 k_x{}^2 + \omega_{cut}{}^2)} \quad \text{where:} \qquad \omega_{cut} = \pi c/W. \tag{6.3.5b}$$

The minimum or *cut-off frequency* $\omega_{cut} = \pi c/W$ is defined. Solving for k_x gives

$$k_x = (\omega^2 / c^2 - \pi^2 / W^2\,)^{1/2}. \tag{6.3.5c}$$

This is the equation for a hyperbola in (ω, ck_x) space which plotted below in Fig. 6.3.2.

$$\omega^2 - c^2 k_x{}^2 = \pi^2 c^2 / W^2 = \omega_{cut}{}^2 \tag{6.3.5d}$$

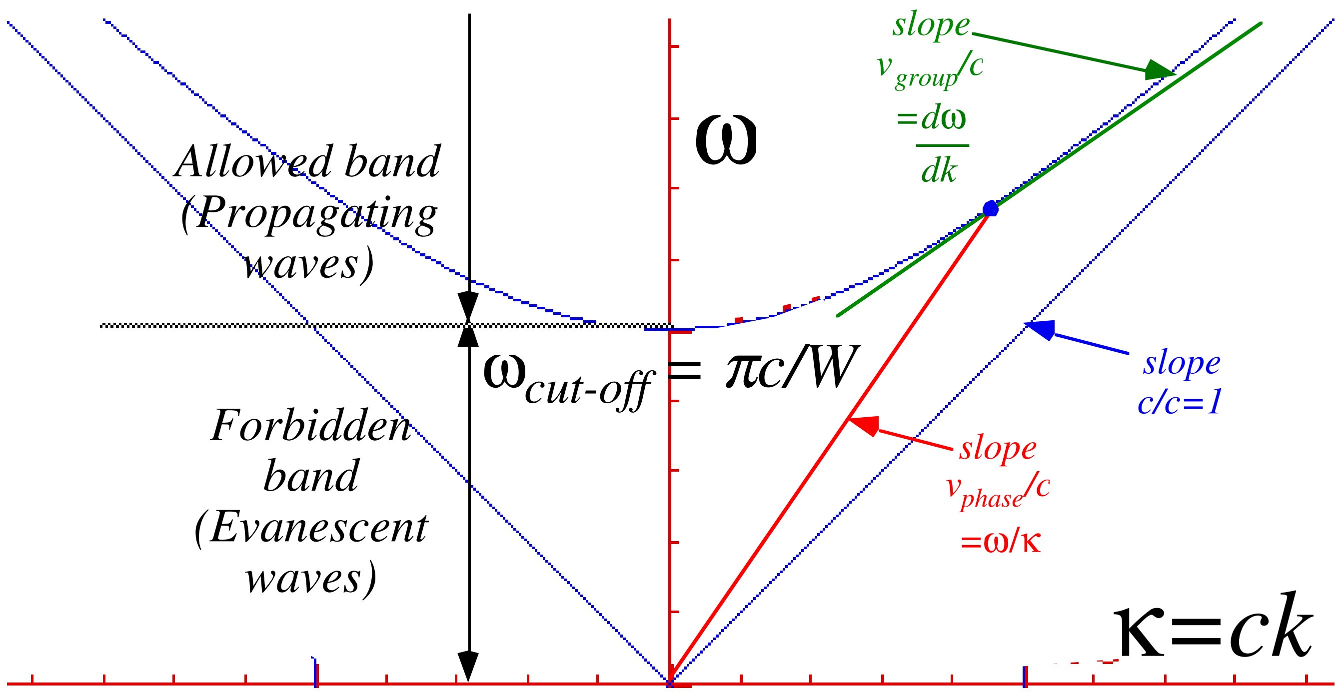

Fig. 6.3.2 Dispersion function for a fundamental TE wave guide mode

The hyperbolic asymptotes are lines of slope equal to the speed of light c. (6.3.5d) is a standard relativistic invariant function. All observers, no matter what their relative x-velocity, agree on how the light travels through space-time in a Hall of Mirrors. This holds only if the mirrors are "perfect" in that their performance does not depend on their x-velocity. You can't tell if a "perfect' mirror was sliding past you! But, if the mirrors' atomic response varies with velocity, as it will when photon frequencies are Doppler shifted into X-ray values, then the dispersion function will cease to be a relativistic invariant. Polarization wave velocity in solids or liquids is rarely, if ever, invariant to the material velocity.

The dispersion relation $\omega(k_x)$ is used to calculate the Hall wave velocities. From the dispersion relation $\omega(k_x)$ in (6.3.6 a) we obtain the phase velocity from (4.4.7) and group velocity from (4.4.6b) for a mono-chromatic wave propagating down the Hall. (However, a group wave cannot be monochromatic!)

$$v_x\,(phase) = \omega/k_x = c\omega/(\omega^2 - \pi^2c^2/W^2)^{1/2} \quad (6.3.7\text{ a})$$

$$v_x\,(group) = d\omega/dk_x = ck_x/(k_x^{\,2} + \pi^2/W^2) = c\,(\omega^2 - \pi^2c^2/W^2)^{1/2}/\omega \quad (6.3.7\text{ b})$$

Using (4.34b) we get the speeds in terms of angle γ and vacuum light speed c.

$$v_x\,(phase) = c/\cos\gamma \quad (6.3.7\text{ c}) \qquad v_x\,(group) = c\cos\gamma \quad (6.3.7\text{ d})$$

As the wavelength is reduced (higher k and ω) v_x *(phase)* and v_x *(group)* approach c which is what light would do anyway if the Hall width W was huge. However, as wavelength grows (lower k and ω) the tipping angle γ grows from zero toward $90°$ in order to match a half wave perfectly to the Hall width W. Then v_x *(phase)* approaches infinity while v_x *(group)* slows to a crawl as the frequency approaches a minimum *cut-off* value $\omega_{cut} = \pi c/W$. This is the *proper frequency* μ of a guided photon, the smallest red-shifted frequency a moving observer could see if he Doppler shifts k_x to zero. The next figures, done by the program *GuideIt* show waves going down a hall at various frequencies.

(b). Rays and wavefronts: Phase and group velocity

Fig. 6.3.3 begins with light entering the Hall of Mirrors at $\gamma=\pm 45°$ to the x-axis. The rays of the $+45°$ wave are being traced as they appear to reflect off the North wall into the $-45°$ wave. Note that the wave amplitude (represented by wavy lines) is maximum in the center of the hallway $(y=0)$ as required by the amplitude factor $[\,2\cos(k\,y\sin\gamma)\,]$ in (6.3.3). The same factor makes the wave identically zero at the North and South walls $(y=\pm W/2)$ according to (6.3.4a).

The wave moves at a speed $v_x\,(phase) = c\sqrt{2}$ according to (6.3.7 c). But, the velocity $v_x\,(group) = c/\sqrt{2}$ is exactly half as fast. This is the velocity of the *rays*. Their progress down the x-axis of the hallway is slower than their actual speed c because they are ricocheting back and forth off the walls as seen in the figures below. This "off-the-wall" explanation of group velocity makes it clear why the group velocity is c times the cosine of the angle γ. It is the x-component of a tipped wave velocity vector. The rays are attached to *wave fronts* of constant phase. On the way up the wave front phase is $2n\pi$ (multiple of 2π) indicated by a thin solid line. The phase changes by π when a ray bounces off a wall, so downward rays are attached to a wave front having a phase of $(2n-1)\pi$, indicated by a thin dotted line. Where solid $(2n\pi)$ fronts meet is a wave *crest.* Where dotted $(n\pi)$ fronts meet is a wave *trough.* A *node* is where fronts of opposite phase meet such as along walls that have a line of nodes.

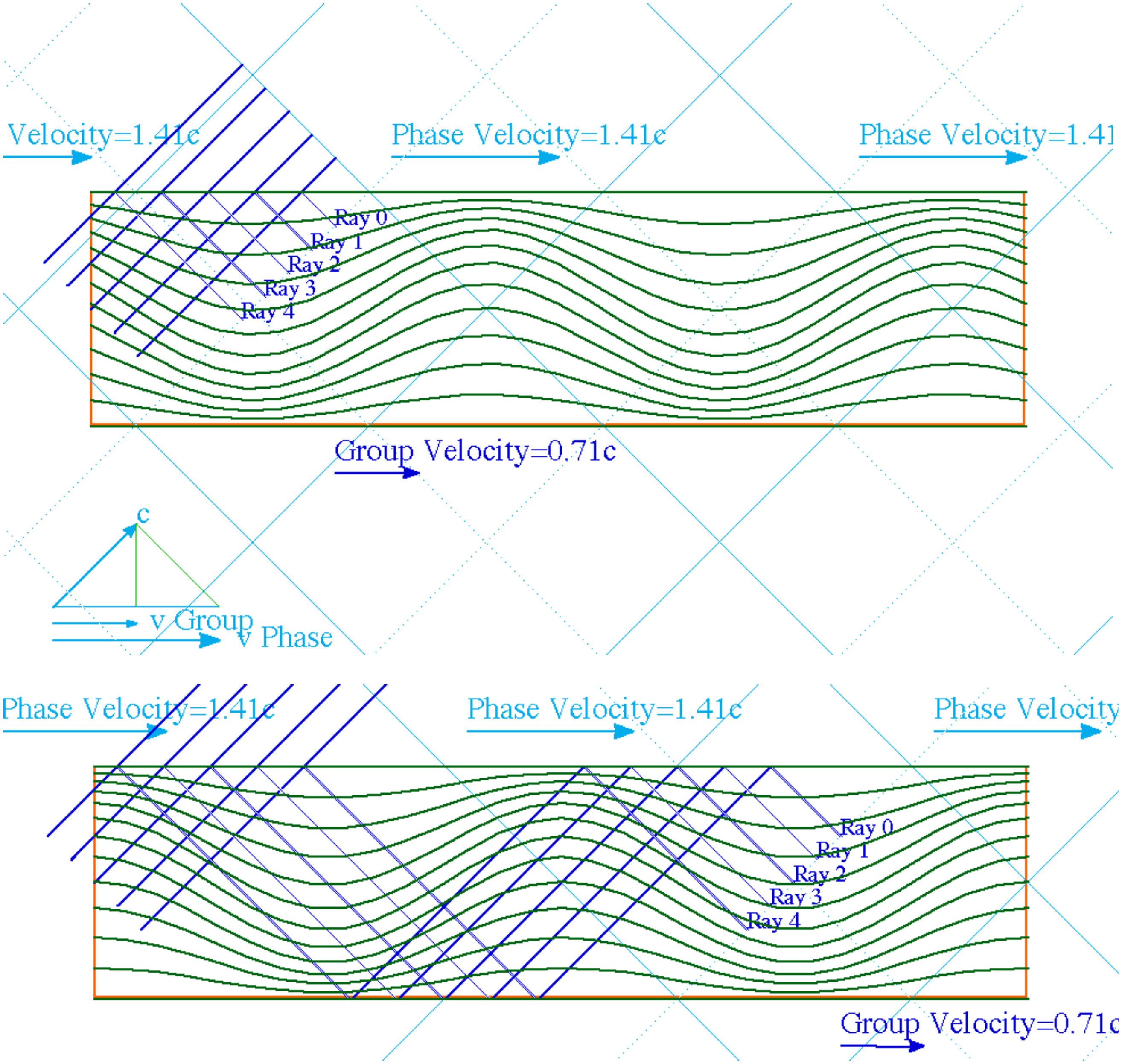

Fig. 6.3.3 Right moving guide wave with $\gamma = 45°$, $V_{phase}=\sqrt{2}c$, $V_{group}=c/\sqrt{2}$.

The x-phase velocity v_x *(phase)* is the speed of the *intersection* of wave fronts with the walls (nodes) or the x-axis (crests and troughs). In the sequence of frames below, note how much faster crests and troughs move than rays. The wave fronts go at velocity c along rays, that is, *perpendicular* to the fronts while rigid diamond-shaped wave patterns go at v_x *(phase)=1.41c* down the x-axis as shown in Fig. 6.3.4. Attached to the diamonds are nodal rectangles (actually *squares* in this example) whose borders lie along the top and bottom walls (as required by the y-boundary conditions (6.3.4a)) and whose vertical sides lie half-way between the crests and troughs. Unlike the diamonds, the nodal squares are observable borders of the interference minima and maxima. The phase diamonds represent unobservable "artistic license."

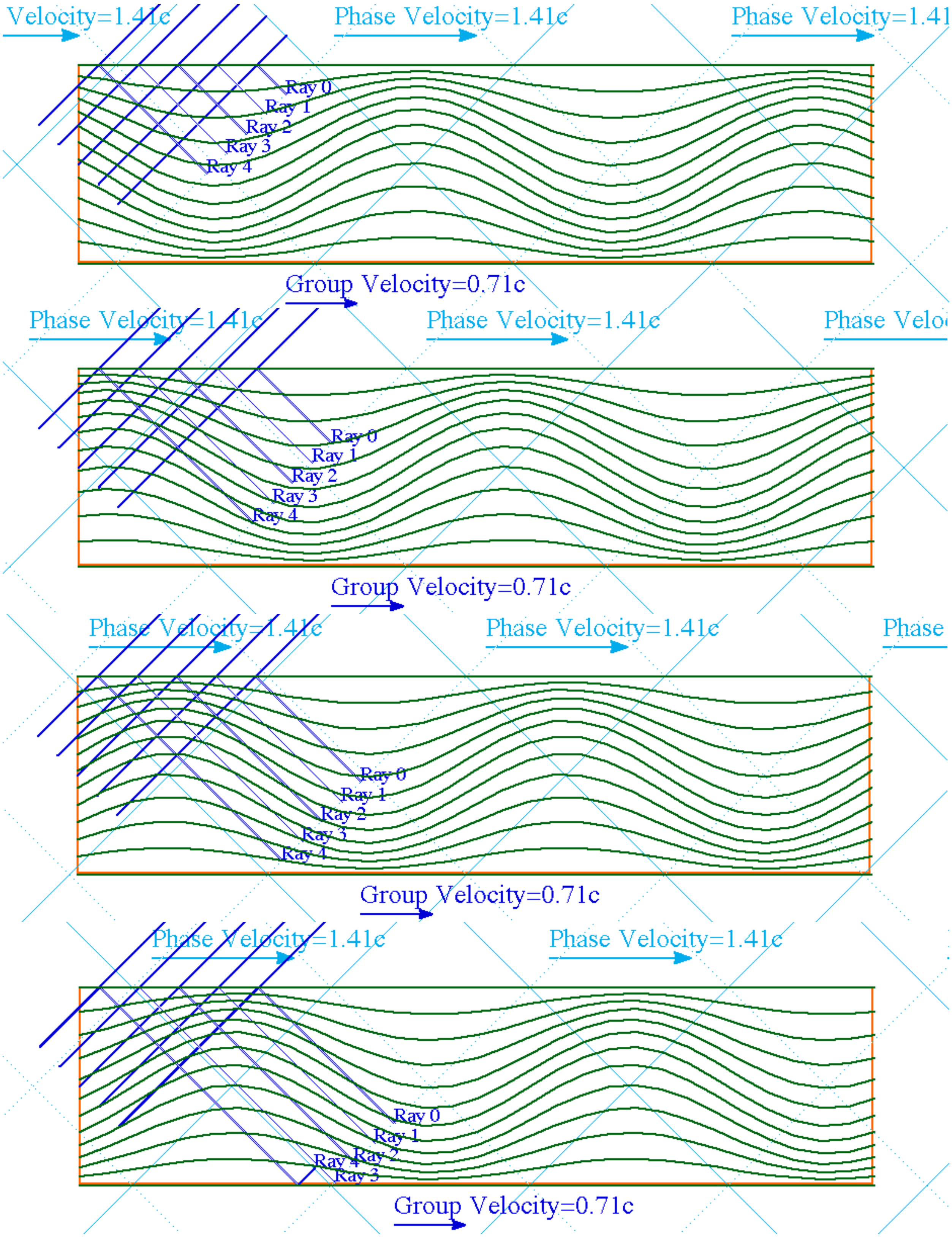

Fig. 6.3.4 Right moving guide wave with $\gamma = 45°$. Rays are half as fast as wave crests.

Higher frequency means a lower γ with two velocities are approaching c as shown in Fig. 6.3.5a where $\gamma = 30°$. As a Hall of mirrors gets much wider than the *~0.5 μm* optical wavelength you can simply look down it with no detectable dispersion. In this limit V_{phase} and v_x *(group)* both converge on c.

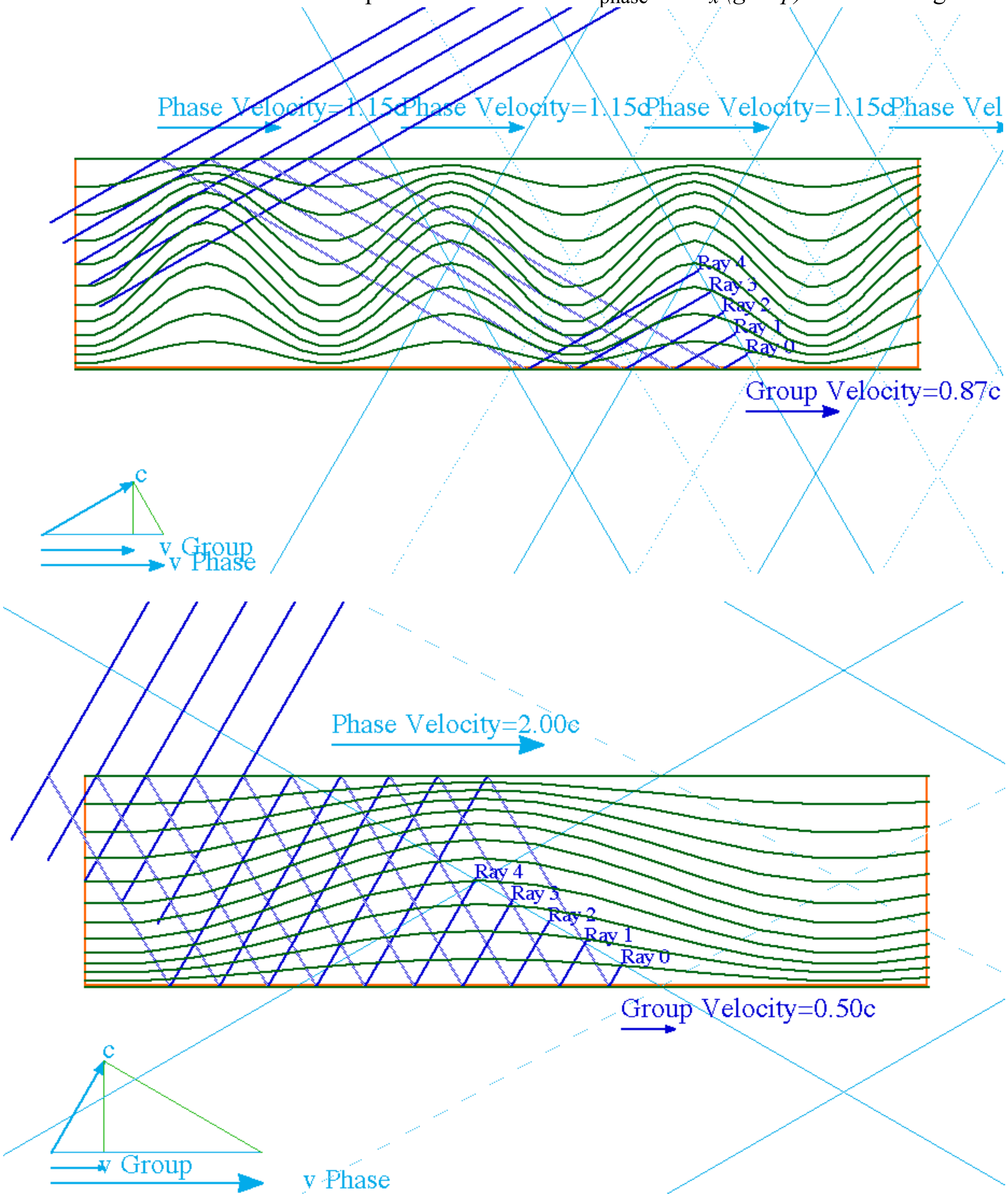

Fig. 6.3.5 Guide waves. *(a)Higher frequency case:* $\gamma = 30°$, $v_{x\ (phase)} = c\sqrt{3}/2c$, $v_{x\ (group)} = c2/\sqrt{3}$.
(b)Lower frequency case: $\gamma = 60°$, $v_{x\ (phase)} = 2c$, $v_{x\ (group)} = c/2$.

Suppose the light frequency is reduced so the wavelength increases and the angle γ increases to $60°$. Then v_x *(phase)* grows to $2c$ while the ray or group velocity reduces to v_x *(group)* $= c \cos 60° = c/2$. That is one-half the speed of light and one-fourth v_x *(phase)* $=2c$ as shown in Fig. 6.3.5b.

Group waves and "messages":(How do I send one?)

Group waves can carry "messages" as was discussed in Sec. 4.4, but this requires at least two different frequency components. The guide waves pictured so far are mono-chromatic and carry no information except a steady "hum" if they are classical waves, or a steady and uniform rain of random counts if we are describing a quantum wave. There is a $|\cos(k\, y \sin \gamma)|^2$ distribution of intensity, but otherwise it's smooth, featureless and motionless. (Quantum mechanics can be really dead, sometimes!)

To send "messages" or "wave-packets" it is necessary to have more than one frequency going in. If the frequencies are close by then AM "lumps" (like Fig. 4.3.3) of increased photon counts will be observed moving down the hall at the velocity v_x *(group)* given by (6.3.7d). As usual, you need many counts to make out even one "lump." (Low-quantum phenomena are elusive, to say the least!)

(c). Evanescent waves

There is an important lower limit to frequency below which waves will not propagate. This happens just when the wave is too big in wavelength to fit even half of it in the wave guide. The limit is indicated in the Fig. 6.3.2 at the bottom of the hyperbolic dispersion function (6.3.5).

For an angular frequency below the so-called *cut-off value*

$$\omega_{cut} = \pi\, c / W \qquad (6.3.8)$$

the wave vector

$$k_x = (\omega^2 - \pi^2 c^2 / W^2)^{1/2}$$

goes to zero and then becomes imaginary. Instead of the usual *propagating* wave

$$\Psi = \exp i(k_x x - \omega t) \qquad (6.3.9)$$

we get a so-called *evanescent* wave

$$\Psi = \exp -(\mu_x x) \exp i(-\omega t) \qquad (6.3.10a)$$

that decays exponentially with the distance x along the wave guide. Its decay rate constant is

$$\mu_x = (\pi^2 c^2 / W^2 - \omega^2)^{1/2} = ik_x\ , \qquad (6.3.10b)$$

and it gets greater as the frequency ω becomes smaller than ω_{cut} .

The cutoff ω_{cut} in (6.3.8) is the bottom of a band of allowed frequencies, and it is the bottom of the lowest of a series of bands for which the an integral number n_y >1 of half waves fit across the hall. A generalization of (6.3.4b) for n_y half waves is

$$k^{(+)}{}_y = k \sin \gamma = n_y\, \pi / W \qquad (n_y = 1, 2, \ldots) \qquad (6.3.11a)$$

This leads to *multiple overlapping bands of dispersion function* $\omega_{ny}(k_x)$.

$$\omega_{ny}(k_x) = kc = c(k_x^2 + k_y^2 + k_z^2)^{1/2} = c(k_x^2 + n_y^2 \pi^2 / W^2)^{1/2} \qquad (6.3.11b)$$

The lowest three of these overlapping hyperbolas (for n_y = 1, 2, and 3) are plotted in Fig. 6.3.6.

(d). Trapped waves and cavity modes: discrete frequencies

When a wave is completely trapped in all the directions it can move, then its spectrum ceases to be continuous and becomes discrete or "quantized." This is what happens to the wave guide modes if the Hall of Mirrors is capped by a pair of doors at, say, $x=0$ and $x=L$, so it becomes a *wave cavity* of length L.

The doors demand the wave electric field be zero at x-boundaries as well as along the walls. The new boundary condition to go with (6.3.11a) is the following.

$$k_x = k\cos\gamma = n_x\pi/L \qquad (n_x = 1, 2, \ldots) \tag{6.3.12a}$$

Now the frequency bands become broken into discrete "quantized" values $\omega_{n_x n_y}$, one for each pair of integers or "quantum numbers" n_x and n_y.

$$\omega_{n_x n_y} = kc = c(k_x^2 + k_y^2 + k_z^2)^{1/2} = c(n_x^2\pi^2/L^2 + n_y^2\pi^2/W^2)^{1/2} \tag{6.3.12b}$$

The frequency values fall where the n_y-hyperbola intersects the n_x-value of k_x in (6.3.12a) as shown in Fig. 6.3.6. These correspond to *cavity modes*. Note: no zero or negative n_x or n_y allowed.

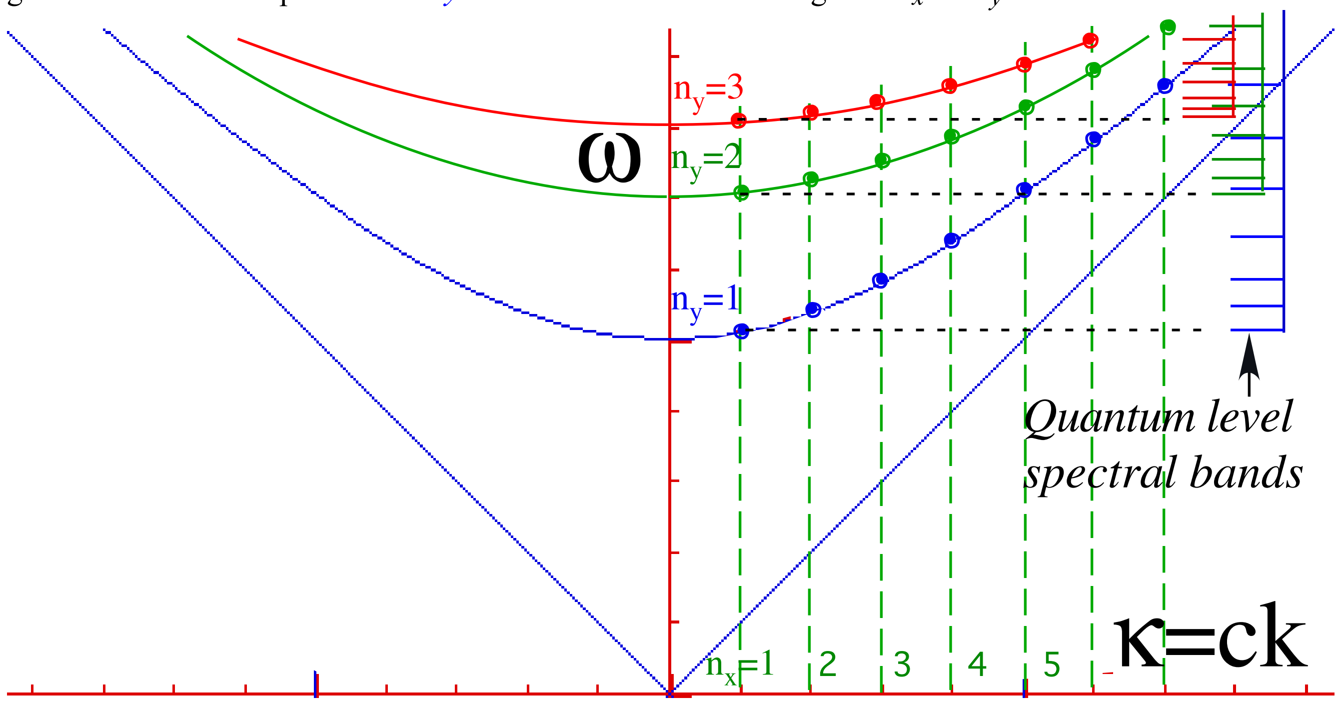

Fig. 6.3.6 Cavity mode dispersion diagram showing overlapping and discrete ω and k values.

Three of the lowest cavity modes for the fundamental ($n_y=1$) dispersion curve corresponding to the x-quantum numbers $n_x = 1, 2$, and 3 are plotted in Fig. 6.3.7 below. These are 2D standing waves. They can be thought of as interference patterns of four moving wave fronts, two oppositely moving pairs for each of the two wavefront lines intersecting at an antinode in each of the figures.

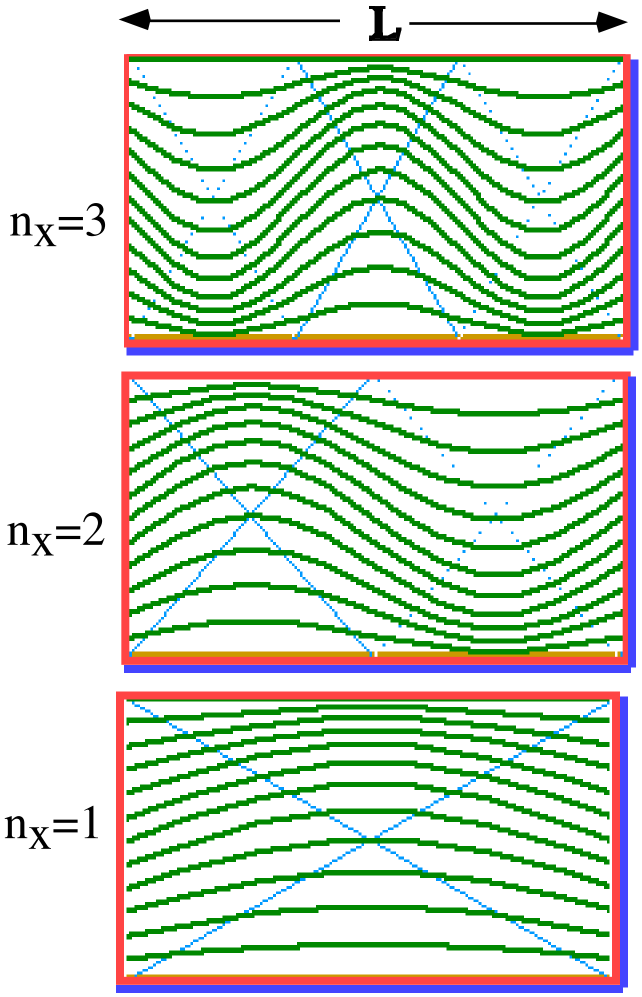

Fig. 6.3.7 Cavity modes for three lowest quantum numbers

It should be noted that the Hall of Mirrors used in the preceding section is a tall hall indeed. It has no floor and no ceiling! Clearly, this is an impractical wave guide with infinite wavelength in the out-of-the-page direction *z*. The hall needs a floor and a ceiling separated by height *H* with boundary conditions.

$$k_z= n_z\, \pi /H \quad (n_z = 1, 2,...) \tag{6.3.13a}$$

This gives new frequency bands corresponding to "quantized" values $\omega_{n_x\, n_y\, n_z}$.

$$\omega_{n_x\, n_y\, n_z} =kc = c(k_x^2 + k_y^2 + k_z^2)^{1/2}= c(n_x^2\pi^2/ L^2 + n_y^2\pi^2/ W^2+ n_z^2\pi^2/ H^2)^{1/2} \tag{6.3.13b}$$

Also, *z*-confinement has the effect of up-shifting the spectrum in (5.1.6) due to the addition of the extra term $n_z^2\pi^2/ H^2$ in (6.3.13b). Since the lowest possible quantum number is $n_z = 1$, we cannot ever ignore it. However, for a tall hall *(H>>W or H>>L)* the resulting shift is small.

Problems for Chapter 6.

Happy Medium

6.1.1 (a) In counter propagating γ-rays of frequency ω_A and ω_B, what speed observer sees a single ω?

(b) In γ-rays of fixed $\mathbf{k}_A$ and $\mathbf{k}_B$, can observer of speed $\mathbf{u}$ see a single ω? If so, find $\mathbf{u}(\mathbf{k}_A, \mathbf{k}_B)$ and $\omega(\mathbf{k}_A, \mathbf{k}_B)$.

Twin Geeks

6.1.2 Two twins decide to split up with one staying in the lab to watch TV while the other goes down the street at velocity u for a day (lab coordinate time) and then spends the next day returning at the same speed.

(a) Plot the two paths on a space-proper-time graph for $u=c/2$. Indicate their difference in age on the graph.

(b) Compute their age difference for $u=c/2$ and $u=0.99c$ and $u=3x10^3$ m/s.

(c) Derive the stellar aberration angles σ for each of the three velocities above.

Transverslongitudinal

6.1.3 The CW speedometer in Fig. 6.1.3 and Fig. 6.1.6 allow ruler-compass Lorentz transformation of optical wave-**k**-vectors that lie along (longitudinal Doppler) and normal (transverse Doppler) to frame velocity **u**.

(a) Construct an accurate plot of both kinds of Doppler shifts for $u=4/5c$.

(b)What about a **k**-vector that starts out neither longitudinal ($0°$ or $180°$) nor transverse ($\pm 180°$)? Construct or plot the $u=4/5c$ transformed vectors from ones that were originally at $\pm 45°$ and $\pm 135°$.

Improper Frequency

6.1.4 Cosmic speedometers in Fig. 6.1.1 through Fig. 6.1.6 are based on light (photons) or other zero-rest-mass particles and CW lasers. Consider the corresponding plots for particles and matter waves?

(a) Consider a proper-frequency *vs.* ck_z plot for a mass-M matter wave of group velocity $\mathbf{u}=u_z\mathbf{e_z}$. Compare it to a proper-time *vs.* coordinate-z plot of a particle of velocity $\mathbf{u}=u_z\mathbf{e_z}$. Do both for $u_z=4/5c$.

(b). Plot a tetrad of length-$(|c\mathbf{k}|=1)$-matter-wave **k**-vectors of the same proper frequency $\mu=1$ in the xz-plane at $0°$, $90°$, $180°$, and $270°$. (Let $c=1$.) The plot them as viewed by an observer with velocity $u_z=4/5c$.

Warped speed

6.2.1 Fig. 6.1.6e show shows a Cartesian xz-plane grid will appear to warp when viewed by an observer who has been boosted by $3/5c$ in the x-direction and then by the same speed in his z-direction.

(a). Obtain the same plot using $4/5c$ speeds, instead.

(b).Does the lab frame see the observer's second boost go $4/5c$ in the lab z-direction? If not, how fast and what direction?

Add it up (Fast!)

6.2.2 Relativistic velocity addition for 1-dimensional travel is given by formulas (4.3.12). The velocity v'_z of something seen by an observer who is moving u_z in lab is added to u_z as follows to give that velocity v_z as seen by the lab.

$$v_z = (u_z + v'_z)/(1 + u_z v'_z/c^2)$$

(a). Explain and derive this from (4.3.12) and from the momentum transformation (5.2.10).

(b) Derive a 3-dimensional generalization which gives (v_x, v_y, v_z) in terms of (v'_x, v'_y, v'_z) and (u_x, u_y, u_z). (Hint: Recall the distinction between velocity dx/dt and derivative $dx/d\tau$ used in momentum (5.3.14b).)

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Listening to a Hall of mirrors

6.3.1. Let the 'Hall of Mirrors' be 4m. wide and lined with polished gold. Suppose a 62.5 MHz radio wave is propagating down its positive x-axis. (Assume floorless hall of Fig. 6.3.1.)

(a) First compute the period_________, angular frequency_________, wavenumber_________, angular wave number_________, and wavelength_________ of this radio wave in a boundary-free vacuum. Use conventional notation and give standard mks units.

(b) Plot the dispersion function ω(k) for this Hall. Then compute the period_________, angular frequency_________, wavenumber_________, angular wave number_________, and wavelength_________ of the 62.5 MHz radio wave along the x-axis of the 'Hall' in the fundamental (lowest) TE mode. Indicate ω and k on your plot as in Fig. 6.3.1.

(c) Compute the phase velocity_______c_ and group velocity_______c_ of this wave in the aforementioned tunnel. Indicate these velocities by secant or tangent lines on your plot. How fast could a radio <u>message</u> go from x=0 to x=24m?_______sec.

Traveling in a Hall of Mirrors

6.3.2 Consider the space-time wave of problem 6.3.1.

(a) Sketch ray paths and plane wavecrest-wave trough lines that define the 62.5 MHz wave as well as some wavy curves representing the actual E-field value inside the Hall. Follow form of Figs.6.3.4 in for at least two full wavelengths at times t=0, then 1/4 , 1/2, and 3/4 period later, but also include a set of four phasors per wavelength to represent the complex wavefunction along the hall center (y=0).

(b) Give the frequency and wavelength for the wave seen by a ship going along the x-axis at velocity v = $^4/_5$c and at v = -$^4/_5$c. Compute the phase and group velocities which the ship would see.

(c) Do the velocity addition formulas (4.3.12) give the right velocities (phase or group) obtained here?

(d) Can the ship go fast enough to see a guide wave Doppler red-shift to below its cutoff? Explain.

Hiding in a Hall of Mirrors

6.3.3 Consider the hall of problem 6.3.1.

(a) Suppose an intense radio (Raser?) of frequency 18.78 MHz is bombarding the end of this tunnel. How far into the tunnel would you have to go so the E-field is less than 5% of what it is at the end?______ (TE fundamental only) Answer the same for the intensity ($\Psi^*\Psi$) instead of E-field?_________

(b) What if the Raser was tuned right at ω_{cutoff}?

Trapped in a Hall of Mirrors

6.3.4 Consider the hall of problem 6.3.1.

(a) Suppose gold doors at *x=0* and *x=24m*. enclose the Hall. Compute the frequency of the lowest TE wave that can resonate in this cavity.__________Give complex wavefunction and sketch crest-trough lines and phasors for this wave.

(b) Compute the cut-off frequency below which radio will not propagate in the tunnel. Give both angular and regular frequency.________________