Appendix 4.A Relativistic Space-time Coordinates

The nature of space-time coordinate transformation will be described here using a fictional spaceship traveling at half the speed of light past two lighthouses. In Fig. 4.A.1 the ship is just passing the Main Lighthouse as it blinks in response to a signal from the North lighthouse located at one light second (about *186,000 miles* or *EXACTLY 299,792,458 meters*) above Main. (Such exactitude is the result of *1970-80* work by Ken Evenson's lab at NIST (National Institute of Standards and Technology in Boulder) and adopted by International Standards Committee in 1984.) Now the speed of light *c* is a constant by civil law as well as physical law! This came about because time and frequency measurement became so much more precise than distance measurement that it was decided to define the meter in terms of *c*.

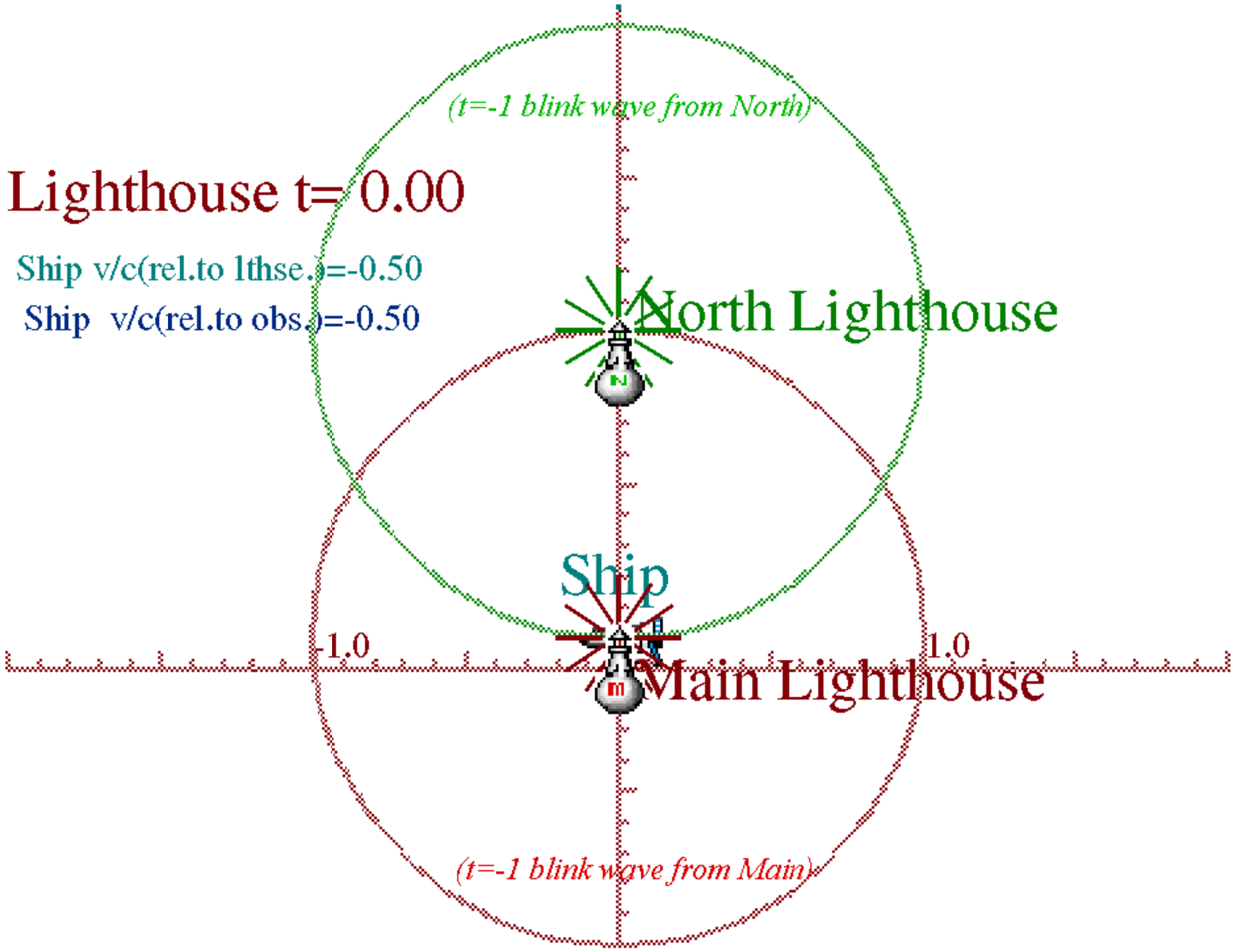

Fig. 4.A.1 Ship passing Main Lighthouse as it blinks at t=0.

This arrangement is a simplified model for a *1Hz* laser resonator. The two lighthouses use each other to maintain a strict one-second time period between blinks. And, strict it must be to do relativistic timing. (Even stricter than *NIST* is the universal agency *BIGANN* or Bureau of Intergalactic Aids to Navigation at Night.) The simulations shown here are done using *RelativIt*.

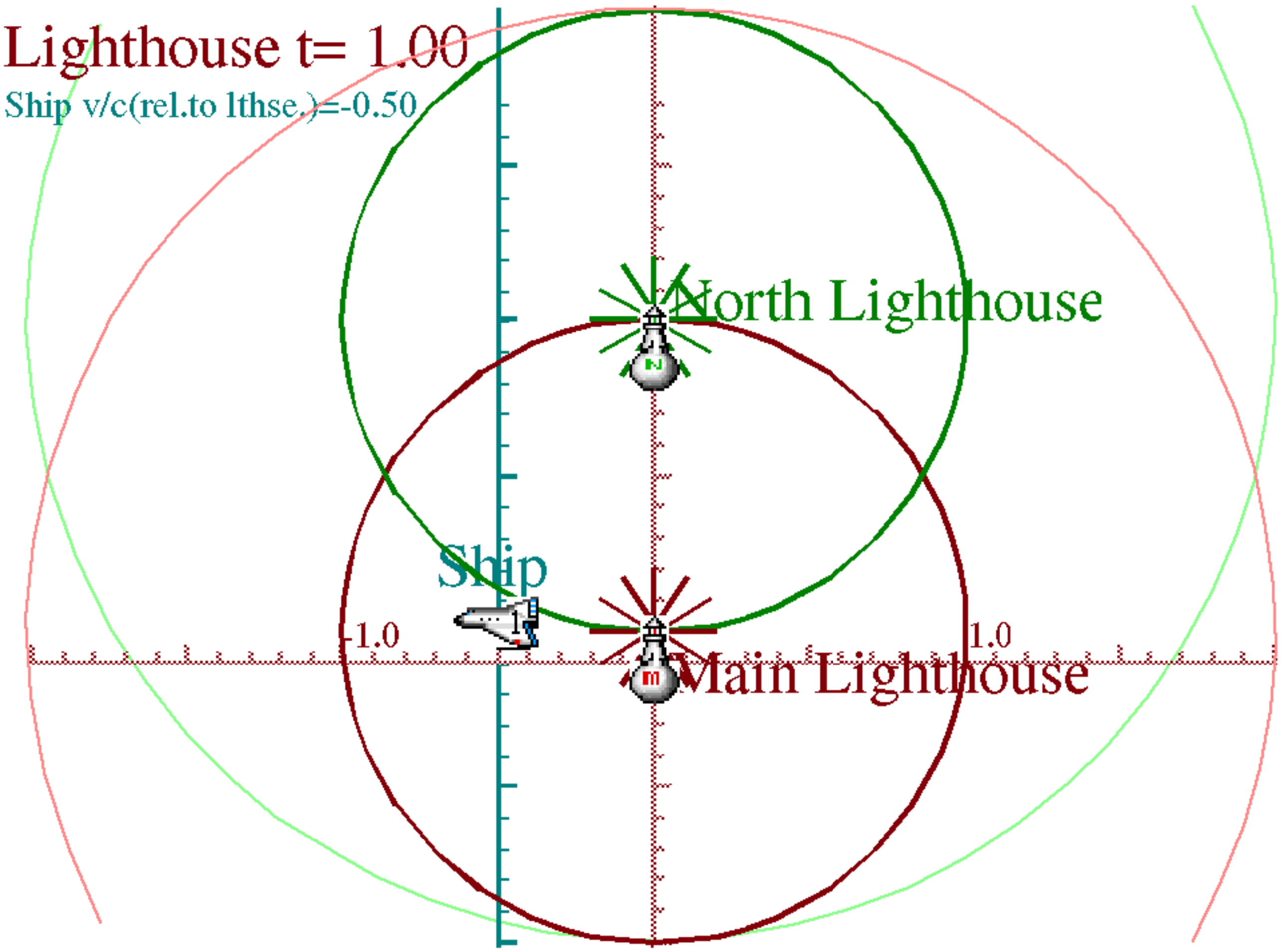

Fig. 4.A.2 Main and North Lighthouses blink each other at precisely t=1.

At precisely *t=1 sec.* the two lighthouses blink again because that is how long it takes their respective *t=0* blink waves to reach each other. This is shown in Fig. 4.A.2. The ship, meanwhile, has only traveled half this far since its speed is *c/2*. Its velocity is -*c/2*, that is, negative, since it is going right to left.

Next, at precisely *t=2 sec.* the two lighthouses blink again. Also, the first (*t=1*) blink catches up to the ship and hits it, that is, the ship sees the first blink. This is shown in Fig. 4.A.3. Much of the discussion will center on two happenings or events labeled *Happening-1* and *Happening-2*. ("Event" is accepted physics terminology. "Happening" is oh-so-60's.)

The coordinates of *Happening-1* are, according to the Lighthouses, $(x_1=-1, ct_1=2)$ while for *Happening-2* they are, according to the Lighthouses, $(x_2=0, ct_2=2)$. Next, we will see how the ship views all this, that is what are ship coordinates (x'_1, ct'_1) and (x'_2, ct'_2) for the events. From that we deduce the essential transformation matrix for all events in special relativity. The ship has a very different transcription of these events as shown in the following figures beginning with Fig. 4.A.4.

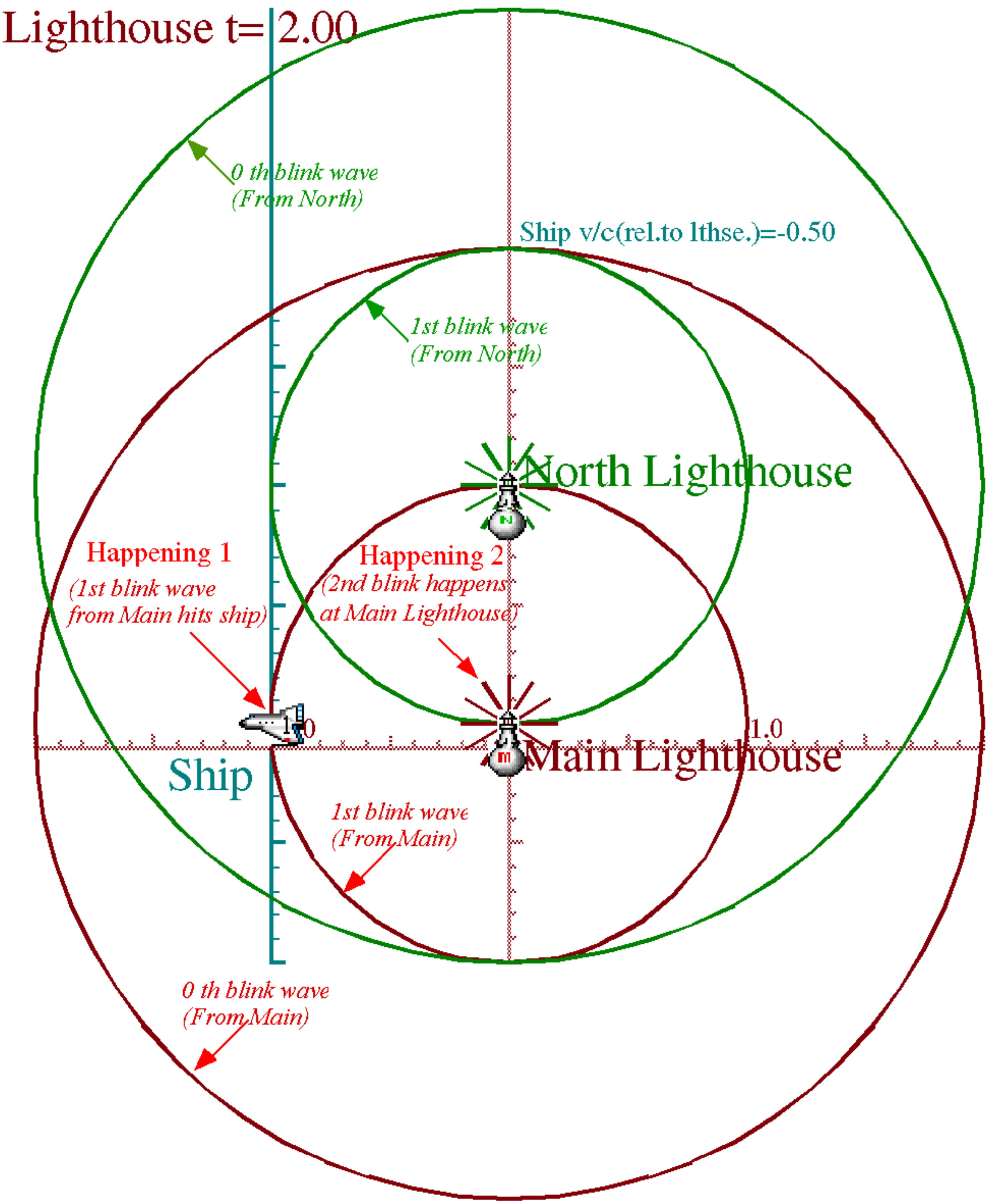

Fig. 4.A.3 Happening 1 (1st blink hits ship) and 2 (2nd blink at Main) both happen at t=2.

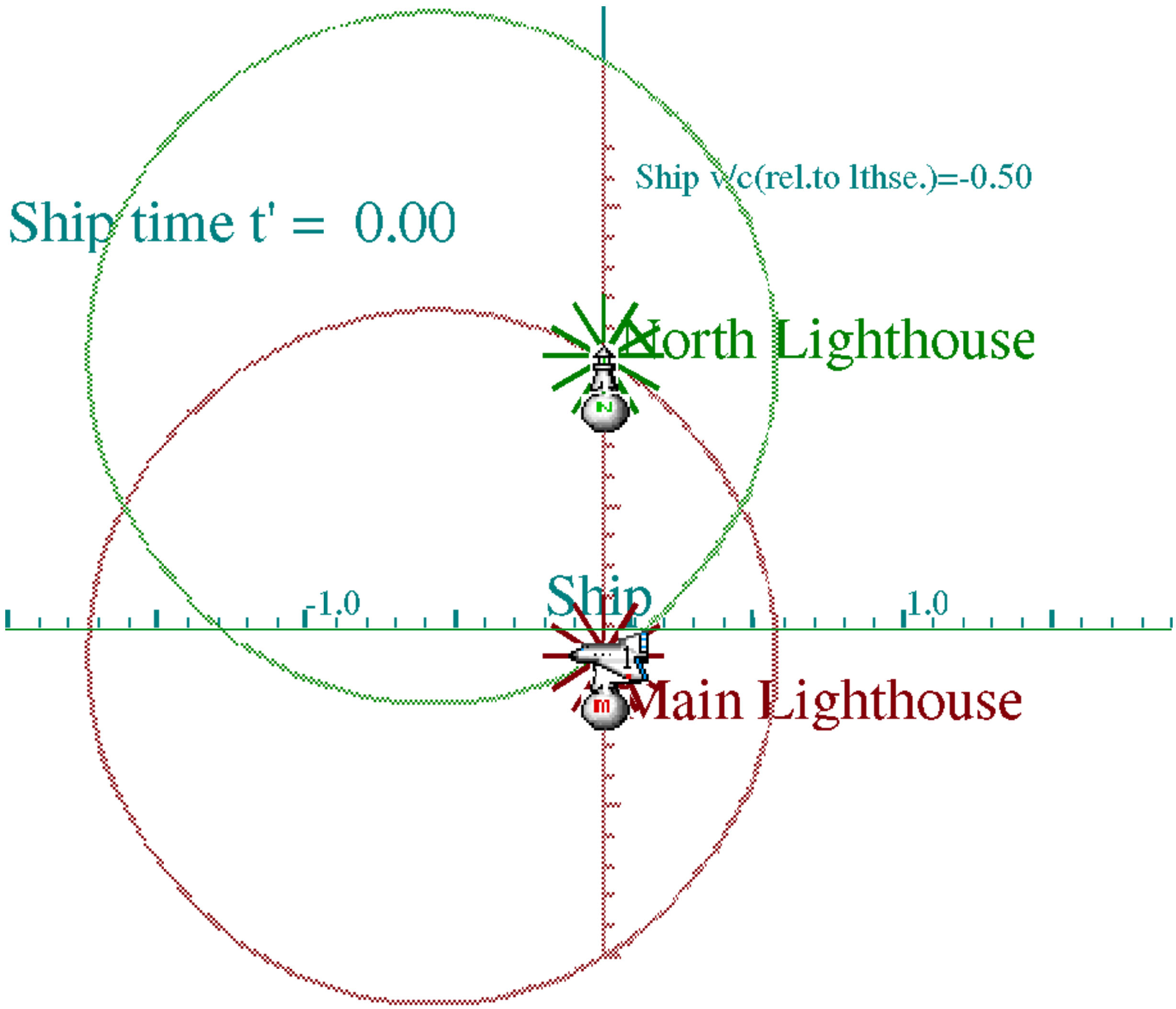

Fig. 4.A.4 Beginning (t′=0) snapshot for ship's view.

Now in Fig. 4.A.4, the ship is stationary and that means that the lighthouses are going in the opposite direction with a positive velocity of $v=+c/2$. Fig. 4.A.4 looks the same as Fig. 4.A.1 except the previous ($t=-1$) blink wave appears to have been "left behind" by the speeding lighthouses. Therein lies a secret of relativity. Snapshots of light pulses always appear to be *circles expanding around the points where they were emitted.* This is true no matter how fast you are going, or, more importantly, no matter how fast the emitter is going. You cannot speed up or slow down light by jerking your laser back and forth!

So Fig. 4.A.4 and several subsequent figures show previous blink waves expanding around points where the lighthouses were when that light was emitted and all expansions take place at a uniform speed of c. It's the law! And, it's one we can live with. Consequences of this law are quite remarkable. We explore consequences shortly including the fact that the ship sees the North blink-wave tipped by a so-called *stellar aberration angle* $\phi=60°$ relative to a vertical North-to-South wave ray track seen by the Main Lighthouse.

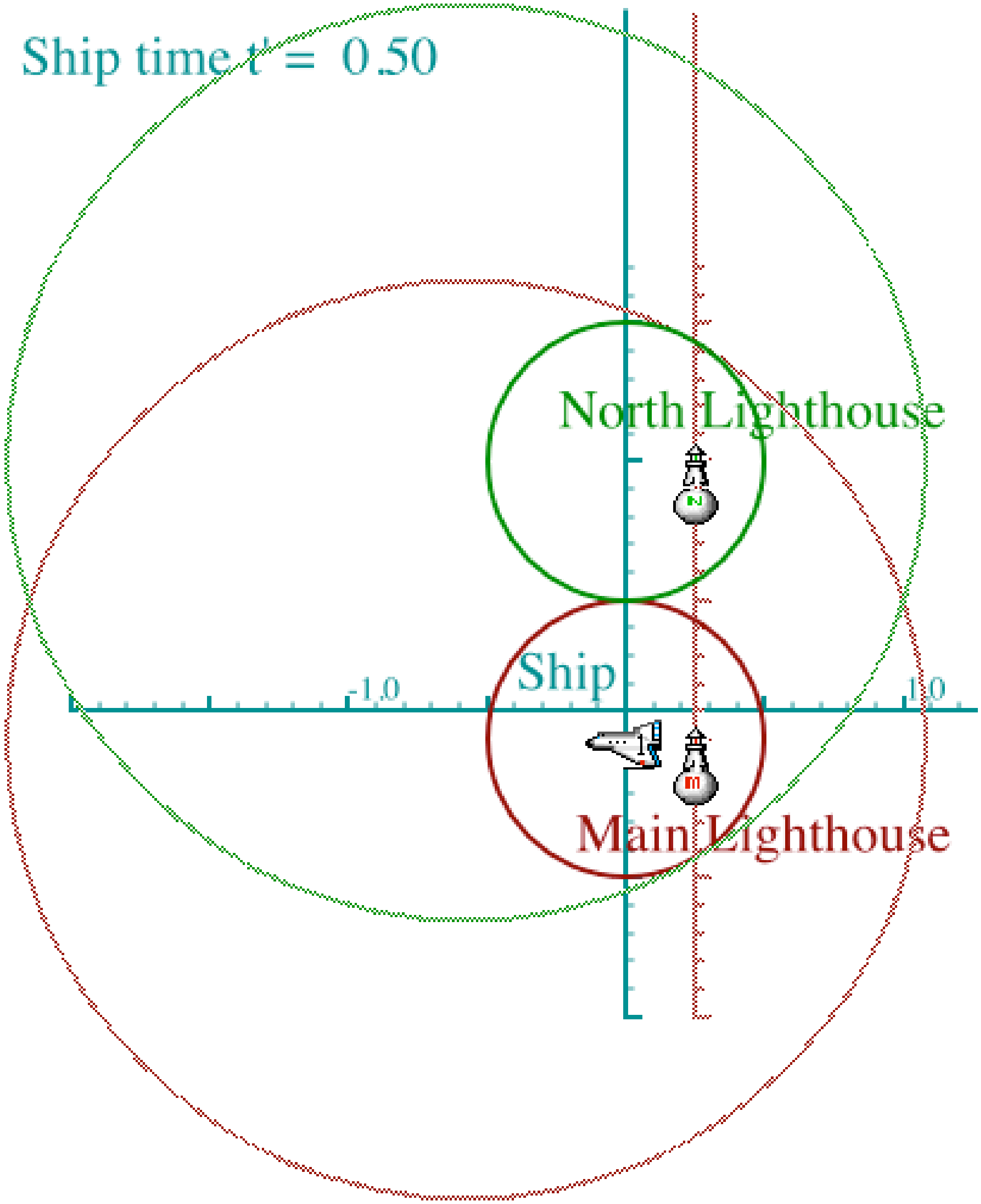

Fig. 4.A.5 Early (t′=0.5) snapshot from ship's view.

The *(t′ =0.5)* view by ship shows the *(t′ =0)* blink waves expanded to exactly half the distance between their emission points. Also, the lighthouses have moved half this distance, that is, a quarter of a light-second, and so Main will not be anywhere near the ship at *(t′ =1.0)* when the *(t′ =0)* blink wave from the North comes down to trigger Main to do its first or *(t=1)* blink. In fact it's *(t′ =1.15)* before the *(t′ =0)* blink from the North finally catches the speeding Main Light to make it blink as shown in the next Fig. 4.A.6.

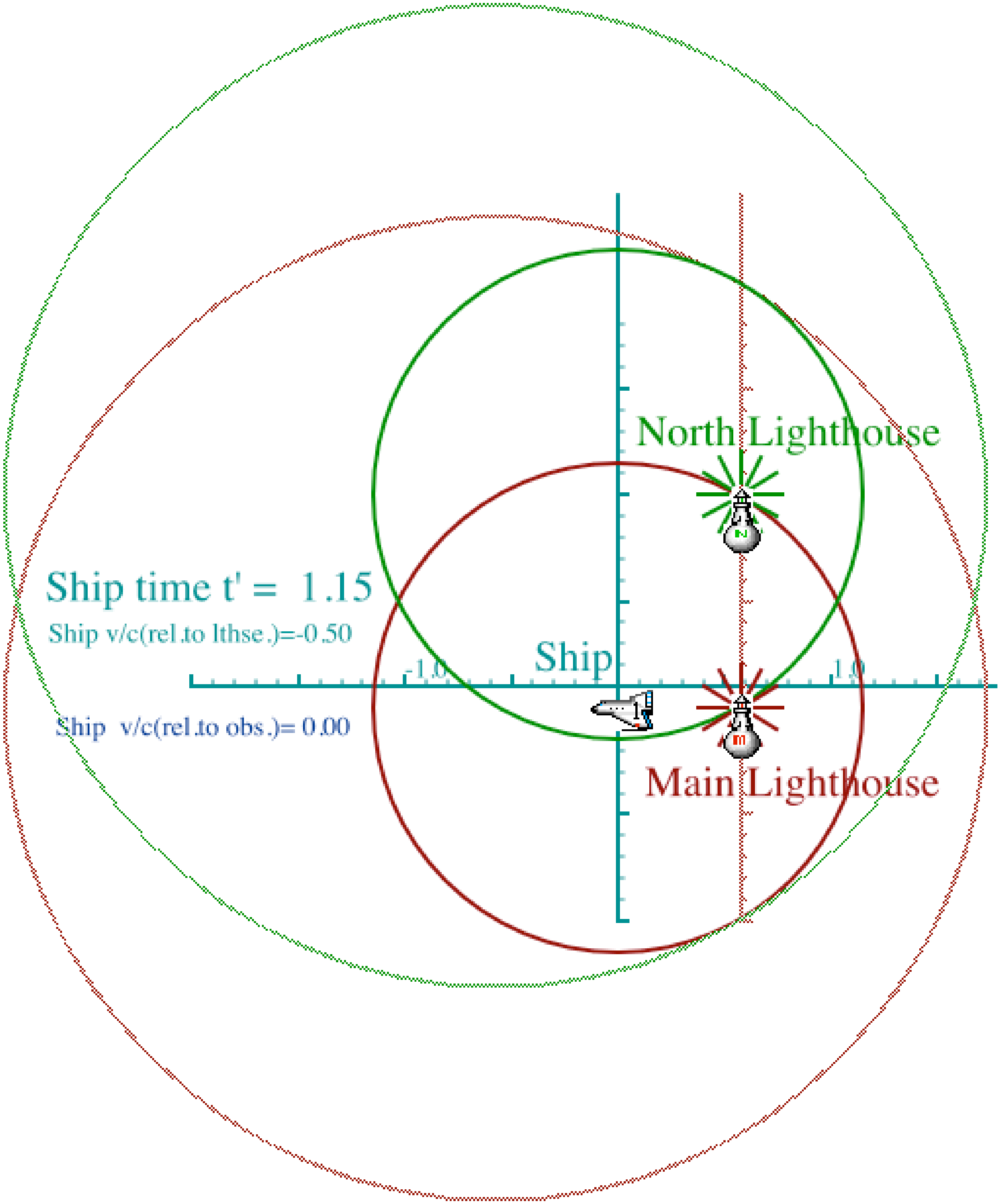

Fig. 4.A.6 Later (t′ =1.15) snapshot from ship's view finally registers the first lighthouse blinks.

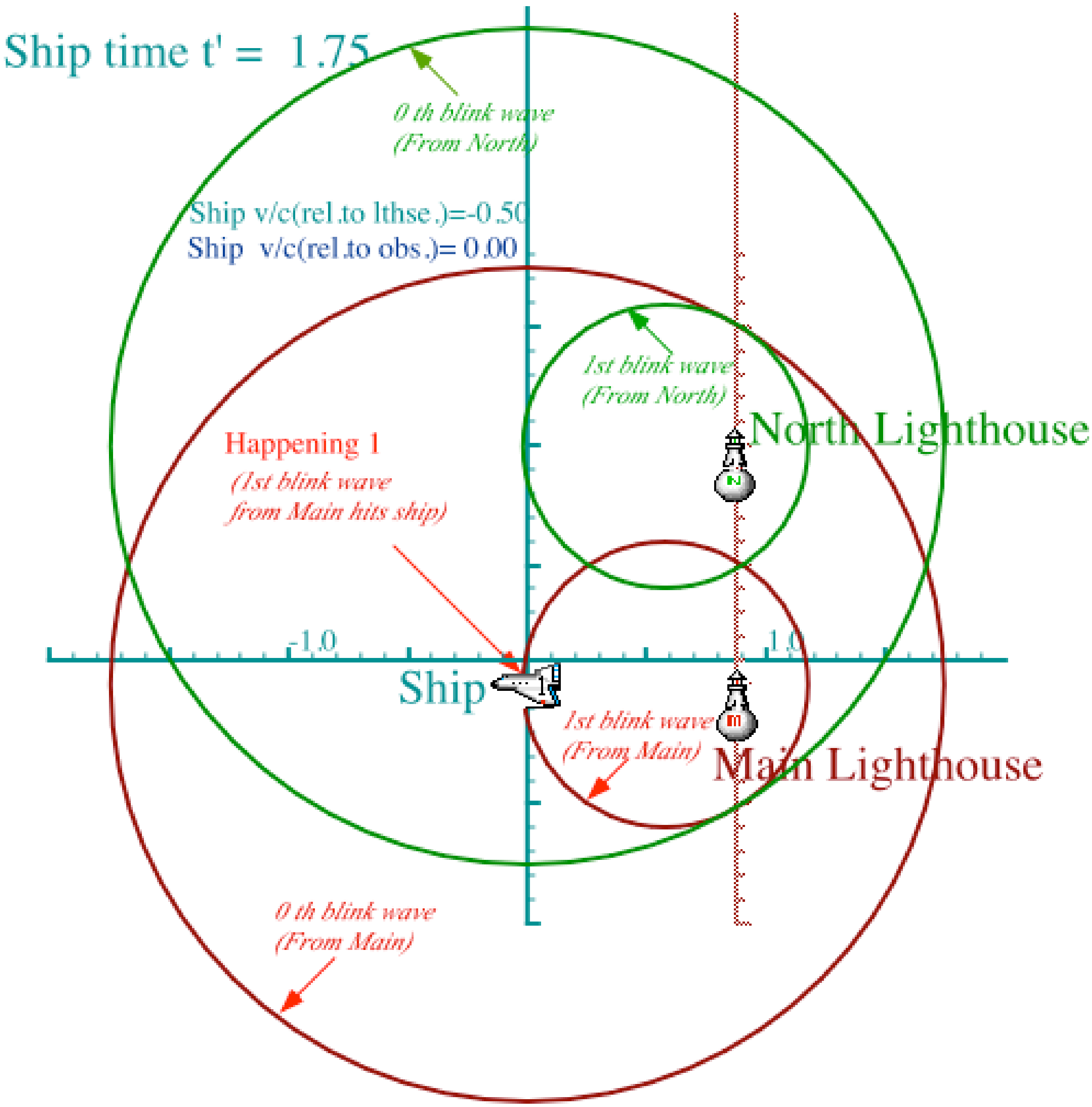

Fig. 4.A.7 Later (t′ =1.75) snapshot from ship's first registers Happening-1.

This shows *Einstein time dilation*. The ship perceives that the lighthouse is running about *15%* late at this speed of *v=c/2*. The next Figs. 4.A.7 and 4.A.8 show something even more surprising to a Newtonian worldview, the *relativity of simultaneity* where, unlike Fig. 4.A.3, *Happening-1* is not simultaneous with *Happening-2*. *Happening-1* (ship hit by 1st blink) happens early at *(t′ =1.75)* and before *Happening-2* (2nd blink) that occurs at *(t′ =2.30)*. (Recall Main cannot blink until a blink from the North hits it so *Happening-2* doesn't happen until *(t′ =2.30)*, or twice the time *(t′ =1.15)* for the first blink as shown in Fig. 4.A.8.)

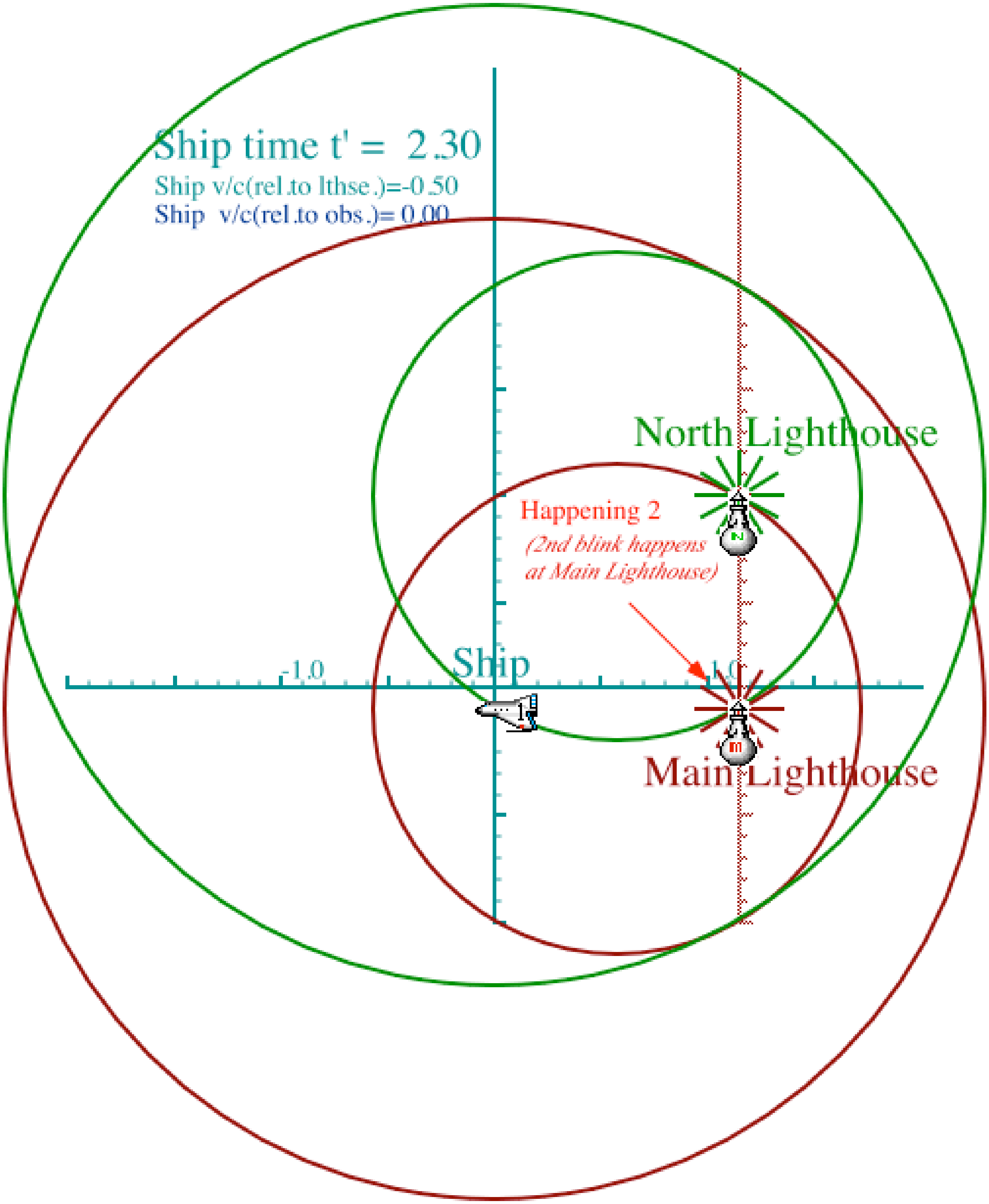

Fig. 4.A.8 Much later (t′ =2.30) snapshot from ship's finally registers Happening-2 .

The time Δ observed between unit-time blinks by a moving ship is called the *Einstein dilation factor Δ*. Its classical derivation follows from a simple right triangle whose altitude is *c* or one light second as shown in Fig. 4.A.9. The triangle base *v*Δ is the distance traveled by the lighthouse before the North blink wave finally hits it after traveling a distance *c*Δ along the hypotenuse as seen by ship. This gives the following.

$$c^2\Delta^2 = c^2 + v^2\Delta^2 \quad \text{or: } \Delta^2\left(c^2 - v^2\right) = c^2 \quad \text{or: } \Delta = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{4.A.1}$$

Note that the ship or any co-moving ship sees the *c*Δ hypotenuse ray tipped in the direction of travel by the *Stellar aberration angle* ϕ whose sine is sin $\phi = v/c$. This is ϕ=*60°* for *v/c=1/2* in Fig. 4.A.9 or Fig. 4.A.4.

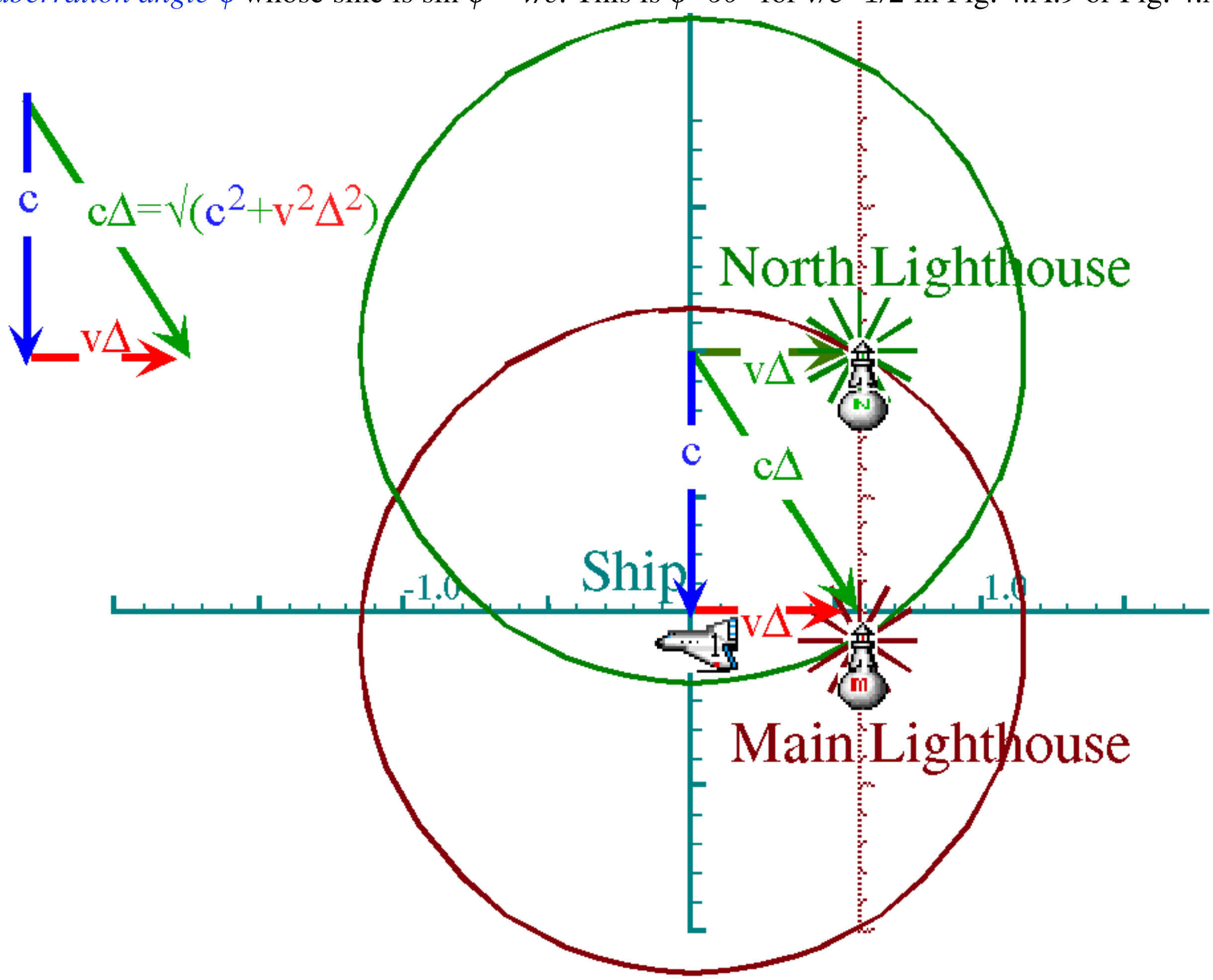

Fig. 4.A.9 Derivation of Einstein time dilation factor Δ or time between blinks .

For the above the lighthouse velocity relative to the ship is *v=c/2*. (4.A.1) gives a time dilation factor of Δ=*1/√0.75 = 1.1547* very close to the *15%* "lateness" in the Fig. 4.A.6 simulation. This lateness grows rapidly and without limit as *v* approaches *c*. For *v=4c/5*, (4.A.1) gives Δ=*5/3 = 1.67* that is a *67%* lateness or *dilation*.

From this we construct an event table to summarize discrepancies or disagreements between space and time coordinates used by the lighthouses and those used by the ship. This is shown below.

Happening 0: Ship passes Main Lighthouse.	Happening 1: Ship gets hit by first blink from Main Lighthouse.	Happening 2: Main Lighthouse blinks second time.
(Lighthouse space) $x = 0$ (Lighthouse time) $t = 0$	$x = -1.00\,c$ $t = 2.00$	$x = 0$ $t = 2.00$
(Ship space) $x' = 0$ (Ship time) $t' = 0$	$x' = 0$ $t' = 1.75$	$x' = c\,\Delta$ $t' = 2\Delta = 2.30$

One of the most important things to remember about the space coordinate x is that each observer frame *carries its own origin (x=0) with it wherever it goes*. If a 'Happening' happens to the Lighthouse then it happens at $x=0$, but if it happens to the ship then it happens at $x'=0$ no matter what the time is. Remembering this saves lots of confusion! Note also: the table above is for a <u>positive</u> lighthouse velocity: $v={}^{c}/_{2}$ relative to the ship. You must always give velocity *as one thing relative to another*. Absolute velocity is *meaningless*.

We need a table like the one above for the case of a general velocity v of the lighthouse relative to the ship. (Note that if we base ourselves in the frame in which the ship is stationary then the lighthouse moves with a positive velocity $v=c/2$.) The zero entries stay the same for any value of v. The times for the second blink are $t=2$ and $t' = 2\Delta$ by definition. The ship's reading for the position of the second blink has to be velocity times travel time or v times 2Δ. ($x' = 2v\Delta$). This becomes $x' = c\Delta$ for $v={}^{c}/_{2}$ as entered above.

The coordinates of *Happening 1* (1st blink hits ship) are found. To hit the ship in the lighthouse frame the 1st blink travels a negative distance $-c$ times $(t-1)$ since it doesn't start from $x=0$ until $t=1$. It hits the ship that has gone that distance starting at $t=0$ from the lighthouse. That distance is $-v$ times t. We equate these.

$$x = -c\,(t - 1) = -v\,t\,, \quad \text{or} \quad t = c/(c-v). \tag{4.A.2}$$

The resulting x and t are entered in the first row under 'Happening 1' in the table below. At this time the ship is located at $x=-vt=-vc/(c-v)$ and that is entered in the table, too.

The lighthouse time for *Happening 1* is based on Fig. 4.A.7. This shows that the 1st blink has to travel the base of a right triangle that is $v\Delta$ long. It starts at time $t'=\Delta$ and goes at rate c for $(t'-\Delta)$ seconds or

$$v\Delta = c(t'-\Delta). \tag{4.A.3}$$

Solving for t' gives the last entry $t' = (v+c)\Delta/c=(1+v/c)/\Delta$ in the *Happening 1*-column of the table.

(4.A.4)

Happening 0: Ship passes Main Lighthouse.	Happening 1: Ship gets hit by first blink from Main Lighthouse.	Happening 2: Main Lighthouse blinks second time.
(Lighthouse space) $x = 0$ (Lighthouse time) $t = 0$	$x = -vc/(c-v)$ $t = c/(c-v)$	$x = 0$ $t = 2.00$
(Ship space) $x' = 0$ (Ship time) $t' = 0$	$x' = 0$ $t' = (v+c)\Delta/c=(1+v/c)/\Delta$	$x' = 2v\Delta$ $t' = 2\Delta$

The last entry $t' = (v+c)\Delta/c = (1+v/c)/\Delta$ in the *Happening 1*-column is the time interval or *period* between hits recorded by the ship as it goes off into the night; the period in this case is *longer* than the BIGANN required blink period of 1 second. On the other hand, before the ship passed the lighthouse it was getting hit in the nose by a fast blink-blink-blink with a *shorter* period than 1 second. The formula for this period found by reversing light velocity c to $-c$ is $t' = (v-c)\Delta/(-c) = (1-v/c)/\Delta$

Fig. 4.A.8 shows that the ship gets hit by blinks a lot more frequently before the lighthouse passes at $t=0$ than after it passes because blink waves are more densely packed in front of the lighthouse than behind it. This frequency down-shift is analogous to what you hear as a car goes by: "..EEEEEEEeooooow..", and is called a *Doppler Shift.* According to blink counters on the ship, the lighthouse period of $\tau_0 = 1$ second LHT (or blink rate of $\nu_0 = 1$ Hz) is increased in period by a factor equal to the ship time $t' = (v+c)\Delta/c = (1+v/c)/\Delta$ for Happening-1, that is, the time the ship sees between blink hits after $t'=0$. The inverse of this is a frequency υ' that that is perceived to suffer a *down-shift* or a *red-shift* from the Lighthouse assigned frequency υ_0=1Hz.

$$\begin{matrix}\textit{Ship Time}\\ \textit{between hits}\\ \textit{(outbound)}\end{matrix} = t' = \tau_0 (v+c)\Delta / c = \tau_0 \frac{1+\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} = \tau_0 \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}, \qquad \begin{matrix}\textit{Outbound}\\ \textit{Observed}\\ \textit{Frequency}\end{matrix} = \upsilon' = 1/\tau' = \upsilon_0 \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} \quad (4.A.5a)$$

An inbound ship sees an *Inverse Doppler* or *blue-shift* an *up-shift* or *increase* in frequency to υ'.

$$\begin{matrix}\textit{Ship Time}\\ \textit{between hits}\\ \textit{(inbound)}\end{matrix} = t' = \tau_0 (c-v)\Delta / c = \tau_0 \frac{1-\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} = \tau_0 \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}, \qquad \begin{matrix}\textit{Outbound}\\ \textit{Observed}\\ \textit{Frequency}\end{matrix} = \upsilon' = 1/\tau' = \upsilon_0 \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} \quad (4.A.5b)$$

Again, the difference between "inbound" and "outbound" cases is a matter of sign difference $\pm c$ of velocity of the light perceived by the ship. The two shifts are inverses of each other as required by time reversal symmetry that underlies relativity and electromagnetism.

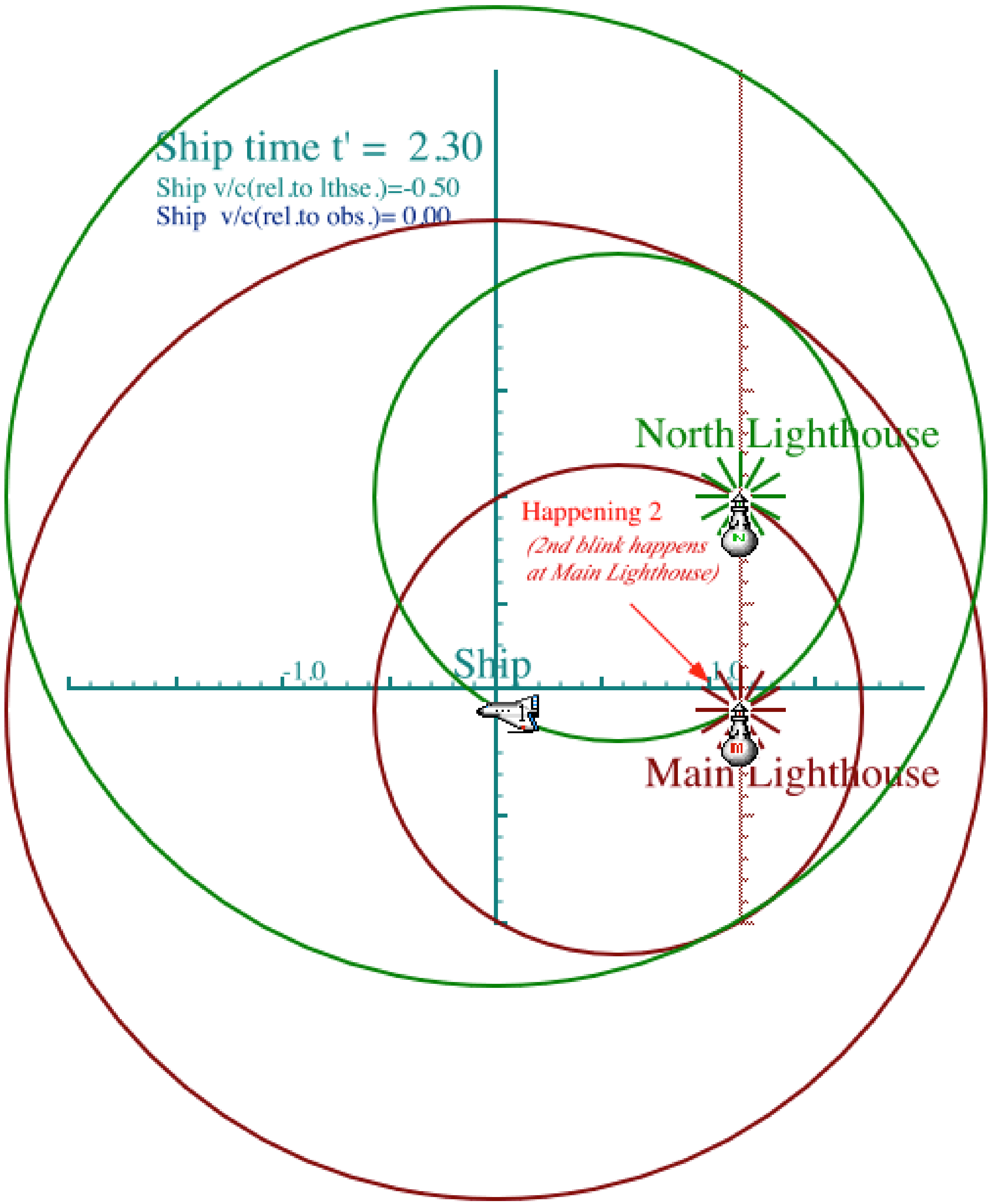

Fig. 4.A..8 Much later (t'=2.30) snapshot from ship's finally registers Happening-2 .

Appendix 4.B Lorentz Transformations and Minkowski Space

The disagreements seen in Table (4.A.4) are analogous to the ones seen in coordinate rotation. Given a rotated grid such as shown in Fig. 4.B.1 one may relate the "disagreements" between a standard US surveyor and a "tipsy" one that headed straight for the saloon. They only agree on the point *(0,0)* of origin.

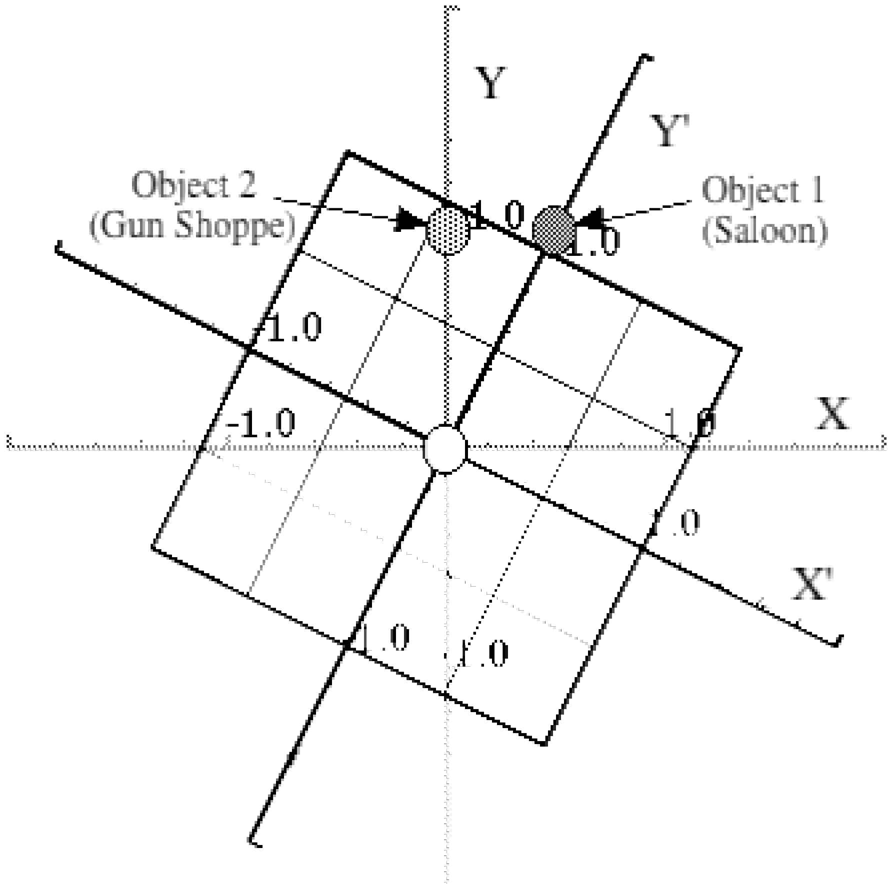

Fig. 4.B.1 Town map according to a "tipsy" surveyor.

Object 0: Town Square.	Object 1: Saloon.	Object 2: Gun Shoppe.
(US surveyor) $x = 0$ $y = 0$	$x = 0.5$ $y = 1.0$	$x = 0$ $y = 1.0$
(2nd surveyor) $x' = 0$ $y' = 0$	$x' = 0$ $y' = 1.1$	$x' = -0.45$ $y' = 0.89$

Before the US surveyor heads for the gun shoppe (and shoots the "non-standard" surveyor) one needs to defuse a potential argument and write a simple coordinate transformation such as derived in Fig. 4.B.2.

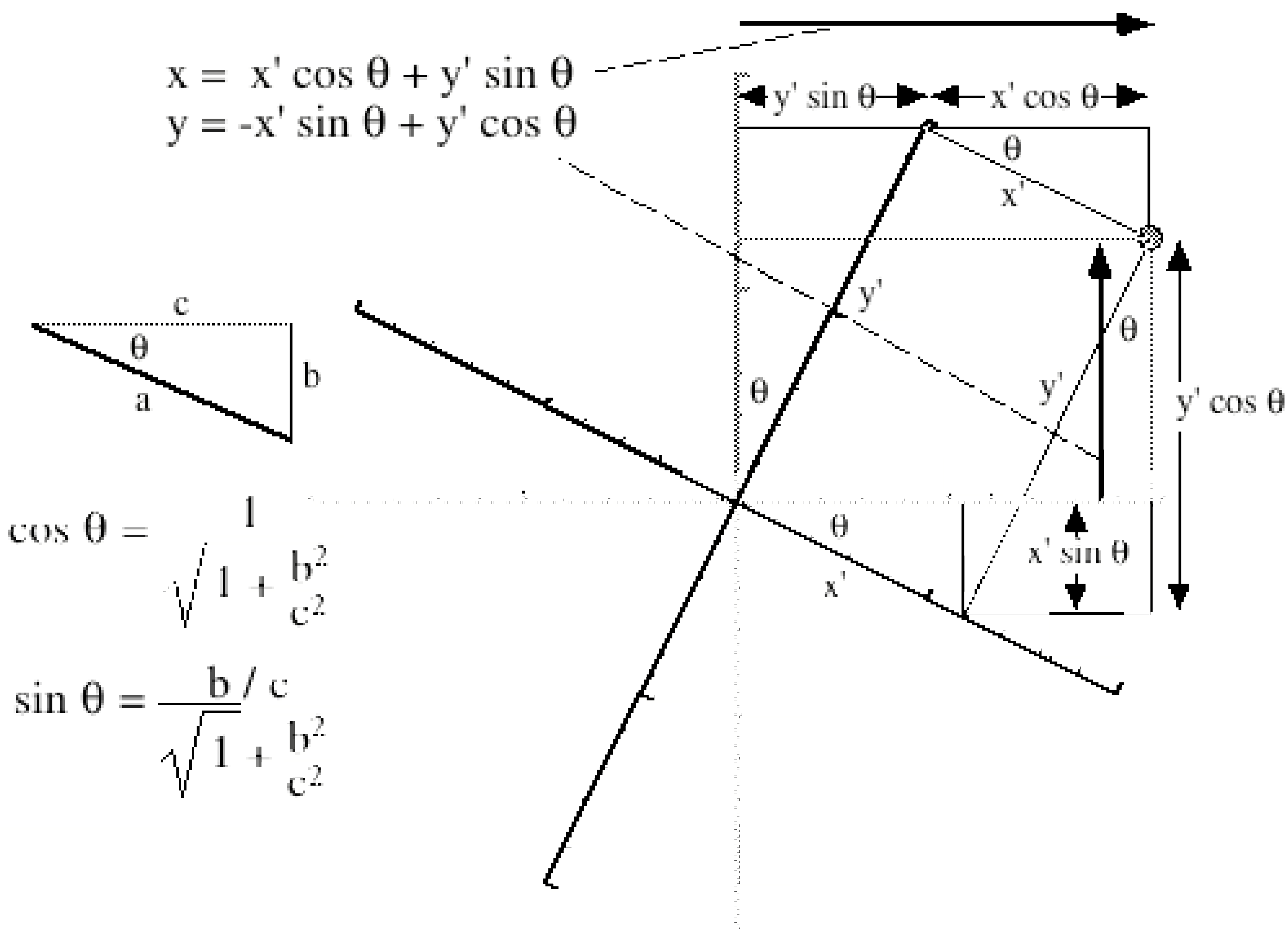

Fig. 4.B.2 The diagram and formulas for reconciliation of the two surveyor's data.

In the notation given above the transformation has a form that is very much like the one we will derive for spacetime. Note that the inverse transform is had by setting angle θ to -θ or slope *(b/c)* to -*(b/c)*.

$$x' = x\cos\theta - y\sin\theta = \frac{x}{\sqrt{1+\frac{b^2}{c^2}}} + \frac{-(b/c)y}{\sqrt{1+\frac{b^2}{c^2}}} \qquad x = x'\cos\theta + y'\sin\theta = \frac{x'}{\sqrt{1+\frac{b^2}{c^2}}} + \frac{(b/c)y'}{\sqrt{1+\frac{b^2}{c^2}}}$$

$$y' = x\sin\theta + y\cos\theta = \frac{(b/c)x}{\sqrt{1+\frac{b^2}{c^2}}} + \frac{y}{\sqrt{1+\frac{b^2}{c^2}}} \qquad y = -x'\sin\theta + y'\cos\theta = \frac{-(b/c)x'}{\sqrt{1+\frac{b^2}{c^2}}} + \frac{y'}{\sqrt{1+\frac{b^2}{c^2}}} \qquad (4.B.1)$$

Remember that a coordinate diagram like Fig. 4.B.2 is a crummy and confusing way to derive this. See Chapter 1 for the better derivations starting from base vectors.

Now we will suppose that the spacetime relations are also a linear transformation.

$$x' = A\,x + B\,ct \qquad (4.B.2a)$$

$$ct' = C\,x + D\,ct \qquad (4.B.2a)$$

We solve for the unknown linear coefficients *A, B, C*, and *D* using the following table from App. 2.A.

Happening 0: Ship passes Main Lighthouse.	Happening 1: Ship gets hit by first blink from Main Lighthouse.	Happening 2: Main Lighthouse blinks second time.
(Lighthouse space) $x = 0$ (Lighthouse time) $t = 0$	$x = -vc/(c-v)$ $t = c/(c-v)$	$x = 0$ $t = 2.00$
(Ship space) $x' = 0$ (Ship time) $t' = 0$	$x' = 0$ $t' = (v+c)\Delta/c$	$x' = 2v\Delta$ $t' = 2\Delta$

(2.A.4)$_{\text{repeated}}$

To do this we stick in the values of (x',ct') and (x,ct) from the Happening Table. For Happening 1 we have

$$A\,x + B\,ct = x', \quad \text{or} \quad A(-vc/(c-v)) + Bc(c/(c-v)) = 0 \quad \text{, or } A = Bc/v \tag{4.B.3a}$$

$$C\,x + D\,ct = ct', \quad \text{or} \quad C(-vc/(c-v)) + Dc(c/(c-v)) = c\Delta(v+c)/c \ , \tag{4.B.3b}$$

and for Happening 2 we have

$$A\,x + B\,ct = x', \quad \text{or} \quad A(0) + Bc(2) = 2v\Delta \ , \tag{4.B.4a}$$

$$C\,x + D\,ct = ct', \quad \text{or} \quad C(0) + Dc(2) = 2c\Delta \ . \tag{4.B.4b}$$

The last two equations immediately give $B=v\Delta/c$ and $D=\Delta$ where you should recall from (4.A.1) that the quantity $\Delta=1/\sqrt{(1-v^2/c^2)}$ is the blink time interval according to the ship. Put these values of B and D back into (4.B.3a-b) to derive $A = \Delta$ and $C= v\Delta/c$. This gives a general formula for converting lighthouse coordinates (x,ct) into ship coordinates (x',ct') or *vice-versa*. It is the *Lorentz Transformation* for rapidity ρ.

$$x' = \frac{x}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{\frac{v}{c}ct}{\sqrt{1-\frac{v^2}{c^2}}} = x\cosh\rho + y\sinh\rho \qquad ct' = \frac{\frac{v}{c}x}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{ct}{\sqrt{1-\frac{v^2}{c^2}}} = x\sinh\rho + y\cosh\rho \tag{4.B.5a}$$

$$x = \frac{x'}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{\frac{v}{c}ct'}{\sqrt{1-\frac{v^2}{c^2}}} = x'\cosh\rho - ct'\sinh\rho \qquad ct = \frac{-\frac{v}{c}x'}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{ct'}{\sqrt{1-\frac{v^2}{c^2}}} = -x'\sinh\rho + ct'\cosh\rho \tag{4.B.5b}$$

To go 'backwards' like (4.B.5b) you only have to switch the sign of velocity v. The use of hyperbolic functions will be explained shortly. For now note that $\cosh^2\rho - \sinh^2\rho = 1$ is satisfied by the $A, B, C,$ and D, that is, $A^2-B^2 = 1$ and $D^2-C^2 = 1$ for all speeds v.

In order to visualize and understand relativity it helps a great deal to plot these transformation equations as coordinate grids. The results are called *Minkowski coordinates* after a Polish mathematicians who happened also to be one of Einstein's math teachers. (It is interesting to note that Einstein himself resisted using these graphs, indeed his papers have precious few figures of any kind.) As seen in Fig. 4.B.3 the Minkowski grids are actually quite striking and not quite as easy to grasp as those of a real rotation.

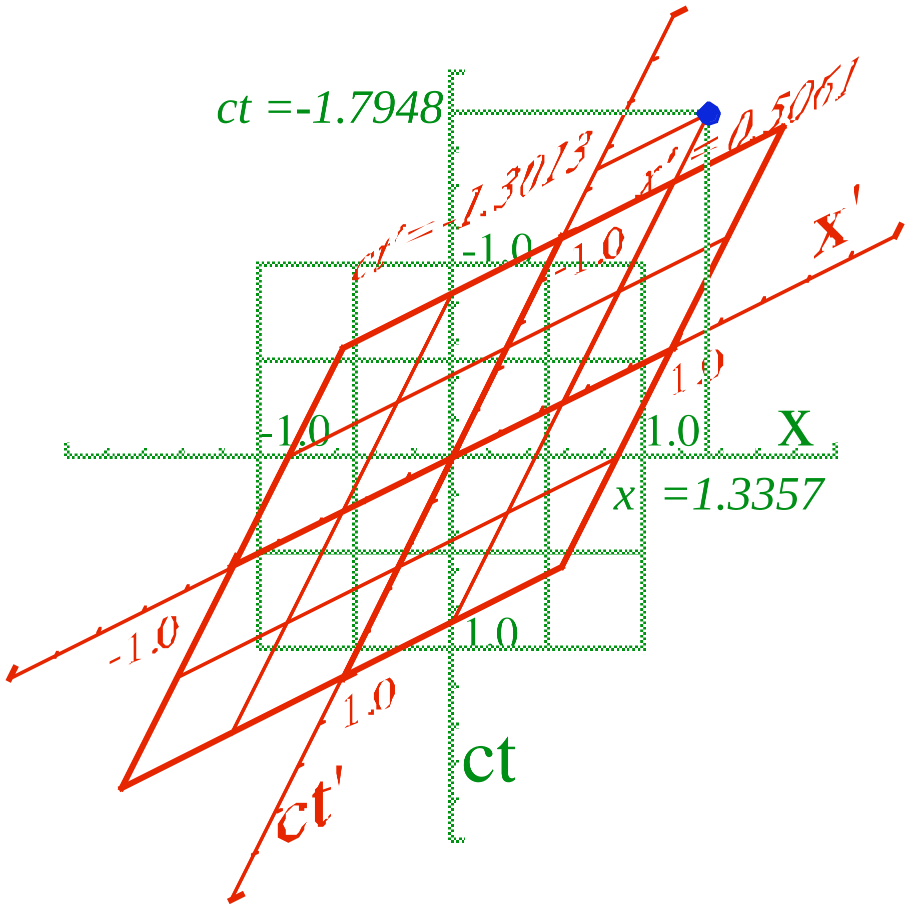

Fig. 4.B.3 Minkowski coordinates (x', ct') for ship going v=-c/2 relative to Lighthouse (x, ct) .

Note that the positive time or future, is down in these graphs. This is the classic *Newtonian convention* in which one plots an *x*-ordinate versus a *t*-abscissa. Note that the *(x′, ct′)* graph gets squeezed relative to the stationary *(x, ct)* graph. The resulting slope of the *ct'* axis is equal to the velocity in *c*-units, that is *v/c*. In this case that slope is *v/c =-1/2*.

This Newtonian slope-to-velocity relation happens because the *ct′* axis is the track of the origin *(x′=0)* of the ship, that is, its *space-time trajectory* or *world line*. As we will see, this slope *v/c* is equal to the hyperbolic tangent *tanh* ρ. However, ρ is called *rapidity* and is not an angle, but an area as will be shown.

A geometric interpretation of Lorentz transformations uses *invariants* of the transformations, functions whose numerical values are unchanged by it so the two protagonists agree on them. In Fig. 4.B.4 we compare the circular invariants of the rotated surveyors with hyperbolic ones of the ship and lighthouse.

The surveyors agree on the distance from town center or origin, that is, the sum of squares of coordinates $(x^2+y^2 = x'^2+y'^2)$. The ship and light houses agree on difference of squares of coordinates $(x^2-ct^2 = x'^2-ct'^2)$ that is, the speed of light *c*. Expanding circles of blink waves trace out cones in space-time as in Fig. 4.B.5. Their *(x,ct)* cross-section are hyperbolic conic sections called *light-cone* sections.

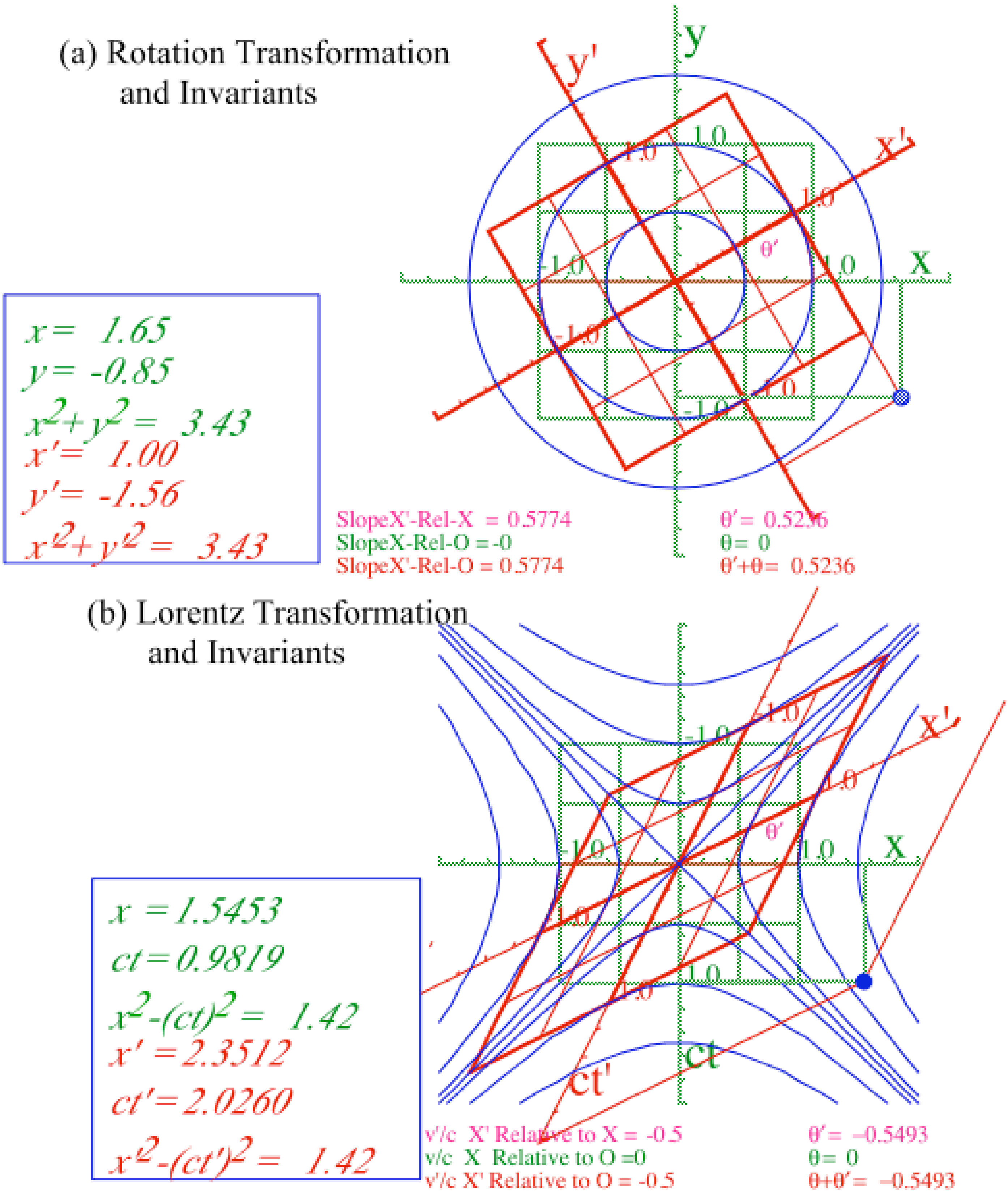

Fig. 4.B.4 Comparison of invariants (a) Rotations invariants are circles. (b) Lorentz invariants are hyperbolas.

Fig. 4.B.5 below is a plot of the North Lighthouse blink waves in *{x, y, ct}* coordinates. Blinks emitted at *t= -1/2, t=0*, and *t=+1/2* seconds trace three concentric *light cones* around the track or *world line* of the North Lighthouse. All observers will see the same cones. They are invariant to one's space-time viewpoint.

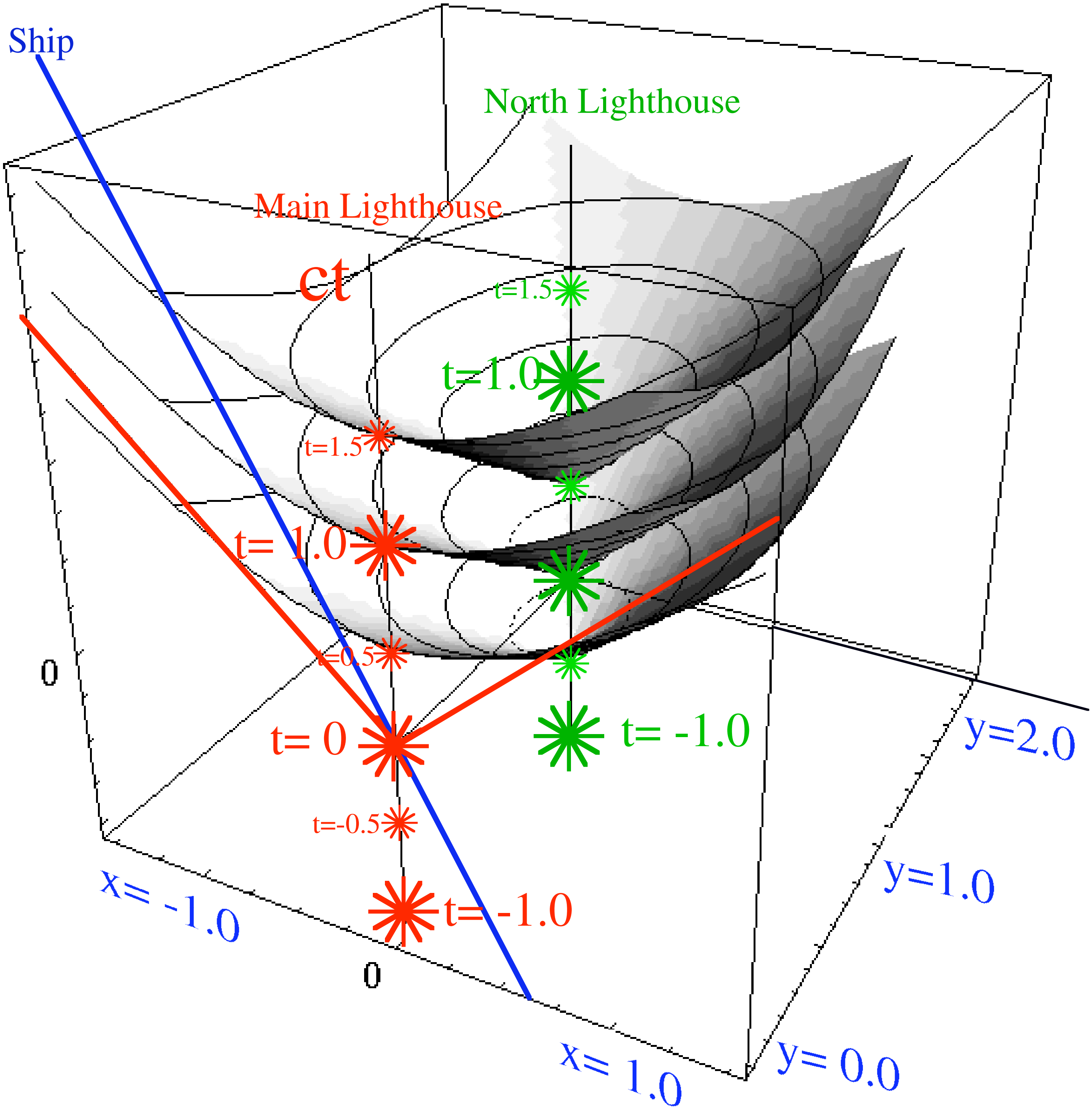

Fig. 4.B.5 Space-Space-Time plot of world lines for Lighthouses. North Lighthouse blink waves trace light cones.

Hyperbolic trigonometry

We are used to circular invariants and circular functions like sine and cosine that go with Cartesian rotation and elementary geometry and trigonometry. Relativistic Lorentz rotations have the transformation equations (4.B.5) in terms of <u>hyperbolic</u> functions sinhρ and coshρ. Invert these relations to get the 'angle' θ=ρ in terms of velocity where *rapidity ρ* is the logarithm of Doppler blue-shift factor in (4.A.5b).

$$\cosh\rho + \sinh\rho = e^{\rho} = \frac{1+\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}, \qquad \text{or:} \qquad \rho = \ln\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} \tag{4.B.6}$$

It turns out that the quantity θ=ρ is not an angle at all but an <u>area</u>. It is the gray area in Fig. 4.B.6 enclosed by the unit hyperbolic invariant $x^2 - (ct)^2 = 1$ and the two x and x' axes. To calculate this area we form a triangle of base $x=\cosh\theta$ and altitude $y=\sinh\theta$ which contains the area as shown below.

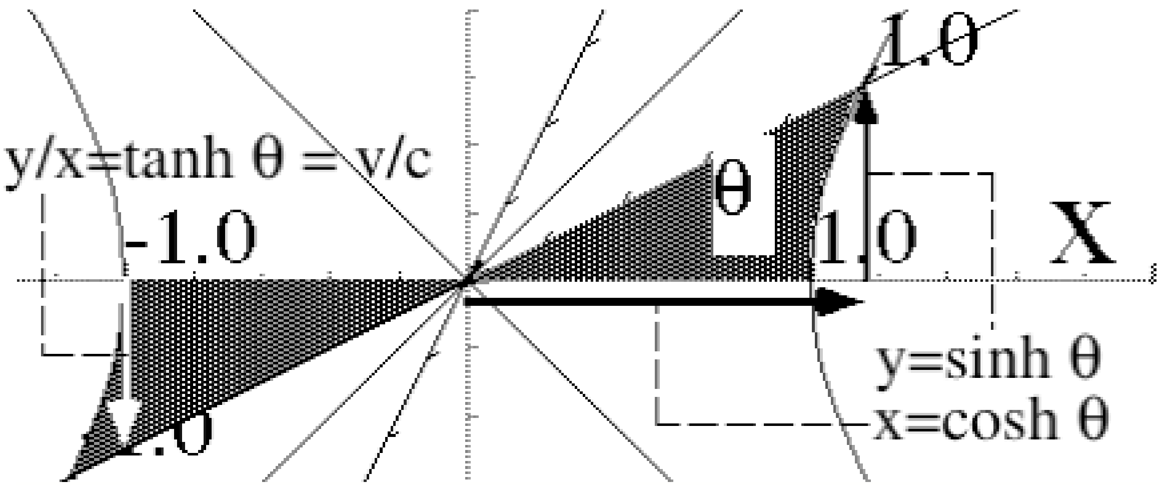

Fig. 4.B.6 Hyperbolic angle-area θ=ρ for unit hyperbola $x^2-(ct)^2=1=\cosh^2\theta - \sinh^2\theta$.

Note that the length of the tangent line between axes is the hyperbolic tangent $\tanh\theta = \sinh\theta/\cosh\theta$.

The desired area is found by subtracting the area under the hyperbola from that of the triangle. This will give us one-half of the gray area shown in the figure. Then $d(\cosh\theta) = \sinh\theta \; d\theta$ is used. Also we have

$$\frac{Area}{2} = \frac{1}{2} base \cdot altitude - \text{area under curve} = \frac{1}{2} xy - \int y\, dx$$
$$\frac{Area}{2} = \frac{1}{2}\sinh\theta\cosh\theta - \int \sinh\theta\, d\left(\cosh\theta\right) \tag{4.B.7}$$

$$\sinh^2\theta = \left(\frac{e^{\theta}-e^{-\theta}}{2}\right)^2 = \frac{1}{4}\left(e^{2\theta}+e^{-2\theta}-2\right) = \frac{\cosh 2\theta - 1}{2} \tag{4.B.8}$$

$$\sinh\theta\cosh\theta = \left(\frac{e^{\theta}-e^{-\theta}}{2}\right)\left(\frac{e^{\theta}+e^{-\theta}}{2}\right) = \frac{1}{4}\left(e^{2\theta}-e^{-2\theta}\right) = \frac{1}{2}\sinh 2\theta \tag{4.B.9}$$

This gives the gray area between hyperbolas subtended by radii.

$$\frac{Area}{2} = \frac{1}{2}\sinh\theta\cosh\theta - \int \sinh^2\theta\, d\theta = \frac{1}{4}\sinh 2\theta - \int \frac{\cosh 2\theta - 1}{2}\, d\theta$$

Using $\int \cosh a\theta\, d\theta = \frac{1}{a}\sinh a\theta$ we derive that the total gray area in Fig. 4.B.6 is equal to θ=*ρ*.

$$Area = \theta = \rho \qquad (4.B.10)$$

Note that the relativistic slope or velocity parameter β=*v/c* is the hyperbolic tangent of this area.

$$\beta = \frac{v}{c} = \frac{\sinh\rho}{\cosh\rho} = \tanh\rho \qquad (4.B.11)$$

Adding relativistic velocities and angles

Suppose, as before, that the ship has a velocity relative to the lighthouse that is half that of light, that is *v'=c/2*. Now suppose there is an observer that sees the lighthouse going at a velocity of *c/2*. What will that observer see for the velocity of the ship? If we simply added the two velocities it would be *0.5c + 0.5c = c*.

However, it does not work that way. As with the space-space tipping transformations *we need to add tangles not slopes*. Consider the plot shown in Fig. 4.B.6 below. The figure shows angle-areas being added to give the correct total area of θ + θ′= *0.5493 + 0.5493 = 1.0986*. The θ=*ρ* are obtained from the hyperbolic tangent relation (4.B.11). θ = $\tanh^{-1}(v/c) = \tanh^{-1}(0.5) = 0.5493$. Then the hyperbolic tangent of the sum is the desired answer: *tanh(1.0986) = 0.8*. The observer will see the ship going at *0.8c* or *4/5* of the speed of light.

A quick way to do relativistic velocity addition is to use the angle addition identity for the hyperbolic tangent. It is similar to the identity for the circular tangent.

$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y} \qquad (4.B.12)$$

Since the relative velocity ratio *u/c* is the hyperbolic tangent of the relative angle θ_u the identity gives:

$$\frac{u'}{c} = \tanh(\theta_u + \theta_v) = \frac{\tanh\theta_u + \tanh\theta_v}{1 + \tanh\theta_u \tanh\theta_v} = \frac{\frac{u}{c} + \frac{v}{c}}{1 + \frac{u}{c}\frac{v}{c}} \qquad (4.B.13)$$

or the *relativistic velocity addition formula*.

$$u' = \frac{u + v}{1 + \frac{uv}{c^2}} \qquad (4.B.14)$$

This is the same result as our previous calculation which added *u=c/2* and *v=c/2* to get *0.8c*.

$$u' = \frac{\frac{c}{2} + \frac{c}{2}}{1 + \frac{1}{4}} = \frac{c}{\frac{5}{4}} = \frac{4c}{5} \qquad (4.B.14)_{\text{example}}$$

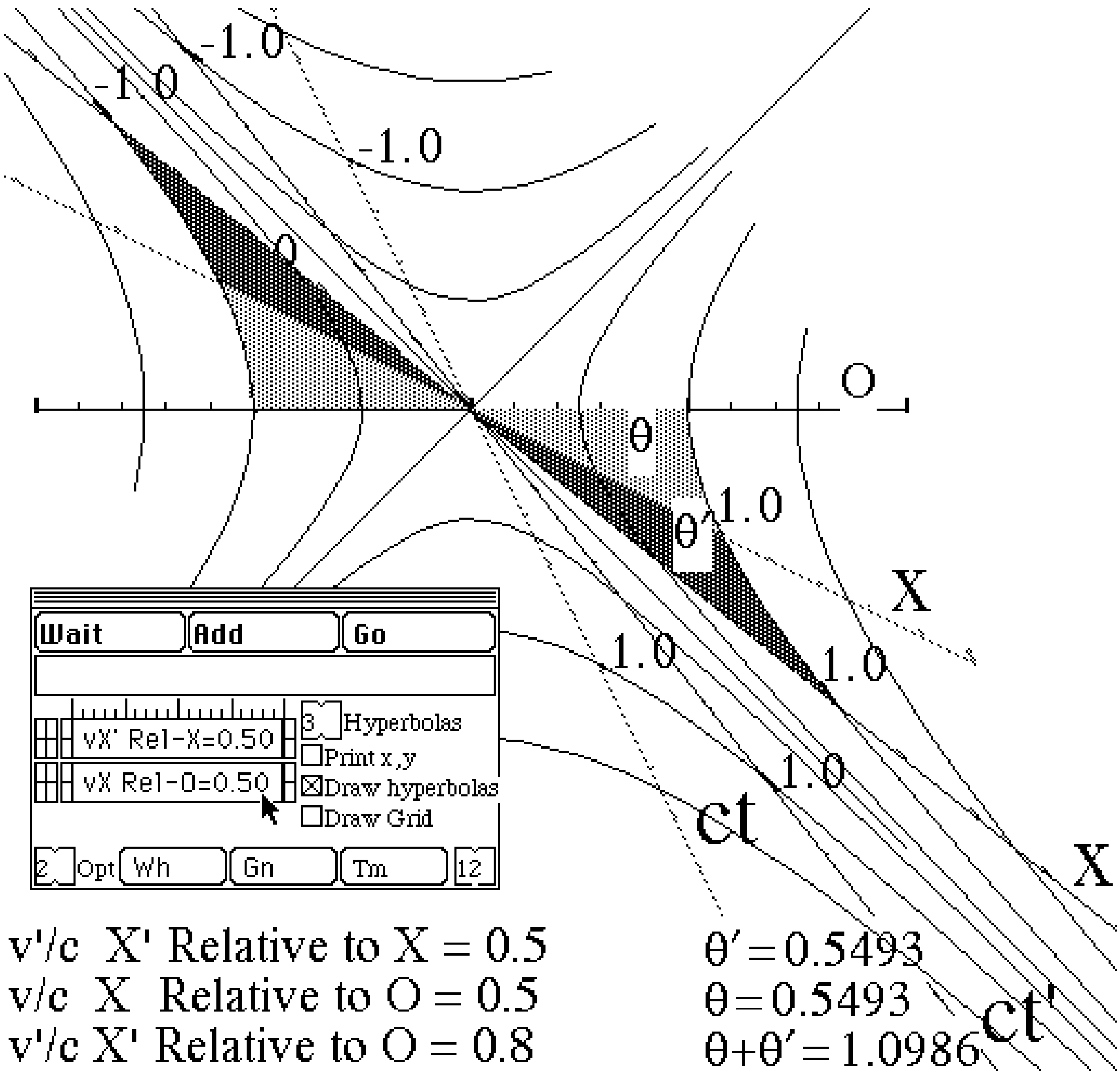

Fig. 4.B.7 Coordinate axis X' tipped by θ′=0.549 relative to X-axis which in turn is tipped by θ =0.549 relative to O-axis.

As you add more and more hyperbolic angle area you approach the speed of light. But there is an infinite amount of angle-area under the hyperbola. No matter how much more speed you add you will never get any closer to the speed of light. It is like a horizon. You can approach it but you can never cross it.

Spacetime graphs such as Fig. 4.B.8 show Doppler effects and more. The blink wave paths are the ±*45°* lines intersecting at blink times of *t= ...-1.0, 0.0, 1.0, 2.0,...*sec. In the upper portion of Fig. 4.B.8 the blink waves from the main lighthouse are seen crossing the ship path, that is the *ct′*-axis or *x′=0*, every half second or so before the ship passes the lighthouse at *t=0=t′*. To be precise, the crossing time is $\sqrt{0.5}/\sqrt{1.5}=0.577$ *sec.* according to Doppler blue-shift formula (2.A.5) But, after passage, it's not until *t′ = 1.73* that the ship encounters another blink hit. This is the red-shift crossing time of $t'=\sqrt{1.5}/\sqrt{0.5}=\sqrt{3}=1.732$ *sec.* The lighthouse claims the first hit (Happening 1) occurs at *t=2* according its clocks, the same time as its second blink (*Happening 2*). This lighthouse moment of *t=2* has a past (*t<2* indicated by gray area) and a future (*t>2* is the white area below the *t=2* line.) The *t=2* line is the space-time location of the lighthouse *x*-axis or "now-line" at this moment. The ship and lighthouse icons are a little misleading. A 3-dimensional object cannot be really drawn on one spatial dimension. Also, note that the North lighthouse lies below the page containing the Main lighthouse in Fig. 4.B.8. This was sketched in Fig. 4.B.5.

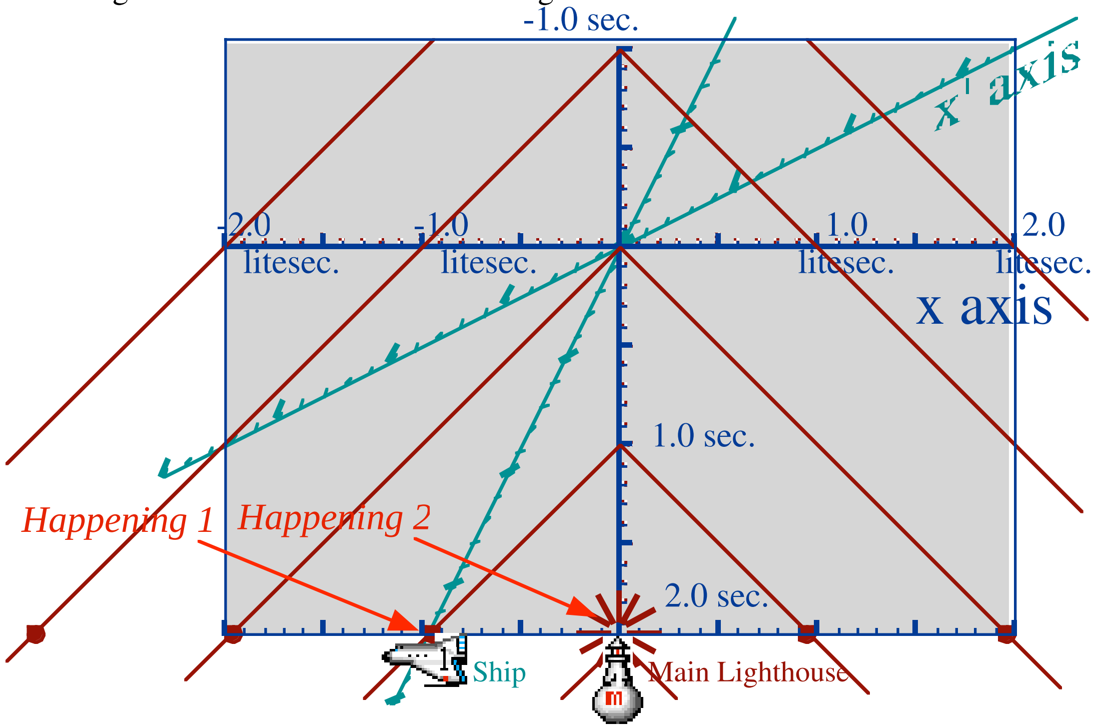

Fig. 4.B.8 Spacetime graph of ship passing lighthouse. (Lighthouse moment t=2 indicated.)

The ship draws its moments differently as seen in Fig. 4.B.9. Here the moment of *Happening 1* is indicated by the ship *x′*-axis at the moment *t′= 1.732 sec*. This ship moment of $t'=\sqrt{3}$ has a past ($t'<\sqrt{3}$ indicated by gray area) and a future ($t'>\sqrt{3}$ is the white area below the $t'=\sqrt{3}$ line.) Note that the ship's past overlaps with the lighthouse future in the leftward direction to which it is traveling, while behind the ship, the lighthouse has regions of its past that correspond to the ship's future. Very strange!

These graphs show why the ship does not regard *Happening 1* and *Happening* 2 to be simultaneous in the way that the lighthouse does. As far as the ship is concerned, points behind it belong to a lighthouse past, and so a 2nd blink (*Happening 2*) will come later, in fact not until *t′ = 2.3 sec.*

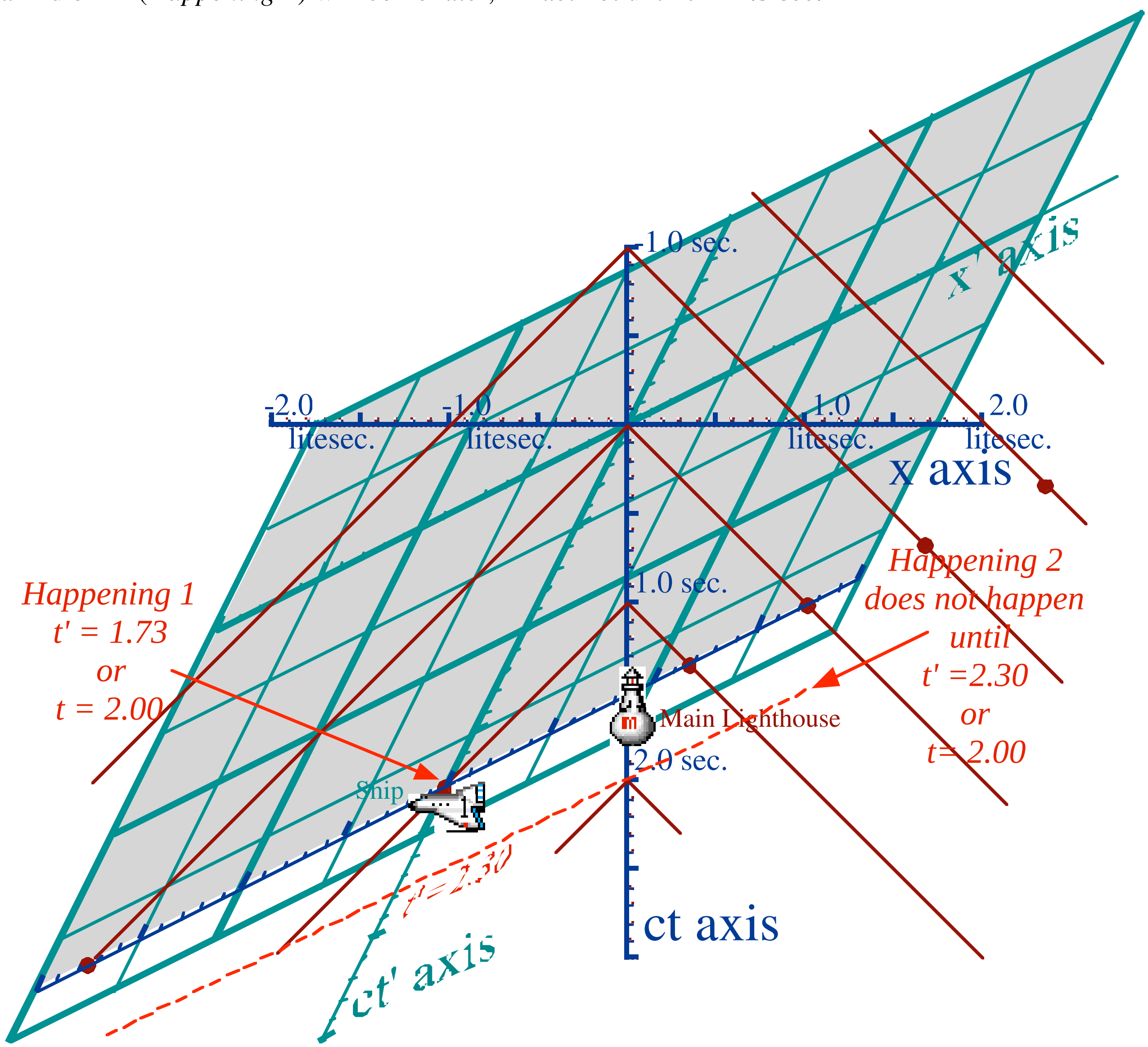

Fig. 4.B.9 Spacetime graph of ship passing lighthouse. (Ship moment t'=1.73 indicated.)

A ship or any observer moving with positive velocity relative to the lighthouse (that is, left to right) will record an opposite time order for *Happening 1* and 2. For such a reference frame, *Happening 2* will come before *Happening 1* since its *x*-axis will tip down to the right in Fig. 4.B.9. This event reversal could present a serious philosophical conundrum if, for example, *Happening 1* caused *Happening 2*. Generally, we prefer causes to precede effects, and this is known as the *causality principle*. Violations of causality are regarded with the same suspicion reserved for violation of energy conservation or the 2nd Law of Thermodynamics. Such violations are tolerated in microscopic quantum fluctuations but not in macroscopic classical averages.

For Happening 1 to actually cause Happening 2, it must send some kind of message, particle, or "force" at a speed greater than light. If a "cause" or particle goes from 1 to 2 it must cut across the light cone!

After Fig. 4.B.7 we noted that hyperbolic asymptotes or light cones were like horizons that one could approach indefinitely but should not expect to cross. This light barrier is considerably more serious than the so-called "sound-barrier." It cannot be broken by ordinary matter by simply having the "right-stuff." Anything that crosses the barrier however briefly pays a great price; it will be seen by many observers to be located at three or more places at one time! Doing this involves (possibly painful) annihilations and recreations as shown below.

Consider a case where Happening 1 comes just a little earlier than Happening 2 as shown in Fig. 4.B.10 so that faster-than-light travel is required to connect or "cause" the second Happening. Then the Ship's view of this is pretty strange as seen in Fig. 4.B.11 where *Happening 2* occurs before *Happening 1*. Any "cause" connecting the two goes "backwards in time." The lighthouse sees the causative particle shown in Fig. 4.B.10 ride down to *Happening 1* then leap faster-than-light to *Happening-2* but the ship finds it at three places during the time between *Happening 2* and *Happening 1* in Fig. 4.B.11. It is as though a particle-anti-particle pair is created at Happening 2 and the anti-particle is annihilated at Happening 1!

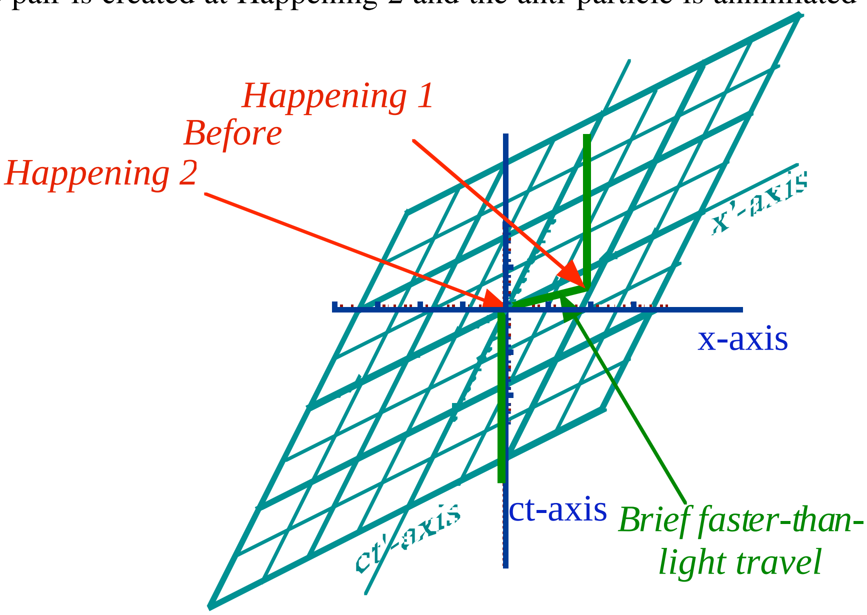

Fig. 4.B.10 Lighthouse plot of Happenings

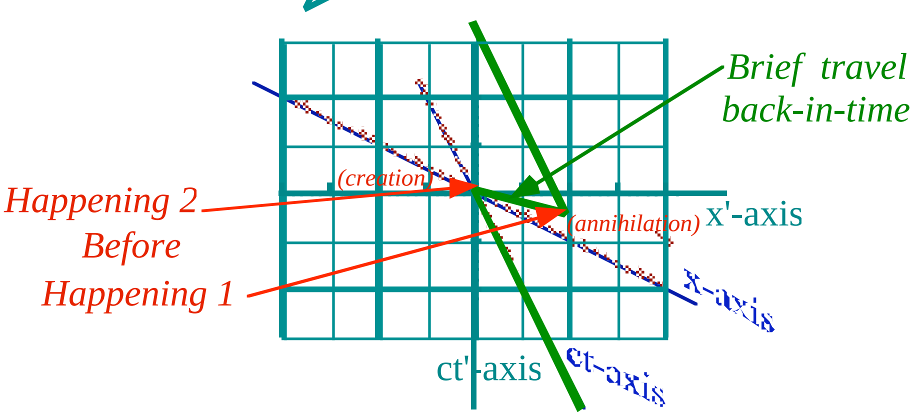

Fig. 4.B.11 Ship plot of two Happenings