Appendix 3.A Matrix Determinants, Adjuncts, and Inverses

Determinants of an $N-by-N$ matrix can be dealt with conveniently using the N-th value Levi-Civita ε-symbol defined below:

$$\varepsilon_{i_1\ i_2\ i_3\ \cdots\ i_N} = \begin{cases} 0: & \text{if any two } i_a \text{ are equal} \\ 1: & \text{if } \{i_1 \ldots i_N\} \text{ is EVEN shuffle of } \{1 \ldots N\} \\ \text{-1}: & \text{if } \{i_1 \ldots i_N\} \text{ is ODD shuffle of } \{1 \ldots N\} \end{cases}$$

(3.A.1)

Then the determinant may be written as a sum over all N^N combination of N integers $\{\ i_1\ \ i_2\ \ \ldots\ \ i_N\ \}$ between 1 and N.

$$\det|M| = \sum_{\{\ i_1\ \ldots\ i_N\ \}}^{N} \varepsilon_{\ i_1\ i_2\ i_3\ \ldots\ i_N} \quad M_{1i_1}\ \ M_{2i_2}\ \ M_{3i_3}\ \ \ldots\ \ M_{Ni_N} \qquad (3.A.2)$$

Only $N\,!$ of these terms actally exist. The non-zero ones are just permutations of $\{\ i_1 = 1\ \ i_2 = 2\ \ i_3 = 3\ \ \ldots\ \ i_N = N\ \}$. Negative (positive) terms belong to odd (even) permutations. (See Appendix 3.B.)

From now on let us imply a sum ($1-to-N$) over any indices-repeated on *only* one side of an equation so we will drop the Σ sign. This is called the *dummy index* sum convention.

The concept of *minor* or *adjunct* component expansions follows easily. Pulling the first component out of (3.A.2) gives (with our sum convention)

$$\det|M| = M_{1i_1} M_{i_1 1}^{ADJ} = M_{11} M_{11}^{ADJ} + M_{12} M_{21}^{ADJ} \ldots + M_{1N} M_{N1}^{ADJ}$$

where adjunct components M^{ADJ} are defined below.

$$\begin{aligned} M_{a1}^{ADJ} &= \varepsilon_{\ a\ i_2\ i_3\ \ldots} M_{2i_2} M_{3i_3} \ldots \\ &= -\varepsilon_{\ i_2\ a\ i_3\ \ldots} M_{2i_2} M_{3i_3} \ldots \qquad (3.A.2) \\ &= \varepsilon_{\ i_2\ i_3\ a\ \ldots} M_{2i_2} M_{3i_3} \ldots \end{aligned}$$

The adjunct component M_{ab}^{ADJ} is just $(-1)^{a+b}$ times the determinant made after crossing out the $b-th$ row and $a-th$ column of matrix M , and it goes to the $a-th$ row and $b-th$ column of the adjunct matrix M^{ADJ} . The determinant $\det|M|$ equals the matrix product of any row of M and the *same* column of M^{ADJ} .

$$\det|M| = M_{1i_1} M_{i_1 1}^{ADJ} = M_{2i_2} M_{i_2 2}^{ADJ} = \dots \tag{3.A.3}$$

where:

$$M_{i_1 1}^{ADJ} = \varepsilon_{i_1 \; i_2 \; i_3} M_{2i_2} M_{3i_3} \cdots$$

$$M_{i_2 2}^{ADJ} = \varepsilon_{i_1 \; i_2 \; i_3} M_{1i_1} M_{3i_3} \cdots$$

$$\vdots$$

A matrix inverse formula follows by showing that the following matrix product involving, for example, the first row of M and the second column of M^{ADJ} is zero.

$$\begin{aligned} M_{1i_1} M_{i_1 2}^{ADJ} &= \varepsilon_{i_1 \; i_2 \; i_3 \; \dots} M_{1i_1} M_{1i_2} M_{3i_3} \cdots \\ &= -\varepsilon_{i_2 \; i_1 \; i_3 \; \dots} M_{1i_1} M_{1i_2} M_{3i_3} \cdots && \text{(Switch two } \varepsilon \text{ indices)} \\ &= -\varepsilon_{i_1 \; i_2 \; i_3 \; \dots} M_{1i_2} M_{1i_1} M_{3i_3} \cdots && \text{(Relabel two sum indices)} \\ &= -\varepsilon_{i_1 \; i_2 \; i_3 \; \dots} M_{1i_1} M_{1i_2} M_{3i_3} \dots = -M_{1i_1} M_{i_1 2}^{ADJ} \\ &= 0 \end{aligned} \tag{3.A.4}$$

Any two equal row factors (the first and second are equal to M_{1i} in (3.A.4) above) in the ε -combination makes it vanish due to ε -antisymmetry. So the following general result holds.

$$M_{ai} M_{ib}^{ADJ} = \delta_{ab} \det|M| \tag{3.A.5}$$

So, for non-singular M (non-zero $\det|M|$) the inverse M^{-1} exists and is defined as follows:

$$M_{ab}^{-1} = \frac{M_{ab}^{ADJ}}{\det|M|}, \tag{3.A.6a}$$

so that

$$M_{ai} M_{ib}^{-1} = \delta_{ab}, \tag{3.A.6b}$$

that is, a matrix product of it with M yields a unit matrix $(\mathbf{1} = MM^{-1})$.

A more complete definition of the determinant used ε-tensors on both sides of the equation to reflect the fact that determinants are antisymmetric to column permutations as well as row permutations.

$$\det\left|M\right|\varepsilon_{abc\ldots} = \varepsilon_{i_1\ i_2\ i_3\ \ldots} M_{i_1 a} M_{i_2 b} M_{i_3 b} \cdots \tag{3.A.7}$$

This helps to expand matrix products and to prove a useful result: the determinant of a matrix product is simply the product of the determinants of the matrix factors. (Remember: repeated indices are being summed.)

$$\begin{aligned}
\det\left|M\cdot N\right|\varepsilon_{abc\ldots} &= \varepsilon_{i_1\ i_2\ i_3\ \ldots}\left(M_{i_1 j_1} N_{j_1 a}\right)\left(M_{i_2 j_2} N_{j_2 b}\right)\left(M_{i_3 j_3} N_{j_3 c}\right)\ldots \\
&= \left(\varepsilon_{i_1\ i_2\ i_3\ \ldots} M_{i_1 j_1} M_{i_2 j_2} M_{i_3 j_3}\cdots\right) N_{j_1 a} N_{j_2 b} N_{j_3 c}\cdots \\
&= \det\left|M\right|\varepsilon_{j_1\ j_2\ j_3\ \ldots} N_{j_1 a} N_{j_2 b} N_{j_3 c}\cdots \\
&= \det\left|M\right|\det\left|N\right|\varepsilon_{abc\ldots}
\end{aligned} \tag{3.A.8}$$

One corollary of (3.A.6b) and (3.A.8) is the following. (We note $\det|\mathbf{1}| = 1$, too.)

$$\det\left|M\right| = \frac{1}{\det\left|M^{-1}\right|} \tag{3.A.9}$$

Appendix 3.B Classification of Permutations

Suppose there is a neatly ordered set of N billiard balls lined up on a rack according to their numbers $\{1,2,3,4,5,6,7,8,..,N\}$. After a game the customers put them back in some permuted order like $\{4,2,8,6,3,7,1,5,...,N\}$. (We'll make it simple here and suppose only the first eight balls are out of order.)

Suppose it's your job to straighten them out. You have only two hands so it's natural to switch two at a time. You could look for the 1-ball and switch it with whatever ball is in the number-1 position. In this case the 4-ball is where the 1-ball should be, so you would switch the 1-and-4 balls. Let's write this as an equation using Dirac notation: (Bold numbers indicate which are being switched.)

$$\left(\mathbf{14}\right)\left|4,2,8,6,3,7,1,5\right\rangle = \left|\mathbf{1},2,8,6,3,7,\mathbf{4},5\right\rangle \tag{3.B.1}$$

The "2-flip" operation (**14**) is called a *transposition or* a *2-shuffle* or a *bicycle.* Using only bicycles we can complete the reordering. Looking for the 2-ball we see it's already in the 2nd- position so we don't need to do anything to it. A 'do-nothing' permutation is written as follows.

$$\left(\mathbf{2}\right)\left|1,2,8,6,3,7,4,5\right\rangle = \left|1,\mathbf{2},8,6,3,7,4,5\right\rangle$$

The operation (**2**) is called an *identity transposition* or a *unicycle*. Combining the preceding two equations gives.

$$\begin{aligned}\left(\mathbf{2}\right)\left(\mathbf{14}\right)\left|4,2,8,6,3,7,1,5\right\rangle &= \left(\mathbf{2}\right)\left|\mathbf{1},2,8,6,3,7,\mathbf{4},5\right\rangle \\ \left(\mathbf{2}\right)\left(\mathbf{14}\right)\left|4,2,8,6,3,7,1,5\right\rangle &= \left|1,\mathbf{2},8,6,3,7,4,5\right\rangle\end{aligned} \tag{3.B.2}$$

Now, the 3-ball needs to go where the 8-ball is currently sitting. So we apply the bicycle (**38**) to this.

$$\left(\mathbf{38}\right)\left(\mathbf{2}\right)\left(\mathbf{14}\right)\left|4,2,8,6,3,7,1,5\right\rangle = \left(\mathbf{38}\right)\left|1,\mathbf{2},8,6,3,7,4,5\right\rangle = \left|1,2,\mathbf{3},6,\mathbf{8},7,4,5\right\rangle \tag{3.B.3}$$

Then the 4-ball is put in the 4-th spot where 6-ball was sitting using bicycle (**46**).

$$\left(\mathbf{46}\right)\left(\mathbf{38}\right)\left(\mathbf{2}\right)\left(\mathbf{14}\right)\left|4,2,8,6,3,7,1,5\right\rangle = \left(\mathbf{46}\right)\left|1,2,\mathbf{3},6,\mathbf{8},7,4,5\right\rangle = \left|1,2,3,\mathbf{4},8,7,\mathbf{6},5\right\rangle \tag{3.B.4}$$

Then the 5-ball is put in the 5-th spot where 8-ball was sitting using bicycle (**58**).

$$\left(\mathbf{58}\right)\left(\mathbf{46}\right)\left(\mathbf{38}\right)\left(\mathbf{2}\right)\left(\mathbf{14}\right)\left|4,2,8,6,3,7,1,5\right\rangle = \left(\mathbf{58}\right)\left|1,2,3,\mathbf{4},8,7,\mathbf{6},5\right\rangle = \left|1,2,3,4,\mathbf{5},7,6,\mathbf{8}\right\rangle \tag{3.B.5}$$

Finally, a (**67**) bicycle finishes the job.

$$\left(\mathbf{67}\right)\left(58\right)\left(46\right)\left(38\right)\left(2\right)\left(14\right)\left|4,2,8,6,3,7,1,5\right\rangle = \left|1,2,3,4,5,\mathbf{6},\mathbf{7},8\right\rangle \tag{3.B.6}$$

Since the whole job took exactly five bicycles this is an *ODD* permutation, and it would get a (-1) sign in an 8-by-8 matrix determinant according to equation (3.A.1). A permutation's *parity* is *EVEN* or *ODD* if it has an even or odd number of bicycles. There are more efficient ways to decompose a permutation but its parity is the same no matter how you do the job.

For example, you may have noticed that we had to move some of the balls more than once. Is there a way to reshuffle while moving each ball just once? The answer is yes if you're able to pick up more than two at a time. This involves permutation *tricycles* (where you pick up three balls at once) or *quadracycles* (where you have to pick up four balls) , and so on.

With a little manual and mathematical dexterity we can rewrite the final equation (3.B.6) in a simpler and ultimately more revealing form. First we note that permutation operations commute with each other if they share no numbers in common. So we can move *(46)* to the left of *(58)* and *(14)* to the left of *(2), (38),* and *(58)* as follows.

$$(67)(58)(46)(38)(2)(14) = (67)(46)(14)\ \ (58)(38)\ \ (2) \qquad (3.B.7)$$

But, that's as far as you can go since *(14)* doesn't commute with *(46)* since both involve the 4-ball. (Try it!)

However, we can combine bicycles that share balls into bigger cycles. For example, two bicycles that share one ball like *(58)(38)* can be read as follows:

First, ball-3 replaces ball-8. (Right operator *(38)* acts first.)
Second, ball-8, in turn displaces ball-5. (Left operator *(58)* acts next.)
Third, ball-5 winds up where ball-8 was <u>after</u> *(38)*. That's where <u>ball-3</u> was <u>before</u> *(38)*.

We write this product as a tricycle

$$(58)(38) = (385) = (538) = (853) \qquad (3.B.8)$$

(385) is read as follows: ..3-displaces-8-displaces-5-displaces-3-.... and is the same as ..
(538) which is read: ..5-displaces-3-displaces-8-displaces-5-.... or....
(853) which is read: ..8-displaces-5-displaces-3-displaces-8-.... .

Note that if a bicycle product shares two balls it becomes a unicycle, that is no operation at all!

$$(85)(58) = (58)(58) = (5)= (8) =...= (1) \qquad (3.B.9)$$

Similarly, a quadracycle is a product of three bicycles such as the following.

$$(67)(46)(14) = (1467) = (4671) = (6714) = (7146) \qquad (3.B.10)$$

So our example permutation has 1 bicycle, 1 tricycle, and 1 quadracycle. Not counting the no-op-unicycle, we see that it is done in only two operations instead of five.

$$(67)(58)(46)(38)(2)(14) = (2)\quad (385)\quad (1467) \qquad (3.B.11)$$

A graphical example of just such a permutaion unraveling is done using a more direct way in Fig. 3.B.1 below. The problem is that it gives the inverse permutation (1764) (358) (2) instead of what we just worked out!

Why?

Welcome to the world of transformation groups! As you will learn if you study this book, every transformation of “things” has to be defined relative to their “pockets.” You may label a transformation using numbers on the things (here, the pool balls) or using numbers on the pockets. As we will see one definition gives the inverse of what the other one gives. This is a very important observation in quantum theory where the “balls” are “particles” and the “pockets” are “states” as will be discussed later.

In the meantime we have already seen a version of this transformational duality in the T-operators or rotation operators that can be defined in “alias” or “alibi” flavors in Section 2.2. A rotation matrix $\langle i|R|k\rangle$ is meaningless unless you specify its bra-kets, that is, its basis. A bra-ket is a two-sided thing, a destination and a point of origin, and all of quantum theory and relativity is concerned with their relative values. Absolutes, one might hope, went out with the absolute monarchs deposed during the 18th century enlightenment.

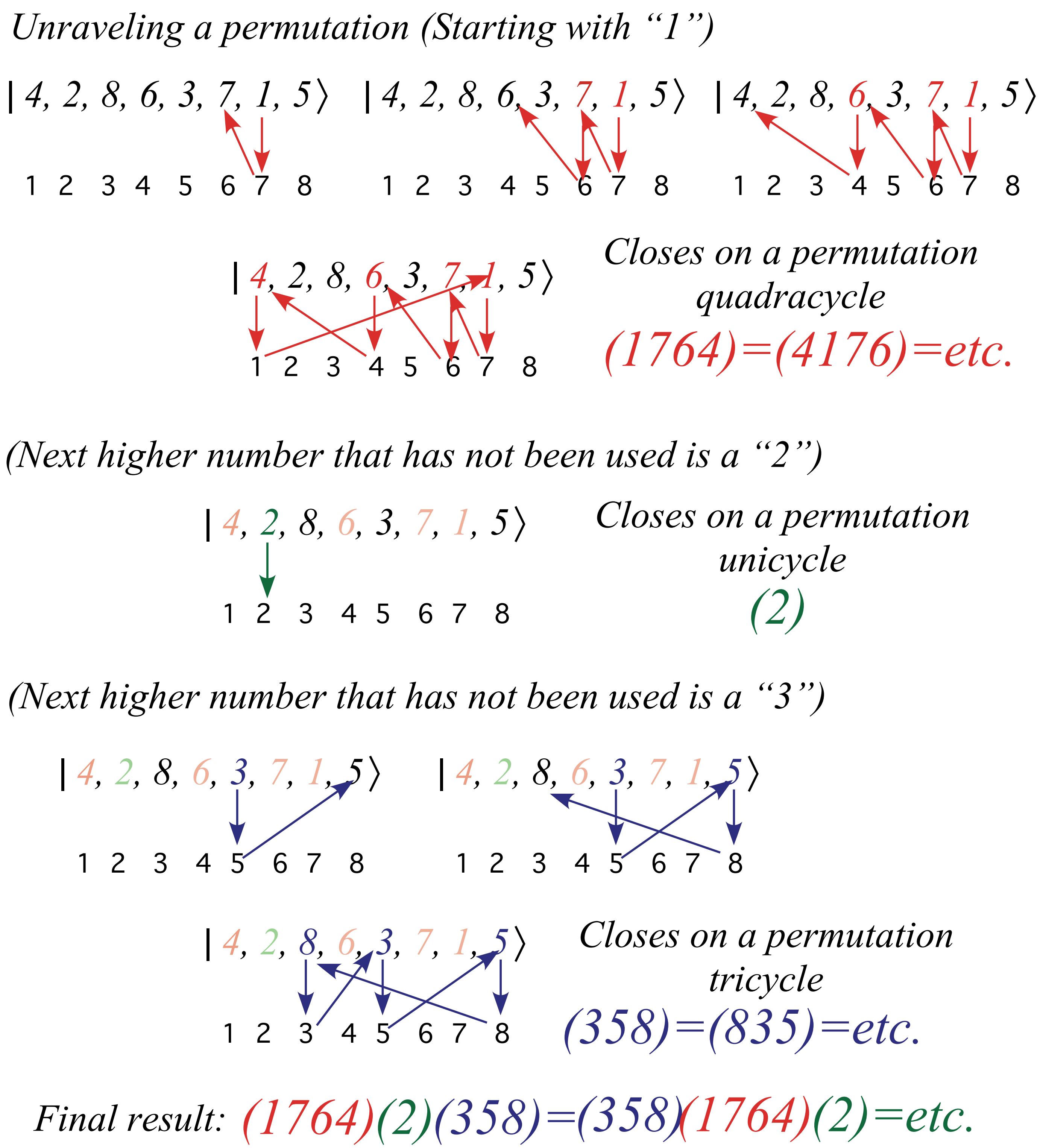

Fig. 3.B.1 Permutation cycle structure unraveling using pocket numbers.

Permutations are classified by the numbers of ν_1 of unicycles, ν_2 of bicycles, ν_3 of tricycles, and so forth. Above we have { $\nu_1 =1, \nu_2 =0, \nu_3 =1, \nu_4 =1, \nu_5 =0, \nu_6 =0,...$ }. Since no ball-number can be repeated in a cycle reduction, the cycle lengths must add up to the number N of balls.

$$\nu_1 + 2\,\nu_2 + 3\,\nu_3 + 4\,\nu_4 + 5\,\nu_5 +...+ N\nu_N = N \qquad (3.B.12)$$

So the number of different classes of permutations is equal to the number of *partitions* of the integer N.
For $N=2$ there are only two classes of two permutations.

Class { $\nu_1 =2, \nu_2 =0$} *corresponding to partition :* $2 = 1 + 1$

One permutation : *(1)(2)*

Class { $\nu_1 =0, \nu_2 =1$} *corresponding to partition* : $2 = 2$

One permutation : *(12)* (3.B.13)

For $N=3$ there are three classes of six permutations.

Class { $\nu_1 =3, \nu_2 =0, \nu_3 =0$} *corresponding to partition* : $3 = 1 + 1+ 1$

One permutation :: *(1)(2)(3)*

Class { $\nu_1 =1, \nu_2 =1, \nu_3 =0$} *corresponding to partition* : $3 = 2 + 1$

Three permutations : *(12)(3), (13)(2), (23)(1)*

Class { $\nu_1 =0, \nu_2 =0, \nu_3 =1$} *corresponding to partition* : $3 = 3$

Two permutations : *(123), (132)* (3.B.14)

The number of permutations in each partition class is given by a relatively simple combinatorial formula. To derive it one needs only consider the redundancy of the cycle labeling which was seen after (3.B.8), for example. Each M-cycle can be written M ways by cycling the numbers as shown in the tri-cycle in (3.B.8). If there are ν_M such M-cycles in a permutation then there are M^{ν_N} such reorderings that do not change the permutation at all. Also, since there are different numbers in each cycle they commute. So there are ν_M ! reorderings of the ν_M commuting cycles that give the same permutation, again. Dividing all these possibilities into N! gives the number of distinct partition class numbers.

$$\textit{Number in partition class } \nu_1\nu_2\nu_3\nu_4\cdots = \frac{N!}{\nu_1!1^{\nu_1}\ \nu_2!2^{\nu_2}\ \nu_3!3^{\nu_3}\ \nu_4!4^{\nu_4}\cdots} \qquad (3.B.15)$$

$$\text{where:} \qquad N = \nu_1 + 2\nu_2 + 3\nu_3 + 4\nu_4 \cdots$$

Exercise: Classify and enumerate the permutations for $N=4$ and $N=5$.

(Check against (3.B.15)

Chapter 3 Problems

Mirror-Mirror (Who's the fairest eigenvector?)

3.1.1 Compute the eigenvectors, eigenvalues, spectral decomposition and d-trans matrix for each of the mirror operations (a) thru (d) in the *Mirror-Mirror* problem. Where possible, tell physical or geometric significance.

(a) Use spectral decomposition's to derive inverse $\mathbf{I}=\mathbf{1}/\mathbf{T}$ and (all) square roots $\mathbf{X}=\sqrt{\mathbf{T}}$ such that $\mathbf{X}^2=\mathbf{T}$. (How many square roots does each have? Are any physically "do-able?")

(b) (extra-credit) Use c-d to invent a "slide rule" that correctly rotates U(2) electron and photon states.

Circle-Squash Switched

3.1.2 The discussion at the beginning of Sec. 1.6 showed that a unit circle is mapped onto an ellipse $\langle\mathbf{r}|\mathbf{T}^{-2}|\mathbf{r}\rangle=1$by matrix $\mathbf{T}=\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$. Consider the same mapping by "switched" matrix $\mathbf{S}=\begin{pmatrix} 1/2 & 1 \\ 1 & 1/2 \end{pmatrix}$.

(a) Find eigenvalues of $\mathbf{S}$ and $\mathbf{S}^{-2}$. Spectrally decompose $\mathbf{S}$ and plot its eigenvectors.

(b) Let $\mathbf{T}^{-1}|\mathbf{r}\rangle=|\mathbf{c}\rangle=\langle\mathbf{r}|\mathbf{T}^{-1}$ or $\mathbf{T}|\mathbf{c}\rangle=|\mathbf{r}\rangle=\langle\mathbf{c}|\mathbf{T}$ so $\langle\mathbf{r}|\mathbf{T}^{-1}|\mathbf{r}\rangle=\langle\mathbf{c}|\mathbf{r}\rangle=\langle\mathbf{c}|\mathbf{T}|\mathbf{c}\rangle$. Suppose all $\mathbf{c}$-vectors lie on a curve $\langle\mathbf{c}|\mathbf{T}|\mathbf{c}\rangle=1$ Discuss curve algebraically and plot this curve and the mapped $\langle\mathbf{r}|\mathbf{T}^{-1}|\mathbf{r}\rangle=1$ curve.

(c) Let $\mathbf{S}^{-1}|\mathbf{r}\rangle=|\mathbf{c}\rangle=\langle\mathbf{r}|\mathbf{S}^{-1}$ or $\mathbf{S}|\mathbf{c}\rangle=|\mathbf{r}\rangle=\langle\mathbf{c}|\mathbf{S}$ so $\langle\mathbf{r}|\mathbf{S}^{-1}|\mathbf{r}\rangle=\langle\mathbf{c}|\mathbf{r}\rangle=\langle\mathbf{c}|\mathbf{S}|\mathbf{c}\rangle$. Suppose all $\mathbf{c}$-vectors lie on a curve $\langle\mathbf{c}|\mathbf{S}|\mathbf{c}\rangle=1$ Discuss curve algebraically and plot this curve and the mapped $\langle\mathbf{r}|\mathbf{S}^{-1}|\mathbf{r}\rangle=1$ curve.

(d) *By conic geometry, derive a map $\mathbf{M}|\mathbf{c}\rangle=|\mathbf{r}\rangle$ of any real vector $|\mathbf{c}\rangle$ by real-symmetric matrix $\mathbf{M}$.

Dagger Your Own Ket

3.1.3 Most quantum matrices have simple relations between eigenvalues ε_m and their conjugates ε_m^*, eigenbras $|\varepsilon_m\rangle$ and kets $\langle\varepsilon_m|$, projectors $\mathbf{P}_m$ and their †-conjugates $(\mathbf{P}_m)^\dagger$, and diagonalizing (*d-tran*) transformations T and their inverses T^{-1} . Let's see what these relations are for...

(a) ...a Hermitian matrix $\mathbf{M} = \mathbf{H}$ such that $\mathbf{H} = \mathbf{H}^\dagger$ by spectrally decomposing and diagonalizing a general *2x2* reflection matrix $\mathbf{H}=\begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}$. (Are its eigenvectors meaningful? Discuss.)

(b) ...a Unitary matrix $\mathbf{M} = \mathbf{U}$ such that $\mathbf{U}^{-1}= \mathbf{U}^\dagger$ by spectrally decomposing and diagonalizing a general *2x2* rotation matrix $\mathbf{U}=\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$. (Are its eigenvectors meaningful? Discuss.)

(c) Find *all* the square-roots of **H** and of **U**. (Test them. There are more than two of each!)

Home on Lagrange

3.1.4 Functional spectral decomposition (3.1.17) is related to Lagrange functional interpolation (3.1.18). Use (3.1.18) to approximate $\sin x$ given only that $\sin 0=0$, $\sin\pi/2=1$, and $\sin\pi=0$. Compare your approximation to order-2 Taylor series approximation of $\sin x$ around $x=\pi/2$.

Bras-ackwards

3.1.5 See if you can work the spectral decomposition ideas backwards by doing the following "inverse" eigenvalue problems. (Hint: Use ket-bras and ⊗. Normalize first!)

(a) Find a Hermitian 3x3 matrix **H** that satisfies.

$$\mathbf{H}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}=\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{H}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}=4\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \quad \mathbf{H}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}=9\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

(b) Write down and test at least one square root $\sqrt{\mathbf{H}}$. (How many square roots are there?)

Cures for Nilpotency

3.1.6 Can a *nilpotent* matrix $\mathbf{N}$ ($\mathbf{N}^m= \mathbf{0}$, $\mathbf{N}^{m-1}$ not zero, integer $m>1$) be Hermitian $\mathbf{N}=\mathbf{N}^\dagger$
(a) ...for $m=2$? , (b) ...for other m? (Experiment with $2x2$ matrices first.)
(c) Use this and exercise *Dagger Your Own Ket* to prove Hermitian matrices must be diagonalizable.

Truly Secular
3.1.7 The coefficients a_k of the general nxn secular equation (3.1.5d) and (3.1.5f) of $\mathbf{M}$ depend on matrix coefficients M_{ij} and on eigenvalues ε_m.
(a) Do they depend on which basis you use to represent $\mathbf{M}$? Why or why not?
(b) For a general $4x4$ matrix $(n=4)$, compute functions $a_k = a_k(\varepsilon_m)$ in an orderly way that clearly shows how they come out for general n.
(c) For a general $4x4$ matrix $(n=4)$, compute functions $a_k = a_k(M_{ij})$ in an orderly way that clearly shows how they come out for general n. Use the ε-expansion in Appendix 1.A and (b) above to help express answer in terms of diagonal minor determinants. (NOTE: This is a "crucial" problem whose solutions belongs in your lab "journal" or equivalent.) May do successively $n= 2, 3,$ until a pattern emerges.

Adjunct Junk
3.1.8 Given (1.A.5) or $\mathbf{A}\,\mathbf{A}^{ADJ}=\mathbf{1}\ (det|\mathbf{A}|)$ with $\mathbf{A} = \mathbf{M} -\lambda\mathbf{1}$ show that $\mathbf{A}^{ADJ}$ has $\mathbf{M}$ eigenkets $|\lambda\rangle$ if λ is an eigenvalue of $\mathbf{M}$. Does $\mathbf{A}^{ADJ}$ also harbor $\mathbf{M}$'s eigenbras? Use $\mathbf{M}=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$ as an example.

Pair'em up
3.1.9 An *nxn pairing matrix* Π has 1 for all n^2 matrix elements $\Pi_{ij}=1$. It's used in superconductivity theory and nuclear structure.
(a) Use 1(c) above to help derive its eigenvalues and spectral decomposition. (Or, you may develop the theory by doing successively $n= 2, 3,$ until the pattern emerges.)
(b) Does the matrix $\Pi+(const.)\mathbf{1}$ have the same eigenvectors? eigenvalues? as Π. Explain.

All Together Now
3.1.10 Show how to do a <u>simultaneous</u> spectral decomposition using the projector splitting technique.. (a) Spectrally decompose $\mathbf{A}=\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, *and* $\mathbf{B}=\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$.

(b) Calculate the "ridiculous function" $\mathbf{B}^{\mathbf{A}}$ of these two matrices.

A Perturbing Problem

3.2.1 Find eigenvalues (to ±1%) of matrix $\mathbf{M}=\begin{pmatrix} 2 & 0.1 & 0.3 \\ 0.1 & 3 & 0.2 \\ 0.3 & 0.2 & 4 \end{pmatrix}$ using perturbation theory.

A Permuted Problem
3.2.2 (a) As in Appendix 3.B show cycle structure of all permutations in symmetric group S_4 and S_5.
(b) Write permutation $\mathbf{(p)}|12345678\rangle=|25386741\rangle$ in cycles. How many $\mathbf{(p)}$ in its S_8 class?

Unit. 1 Review Topics and Formulas

Transformation Matrix
General $n \times n$ T Matrix

$$\begin{array}{cc} \text{Bras:} & \begin{array}{c} \langle 1| \\ \langle 2| \\ \langle 3| \\ \end{array} \end{array} \begin{pmatrix} \langle 1|1'\rangle & \langle 1|2'\rangle & \langle 1|2'\rangle & \cdots \\ \langle 2|1'\rangle & \langle 2|2'\rangle & \langle 2|3'\rangle & \cdots \\ \langle 3|1'\rangle & \langle 3|2'\rangle & \langle 3|3'\rangle & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

$$\text{Kets}: \quad |1'\rangle \quad |2'\rangle \quad |3'\rangle \quad \cdots$$

Transformation Matrix
1.Photon Polarization

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Transformation Matrix
2.Electron Polarization

$$\begin{pmatrix} \langle \uparrow|\uparrow'\rangle & \langle \uparrow|\downarrow'\rangle \\ \langle \downarrow|\uparrow'\rangle & \langle \downarrow|\downarrow'\rangle \end{pmatrix} = \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}$$

Axiom 1: The absolute square $|\langle j|k'\rangle|^2 = \langle j|k'\rangle^* \langle j|k'\rangle$ *gives the probability for state-j of a system in state-k'=1' to n' from one sorter and then forced to choose between states j=1 to n by another sorter.*

Axiom 2: The complex conjugate of an amplitude gives its reverse: $\langle j|k'\rangle^* = \langle k'|j\rangle$

Axiom 3: If identical analyzers are used twice or more the amplitude for a passed state-k is one,

and for all others it is forever zero: $\langle j|k\rangle = \delta_{jk} = \begin{Bmatrix} 1 \text{ if: } j{=}k \\ 0 \text{ if: } j \neq k \end{Bmatrix} = \langle j'|k'\rangle$ *(ORTHONORMALITY)*

Axiom 4. Ideal sorting followed by ideal recombination of amplitudes has no effect:

$$\langle j''|m'\rangle = \sum_{k=1}^{n} \langle j''|k\rangle\langle k|m'\rangle \quad \Rightarrow \quad \mathbf{1} = \sum_{k=1}^{n} |k\rangle\langle k| = \sum_{k=1}^{n} \mathbf{P}_k \qquad \textit{(COMPLETNESS)}$$

The *secular equation* $\det|\mathbf{M} - \varepsilon\mathbf{1}| = 0 = (-1)^n \left(\varepsilon^n + a_1\varepsilon^{n-1} + a_2\varepsilon^{n-2} + \ldots + a_{n-1}\varepsilon + a_n\right)$ where:

$$a_1 = -Trace\mathbf{M}, \cdots, \; a_k = (-1)^k \sum \text{diagonal k-by-k minors of } \mathbf{M}, \cdots, \; a_n = (-1)^n \det|\mathbf{M}|$$

The *Hamilton-Cayley (HC) equation* $\mathbf{0} = (\mathbf{M} - \varepsilon_1\mathbf{1})(\mathbf{M} - \varepsilon_2\mathbf{1})\cdots(\mathbf{M} - \varepsilon_n\mathbf{1})$

Projection operators: $\mathbf{P}_k = \dfrac{\prod_{j\neq k} (\mathbf{M} - \varepsilon_j\mathbf{1})}{\prod_{j\neq k} (\varepsilon_k - \varepsilon_j)}$ are *eigenoperators* for $\mathbf{M}$ such that: $\mathbf{P}_k \mathbf{M} = \mathbf{M}\mathbf{P}_k = \varepsilon_k \mathbf{P}_k$

$\mathbf{P}_k$ satisfy *projector orthonormality* $\mathbf{P}_j\mathbf{P}_k = \delta_{jk}\mathbf{P}_k$ and *projector completeness* $\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$
and: *spectral decompositions of an operator* $\mathbf{M}$. $\mathbf{M} = \varepsilon_1\mathbf{P}_1 + \varepsilon_2\mathbf{P}_2 + \ldots + \varepsilon_n\mathbf{P}_n$

$f(\mathbf{M}) = f(\varepsilon_1)\mathbf{P}_1 + f(\varepsilon_2)\mathbf{P}_2 + \ldots + f(\varepsilon_n)\mathbf{P}_n$

The old "**1=1·1** *trick*" will be used later in symmetry analysis and spectroscopic theory

$$\mathbf{1}{=}\mathbf{1}\cdot\mathbf{1} = \left(\mathbf{P}_1^G + \mathbf{P}_{-1}^G\right)\left(\mathbf{P}_2^H + \mathbf{P}_{-2}^H\right) = \mathbf{1} = \left(\mathbf{P}_1^G\mathbf{P}_2^H + \mathbf{P}_1^G\mathbf{P}_{-2}^H + \mathbf{P}_{-1}^G\mathbf{P}_2^H + \mathbf{P}_{-1}^G\mathbf{P}_{-2}^H\right)$$

Perturbation expansion for eigenvalue nearest $E_1 = H_{11}$: (For order higher than 2: Caution and good luck!)

$$\lambda = E_1 + \sum_{j\neq 1}^{N} \frac{H_{1j}H_{j1}}{(E_1 - E_j)} + \sum_{j\neq 1}^{N}\sum_{k\neq 1,j}^{N} \frac{H_{1j}H_{jk}H_{k1}}{(E_1 - E_j)(E_1 - E_k)} + \sum_{j=1}^{N}\sum_{k\neq j}^{N}\sum_{\ell\neq j,k}^{N} \frac{H_{1j}H_{jk}H_{k\ell}H_{\ell 1}}{(E_1 - E_j)(E_1 - E_k)(E_1 - E_\ell)} \cdots$$

Unit 1 Exam **Quantum Mechanics 5413** *Dirac Notation and Matrix Algebra*

(40 pts)

1. We're given the following base state definitions of transformation operator **R**.

$|1'\rangle = \mathbf{R}|1\rangle = |2\rangle$,

$|2'\rangle = \mathbf{R}|2\rangle = |3\rangle$,

$|3'\rangle = \mathbf{R}|3\rangle = |4\rangle$,

$|4'\rangle = \mathbf{R}|4\rangle = |1\rangle$,

(a) Write down a matrix representation for **R** in the $\{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}$ basis in Dirac notation and numerically.

(b) Use (a) to compute a representation of $\mathbf{R}^2$, $\mathbf{R}^3$, and $\mathbf{R}^4$,too.

(c) Write down a matrix representation for **R** in the $\{|1'\rangle, |2'\rangle, |3'\rangle, |4'\rangle\}$ basis in Dirac notation and numerically. Is it different from the result in (a)? Why or why not?

(d) By examining powers $\mathbf{R}^p$ deduce the Hamilton Cayley equation and secular equation of **R** matrix.

(e) Write down minimal equation and eigenvalues for **R** .

(f) Can matrix **R** be spectrally decomposed and diagonalized? How do you tell?

(g) Can <u>all</u> the matrices $\mathbf{R}$, $\mathbf{R}^2$, $\mathbf{R}^3$ ..., $\mathbf{R}^p$ be <u>simultaneously</u> decomposed by a single set of projectors and transformation matrix. How do you know?

(h) If (f) is "Yes" do spectral decomposition and diagonalization of **R** .

(i) If (g) is "Yes" do spectral decomposition and diagonalization of $\mathbf{R}^p$ for any power p.

(30 pts)

2. The results from the preceding problem may help to spectrally decompose the following general type of matrix. If so, explain why and use the results to find its eigenvectors and eigenvalues in terms of constant parameters *A, B, C* and *D*. If not, explain why not.

$$\mathbf{H} = \begin{pmatrix} D & A & B & C \\ C & D & A & B \\ B & C & D & A \\ A & B & C & D \end{pmatrix}$$ Sketch levels for case $A=B=C = 0.2$ and $D=1$.

(20 pts)

3. (a) Write a 2nd order perturbation expression for the eigenvalues of

$$\mathbf{H} = \begin{pmatrix} D_1 & A & B & C \\ C & D_2 & A & B \\ B & C & D_3 & A \\ A & B & C & D_4 \end{pmatrix}$$

in terms of parameters *A, B,* and *C* for $D_1 = 1, D_2 = 2, D_3 = 3, D_4 = 4$.

(b) Sketch levels for case $A=B=C = 0.2$.

(c) Can you use your expression (a) on the matrix in problem 2? Why or why not? Explain while giving a brief discussion of the requirements for a valid perturbation result.

(10 pts)

4. The transformations $\mathbf{R}$, $\mathbf{R}^2$, $\mathbf{R}^3$, in Problem 1 also behave like *permutations*. How?

(a) Give the cycle structure and notation for each. How many distinct $\mathbf{R}^p$ (p integral) exist ?

(b) How many different permutations can you make by considering <u>all</u> possible arrangements of numbers 1,2,3, and 4 in the cycles of **R**.? Classify them by cycles.