Relativity of transverse waves and 4-vectors
(Ch. 2-5 of CMwBang-Unit 8  Ch. 6 of QTforCA Unit 2 )

Reviewing “Relawavity” geometry
Reviewing the stellar aberration angle $\sigma$ vs. rapidity $\rho$
Pattern recognition: “Occam’s Sword”

Introducing per-spacetime 4-vector $(\omega_0, \omega_x, \omega_y, \omega_z) = (\omega, ck_x, ck_y, ck_z)$ transformation

More details of Lorentz boost of North-South-East-West plane-wave 4-vectors $(\omega_0, \omega_x, \omega_y, \omega_z)$
- Thales-like construction of Lorentz boost in 2D and 3D
- The spectral ellipsoid

Combination and interference of 4-vector plane waves (Idealized polarization case)
- Combination group and phase waves define 4D Minkowski coordinates
- Combination group and phase waves define wave guide dynamics
  - Waveguide dispersion and geometry
    - 1st-quantized cavity modes
      - (And introducing 2nd-quantized cavity modes)
- Lorentz symmetry effects
  - How it makes momentum and energy be conserved
Reviewing “Relawavity” geometry
Reviewing the stellar aberration angle $\sigma$ vs. rapidity $\rho$
Pattern recognition: “Occam’s Sword”
Reviewing “Relawavity” geometry

Energy ($E$)

\[ \frac{\nu}{c} = \beta = 0.600 \]

Doppler blue shift factor = $b = 2.000$

Doppler red shift factor = $r = 0.500$

$\nu = 0.540 = 30.964^\circ$

$\theta = 0.693$

$\sigma = 0.644 = 36.870^\circ$

Coordinate angle $\nu = \tan(u/c)$

Stellar aberration angle $\sigma = \sinh(u/c)$

Momentum

$cp = B \sinh(\theta)$

Hamiltonian

$H(p) = B \cosh(\theta)$

Lagrangian

$L(u) = B \sech(\theta)$

Group Velocity

$\frac{u}{c} = B \tanh(\theta)$

Phase Velocity

$\frac{c}{u} = B \coth(\theta)$

DeBroglie Wavelength

$\frac{\lambda}{c} = B \text{csch}(\theta)$

Phase Velocity

$\frac{c}{u} = B \coth(\theta)$

DeBroglie Wavelength

$\frac{\lambda}{c} = B \text{csch}(\theta)$

Thursday, March 6, 2014
Reviewing “Relawavity” geometry
Reviewing the stellar aberration angle $\sigma$ vs. rapidity $\rho$
Pattern recognition: “Occam’s Sword”
Pattern recognition: “Occam’s Sword”

Fig. 5.5
Relativistic wave mechanics geometry.
(a) Overview.
(b-d) Details of contacting tangents.

(c) Basic construction given \( u/c = 45/53 \)
(d) \( u/c = 3/5 \)

from: Fig. 8.5.5
QTforCA
Unit 8 Ch.5

Thursday, March 6, 2014
(a) Geometry of relativistic transformation and wave based mechanics

(b) Tangent geometry \((u/c=3/5)\)

**Velocity aberration angle** \(\phi\)

**Momentum** \(\rho\)

\(H = \sinh \rho\)

\(B = \cosh \rho\)

\(M_c = B \tanh \rho\)

\(H = \frac{5}{4}\)

\(-L = \frac{4}{5}\)

\(u/c = 1\)

\(e^\rho = 2/7\)

\(e^\rho = 1/2\)

\(cp = 45/28\)

\(cp = 3/4\)

\(\sigma\)

**Fig. 5.5**

Relativistic wave mechanics geometry.

(a) Overview.

(b-d) Details of contacting tangents.
Pattern recognition: “Occam’s Sword”

Fig. 5.10 CW cosmic speedometer.
Geometry of boosted counter-propagating waves.
Pattern recognition: “Occam’s Sword”

Fig. 5.10 CW cosmic speedometer.
Geometry of boosted counter-propagating waves.

\[ \sinh \rho = \tan \sigma \]
\[ \tanh \rho = \sin \sigma \]

\[ c = 1 \]

\[ e^\rho = \sinh \rho + \cosh \rho \]
Fig. 5.10 CW cosmic speedometer.
Geometry of boosted counter-propagating waves.

Pattern recognition: “Occam’s Sword”
Fig. 5.10 CW cosmic speedometer.
Geometry of boosted counter-propagating waves.

Pattern recognition: “Occam’s Sword”
Pattern recognition: “Occam’s Sword”

**Fig. 5.10** CW cosmic speedometer.
Geometry of boosted counter-propagating waves.
Pattern recognition: “Occam’s Sword”

Fig. 5.10 CW cosmic speedometer.
Geometry of boosted counter-propagating waves.

Relativity geometry has geometric series!
(Surprise, surprise, surprise, etc.)

\[
\sinh \rho = \tan \sigma \\
\tanh \rho = \sin \sigma \\
\cosh \rho = \sec \sigma \\
\sech \rho = \cos \sigma \\
\cosh^2 \rho = \sec^2 \sigma \\
\sech^2 \rho = \cos^2 \sigma \\
\text{etc.}
\]
Introducing per-spacetime 4-vector \((\omega_0, \omega_x, \omega_y, \omega_z) = (\omega, c k_x, c k_y, c k_z)\) transformation

More details of Lorentz boost of North-South-East-West plane-wave 4-vectors \((\omega_0, \omega_x, \omega_y, \omega_z)\)

Thales-like construction of Lorentz boost in 2D and 3D

The spectral ellipsoid
Per-spacetime 4-vector \((\omega_0, \omega_x, \omega_y, \omega_z) = (\omega, ck_x, ck_y, ck_z)\) transformation

Suppose starlight in lighthouse frame is straight down x-axis: \((\omega_\downarrow, ck_{x\downarrow}, ck_{y\downarrow}, ck_{y\downarrow}) = (\omega_0, -\omega_0, 0, 0)\)
Per-spacetime 4-vector \((\omega_0, \omega_x, \omega_y, \omega_z) = (\omega, ck_x, ck_y, ck_z)\) transformation

(a) Laser frame \(\omega_0\)

Suppose starlight in lighthouse frame is straight down x-axis: \((\omega_\downarrow, ck_{x_\downarrow}, ck_{y_\downarrow}, ck_{z_\downarrow}) = (\omega_0, -\omega_0, 0, 0)\)

"South"

"West"

from: Fig. 6.1.3
QT for CA
Unit 8 Ch. 6
Per-spacetime 4-vector \((\omega_0, \omega_x, \omega_y, \omega_z) = (\omega, ck_x, ck_y, ck_z)\) transformation

(a) Laser frame \(\omega_0\)

Suppose starlight in lighthouse frame is straight down x-axis:
\[
(\omega, ck_x, ck_y, ck_z) = (\omega_0, -\omega_0, 0, 0)
\]

from: Fig. 6.1.3
QTforCA
Unit 8 Ch.6
Per-spacetime 4-vector \((\omega_0, \omega_x, \omega_y, \omega_z) = (\omega, ck_x, ck_y, ck_z)\) transformation

(a) Laser frame \(\omega_0\)

\[
\begin{align*}
\text{up } +x\text{-axis: } & (\omega^\uparrow, ck_x^\uparrow, ck_y^\uparrow, ck_z^\uparrow) = (\omega_0, +\omega_0, 0, 0) \\
\text{along } +z\text{-axis: } & (\omega^\rightarrow, ck_x^\rightarrow, ck_y^\rightarrow, ck_z^\rightarrow) = (\omega_0, 0, 0, +\omega_0) \\
\text{along } -z\text{-axis: } & (\omega^\leftarrow, ck_x^\leftarrow, ck_y^\leftarrow, ck_z^\leftarrow) = (\omega_0, 0, 0, -\omega_0) \\
\text{down } -x\text{-axis: } & (\omega^\downarrow, ck_x^\downarrow, ck_y^\downarrow, ck_z^\downarrow) = (\omega_0, -\omega_0, 0, 0)
\end{align*}
\]

Suppose starlight in lighthouse frame is straight down x-axis: \((\omega^\downarrow, ck_x^\downarrow, ck_y^\downarrow, ck_z^\downarrow) = (\omega_0, -\omega_0, 0, 0)\)

from: Fig. 6.1.3
QTforCA
Unit 8 Ch.6
Suppose starlight in lighthouse frame is straight down  \( x \)-axis: 

\[
\begin{pmatrix}
\omega_0, ck_x, ck_y, ck_z
\end{pmatrix} = (\omega_0, -\omega_0, 0, 0)
\]

+ \( \rho_z \)-rapidity ship frame sees starlight Lorentz transformed to: 

\[
\begin{pmatrix}
\omega'_0, ck'_x, ck'_y, ck'_z
\end{pmatrix} = \begin{pmatrix}
\omega_0 \cosh \rho_z, -\omega_0, 0, -\omega_0 \sinh \rho_z
\end{pmatrix}
\]
Per-spacetime 4-vector \((\omega_0, \omega_x, \omega_y, \omega_z) = (\omega, c k_x, c k_y, c k_z)\) transformation

(a) Laser frame \(\omega_0\)

(b) \(z\)-\((\rightarrow\text{Moving})\) ship

Suppose starlight in lighthouse frame is straight down \(x\)-axis: \((\omega_\downarrow, c k_{x \downarrow}, c k_{y \downarrow}, c k_{z \downarrow}) = (\omega_0, -\omega_0, 0, 0)\)

\(+\rho_z\) -rapidity ship frame sees starlight Lorentz transformed to: \((\omega'_\downarrow, c k'_{x \downarrow}, c k'_{y \downarrow}, c k'_{z \downarrow}) = (\omega_0 \cosh \rho_z, -\omega_0, 0, -\omega_0 \sinh \rho_z)\)

\[
\begin{pmatrix}
\omega'_\downarrow \\
ck'_{x \downarrow} \\
ck'_{y \downarrow} \\
ck'_{z \downarrow}
\end{pmatrix} =
\begin{pmatrix}
\cosh \rho_z & - \sinh \rho_z \\
- \sinh \rho_z & \cosh \rho_z \\
1 & 1 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
\omega_\downarrow \\
ck_{x \downarrow} \\
ck_{y \downarrow} \\
ck_{z \downarrow}
\end{pmatrix} =
\begin{pmatrix}
\cosh \rho_z & 0 & - \sinh \rho_z & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & - \sinh \rho_z & \cosh \rho_z
\end{pmatrix}
\begin{pmatrix}
\omega_0 \\
\omega_0 \cosh \rho_z \\
\omega_0 \\
-\omega_0 \sinh \rho_z
\end{pmatrix}
\]

from: Fig. 6.1.3
(modified)
QTforCA
Unit 8 Ch.6
Per-spacetime 4-vector \((\omega_0, \omega_x, \omega_y, \omega_z) = (\omega, ck_x, ck_y, ck_z)\) transformation

\[(a) \text{ Laser frame } \omega_0 \quad (b) \text{ z-(→ Moving) ship}\]

Suppose starlight in lighthouse frame is straight down x-axis: \((\omega_\downarrow, ck_{x\downarrow}, ck_{y\downarrow}, ck_{z\downarrow}) = (\omega_0, -\omega_0, 0, 0)\)

+\(\rho_z\) -rapidity ship frame sees starlight Lorentz transformed to: \((\omega'_\downarrow, ck'_{x\downarrow}, ck'_{y\downarrow}, ck'_{z\downarrow}) = (\omega_0 \cosh \rho_z, -\omega_0, 0, -\omega_0 \sinh \rho_z)\)

\[
\begin{pmatrix}
\omega'_\downarrow \\
ck'_{x\downarrow} \\
ck'_{y\downarrow} \\
cck'_{z\downarrow}
\end{pmatrix} = \begin{pmatrix}
\cosh \rho_z & \cdot & -\sinh \rho_z \\
\cdot & 1 & \cdot \\
\cdot & \cdot & 1 \\
-\sinh \rho_z & \cdot & \cosh \rho_z
\end{pmatrix}\begin{pmatrix}
\omega_\downarrow \\
ck_{x\downarrow} \\
ck_{y\downarrow} \\
cck_{z\downarrow}
\end{pmatrix} = \begin{pmatrix}
\cosh \rho_z & \cdot & -\sinh \rho_z \\
\cdot & 1 & \cdot \\
\cdot & \cdot & 1 \\
-\sinh \rho_z & \cdot & \cosh \rho_z
\end{pmatrix}\begin{pmatrix}
\omega_0 \\
-\omega_0 \\
0 \\
-\omega_0 \sinh \rho_z
\end{pmatrix} = \begin{pmatrix}
\omega_0 \cosh \rho_z \\
-\omega_0 \\
0 \\
-\omega_0 \sinh \rho_z
\end{pmatrix}
\]

After the 4-vector transformation, \(\omega_0 = \omega_\downarrow\) is transverse Doppler shifted to \(\omega_0 \cosh \rho_z\), while \(ck_z = 0\) becomes \(ck'_z = -\omega_0 \sinh \rho_z\).

(The x-component is unchanged: \(ck'_x = -\omega_0 = ck_x\) and so is y-component: \(ck'_y = -\omega_0 = ck_y\).)

\[\text{from:Fig. 6.1.3 (modified)}
\]
\[\text{QTforCA}
\]
\[\text{Unit 8 Ch.6}\]
Per-spacetime 4-vector \((\omega_0, \omega_x, \omega_y, \omega_z) = (\omega, c_kx, ck_y, ck_z)\) transformation

(a) Laser frame \(\omega_0\)

(b) \(z\)-(→ Moving) ship

Suppose starlight in lighthouse frame is straight down x-axis: \((\omega_\downarrow, c_kx_\downarrow, ck_y_\downarrow, ck_z_\downarrow) = (\omega_0, -\omega_0, 0, 0)\) + \(\rho_z\)-rapidity ship frame sees starlight Lorentz transformed to: \((\omega'_\downarrow, c_kx'_\downarrow, ck_y'_\downarrow, ck_z'_\downarrow) = (\omega_0 \cosh \rho_z, -\omega_0, 0, -\omega_0 \sinh \rho_z)\)

\[
\begin{pmatrix}
\omega'_\downarrow \\
c_kx'_\downarrow \\
ck_y'_\downarrow \\
cck_z'_\downarrow
\end{pmatrix} =
\begin{pmatrix}
\cosh \rho_z & -\sinh \rho_z \\
1 & 1 \\
1 & 1 \\
-\sinh \rho_z & \cosh \rho_z
\end{pmatrix}
\begin{pmatrix}
\omega_\downarrow \\
c_kx_\downarrow \\
ck_y_\downarrow \\
cck_z_\downarrow
\end{pmatrix} =
\begin{pmatrix}
\cosh \rho_z & -\sinh \rho_z \\
1 & 1 \\
1 & 1 \\
-\sinh \rho_z & \cosh \rho_z
\end{pmatrix}
\begin{pmatrix}
\omega_0 \\
-\omega_0 \\
0 \\
-\omega_0 \sinh \rho_z
\end{pmatrix} =
\begin{pmatrix}
\omega_0 \cosh \rho_z \\
-\omega_0 \\
0 \\
-\omega_0 \sinh \rho_z
\end{pmatrix}
\]

After the 4-vector transformation, \(\omega_0 = \omega_\downarrow\) is transverse Doppler shifted to \(\omega_0 \cosh \rho_z\), while \(ck_z = 0\) becomes \(ck_z' = -\omega_0 \sinh \rho_z\). (The \(x\)-component is unchanged: \(ck_x' = -\omega_0 = ck_x\) and so is \(y\)-component: \(ck_y' = -\omega_0 = ck_y\).)

from: Fig. 6.1.3 (modified)

QT for CA

Unit 8 Ch.6
**Per-spacetime 4-vector** \((\omega_0, \omega_x, \omega_y, \omega_z) = (\omega, ck_x, ck_y, ck_z)\) transformation

(a) Laser frame \(\omega_0\)

\[
\begin{pmatrix}
\omega_0 \\
\omega_0 \\
\omega_0 \\
\omega_0 \\
\end{pmatrix}
\]

(b) z-(→ Moving) ship

\[
\begin{pmatrix}
\omega_0 \\
\omega_0 e^\rho \\
\omega_0 e^{-\rho} \\
\omega_0 \\
\end{pmatrix}
\]

Suppose starlight in lighthouse frame is straight down x-axis : \(\left(\omega_\downarrow, ck_\downarrow, ck_\downarrow, ck_\downarrow\right) = (\omega_0, -\omega_0, 0, 0)\)

\[+ \rho_z\text{-rapidity ship frame sees starlight Lorentz transformed to } \left(\omega'_\downarrow, ck'_\downarrow, ck'_\downarrow, ck'_\downarrow\right) = (\omega_0 \cosh \rho_z, -\omega_0, 0, -\omega_0 \sinh \rho_z)\]

\[
\begin{pmatrix}
\omega'_\downarrow \\
ck'_{\downarrow} \\
ck'_{\downarrow} \\
ck'_{\downarrow} \\
\end{pmatrix} = \begin{pmatrix}
\cosh \rho_z & -\sinh \rho_z \\
1 & 1 \\
-\sinh \rho_z & \cosh \rho_z \\
\end{pmatrix} \begin{pmatrix}
\omega \downarrow \\
ck_{\downarrow} \\
ck_{\downarrow} \\
ck_{\downarrow} \\
\end{pmatrix} = \begin{pmatrix}
\cosh \rho_z & -\sinh \rho_z \\
1 & 1 \\
-\sinh \rho_z & \cosh \rho_z \\
\end{pmatrix} \begin{pmatrix}
\omega_0 \\
-\omega_0 \\
0 \\
-\omega_0 \sinh \rho_z \\
\end{pmatrix} = \begin{pmatrix}
\omega_0 \cosh \rho_z \\
-\omega_0 \\
0 \\
-\omega_0 \sinh \rho_z \\
\end{pmatrix} = \begin{pmatrix}
\omega_0 \sec \sigma \\
-\omega_0 \\
0 \\
-\omega_0 \tan \sigma \\
\end{pmatrix}
\]

After the 4-vector transformation, \(\omega_0=\omega_\downarrow\) is **transverse Doppler shifted** to \(\omega_0 \cosh \rho_z\), while \(ck_z=0\) becomes \(ck'_z = -\omega_0 \sinh \rho_z\).

(The \(x\)-component is unchanged: \(ck'_x = -\omega_0 = ck_x\) and so is \(y\)-component: \(ck'_y = -\omega_0 = ck_y\).)

Recall hyperbolic invariant to Lorentz transform: \(\omega^2-c^2 k^2 = \omega'^2 - c^2 k'^2 (=0 \text{ for } 1\text{-CW light})\)

*The 4-vector form of this is: \(\omega^2-c^2 \mathbf{k} \cdot \mathbf{k} = \omega'^2 - c^2 \mathbf{k}' \cdot \mathbf{k}' (=0 \ "\ ")*
Fig. 5.10 CW cosmic speedometer:
Geometry of Lorentz boost of counter-propagating waves.
Fig. 5.10 CW cosmic speedometer:
Geometry of Lorentz boost of counter-propagating waves.

If starlight is horizontal right-moving $\mathbf{k} \rightarrow$ wave then ship going $\mathbf{u}$ along $z$-axis sees:

\[
\begin{pmatrix}
\omega' \\
ck'_{x} \\
ck'_{y} \\
ck'_{z}
\end{pmatrix} =
\begin{pmatrix}
\cosh \rho_z & -\sinh \rho_z \\
\cdot & 1 & \cdot \\
\cdot & 1 & \cdot \\
-\sinh \rho_z & \cosh \rho_z
\end{pmatrix}
\begin{pmatrix}
\omega_0 \\
0 \\
0 \\
+\omega_0
\end{pmatrix}
= \omega_0
\begin{pmatrix}
\cosh \rho_z - \sinh \rho_z \\
0 \\
0 \\
-\sinh \rho_z + \cosh \rho_z
\end{pmatrix}
= \omega_0 e^{-\rho_z}
\begin{pmatrix}
0 \\
0 \\
0 \\
-\omega_0 e^{-\rho_z}
\end{pmatrix}
\]

If starlight is horizontal left-moving $\mathbf{k} \leftarrow$ wave then ship going $\mathbf{u}$ along $z$-axis sees:

\[
\begin{pmatrix}
\omega' \\
ck'_{x} \\
ck'_{y} \\
ck'_{z}
\end{pmatrix} =
\begin{pmatrix}
\cosh \rho_z & -\sinh \rho_z \\
\cdot & 1 & \cdot \\
\cdot & 1 & \cdot \\
-\sinh \rho_z & \cosh \rho_z
\end{pmatrix}
\begin{pmatrix}
\omega_0 \\
0 \\
0 \\
-\omega_0
\end{pmatrix}
= \omega_0
\begin{pmatrix}
\cosh \rho_z + \sinh \rho_z \\
0 \\
0 \\
-\sinh \rho_z - \cosh \rho_z
\end{pmatrix}
= \omega_0 e^{+\rho_z}
\begin{pmatrix}
0 \\
0 \\
0 \\
-\omega_0 e^{+\rho_z}
\end{pmatrix}
\]

The usual longitudinal Doppler blue shifts $e^{+\rho_z}$ or Doppler red shifts $e^{-\rho_z}$ appear on both $k$-vector and frequency $\omega_0$. 

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More details of Lorentz boost of North-South-East-West plane-wave 4-vectors \((\omega_0, \omega_x, \omega_y, \omega_z)\)
Thales-like construction of Lorentz boost in 2D and 3D
The spectral ellipsoid
More details of Lorentz boost of North-South-East-West plane-wave 4-vectors ($\omega_0, \omega_x, \omega_y, \omega_z$)

\[ \sigma = 30^\circ = 0.524 \]
\[ \rho = 0.549 \]
\[ e^\rho = \sqrt{3} \]
\[ e^{-\rho} = 1/\sqrt{3} \]

South starlight in lighthouse frame is straight down x-axis: \((\omega_\downarrow, \mathbf{c}_x\downarrow, \mathbf{c}_y\downarrow, \mathbf{c}_z\downarrow) = (\omega_0, -\omega_0, 0, 0)\)

\[ \rho_z\text{-rapidity ship frame sees starlight Lorentz transformed to:} \left( \omega'_\downarrow, \mathbf{c}_x'\downarrow, \mathbf{c}_y'\downarrow, \mathbf{c}_z'\downarrow \right) = \left( \omega_0 \cosh \rho_z, -\omega_0, 0, -\omega_0 \sinh \rho_z \right) \]

\[
\begin{align*}
\begin{pmatrix}
\omega'_\downarrow \\
c_{x'}\downarrow \\
c_{y'}\downarrow \\
c_{z'}\downarrow \\
\end{pmatrix}
&= 
\begin{pmatrix}
\cosh \rho_z & -\sinh \rho_z \\
1 & 0 \\
0 & 1 \\
-\sinh \rho_z & \cosh \rho_z \\
\end{pmatrix}
\begin{pmatrix}
\omega_\downarrow \\
c_{x}\downarrow \\
c_{y}\downarrow \\
c_{z}\downarrow \\
\end{pmatrix}
= 
\begin{pmatrix}
\cosh \rho_z & -\sinh \rho_z \\
1 & 0 \\
0 & 1 \\
-\sinh \rho_z & \cosh \rho_z \\
\end{pmatrix}
\begin{pmatrix}
\omega_0 \\
-\omega_0 \\
0 \\
-\omega_0 \sinh \rho_z \\
\end{pmatrix}

&= 
\begin{pmatrix}
\omega_0 \cosh \rho_z \\
-\omega_0 \\
0 \\
-\omega_0 \sinh \rho_z \\
\end{pmatrix}
\begin{pmatrix}
\omega_0 \sec \sigma \\
-\omega_0 \\
0 \\
-\omega_0 \tan \sigma \\
\end{pmatrix}
\end{align*}
\]
More details of Lorentz boost of North-South-East-West plane-wave 4-vectors \((\omega_0, \omega_x, \omega_y, \omega_z)\)

\[
\sigma = 30^\circ = 0.524 \\
\rho = 0.549 \\
e^\rho = \sqrt{3} \\
e^{-\rho} = 1/\sqrt{3}
\]

\[
u/c = \sin \sigma = 1/2 \\
u/c = \tanh \rho = 1/2
\]

\[
\omega_0 \sec \sigma = \omega_0 \sinh \rho = \omega_0/\sqrt{3}
\]

For ship going \(u = c \tanh \rho\) along \(z\)-axis

**West** starlight \((\omega_0, 0, 0, -\omega_0)\) is **blue shifted** by \(e^\rho = \cosh \rho + \sinh \rho\)

\[
\begin{bmatrix}
\omega' \\
ck'_x \\
ck'_y \\
ck'_z
\end{bmatrix}
= \omega_0
\begin{bmatrix}
\cosh \rho_z + \sinh \rho_z \\
0 \\
0 \\
-\sinh \rho_z - \cosh \rho_z
\end{bmatrix}
= \omega_0 e^{+\rho_z}
\]

**Blue shift factor is** \(e^\rho = \cosh \rho + \sinh \rho = \sec \sigma + \tan \sigma\)

and **East** starlight \((\omega_0, 0, 0, +\omega_0)\) is **red shifted** by \(e^{-\rho} = \cosh \rho - \sinh \rho\)

\[
\begin{bmatrix}
\omega' \\
ck'_x \\
ck'_y \\
ck'_z
\end{bmatrix}
= \omega_0
\begin{bmatrix}
\cosh \rho_z - \sinh \rho_z \\
0 \\
0 \\
-\sinh \rho_z + \cosh \rho_z
\end{bmatrix}
= \omega_0 e^{-\rho_z}
\]

**Red shift factor is** \(e^{-\rho} = \cosh \rho - \sinh \rho = \sec \sigma - \tan \sigma\)
Faster Lorentz boost of
North-South-East-West
plane-wave 4-vectors ($\omega_0, \omega_x, \omega_y, \omega_z$)

Lorentz boost by $\sigma=60^\circ$ or $e^{+\rho}=2+\sqrt{3}$

Lighthouse view ($\omega, c\mathbf{k}$)
of wave-vectors

Ship-frame view ($\omega', c\mathbf{k}'$)
of wave-vectors

$u/c = \sin \sigma = \sqrt{3}/2$
$u/c = \tanh \rho = \sqrt{3}/2$

$\sigma=60^\circ=1.047$
$\rho=1.317$
$e^\rho=2+\sqrt{3}$
$e^{-\rho}=2-\sqrt{3}$

Red shift
$\omega e^{-\rho} = \omega_0(2-\sqrt{3})$

Blue shift
$\omega e^{+\rho} = \omega_0(2+\sqrt{3})$
More details of Lorentz boost of *North-South-East-West plane-wave 4-vectors* \((\omega_0, \omega_x, \omega_y, \omega_z)\)

*Thales-like construction of Lorentz boost in 2D and 3D*

*The spectral ellipsoid*
Faster Lorentz boost of North-South-East-West plane-wave 4-vectors ($\omega_0, \omega_x, \omega_y, \omega_z$)

Lorentz boost by $\sigma=60^\circ$ or $e^\rho=2+\sqrt{3}$

How does Lorentz boost affect vector of arbitrary $\theta$?

Lighthouse view ($\omega, c \mathbf{k}$) of wave-vectors

Ship-frame view ($\omega', c \mathbf{k}'$) of wave-vectors

$u/c = \sin \sigma = \sqrt{3}/2$
$u/c = \tanh \rho = \sqrt{3}/2$

$\omega_0 \tan \sigma = \omega_0 \sinh \rho$
$\omega_0 \sec \sigma = \omega_0 \cosh \rho$

Red shift
$\omega_0 e^\rho = \omega_0 (2+\sqrt{3})$

Blue shift
$\omega_0 e^{-\rho} = \omega_0 (2-\sqrt{3})$

$\sigma=60^\circ=1.047$
$\rho=1.317$
$e^\rho=2+\sqrt{3}$
$e^{-\rho}=2-\sqrt{3}$
Faster Lorentz boost of North-South-East-West plane-wave 4-vectors \((\omega_0, \omega_x, \omega_y, \omega_z)\)

Lorentz boost by \(\sigma=60^\circ\) or \(e^{i\rho} = 2 + \sqrt{3}\)

How does Lorentz boost affect vector of arbitrary \(\theta\)?

Lighthouse view \((\omega, ck)\) of wave-vectors

Ship-frame view \((\omega', ck')\) of wave-vectors

\[
\begin{align*}
\begin{pmatrix}
\omega_0' \\
\omega_x' \\
\omega_y' \\
\omega_z'
\end{pmatrix}
&= \begin{pmatrix}
\cosh \rho_z & -\sinh \rho_z & 0 \\
1 & 0 & 0 \\
-\sinh \rho_z & \cosh \rho_z & 0
\end{pmatrix}
\begin{pmatrix}
\omega_0 \\
\omega_0 \cos \theta \\
0
\end{pmatrix} = \begin{pmatrix}
\cosh \rho_z + \sinh \rho_z \sin \theta \\
\omega_0 \cos \theta \\
-\sinh \rho_z - \cosh \rho_z \sin \theta
\end{pmatrix} = \begin{pmatrix}
\sec \sigma + \tan \sigma \sin \theta \\
\cos \theta \\
-\tan \sigma - \sec \sigma \sin \theta
\end{pmatrix}
\end{align*}
\]

Let lab starlight ray at polar angle \(\theta\) have \(k^\wedge \theta = \omega_0 (1, \cos \theta, 0, -\sin \theta)\). Then ship going \(u\) along \(z\)-axis sees:

\[
\begin{align*}
u/c = \sin \sigma &= \sqrt{3}/2 \\
u/c = \tanh \rho &= \sqrt{3}/2
\end{align*}
\]
Faster Lorentz boost of North-South-East-West plane-wave 4-vectors \((\omega_0, \omega_x, \omega_y, \omega_z)\)

\[ \sigma = 60^\circ \text{ or } e^{i\rho} = 2 + \sqrt{3} \]

How does Lorentz boost affect vector of arbitrary \(\theta\)?

Lighthouse view \((\omega, c\mathbf{k})\) of wave-vectors

Ship-frame view \((\omega', c\mathbf{k}')\) of wave-vectors

Let lab starlight ray at polar angle \(\theta\) have \(\mathbf{k} \uparrow \theta = \omega_0 (1, \cos \theta, 0, -\sin \theta)\). Then ship going \(u\) along \(z\)-axis sees:

\[
\begin{pmatrix}
\omega'_x \\
\omega'_y \\
\omega'_z \\
ck'_x \\
ck'_y \\
ck'_z
\end{pmatrix} =
\begin{pmatrix}
\cosh \rho_z & \cdot & -\sinh \rho_z \\
\cdot & 1 & \cdot \\
\cdot & \cdot & 1 \\
-\sinh \rho_z & \cdot & \cosh \rho_z
\end{pmatrix}
\begin{pmatrix}
\omega_0 \\
\omega_0 \cos \theta \\
0 \\
-\omega_0 \sin \theta
\end{pmatrix} =
\begin{pmatrix}
\cosh \rho_z + \sinh \rho_z \sin \theta \\
\omega_0 \cos \theta \\
0 \\
-\sinh \rho_z - \cosh \rho_z \sin \theta
\end{pmatrix} =
\begin{pmatrix}
\sec \sigma + \tan \sigma \sin \theta \\
\cos \theta \\
0 \\
-\tan \sigma - \sec \sigma \sin \theta
\end{pmatrix}
\]
Faster Lorentz boost of North-South-East-West plane-wave 4-vectors \((\omega_0, \omega_x, \omega_y, \omega_z)\)

\(\sigma = 60^\circ\) or \(e^{\sigma p} = 2 + \sqrt{3}\)

How does Lorentz boost affect vector of arbitrary \(\theta\)?

Lighthouse view \((\omega, c k)\)

of wave-vectors

Ship-frame view \((\omega', c k')\)

of wave-vectors

Let lab starlight ray at polar angle \(\theta\) have \(k' \uparrow \theta = \omega_0 (1, \cos \theta, 0, -\sin \theta)\). Then ship going \(u\) along \(z\)-axis sees:

\[
\begin{pmatrix}
\omega'_{\circ \theta} \\
ck'_{x \uparrow \theta} \\
ck'_{y \uparrow \theta} \\
ck'_{z \uparrow \theta}
\end{pmatrix} =
\begin{pmatrix}
\cosh \rho_z & -\sinh \rho_z & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\sinh \rho_z & \cosh \rho_z & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\omega_0 \\
0 \\
0 \\
-\omega_0 \sin \theta
\end{pmatrix} =
\begin{pmatrix}
\cosh \rho_z + \sinh \rho_z \sin \theta & \cosh \rho_z - \sinh \rho_z \sin \theta & \sec \sigma + \tan \sigma \sin \theta \\
\omega_0 \cos \theta & 0 & \cos \theta \\
0 & 0 & \cosh \rho_z - \sinh \rho_z \sin \theta \\
-\omega_0 \sin \theta & -\omega_0 \sin \theta & -\tan \sigma - \sec \sigma \sin \theta
\end{pmatrix} =
\begin{pmatrix}
\omega_0 \\
\omega_0 \cos \theta \\
0 \\
\omega_0 \cosh \rho_z - \sinh \rho_z \sin \theta
\end{pmatrix}
\]
Faster Lorentz boost of North-South-East-West plane-wave 4-vectors \((\omega_0, \omega_x, \omega_y, \omega_z)\)

Lorentz boost by \(\sigma=60^\circ\) or \(e^{i\rho}=2+\sqrt{3}\)

How does Lorentz boost affect vector of arbitrary \(\theta\)?

Lighthouse view \((\omega, c\mathbf{k})\) of wave-vectors

Ship-frame view \((\omega', c\mathbf{k}')\) of wave-vectors

Let lab starlight ray at polar angle \(\theta\) have \(\mathbf{k}^\uparrow = \omega_0 (1, \cos \theta, 0, -\sin \theta)\). Then ship going \(u\) along \(z\)-axis sees:

\[
\begin{pmatrix}
\omega'_{x}\theta \\
\omega'_{y}\theta \\
\omega'_{z}\theta \\
\end{pmatrix} =
\begin{pmatrix}
\cosh \rho_z & -\sinh \rho_z & \\
1 & 1 & \\
-\sinh \rho_z & \cosh \rho_z & \\
\end{pmatrix}
\begin{pmatrix}
\omega_0 \\
\omega_0 \cos \theta \\
-\omega_0 \sin \theta \\
\end{pmatrix} =
\begin{pmatrix}
\cosh \rho_z + \sinh \rho_z \sin \theta \\
\omega_0 \cos \theta \\
-\sinh \rho_z - \cosh \rho_z \sin \theta \\
\end{pmatrix} =
\begin{pmatrix}
\sec \sigma + \tan \sigma \sin \theta \\
\cos \theta \\
0 \\
\end{pmatrix} =
\begin{pmatrix}
\sec \sigma + \tan \sigma \sin \theta \\
\cos \theta \\
0 \\
\end{pmatrix} =
\begin{pmatrix}
\sec \sigma + \tan \sigma \sin \theta \\
\cos \theta \\
0 \\
\end{pmatrix}.
Faster Lorentz boost of North-South-East-West plane-wave 4-vectors \((\omega_0, \omega_x, \omega_y, \omega_z)\)

\(\text{Lorentz boost by } \sigma = 60^\circ \text{ or } e^{i\rho} = 2 + \sqrt{3}\)

How does Lorentz boost affect vector of arbitrary \(\theta\)?

Lighthouse view \((\omega, c\mathbf{k})\) of wave-vectors

Ship-frame view \((\omega', c\mathbf{k}')\) of wave-vectors

Let lab starlight ray at polar angle \(\theta\) have \(\mathbf{k} \uparrow \theta = \omega_0 (1, \cos \theta, 0, -\sin \theta)\). Then ship going \(u\) along \(z\)-axis sees:

\[
\begin{pmatrix}
\omega'_{\theta} \\
ck'_{x \uparrow \theta} \\
ck'_{y \uparrow \theta} \\
ck'_{z \uparrow \theta}
\end{pmatrix} =
\begin{pmatrix}
cosh \rho_z & -\sinh \rho_z \\
1 & 0 \\
1 & 0 \\
-\sinh \rho_z & \cosh \rho_z
\end{pmatrix}
\begin{pmatrix}
\omega_0 \\
\omega_0 \cos \theta \\
0 \\
-\omega_0 \sin \theta
\end{pmatrix} =
\begin{pmatrix}
cosh \rho_z + \sinh \rho_z \sin \theta \\
\cos \theta \\
0 \\
-\sinh \rho_z - \cosh \rho_z \sin \theta
\end{pmatrix} =
\begin{pmatrix}
\sec \sigma + \tan \sigma \sin \theta \\
\cos \theta \\
0 \\
-\tan \sigma - \sec \sigma \sin \theta
\end{pmatrix}
\]
Space-Time Geometry

Multiply segments by \( \cosh \rho = \sec \sigma = \sqrt{1 - \frac{v^2}{c^2}} \)
to recover dimensions in \((ck, \omega)\) plot

\[
\sqrt{(c-v)(c+v)} = c \sqrt{1 - \frac{v^2}{c^2}} = \text{sech} \rho = c \cos \sigma
\]

\[
v = c \tanh \rho = c \sin \sigma
\]

\[
c \sin \sigma
\]

\[
\]
x-Space-y-Space Plot of wavefronts dropped by CW or PW source moving at $u=4c/5$

Multiply segments by $\cosh \rho = \sec \sigma = 1/\sqrt{1-v^2/c^2}$ to recover dimensions in $(ck, \omega)$ plot.

$\sqrt{(c-v)(c+v)} = c\sqrt{1-v^2/c^2} = \cosh \rho = c \cos \sigma$

$\nabla = 4c/5$

$\nabla = 5c/4$

$\nabla = c$

$\sigma$

$V = c \tanh \rho = c \sin \sigma$

$V = c \sin \sigma$

$c \sin \sigma$

$c - V$

$c + V$

$c$

$V$

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$\sqrt{\frac{c-v}{c+v}} = c \sqrt{1 - \frac{v^2}{c^2}} = c \tanh \rho = c \sin \sigma$

Multiply segments by $\cosh \rho = \sec \sigma = 1/\sqrt{1 - v^2/c^2}$
to recover dimensions in $(ck, \omega)$ plot

$x$-Space-$y$-Space Plot of wavefronts dropped by CW or PW source moving at $u=4c/5$
\[ v/c = \beta = 0.800 \]

Doppler blue shift factor = \( b = 3.001 \)

Doppler red shift factor = \( r = 0.333 \)

\[ v = 0.675 = 38.665^\circ \]

\[ q = 1.099 \]

\[ \sigma = 0.928 = 53.143^\circ \]
Combination and interference of 4-vector plane waves (Idealized polarization case)

Combination group and phase waves define 4D Minkowski coordinates

Combination group and phase waves define wave guide dynamics

Waveguide dispersion and geometry

$1^{st}$-quantized cavity modes

(And introducing $2^{nd}$-quantized cavity modes)
Combination and interference of 4-vector plane waves (Idealized amplitude case)

\[ \Psi_{\rightarrow, \omega \rightarrow, k \rightarrow; \leftarrow, \omega \leftarrow, k \leftarrow}(r, t) = A_{\rightarrow}e^{i(k \cdot r - \omega \rightarrow t)} + A_{\leftarrow}e^{i(k \cdot r - \omega \leftarrow t)} \]

2-CW-single-plane-polarized case: \( \Psi_k(r, t) = e^{i(k \cdot r - \omega \rightarrow t)} + e^{i(k \cdot r - \omega \leftarrow t)} \)  
Idealized: Equal amplitudes and single plane polarization

Factored into phase and group factors:

\[ e^{i(k \cdot r - \omega \rightarrow t)} = e^{i\left(\frac{(k_{\rightarrow} + k_{\leftarrow}) \cdot r - (\omega_{\rightarrow} + \omega_{\leftarrow}) t}{2}\right)}2\cos\left(\frac{2(k_{\rightarrow} - k_{\leftarrow}) \cdot r - (\omega_{\rightarrow} - \omega_{\leftarrow}) t}{2}\right) = e^{i(\bar{K} \cdot r - \bar{\omega} t)}2\cos(\bar{K} \cdot r - \bar{\omega} t) \]

\[ \Phi_{\rightarrow, \omega \rightarrow, k \rightarrow; \leftarrow, \omega \leftarrow, k \leftarrow}(r, t) = \frac{(k_{\rightarrow} + k_{\leftarrow}) \cdot r - (\omega_{\rightarrow} + \omega_{\leftarrow}) t}{2} = \bar{K} \]
\[ \frac{(k_{\rightarrow} - k_{\leftarrow}) \cdot r - (\omega_{\rightarrow} - \omega_{\leftarrow}) t}{2} = \bar{\omega} \]

Fig. 6.1.1 Sketch of a 1-CW-single-plane-polarized plane wavefunction \( \Psi_k(r, t) = Ae^{i\Phi} = Ae^{i(k \cdot r - \omega t)} \) with wavevector \( k \).
Combination and interference of 4-vector plane waves (Idealized amplitude case)

$$\Psi_{A \rightarrow, \omega \rightarrow, k \rightarrow; A \leftarrow, \omega \leftarrow, k \leftarrow} (r, t) = A \rightarrow e^{i(k \cdot r - \omega \cdot t)} + A \leftarrow e^{i(k \cdot r - \omega \cdot t)}$$

2-CW-single-plane-polarized case: $\Psi_k (r, t) = e^{i(k \cdot r - \omega \cdot t)} + e^{i(k \cdot r - \omega \cdot t)}$

Factored into phase and group factors:

$$\frac{e^{i(k \cdot r - \omega \cdot t)}}{2} + \frac{e^{i(k \cdot r - \omega \cdot t)}}{2} = e^{i(\overline{K} \cdot r - \overline{\omega} t)} 2 \cos \left( \frac{k \cdot r - \omega t}{2} \right)$$

Phase $(k, \omega)$

$$\frac{k \cdot r - \omega \cdot t}{2} = \overline{K}$$

Group $(k, \omega)$

$$\frac{k \cdot r - \omega \cdot t}{2} = \overline{\omega}$$

Fig. 6.1.1 Sketch of a 1-CW-single-plane-polarized plane wavefunction $\Psi_k (r, t) = A e^{i \Phi} = A e^{i(k \cdot r - \omega t)}$ with wavevector $k$.

Individual laser 4-vectors reside on light cone or null-invariant.

$$c^2 k' \cdot k' - \omega^2 = c^2 k \cdot k - \omega^2 = c^2 k_0^2 - \omega_0^2 = 0$$

$$c^2 k' \cdot k' - \omega^2 = c^2 k \cdot k - \omega^2 = c^2 k_0^2 - \omega_0^2 = 0$$

from: Fig. 6.1.1

QTforCA

Unit 2 Ch.6

Thursday, March 6, 2014
Combination and interference of 4-vector plane waves (Idealized amplitude case)

\[
\Psi_{A \rightarrow \omega \rightarrow \mathbf{k} \leftarrow \omega \leftarrow \mathbf{k} \leftarrow \omega} (\mathbf{r}, t) = A e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} + A e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}
\]

2-CW-single-plane-polarized case: \( \Psi_k (\mathbf{r}, t) = e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} + e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \), Idealized: Equal amplitudes and single plane polarization

Factored into phase and group factors:

\[
\Psi_k (\mathbf{r}, t) = e^{i\left(\frac{\mathbf{k} \cdot \mathbf{r} - \omega t}{2}\right)} + e^{i\left(\frac{\mathbf{k} \cdot \mathbf{r} - \omega t}{2}\right)}
\]

Phase \((k, \omega)\)
\[
\frac{k_+ + k_-}{2} = \mathbf{K},
\]

Group \((k, \omega)\)
\[
\frac{\omega_+ + \omega_-}{2} = \mathbf{\Omega},
\]

\[
\mathbf{k} = \frac{k_+ - k_-}{2},
\]

\[
\mathbf{\Omega} = \frac{\omega_+ - \omega_-}{2}.
\]

from: Fig. 6.1.1

Fig. 6.1.1 Sketch of a 1-CW-single-plane-polarized plane wavefunction \( \Psi_k (\mathbf{r}, t) = A e^{i\mathbf{k} \cdot \mathbf{r}} = A e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \) with wavevector \( \mathbf{k} \).

Individual laser 4-vectors reside on light cone or null-invariant.

\[
c^2 \mathbf{k}_+ \cdot \mathbf{k}_- - \omega_+^2 = c^2 \mathbf{k}_- \cdot \mathbf{k}_+ - \omega_-^2 = c^2 k_0^2 - \omega_0^2 = 0
\]

\[
c^2 \mathbf{k}_- \cdot \mathbf{k}_+ - \omega_-^2 = c^2 \mathbf{k}_+ \cdot \mathbf{k}_- - \omega_+^2 = c^2 k_0^2 - \omega_0^2 = 0
\]

Sum and difference vectors are not on the light cone.

\[
\mathbf{\Omega}^2 - c^2 \mathbf{K}_+ \cdot \mathbf{K}_- = \mathbf{\Omega}^2 - c^2 \mathbf{K} \cdot \mathbf{K} = \omega_0^2 - 0 = c^2 k_0^2
\]

\[
\mathbf{\bar{\omega}}^2 - c^2 \mathbf{\bar{K}_+} \cdot \mathbf{\bar{K}_-} = \mathbf{\bar{\omega}}^2 - c^2 \mathbf{\bar{K}} \cdot \mathbf{\bar{K}} = 0 - c^2 k_0^2 \cdot \mathbf{k}_0 = -c^2 k_0^2
\]
Combination group and phase define 4D Minkowski coordinates (Idealized amplitude case)

Future work: More efficient mapping Lorentz-Group operators and coordinate frames

Fig. 6.2.1 Examples of sequential relativistic transformations of a tetrad of light wavevectors.
Combination and interference of 4-vector plane waves (Idealized polarization case)

Combination group and phase waves define 4D Minkowski coordinates

Combination group and phase waves define wave guide dynamics

Waveguide dispersion and geometry

1\textsuperscript{st}-quantized cavity modes

(And introducing 2\textsuperscript{nd}-quantized cavity modes)
2-Dimensional wave mechanics: guided waves and dispersion in the “Hall of Mirrors”

Any two or three-dimensional wave will be seen to exceed the \( c \)-limit when it approaches an axis obliquely. It happens for plane waves. The phase velocities along coordinate axes are given by

\[
\begin{align*}
    u_x &= \frac{\omega}{k_x}, \\
    u_y &= \frac{\omega}{k_y}, \\
    u_z &= \frac{\omega}{k_z}.
\end{align*}
\]

Waveguide dispersion and geometry

\[ u_x = \frac{\omega}{k_x} \quad \text{very fast!} \]

\[ u_x \approx \infty \quad \text{as} \quad k_x \to 0 \]
2-Dimensional wave mechanics: guided waves and dispersion in the “Hall of Mirrors”

Any two or three-dimensional wave will be seen to exceed the $c$-limit when it approaches an axis obliquely. It happens for plane waves. The phase velocities along coordinate axes are given by

$$u_x = \frac{\omega}{k_x}, \quad u_y = \frac{\omega}{k_y}, \quad u_z = \frac{\omega}{k_z}.$$ 

Each of the components $(k_x, k_y, k_z)$ must be less than or equal to magnitude $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$. Thus, all the component phase velocities equal or exceed the phase velocity $\frac{\omega}{k}$ which is $c$ for light!

A **water** waves exceeds $c$ if it breaks parallel to shore so 'break-line" moves infinitely fast with $k_x = 0$. 

$\omega$
2-Dimensional wave mechanics: guided waves and dispersion in the “Hall of Mirrors”

Any two or three-dimensional wave will be seen to exceed the \( c \)-limit when it approaches an axis obliquely. It happens for plane waves. The phase velocities along coordinate axes are given by

\[
\begin{align*}
    u_x &= \frac{\omega}{k_x}, \\
    u_y &= \frac{\omega}{k_y}, \\
    u_z &= \frac{\omega}{k_z}.
\end{align*}
\]

Each of the components \((k_x, k_y, k_z)\) must be less than or equal to magnitude \( k = \sqrt{k_x^2 + k_y^2 + k_z^2} \).

Thus, all the component phase velocities equal or exceed the phase velocity \( \frac{\omega}{k} \) which is \( c \) for light!

A water waves exceeds \( c \) if it breaks parallel to shore so 'break-line" moves infinitely fast with \( k_x = 0 \).

Consider 'Hall of Mirrors" with two parallel mirrors on either side of the \( x \)-axis be separated by a distance \( y=W \).

The South wall will be at \( y=-W/2 \) and the North wall at \( y=W/2 \). (\( z \)-axis or "up" is into the page here.)

The Hall should have a floor and ceiling at \( z=\pm H/2 \) as discussed later. Here waves move in \( xy \)-plane only.

Consider \( k^{(+)} = (k^{(+)}_x, k^{(+)}_y, 0) \).

\[
\begin{align*}
    k^{(+)} &= (k^{(+)}_x, k^{(+)}_y, 0) = (k \cos \gamma, k \sin \gamma, 0)
\end{align*}
\]

Suppose input \( k \)-vector \( k^{(+)} \) enters at angle \( +\gamma \).

\( u_x \) approaches \( \infty \) as \( k_x \) approaches 0 very fast!
2-Dimensional wave mechanics: guided waves and dispersion in the “Hall of Mirrors”

Any two or three-dimensional wave will be seen to exceed the c-limit when it approaches an axis obliquely. It happens for plane waves. The phase velocities along coordinate axes are given by

\[ u_x = \frac{\omega}{k_x}, \quad u_y = \frac{\omega}{k_y}, \quad u_z = \frac{\omega}{k_z}. \]

Each of the components \((k_x, k_y, k_z)\) must be less than or equal to magnitude \(k = \sqrt{k_x^2 + k_y^2 + k_z^2}\).

Thus, all the component phase velocities equal or exceed the phase velocity \(\frac{\omega}{k}\) which is \(c\) for light! A water waves exceeds \(c\) if it breaks parallel to shore so "break-line" moves infinitely fast with \(k_x = 0\).

Consider 'Hall of Mirrors" with two parallel mirrors on either side of the x-axis be separated by a distance \(y=W\). The South wall will be at \(y=-W/2\) and the North wall at \(y=W/2\). (z-axis or "up" is into the page here.) The Hall should have a floor and ceiling at \(z=\pm H/2\) as discussed later. Here waves move in \(xy\)-plane only.

![Fig. 6.3.1 "Hall of mirrors" model for an optical wave guide of width \(W\).](image)

\[ \mathbf{k}(+) = (k_x, k_y, 0) = (k \cos \gamma, k \sin \gamma, 0) \]

Suppose input \(\mathbf{k}\)-vector \(\mathbf{k}(+)\) enters at angle \(+\gamma\).

\[ \mathbf{k}(-) = (k(-)_x, k(-)_y, 0) = (k \cos \gamma, -k \sin \gamma, 0) \]

\(y\)-reflected mirror image has \(\mathbf{k}\)-vector \(\mathbf{k}(-)\) at angle \(-\gamma\).
Waveguide dispersion and geometry

2-Dimensional wave mechanics: guided waves and dispersion in the “Hall of Mirrors”

Any two or three-dimensional wave will be seen to exceed the $c$-limit when it approaches an axis obliquely. It happens for plane waves. The phase velocities along coordinate axes are given by

\[ u_x = \frac{\omega}{k_x}, \quad u_y = \frac{\omega}{k_y}, \quad u_z = \frac{\omega}{k_z}. \]

Each of the components $(k_x, k_y, k_z)$ must be less than or equal to magnitude $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$.

Thus, all the component phase velocities equal or exceed the phase velocity $\frac{\omega}{k}$ which is $c$ for light!

A water waves exceeds $c$ if it breaks parallel to shore so 'break-line" moves infinitely fast with $k_x = 0$.

Consider 'Hall of Mirrors" with two parallel mirrors on either side of the $x$-axis be separated by a distance $y=W$.

The South wall will be at $y=-W/2$ and the North wall at $y=W/2$. ($z$-axis or "up" is into the page here.)

The Hall should have a floor and ceiling at $z=\pm H/2$ as discussed later. Here waves move in $xy$-plane only.

![Fig. 6.3.1 "Hall of mirrors" model for an optical wave guide of width $W$.](image)

\[ E(r,t) = \exp i(k^{(+)} \cdot r - \omega t) + \exp i(k^{(-)} \cdot r - \omega t) \]

\[ = \exp i(kx \cos \gamma + ky \sin \gamma - \omega t) + \exp i(kx \cos \gamma - ky \sin \gamma - \omega t) \]

\[ = \exp i(kx \cos \gamma - \omega t) \left[ \exp i(ky \sin \gamma) + \exp i(-ky \sin \gamma) \right] \]

\[ = e^{i(kx \cos \gamma - \omega t)} [2 \cos(ky \sin \gamma)] \]

\[ \text{guide phase wave and group wave} \]

Suppose input $k$-vector $k^{(+)}$ enters at angle $+\gamma$.

\[ k^{(+)} = (k^{(+)}_x, k^{(+)}_y, 0) = (k \cos \gamma, k \sin \gamma, 0) \]

$y$-reflected mirror image has $k$-vector $k^{(-)}$ at angle -$\gamma$.

\[ k^{(-)} = (k^{(-)}_x, k^{(-)}_y, 0) = (k \cos \gamma, -k \sin \gamma, 0) \]
Any two or three-dimensional wave will be seen to exceed the \(c\)-limit when it approaches an axis obliquely. It happens for plane waves. The phase velocities along coordinate axes are given by

\[
\begin{align*}
    u_x &= \frac{\omega}{k_x}, & u_y &= \frac{\omega}{k_y}, & u_z &= \frac{\omega}{k_z}.
\end{align*}
\]

Each of the components \((k_x, k_y, k_z)\) must be less than or equal to magnitude

\[
k = \sqrt{k_x^2 + k_y^2 + k_z^2}.
\]

Thus, all the component phase velocities equal or exceed the phase velocity \(\frac{\omega}{k}\) which is \(c\) for light!

A water waves exceeds \(c\) if it breaks parallel to shore so 'break-line' moves infinitely fast with \(k_x = 0\).

Consider 'Hall of Mirrors' with two parallel mirrors on either side of the \(x\)-axis be separated by a distance \(y=W\). The South wall will be at \(y=-W/2\) and the North wall at \(y=W/2\). \((z\)-axis or "up" is into the page here.) The Hall should have a floor and ceiling at \(z=\pm H/2\) as discussed later. Here waves move in \(xy\)-plane only.

Assume Transverse Electric-mode. It always has \(E\) polarized parallel to \(xz\) plane.

Suppose input \(k\)-vector \(\mathbf{k}(+)\) enters at angle \(+\gamma\).

\[
\mathbf{k}(+) = (k(+)x, k(+)y, 0) = (k \cos \gamma, k \sin \gamma, 0)
\]

\(y\)-reflected mirror image has \(k\)-vector \(\mathbf{k}(-)\) at angle \(-\gamma\).

\[
\mathbf{k}(-) = (k(-)x, k(-)y, 0) = (k \cos \gamma, -k \sin \gamma, 0)
\]

TE boundary conditions make group be zero on metal walls \(y=\pm W/2\).

\[
0 = 2 \cos \left(\frac{W}{W} \sin \gamma\right), \text{ or: } k(W/2) \sin \gamma = \pi/2, \text{ or: } \sin \gamma = \pi/(kW)
\]
Combination and interference of 4-vector plane waves (Idealized polarization case)
Combination group and phase waves define 4D Minkowski coordinates
Combination group and phase waves define wave guide dynamics

Waveguide dispersion and geometry

1st-quantized cavity modes
(And introducing 2nd-quantized cavity modes)
Waveguide dispersion and geometry

Assume T\text{ranverseE}lectric-mode. It always has \textbf{E} polarized parallel to \textit{xz} plane.

Suppose input \textbf{k}-vector \textbf{k}(-) enters at angle +\gamma.
\[
\textbf{k}(+) = (k(+)_x, k(+)_y, 0) = (k \cos \gamma, k \sin \gamma, 0)
\]

\(y\)-reflected mirror image has \textbf{k}-vector \textbf{k}(-) at angle -\gamma.
\[
\textbf{k}(-) = (k(-)_x, k(-)_y, 0) = (k \cos \gamma, -k \sin \gamma, 0).
\]

TE boundary conditions make group be zero on metal walls \(y=\pm W/2\).

\[
0 = 2 \cos (k(W/2) \sin \gamma), \text{ or: } k(W/2) \sin \gamma = \pi/2, \text{ or: } \sin \gamma = \pi/(kW)
\]

Condition \(k(+)_y = k \sin \gamma = \pi/W\) gives dispersion function \(\omega(k_x)\) or \(\omega\) vs. \(k_x\) relation
\[
\omega = \frac{ck}{\sqrt{k_x^2 + k_y^2 + k_z^2}}^{1/2}
\]

---

Fig. 6.3.1 "Hall of mirrors" model for an optical wave guide of width \(W\).

\(\gamma\) = "stellar" angle \(\sigma\)

\(y\)-reflected mirror image has \textbf{k}-vector \textbf{k}(-) at angle -\gamma.
\[
\textbf{k}(-) = (k(-)_x, k(-)_y, 0) = (k \cos \gamma, -k \sin \gamma, 0).
\]

Condition \(k(+)_y = k \sin \gamma = \pi/W\) gives dispersion function \(\omega(k_x)\) or \(\omega\) vs. \(k_x\) relation
\[
\omega = \frac{ck}{\sqrt{k_x^2 + k_y^2 + k_z^2}}^{1/2}
\]
Waveguide dispersion and geometry

Assume \( \text{T} \text{ransverse E} \text{l ectric-mode.} \)

It always has \( \mathbf{E} \) polarized parallel to \( xz \) plane.

Suppose input \( \mathbf{k} \)-vector \( \mathbf{k}(-) \) enters at angle \( +\gamma \).

\[
\mathbf{k}(+)= (k_x(+), k_y(+), 0) = (k \cos \gamma, k \sin \gamma, 0)
\]

\( y \)-reflected mirror image has \( \mathbf{k} \)-vector \( \mathbf{k}(-) \) at angle \( -\gamma \).

\[
\mathbf{k}(-)= (k_x(-), k_y(-), 0) = (k \cos \gamma, -k \sin \gamma, 0)
\]

TE boundary conditions make group be zero on metal walls \( y=\pm W/2 \).

\[
0=2 \cos (k(W/2) \sin \gamma), \text{ or: } k(W/2) \sin \gamma = \pi/2, \text{ or: } \sin \gamma = \pi/(kW)
\]

Condition \( k_x(+) = k \sin \gamma = \pi/W \) gives dispersion function \( \omega(k_x) \) or \( \omega \ vs. \ k_x \) relation

\[
\omega = kc = c(k_x^2 + k_y^2 + k_z^2)^{1/2} = c(k_x^2 + \pi^2/W^2)^{1/2} = \sqrt{c^2k_x^2 + \omega_{cut}^2}
\]

where: \( \omega_{cut} = \pi c/W \).

---

Fig. 6.3.1 "Hall of mirrors" model for an optical waveguide of width \( W \).

**Condition k**(+)\( y = k \sin \gamma = \pi/W \) gives dispersion function \( \omega(k_x) \) or \( \omega \ vs. \ k_x \) relation
Waveguide dispersion and geometry

\[ \mathbf{E}(\mathbf{r},t) = \exp(i\mathbf{k}(+) \cdot \mathbf{r} - \omega t) + \exp(i\mathbf{k}(-) \cdot \mathbf{r} - \omega t) = \exp(i(kx \cos \gamma + ky \sin \gamma - \omega t) + \exp(i(kx \cos \gamma - ky \sin \gamma - \omega t) = \exp(i(kx \cos \gamma - \omega t) [\exp(i(ky \sin \gamma) + \exp(i(-ky \sin \gamma))] = e^{i(kx \cos \gamma - \omega t)} [2\cos(ky \sin \gamma)] \]

Y-reflected mirror image has \( \mathbf{k} \)-vector \( \mathbf{k}(-) \) at angle \(-\gamma\).

\[ \mathbf{k}(-) = (k(-)x, k(-)y, 0) = (k \cos \gamma, -k \sin \gamma, 0) \]

**Fig. 6.3.2** Dispersion function for a fundamental TE waveguide mode

**Waveguide dispersion and geometry**

**Condition** \( k(+)y = k \sin \gamma = \pi/W \) gives dispersion function \( \omega(k_x) \) or \( \omega \) vs. \( k_x \) relation

\[ \omega = kc = c(k_x^2 + ky^2 + k_z^2)^{1/2} = c(k_x^2 + \pi^2/W^2)^{1/2} = \sqrt{c^2k_x^2 + \omega_{cut}^2} \]

where: \( \omega_{cut} = \pi c/W \)

**Fig. 6.3.2** Dispersion function for a fundamental TE waveguide mode
Waveguide dispersion and geometry

\[ \omega = kc = c \sqrt{(k_x^2 + k_y^2 + k_z^2)} = c \sqrt{(k_x^2 + \pi^2/W^2)} = \sqrt{(c^2 k_x^2 + \omega_{cut}^2)} \]

Fig. 6.3.2 Thales geometry of cavity or waveguide mode

\[ \omega = \frac{d\omega}{dk} \]

\[ \omega_{cut-off} = \frac{\pi c}{W} \]

\[ \sigma = \text{stellar ab. angle} \]

\[ \kappa = ck \]

\[ \omega = \omega_{cut-off} = \omega_{cut-off} \]

\[ \omega = \omega_{cut-off} \cos\varphi \]

\[ \omega = \omega_{cut-off} \sec\sigma \]

\[ \lambda_x = \text{wavevector} \]

\[ r = e^{-\rho} \]

\[ \rho = \text{Doppler red-shift} \]

\[ b = e^{+\rho} \]

\[ \text{Doppler blue-shift} \]

\[ \text{Group velocity} \]

\[ \text{Phase velocity} \]

\[ \tau = \sin\varphi = \tan\sigma \]

\[ \varphi = \text{red-shift} \]

\[ \sigma = \text{stellar ab. angle} \]

\[ \kappa = ck \]

\[ \omega = \frac{d\omega}{dk} \]

\[ \omega_{cut-off} = \frac{\pi c}{W} \]

\[ \lambda_x = \text{wavevector} \]

\[ r = e^{-\rho} \]

\[ \rho = \text{Doppler red-shift} \]

\[ b = e^{+\rho} \]

\[ \text{Doppler blue-shift} \]

\[ \text{Group velocity} \]

\[ \text{Phase velocity} \]

\[ \tau = \sin\varphi = \tan\sigma \]

\[ \sigma = \text{red-shift} \]

\[ \kappa = ck \]

\[ \omega = \frac{d\omega}{dk} \]

\[ \omega_{cut-off} = \frac{\pi c}{W} \]

\[ \lambda_x = \text{wavevector} \]

\[ r = e^{-\rho} \]

\[ \rho = \text{Doppler red-shift} \]

\[ b = e^{+\rho} \]

\[ \text{Doppler blue-shift} \]

\[ \text{Group velocity} \]

\[ \text{Phase velocity} \]

\[ \tau = \sin\varphi = \tan\sigma \]

\[ \sigma = \text{red-shift} \]
Fig. 6B.8 Thales geometry of cavity or waveguide mode

(Lecture 28 ends here)
**Fig. 6.3.4** Right moving guide wave with $\sigma = 45^\circ$, $V_{\text{phase}} = \sqrt{2}c$, $V_{\text{group}} = c/\sqrt{2}$.

Waveguide dispersion and geometry

Rare case!
Aberration angle is $\sigma = 45^\circ$

from:Fig. 6.3.4
QTforCA
Unit 2 Ch.6

Thursday, March 6, 2014
Waveguide dispersion and geometry

Fig. 6.3.4 Right moving guide wave with $\sigma = 45^\circ$, $V_{\text{phase}} = \sqrt{2}c$, $V_{\text{group}} = c/\sqrt{2}$.

Rare case!
Aberration angle is $\sigma = 45^\circ$

from: Fig. 6.3.4
QT for CA
Unit 2 Ch. 6
Waveguide dispersion and geometry

Fig. 6.3.5 Guide waves. (a) Higher frequency case: $\sigma = 30^\circ$, $u_x$ (phase) = $c\sqrt{3}/2c$, $u_x$ (group) = $c2/\sqrt{3}$. (b) Lower frequency case: $\sigma = 60^\circ$, $u_x$ (phase) = $2c$, $u_x$ (group) = $c/2$

\[ k_x = \sqrt{(\omega^2/c^2 - \pi^2/W^2)} \]
\[ \omega = kc = \sqrt{c^2 k_x^2 + \omega_{cut}^2} \]

\[ u_x (phase) = \omega / k_x = c\omega / \sqrt{(\omega^2 - \pi^2c^2/W^2)} \]
\[ = c / \cos \gamma = c / \sin \sigma = c \csc \sigma \]

\[ u_x (group) = d\omega / dk_x = ck_x \sqrt{k_x^2 + \pi^2/W^2} \]
\[ = c (\omega^2 - \pi^2c^2/W^2)^{1/2} / \omega = c \cos \gamma = c \sin \sigma \]
Combination and interference of 4-vector plane waves (Idealized polarization case)
Combination group and phase waves define 4D Minkowski coordinates
Combination group and phase waves define wave guide dynamics
Waveguide dispersion and geometry

\[ I^{st}-quantized \text{ cavity modes} \]
\[ (And \ introducing \ 2^{nd}-quantized \text{ cavity modes}) \]
Hall of Mirrors capped by a pair of doors at \( x=0 \) and \( x=L \) becomes a *wave cavity* of length \( L \). The doors demand the wave electric field be zero at \( x \)-boundaries as well as along the walls. New boundary conditions:

\[
k_x = k \cos \gamma = n_x \pi / L \quad (n_x = 1, 2, ...)
\]

Frequency bands are broken into discrete "quantized" values \( \omega n_x n_y \), one for each pair of integers or "quantum numbers" \( n_x \) and \( n_y \).

\[
\omega n_x n_y = kc = c \sqrt{ \left( k_x^2 + k_y^2 + k_z^2 \right) = c \sqrt{ \left( n_x^2 \pi^2 / L^2 + n_y^2 \pi^2 / W^2 \right) }}
\]

*Fig. 6.3.6 Cavity mode dispersion diagram showing overlapping and discrete \( \omega \) and \( k \) values.*

*Fig. 6.3.7 Cavity modes for three lowest quantum numbers*
**Quantized Amplitude Counting “photon” number**

Planck’s relation $E = N\hbar \nu$ began as a tenative axiom to explain low-T light. Then he tried to disavow it! Einstein picked it up in his 1905 paper. Since then its use has grown enormously and continues to amaze, amuse (or bewilder) all who study it. A current view is that it represents the quantization of optical field amplitude. We picture this below as $N$-photon wave states for each box-mode of $m$ wave kinks.
Quantized Amplitude Counting “photon” number

Planck’s relation $E = N\hbar\nu$ began as a tenative axiom to explain low-T light. Then he tried to disavow it! Einstein picked it up in his 1905 paper. Since then its use has grown enormously and continues to amaze, amuse (or bewilder) all who study it. A current view is that it represents the quantization of optical field amplitude. We picture this below as $N$-photon wave states for each box-mode of $m$ wave kinks.

Quantum field definitions have been called “2nd quantization” or “wave-waves”

NOTE: We’re using “false-color” here.

These are the 1st excited or fundamental transition levels

These are the fundamental “zero-point” or “vacuum” levels
Lorentz symmetry effects
How it makes momentum and energy be conserved

A strength (and also, weakness) of CW axioms (1.1-2) is that they are symmetry principles due to the Lorentz-Poincare isotropy of space-time (invariance to space-time translation \( T(\delta, \tau) \) in the vacuum). Operator \( T \) has plane wave eigenfunctions \( \psi_{k,\omega} = A e^{i(kx-\omega t)} \) with roots-of-unity eigenvalues \( e^{i(k\delta-\omega \tau)} \).

\[
\langle \psi_{k,\omega} | T^\dagger = \langle \psi_{k,\omega} | e^{-i(k\delta-\omega \tau)} \tag{5.18a}
\]

\[
T | \psi_{k,\omega} \rangle = e^{i(k\delta-\omega \tau)} \langle \psi_{k,\omega} \rangle \tag{5.18b}
\]

This also applies to 2-part or “2-particle” product states \( \Psi_{K,\Omega} = \psi_{k_1,\omega_1} \psi_{k_2,\omega_2} \) where exponents add \((k,\omega)\)-values of each constituent to \( K=k_1+k_2 \) and \( \Omega=\omega_1+\omega_2 \), and \( T(\delta,\tau) \)-eigenvalues also have that form \( e^{i(K\delta-\Omega \tau)} \).

Matrix \( \langle \Psi'_{K',\Omega'} | U | \Psi_{K,\Omega} \rangle \) of \( T \)-symmetric evolution \( U \) is zero unless \( K'=k'_1+k'_2 = K \) and \( \Omega'=\omega'_1+\omega'_2 = \Omega \).

\[
\langle \Psi'_{K',\Omega'} | U | \Psi_{K,\Omega} \rangle = \langle \Psi'_{K',\Omega'} | T^\dagger(\delta,\tau) U T(\delta,\tau) | \Psi_{K,\Omega} \rangle \quad (\text{if } UT = TU \text{ for all } \delta \text{ and } \tau)
\]

\[
e^{-i(K'\delta-\Omega' \tau)} e^{i(K\delta-\Omega \tau)} \langle \Psi'_{K',\Omega'} | U | \Psi_{K,\Omega} \rangle = 0 \quad \text{unless: } K' = K \text{ and: } \Omega' = \Omega
\]

That’s momentum \((P=hK)\) and energy \((E=hW)\) conservation!