Relativity of lightwaves and Lorentz-Minkowski coordinates IV.

(Ch. 0-3 of Unit 8)

More connections to conventional approach to relativity and old-fashioned formulas

Catching up to light (Coyote finally triumphs! Rest-frame at last.)

The most old-fashioned formula of all: Thales & Euclid means

Galileo wins one! (...in gauge space) That “old-time” relativity (Circa 600BCE- 1905CE)

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Suppose you see two counter-propagating laser beams $\omega_R \rightarrow$ going right and $\omega_L \leftarrow$ going left.

$(\omega_{R \rightarrow}, ck_{R \rightarrow})$ meets $(\omega_{L \leftarrow}, -ck_{L \leftarrow})$

$= (4, +4c)$

$= (1, -1c)$
Catching up to light (Coyote finally triumphs! Rest-frame at last.)

Suppose you see two counter-propagating laser beams $\omega_R \rightarrow$ going right and $\omega_L \leftarrow$ going left.

$Q_1$: How fast do you go to "catch up" to see both as the same color (frequency $\omega$)?

$\left( \omega_{R \rightarrow}, \ c k_{R \rightarrow} \right)$ meets $\left( \omega_{L \leftarrow}, -c k_{L \leftarrow} \right)$

$= \left( 4, +4c \right) \quad = \left( 1, -1c \right)$
Catching up to light (Coyote finally triumphs! Rest-frame at last.)

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$Q_1$: How fast do you go to “catch up” to see both as the same color (frequency $\omega_A$)?

$Q_2$: What is that color (frequency $\omega_A$)?

$\left(\omega_{R \rightarrow}, ck_{R \rightarrow}\right)$ meets $\left(\omega_{L \leftarrow}, -ck_{L \leftarrow}\right)$

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“Jeopardy” answers:

**A1:** How fast is the group velocity?
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“Jeopardy” answers:

**A1:** How fast is the group velocity?

\[
\frac{V_{\text{group}}}{c} = \frac{\omega_{R\rightarrow} - \omega_{L\leftarrow}}{ck_{R\rightarrow} - ck_{L\leftarrow}} = \frac{\omega_{R\rightarrow} - \omega_{L\leftarrow}}{\omega_{R\rightarrow} + \omega_{L\leftarrow}} = \frac{4 - 1}{4 + 1} = \frac{3}{5}
\]
Catching up to light (Coyote finally triumphs! Rest-frame at last.)

Suppose you see two counter-propagating laser beams $\omega_R \rightarrow$ going right and $\omega_L \leftarrow$ going left.

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**A2:** What is the geometric mean of $\omega_R \rightarrow$ and $\omega_L \leftarrow$?
Catching up to light (Coyote finally triumphs! Rest-frame at last.)

Suppose you see two counter-propagating laser beams $\omega_{R\rightarrow}$ going right and $\omega_{L\leftarrow}$ going left.

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A2: What is the geometric mean of $\omega_{R\rightarrow}$ and $\omega_{L\leftarrow}$? 

$$\omega_A = \sqrt{\omega_{R\rightarrow} \cdot \omega_{L\leftarrow}} = \sqrt{4 \cdot 1} = 2$$
Catching up to light (Coyote finally triumphs! Rest-frame at last.)

Suppose you see two counter-propagating laser beams $\omega_{R\rightarrow}$ going right and $\omega_{L\leftarrow}$ going left.

Q1: How fast do you go to “catch up” to see both as the same color (frequency $\omega_A$)?

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$A_2$: What is the geometric mean of $\omega_{R\rightarrow}$ and $\omega_{L\leftarrow}$?

$$\omega_A = \sqrt{\omega_{R\rightarrow} \cdot \omega_{L\leftarrow}} = \sqrt{4 \cdot 1} = 2$$

If you accelerate to $V_{\text{group}} = \frac{3}{5}c$ then you see...

$(\omega_{A\rightarrow}, ck_{A\rightarrow})$ meets $(\omega_{A\leftarrow}, -ck_{A\leftarrow})$

$$= (2, +2c)$$

...a standing wave... (assuming equal amplitudes, coherence, etc.)
Catching up to light (Coyote finally triumphs! Rest-frame at last.)

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$$\left( \omega_A \rightarrow , c k_A \rightarrow \right) \text{ meets } \left( \omega_A \leftarrow , -c k_A \leftarrow \right) = (2, +2c) = (2, -2c)$$

G

P

$\omega_{\text{phase}}$, $ck_{\text{phase}}$

$\omega_{\text{group}}$, $ck_{\text{group}}$

$\omega_{\text{group}}$, $ck_{\text{group}}$

$\omega_{\text{group}}$, $ck_{\text{group}}$

$(2, 0c)$

$(0, 2c)$
Catching up to light (Coyote finally triumphs! Rest-frame at last.)
The most old-fashioned form ula of all: Thales & Euclid means
Galileo wins one! (...in gauge space) That “old-time” relativity (Circa 600BCE- 1905CE)
Euclid’s 3-means (300 BC)
Geometric “heart” of wave mechanics

Thales (580BC) rectangle-in-circle
Relates to wave interference by (Galilean)
phasor angular velocity addition

geometric mean: $2^{1/2}$

$[1 \cdot 4]^{1/2} = 2$

difference mean:

$\frac{1}{2} [4 - 1] = \frac{3}{2}$

(HALF-DIFFERENCE)

arithmetical mean:

$\frac{1}{2} [4 + 1] = \frac{5}{2} = \frac{5}{2}$

(HALF-SUM)

frequency

Linear velocity $V_{group}/c = u/c$

is (HALF-DIFF./HALF-SUM) $= \frac{3}{5}$

Sites for animation:

http://www.uark.edu/ua/pirelli/php/means_1.php
http://www.uark.edu/ua/pirelli/php/half_sum_2.php

Fig. 3.3a Euclidian mean geometry for counter-moving waves of frequency 1 and 4. (300THz units).
Fig. 3.1 Wave phasor addition. (a) Each phasor in a wave array is a sum (b) of two component phasors.

(b) Typical Phasor Sum:
Red phasor B
\[ \psi_B = e^{i\beta} \]

PLUS
\[ \psi_A = e^{i\alpha} \]

Green phasor A

EQUALS: \[ \psi_{A+B} = \psi_A + \psi_B \]

(c) Phasor-relative views
A moves relative to B
Sum: \[ \psi_{A+B} = \psi_A + \psi_B \]

B moves relative to A
Difference: \[ \psi_{A-B} = \psi_A - \psi_B \]

Galileo’s revenge!
Galileo wins one (in gauge space)
Now we use Galilean relativity to add angular velocity, that is frequency \( \omega_a \) and \( \omega_b \), in phasor or “gauge” space. No “c-limit” evident. (So far at 18-fig. precision.)

Sites for animations:
http://www.uark.edu/ua/pirelli/php/means_1.php
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Comparing Ship and Lighthouse views: Happening tables

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<tr>
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<td>(Lighthouse time) $t = 0$</td>
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Fig. 2.A.3 Happening 1 (1st blink hits ship) and 2 (2nd blink at Main) both happen at $t=2$. 

The ship and lighthouse saga
Happening 0.5: Main Lite blinks first time.

| Lighthouse:   | $x = 0$          |
| Lighthouse:   | $t = 1.00$       |
| Ship:         | $x' = 0$         |
| Ship:         | $t' = \Delta = ??$ |

Happening 2: Main Lighthouse blinks second time.

| Lighthouse space | $x = -1.00 c$ |
| Lighthouse time  | $t = 2.00$    |
| Ship space       | $x' = 0$      |
| Ship time        | $t' = 1.75$   |

Comparing Ship and Lighthouse views: Happening tables

**Happening 0:** Ship passes Main Lighthouse.

**Happening 1:** Ship gets hit by first blink from Main Lighthouse.

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Fig. 2A.3 Happening 1 (1st blink hits ship) and 2 (2nd blink at Main) both happen at $t = 2$. 
**The ship and lighthouse saga**

### Happening 0.5:
Main Lite blinks first time.

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### Ship Time t' = Δ = ???

### Happening 0:
Ship passes Main Lighthouse.

- (Lighthouse space): x = 0
- (Lighthouse time): t = 0
- (Ship space): x' = 0
- (Ship time): t' = 0

### Happening 1: Ship gets hit by first blink from Main Lighthouse.

- (Lighthouse space): x = -1.00 c
- (Lighthouse time): t = 2.00
- (Ship space): x' = 0
- (Ship time): t' = 1.75

### Happening 2: Main Lighthouse blinks second time.

- (Lighthouse space): x = 0
- (Lighthouse time): t = 2.00
- (Ship space): x' = c Δ
- (Ship time): t' = 2Δ = 2.30

**Fig. 2.A.3** Happening 1 (1st blink hits ship) and 2 (2nd blink at Main) both happen at t=2.
The ship and lighthouse saga

Happening 0.5:
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Ship Time t' = Δ = ???

\[
c^2 \Delta^2 = c^2 + v^2 \Delta^2 \\
\left(c^2 - v^2\right) \Delta^2 = c^2
\]

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The ship and lighthouse saga

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Ship Time $t' = \Delta = 1/\sqrt{(1 - v^2/c^2)} = \cosh \rho$

$$c^2 \Delta^2 = c^2 + v^2 \Delta^2$$

$$(c^2 - v^2) \Delta^2 = c^2$$

$$\Delta^2 = \frac{c^2}{(c^2 - v^2)} = \frac{1}{1 - \frac{v^2}{c^2}}$$

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Fig. 2.A.3 Happening 1 (1st blink hits ship) and 2 (2nd blink at Main) both happen at $t=2$.  

Friday, February 7, 2014
The ship and lighthouse saga

Happening 0.5: Main Lite blinks first time.

Lighthouse: \( x = 0 \)
Lighthouse: \( t = 1.00 \)

Ship: \( x' = 0 \)
Ship: \( t' = \Delta = 1.15 \)

\[ \Delta = \frac{c^2}{c^2 - v^2} = \frac{1}{1 - \frac{v^2}{c^2}} \]

\[ c^2 \Delta^2 = c^2 + v^2 \Delta^2 \]
\[ (c^2 - v^2) \Delta^2 = c^2 \]

\[ \Delta^2 = \frac{c^2}{c^2 - v^2} = \frac{1}{1 - \frac{v^2}{c^2}} \]

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Fig. 2.A.3 Happening 1 (1st blink hits ship) and 2 (2nd blink at Main) both happen at \( t=2 \).
Ship v/c (rel. to lighthouse) = -0.50

Lighthouse t = 2.00

0th blink wave (From North)

1st blink wave (From North)

1st blink wave (From Main)

0th blink wave (From Main)

Ship

Main Lighthouse

North Lighthouse

Happening 1:
(1st blink wave from Main hits ship)

Happening 2:
(2nd blink happens at Main Lighthouse)

Comparing Ship and Lighthouse views:

Happening tables

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For u/c = 1/2

\[ \Delta = 1/\sqrt{1 - 1/4} = 2/\sqrt{3} = 1.15 \ldots \]

\[ \text{Ship Time } t' = \Delta = 1/\sqrt{(1-v^2/c^2)} = \cosh \rho = 1.15 \]

Fig. 2.A.3 Happening 1 (1st blink hits ship) and 2 (2nd blink at Main) both happen at t = 2.

Friday, February 7, 2014
Ship v/c (rel. to lighthouse) = -0.50

Ship 1.0 -1.0
Main Lighthouse

Lighthouse t = 2.00

North Lighthouse
1st blink wave
From North

0th blink wave
From North

1st blink wave
From North

Happening 1 (1st blink wave from Main hits ship)

Happening 2 (2nd blink happens at Main Lighthouse)

Comparing Ship and Lighthouse views:

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<td>x = 0</td>
</tr>
<tr>
<td>(Lighthouse time) t = 0</td>
<td>t = c/(c-v)</td>
<td>t = 2.00</td>
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<td>x' = 0</td>
<td>x' = 2vΔ</td>
</tr>
<tr>
<td>(Ship time) t' = 0</td>
<td>t' = (v+c)Δ/c</td>
<td>t' = 2Δ</td>
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</table>

Fig. 2A.3 Happening 1 (1st blink hits ship) and 2 (2nd blink at Main) both happen at t=2.

Lecture 24 ended here
That “old-time” relativity (Circa 600BCE- 1905CE)

(“Bouncing-photons” in smoke & mirrors and Thales, again)

The Ship and Lighthouse saga

Light-conic-sections make invariants

A politically incorrect analogy of rotational transformation and Lorentz transformation

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How Minkowski’s space-time graphs help visualize relativity

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Fig. 2.B.5 Space-Space-Time plot of world lines for Lighthouses. North Lighthouse blink waves trace light cones.
That “old-time” relativity (Circa 600BCE- 1905CE)

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Fig. 2.B.1 Town map according to a "tipsy" surveyor.

<table>
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<tr>
<th></th>
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<tbody>
<tr>
<td>(US surveyor)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x = 0$</td>
<td>$x = 0.5$</td>
<td>$x = 0$</td>
</tr>
<tr>
<td>$y = 0$</td>
<td>$y = 1.0$</td>
<td>$y = 1.0$</td>
</tr>
<tr>
<td>(French surveyor)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x' = 0$</td>
<td>$x' = 0$</td>
<td>$x' = -0.45$</td>
</tr>
<tr>
<td>$y' = 0$</td>
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**A politically incorrect analogy of rotational transformation and Lorentz transformation**

Fig. 2.B.1 Town map according to a "tipsy" surveyor.  
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$$\cos \theta = \frac{1}{\sqrt{1 + \frac{b^2}{c^2}}}$$

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$$x' = x \cos \theta - y \sin \theta = \frac{x}{\sqrt{1 + \frac{b^2}{c^2}}} + \frac{-\left(b/c\right)y}{\sqrt{1 + \frac{b^2}{c^2}}}$$

$$y' = x \sin \theta + y \cos \theta = \frac{\left(b/c\right)x}{\sqrt{1 + \frac{b^2}{c^2}}} + \frac{y}{\sqrt{1 + \frac{b^2}{c^2}}}$$
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**Reminder:** Component-based derivation is clumsy!

\[
\begin{align*}
    x &= x' \cos \theta + y' \sin \theta \\
    y &= -x' \sin \theta + y' \cos \theta
\end{align*}
\]

\[
\begin{align*}
    \cos \theta &= \frac{1}{\sqrt{1 + \frac{b^2}{c^2}}} \\
    \sin \theta &= \frac{b / c}{\sqrt{1 + \frac{b^2}{c^2}}}
\end{align*}
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\end{align*}
\]

Instead, use Dirac unit vectors \(|x\rangle, |y\rangle\) and \(|x'\rangle, |y'\rangle\)

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\end{align*}
\]

- **Object 0:** Town Square.
- **Object 1:** Saloon.
- **Object 2:** Gun Shoppe.

**US surveyor**  
\[
\begin{array}{c|c|c}
(US surveyor) & x = 0 & x = 0.5 \\
& y = 0 & y = 1.0 \\
\end{array}
\]

**2nd surveyor**  
\[
\begin{array}{c|c|c}
(2nd surveyor) & x' = 0 & x' = -0.45 \\
& y' = 0 & y' = 0.89 \\
\end{array}
\]

Instead, use Dirac unit vectors \(|x\rangle, |y\rangle\) and \(|x'\rangle, |y'\rangle\).

You may apply (Jacobian) transform matrix:
\[
\begin{pmatrix}
|x\rangle & |x'\rangle \\
|y\rangle & |y'\rangle
\end{pmatrix} =
\begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}
\]

or the inverse (Kajobian) transformation:
\[
\begin{pmatrix}
|x\rangle & |x'\rangle \\
|y\rangle & |y'\rangle
\end{pmatrix} =
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\]

to any vector \(\mathbf{V} = |\mathbf{V}\rangle = |x\rangle\langle x| + |y\rangle\langle y|
\]
\[
= |x'\rangle\langle x'| + |y'\rangle\langle y'|
\]

Friday, February 7, 2014
A politically incorrect analogy of rotational transformation and Lorentz transformation

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Instead, use Dirac unit vectors $|x\rangle,|y\rangle$ and $|x'\rangle,|y'\rangle$

You may apply (Jacobian) transform matrix:

\[
\begin{pmatrix}
\langle x|x' \rangle & \langle x|y' \rangle \\
\langle y|x' \rangle & \langle y|y' \rangle \\
\end{pmatrix}
\begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta \\
\end{pmatrix}
\]

or the inverse (Kajobian) transformation:

\[
\begin{pmatrix}
\langle x'|x \rangle & \langle x'|y \rangle \\
\langle y'|x \rangle & \langle y'|y \rangle \\
\end{pmatrix}
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta \\
\end{pmatrix}
\]

to any vector $V = |V\rangle = |x\rangle \langle x|V\rangle + |y\rangle \langle y|V\rangle$

\[
= |x\rangle \langle x'|V\rangle + |y\rangle \langle y'|V\rangle
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**Reminder:** Component-based derivation is clumsy!

Forget this!! It's too clumsy to generalize to 3D, 4D, ...

Instead, use Dirac unit vectors $|x\rangle,|y\rangle$ and $|x'\rangle,|y'\rangle$.

You may apply (Jacobian) transform matrix:

\[
\begin{pmatrix}
\langle x|x'\rangle & \langle x|y'\rangle \\
\langle y|x'\rangle & \langle y|y'\rangle
\end{pmatrix} = \begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}
\]

or the inverse (Kajobian) transformation:

\[
\begin{pmatrix}
\langle x'|x\rangle & \langle x'|y\rangle \\
\langle y'|x\rangle & \langle y'|y\rangle
\end{pmatrix} = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\]

for any vector $\hat{V} = |V\rangle = \langle x|x\rangle + \langle y|y\rangle$:

\[
\hat{V} = \langle x|x\rangle |x\rangle + \langle y|y\rangle |y\rangle = |x\rangle\langle x| + |y\rangle\langle y|
\]

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(Jacobian) transformation $\{V_x V_y\}$ from $\{V_x V_y\}$:

\[
\begin{pmatrix}
V_x \\
V_y
\end{pmatrix} = \begin{pmatrix}
\langle x|x\rangle & \langle x|y\rangle \\
\langle y|x\rangle & \langle y|y\rangle
\end{pmatrix} \begin{pmatrix}
V_x \\
V_y
\end{pmatrix}
\]

### Terms from the text:
- **X**, **Y**: Cartesian coordinates.
- **X'**, **Y'**: rotated coordinates.
- **V_x**, **V_y**: vector components.
- **|x⟩, ⟨x|**: Dirac notation.
- **e_x, e_y**: unit vectors.
- **θ**: rotation angle.
- **cos θ**, **sin θ**: trigonometric functions.
- **Jacobian**, **Kajobian**: transformation matrices.

\[
\begin{align*}
\cos \theta &= \frac{1}{\sqrt{1 + \frac{b^2}{c^2}}} \\
\sin \theta &= \frac{b/c}{\sqrt{1 + \frac{b^2}{c^2}}}
\end{align*}
\]
**PLEASE!**

Do NOT ever write this:

\[ e_x = |x'\rangle = \cos \theta |x\rangle - \sin \theta |y\rangle \]
\[ e_y = |y'\rangle = \sin \theta |x\rangle + \cos \theta |y\rangle \]

like this:

\[
\begin{pmatrix}
  e_x \\
  e_y
\end{pmatrix} =
\begin{pmatrix}
  |x'\rangle \\
  |y'\rangle
\end{pmatrix} =
\begin{pmatrix}
  \cos \theta & -\sin \theta \\
  \sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
  |x\rangle \\
  |y\rangle
\end{pmatrix}
\]
PLEASE!

Do NOT ever write this:

\[
\begin{align*}
e_x' &= |x'angle = \cos \theta |x\rangle - \sin \theta |y\rangle \equiv \mathbf{R} |x\rangle \\
e_y' &= |y'angle = \sin \theta |x\rangle + \cos \theta |y\rangle \equiv \mathbf{R} |y\rangle
\end{align*}
\]

(This is a useful abstract definition.)

Here is a matrix representation of abstract definitions: \(|x'\rangle \equiv \mathbf{R} |x\rangle, \ |y'\rangle \equiv \mathbf{R} |y\rangle\)

\[
\begin{pmatrix}
V_x & V_x' \\
V_y & V_y'
\end{pmatrix} = \begin{pmatrix}
|x\rangle & |x'\rangle \\
|y\rangle & |y'\rangle
\end{pmatrix} \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}
|x\rangle \\
|y\rangle
\end{pmatrix}
\]
(a) Rotation Transformation and Invariants

\[ x' = x \cos \theta - y \sin \theta = \frac{x}{\sqrt{1 + \left(\frac{b}{c}\right)^2}} + \frac{-\left(\frac{b}{c}\right)y}{\sqrt{1 + \left(\frac{b}{c}\right)^2}} \]

\[ y' = x \sin \theta + y \cos \theta = \frac{\left(\frac{b}{c}\right)x}{\sqrt{1 + \left(\frac{b}{c}\right)^2}} + \frac{y}{\sqrt{1 + \left(\frac{b}{c}\right)^2}} \]

(b) Lorentz Transformation and Invariants

\[ x' = \frac{x}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{-\frac{v}{c} ct}{\sqrt{1 - \frac{v^2}{c^2}}} = x \cosh \rho + y \sinh \rho \]

\[ ct' = \frac{ct}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{\frac{v}{c} x}{\sqrt{1 - \frac{v^2}{c^2}}} = x \sinh \rho + y \cosh \rho \]
That “old-time” relativity (Circa 600BCE- 1905CE)

(“Bouncing-photons” in smoke & mirrors and Thales, again)
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\[ y/x = \tanh \theta = v/c \]

\[ y = \sinh \rho \]
\[ x = \cosh \rho \]

Area \[2 = \frac{1}{2} \text{ base} \times \text{ altitude} - \text{ area under curve} = \frac{1}{2} xy - \int y \, dx \]

The “Area” being calculated is the total Gray Area between hyperbola pairs, X axis, and sloping u-line

2005 Web version:

www.uark.edu/ua/pirelli/php/complexphasisors_1.php
The straight scoop on “angle” and “rapidity” (They both are area!)

\[ y/x = \tanh \theta = v/c \]

\[ y = \sinh \rho \]
\[ x = \cosh \rho \]

\[ \frac{\text{Area}}{2} = \frac{1}{2} \text{base} \times \text{altitude} - \text{area under curve} = \frac{1}{2} xy - \int y \, dx \]

The “Area” being calculated is the total Gray Area between hyperbola pairs, X axis, and sloping u-line

2005 Web version:
www.uark.edu/ua/pirelli/php/complex_phasors_1.php

2014...Web-app versions:
http://www.uark.edu/ua/modphys/markup/RelaWavityWeb.html
The straight scoop on "angle" and "rapidity" (They both are area!)

\[
x = \cosh \theta \\
y = \sinh \theta \\
y/x = \tanh \theta = \frac{v}{c}
\]

\[
A_{\text{area}} = \int_{x}^{y} (x - y) \, dx
\]

The "Area" being calculated is the total Gray Area between hyperbola pairs, X axis, and sloping u-line.

Circular Functions
- \( m_{\angle}(\sigma) = 0.8582 \)
- \( \text{Length}(\sigma) = 0.8582 \)
- \( \text{Area}(\sigma) = 0.8582 \)

Hyperbolic Functions
- \( q = 0.9884 \)
- \( \text{Area}(q) = 0.9884 \)

- \( \sin(\sigma) = 0.7567 \)
- \( \tan(\sigma) = 1.1574 \)
- \( \sec(\sigma) = 1.5295 \)
- \( \tanh(\sigma) = 0.7567 \)
- \( \sinh(\sigma) = 1.1574 \)
- \( \cosh(\sigma) = 1.5295 \)
The straight scoop on “angle” and “rapidity” (They’re area!)

\[ x = \cosh \theta \]
\[ y = \sinh \theta \]
\[ \frac{y}{x} = \tanh \theta = \frac{v}{c} \]

The “Area” being calculated is the total Gray Area between hyperbola pairs, X axis, and sloping u-line

Useful hyperbolic identities

\[
\frac{\text{Area}}{2} = \frac{1}{2} \text{base} \times \text{altitude} - \text{area under curve} = \frac{1}{2} xy - \int y \, dx
\]

\[
\frac{\text{Area}}{2} = \frac{1}{2} \sinh \rho \cosh \rho - \int \sinh \rho \, d(\cosh \rho)
\]

\[
\sinh^2 \rho = \left( \frac{e^\rho - e^{-\rho}}{2} \right)^2 = \frac{1}{4} \left( e^{2\rho} + e^{-2\rho} - 2 \right) = \frac{\cosh 2\rho - 1}{2}
\]

\[
\sinh \rho \cosh \rho = \left( \frac{e^\rho - e^{-\rho}}{2} \right) \left( \frac{e^\rho + e^{-\rho}}{2} \right) = \frac{1}{4} \left( e^{2\rho} - e^{-2\rho} \right) = \frac{1}{2} \sinh 2\rho
\]
The straight scoop on “angle” and “rapidity” (They’re area!)

\[ x = \cosh \theta \]
\[ y = \sinh \theta \]
\[ \frac{y}{x} = \tanh \theta = \frac{v}{c} \]

The “Area” being calculated is the total Gray Area between hyperbola pairs, X axis, and sloping u-line

\[ \text{Area} = \frac{1}{2} \text{base} \cdot \text{altitude} - \text{area under curve} = \frac{1}{2} xy - \int y \, dx \]

\[ \frac{\text{Area}}{2} = \frac{1}{2} \sinh \rho \cosh \rho - \int \sinh \rho \, d(\cosh \rho) \]

\[ \int \cosh a\rho \, d\rho = \frac{1}{a} \sinh a\rho \]

Useful hyperbolic identities

\[ \sinh^2 \rho = \frac{1}{4} \left( e^{2\rho} + e^{-2\rho} - 2 \right) = \frac{1}{4} (e^{2\rho} + e^{-2\rho} - 2) = \frac{\cosh 2\rho - 1}{2} \]

\[ \sinh \theta \cosh \theta = \frac{1}{4} \left( e^{2\theta} - e^{-2\theta} \right) \left( e^{2\theta} + e^{-2\theta} \right) = \frac{1}{4} \left( e^{2\theta} - e^{-2\theta} \right) = \frac{1}{2} \sinh 2\theta \]
The straight scoop on “angle” and “rapidity” (They’re area!)

\[ \theta = \tanh^{-1} \left( \frac{v}{c} \right) \]

\[ y = \sinh \rho \]
\[ x = \cosh \rho \]

\[ \frac{\text{Area}}{2} = \frac{1}{2} \sinh \rho \cosh \rho - \int \sinh \rho \, d(\cosh \rho) \]

The “Area” being calculated is the total Gray Area between hyperbola pairs, X axis, and sloping u-line

Useful hyperbolic identities

\[ \sinh^2 \rho = \frac{\left( e^\rho - e^{-\rho} \right)^2}{2} = \frac{1}{4} \left( e^{2\rho} + e^{-2\rho} - 2 \right) = \frac{\cosh 2\rho - 1}{2} \]

\[ \sinh \rho \cosh \rho = \frac{\left( e^\rho - e^{-\rho} \right) \left( e^\rho + e^{-\rho} \right)}{2} = \frac{1}{4} \left( e^{2\rho} - e^{-2\rho} \right) = \frac{1}{2} \sinh 2\rho \]

\[ \int \cosh a\theta \, d\theta = \frac{1}{a} \sinh a\theta \]

Amazing result: \( \text{Area} = \rho \) is rapidity
That “old-time” relativity (Circa 600BCE- 1905CE)

(“Bouncing-photons” in smoke & mirrors and Thales, again)

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From Lect. 22 p. 27 or eq. (3.6) in Ch. 3 of Unit 2:

Evenson axiom requires geometric Doppler transform: \( e^{\rho_{AB}} \cdot e^{\rho_{BC}} = e^{\rho_{AC}} = e^{\rho_{AB} + \rho_{BC}} \)

Easy to combine frame velocities using rapidity addition:

\[ \rho_{u+v} = \rho_u + \rho_v \]

\[ \frac{v_A}{v_B} = b_{AB} = e^{\rho_{AB}} = 2 \]
\[ \rho_{AB} = \ln(2) = 0.69 \]

\[ \frac{v_B}{v_C} = b_{BC} = e^{\rho_{BC}} = 1/4 \]
\[ \rho_{BC} = \ln(1/4) = -1.38 \]

\[ \frac{v_C}{v_A} = b_{CA} = e^{\rho_{CA}} = 2 \]
\[ \rho_{CA} = \ln(2) = 0.69 \]

\[ \rho_{AB} + \rho_{BC} = \rho_{AC} = -\rho_{CA} \]
\[ 0.69 - 1.38 = -0.69 \]
Galilean velocity addition becomes \textit{rapidity} addition

From Lect. 22 p. 27 or eq. (3.6) in Ch. 3 of Unit 2:

Evenson axiom requires \textit{geometric} Doppler transform: \( e^{\rho_{AB}} \cdot e^{\rho_{BC}} = e^{\rho_{AC}} = e^{\rho_{AB} + \rho_{BC}} \)

Easy to combine frame velocities using \textit{rapidity} addition: \( \rho_{u+v} = \rho_u + \rho_v \)

\[
\frac{u'}{c} = \frac{\tanh(\rho_u + \rho_v)}{c} = \frac{\tanh \rho_u + \tanh \rho_v}{1 + \tanh \rho_u \tanh \rho_v} = \frac{\frac{u}{c} + \frac{v}{c}}{1 + \frac{u}{c} \cdot \frac{v}{c}}
\]

or: \( u' = \frac{u + v}{1 + \frac{u \cdot v}{c^2}} \)

\[\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}\]
**Galilean velocity addition becomes rapidity addition**

*From Lect. 22 p. 27 or eq. (3.6) in Ch. 3 of Unit 2:*

Evenson axiom requires geometric Doppler transform: $e^{\rho_{AB}} \cdot e^{\rho_{BC}} = e^{\rho_{AC}} = e^{\rho_{AB} + \rho_{BC}}$

Easy to combine frame velocities using rapidity addition: \[ \rho_{u+v} = \rho_u + \rho_v \]

\[ \frac{u'}{c} = \tanh(\rho_u + \rho_v) = \frac{\tanh \rho_u + \tanh \rho_v}{1 + \tanh \rho_u \tanh \rho_v} = \frac{u + v}{c + v} \]

or: \[ u' = \frac{u + v}{1 + \frac{u \cdot v}{c^2}} \]

No longer does $(1/2+1/2)c$ equal $(1)c$...

Relativistic result is: \[ \frac{1}{2} + \frac{1}{2} = \frac{1}{1 + \frac{1}{2 \cdot 2}} = \frac{1}{1 + \frac{1}{4}} = \frac{1}{5} c = \frac{4}{5} c \]
**Galilean velocity addition becomes rapidity addition**

From Lect. 22 p. 27 or eq. (3.6) in Ch. 3 of Unit 2:

Evenson axiom requires geometric Doppler transform: \( e^{\rho_{AB} \cdot e^{\rho_{BC}}} = e^{\rho_{AC}} = e^{\rho_{AB} + \rho_{BC}} \)

Easy to combine frame velocities using rapidity addition: 

\[
\frac{u'}{c} = \tanh(\rho_u + \rho_v) = \frac{\tanh \rho_u + \tanh \rho_v}{1 + \tanh \rho_u \tanh \rho_v} = \frac{u + v}{c} \]

or: \[ u' = \frac{u + v}{1 + \frac{u \cdot v}{c^2}} \]

No longer does \((1/2 + 1/2)c\) equal \((1)c\)…

Relativistic result is:

\[
\frac{1 + \frac{1}{2}}{1 + \frac{1}{2}} c = \frac{1}{1 + \frac{1}{4}} c = \frac{1}{5} c = \frac{4}{5} c
\]

…but, \((1/2 + 1)c\) does equal \((1)c\)…

\[
\frac{1 + \frac{1}{2}}{1 + \frac{1}{2}} c = \frac{1}{2 + 1} c = c
\]
<table>
<thead>
<tr>
<th>a) Space-time ( (c\tau', x') ) geometry of CW( \phi )-paths</th>
</tr>
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<tbody>
<tr>
<td>( c \cdot \tau = \lambda' )</td>
</tr>
<tr>
<td>units: ( \lambda_A = c\tau_A = \frac{1}{2} \text{micron} )</td>
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<table>
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</thead>
<tbody>
<tr>
<td>Frequency: ( \nu' = 2\pi \cdot \omega' )</td>
</tr>
<tr>
<td>units: ( \nu_A = 600 \text{THz} )</td>
</tr>
</tbody>
</table>

**Fig. 7 SRQM by R&C**
Fig. 7 SRQM by R&C
Space-time \( (c\tau, x') \) geometry of CW \( \phi \) -paths

- Time-Period \( c\tau = \lambda' \)
- Units: \( \lambda_A = c\tau_A = \frac{1}{2} \text{micron} \)

- "seconds per wave"

Space-Wavelength \( x' \)
- Units: \( \lambda_A = \frac{1}{2} \text{micron} \)

- "meters per wave"

(b) Per-space-time \( (\nu', c\kappa') \) geometry of CW point vectors

- Frequency: \( \nu' = 2\pi \nu' \)
- Units: \( \nu_A = 600\text{THz} \)

- "waves per second"

- Slope: \( V_{\text{group}} = c = \tanh \rho = \frac{3}{5} \)

- Slope: \( V_{\text{phase}} = c = \coth \rho = \frac{5}{3} \)

\[ c\cdot\text{Wave Number} \ c\cdot\kappa' = c\cdot\kappa/2\pi \]
- Units: \( \nu_A = c\cdot\kappa_A = 600\text{THz} \)

- "waves per meter"

- \( e^{-\rho} = \frac{1}{2} \)

- \( e^{+\rho} = 2 \)

\[
\begin{align*}
\nu'_{\text{phase}} &= \frac{\nu' + \nu}{2} = \nu_A \\
\nu'_{\text{group}} &= \frac{\nu' + \nu - \nu}{2} = \nu_A
\end{align*}
\]

\[
\begin{align*}
\nu'_{\text{phase}} &= \nu_A \begin{pmatrix} \cosh \rho \\ \sinh \rho \end{pmatrix} = \nu_A \begin{pmatrix} 5 \\ 4 \end{pmatrix} \\
\nu'_{\text{group}} &= \nu_A \begin{pmatrix} \sinh \rho \\ \cosh \rho \end{pmatrix} = \nu_A \begin{pmatrix} 3 \\ 4 \end{pmatrix}
\end{align*}
\]

\[ \text{Fig. 7 SRQMbyR&C} \]
Fig. 7 SRQMbyR&C
Fig. 7 SRQMbyR&C
Fig. 7 SRQM by R&C
A. Transformations and phase invariance

A laser phasor sketched in Fig.4 should be taken seriously as a gauge of time (clock) and of space (metric ruler) by giving time (wave period $\tau$) and distance (wavelength $\lambda$) in Fig.7a. A time-stamp reading of phase $\phi$ at a particular space-time point should be equal for Alice and Bob in spite of having unequal readings $(x, t)$ and $(x', t')$ for that point and unequal frequency-wavevector readings $(\omega, k)$ and $(\omega', k')$ for a laser group-wave or its phase-wave.

\[
\phi'_{\text{phase}} \equiv k'_{\text{phase}} x' - \omega'_{\text{phase}} t' = k_{\text{phase}} x - \omega_{\text{phase}} t \equiv \phi_{\text{phase}}
\]

\[
\phi'_{\text{group}} \equiv k'_{\text{group}} x' - \omega'_{\text{group}} t' = k_{\text{group}} x - \omega_{\text{group}} t \equiv \phi_{\text{group}}
\]
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$$\phi'_{\text{phase}} \equiv k'_{\text{phase}} x' - \omega'_{\text{phase}} t' = k_{\text{phase}} x - \omega_{\text{phase}} t \equiv \phi_{\text{phase}}$$

$$\phi'_{\text{group}} \equiv k'_{\text{group}} x' - \omega'_{\text{group}} t' = k_{\text{group}} x - \omega_{\text{group}} t \equiv \phi_{\text{group}}$$

Bob’s $(\omega', k')$ components are in (14) and (15). Alice’s $(\omega, k)$ are the same with $\rho = 0$. An Einstein-Lorentz Transformation (ELT) of Bob’s $(x', t')$ to Alice’s $(x, t)$ follows.

$$\phi_{\text{phase}} \equiv x' \frac{\omega_A}{c} \cosh \rho - t' \omega_A \sinh \rho = 0 \cdot x - \omega_A t \quad \Rightarrow \quad ct = \frac{c t'}{\cosh \rho} - x' \sinh \rho$$

$$\phi_{\text{group}} \equiv x' \frac{\omega_A}{c} \sinh \rho - t' \omega_A \cosh \rho = \frac{\omega_A}{c} x - 0 \cdot t \quad \Rightarrow \quad x = -\frac{c t'}{\sinh \rho} + x' \cosh \rho$$

Friday, February 7, 2014
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$$\phi_{\text{phase}} \equiv x' \frac{\omega_A}{c} \cosh \rho - t' \omega_A \sinh \rho = 0 \cdot x - \omega_A t \quad \Rightarrow \quad ct = ct' \cosh \rho - x' \sinh \rho$$

$$\phi_{\text{group}} \equiv x' \frac{\omega_A}{c} \sinh \rho - t' \omega_A \cosh \rho = \frac{\omega_A}{c} x - 0 \cdot t \quad \Rightarrow \quad x = -ct' \sinh \rho + x' \cosh \rho$$

The ELT matrix form and its inverse complete the space-time side of Fig.7.

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \cosh \rho & -\sinh \rho \\ -\sinh \rho & \cosh \rho \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix} \Rightarrow \begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh \rho & +\sinh \rho \\ +\sinh \rho & \cosh \rho \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$
A. Transformations and phase invariance

A laser phasor sketched in Fig. 4 should be taken seriously as a gauge of time (clock) and of space (metric ruler) by giving time (wave period \( \tau \)) and distance (wavelength \( \lambda \)) in Fig. 7a. A time-stamp reading of phase \( \phi \) at a particular space-time point should be equal for Alice and Bob in spite of having unequal readings \((x, t)\) and \((x', t')\) for that point and unequal frequency-wavevector readings \((\omega, k)\) and \((\omega', k')\) for a laser group-wave or its phase-wave.

\[
\begin{align*}
\phi'_{\text{phase}} &\equiv k'_{\text{phase}} x' - \omega'_{\text{phase}} t' = k_{\text{phase}} x - \omega_{\text{phase}} t \equiv \phi_{\text{phase}} \\
\phi'_{\text{group}} &\equiv k'_{\text{group}} x' - \omega'_{\text{group}} t' = k_{\text{group}} x - \omega_{\text{group}} t \equiv \phi_{\text{group}}
\end{align*}
\]

Direct derivation of ELT uses base vectors \( \mathbb{P}' \) and \( \mathbb{G}' \) or \( \mathbb{P} \) and \( \mathbb{G} \) in (14) and (15).

\[
\begin{align*}
\mathbb{P}' &= \begin{pmatrix}
\omega'_{\text{phase}} \\
ck'_{\text{phase}}
\end{pmatrix} = \omega_A \begin{pmatrix}
cosh \rho \\
sinh \rho
\end{pmatrix} = \begin{pmatrix}
\omega_A \\
0
\end{pmatrix} \cosh \rho + \begin{pmatrix}
0 \\
\omega_A
\end{pmatrix} \sinh \rho = \mathbb{P} \cosh \rho + \mathbb{G} \sinh \rho \\
\mathbb{G}' &= \begin{pmatrix}
\omega'_{\text{group}} \\
ck'_{\text{group}}
\end{pmatrix} = \omega_A \begin{pmatrix}
sinh \rho \\
cosh \rho
\end{pmatrix} = \begin{pmatrix}
\omega_A \\
0
\end{pmatrix} \sinh \rho + \begin{pmatrix}
0 \\
\omega_A
\end{pmatrix} \cosh \rho = \mathbb{P} \sinh \rho + \mathbb{G} \cosh \rho
\end{align*}
\]
That “old-time” relativity (Circa 600BCE-1905CE)

(“Bouncing-photons” in smoke & mirrors and Thales, again)

The Ship and Lighthouse saga
Light-conic-sections make invariants

A politically incorrect analogy of rotational transformation and Lorentz transformation

The straight scoop on “angle” and “rapidity” (They’re area!)

Galilean velocity addition becomes rapidity addition

Introducing the “Sin-Tan Rosetta Stone” (Thanks, Thales!)

Introducing the stellar aberration angle $\sigma$ vs. rapidity $\rho$

How Minkowski’s space-time graphs help visualize relativity

Group vs. phase velocity and tangent contacts
Introducing the “Sin-Tan Rosetta Stone”  

NOTE: Angle $\phi$ is now called stellar aberration angle $\sigma$  

---  

Fig. 5.4  
in Unit 8  

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(a) Circular Functions  
(plane geometry)  

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www.uark.edu/ua/pirelli/php/complexphasors_1.php
Introducing the “Sin-Tan Rosetta Stone”

(a) Circular Functions
(plane geometry)

NOTE: Angle $\phi$ is now called stellar aberration angle $\sigma$
Circular Functions

\[ \sin(\sigma) = 0.7567 \]
\[ \tan(\sigma) = 1.1574 \]
\[ \sec(\sigma) = 1.5295 \]
\[ \cos(\sigma) = 0.6538 \]
\[ \cot(\sigma) = 0.8640 \]
\[ \csc(\sigma) = 1.3216 \]

The angle \( \phi \) is now called the stellar aberration angle.
That “old-time” relativity (Circa 600BCE- 1905CE)
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How Minkowski’s space-time graphs help visualize relativity
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Together, rapidity $\rho = \ln b$ and stellar aberration angle $\sigma$ are parameters of relative velocity.

The rapidity $\rho = \ln b$ is based on longitudinal wave Doppler shift $b = e^\rho$ defined by $u/c = \tanh(\rho)$.
At low speed: $u/c \sim \rho$.

The stellar aberration angle $\sigma$ is based on the transverse wave rotation $R = e^{i\sigma}$ defined by $u/c = \sin(\sigma)$.
At low speed: $u/c \sim \sigma$.

Fig. 5.6 Epstein’s cosmic speedometer with aberration angle $\sigma$ and transverse Doppler shift $\cosh \nu$. 
That “old-time” relativity \( (\text{Circa} \ 600\text{BCE-} \ 1905\text{CE}) \)

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How Minkowski’s space-time graphs help visualize relativity

Group vs. phase velocity and tangent contacts
How Minkowski’s space-time graphs help visualize relativity

Note that in Lighthouse frame Happening 1 is simultaneous with Happening 2 at t=2.00sec.

Space-space Animation of Two Relativistic Ships Passing Two

2005 Web versions:
www.uark.edu/ua/pirelli/php/lighthouse_scenarios.php

Friday, February 7, 2014
2014...Web-app versions:

http://www.uark.edu/ua/modphys/markup/RelativItWeb.html
How Minkowski’s space-time graphs help visualize relativity (Here: \( r = \tanh(1/2) = 0.549 \), \( \rho = \tanh(1/2) = 0.55 \), and: \( \sigma = \sin(1/2) = 0.52 \text{ or } 30^\circ \)).

Note that in Lighthouse frame Happening 1 is simultaneous with Happening 2 at \( t = 2.00 \) sec.

...but, in Ship frame Happening 1 is at \( t' = 1.74 \) and Happening 2 is at \( t' = 2.30 \) sec.

Happening 1: Ship 1 is hit by Blink 1
Happening 2: Lighthouse emits Blink 2

Happening 1: Ship 1 is hit by Blink 1
Happening 2: Lighthouse emits Blink 2

(Here: \( \rho = \tanh(1/2) = 0.55 \),
and: \( \sigma = \sin(1/2) = 0.52 \text{ or } 30^\circ \))
How Minkowski’s space-time graphs help visualize relativity

Note that in Lighthouse frame Happening 1 is simultaneous with Happening 2 at t=2.00sec.

...but, in Ship frame Happening 1 is at t' = 1.74 and Happening 2 is at t' = 2.30sec.

That is t' = 2.30 ship time

www.uark.edu/ua/pirelli/php/lighthouse_scenarios.php
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NOTE: Angle $\phi$ is now called stellar aberration angle $\sigma$

(a) Circular Functions

(plane geometry)

(circular arc area
$\phi = 0.8934 = \text{angle}$
$\sin \phi = 0.7792$
$\cos \phi = 0.6267$
$\tan \phi = 1.2433$
$csc \phi = 1.2833$
$sec \phi = 1.5955$
$cot \phi = 0.8043$

(b) Hyperbolic Functions

(spacetime geometry)

(ct (time))

Hyperbolic arc area
$\rho = 1.0434 = \text{rapidity}$
$\sinh \rho = 1.2433$
$cosh \rho = 1.5955$
$tanh \rho = 0.7792$
$csch \rho = 0.8043$
$sech \rho = 0.6267$
$coth \rho = 1.2833$

Fig. 5.4 in Unit 8

2005 Web version:
https://www.uark.edu/ua/pirelli/php/hyper_construct.php
Introducing the “Sin-Tan Rosetta Stone”  

NOTE: Angle $\phi$ is now called stellar aberration angle $\sigma$

(a) Circular Functions  
(plane geometry)

(circular arc area  
$\sigma = 0.8934 = \text{angle}$

$\sin \sigma = 0.7792$

$\cos \sigma = 0.6267$

$tan \sigma = 1.2433$

$csc \sigma = 1.2833$

$sec \sigma = 1.5955$

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Hyperbolic arc area  
$\rho = 1.0434 = \text{rapidity}$

$\sinh \rho = 1.2433$

$\cosh \rho = 1.5955$

$tanh \rho = 0.7792$

$csch \rho = 0.8043$

$sech \rho = 0.6267$

$coth \rho = 1.2833$

2005 Web version:  
https://www.uark.edu/ua/rireli/php/hyper_constrct.php
Hyperbolic Functions

\( q = 1.1714 \)

Area\( (q) = 1.1714 \)
\( \sinh(q) = 1.4582 \)
\( \cosh(q) = 1.7682 \)
\( \tanh(q) = 0.8247 \)
\( \csch(q) = 0.6858 \)
\( \sech(q) = 0.5656 \)
\( \coth(q) = 1.2125 \)

Circular Functions

\( m_\phi(\sigma) = 0.9697 \)
Length\( (\sigma) = 0.9697 \)
Area\( (\sigma) = 0.9697 \)
\( \sin(\sigma) = 0.8247 \)
\( \cos(\sigma) = 0.5656 \)
\( \tan(\sigma) = 1.4582 \)
\( \csc(\sigma) = 1.2125 \)
\( \sec(\sigma) = 1.7682 \)
\( \cot(\sigma) = 0.6858 \)

2014... Web-app versions:
http://www.uark.edu/ua/modphys/markup/RelativityWeb.html
Laser frequency = $B = 2 = 600$ THz
Doppler blue shift factor = $b = 2.005$
Doppler red shift factor = $r = 0.499$
$q = 0.696$

CW Light Axioms
All colors go $c$: $\omega/k = c$ or L&R on diagonals
Time Reversal ($r \leftrightarrow b$): $r = 1/b$

2014...Web-app versions:
http://www.uark.edu/ua/modphys/markup/RelaWavityWeb.html
Frequency $B = 2 = 600 \text{THz}$
plasma blue shift factor $b = 2.005$
plasma red shift factor $r = 0.499$
$0.696$

Light Axioms
All colors go $c$: $\omega/k = c$ or $L&L$ on diagonals
Time Reversal ($r \leftrightarrow b$): $r = 1/b$

$\text{Per-Time (}\omega\text{)}$

$B\sinh(q)$
$B\cosh(q)$

Red shift
Blue shift
Per-Time ($\omega$)

Frequency $= B = 2 = 600\text{THz}$

Doppler blue shift factor $= b = 2.005$

Doppler red shift factor $= r = 0.499$

0.696

Light Axioms

All colors go $c$: $\omega/k = c$ or L&R on diagonals

Time Reversal ($r \leftrightarrow b$): $r = 1/b$

Red shift

Blue shift
\[ v/c = \beta = 0.600 \]
\[ \text{Doppler blue shift factor} = b = 2.000 \]
\[ \text{Doppler red shift factor} = r = 0.500 \]
\[ v = 0.540 = 30.964^\circ \]
\[ q = 0.693 \]
\[ \sigma = 0.644 = 36.870^\circ \]
$v/c = \beta = 0.600$
Doppler blue shift factor = $b = 2.000$
Doppler red shift factor = $r = 0.500$
$v = 0.540 \approx 30.964^\circ$
$q = 0.693$
$\sigma = 0.644 \approx 36.870^\circ$
\[ v/c = \beta = 0.600 \]
Doppler blue shift factor = \( b = 2.000 \)
Doppler red shift factor = \( r = 0.500 \)
\[ \nu = 0.540 = 30.964^\circ \]
\[ q = 0.693 \]
\[ \sigma = 0.644 = 36.870^\circ \]