AMOP Lecture 13
Tue. 4.01 2014

Based on QTCA Lectures 7, 23-25
Group Theory in Quantum Mechanics

Quantum theory of harmonic oscillators $U(1)\subset U(2)\subset U(3)$...

(Int.J.Mol.Sci, 14, 714(2013) p.755-774 , QTCA Unit 7 Ch. 21-22, PSDS - Ch. 8)

Review : *1-D* $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(1) representations*

Review : *2-D Classical and semi-classical harmonic oscillator ABCD-analysis*

U(2) vs R(3): 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

Spinor-complex variable analogies: arithmetic, vector algebra, operator calculus

2-D $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(2) representations and R(3) angular momentum operators*

- *2D-Oscillator basics*
  - *Commutation relations*
  - *Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry*
    - *Anti-commutation relations*
  - *Two-dimensional (or 2-particle) base states: ket-kets and bra-bras*
    - *Outer product arrays*
    - *Entangled 2-particle states*
- *Two-particle (or 2-dimensional) matrix operators*
  - *U(2) Hamiltonian and irreducible representations*
  - *2D-Oscillator eigensolutions*

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Review : *Creation-Destruction* $\mathbf{a}^\dagger\mathbf{a}$ *algebra*

Define $$\mathbf{a} = \frac{(\mathbf{X}+i\mathbf{P})}{\sqrt{\hbar\omega}} = \frac{\left(\sqrt{M\omega}\,\mathbf{x} + i\mathbf{p}/\sqrt{M\omega}\right)}{\sqrt{2\hbar}}$$ *Destruction operator*

and $$\mathbf{a}^\dagger = \frac{(\mathbf{X}-i\mathbf{P})}{\sqrt{\hbar\omega}} = \frac{\left(\sqrt{M\omega}\,\mathbf{x} - i\mathbf{p}/\sqrt{M\omega}\right)}{\sqrt{2\hbar}}$$ *Creation Operator*

Commutation relations between $\mathbf{a} = (\mathbf{X}+i\mathbf{P})/2$ and $\mathbf{a}^\dagger = (\mathbf{X}-i\mathbf{P})/2$ with $\mathbf{X}\equiv\sqrt{M\omega}\mathbf{x}/\sqrt{2}$ and $\mathbf{P}\equiv\mathbf{p}/\sqrt{2M}$:

$$\left[\mathbf{a},\mathbf{a}^\dagger\right] \equiv \mathbf{a}\mathbf{a}^\dagger - \mathbf{a}^\dagger\mathbf{a} = \frac{1}{2\hbar}\left(\sqrt{M\omega}\,\mathbf{x} + i\mathbf{p}/\sqrt{M\omega}\right)\left(\sqrt{M\omega}\,\mathbf{x} - i\mathbf{p}/\sqrt{M\omega}\right) - \frac{1}{2\hbar}\left(\sqrt{M\omega}\,\mathbf{x} - i\mathbf{p}/\sqrt{M\omega}\right)\left(\sqrt{M\omega}\,\mathbf{x} + i\mathbf{p}/\sqrt{M\omega}\right)$$

$$\left[\mathbf{a},\mathbf{a}^\dagger\right] = \frac{2i}{2\hbar}(\mathbf{p}\mathbf{x} - \mathbf{x}\mathbf{p}) = \frac{-i}{\hbar}\left[\mathbf{x},\mathbf{p}\right] = \mathbf{1}$$

$$\left[\mathbf{a},\mathbf{a}^\dagger\right] = \mathbf{1}$$ or $$\mathbf{a}\mathbf{a}^\dagger = \mathbf{a}^\dagger\mathbf{a} + \mathbf{1}$$

$$\left[\mathbf{x},\mathbf{p}\right] \equiv \mathbf{x}\mathbf{p} - \mathbf{p}\mathbf{x} = \hbar i\,\mathbf{1}$$

Review : *Wavefunction creationism (1st Excited state)*

1st excited state wavefunction $\psi_1(x) = \langle x|1\rangle$

$$\langle x|\mathbf{a}^\dagger|0\rangle = \langle x|1\rangle = \psi_1(x)$$

Expanding the creation operator

$$\langle x|\mathbf{a}^\dagger|0\rangle = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{M\omega}\,\langle x|\mathbf{x}|0\rangle - i\langle x|\mathbf{p}|0\rangle/\sqrt{M\omega}\right) = \langle x|1\rangle = \psi_1(x)$$

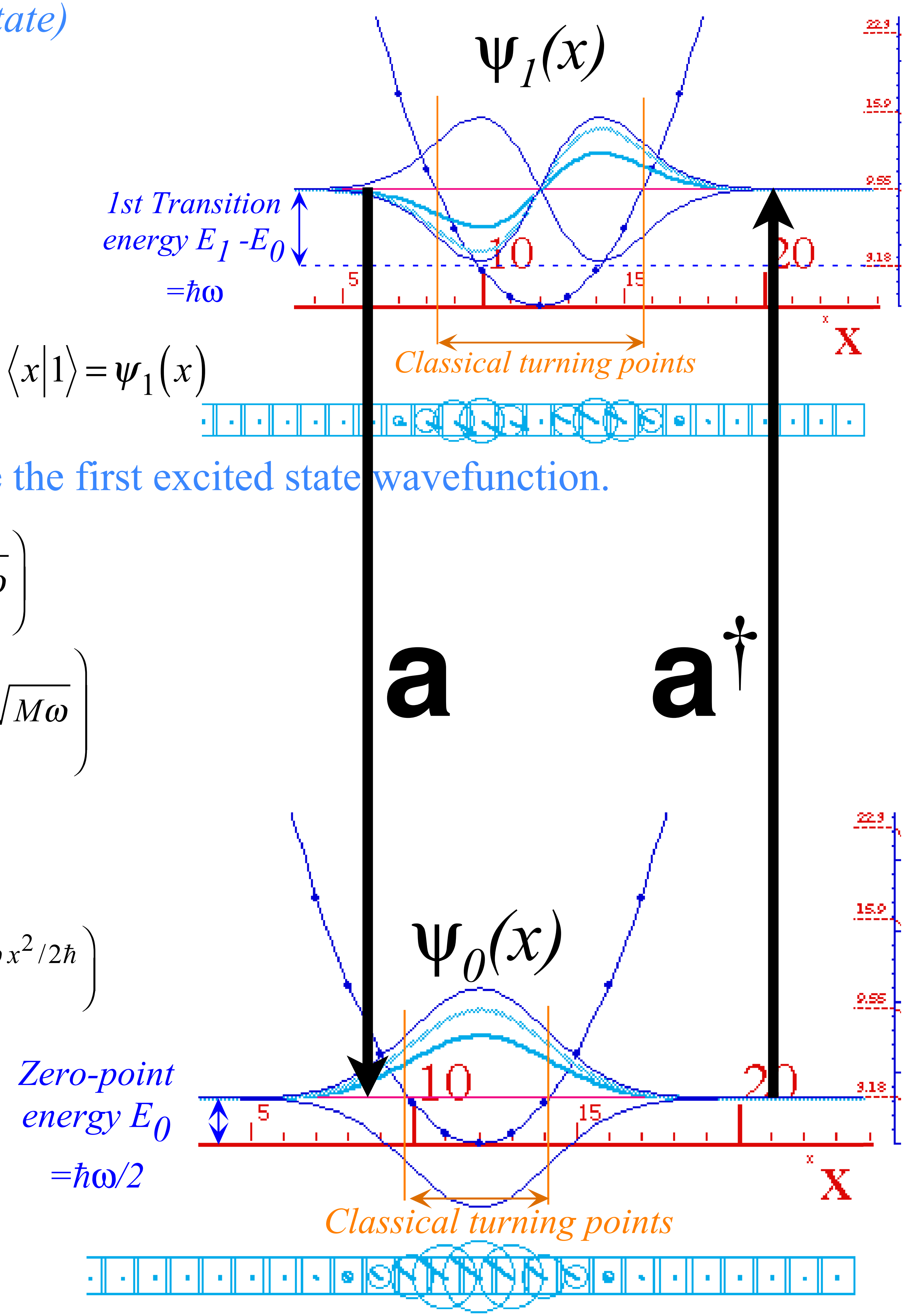
The operator coordinate representations generate the first excited state wavefunction.

$$\langle x|1\rangle = \psi_1(x) = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{M\omega}\,x\psi_0(x) - i\frac{\hbar}{i}\frac{\partial\psi_0(x)}{\partial x}/\sqrt{M\omega}\right)$$

$$= \frac{1}{\sqrt{2\hbar}}\left(\sqrt{M\omega}\,x\frac{e^{-M\omega x^2/2\hbar}}{const.} - i\frac{\hbar}{i}\frac{\partial}{\partial x}\frac{e^{-M\omega x^2/2\hbar}}{const.}/\sqrt{M\omega}\right)$$

$$= \frac{1}{\sqrt{2\hbar}}\frac{e^{-M\omega x^2/2\hbar}}{const.}\left(\sqrt{M\omega}\,x + i\frac{\hbar}{i}\frac{M\omega x}{\hbar}/\sqrt{M\omega}\right)$$

$$= \frac{\sqrt{M\omega}}{\sqrt{2\hbar}}\frac{e^{-M\omega x^2/2\hbar}}{const.}(2x) = \left(\frac{M\omega}{\pi\hbar}\right)^{3/4}\sqrt{2\pi}\left(x\,e^{-M\omega x^2/2\hbar}\right)$$



Review : *Matrix $\langle \mathbf{a}^n \mathbf{a}^{\dagger n} \rangle$ calculation*

Derive normalization for n^{th} state obtained by $(\mathbf{a}^\dagger)^n$ operator:

Use: $\mathbf{a}^n \mathbf{a}^{\dagger n} = n!\left(\mathbf{1} + n\mathbf{a}^\dagger \mathbf{a} + \frac{n(n-1)}{2!\cdot 2!}\mathbf{a}^{\dagger 2}\mathbf{a}^2 + \ldots \right)$

$$|n\rangle = \frac{\mathbf{a}^{\dagger n}|0\rangle}{const.}, \quad \text{where:} \quad 1 = \langle n|n\rangle = \frac{\langle 0|\mathbf{a}^n \mathbf{a}^{\dagger n}|0\rangle}{(const.)^2} = n!\frac{\langle 0|\mathbf{1} + n\mathbf{a}^\dagger \mathbf{a} + ..|0\rangle}{(const.)^2} = \frac{n!}{(const.)^2}$$

$$|n\rangle = \frac{\mathbf{a}^{\dagger n}|0\rangle}{\sqrt{n!}} \quad \textit{Root-factorial normalization}$$

Use: $\mathbf{a}\mathbf{a}^{\dagger n} = n\mathbf{a}^{\dagger n-1} + \mathbf{a}^{\dagger n}\mathbf{a}$

Apply creation $\mathbf{a}^\dagger$:

$$\mathbf{a}^\dagger|n\rangle = \frac{\mathbf{a}^{\dagger n+1}|0\rangle}{\sqrt{n!}} = \sqrt{n+1}\frac{\mathbf{a}^{\dagger n+1}|0\rangle}{\sqrt{(n+1)!}}$$

Apply destruction $\mathbf{a}$:

$$\mathbf{a}|n\rangle = \frac{\mathbf{a}\mathbf{a}^{\dagger n}|0\rangle}{\sqrt{n!}} = \frac{(n\mathbf{a}^{\dagger n-1} + \mathbf{a}^{\dagger n}\mathbf{a})|0\rangle}{\sqrt{n!}} = \sqrt{n}\frac{\mathbf{a}^{\dagger n-1}|0\rangle}{\sqrt{(n-1)!}}$$

$$\mathbf{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \qquad \mathbf{a}|n\rangle = \sqrt{n}|n-1\rangle$$

Feynman's mnemonic rule: Larger of two quanta goes in radical factor

$$\langle \mathbf{a}^\dagger \rangle = \begin{pmatrix} \cdot & & & & & \\ 1 & \cdot & & & & \\ & \sqrt{2} & \cdot & & & \\ & & \sqrt{3} & \cdot & & \\ & & & \sqrt{4} & \cdot & \\ & & & & \ddots & \cdot \end{pmatrix} \qquad \langle \mathbf{a} \rangle = \begin{pmatrix} \cdot & 1 & & & & \\ & \cdot & \sqrt{2} & & & \\ & & \cdot & \sqrt{3} & & \\ & & & \cdot & \sqrt{4} & \\ & & & & \cdot & \ddots \\ & & & & & \cdot \end{pmatrix}$$

Use: $\mathbf{a}\mathbf{a}^{\dagger n} = n\mathbf{a}^{\dagger n-1} + \mathbf{a}^{\dagger n}\mathbf{a}$

Number operator and Hamiltonian operator

Number operator $\mathbf{N}=\mathbf{a}^\dagger\mathbf{a}$ counts quanta.

$$\mathbf{a}^\dagger\mathbf{a}|n\rangle = \frac{\mathbf{a}^\dagger\mathbf{a}\mathbf{a}^{\dagger n}|0\rangle}{\sqrt{n!}} = n\frac{\mathbf{a}^\dagger\mathbf{a}^{\dagger n-1}|0\rangle}{\sqrt{n!}} = n\frac{\mathbf{a}^{\dagger n}|0\rangle}{\sqrt{n!}} = n|n\rangle$$

Hamiltonian operator

$$\mathbf{H}\,|n\rangle = \hbar\omega\, \mathbf{a}^\dagger\mathbf{a}\,|n\rangle + \hbar\omega/2\,\mathbf{1}\,|n\rangle = \hbar\omega(n+1/2)|n\rangle$$

$$\langle \mathbf{H} \rangle = \hbar\omega\langle \mathbf{a}^\dagger\mathbf{a} + \tfrac{1}{2}\mathbf{1}\rangle = \hbar\omega\begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 2 & & \\ & & & 3 & \\ & & & & \ddots \end{pmatrix} + \hbar\omega\begin{pmatrix} 1/2 & & & & \\ & 1/2 & & & \\ & & 1/2 & & \\ & & & 1/2 & \\ & & & & \ddots \end{pmatrix}$$

Hamiltonian operator is $\hbar\omega$ $\mathbf{N}$ plus zero-point energy $\mathbf{1}\hbar\omega/2$.

Review : *Expectation values of position, momentum, and uncertainty for eigenstate $|n\rangle$*

Operator for position $\mathbf{x}$: $\sqrt{\frac{M\omega}{2\hbar}}\mathbf{x} = \frac{\mathbf{a}+\mathbf{a}^\dagger}{2}$

expectation for position $\langle \mathbf{x} \rangle$:

$$\bar{\mathbf{x}}\,|_n = \langle n|\mathbf{x}|n\rangle = \sqrt{\frac{\hbar}{2M\omega}}\langle n|\left(\mathbf{a}+\mathbf{a}^\dagger\right)|n\rangle = 0$$

expectation for (position)^2 $\langle \mathbf{x}^2 \rangle$:

$$\overline{\mathbf{x}^2}\,|_n = \langle n|\mathbf{x}^2|n\rangle = \frac{\hbar}{2M\omega}\langle n|\left(\mathbf{a}+\mathbf{a}^\dagger\right)^2|n\rangle$$

$$= \frac{\hbar}{2M\omega}\langle n|\left(\mathbf{a}^2+\mathbf{a}^\dagger\mathbf{a}+\mathbf{a}\mathbf{a}^\dagger+\mathbf{a}^{\dagger 2}\right)|n\rangle$$

$$= \frac{\hbar}{2M\omega}\quad(2n+1)$$

Use:
$\mathbf{a}\mathbf{a}^\dagger = \mathbf{1}+\mathbf{a}^\dagger\mathbf{a}$

Operator for momentum $\mathbf{p}$: $\sqrt{\frac{1}{2\hbar M\omega}}\,\mathbf{p} = \frac{\mathbf{a}-\mathbf{a}^\dagger}{2i}$

expectation for momentum $\langle \mathbf{p} \rangle$:

$$\bar{\mathbf{p}}\,|_n = \langle n|\mathbf{p}|n\rangle = i\sqrt{\frac{\hbar M\omega}{2}}\langle n|\left(\mathbf{a}^\dagger-\mathbf{a}\right)|n\rangle = 0$$

expectation for (momentum)^2 $\langle \mathbf{p}^2 \rangle$:

$$\overline{\mathbf{p}^2}\,|_n = \langle n|\mathbf{p}^2|n\rangle = i^2\frac{\hbar M\omega}{2}\langle n|\left(\mathbf{a}^\dagger-\mathbf{a}\right)^2|n\rangle$$

$$= -\frac{\hbar M\omega}{2}\langle n|\left(\mathbf{a}^{\dagger 2}-\mathbf{a}^\dagger\mathbf{a}-\mathbf{a}\mathbf{a}^\dagger+\mathbf{a}^2\right)|n\rangle$$

$$= \frac{\hbar M\omega}{2}\quad(2n+1)$$

Uncertainty or *standard deviation* Δq of a statistical quantity q is its root mean-square difference.

$$(\Delta q)^2 = \overline{(q-\bar{q})^2} \qquad \text{or:}\quad \Delta q = \sqrt{\overline{(q-\bar{q})^2}}$$

$$\Delta x|_n = \sqrt{\overline{\mathbf{x}^2}} = \sqrt{\frac{\hbar(2n+1)}{2M\omega}} \qquad\qquad \Delta p|_n = \sqrt{\overline{\mathbf{p}^2}} = \sqrt{\frac{\hbar M\omega(2n+1)}{2}}$$

Heisenberg uncertainty product for the n-quantum eigenstate $|n\rangle$

$$(\Delta x\cdot\Delta p)|_n = \sqrt{\overline{\mathbf{x}^2}}\sqrt{\overline{\mathbf{p}^2}} = \sqrt{\frac{\hbar(2n+1)}{2M\omega}}\sqrt{\frac{\hbar M\omega(2n+1)}{2}}$$

$$\boxed{(\Delta x\cdot\Delta p)|_n = \hbar\left(n+\frac{1}{2}\right)}$$

Heisenberg minimum uncertainty product occurs for the *0*-quantum (ground) eigenstate.

$$\boxed{(\Delta x\cdot\Delta p)|_0 = \frac{\hbar}{2}}$$

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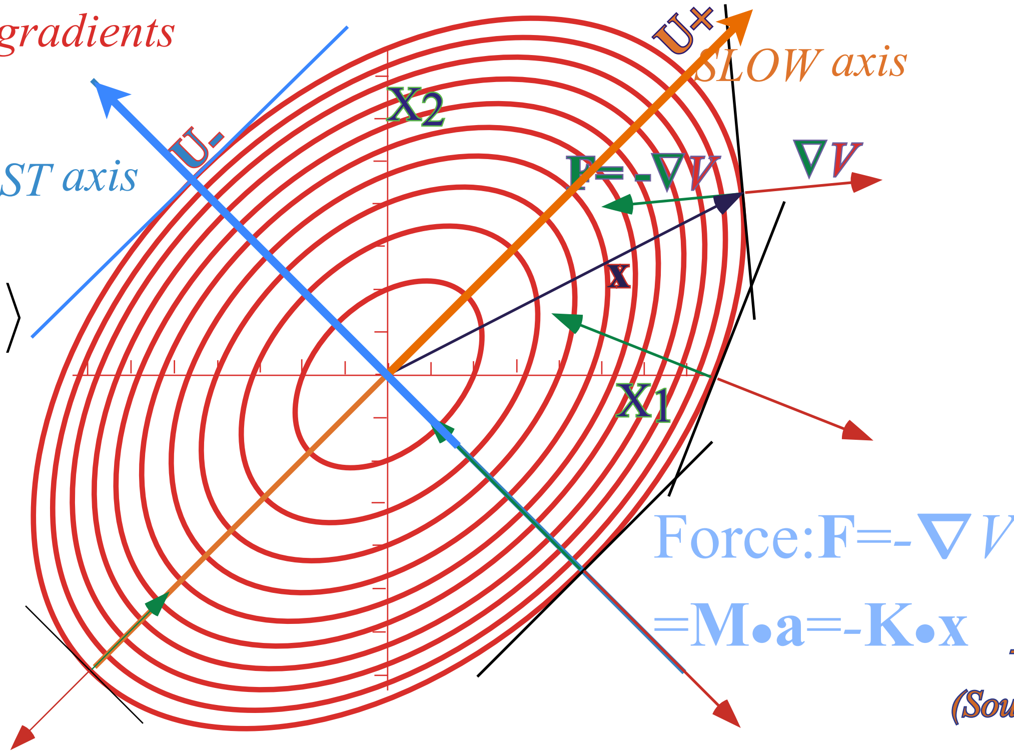
2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical V-contours (Here: $k_1 = k = k_2$)

$$V=\frac{1}{2}(k+k_{12})x_1^2-k_{12}x_1x_2+\frac{1}{2}(k+k_{12})x_2^2=\frac{1}{2}\langle\mathbf{x}|\mathbf{K}|\mathbf{x}\rangle=\frac{1}{2}\mathbf{x}\bullet\mathbf{K}\bullet\mathbf{x}=\begin{pmatrix} x_1 & x_2\end{pmatrix}\begin{pmatrix} k+k_{12} & -k_{12}\\ -k_{12} & k+k_{12}\end{pmatrix}\begin{pmatrix} x_1\\ x_2\end{pmatrix}$$

(a) PE Contours and gradients



Review:

What direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ is the *same* as $\mathbf{K}|\mathbf{x}\rangle$??
Not most directions!
Only extremal axes work. (major or minor axes)

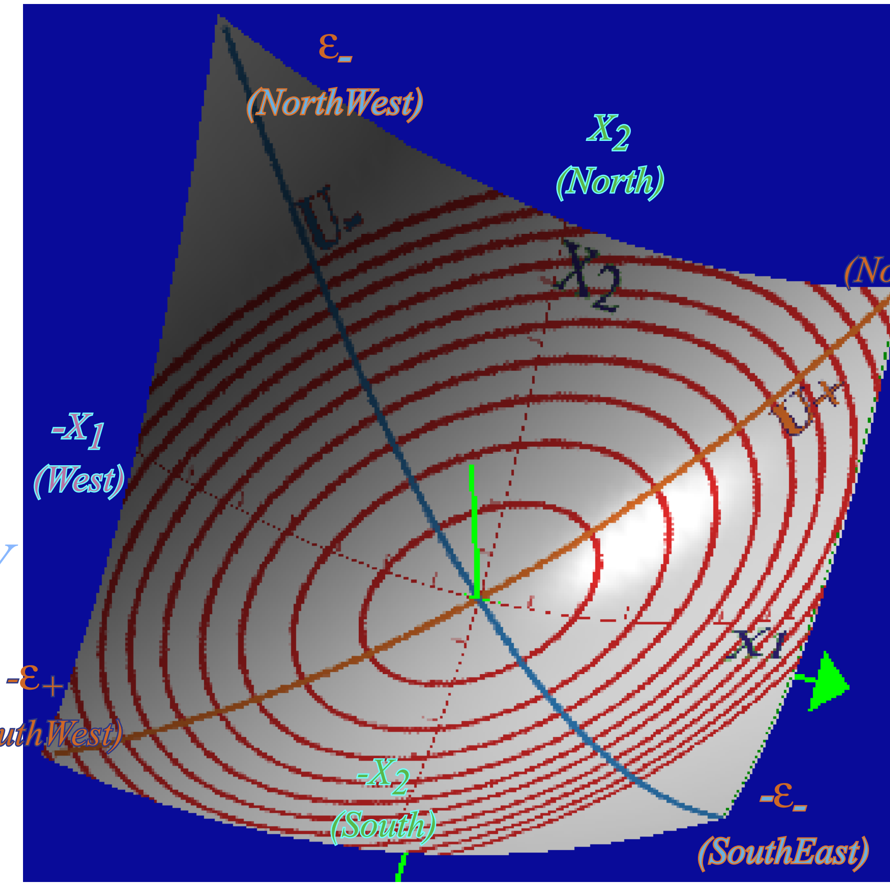
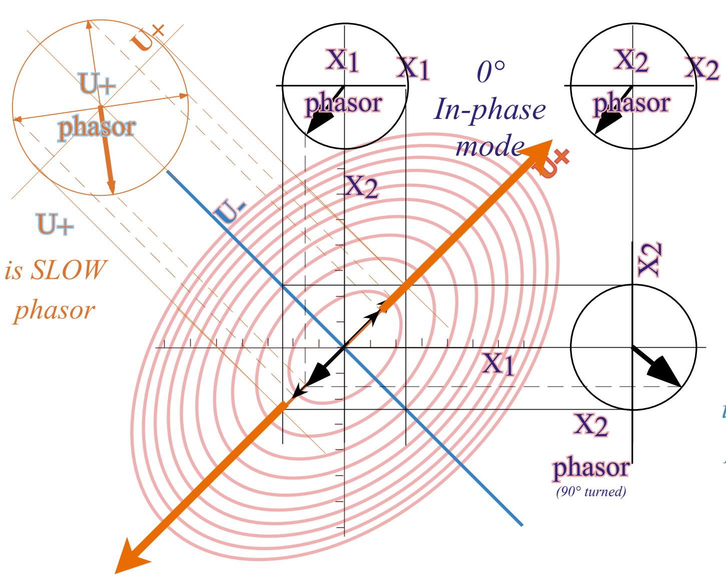
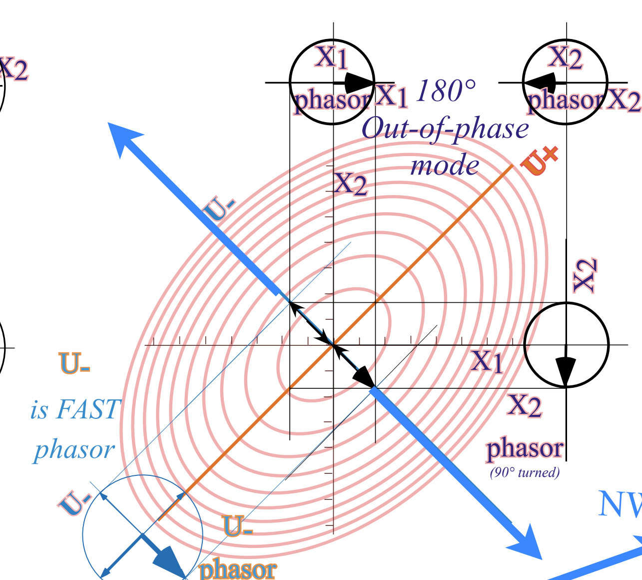


Fig. 3.3.4 Plot of potential function $V(x_1,x_2)$ showing elliptical $V(x_1,x_2)$=const. level curves.

(b) Symmetric U+ Coordinate SLOW Mode



(c) Anti-symmetric U- Coordinate FAST Mode



With Bilateral symmetry ($k_1 = k = k_2$) the extremal axes lie at ±45°

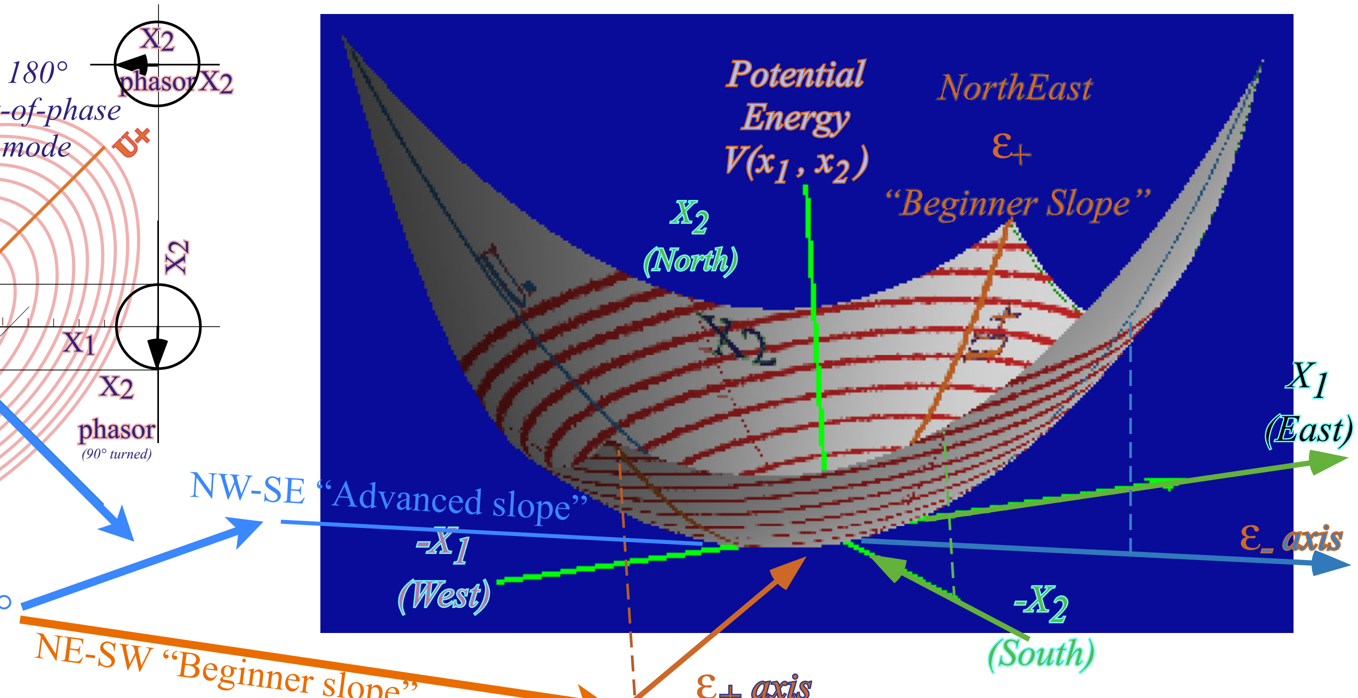
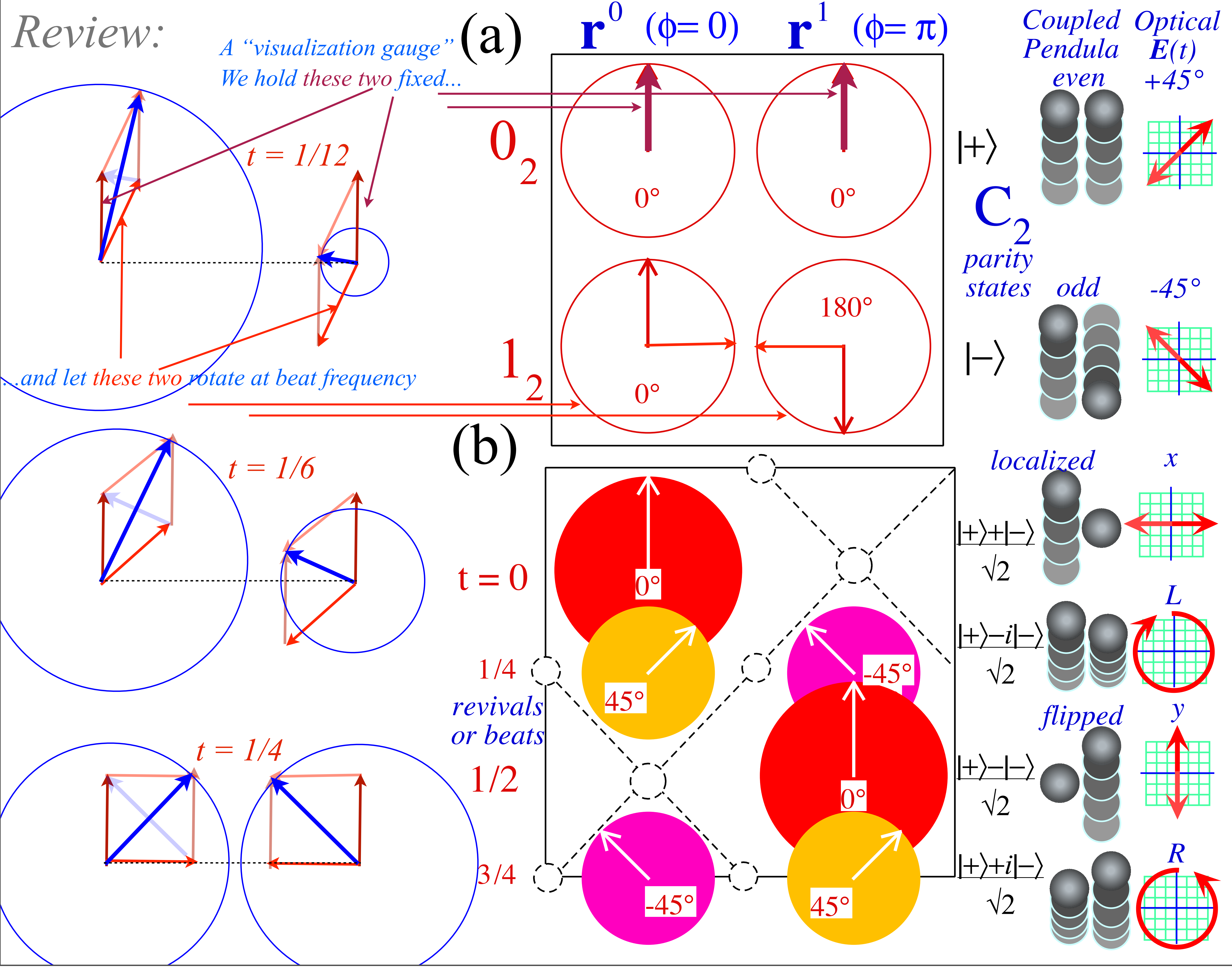


Fig. 3.3.5 Topography lines of potential function $V(x_1,x_2)$ and orthogonal ε_+ and ε_- normal mode slopes

2D-HO beats and mixed mode geometry

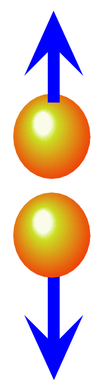


Some of the most famous 2-state systems and their two-complex-component coordinates.

Review:

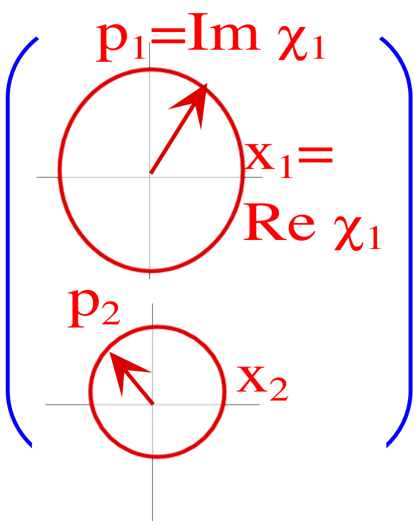
(a) Electron Spin-1/2-Polarization

Spin-up $|1\rangle = |\uparrow\rangle$



Spin-dn $|2\rangle = |\downarrow\rangle$

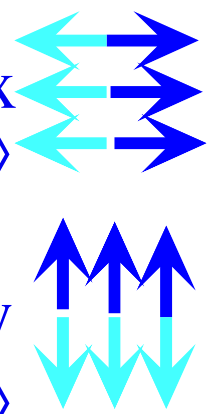
$$|\chi\rangle = \begin{pmatrix} \chi\uparrow \\ \chi\downarrow \end{pmatrix} = \begin{pmatrix} \langle\uparrow|\chi\rangle \\ \langle\downarrow|\chi\rangle \end{pmatrix} =$$



$$= |\uparrow\rangle\langle\uparrow|\Psi\rangle + |\downarrow\rangle\langle\downarrow|\Psi\rangle$$

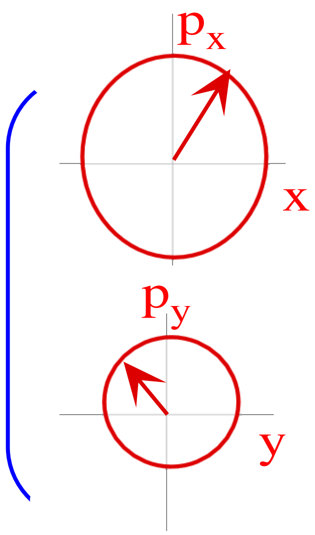
(b) Photon Spin-1-Polarization

Plane-x $|1\rangle = |x\rangle$



Plane-y $|2\rangle = |y\rangle$

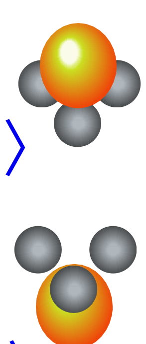
$$|\psi\rangle = \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix} = \begin{pmatrix} \langle x|\psi\rangle \\ \langle y|\psi\rangle \end{pmatrix} =$$



$$= |x\rangle\langle x|\psi\rangle + |y\rangle\langle y|\psi\rangle$$

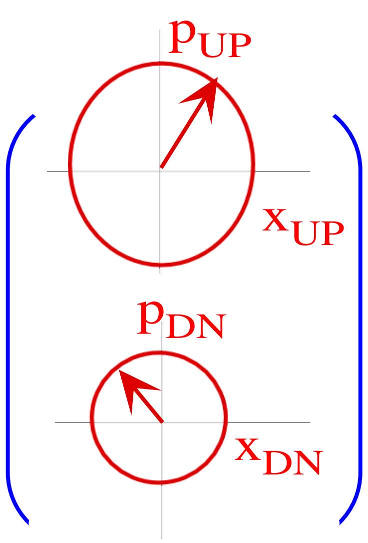
(c) Ammonia (NH_3) Inversion States

N-UP $|1\rangle = |UP\rangle$



N-DN $|2\rangle = |DN\rangle$

$$|\nu\rangle = \begin{pmatrix} \nu_{UP} \\ \nu_{DN} \end{pmatrix} = \begin{pmatrix} \langle UP|\nu\rangle \\ \langle DN|\nu\rangle \end{pmatrix} =$$



$$= |UP\rangle\langle UP|\nu\rangle + |DN\rangle\langle DN|\nu\rangle$$

Fig. 10.5.1
QTCA Unit 3 Chapter 10

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$U(2)$ vs $R(3)$: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ vs. Classical 2D-HO: $\partial^2_t \mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

Review:

$$i\hbar|\dot{\Psi}(t)\rangle = \mathbf{H}|\Psi(t)\rangle \qquad |\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$$

First start with 2-by-2 Hermitian (self-conjugate) matrix

$$\mathbf{H} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H}^\dagger$$

that operates on 2-D complex Dirac ket vector $|\Psi\rangle$.

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Separate real x_k and imaginary p_k parts of Ψ_k amplitudes to convert the complex 1st-order equation $i\partial_t\Psi = \mathbf{H}\Psi$ into pairs of ***real*** 1st-order differential equations.

Then start with classical Hamiltonian. (Designed to give same result.)

$$H_c = \frac{A}{2}\left(p_1^2 + x_1^2\right) + B\left(x_1x_2 + p_1p_2\right) + C\left(x_1p_2 - x_2p_1\right) + \frac{D}{2}\left(p_2^2 + x_2^2\right)$$

Then Hamilton's equations of motion are the following.

$$\dot{x}_1 = \frac{\partial H_c}{\partial p_1} = Ap_1 + Bp_2 - Cx_2 \qquad \dot{p}_1 = -\frac{\partial H_c}{\partial x_1} = -\left(Ax_1 + Bx_2 + Cp_2\right)$$

$$\dot{x}_2 = \frac{\partial H_c}{\partial p_2} = Bp_1 + Dp_2 + Cx_1 \qquad \dot{p}_2 = -\frac{\partial H_c}{\partial x_2} = -\left(Bx_1 + Dx_2 - Cp_1\right)$$

$$\dot{x}_1 = Ap_1 + Bp_2 - Cx_2 \qquad \dot{p}_1 = -Ax_1 - Bx_2 - Cp_2$$

$$\dot{x}_2 = Bp_1 + Dp_2 + Cx_1 \qquad \dot{p}_2 = -Bx_1 - Dx_2 + Cp_1$$

QM vs. Classical Equations are identical

Finally a 2nd time derivative (Assume *constant* A, B, D, and *let* $C=0$) gives 2nd-order classical Newton-Hooke-like equation: $|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$

$$\ddot{x}_1 = A\dot{p}_1 + B\dot{p}_2 - C\dot{x}_2$$
$$= -A\left(Ax_1 + Bx_2 + Cp_2\right) - B\left(Bx_1 + Dx_2 - Cp_1\right) - C\left(Bp_1 + Dp_2 + Cx_1\right)$$
$$= -\left(A^2 + B^2 + C^2\right)x_1 - \left(AB + BD\right)x_2 - C\left(A + D\right)p_2$$

$$\ddot{x}_2 = B\dot{p}_1 + D\dot{p}_2 + C\dot{x}_1$$
$$= -B\left(Ax_1 + Bx_2 + Cp_2\right) - D\left(Bx_1 + Dx_2 - Cp_1\right) + C\left(Ap_1 + Bp_2 - Cx_2\right)$$
$$= -\left(AB + BD\right)x_1 - \left(B^2 + D^2 + C^2\right)x_2 + C\left(A + D\right)p_1$$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} A^2 + B^2 & AB + BD \\ AB + BD & B^2 + D^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

For C=0 Is form of 2D Hooke harmonic oscillator

$$\frac{\partial^2}{\partial t^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \equiv \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Here is an operator view of the QM-Classical connection: Take Schrodinger operator $i\partial_t = \mathbf{H}$ (with $C=0$) and square it!

$$i\frac{\partial}{\partial t} = \begin{pmatrix} A & B \\ B & D \end{pmatrix} \Rightarrow \left(i\frac{\partial}{\partial t}\right)^2 = \begin{pmatrix} A & B \\ B & D \end{pmatrix}^2 \Rightarrow -\frac{\partial^2}{\partial t^2} = \begin{pmatrix} A^2 + B^2 & AB + BD \\ AB + BD & B^2 + D^2 \end{pmatrix}$$

Conclusion: 2-state Schro-equation $i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ is like "square-root" of Newton-Hooke. $\sqrt{\;|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle}$

Review:

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} A^2+B^2 & AB+BD \\ AB+BD & B^2+D^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

For C=0
Is form of 2D Hooke harmonic oscillator

$$\frac{\partial^2}{\partial t^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \equiv \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Here is an operator view of the QM-Classical connection: Take Schrodinger operator $i\partial_t = \mathbf{H}$ (with $C=0$) and square it!

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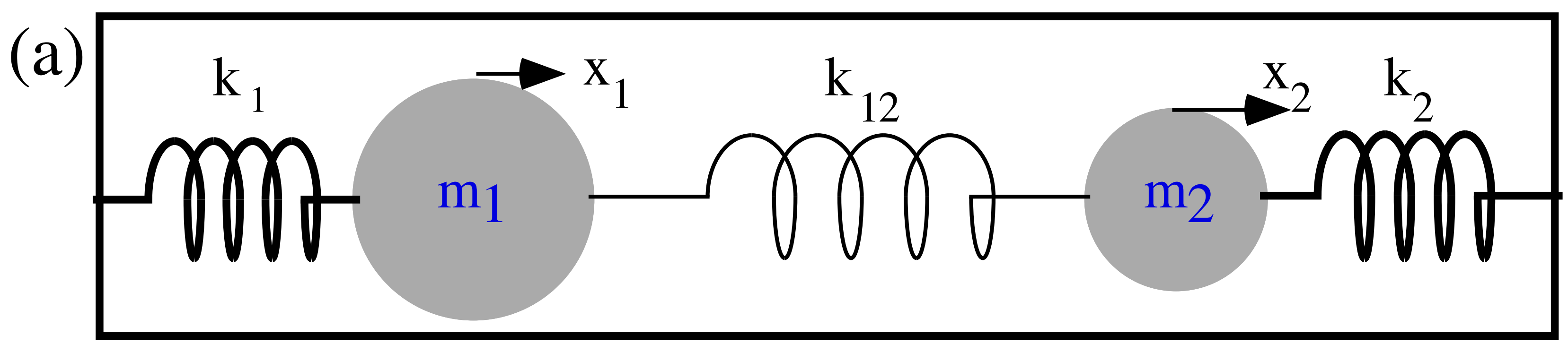
Conclusion: 2-state Schro-equation $i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ is like "square-root" of Newton-Hooke. $\sqrt{\;|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle}$

$$i\frac{\partial}{\partial t} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} \Rightarrow \left(i\frac{\partial}{\partial t}\right)^2 = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}^2 \Rightarrow -\frac{\partial^2}{\partial t^2} = \begin{pmatrix} A^2+B^2+C^2 & AB+BD-iAC-iCD \\ AB+BD+iAC+iCD & B^2+C^2+D^2 \end{pmatrix}$$

General case for $C \neq 0$

$$-\ddot{x}_1 = K_{11}x_1 + K_{12}x_2$$
$$-\ddot{x}_2 = K_{21}x_1 + K_{22}x_2$$



$$m_1K_{11} = A^2+B^2 = k_1+k_{12}, \qquad m_1K_{12} = AB+BD = -k_{12},$$
$$m_2K_{21} = AB+BD = -k_{12}, \qquad m_2K_{22} = B^2+D^2 = k_2+k_{12}.$$

Review : *1-D* $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(1) representations*

2-D Classical and semi-classical harmonic oscillator ABCD-analysis

U(2) vs R(3): 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

2-D $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(2) representations and R(3) angular momentum operators*

2D-Oscillator basics

Commutation relations

Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry

Anti-commutation relations

Two-dimensional (or 2-particle) base states: ket-kets and bra-bras

Outer product arrays

Entangled 2-particle states

Two-particle (or 2-dimensional) matrix operators

U(2) Hamiltonian and irreducible representations

2D-Oscillator eigensolutions

ABCD Symmetry operator analysis and U(2) spinors

Decompose the Hamiltonian operator **H** into four *ABCD symmetry operators*
(Labeled to provide dynamic mnemonics as well as colorful analogies)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + D\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = A\mathbf{e}_{11} + B\boldsymbol{\sigma}_B + C\boldsymbol{\sigma}_C + D\mathbf{e}_{22}$$

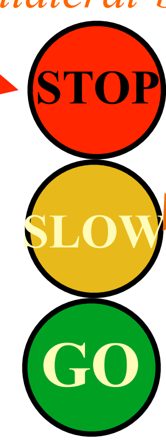
Review:

$$= \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{H} = \frac{A-D}{2}\boldsymbol{\sigma}_A + B\,\boldsymbol{\sigma}_B + C\,\boldsymbol{\sigma}_C + \frac{A+D}{2}\boldsymbol{\sigma}_0$$

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (complex, circular, chiral, cyclotron, Coriolis, centrifugal, curly, and circulating-current-carrying...)*

Motivation for coloring scheme:
The Traffic Signal



C≠0: Moving waves or "Galloping" waves

Standing waves *C=0*

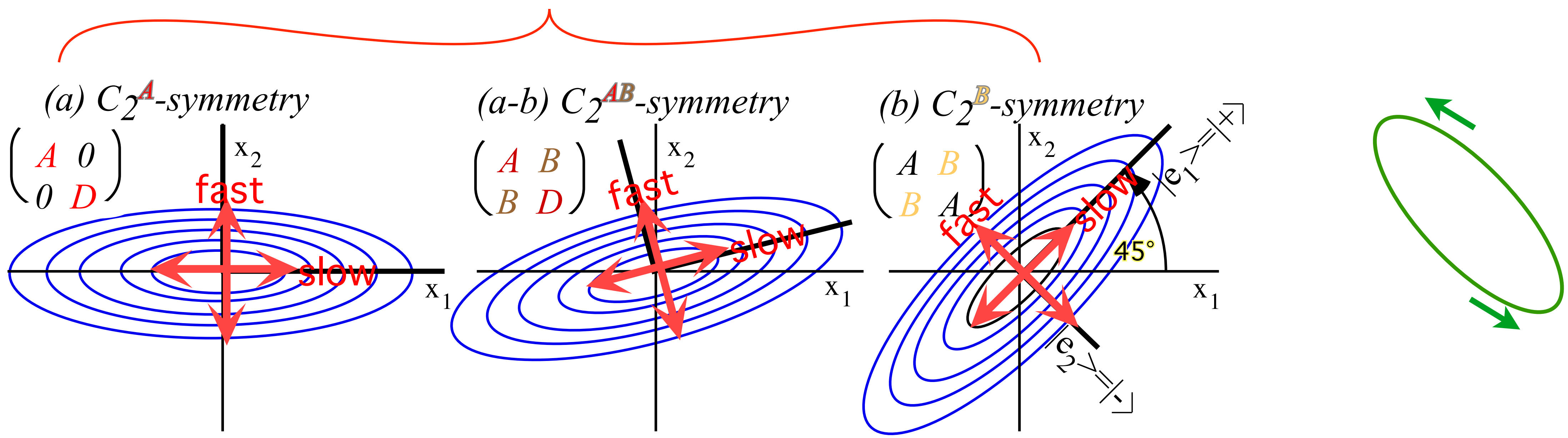


Fig. 10.1.2 Potentials for (a) C_2^A-asymmetric-diagonal, (ab) C_2^{AB}-mixed , (b) C_2^B-bilateral (c) C_2^C-cicular U(2)system.

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

Review:

$$\left|\Psi(t)\right\rangle = e^{-i\mathbf{H}\cdot t}\left|\Psi(0)\right\rangle$$

From QTCA Lecture 7

ABCD Time evolution operator

Hamilton generalized Euler's expansion $e^{-i\omega t} = \cos\omega t - i\sin\omega t$ so matrix exponential becomes powerful.

$$e^{-i\mathbf{H}\cdot t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D\end{pmatrix}\cdot t} = e^{-i\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\cdot t - iB\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\cdot t - iC\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\cdot t - i\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\cdot t}$$

$\sigma_A = \sigma_Z$ $\quad\sigma_B = \sigma_X$ $\quad\sigma_C = \sigma_Y$

$$= e^{-i\sigma_\varphi \varphi} e^{-i\omega_0 \cdot t} = e^{-i\vec{\sigma}\bullet\vec{\omega}\cdot t} e^{-i\omega_0\cdot t} \text{ where: } \vec{\varphi} = \begin{pmatrix}\varphi_A \\ \varphi_B \\ \varphi_C\end{pmatrix} = \vec{\omega}\cdot t = \begin{pmatrix}\omega_A \\ \omega_B \\ \omega_C\end{pmatrix}\cdot t = \begin{pmatrix}\frac{A-D}{2} \\ B \\ C\end{pmatrix}\cdot t \text{ and: } \omega_0 = \frac{A+D}{2}$$

$$= (\mathbf{1}\cos\varphi - i\sigma_\varphi \sin\varphi) e^{-i\omega_0\cdot t}$$

Symmetry relations make spinors $\sigma_X = \sigma_B$, $\sigma_Y = \sigma_C$, and $\sigma_Z = \sigma_A$ or quaternions **i**$=-i\sigma_X$, **j**$=-i\sigma_Y$, and **k**$=-i\sigma_Z$ powerful.

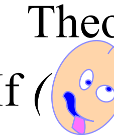
Hamilton is able to generalize Euler's complex rotation operators $e^{+i\varphi}$ and $e^{-i\varphi}$. (Recall Euler - DeMoivre Theorem.)

$$e^{-i\varphi} = 1 + (-i\varphi) + \frac{1}{2!}(-i\varphi)^2 + \frac{1}{3!}(-i\varphi)^3 + \frac{1}{4!}(-i\varphi)^4 \cdots = [1 - \frac{1}{2!}\varphi^2 + \frac{1}{4!}\varphi^4 \cdots] = [\cos\varphi]$$

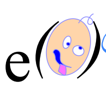
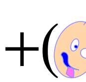
$$- i(\varphi + \frac{1}{3!}\varphi^3 \cdots) \quad - i(\sin\varphi)$$

Note even powers of *(-i)* are ± 1 and odd powers of *(-i)* are $\pm i$.: $(-i)^0 = +1,\ (-i)^1 = -i,\ (-i)^2 = -1,\ (-i)^3 = +i,\ (-i)^4 = +1,\ (-i)^5 = -i,\ etc.$

The Crazy Thing Theorem:

If ()^2= -**1**

Then:

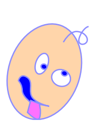
e$^{(\text{face})\varphi}$  $=\mathbf{1}\cos\varphi + ($  $)\sin\varphi$

Hamilton replaces *(-i)* with $-i\sigma_\varphi$ in the $e^{-i\varphi}$ power series above to get a sequence of terms just like it.

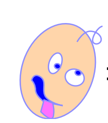
$$(-i\sigma_\varphi)^0 = +\mathbf{1},\ (-i\sigma_\varphi)^1 = -i\sigma_\varphi,\ (-i\sigma_\varphi)^2 = -\mathbf{1},\ (-i\sigma_\varphi)^3 = +i\sigma_\varphi,\ (-i\sigma_\varphi)^4 = +\mathbf{1},\ (-i\sigma_\varphi)^5 = -i\sigma_\varphi,\ etc.$$

This allows Hamilton to generalize Euler's rotation $e^{-i\varphi}$ to $e^{-i\sigma_\varphi\varphi}$ for any $\sigma_\varphi\varphi = (\boldsymbol{\sigma}\bullet\vec{\varphi}) = \varphi_A\sigma_A + \varphi_B\sigma_B + \varphi_C\sigma_C \doteq (\boldsymbol{\sigma}\bullet\hat{\varphi})\varphi$

$e^{-i\varphi} = 1\ \cos\varphi \ -\ i\ \sin\varphi$ *generalizes to:* $e^{-i\sigma_\varphi\varphi} = \mathbf{1}\cos\varphi \ -\ i\ \sigma_\varphi \sin\varphi$

Here:  = *-i*

Crazy thing is just -√-1

Here:  $= -i\sigma_\varphi = -i(\boldsymbol{\sigma}\bullet\hat{\varphi}) = -i\frac{(\boldsymbol{\sigma}\bullet\vec{\varphi})}{\varphi}$

"Crazy-Thing"-Theorem vs Lorentz

Use projectors to derive regular rotations and Lorentz rotations

Symmetry *product table* gives C_2 group representations in *group basis* $\{|0\rangle=\mathbf{1}|0\rangle\equiv|\mathbf{1}\rangle\ ,\ |1\rangle=\sigma_B|0\rangle\equiv|\sigma_B\rangle\}$

$$\begin{pmatrix} \langle \mathbf{1}|\mathbf{1}|\mathbf{1}\rangle & \langle \mathbf{1}|\mathbf{1}|\sigma_B\rangle \\ \langle \sigma_B|\mathbf{1}|\mathbf{1}\rangle & \langle \sigma_B|\mathbf{1}|\sigma_B\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} \langle \mathbf{1}|\sigma_B|\mathbf{1}\rangle & \langle \mathbf{1}|\sigma_B|\sigma_B\rangle \\ \langle \sigma_B|\sigma_B|\mathbf{1}\rangle & \langle \sigma_B|\sigma_B|\sigma_B\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Minimal equation of σ_B is: $\sigma_B{}^2=1$

or: $\sigma_B{}^2-1=0=(\sigma_B-1)(\sigma_B+1)$

with eigenvalues:

$\{\chi^+(\sigma_B)=+1,\ \chi^-(\sigma_B)=-1\}$

Spectral decomposition of $C_2(\sigma_B)$ into $\{\mathbf{P}^+,\mathbf{P}^-\}$

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

$\mathbf{P}^\pm$-projectors:

$$\mathbf{P}^+ = \frac{\mathbf{1}+\sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\mathbf{P}^- = \frac{\mathbf{1}-\sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Regular rotation $R_B(\varphi)=e^{-i\varphi\sigma_B}$

$$\begin{aligned} R_B(\varphi) = e^{-i\varphi\sigma_B} &= e^{-i\varphi\chi^+(\sigma_B)}\ \mathbf{P}^+ + e^{-i\varphi\chi^-(\sigma_B)}\ \mathbf{P}^- \\ &= e^{-i\varphi(+1)}\ \mathbf{P}^+ + e^{-i\varphi(-1)}\ \mathbf{P}^- \\ &= e^{-i\varphi}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + e^{+i\varphi}\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} e^{-i\varphi}+e^{+i\varphi} & e^{-i\varphi}-e^{+i\varphi} \\ e^{-i\varphi}-e^{+i\varphi} & e^{-i\varphi}+e^{+i\varphi} \end{pmatrix} \end{aligned}$$

Review:

Calculation agrees with "Crazy-thing" Theorem

$$= \begin{pmatrix} \cos\varphi & -i\sin\varphi \\ -i\sin\varphi & \cos\varphi \end{pmatrix} = \mathbf{1}\cos\varphi - i\boldsymbol{\sigma}_B\sin\varphi$$

Lorentz rotation $L_B(\rho)=e^{-\rho\sigma_B}$

$$\begin{aligned} L_B(\rho) = e^{-\rho\sigma_B} &= e^{-\rho\chi^+(\sigma_B)}\ \mathbf{P}^+ + e^{-\rho\chi^-(\sigma_B)}\ \mathbf{P}^- \\ &= e^{-\rho(+1)}\ \mathbf{P}^+ + e^{-\rho(-1)}\ \mathbf{P}^- \\ &= e^{-\rho}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + e^{+\rho}\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} e^{-\rho}+e^{+\rho} & e^{-\rho}-e^{+\rho} \\ e^{-\rho}-e^{+\rho} & e^{-\rho}+e^{+\rho} \end{pmatrix} \\ &= \begin{pmatrix} \cosh\rho & -\sinh\rho \\ -\sinh\rho & \cosh\rho \end{pmatrix} = \mathbf{1}\cosh\rho - \boldsymbol{\sigma}_B\sinh\rho \end{aligned}$$

Review:

Comparing Lorentz rotations

Lorentz rotation $L_A(\rho)=e^{-\rho\sigma_A}$

$$L_A(\rho)=e^{-\rho\begin{pmatrix}1&0\\0&-1\end{pmatrix}}$$

$$=\begin{pmatrix}e^{-\rho}&0\\0&e^{+\rho}\end{pmatrix}$$

$$=\mathbf{1}\cosh\rho-\boldsymbol{\sigma}_A\sinh\rho$$

Lorentz rotation $L_B(\rho)=e^{-\rho\sigma_B}$

$$L_B(\rho)=e^{-\rho\begin{pmatrix}0&1\\1&0\end{pmatrix}}$$

$$=\begin{pmatrix}\cosh\rho&-\sinh\rho\\-\sinh\rho&\cosh\rho\end{pmatrix}$$

Lorentz rotation $L_C(\rho)=e^{-\rho\sigma_C}$

$$L_C(\rho)=e^{-\rho\begin{pmatrix}0&-i\\i&0\end{pmatrix}}$$

$$=\begin{pmatrix}\cosh\rho&+i\sinh\rho\\-i\sinh\rho&\cosh\rho\end{pmatrix}$$

$$=\mathbf{1}\cosh\rho-\boldsymbol{\sigma}_C\sinh\rho$$

Comparing regular rotations

Regular rotation $R_A(\varphi)=e^{-i\varphi\sigma_A}$

$$e^{-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\varphi_A}$$

$$=\begin{pmatrix}1&0\\0&1\end{pmatrix}\cos\varphi_A-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\sin\varphi_A$$

$$=\begin{pmatrix}\cos\varphi_A-i\sin\varphi_A&0\\0&\cos\varphi_A+i\sin\varphi_A\end{pmatrix}$$

$$=\begin{pmatrix}e^{-i\varphi_A}&0\\0&e^{i\varphi_A}\end{pmatrix}$$

Example A:
A or Z
rotation

Regular rotation $R_B(\varphi)=e^{-i\varphi\sigma_B}$

$$e^{-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\varphi_B}$$

$$=\begin{pmatrix}1&0\\0&1\end{pmatrix}\cos\varphi_B-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\sin\varphi_B$$

$$=\begin{pmatrix}\cos\varphi_B&-i\sin\varphi_B\\-i\sin\varphi_B&\cos\varphi_B\end{pmatrix}$$

Example B:
B or X
rotation

Regular rotation $R_C(\varphi)=e^{-i\varphi\sigma_C}$

$$e^{-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\varphi_C}$$

$$=\begin{pmatrix}1&0\\0&1\end{pmatrix}\cos\varphi_C-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\sin\varphi_C$$

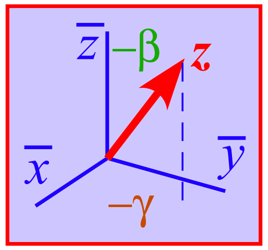
$$=\begin{pmatrix}\cos\varphi_C&-\sin\varphi_C\\\sin\varphi_C&\cos\varphi_C\end{pmatrix}$$

Example C:
C or Y
rotation

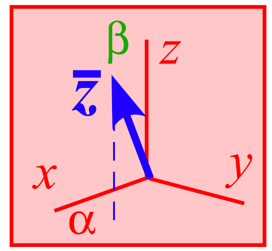
Euler's rotation state definition using rotations **R**(α,0,0), **R**(0,β,0),and **R**(0,0,γ)

Spin-1 (3D-real vector) case

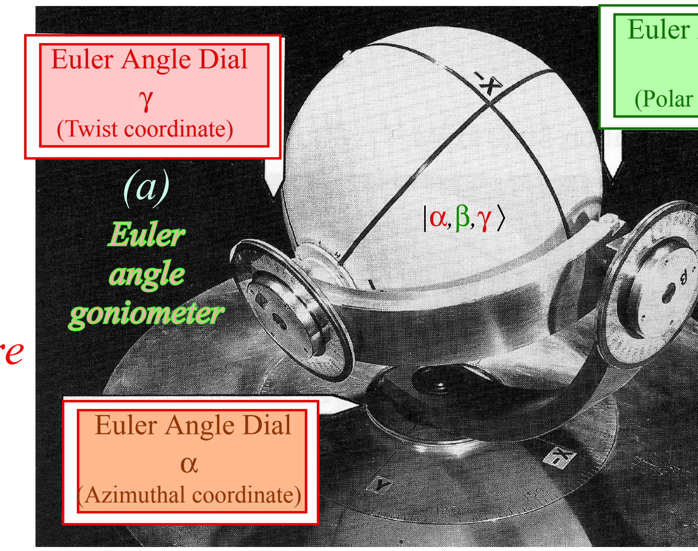
Review:



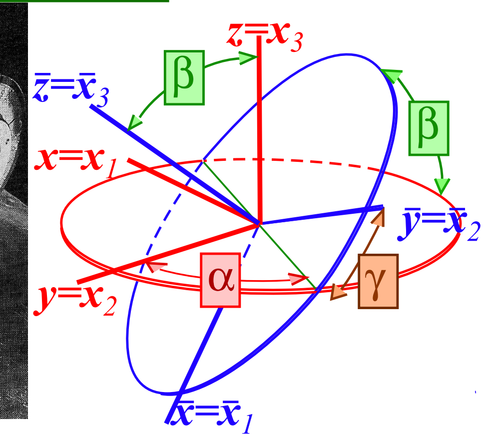
BOD frame view
Polar angles of LAB zenith $z=x_3$ *are*
(azimuth angle=–γ, polar angle=–β)

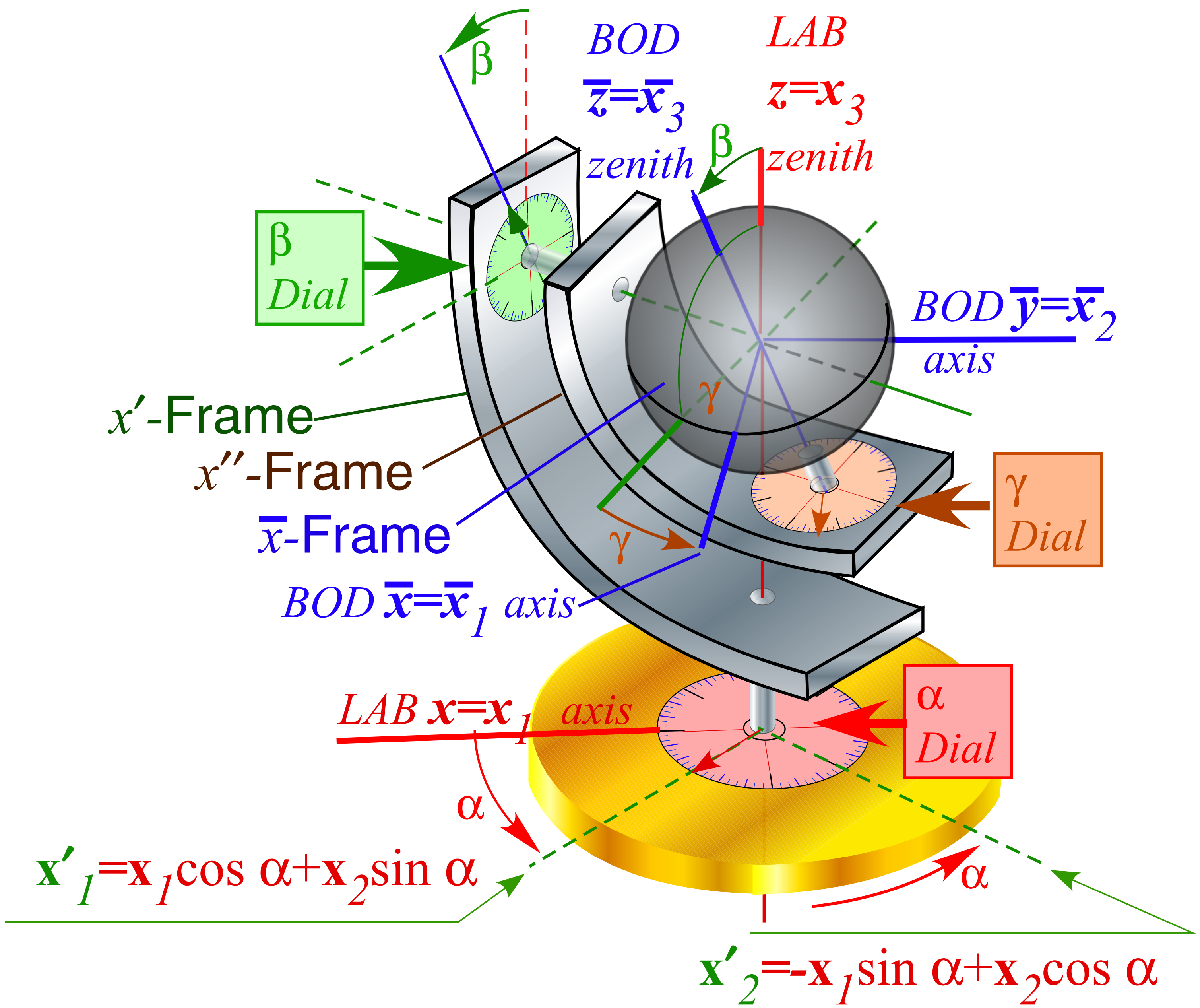


LAB frame view
Polar angles of BOD zenith $\bar{z}=\bar{x}_3$ *are*
(azimuth angle=α, polar angle=β)



An astronomer's diagram





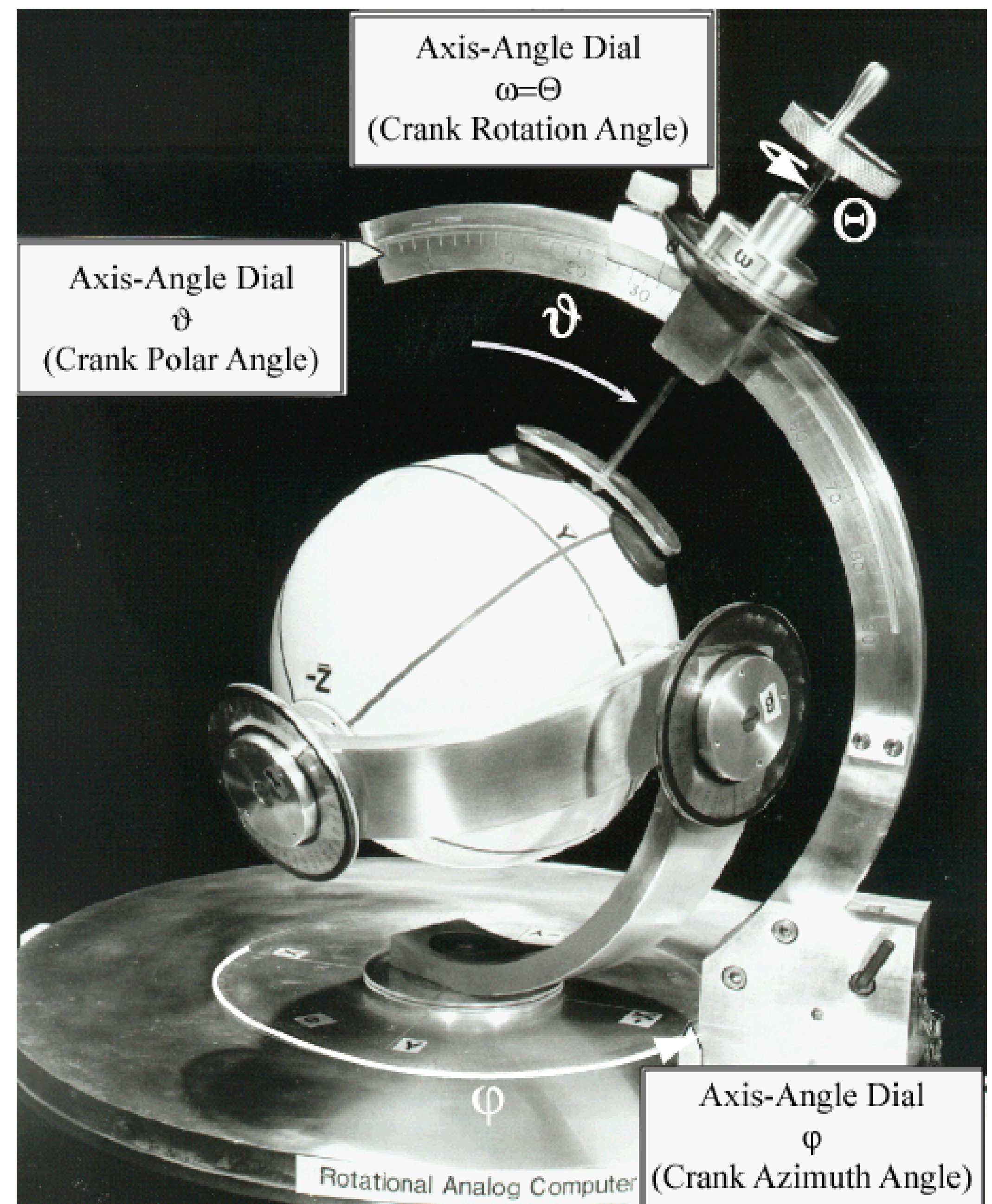


Fig. 10.A.3-4 Mechanical device demonstrating Euler angles (α,β,γ)

Euler's rotation state definition using rotations **R**$(\alpha,0,0)$, **R**$(0,\beta,0)$, *and* **R**$(0,0,\gamma)$

Review:

Spin-1/2 (2D-complex spinor) case

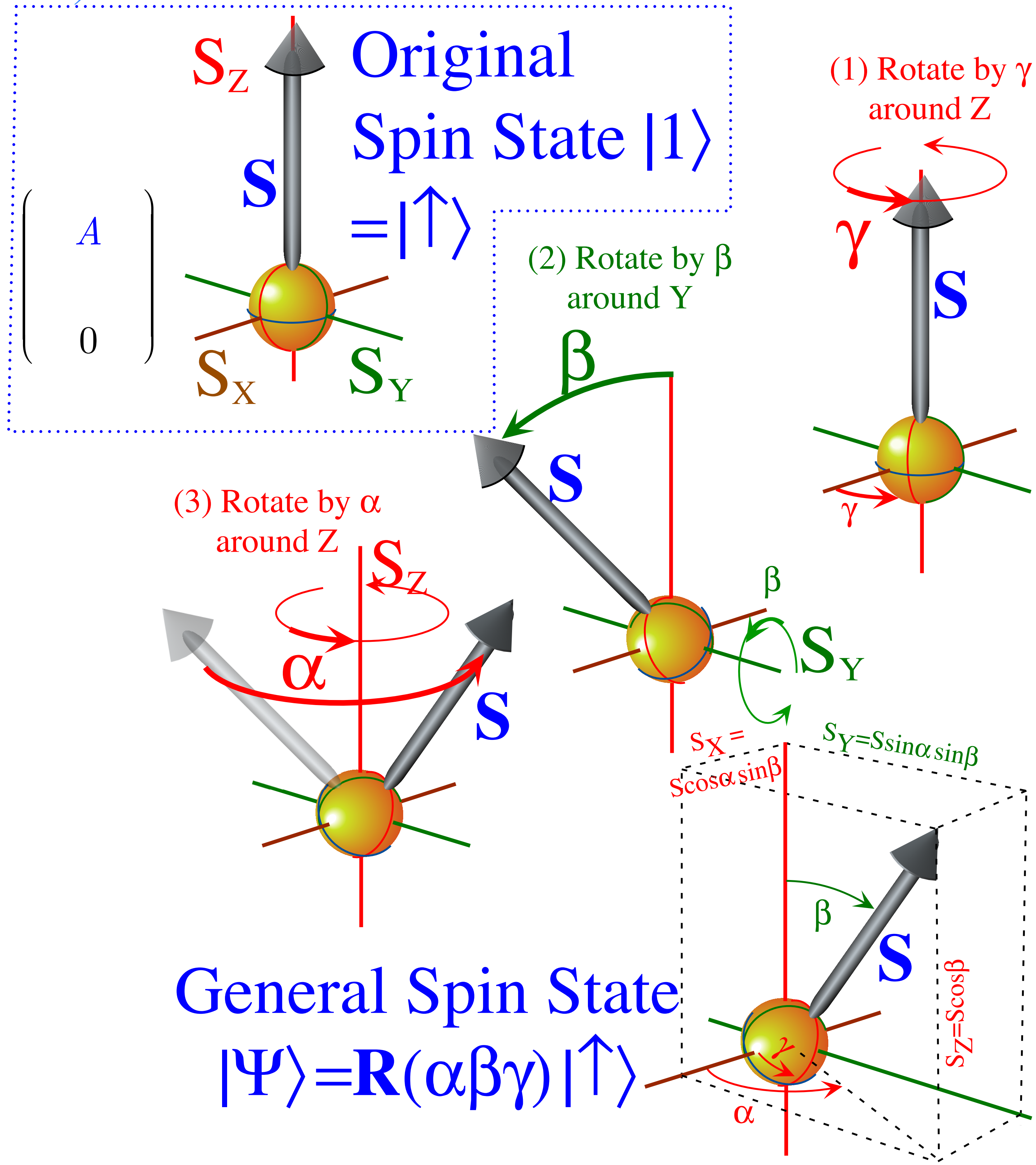
$$|a\rangle = \mathbf{R}(\alpha\beta\gamma)|\uparrow\rangle$$

$$= \mathbf{R}[\alpha \text{ about } Z]\cdot\mathbf{R}[\beta \text{ about } Y]\cdot\mathbf{R}[\gamma \text{ about } Z]|\uparrow\rangle$$

$$= \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}\begin{pmatrix} A \\ 0 \end{pmatrix}$$

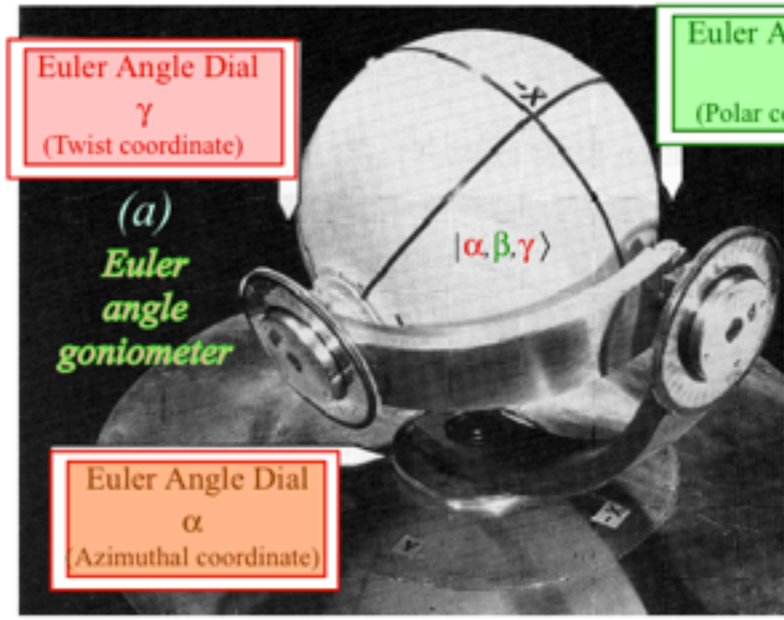
$$= \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} A \\ 0 \end{pmatrix}$$

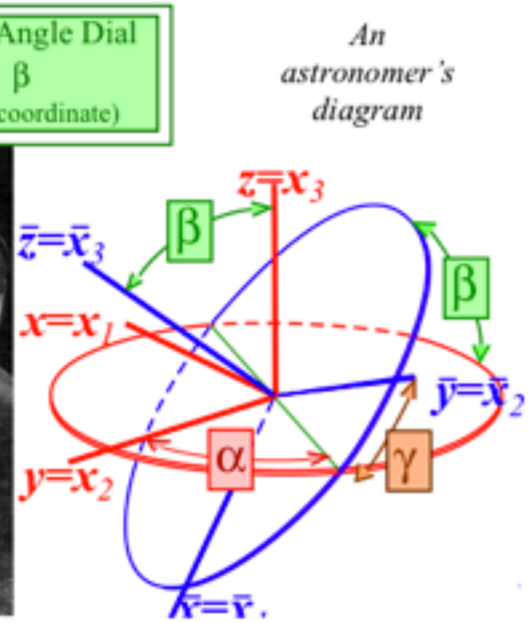
$$= A\begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

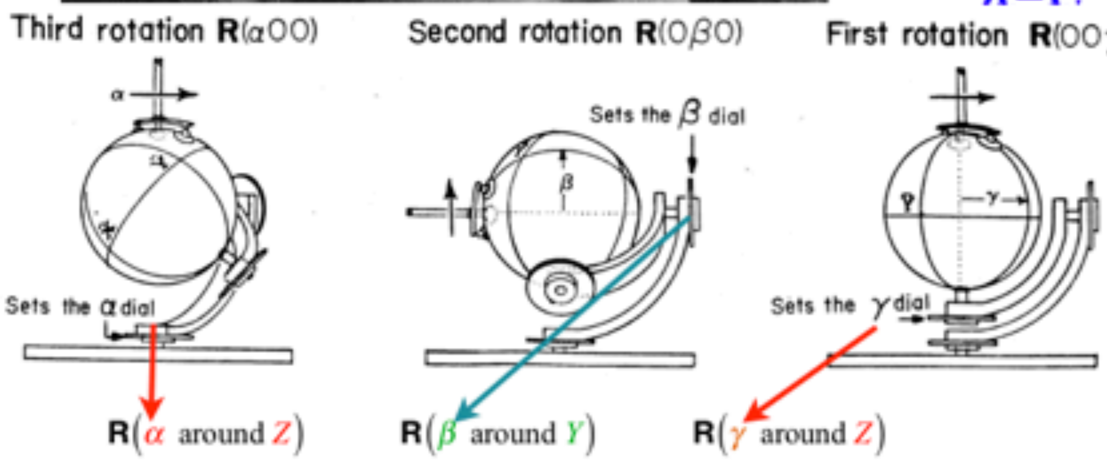


Recall from Lecture 12 p. 117:

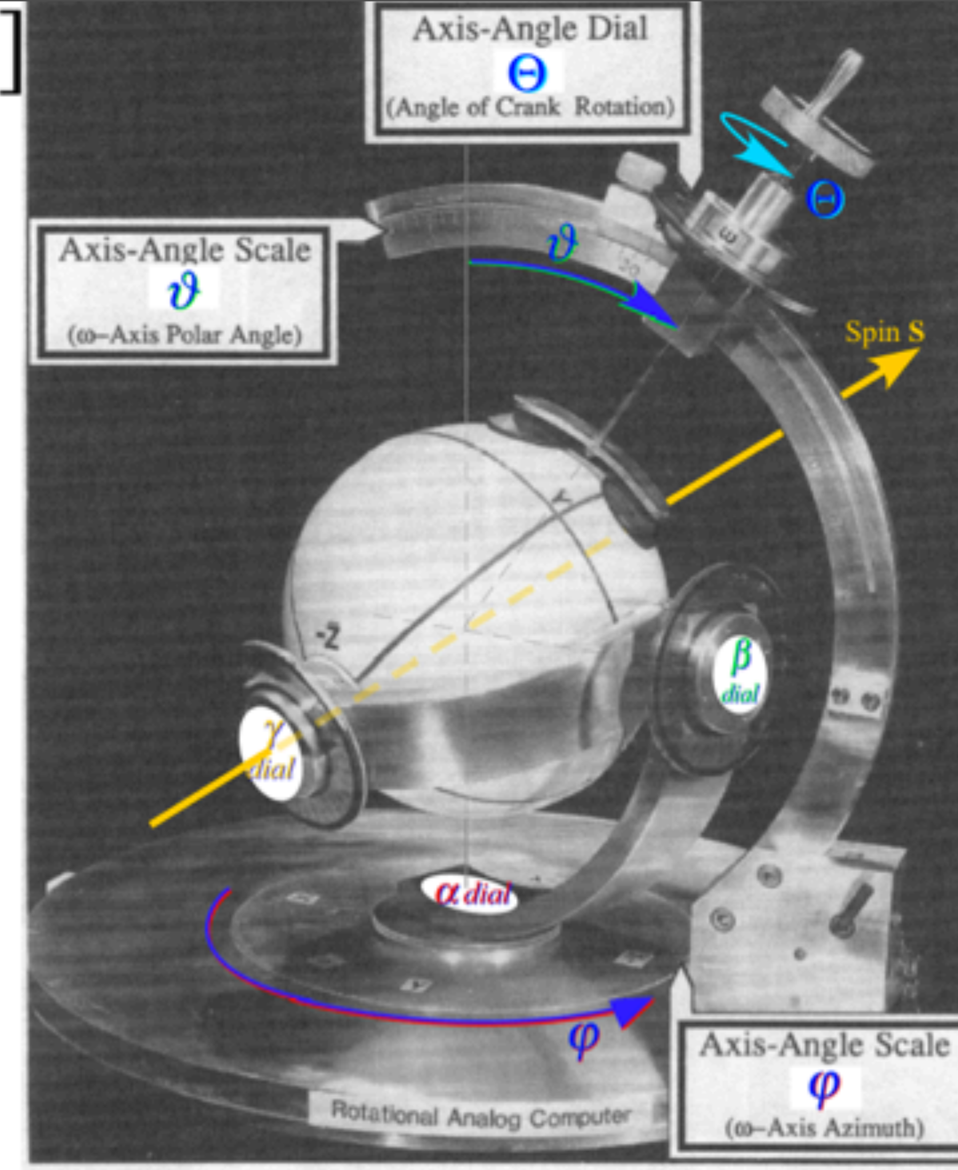
Euler R(αβγ) versus Darboux R[φϑΘ]







Review:
Recall Lecture 12 p.131:



$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}}\end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}}\end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

$$\mathbf{R}[\vec{\Theta}]=\begin{pmatrix}\cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}(\hat{\Theta}_X-i\hat{\Theta}_Y) \\ -i\sin\frac{\Theta}{2}(\hat{\Theta}_X+i\hat{\Theta}_Y) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2}\end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

$$=\cos\frac{\Theta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$=\begin{pmatrix}\cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta) \\ \sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2}\end{pmatrix}$$

Euler R(αβγ) is simpler to form than Θ-axis *Darboux* R[φϑΘ].

Euler *state definition* lets us relate R(αβγ) to R[φϑΘ] ...

|αβγ⟩= R(αβγ)|000⟩   *(αβγ make better coordinates)*

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}x_1+ip_1\\x_2+ip_2\end{pmatrix}$$

$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$

$-p_2 = \sin[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_X \sin\Theta/2 = \cos\varphi\ \sin\vartheta\ \sin\Theta/2$

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$$= A\left(\mathbf{a}_1^\dagger\mathbf{a}_1+\mathbf{1}/2\right)+(B-iC)\mathbf{a}_1^\dagger\mathbf{a}_2 + (B+iC)\mathbf{a}_2^\dagger\mathbf{a}_1+D\left(\mathbf{a}_2^\dagger\mathbf{a}_2+\mathbf{1}/2\right)$$

$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$$

2D-Oscillator basics

First rewrite a classical 2-D Hamiltonian (10.1.3a) with a thick-tip pen! (They're **operators** now!)

$$\mathbf{H} = \frac{A}{2}\left(\mathbf{p}_1^2 + \mathbf{x}_1^2\right) + B\left(\mathbf{x}_1\mathbf{x}_2 + \mathbf{p}_1\mathbf{p}_2\right) + C\left(\mathbf{x}_1\mathbf{p}_2 - \mathbf{x}_2\mathbf{p}_1\right) + \frac{D}{2}\left(\mathbf{p}_2^2 + \mathbf{x}_2^2\right)$$

(Mass factors $\sqrt{M}$, spring constants K_{ij}, and Planck $\hbar$ constants are absorbed into A, B, C, and D constants used in Lecture 12.)

Define **a** and **a**† **operators**

$\mathbf{a}_1 = (\mathbf{x}_1 + i\,\mathbf{p}_1)/\sqrt{2}$ $\quad \mathbf{a}^\dagger_1 = (\mathbf{x}_1 - i\,\mathbf{p}_1)/\sqrt{2}$ $\quad \mathbf{a}_2 = (\mathbf{x}_2 + i\,\mathbf{p}_2)/\sqrt{2}$ $\quad \mathbf{a}^\dagger_2 = (\mathbf{x}_2 - i\,\mathbf{p}_2)/\sqrt{2}$

$\mathbf{x}_1 = (\mathbf{a}^\dagger_1 + \mathbf{a}_1)/\sqrt{2}$ $\quad \mathbf{p}_1 = i\,(\mathbf{a}^\dagger_1 - \mathbf{a}_1)/\sqrt{2}$ $\quad \mathbf{x}_2 = (\mathbf{a}^\dagger_2 + \mathbf{a}_2)/\sqrt{2}$ $\quad \mathbf{p}_2 = i\,(\mathbf{a}^\dagger_2 - \mathbf{a}_2)/\sqrt{2}$

Each system dimension $\mathbf{x}_1$ and $\mathbf{x}_2$ is assumed orthogonal, neither being constrained by the other. This includes an axiom of *inter-dimensional commutivity*.

$$[\,\mathbf{x}_1, \mathbf{p}_2] = \mathbf{0} = [\,\mathbf{x}_2, \mathbf{p}_1], \quad [\,\mathbf{a}_1, \mathbf{a}^\dagger_2] = \mathbf{0} = [\,\mathbf{a}_2, \mathbf{a}^\dagger_1]$$

Commutation relations within space-1 or space-2 space are those of a 1D-oscillator.

$$[\,\mathbf{a}_1, \mathbf{a}^\dagger_1] = \mathbf{1}, \quad [\,\mathbf{a}_2, \mathbf{a}^\dagger_2] = \mathbf{1}$$

This applies in general to N-dimensional oscillator problems.

$$[\,\mathbf{a}_m, \mathbf{a}_n] = \mathbf{a}_m\mathbf{a}_n - \mathbf{a}_n\mathbf{a}_m = \mathbf{0} \qquad [\,\mathbf{a}_m, \mathbf{a}^\dagger_n] = \mathbf{a}_m\mathbf{a}^\dagger_n - \mathbf{a}^\dagger_n\mathbf{a}_m = \delta_{mn}\mathbf{1} \qquad [\,\mathbf{a}^\dagger_m, \mathbf{a}^\dagger_n] = \mathbf{a}^\dagger_m\mathbf{a}^\dagger_n - \mathbf{a}^\dagger_n\mathbf{a}^\dagger_m = \mathbf{0}$$

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$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix}$$

Both are elementary "place-holders" for parameters H_{mn} or A, $B \pm iC$, and D.

$$|m\rangle\langle n| \rightarrow \left(\mathbf{a}_m^\dagger\mathbf{a}_n + \mathbf{a}_n\mathbf{a}_m^\dagger\right)/2 = \mathbf{a}_m^\dagger\mathbf{a}_n + \delta_{m,n}\mathbf{1}/2$$

Review : *1-D* $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(1) representations*

2-D Classical and semi-classical harmonic oscillator ABCD-analysis

U(2) vs R(3): 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

2-D $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(2) representations and R(3) angular momentum operators*

2D-Oscillator basics

Commutation relations

➡ *Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry* ⬅

Anti-commutation relations

Two-dimensional (or 2-particle) base states: ket-kets and bra-bras

Outer product arrays

Entangled 2-particle states

Two-particle (or 2-dimensional) matrix operators

U(2) Hamiltonian and irreducible representations

2D-Oscillator eigensolutions

Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry

Commutivity is known as *Bose symmetry.* Bose and Einstein discovered this symmetry of light quanta. $(\mathbf{a}_m, \mathbf{a}^\dagger{}_n)$ operators called *Boson operators* create or destroy *quanta* or "particles" known as *Bosons*.

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2-D **a†a** *algebra of U(2) representations and R(3) angular momentum operators*

- *2D-Oscillator basics*
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Anti-commutivity is named *Fermi-Dirac symmetry* or *anti-symmetry*. It is found in electron waves.

Fermi operators ($\mathbf{c}_m$,$\mathbf{c}_n$) are defined to create *Fermions* and use *anti-commutators* $\{\mathbf{A},\mathbf{B}\} = \mathbf{AB}+\mathbf{BA}$.

$\{\mathbf{c}_m,\mathbf{c}_n\}=\mathbf{c}_m\mathbf{c}_n+\mathbf{c}_n\mathbf{c}_m=\mathbf{0}$ $\quad$ $\{\mathbf{c}_m,\mathbf{c}^\dagger{}_n\}=\mathbf{c}_m\mathbf{c}^\dagger{}_n+\mathbf{c}^\dagger{}_n\mathbf{c}_m=\delta_{mn}\mathbf{1}$ $\quad$ $\{\mathbf{c}^\dagger{}_m,\mathbf{c}^\dagger{}_n\}=\mathbf{c}^\dagger{}_m\mathbf{c}^\dagger{}_n+\mathbf{c}^\dagger{}_n\mathbf{c}^\dagger{}_m=\mathbf{0}$

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Fermi $\mathbf{c}^\dagger{}_n$ has a rigid birth-control policy; they are allowed just one Fermion or else, none at all.

Creating two Fermions of the same type is punished by death. This is because $x=-x$ implies $x=0$.

$$\mathbf{c}^\dagger{}_m\mathbf{c}^\dagger{}_m\,|0\rangle = -\,\mathbf{c}^\dagger{}_m\mathbf{c}^\dagger{}_m\,|0\rangle = \mathbf{0}$$

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That no two indistinguishable Fermions can be in the same state, is called the *Pauli exclusion principle*.

Quantum numbers of $n=0$ and $n=1$ are the only allowed eigenvalues of the number operator $\mathbf{c}^\dagger_m\mathbf{c}_m$.

$$\mathbf{c}^\dagger_m\mathbf{c}_m\,|0\rangle = \mathbf{0}\ ,\ \ \mathbf{c}^\dagger_m\mathbf{c}_m\,|1\rangle = |1\rangle\ ,\ \ \mathbf{c}^\dagger_m\mathbf{c}_m\,|n\rangle = \mathbf{0}\ \text{ for: } n>1$$

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2-D **a†a** *algebra of U(2) representations and R(3) angular momentum operators*

- *2D-Oscillator basics*
  - *Commutation relations*
  - *Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry*
    - *Anti-commutation relations*
  - *Two-dimensional (or 2-particle) base states: ket-kets and bra-bras*
    - *Outer product arrays*
    - *Entangled 2-particle states*
- *Two-particle (or 2-dimensional) matrix operators*
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Two-dimensional (or 2-particle) base states: ket-kets and bra-bras

A state for a particle in two-dimensions (or two one-dimensional particles) is a *"ket-ket"* $|n_1\rangle|n_2\rangle$
It is outer product of the kets for each single dimension or particle.
The dual description is done similarly using *"bra-bras"* $\langle n_2|\langle n_1| = (|n_1\rangle|n_2\rangle)^\dagger$

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Scalar product is defined so that each kind of particle or dimension will "find" each other and ignore the presence of other kind(s).

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Probability axiom-1 gives correct probability for finding particle-1 at x_1 *and* particle-2 at x_2,
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Product of individual probabilities $\left|\langle x_1|\Psi_1\rangle\right|^2$ and $\left|\langle x_2|\Psi_2\rangle\right|^2$ respects standard Bayesian probability theory.

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Must ask a perennial modern question: *"How are these structures stored in a computer program?"*
The usual answer is in *outer product* or *tensor arrays*. Next pages show sketches of these objects.

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Outer product arrays

Start with an elementary ket basis for each dimension or particle type-1 and type-2.

$Type-1$ $\qquad\qquad\qquad\qquad$ $Type-2$ $\qquad\qquad\qquad\qquad$ $\cdots$

$$\left|0_1\right\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \left|1_1\right\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \left|2_1\right\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{pmatrix}, \cdots \qquad \left|0_2\right\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \left|1_2\right\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \left|2_2\right\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{pmatrix}, \cdots$$

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$$\begin{array}{lll} Type-1 & Type-2 & \cdots \\ \left|0_1\right\rangle = \begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}, \left|1_1\right\rangle = \begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}, \left|2_1\right\rangle = \begin{pmatrix}0\\0\\1\\\vdots\end{pmatrix}, \cdots & \left|0_2\right\rangle = \begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}, \left|1_2\right\rangle = \begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}, \left|2_2\right\rangle = \begin{pmatrix}0\\0\\1\\\vdots\end{pmatrix}, \cdots & \end{array}$$

Outer products are constructed for the states that might have non-negligible amplitudes.

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\end{array}
$$

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shorthand *big-bra-big-ket* notation

$$|\Psi\rangle = \begin{pmatrix} \langle 0_1 0_2|\Psi\rangle \\ \langle 0_1 1_2|\Psi\rangle \\ \langle 0_1 2_2|\Psi\rangle \\ \vdots \\ \hline \langle 1_1 0_2|\Psi\rangle \\ \langle 1_1 1_2|\Psi\rangle \\ \langle 1_1 2_2|\Psi\rangle \\ \vdots \\ \hline \langle 2_1 0_2|\Psi\rangle \\ \langle 2_1 1_2|\Psi\rangle \\ \langle 2_1 2_2|\Psi\rangle \\ \vdots \end{pmatrix} = \begin{pmatrix} \Psi_{00} \\ \Psi_{01} \\ \Psi_{02} \\ \vdots \\ \hline \Psi_{10} \\ \Psi_{11} \\ \Psi_{12} \\ \vdots \\ \hline \Psi_{20} \\ \Psi_{21} \\ \Psi_{22} \\ \vdots \end{pmatrix}$$

Review : *1-D* $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(1) representations*

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U(2) vs R(3):2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

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2D-Oscillator basics

Commutation relations

Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry

Anti-commutation relations

Two-dimensional (or 2-particle) base states: ket-kets and bra-bras

Outer product arrays

→ *Entangled 2-particle states* ←

Two-particle (or 2-dimensional) matrix operators

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2-D **a**†**a** *algebra of U(2) representations and R(3) angular momentum operators*

- *2D-Oscillator basics*
  - *Commutation relations*
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Two-particle (or 2-dimensional) matrix operators

When 2-particle operator $\mathbf{a}_k$ acts on a 2-particle state, $\mathbf{a}_k$ "finds" its type-k state but ignores the others.

$$\mathbf{a}_1^\dagger|n_1 n_2\rangle = \mathbf{a}_1^\dagger|n_1\rangle|n_2\rangle = \sqrt{n_1+1}|n_1+1\ n_2\rangle \qquad \mathbf{a}_2^\dagger|n_1 n_2\rangle = |n_1\rangle\mathbf{a}_2^\dagger|n_2\rangle = \sqrt{n_2+1}|n_1\ n_2+1\rangle$$

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	$\|00\rangle$	$\|01\rangle$	$\|02\rangle$	$\cdots$	$\|10\rangle$	$\|11\rangle$	$\|12\rangle$	$\cdots$	$\|20\rangle$	$\|21\rangle$	$\|22\rangle$	$\cdots$
$\langle 00\|$	0			$\cdots$	$\cdot$			$\cdots$				$\cdots$
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$\langle 02\|$			$2D$	$\cdots$		$\sqrt{2}(B+iC)$	$\cdot$	$\cdots$				$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$				$\cdots$
$\langle 10\|$	$\cdot$	$B-iC$		$\cdots$	A			$\cdots$	$\cdot$			$\cdots$
$\langle 11\|$		$\cdot$	$\sqrt{2}(B-iC)$	$\cdots$		$A+D$		$\cdots$	$\sqrt{2}(B+iC)$	$\cdot$		$\cdots$
$\langle 12\|$			$\cdot$	$\cdots$			$A+2D$	$\cdots$		$\sqrt{4}(B+iC)$	$\cdot$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\langle 20\|$					$\cdot$	$\sqrt{2}(B-iC)$		$\cdots$	$2A$			$\cdots$
$\langle 21\|$						$\cdot$	$\sqrt{4}(B-iC)$	$\cdots$		$2A+D$		$\cdots$
$\langle 22\|$							$\cdot$	$\cdots$			$2A+2D$	$\cdots$
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	$\|00\rangle$	$\|01\rangle$	$\|02\rangle$	$\cdots$	$\|10\rangle$	$\|11\rangle$	$\|12\rangle$	$\cdots$	$\|20\rangle$	$\|21\rangle$	$\|22\rangle$	$\cdots$
$\langle 00\|$	0			$\cdots$	$\cdot$			$\cdots$				$\cdots$
$\langle 01\|$		D		$\cdots$	$B+iC$	$\cdot$		$\cdots$				$\cdots$
$\langle 02\|$			$2D$	$\cdots$		$\sqrt{2}(B+iC)$	$\cdot$	$\cdots$				$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$				$\cdots$
$\langle 10\|$	$\cdot$	$B-iC$		$\cdots$	A			$\cdots$	$\cdot$			$\cdots$
$\langle 11\|$		$\cdot$	$\sqrt{2}(B-iC)$	$\cdots$		$A+D$		$\cdots$	$\sqrt{2}(B+iC)$	$\cdot$		$\cdots$
$\langle 12\|$			$\cdot$	$\cdots$			$A+2D$	$\cdots$		$\sqrt{4}(B+iC)$	$\cdot$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\langle 20\|$					$\cdot$	$\sqrt{2}(B-iC)$		$\cdots$	$2A$			$\cdots$
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$\langle 22\|$								$\cdots$			$2A+2D$	$\cdots$
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	$\|00\rangle$	$\|01\rangle$	$\|02\rangle$	$\cdots$	$\|10\rangle$	$\|11\rangle$	$\|12\rangle$	$\cdots$	$\|20\rangle$	$\|21\rangle$	$\|22\rangle$	$\cdots$
$\langle 00\|$	0			$\cdots$	$\cdot$			$\cdots$				$\cdots$
$\langle 01\|$		D		$\cdots$	$B+iC$	$\cdot$		$\cdots$				$\cdots$
$\langle 02\|$			$2D$	$\cdots$		$\sqrt{2}(B+iC)$	$\cdot$	$\cdots$				$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$				$\cdots$
$\langle 10\|$	$\cdot$	$B-iC$		$\cdots$	A			$\cdots$	$\cdot$			$\cdots$
$\langle 11\|$		$\cdot$	$\sqrt{2}(B-iC)$	$\cdots$		$A+D$		$\cdots$	$\sqrt{2}(B+iC)$	$\cdot$		$\cdots$
$\langle 12\|$			$\cdot$	$\cdots$			$A+2D$	$\cdots$		$\sqrt{4}(B+iC)$	$\cdot$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\langle 20\|$					$\cdot$	$\sqrt{2}(B-iC)$		$\cdots$	$2A$			$\cdots$
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$\langle 22\|$							$\cdot$	$\cdots$			$2A+2D$	$\cdots$
$\vdots$					$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Review : *1-D* **a†a** *algebra of U(1) representations*

2-D Classical and semi-classical harmonic oscillator ABCD-analysis

U(2) vs R(3): 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2{}_t\mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry (**σ***-expansion in ABCD-Types)* $\mathbf{H} = \omega_\mu \boldsymbol{\sigma}_\mu$

2-D **a†a** *algebra of U(2) representations and R(3) angular momentum operators*

- *2D-Oscillator basics*
  - *Commutation relations*
  - *Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry*
    - *Anti-commutation relations*
  - *Two-dimensional (or 2-particle) base states: ket-kets and bra-bras*
    - *Outer product arrays*
    - *Entangled 2-particle states*
- *Two-particle (or 2-dimensional) matrix operators*
  - → *U(2) Hamiltonian and irreducible representations* ←
  - *2D-Oscillator eigensolutions*

2-dimensional HO Hamiltonian matrices: U(2) irreducible representations

$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 02\rangle$	$\cdots$	$\vert 10\rangle$	$\vert 11\rangle$	$\vert 12\rangle$	$\cdots$	$\vert 20\rangle$	$\vert 21\rangle$	$\vert 22\rangle$	$\cdots$
$\langle 00\vert$	0			$\cdots$	$\cdot$			$\cdots$				$\cdots$
$\langle 01\vert$		D		$\cdots$	$B+iC$	$\cdot$		$\cdots$				$\cdots$
$\langle 02\vert$			$2D$	$\cdots$		$\sqrt{2}(B+iC)$	$\cdot$	$\cdots$				$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$				$\cdots$
$\langle 10\vert$	$\cdot$	$B-iC$		$\cdots$	A			$\cdots$	$\cdot$			$\cdots$
$\langle 11\vert$		$\cdot$	$\sqrt{2}(B-iC)$	$\cdots$		$A+D$		$\cdots$	$\sqrt{2}(B+iC)$	$\cdot$		$\cdots$
$\langle 12\vert$			$\cdot$	$\cdots$			$A+2D$	$\cdots$		$\sqrt{4}(B+iC)$	$\cdot$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\langle 20\vert$					$\cdot$	$\sqrt{2}(B-iC)$		$\cdots$	$2A$			$\cdots$
$\langle 21\vert$						$\cdot$	$\sqrt{4}(B-iC)$	$\cdots$		$2A+D$		$\cdots$
$\langle 22\vert$							$\cdot$	$\cdots$			$2A+2D$	$\cdots$
$\vdots$					$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

"Big-Endian" indexing (*...01,02,..10,11 ... 20,21...*)

Rearrangement of rows and columns brings the matrix to a block-diagonal form.

2-dimensional HO Hamiltonian matrices: U(2) irreducible representations

$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 02\rangle$	$\cdots$	$\vert 10\rangle$	$\vert 11\rangle$	$\vert 12\rangle$	$\cdots$	$\vert 20\rangle$	$\vert 21\rangle$	$\vert 22\rangle$	$\cdots$
$\langle 00\vert$	0			$\cdots$	$\cdot$			$\cdots$				$\cdots$
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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$				$\cdots$
$\langle 10\vert$	$\cdot$	$B-iC$		$\cdots$	A			$\cdots$	$\cdot$			$\cdots$
$\langle 11\vert$		$\cdot$	$\sqrt{2}(B-iC)$	$\cdots$		$A+D$		$\cdots$	$\sqrt{2}(B+iC)$	$\cdot$		$\cdots$
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Base states $|n_1\rangle|n_2\rangle$ with the same ***total quantum number*** $\nu = n_1 + n_2$ define each block.

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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$				$\cdots$
$\langle 10\vert$	$\cdot$	$B-iC$		$\cdots$	A			$\cdots$	$\cdot$			$\cdots$
$\langle 11\vert$		$\cdot$	$\sqrt{2}(B-iC)$	$\cdots$		$A+D$		$\cdots$	$\sqrt{2}(B+iC)$	$\cdot$		$\cdots$
$\langle 12\vert$			$\cdot$	$\cdots$			$A+2D$	$\cdots$		$\sqrt{4}(B+iC)$	$\cdot$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\langle 20\vert$					$\cdot$	$\sqrt{2}(B-iC)$		$\cdots$	$2A$			$\cdots$
$\langle 21\vert$						$\cdot$	$\sqrt{4}(B-iC)$	$\cdots$		$2A+D$		$\cdots$
$\langle 22\vert$							$\cdot$	$\cdots$			$2A+2D$	$\cdots$
$\vdots$					$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

"Big-Endian" indexing (...01,02,..10,11 ... 20,21...)

Rearrangement of rows and columns brings the matrix to a block-diagonal form.

Base states $|n_1\rangle|n_2\rangle$ with the same *total quantum number* $\upsilon = n_1 + n_2$ define each block.

$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 10\rangle$	$\vert 02\rangle$	$\vert 11\rangle$	$\vert 20\rangle$	$\vert 03\rangle$	$\vert 12\rangle$	$\vert 21\rangle$	$\vert 30\rangle$	$\cdots$
$\langle 00\vert$	0										
$\langle 01\vert$		D	$B+iC$								
$\langle 10\vert$		$B-iC$	A								
$\langle 02\vert$				$2D$	$\sqrt{2}(B+iC)$						
$\langle 11\vert$				$\sqrt{2}(B-iC)$	$A+D$	$\sqrt{2}(B+iC)$					
$\langle 20\vert$					$\sqrt{2}(B-iC)$	$2A$					
$\langle 03\vert$							$3D$	$\sqrt{3}(B+iC)$			
$\langle 12\vert$							$\sqrt{3}(B-iC)$	$A+2D$	$\sqrt{4}(B+iC)$		
$\langle 21\vert$								$\sqrt{4}(B-iC)$	$2A+D$	$\sqrt{3}(B+iC)$	
$\langle 30\vert$									$\sqrt{3}(B-iC)$	$3A$	
$\vdots$											

Vacuum (ν=0)

Fundamental (ν=1) vibrational sub-space

Overtone (ν=2) vibrational sub-space

Overtone (ν=3) vibrational sub-space

"Big-Endian" indexing (...01,02,..10,11 ... 20,21...)

$$\mathbf{H}^A = A\left(\mathbf{a}_1^\dagger \mathbf{a}_1 + \mathbf{1}/2\right) + D\left(\mathbf{a}_2^\dagger \mathbf{a}_2 + \mathbf{1}/2\right)$$

$$\varepsilon^A_{n_1 n_2} = A\left(n_1 + \frac{1}{2}\right) + D\left(n_2 + \frac{1}{2}\right) = \frac{A+D}{2}(n_1 + n_2 + 1) + \frac{A-D}{2}(n_1 - n_2)$$

Review : *1-D* **a†a** *algebra of U(1) representations*

2-D Classical and semi-classical harmonic oscillator ABCD-analysis

U(2) vs R(3): 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

2-D **a†a** *algebra of U(2) representations and R(3) angular momentum operators*

- *2D-Oscillator basics*
  - *Commutation relations*
  - *Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry*
    - *Anti-commutation relations*
  - *Two-dimensional (or 2-particle) base states: ket-kets and bra-bras*
    - *Outer product arrays*
    - *Entangled 2-particle states*
- *Two-particle (or 2-dimensional) matrix operators*
  - *U(2) Hamiltonian and irreducible representations*
  - → *2D-Oscillator eigensolutions* ←

2D-Oscillator eigensolutions

"Little-Endian" indexing (... *10, 01, ...20,11,21...*)

Fundamental eigenstates

The first step is to diagonalize the fundamental 2-by-2 matrix .

$$\langle \mathbf{H} \rangle_{\upsilon=1}^{Fundamental} = \begin{array}{c|cc} n_1, n_2 & |1,0\rangle & |0,1\rangle \\ \hline \langle 1,0| & A & B - iC \\ \langle 0,1| & B + iC & D \end{array} + \frac{A+D}{2}\mathbf{1}$$

2D-Oscillator eigensolutions

"Little-Endian" indexing (*... 10, 01, ...20,11,21...*)

Fundamental eigenstates

The first step is to diagonalize the fundamental 2-by-2 matrix .

$\langle \mathbf{H} \rangle_{\upsilon=1}^{Fundamental} =$

n_1, n_2	$\vert 1,0\rangle$	$\vert 0,1\rangle$
$\langle 1,0\vert$	A	$B-iC$
$\langle 0,1\vert$	$B+iC$	D

$+\frac{A+D}{2}\mathbf{1}$

Recall decomposition of **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} + \frac{A+D}{2}\mathbf{1} = (A+D)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{2} + 2C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{1}{2} + (A-D)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2}$$

2D-Oscillator eigensolutions

"Little-Endian" indexing (... 10, 01, ...20,11,21...)

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in terms of Jordan-Pauli spin operators.

$$\mathbf{H} = \Omega_0\mathbf{1} + \mathbf{\Omega}\bullet\vec{\mathbf{S}} = \Omega_0\mathbf{1} + \Omega_B\mathbf{S}_B + \Omega_C\mathbf{S}_C + \Omega_A\mathbf{S}_A \quad (ABC \text{ Optical vector notation})$$

$$= \Omega_0\mathbf{1} + \Omega_X\mathbf{S}_X + \Omega_Y\mathbf{S}_Y + \Omega_Z\mathbf{S}_Z \quad (XYZ \text{ Electron spin notation})$$

2D-Oscillator eigensolutions

"Little-Endian" indexing (*... 10, 01, ...20,11,21...*)

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The first step is to diagonalize the fundamental 2-by-2 matrix .

$\langle \mathbf{H} \rangle_{\upsilon=1}^{Fundamental} =$

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$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} + \frac{A+D}{2}\mathbf{1} = (A+D)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{2} + 2C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{1}{2} + (A-D)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2}$$

in terms of Jordan-Pauli spin operators.

$$\mathbf{H} = \Omega_0 \mathbf{1} + \boldsymbol{\Omega} \bullet \vec{\mathbf{S}} = \Omega_0 \mathbf{1} + \Omega_B \mathbf{S}_B + \Omega_C \mathbf{S}_C + \Omega_A \mathbf{S}_A \quad \textit{(ABC Optical vector notation)}$$

$$= \Omega_0 \mathbf{1} + \Omega_X \mathbf{S}_X + \Omega_Y \mathbf{S}_Y + \Omega_Z \mathbf{S}_Z \quad \textit{(XYZ Electron spin notation)}$$

Frequency eigenvalues $\omega_\pm$ of $\mathbf{H}\text{-}\Omega_0\mathbf{1}/2$ and *fundamental transition frequency* $\Omega = \omega_+ - \omega_-$:

$$\omega_\pm = \frac{\Omega_0 \pm \Omega}{2} = \frac{A+D \pm \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}}{2} = \frac{A+D}{2} \pm \sqrt{\left(\frac{A-D}{2}\right)^2 + B^2 + C^2}$$

2D-Oscillator eigensolutions

"Little-Endian" indexing (... 10, 01, ...20,11,21...)

Fundamental eigenstates

The first step is to diagonalize the fundamental 2-by-2 matrix .

$$\langle \mathbf{H} \rangle^{Fundamental}_{\upsilon=1} = \begin{array}{c|cc} n_1,n_2 & |1,0\rangle & |0,1\rangle \\ \hline \langle 1,0| & A & B-iC \\ \langle 0,1| & B+iC & D \end{array} + \frac{A+D}{2}\mathbf{1}$$

Recall decomposition of **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} + \frac{A+D}{2}\mathbf{1} = (A+D)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{2} + 2C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{1}{2} + (A-D)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2}$$

in terms of Jordan-Pauli spin operators.

$$\mathbf{H} = \Omega_0 \mathbf{1} + \mathbf{\Omega} \bullet \vec{\mathbf{S}} = \Omega_0 \mathbf{1} + \Omega_B \mathbf{S}_B + \Omega_C \mathbf{S}_C + \Omega_A \mathbf{S}_A \quad (ABC \text{ Optical vector notation})$$

$$= \Omega_0 \mathbf{1} + \Omega_X \mathbf{S}_X + \Omega_Y \mathbf{S}_Y + \Omega_Z \mathbf{S}_Z \quad (XYZ \text{ Electron spin notation})$$

Frequency eigenvalues $\omega_\pm$ of $\mathbf{H}\text{-}\Omega_0\mathbf{1}/2$ and *fundamental transition frequency* $\Omega = \omega_+ - \omega_-$:

$$\omega_\pm = \frac{\Omega_0 \pm \Omega}{2} = \frac{A+D \pm \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}}{2} = \frac{A+D}{2} \pm \sqrt{\left(\frac{A-D}{2}\right)^2 + B^2 + C^2}$$

Polar angles (φ,ϑ) of $+\mathbf{\Omega}$-vector (or polar angles $(\varphi,\vartheta\pm\pi)$ of $-\mathbf{\Omega}$-vector) gives **H** eigenvectors.

$$|\omega_+\rangle = \begin{pmatrix} e^{-i\varphi/2}\cos\frac{\vartheta}{2} \\ e^{i\varphi/2}\sin\frac{\vartheta}{2} \end{pmatrix}, \quad |\omega_-\rangle = \begin{pmatrix} -e^{-i\varphi/2}\sin\frac{\vartheta}{2} \\ e^{i\varphi/2}\cos\frac{\vartheta}{2} \end{pmatrix} \quad \text{where: } \begin{cases} \cos\vartheta = \frac{A\text{-}D}{\Omega} \\ \tan\varphi = \frac{C}{B} \end{cases}$$

Recall from Lecture 12 p. 117 and p.131:

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ *($\alpha\beta\gamma$ make better coordinates)*

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

2D-Oscillator eigensolutions

Fundamental eigenstates

The first step is to diagonalize the fundamental 2-by-2 matrix .

$\langle \mathbf{H} \rangle_{\upsilon=1}^{Fundamental} =$

n_1,n_2	$\lvert 1,0\rangle$	$\lvert 0,1\rangle$
$\langle 1,0 \rvert$	A	$B-iC$
$\langle 0,1 \rvert$	$B+iC$	D

$+\frac{A+D}{2}\mathbf{1}$

Recall decomposition of **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}+\frac{A+D}{2}\mathbf{1}=(A+D)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+2B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{2}+2C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{1}{2}+(A-D)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2}$$

in terms of Jordan-Pauli spin operators.

$$\mathbf{H}=\Omega_0\mathbf{1}+\mathbf{\Omega}\bullet\vec{\mathbf{S}}=\Omega_0\mathbf{1}+\Omega_B\mathbf{S}_B+\Omega_C\mathbf{S}_C+\Omega_A\mathbf{S}_A \quad (ABC\ \textit{Optical vector notation})$$

$$=\Omega_0\mathbf{1}+\Omega_X\mathbf{S}_X+\Omega_Y\mathbf{S}_Y+\Omega_Z\mathbf{S}_Z \quad (XYZ\ \textit{Electron spin notation})$$

Frequency eigenvalues $\omega_\pm$ of $\mathbf{H}$-$\Omega_0\mathbf{1}/2$ and *fundamental transition frequency* $\Omega = \omega_+ - \omega_-$:

$$\omega_\pm=\frac{\Omega_0\pm\Omega}{2}=\frac{A+D\pm\sqrt{(2B)^2+(2C)^2+(A-D)^2}}{2}=\frac{A+D}{2}\pm\sqrt{\left(\frac{A-D}{2}\right)^2+B^2+C^2}$$

Polar angles (φ,ϑ) of $+\mathbf{\Omega}$-vector (or polar angles $(\varphi,\vartheta\pm\pi)$ of $-\mathbf{\Omega}$-vector) gives **H** eigenvectors.

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More important for the general solution, are the *eigen-creation operators* $\mathbf{a}^\dagger_+$ *and* $\mathbf{a}^\dagger_-$ defined by

$$\mathbf{a}_+^\dagger=e^{-i\varphi/2}\left(\cos\frac{\vartheta}{2}\mathbf{a}_1^\dagger+e^{i\varphi}\sin\frac{\vartheta}{2}\mathbf{a}_2^\dagger\right),\quad \mathbf{a}_-^\dagger=e^{-i\varphi/2}\left(-\sin\frac{\vartheta}{2}\mathbf{a}_1^\dagger+e^{i\varphi}\cos\frac{\vartheta}{2}\mathbf{a}_2^\dagger\right)$$

2D-Oscillator eigensolutions

"Little-Endian" indexing (... 10, 01, ...20,11,21...)

Fundamental eigenstates

The first step is to diagonalize the fundamental 2-by-2 matrix .

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n_1,n_2	$\|1,0\rangle$	$\|0,1\rangle$
$\langle 1,0\|$	A	$B-iC$
$\langle 0,1\|$	$B+iC$	D

$+\frac{A+D}{2}\mathbf{1}$

Recall decomposition of **H**

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in terms of Jordan-Pauli spin operators.

$$\mathbf{H} = \Omega_0 \mathbf{1} + \mathbf{\Omega} \bullet \vec{\mathbf{S}} = \Omega_0 \mathbf{1} + \Omega_B \mathbf{S}_B + \Omega_C \mathbf{S}_C + \Omega_A \mathbf{S}_A \quad (ABC \text{ Optical vector notation})$$

$$= \Omega_0 \mathbf{1} + \Omega_X \mathbf{S}_X + \Omega_Y \mathbf{S}_Y + \Omega_Z \mathbf{S}_Z \quad (XYZ \text{ Electron spin notation})$$

Frequency eigenvalues $\omega_\pm$ of $\mathbf{H}\text{-}\Omega_0\mathbf{1}/2$ and *fundamental transition frequency* $\Omega = \omega_+ - \omega_-$:

$$\omega_\pm = \frac{\Omega_0 \pm \Omega}{2} = \frac{A+D \pm \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}}{2} = \frac{A+D}{2} \pm \sqrt{\left(\frac{A-D}{2}\right)^2 + B^2 + C^2}$$

Polar angles (φ,ϑ) of $+\mathbf{\Omega}$-vector (or polar angles $(\varphi,\vartheta\pm\pi)$ of $-\mathbf{\Omega}$-vector) gives **H** eigenvectors.

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More important for the general solution, are the *eigen-creation operators* $\mathbf{a}^\dagger_+$ *and* $\mathbf{a}^\dagger_-$ defined by

$$\mathbf{a}_+^\dagger = e^{-i\varphi/2}\left(\cos\frac{\vartheta}{2}\mathbf{a}_1^\dagger + e^{i\varphi}\sin\frac{\vartheta}{2}\mathbf{a}_2^\dagger\right), \quad \mathbf{a}_-^\dagger = e^{-i\varphi/2}\left(-\sin\frac{\vartheta}{2}\mathbf{a}_1^\dagger + e^{i\varphi}\cos\frac{\vartheta}{2}\mathbf{a}_2^\dagger\right)$$

$\mathbf{a}_\pm^\dagger$ create **H** eigenstates directly from the ground state.

$$\mathbf{a}_+^\dagger|0\rangle = |\omega_+\rangle, \quad \mathbf{a}_-^\dagger|0\rangle = |\omega_-\rangle$$

Setting $(B=0=C)$ and $(A=\omega_+)$ and $(D=\omega_-)$ gives diagonal block matrices.

$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 10\rangle$	$\vert 02\rangle$	$\vert 11\rangle$	$\vert 20\rangle$	$\vert 03\rangle$	$\vert 12\rangle$	$\vert 21\rangle$	$\vert 30\rangle$	$\cdots$
$\langle 00\vert$	0										
$\langle 01\vert$		ω_-									
$\langle 10\vert$			ω_+								
$\langle 02\vert$				$2\omega_-$							
$\langle 11\vert$					$\omega_+ + \omega_-$						
$\langle 20\vert$						$2\omega_+$					
$\langle 03\vert$							$3\omega_-$				
$\langle 12\vert$								$\omega_+ + 2\omega_-$			
$\langle 21\vert$									$2\omega_+ + \omega_-$		
$\langle 30\vert$										$3\omega_+$	
$\vdots$											

$$\omega_+ - \omega_- = \Omega$$
$$= \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}$$
$$= A - D$$

$$\mathbf{H}^A = A\left(\mathbf{a}_1^\dagger \mathbf{a}_1 + \mathbf{1}/2\right) + D\left(\mathbf{a}_2^\dagger \mathbf{a}_2 + \mathbf{1}/2\right)$$

Setting (B=0=C) and (A=ω_+) and (D=ω_-) gives diagonal block matrices.

$$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$$

	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 10\rangle$	$\vert 02\rangle$	$\vert 11\rangle$	$\vert 20\rangle$	$\vert 03\rangle$	$\vert 12\rangle$	$\vert 21\rangle$	$\vert 30\rangle$	$\cdots$
$\langle 00\vert$	0										
$\langle 01\vert$		ω_-									
$\langle 10\vert$			ω_+								
$\langle 02\vert$				$2\omega_-$							
$\langle 11\vert$					$\omega_+ + \omega_-$						
$\langle 20\vert$						$2\omega_+$					
$\langle 03\vert$							$3\omega_-$				
$\langle 12\vert$								$\omega_+ + 2\omega_-$			
$\langle 21\vert$									$2\omega_+ + \omega_-$		
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$\vdots$											

$$\omega_+ - \omega_- = \Omega$$
$$= \sqrt{(2B)^2 + (2C)^2 + (A - D)^2}$$
$$= A - D$$

$$\mathbf{H}^A = A\left(\mathbf{a}_1^\dagger \mathbf{a}_1 + \mathbf{1}/2\right) + D\left(\mathbf{a}_2^\dagger \mathbf{a}_2 + \mathbf{1}/2\right)$$

$$\varepsilon^A_{n_1 n_2} = A\left(n_1 + \frac{1}{2}\right) + D\left(n_2 + \frac{1}{2}\right) = \frac{A+D}{2}\left(n_1 + n_2 + 1\right) + \frac{A-D}{2}\left(n_1 - n_2\right)$$

Setting $(B=0=C)$ and $(A=\omega_+)$ and $(D=\omega_-)$ gives diagonal block matrices.

$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 10\rangle$	$\vert 02\rangle$	$\vert 11\rangle$	$\vert 20\rangle$	$\vert 03\rangle$	$\vert 12\rangle$	$\vert 21\rangle$	$\vert 30\rangle$	$\cdots$
$\langle 00\vert$	0										
$\langle 01\vert$		ω_-									
$\langle 10\vert$			ω_+								
$\langle 02\vert$				$2\omega_-$							
$\langle 11\vert$					$\omega_+ + \omega_-$						
$\langle 20\vert$						$2\omega_+$					
$\langle 03\vert$							$3\omega_-$				
$\langle 12\vert$								$\omega_+ + 2\omega_-$			
$\langle 21\vert$									$2\omega_+ + \omega_-$		
$\langle 30\vert$										$3\omega_+$	
$\vdots$											

$$\omega_+ - \omega_- = \Omega$$
$$= \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}$$
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$$\mathbf{H}^A = A\left(\mathbf{a}_1^\dagger \mathbf{a}_1 + \mathbf{1}/2\right) + D\left(\mathbf{a}_2^\dagger \mathbf{a}_2 + \mathbf{1}/2\right)$$

$$\varepsilon^A_{n_1 n_2} = A\left(n_1 + \frac{1}{2}\right) + D\left(n_2 + \frac{1}{2}\right) = \frac{A+D}{2}(n_1 + n_2 + 1) + \frac{A-D}{2}(n_1 - n_2)$$
$$= \Omega_0 (n_1 + n_2 + 1) + \frac{\Omega}{2}(n_1 - n_2) = \Omega_0(\upsilon + 1) + \Omega\, m$$

Setting $(B=0=C)$ and $(A=\omega_+)$ and $(D=\omega_-)$ gives diagonal block matrices.

$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 10\rangle$	$\vert 02\rangle$	$\vert 11\rangle$	$\vert 20\rangle$	$\vert 03\rangle$	$\vert 12\rangle$	$\vert 21\rangle$	$\vert 30\rangle$	$\cdots$
$\langle 00\vert$	0										
$\langle 01\vert$		ω_-									
$\langle 10\vert$			ω_+								
$\langle 02\vert$				$2\omega_-$							
$\langle 11\vert$					$\omega_+ + \omega_-$						
$\langle 20\vert$						$2\omega_+$					
$\langle 03\vert$							$3\omega_-$				
$\langle 12\vert$								$\omega_+ + 2\omega_-$			
$\langle 21\vert$									$2\omega_+ + \omega_-$		
$\langle 30\vert$										$3\omega_+$	
$\vdots$											

$$\omega_+ - \omega_- = \Omega$$
$$= \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}$$
$$= A - D$$

$$\mathbf{H}^A = A\left(\mathbf{a}_1^\dagger \mathbf{a}_1 + \mathbf{1}/2\right) + D\left(\mathbf{a}_2^\dagger \mathbf{a}_2 + \mathbf{1}/2\right)$$

$$\varepsilon^A_{n_1 n_2} = A\left(n_1 + \frac{1}{2}\right) + D\left(n_2 + \frac{1}{2}\right) = \frac{A+D}{2}(n_1 + n_2 + 1) + \frac{A-D}{2}(n_1 - n_2)$$
$$= \Omega_0 (n_1 + n_2 + 1) + \frac{\Omega}{2}(n_1 - n_2) = \Omega_0 (\upsilon + 1) + \Omega\, m$$

Define *total quantum number* $\upsilon = 2j$ and half-difference or *asymmetry quantum number m*

$$\upsilon = n_1 + n_2 = 2j \qquad j = \frac{n_1 + n_2}{2} = \frac{\upsilon}{2} \qquad m = \frac{n_1 - n_2}{2}$$

Setting ($B=0=C$) and ($A=\omega_+$) and ($D=\omega_-$) gives diagonal block matrices.

$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

	$\|00\rangle$	$\|01\rangle$	$\|10\rangle$	$\|02\rangle$	$\|11\rangle$	$\|20\rangle$	$\|03\rangle$	$\|12\rangle$	$\|21\rangle$	$\|30\rangle$	$\cdots$
$\langle 00\|$	0										
$\langle 01\|$		ω_-									
$\langle 10\|$			ω_+								
$\langle 02\|$				$2\omega_-$							
$\langle 11\|$					$\omega_+ + \omega_-$						
$\langle 20\|$						$2\omega_+$					
$\langle 03\|$							$3\omega_-$				
$\langle 12\|$								$\omega_+ + 2\omega_-$			
$\langle 21\|$									$2\omega_+ + \omega_-$		
$\langle 30\|$										$3\omega_+$	
$\vdots$											

$$\omega_+ - \omega_- = \Omega = \sqrt{(2B)^2 + (2C)^2 + (A-D)^2} = A - D$$

$$\mathbf{H}^A = A\left(\mathbf{a}_1^\dagger \mathbf{a}_1 + \mathbf{1}/2\right) + D\left(\mathbf{a}_2^\dagger \mathbf{a}_2 + \mathbf{1}/2\right)$$

$$\varepsilon^A_{n_1 n_2} = A\left(n_1 + \frac{1}{2}\right) + D\left(n_2 + \frac{1}{2}\right) = \frac{A+D}{2}(n_1 + n_2 + 1) + \frac{A-D}{2}(n_1 - n_2)$$

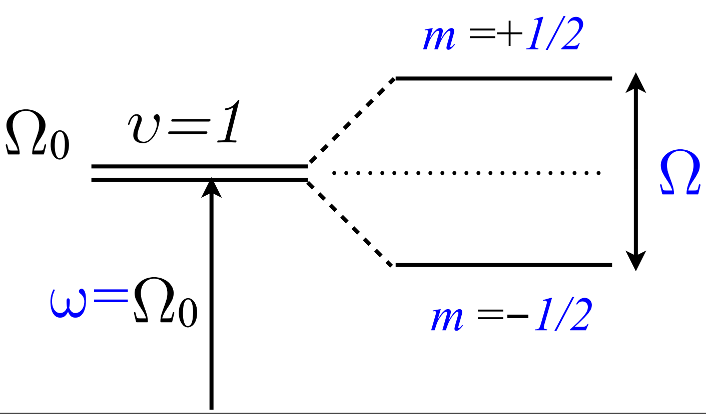
$$= \Omega_0 (n_1 + n_2 + 1) + \frac{\Omega}{2}(n_1 - n_2) = \Omega_0(\upsilon + 1) + \Omega\, m$$

Define *total quantum number* $\upsilon=2j$ and half-difference or *asymmetry quantum number m*

$$\upsilon = n_1 + n_2 = 2j \qquad j = \frac{n_1 + n_2}{2} = \frac{\upsilon}{2} \qquad m = \frac{n_1 - n_2}{2}$$

$\upsilon+1=2j+1$ multiplies *base frequency* $\omega=\Omega_0$

m multiplies *beat frequency* Ω



$$\omega_+ = \Omega_0 + \Omega(+\tfrac{1}{2})$$

$$\omega_- = \Omega_0 + \Omega(-\tfrac{1}{2})$$

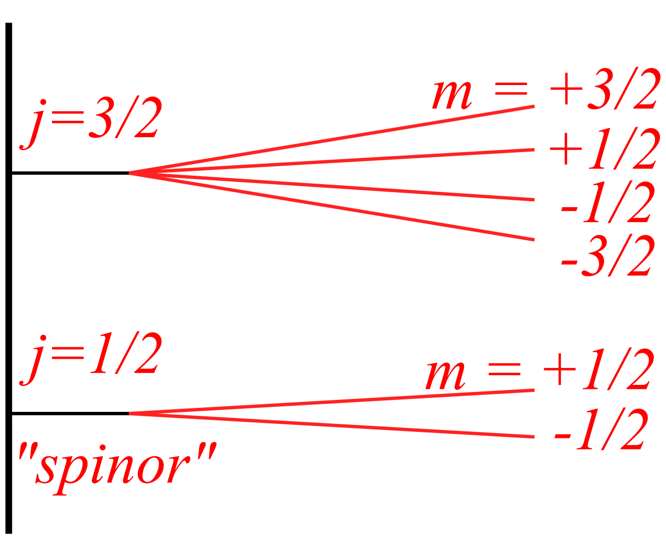
Setting (B=0=C) and (A=ω_+) and (D=ω_-) gives diagonal block matrices.

$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

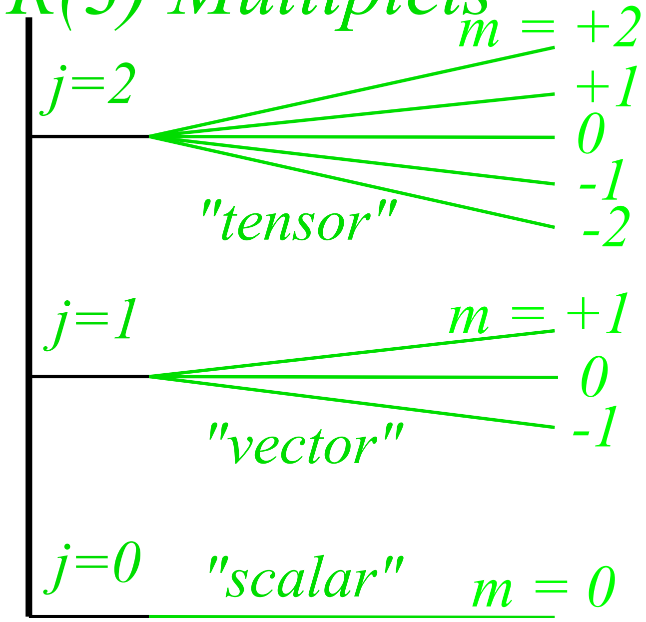
	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 10\rangle$	$\vert 02\rangle$	$\vert 11\rangle$	$\vert 20\rangle$	$\vert 03\rangle$	$\vert 12\rangle$	$\vert 21\rangle$	$\vert 30\rangle$	$\cdots$
$\langle 00\vert$	0										
$\langle 01\vert$		ω_-									
$\langle 10\vert$			ω_+								
$\langle 02\vert$				$2\omega_-$							
$\langle 11\vert$					$\omega_+ + \omega_-$						
$\langle 20\vert$						$2\omega_+$					
$\langle 03\vert$							$3\omega_-$				
$\langle 12\vert$								$\omega_+ + 2\omega_-$			
$\langle 21\vert$									$2\omega_+ + \omega_-$		
$\langle 30\vert$										$3\omega_+$	
$\vdots$											

$$\omega_+ - \omega_- = \Omega$$
$$= \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}$$
$$= A - D$$

SU(2) Multiplets



R(3) Multiplets

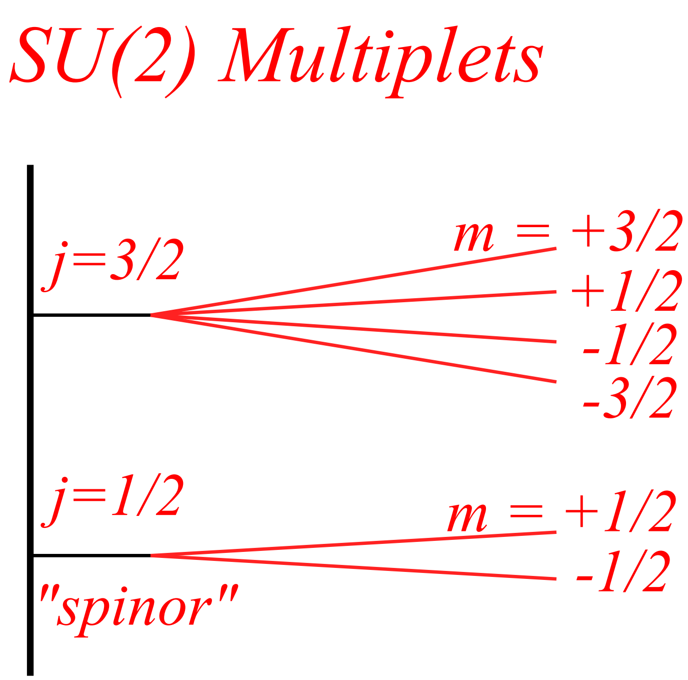


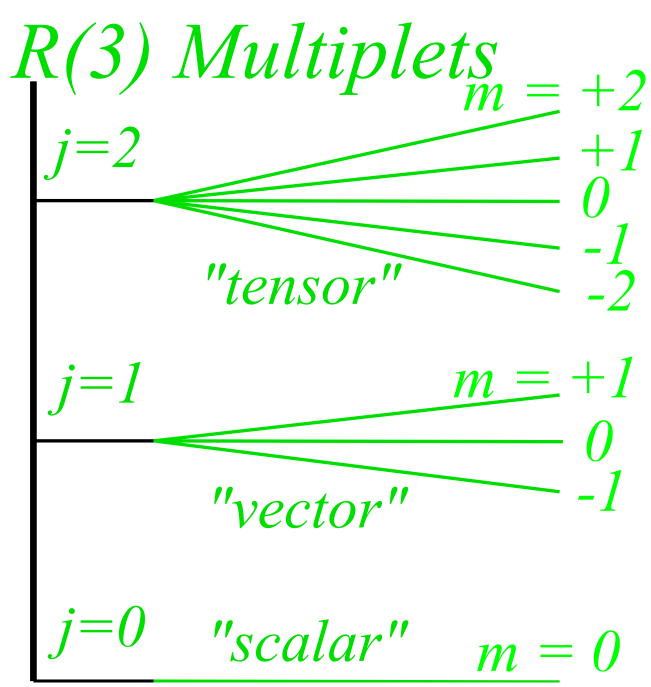
Setting $(B=0=C)$ and $(A=\omega_+)$ and $(D=\omega_-)$ gives diagonal block matrices.

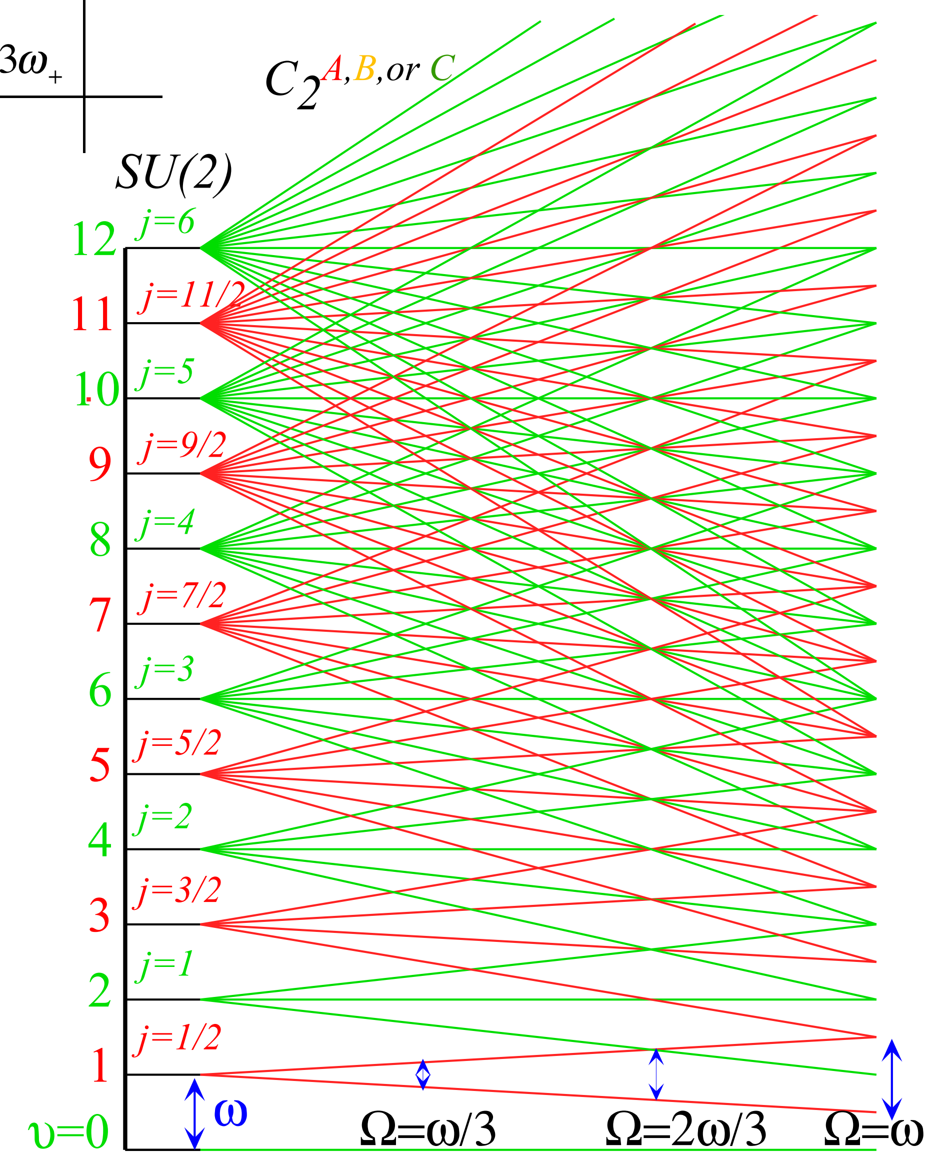
$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 10\rangle$	$\vert 02\rangle$	$\vert 11\rangle$	$\vert 20\rangle$	$\vert 03\rangle$	$\vert 12\rangle$	$\vert 21\rangle$	$\vert 30\rangle$	$\cdots$
$\langle 00\vert$	0										
$\langle 01\vert$		ω_-									
$\langle 10\vert$			ω_+								
$\langle 02\vert$				$2\omega_-$							
$\langle 11\vert$					$\omega_+ + \omega_-$						
$\langle 20\vert$						$2\omega_+$					
$\langle 03\vert$							$3\omega_-$				
$\langle 12\vert$								$\omega_+ + 2\omega_-$			
$\langle 21\vert$									$2\omega_+ + \omega_-$		
$\langle 30\vert$										$3\omega_+$	
$\vdots$											

$$\omega_+ - \omega_- = \Omega$$
$$= \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}$$
$$= A - D$$







Structure of U(2)

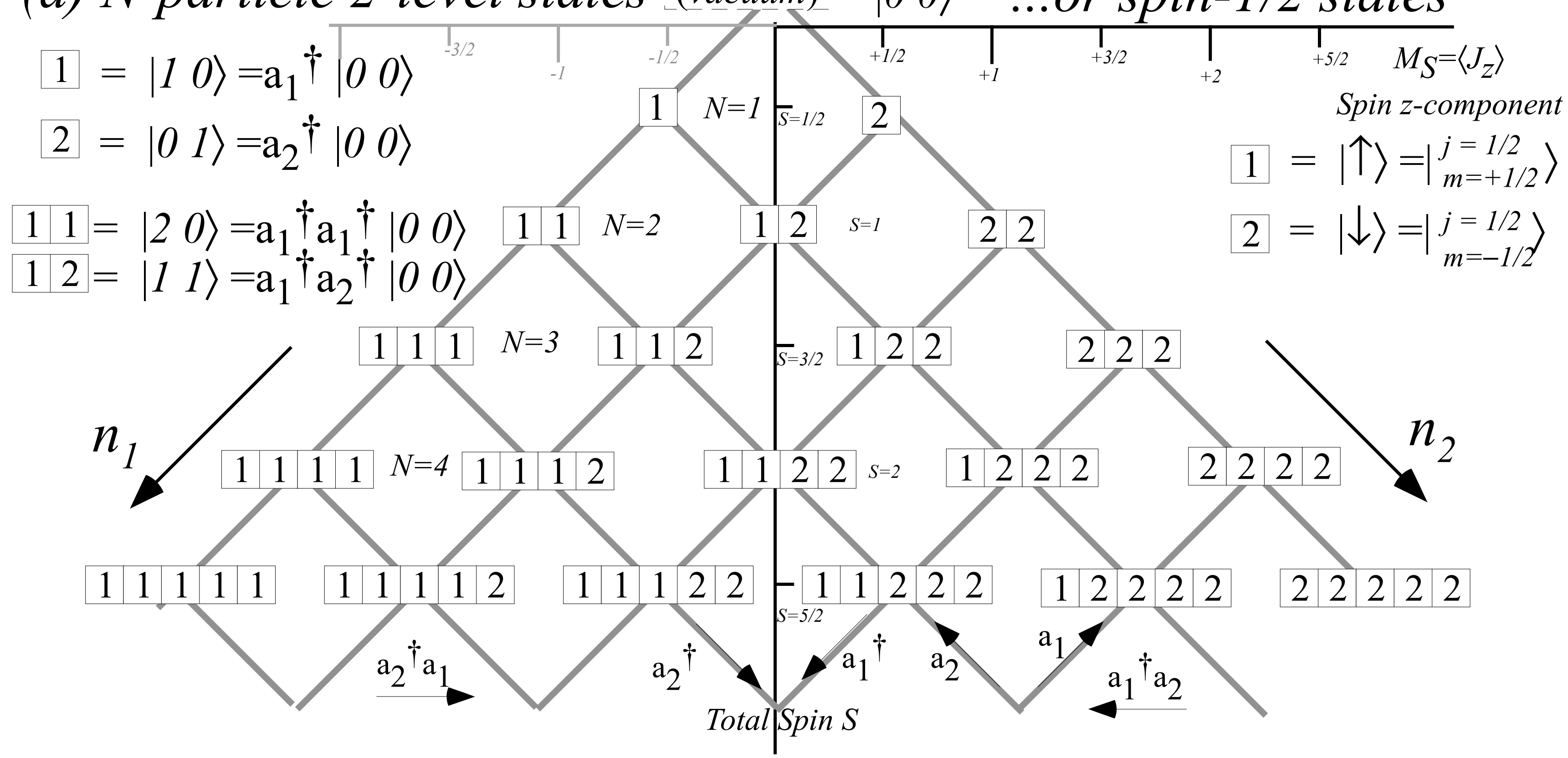
$\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle = |n_1 n_2\rangle$

$j=0$	$\left\|\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right\rangle = \|00\rangle$	"scalar"
$j=\frac{1}{2}$	$\left\|\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right\rangle = \|10\rangle = \|\uparrow\rangle$ $\left\|\begin{smallmatrix} 1/2 \\ -1/2 \end{smallmatrix}\right\rangle = \|01\rangle = \|\downarrow\rangle$	"spinor"
$j=1$	$\left\|\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right\rangle = \|20\rangle$ $\left\|\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right\rangle = \|11\rangle$ $\left\|\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right\rangle = \|02\rangle$	"3-vector"
$j=\frac{3}{2}$	$\left\|\begin{smallmatrix} 3/2 \\ 1/2 \end{smallmatrix}\right\rangle = \|30\rangle$ $\left\|\begin{smallmatrix} 3/2 \\ 1/2 \end{smallmatrix}\right\rangle = \|21\rangle$ $\left\|\begin{smallmatrix} 3/2 \\ -1/2 \end{smallmatrix}\right\rangle = \|12\rangle$ $\left\|\begin{smallmatrix} 3/2 \\ -3/2 \end{smallmatrix}\right\rangle = \|03\rangle$	"4-spinor"
$j=2$	$\left\|\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right\rangle = \|40\rangle$ $\left\|\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right\rangle = \|31\rangle$ $\left\|\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right\rangle = \|22\rangle$ $\left\|\begin{smallmatrix} 2 \\ -1 \end{smallmatrix}\right\rangle = \|13\rangle$ $\left\|\begin{smallmatrix} 2 \\ -2 \end{smallmatrix}\right\rangle = \|04\rangle$	"tensor"
	$\vdots$	

$$j = \frac{\upsilon}{2} = \frac{n_1 + n_2}{2} \qquad n_1 = j + m = 2\upsilon + m$$

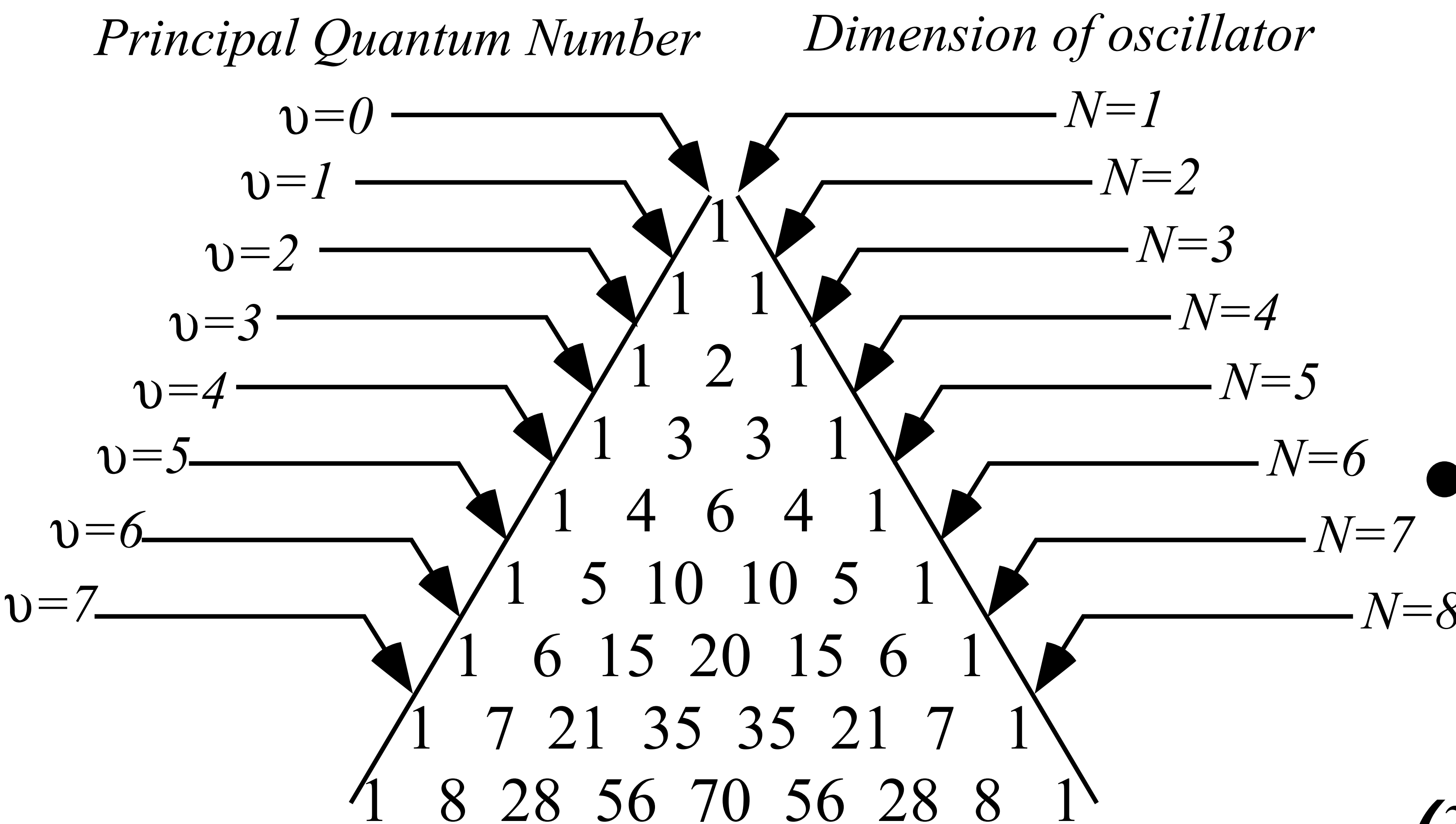
$$m = \frac{n_1 - n_2}{2} \qquad n_2 = j - m = 2\upsilon - m$$

(a) N-particle 2-level states (vacuum) = $|0\ 0\rangle$ ...or spin-1/2 states

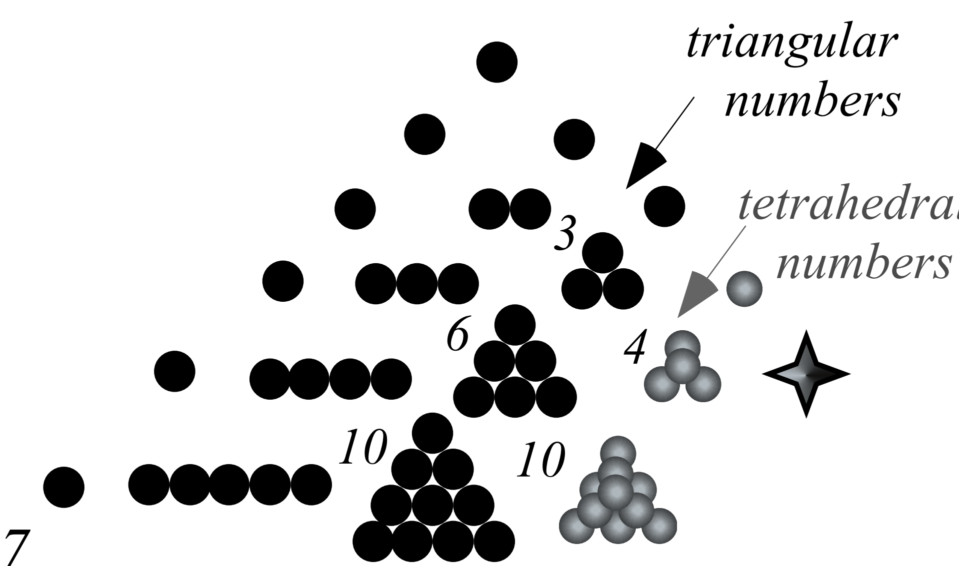


Introducing U(N)

(a) N-D Oscillator Degeneracy ℓ of quamtum level υ

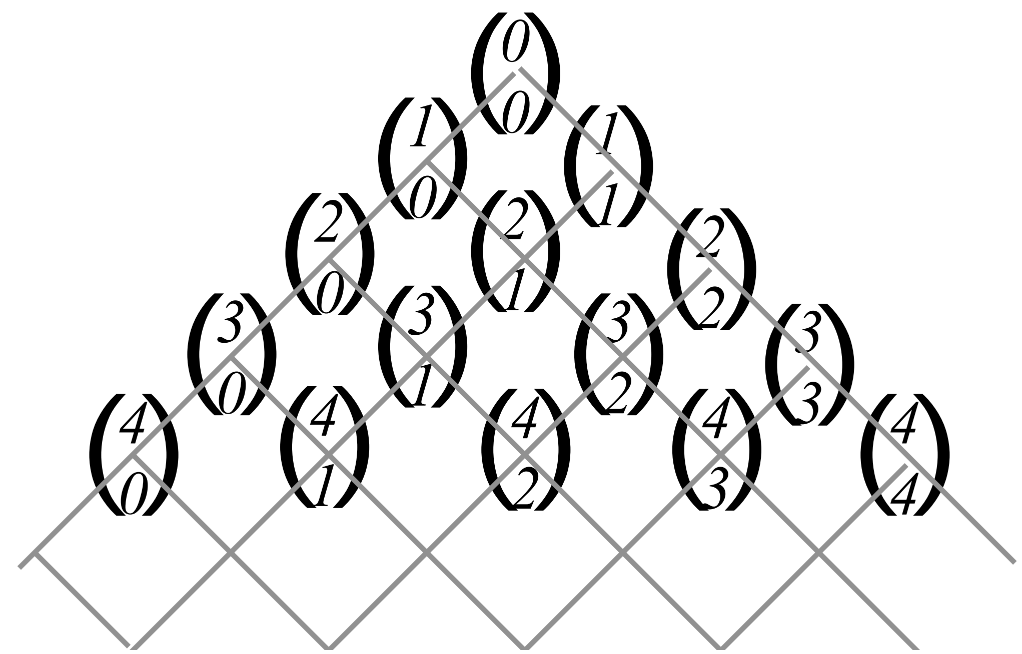


(b) Stacking numbers



(c) Binomial coefficients

$$\frac{(N-1+\upsilon)!}{(N-1)!\,\upsilon!} = \binom{N-1+\upsilon}{\upsilon} = \binom{N-1+\upsilon}{N-1}$$



Introducing U(3)

(b) N-particle 3-level states ...or spin-1 states

$\boxed{1} = |1\ 0\ 0\rangle = a_1^\dagger\, |0\ 0\ 0\rangle$

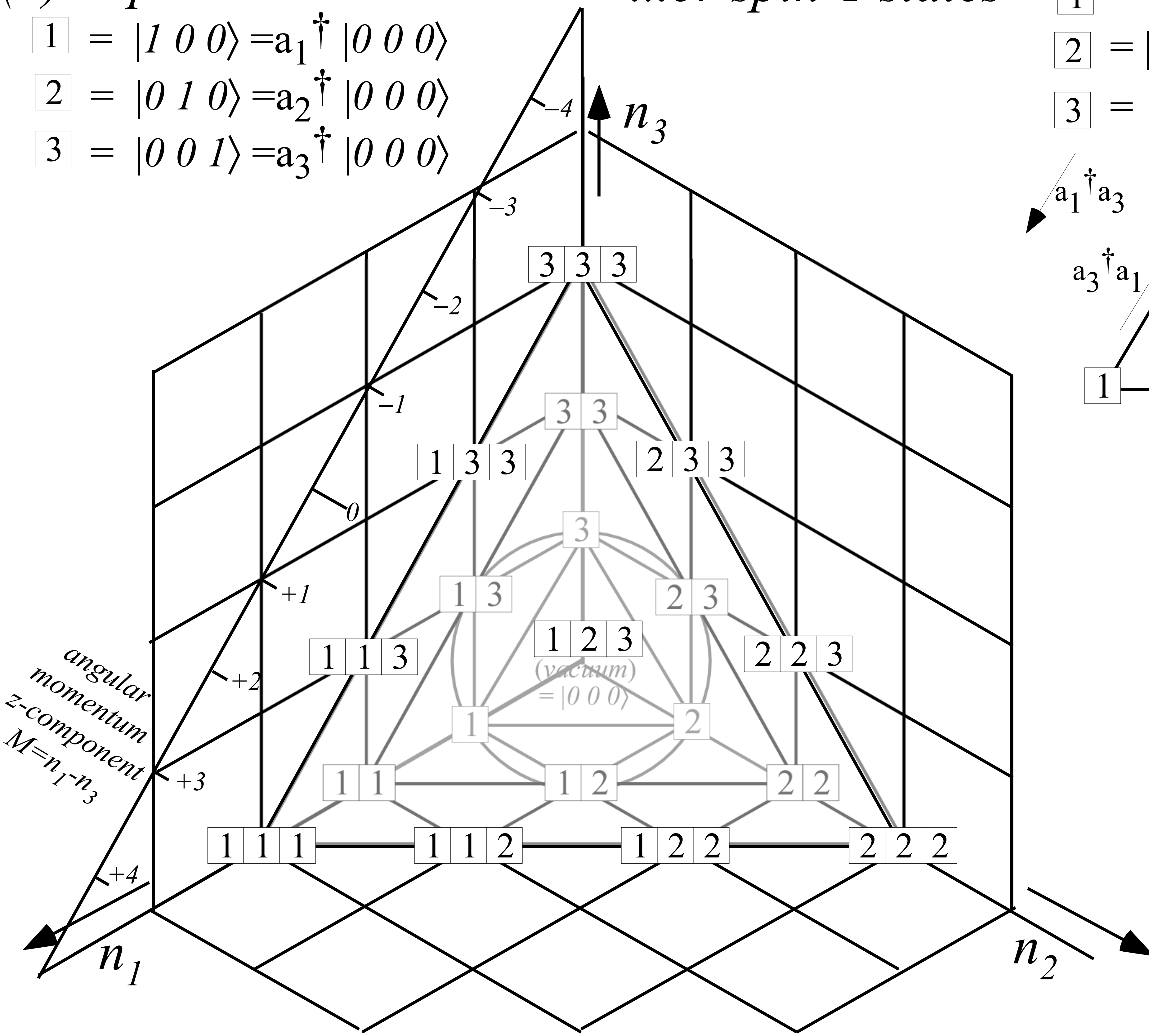
$\boxed{2} = |0\ 1\ 0\rangle = a_2^\dagger\, |0\ 0\ 0\rangle$

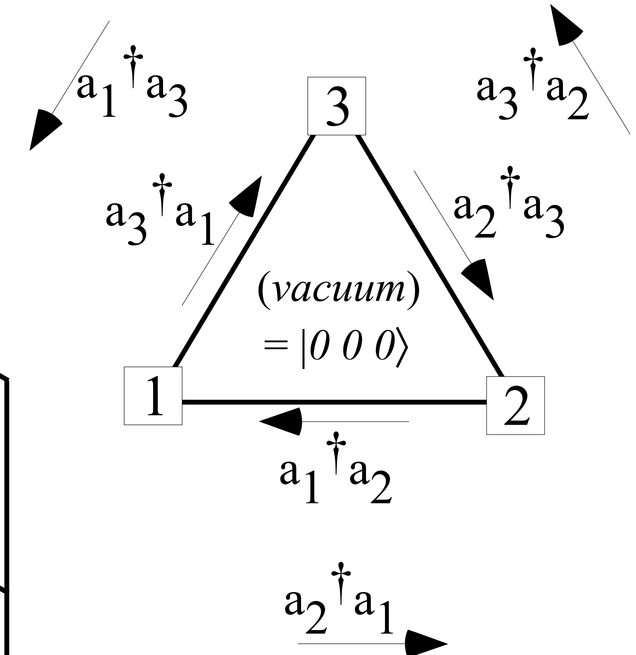
$\boxed{3} = |0\ 0\ 1\rangle = a_3^\dagger\, |0\ 0\ 0\rangle$

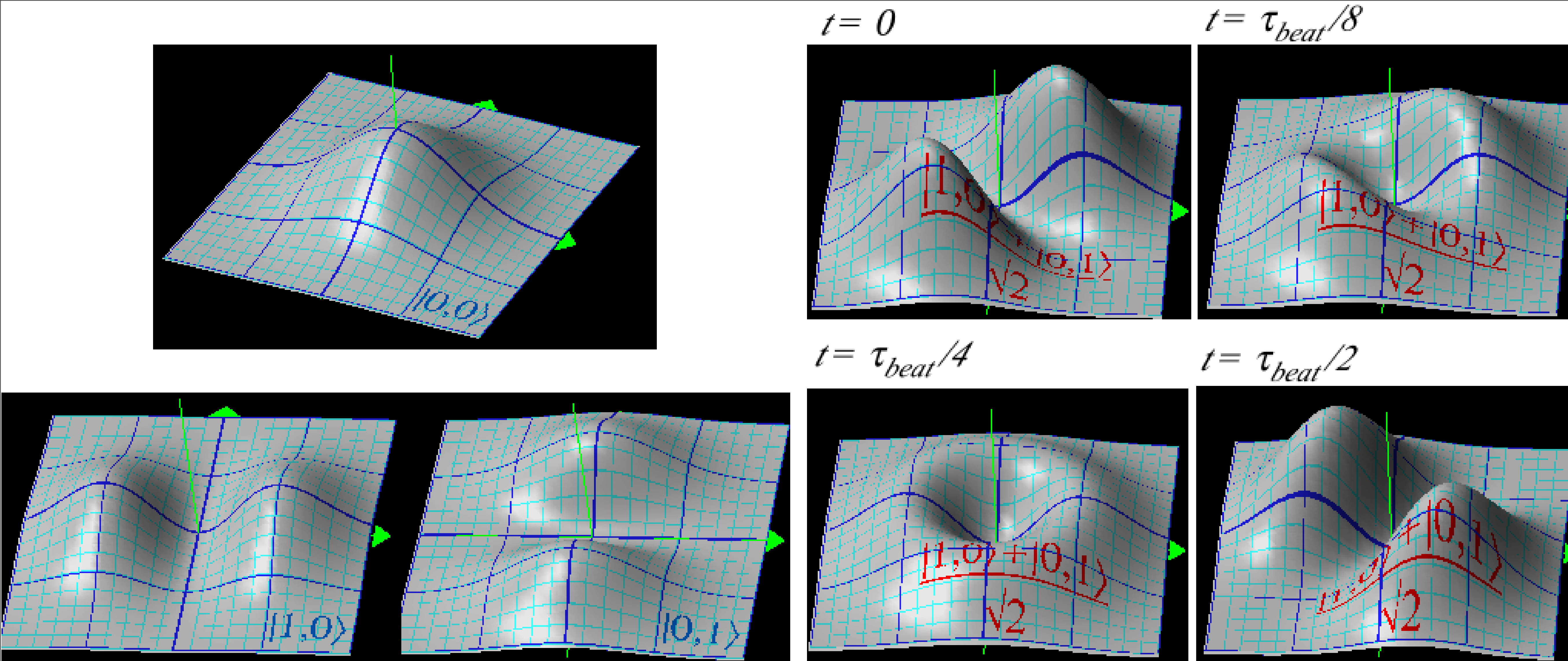
$\boxed{1} = |\uparrow\rangle = |{}^{j=1}_{m=+1}\rangle$

$\boxed{2} = |\leftrightarrow\rangle = |{}^{j=1}_{m=0}\rangle$

$\boxed{3} = |\downarrow\rangle = |{}^{j=1}_{m=-1}\rangle$







$$\Psi\left(x_1,x_2,t\right)=\frac{1}{2}\left|\psi_{10}\left(x_1,x_2\right)e^{-i\omega_{10}t}+\psi_{01}\left(x_1,x_2\right)e^{-i\omega_{01}t}\right|^2 e^{-\left(x_1^2+x_2^2\right)}=\frac{e^{-\left(x_1^2+x_2^2\right)}}{2\pi}\left|\sqrt{2}x_1e^{-i\omega_{10}t}+\sqrt{2}x_1e^{-i\omega_{01}t}\right|^2$$

$$=\frac{e^{-\left(x_1^2+x_2^2\right)}}{\pi}\left(x_1^2+x_2^2+2x_1x_2\cos\left(\omega_{10}-\omega_{01}\right)t\right)=\frac{e^{-\left(x_1^2+x_2^2\right)}}{\pi}\begin{cases}\left|x_1+x_2\right|^2 & for\text{: } t\text{=}0\\ x_1^2+x_2^2 & for\text{: } t\text{=}\tau_{\text{beat}}/4\\ \left|x_1-x_2\right|^2 & for\text{: } t\text{=}\tau_{\text{beat}}/2\end{cases}\qquad(21.1.30)$$