AMOP reference links on following page

2.05.18 class 7.0: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

Symmetry group $\mathscr{G}$=U(1) representations, 1D HO Hamiltonian $\mathbf{H}=\hbar\omega\mathbf{a}^\dagger\mathbf{a}$ operators, 1D HO wave eigenfunctions Ψ_n, and coherent α-states

Factoring 1D-HO Hamiltonian $\mathbf{H}=\mathbf{p}^2+\mathbf{x}^2$

Creation-Destruction $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(1) operatorss*

Eigenstate creationism (and destructionism)

Vacuum state $|0\rangle$, *1st excited state* $|1\rangle$, $|2\rangle$, ...

Normal ordering for matrix calculation (creation $\mathbf{a}^\dagger$ *on left, destruction* $\mathbf{a}$ *on right)*

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$\langle\mathbf{a}^n\mathbf{a}^{\dagger n}\rangle$ *operator calculations*

Number operator and Hamiltonian operator

Expectation values of position, momentum, and uncertainty for eigenstate $|n\rangle$

Harmonic oscillator beat dynamics of mixed states

Oscillator coherent states ("Shoved" and "kicked" states)

Translation operators vs. boost operators, boost-translation combinations

Time evolution of a coherent state $|\alpha\rangle$

Properties of coherent states and "squeezed" states

AMOP reference links *(Updated list given on 2nd page of each class presentation)*

Web Resources - front page

UAF Physics UTube channel

2014 AMOP

2017 Group Theory for QM

2018 AMOP

Frame Transformation Relations And Multipole Transitions In Symmetric Polyatomic Molecules - RMP-1978 (Alt Scanned version)
Rotational energy surfaces and high- J eigenvalue structure of polyatomic molecules - Harter - Patterson - 1984
Galloping waves and their relativistic properties - ajp-1985-Harter
Asymptotic eigensolutions of fourth and sixth rank octahedral tensor operators - Harter-Patterson-JMP-1979
Nuclear spin weights and gas phase spectral structure of 12C60 and 13C60 buckminsterfullerene -Harter-Reimer-Cpl-1992 - (Alt1, Alt2 Erratum)

Theory of hyperfine and superfine levels in symmetric polyatomic molecules.

I) Trigonal and tetrahedral molecules: Elementary spin-1/2 cases in vibronic ground states - PRA-1979-Harter-Patterson (Alt scan)
II) Elementary cases in octahedral hexafluoride molecules - Harter-PRA-1981 (Alt scan)

Rotation-vibration scalar coupling zeta coefficients and spectroscopic band shapes of buckminsterfullerene - Weeks-Harter-CPL-1991 (Alt scan)
Fullerene symmetry reduction and rotational level fine structure/ the Buckyball isotopomer 12C 13C59 - jcp-Reimer-Harter-1997 (HiRez)
Molecular Eigensolution Symmetry Analysis and Fine Structure - IJMS-harter-mitchell-2013

Rotation–vibration spectra of icosahedral molecules.

I) Icosahedral symmetry analysis and fine structure - harter-weeks-jcp-1989
II) Icosahedral symmetry, vibrational eigenfrequencies, and normal modes of buckminsterfullerene - weeks-harter-jcp-1989
III) Half-integral angular momentum - harter-reimer-jcp-1991

QTCA Unit 10 Ch 30 - 2013
AMOP Ch 32 Molecular Symmetry and Dynamics - 2019
AMOP Ch 0 Space-Time Symmetry - 2019

RESONANCE AND REVIVALS

I) QUANTUM ROTOR AND INFINITE-WELL DYNAMICS - ISMSLi2012 (Talk) https://kb.osu.edu/dspace/handle/1811/52324
II) Comparing Half-integer Spin and Integer Spin - Alva-ISMS-Ohio2013-R777 (Talks)
III) Quantum Resonant Beats and Revivals in the Morse Oscillators and Rotors - (2013-Li-Diss)

Rovibrational Spectral Fine Structure Of Icosahedral Molecules - Cpl 1986 (Alt Scan)
Gas Phase Level Structure of C60 Buckyball and Derivatives Exhibiting Broken Icosahedral Symmetry - reimer-diss-1996
Resonance and Revivals in Quantum Rotors - Comparing Half-integer Spin and Integer Spin - Alva-ISMS-Ohio2013-R777 (Talk)
Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-cpl-2013
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - harter - jms - 2001

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Relawavity eq(57) p.63

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Proof: *QED:*

$$\langle x|\mathbf{xp}-\mathbf{px}|\psi\rangle = \frac{\hbar}{i}\left(x\frac{\partial}{\partial x}\psi(x) - \frac{\partial}{\partial x}x\psi(x)\right) = \frac{\hbar}{i}\left(x\frac{\partial}{\partial x}\psi(x) - x\frac{\partial}{\partial x}\psi(x) - \frac{\partial x}{\partial x}\psi(x)\right) = -\frac{\hbar}{i}\psi(x) = \hbar i\psi(x) = \langle x|[\mathbf{x},\mathbf{p}]|\psi\rangle$$

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$$\mathbf{a} = \frac{(\mathbf{X}+i\mathbf{P})}{\sqrt{\hbar\omega}} = \frac{\left(\sqrt{M\omega}\,\mathbf{x}+i\mathbf{p}/\sqrt{M\omega}\right)}{\sqrt{2\hbar}}$$

Destruction operator

and

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Creation Operator

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$$\left[\mathbf{a},\mathbf{a}^\dagger\right] \equiv \mathbf{a}\mathbf{a}^\dagger-\mathbf{a}^\dagger\mathbf{a}=\frac{1}{2\hbar}\left(\sqrt{M\omega}\,\mathbf{x}+i\mathbf{p}/\sqrt{M\omega}\right)\left(\sqrt{M\omega}\,\mathbf{x}-i\mathbf{p}/\sqrt{M\omega}\right)-\frac{1}{2\hbar}\left(\sqrt{M\omega}\,\mathbf{x}-i\mathbf{p}/\sqrt{M\omega}\right)\left(\sqrt{M\omega}\,\mathbf{x}+i\mathbf{p}/\sqrt{M\omega}\right)$$

Creation-Destruction $\mathbf{a}^\dagger\mathbf{a}$ *algebra*

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$$\mathbf{a} = \frac{(\mathbf{X} + i\mathbf{P})}{\sqrt{\hbar\omega}} = \frac{\left(\sqrt{M\omega}\,\mathbf{x} + i\mathbf{p}/\sqrt{M\omega}\right)}{\sqrt{2\hbar}}$$
Destruction operator

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Creation Operator

Commutation relations between $\mathbf{a} = (\mathbf{X}+i\mathbf{P})/2$ and $\mathbf{a}^\dagger = (\mathbf{X}-i\mathbf{P})/2$ with $\mathbf{X}\equiv\sqrt{M\omega}\mathbf{x}/\sqrt{2}$ and $\mathbf{P}\equiv\mathbf{p}/\sqrt{2M}$:

$$\left[\mathbf{a},\mathbf{a}^\dagger\right] \equiv \mathbf{a}\mathbf{a}^\dagger - \mathbf{a}^\dagger\mathbf{a} = \frac{1}{2\hbar}\left(\sqrt{M\omega}\,\mathbf{x} + i\mathbf{p}/\sqrt{M\omega}\right)\left(\sqrt{M\omega}\,\mathbf{x} - i\mathbf{p}/\sqrt{M\omega}\right) - \frac{1}{2\hbar}\left(\sqrt{M\omega}\,\mathbf{x} - i\mathbf{p}/\sqrt{M\omega}\right)\left(\sqrt{M\omega}\,\mathbf{x} + i\mathbf{p}/\sqrt{M\omega}\right)$$

$$\left[\mathbf{a},\mathbf{a}^\dagger\right] = \frac{2i}{2\hbar}\left(\mathbf{p}\mathbf{x} - \mathbf{x}\mathbf{p}\right) = \frac{-i}{\hbar}\left[\mathbf{x},\mathbf{p}\right]$$

Creation-Destruction $\mathbf{a}^\dagger\mathbf{a}$ *algebra*

Define
$$\mathbf{a} = \frac{(\mathbf{X} + i\mathbf{P})}{\sqrt{\hbar\omega}} = \frac{\left(\sqrt{M\omega}\,\mathbf{x} + i\mathbf{p}/\sqrt{M\omega}\right)}{\sqrt{2\hbar}}$$
Destruction operator

and
$$\mathbf{a}^\dagger = \frac{(\mathbf{X} - i\mathbf{P})}{\sqrt{\hbar\omega}} = \frac{\left(\sqrt{M\omega}\,\mathbf{x} - i\mathbf{p}/\sqrt{M\omega}\right)}{\sqrt{2\hbar}}$$
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$$\left[\mathbf{a},\mathbf{a}^\dagger\right] = \frac{2i}{2\hbar}(\mathbf{px} - \mathbf{xp}) = \frac{-i}{\hbar}[\mathbf{x},\mathbf{p}] = \mathbf{1}$$

Recall *commutator* $[\mathbf{x}\ ,\ \mathbf{p}]$ *relation*: $[\mathbf{x},\mathbf{p}] \equiv \mathbf{xp} - \mathbf{px} = \hbar i\,\mathbf{1}$

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1D-HO Hamiltonian in terms of $\mathbf{a}^{\dagger}\mathbf{a}$ *operator*

Recall: $\mathbf{H}(\mathbf{x},\mathbf{p}) = \hbar\omega\,(\mathbf{a}^{\dagger}\mathbf{a} + \mathbf{a}\mathbf{a}^{\dagger})/2$

Recall *commutator* [**x** , **p**] *relation*: $\left[\mathbf{x},\mathbf{p}\right] \equiv \mathbf{x}\mathbf{p} - \mathbf{p}\mathbf{x} = \hbar i\,\mathbf{1}$

Creation-Destruction $\mathbf{a}^\dagger\mathbf{a}$ *algebra*

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1D-HO Hamiltonian in terms of $\mathbf{a}^\dagger\mathbf{a}$ *operator*

$$\mathbf{H}(\mathbf{x},\mathbf{p}) = \hbar\omega\,(\mathbf{a}^\dagger\mathbf{a} + \mathbf{a}\mathbf{a}^\dagger)/2 \quad = \hbar\omega\,(\mathbf{a}^\dagger\mathbf{a} + \mathbf{a}^\dagger\mathbf{a} + \mathbf{1})/2$$

Recall *commutator* [$\mathbf{x}$, $\mathbf{p}$] *relation*: $[\mathbf{x},\mathbf{p}] \equiv \mathbf{xp} - \mathbf{px} = \hbar i\,\mathbf{1}$

Creation-Destruction $\mathbf{a^\dagger a}$ *algebra*

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Destruction operator

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or

$$\mathbf{a}\,\mathbf{a}^\dagger = \mathbf{a}^\dagger\mathbf{a} + \mathbf{1}$$

1D-HO Hamiltonian in terms of $\mathbf{a^\dagger a}$ *operator*

$$\mathbf{H}(\mathbf{x},\mathbf{p}) = \hbar\omega\,(\mathbf{a}^\dagger\mathbf{a} + \mathbf{aa}^\dagger)/2 = \hbar\omega\,(\mathbf{a}^\dagger\mathbf{a} + \mathbf{a}^\dagger\mathbf{a} + \mathbf{1})/2 = \hbar\omega\,\mathbf{a}^\dagger\mathbf{a} + \mathbf{1}\hbar\omega/2$$

Recall *commutator* [**x** , **p**] *relation*: $[\mathbf{x},\mathbf{p}] \equiv \mathbf{xp}-\mathbf{px} = \hbar i\,\mathbf{1}$

Symmetry group $\mathscr{G}$=U(1) representations, 1D HO Hamiltonian $\mathbf{H}=\hbar\omega\mathbf{a}^\dagger\mathbf{a}$ operators, 1D HO wave eigenfunctions Ψ_n, and coherent α-states

Factoring 1D-HO Hamiltonian $\mathbf{H}=\mathbf{p}^2+\mathbf{x}^2$

Creation-Destruction $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(1) operators*

Eigenstate creationism (and destructionism)

Vacuum state $|0>$, *1st excited state* $|1>$, $|2>$, ...

Normal ordering for matrix calculation (creation $\mathbf{a}^\dagger$ *on left, destruction* $\mathbf{a}$ *on right)*

Commutator derivative identities, Binomial expansion identities

$\langle\mathbf{a}^n\mathbf{a}^{\dagger n}\rangle$ *operator calculations*

Number operator and Hamiltonian operator

Expectation values of position, momentum, and uncertainty for eigenstate $|n\rangle$

Harmonic oscillator beat dynamics of mixed states

Oscillator coherent states ("Shoved" and "kicked" states)

Translation operators vs. boost operators, boost-translation combinations

Time evolution of a coherent state $|\alpha>$

Properties of coherent states and "squeezed" states

Eigenstate creationism (and destructionism)

Given 1D-HO Hamiltonian: $\mathbf{H}(\mathbf{x},\mathbf{p}) = \hbar\omega\, \mathbf{a}^\dagger \mathbf{a} + \mathbf{1}\hbar\omega/2$ *and commutation:* $\left[\mathbf{a},\mathbf{a}^\dagger\right] = \mathbf{1}$ or $\mathbf{a}\mathbf{a}^\dagger = \mathbf{a}^\dagger\mathbf{a} + \mathbf{1}$

Define *ground state* $|0\rangle$ as the eigenstate of $\mathbf{H}(\mathbf{x},\mathbf{p})$ with the *zero point eigenvalue* $E_0 = \hbar\omega/2$.

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Action by **a** on ground ket $|0\rangle$ (or $\mathbf{a}^\dagger$ on ground bra $\langle 0|$) gives *nothing* (zero vectors $\boldsymbol{0}$).

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Proof:

$$\mathbf{H}(\mathbf{x},\mathbf{p})\,\mathbf{a}^\dagger|0\rangle = \hbar\omega\,\mathbf{a}^\dagger\mathbf{a}\,\mathbf{a}^\dagger|0\rangle \qquad + \hbar\omega/2\,\mathbf{a}^\dagger|0\rangle$$

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QED:

$$\mathbf{H}(\mathbf{x},\mathbf{p})\,|1\rangle = (\hbar\omega + \hbar\omega/2)\,|1\rangle = E_1\,|1\rangle$$ where: $E_1 = \hbar\omega + E_0$

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One-quantum or *1st excited eigenket* $|1\rangle = \mathbf{a}^\dagger|0\rangle$

Eigenstate creationism (and destructionism)

Given 1D-HO Hamiltonian: $\mathbf{H}(\mathbf{x},\mathbf{p}) = \hbar\omega\, \mathbf{a}^\dagger \mathbf{a} + \mathbf{1}\hbar\omega/2$ *and commutation:* $[\mathbf{a},\mathbf{a}^\dagger] = \mathbf{1}$ or $\mathbf{a}\mathbf{a}^\dagger = \mathbf{a}^\dagger\mathbf{a} + \mathbf{1}$

Define *ground state* $|0\rangle$ as the eigenstate of $\mathbf{H}(\mathbf{x},\mathbf{p})$ with the *zero point eigenvalue* $E_0 = \hbar\omega/2$.

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One-quantum or *1st excited eigenket* $|1\rangle = \mathbf{a}^\dagger|0\rangle$

For kets, $\mathbf{a}^\dagger$ is *creation operator* while $\mathbf{a}$ is *destruction operator*.

$$\mathbf{a}|1\rangle = \mathbf{a}\mathbf{a}^\dagger|0\rangle = (\mathbf{a}^\dagger\mathbf{a} + \mathbf{1})|0\rangle = |0\rangle$$

Eigenstate creationism (and destructionism)

Given 1D-HO Hamiltonian: $\mathbf{H}(\mathbf{x},\mathbf{p}) = \hbar\omega\, \mathbf{a}^\dagger \mathbf{a} + \mathbf{1}\hbar\omega/2$ *and commutation:* $[\mathbf{a},\mathbf{a}^\dagger] = \mathbf{1}$ or $\mathbf{a}\mathbf{a}^\dagger = \mathbf{a}^\dagger\mathbf{a} + \mathbf{1}$

Define *ground state* $|0\rangle$ as the eigenstate of $\mathbf{H}(\mathbf{x},\mathbf{p})$ with the *zero point eigenvalue* $E_0 = \hbar\omega/2$.

$$\mathbf{H}(\mathbf{x},\mathbf{p})\,|0\rangle = \hbar\omega/2\,|0\rangle \qquad \langle 0|\,\mathbf{H}(\mathbf{x},\mathbf{p}) = \hbar\omega/2\,\langle 0|$$

Action by $\mathbf{a}$ on ground ket $|0\rangle$ (or $\mathbf{a}^\dagger$ on ground bra $\langle 0|$) gives *nothing* (zero vectors $\boldsymbol{0}$).

$$\mathbf{a}\,|0\rangle = \boldsymbol{0} \qquad \langle 0|\,\mathbf{a}^\dagger = \boldsymbol{0}$$

But, $\mathbf{a}^\dagger$ acts on ground ket to give $|1\rangle = \mathbf{a}^\dagger|0\rangle$ with $\mathbf{H}$ eigenvalue $E_1 = \hbar\omega + E_0$. ($|1\rangle = \mathbf{a}^\dagger|0\rangle$, $\langle 0|\mathbf{a} = \langle 1|$.)

Proof:

$$\mathbf{H}(\mathbf{x},\mathbf{p})\,\mathbf{a}^\dagger|0\rangle = \hbar\omega\,\mathbf{a}^\dagger\mathbf{a}\,\mathbf{a}^\dagger|0\rangle + \hbar\omega/2\,\mathbf{a}^\dagger|0\rangle$$

$$\mathbf{H}(\mathbf{x},\mathbf{p})\,\mathbf{a}^\dagger|0\rangle = \hbar\omega\,\mathbf{a}^\dagger(\mathbf{a}^\dagger\mathbf{a} + \mathbf{1})|0\rangle + \hbar\omega/2\,\mathbf{a}^\dagger|0\rangle$$

$$= \hbar\omega\,\mathbf{a}^\dagger|0\rangle + \boldsymbol{0} + \hbar\omega/2\,\mathbf{a}^\dagger|0\rangle$$

QED:

$$\mathbf{H}(\mathbf{x},\mathbf{p})\,|1\rangle = (\hbar\omega + \hbar\omega/2)\,|1\rangle = E_1\,|1\rangle$$ where: $E_1 = \hbar\omega + E_0$

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Coordinate representation of the "nothing" equation $\langle x|\,\mathbf{a}|0\rangle = \boldsymbol{0}$ with: $\mathbf{p} = \mathbf{hk} = \frac{\hbar}{i}\frac{\partial}{\partial x}$

$$\langle x|\mathbf{a}|0\rangle = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{M\omega}\,\langle x|\mathbf{x}|0\rangle + i\langle x|\mathbf{p}|0\rangle / \sqrt{M\omega}\right) = 0$$

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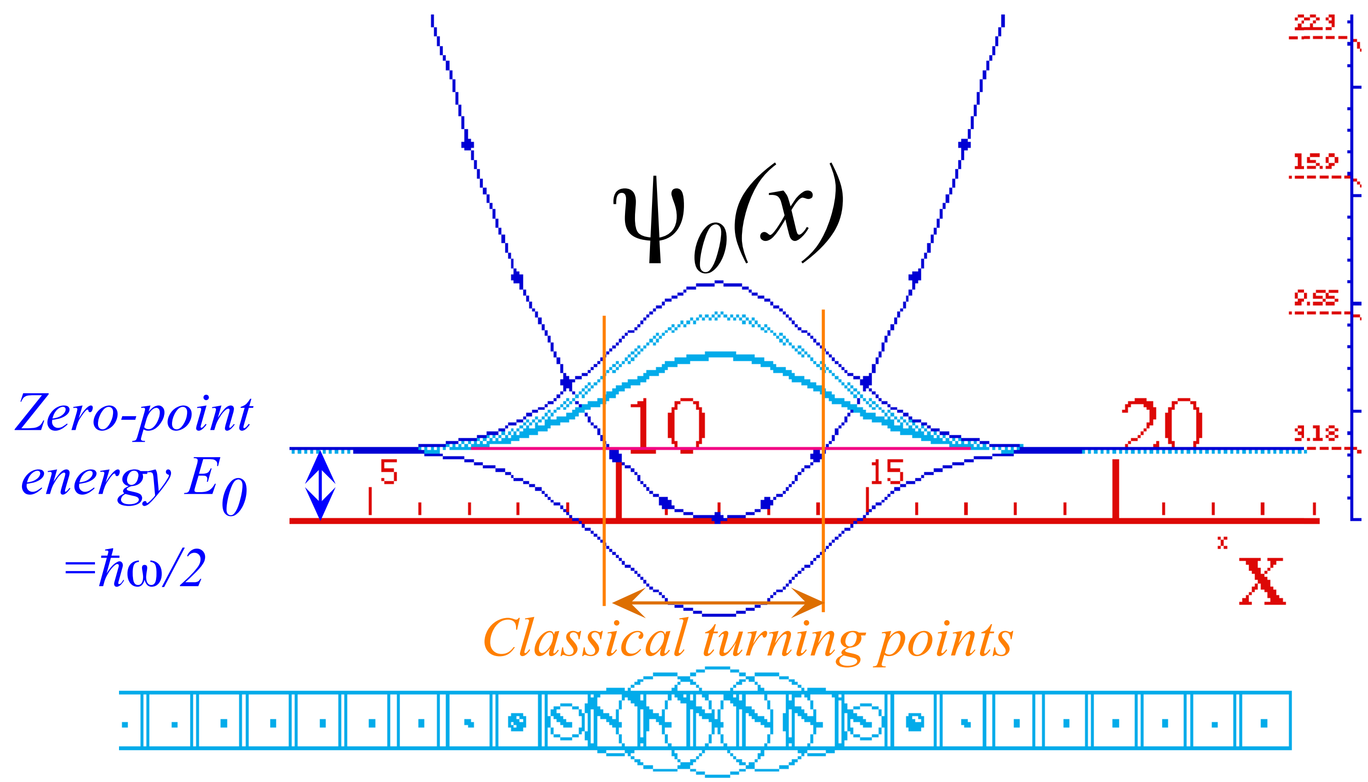

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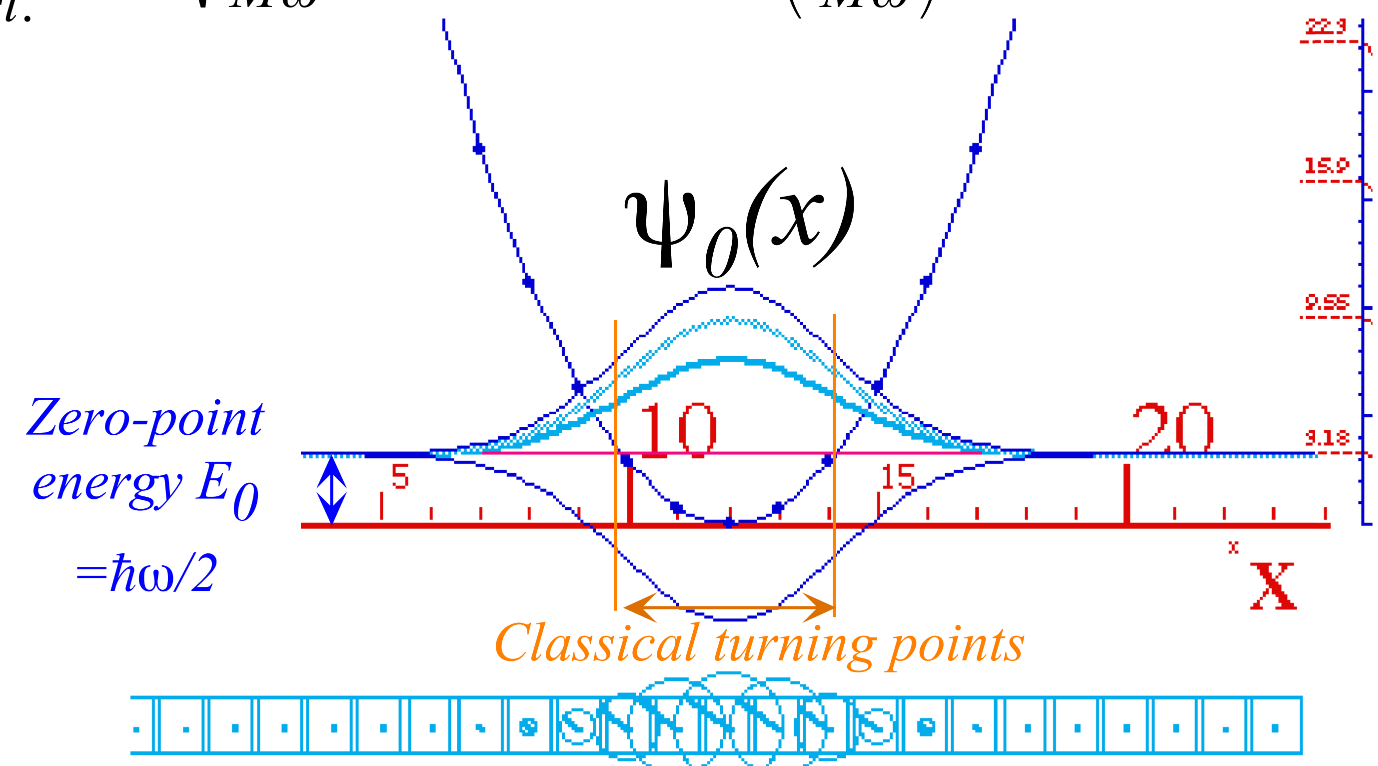
The normalization *const.* is evaluated using a standard Gaussian integral: $\int_{-\infty}^{\infty}dx\,e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$

$$\langle\psi_0|\psi_0\rangle = 1 = \int_{-\infty}^{\infty}dx\,\frac{e^{-M\omega x^2 2/2\hbar}}{const.^2} = \sqrt{\frac{\pi\hbar}{M\omega}} / const.^2 \Rightarrow const. = \left(\frac{\pi\hbar}{M\omega}\right)^{1/4}$$



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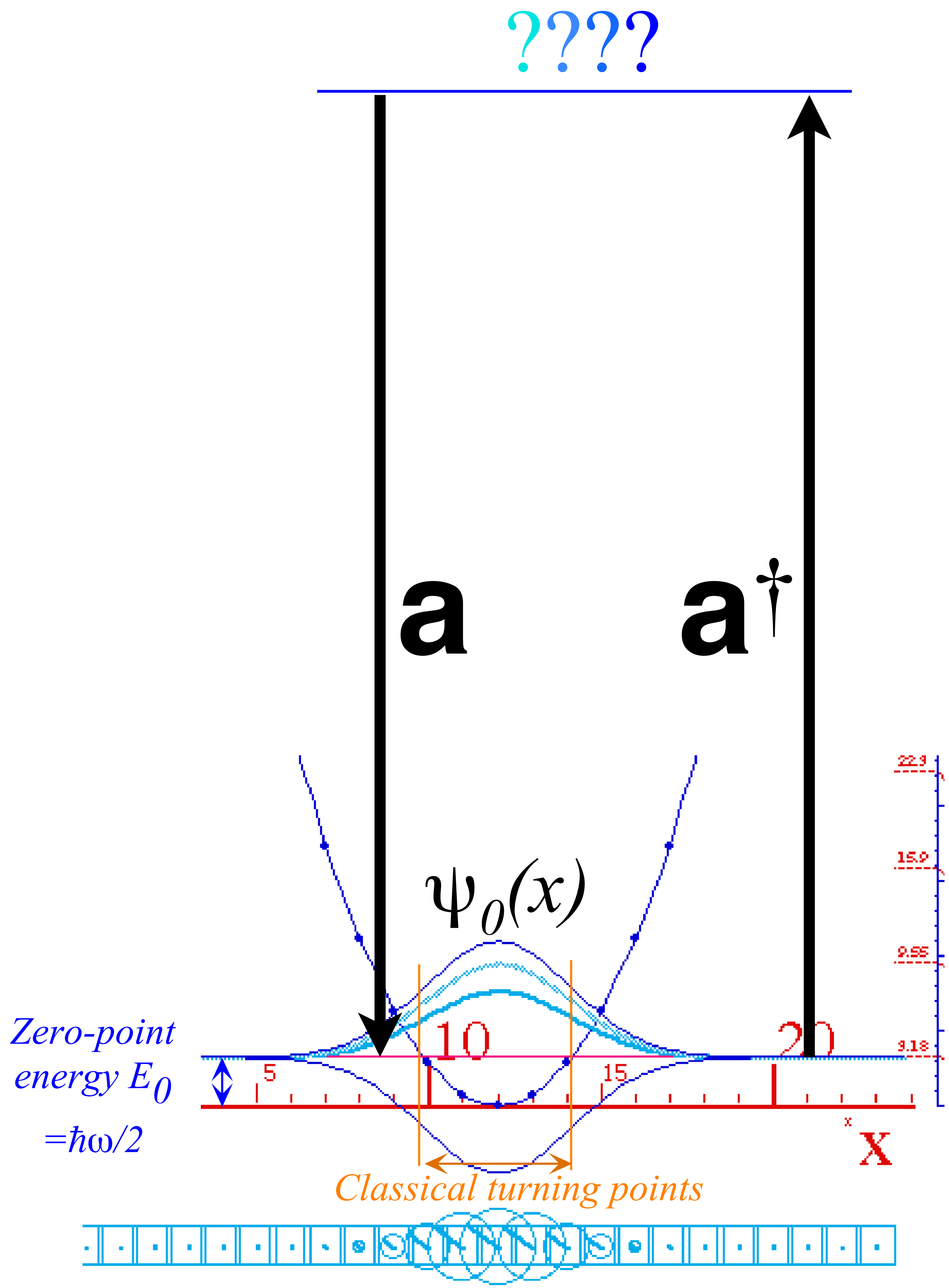
1st excited state wavefunction $\psi_1(x) = \langle x | 1 \rangle$

$$\langle x | \mathbf{a}^\dagger | 0 \rangle = \langle x | 1 \rangle = \psi_1(x)$$



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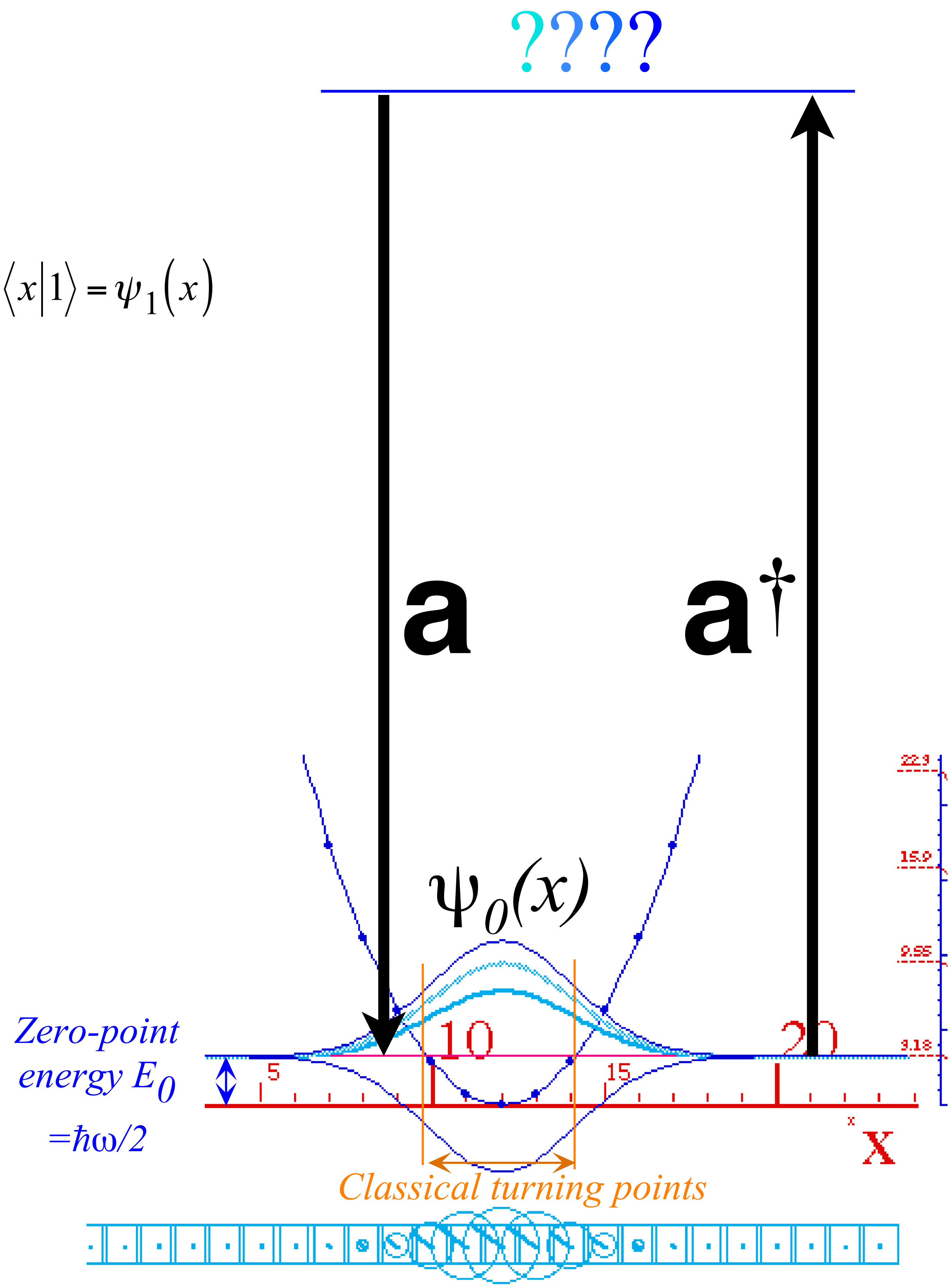

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The operator coordinate representations generate the first excited state wavefunction.

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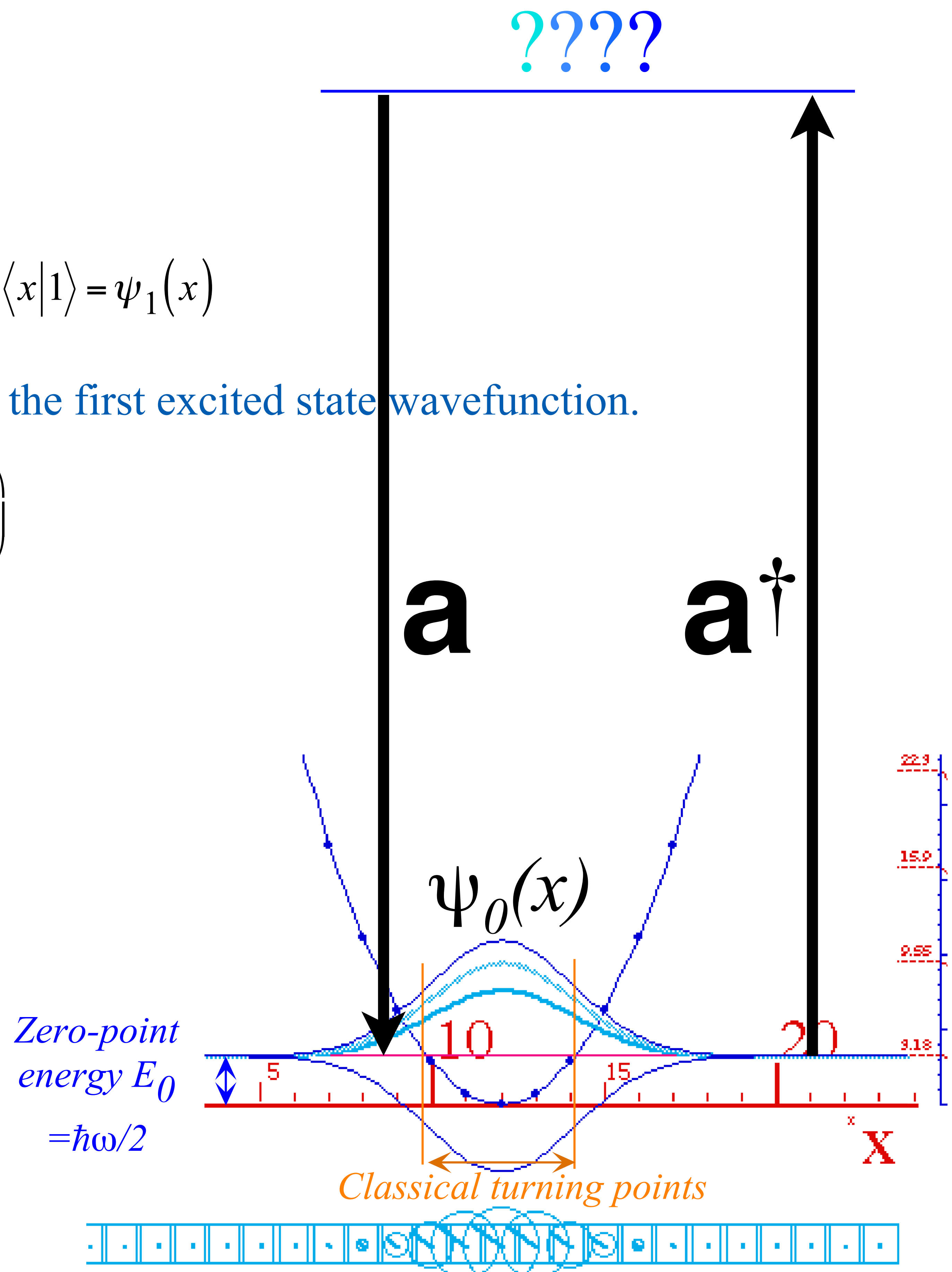

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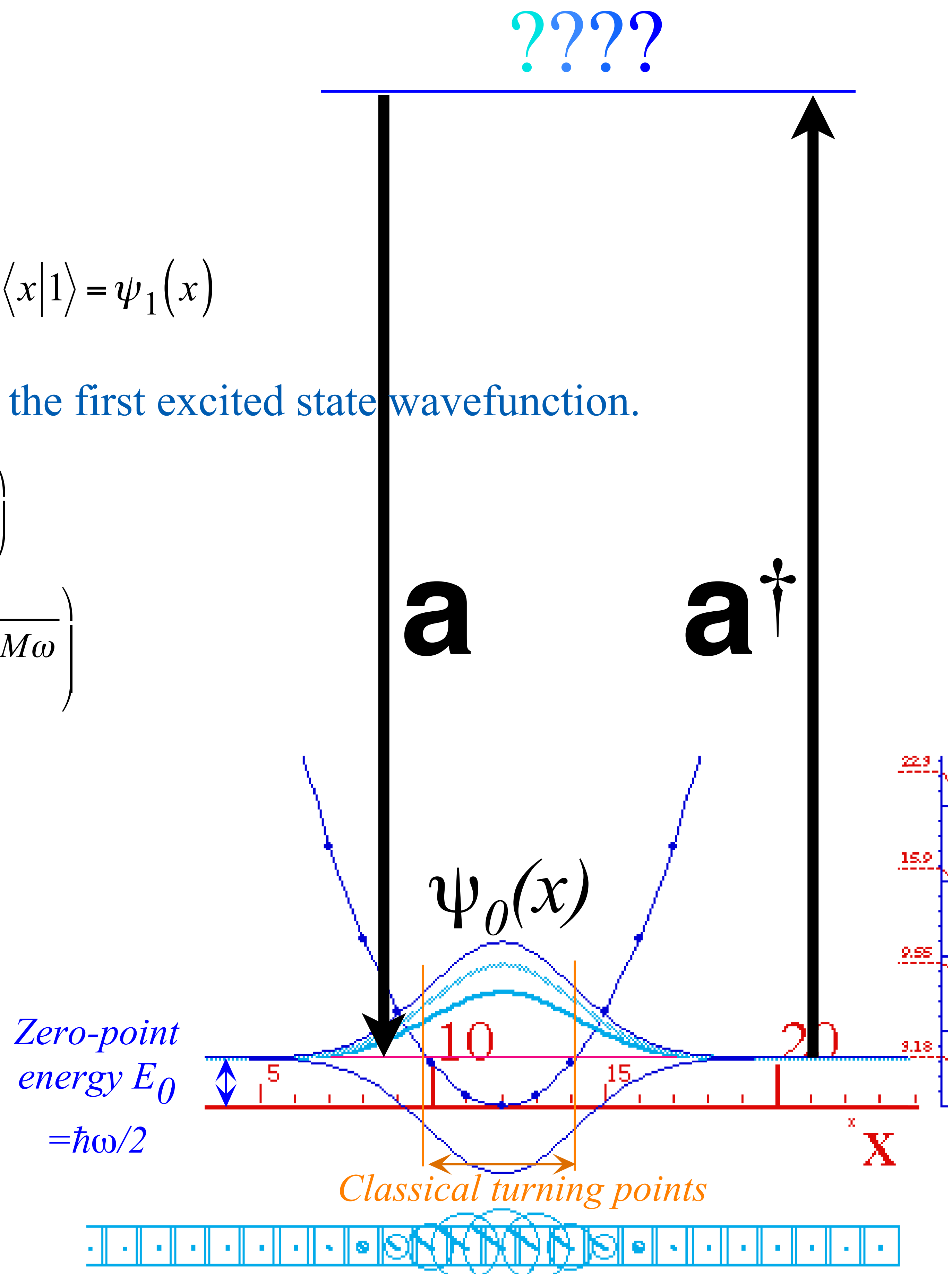

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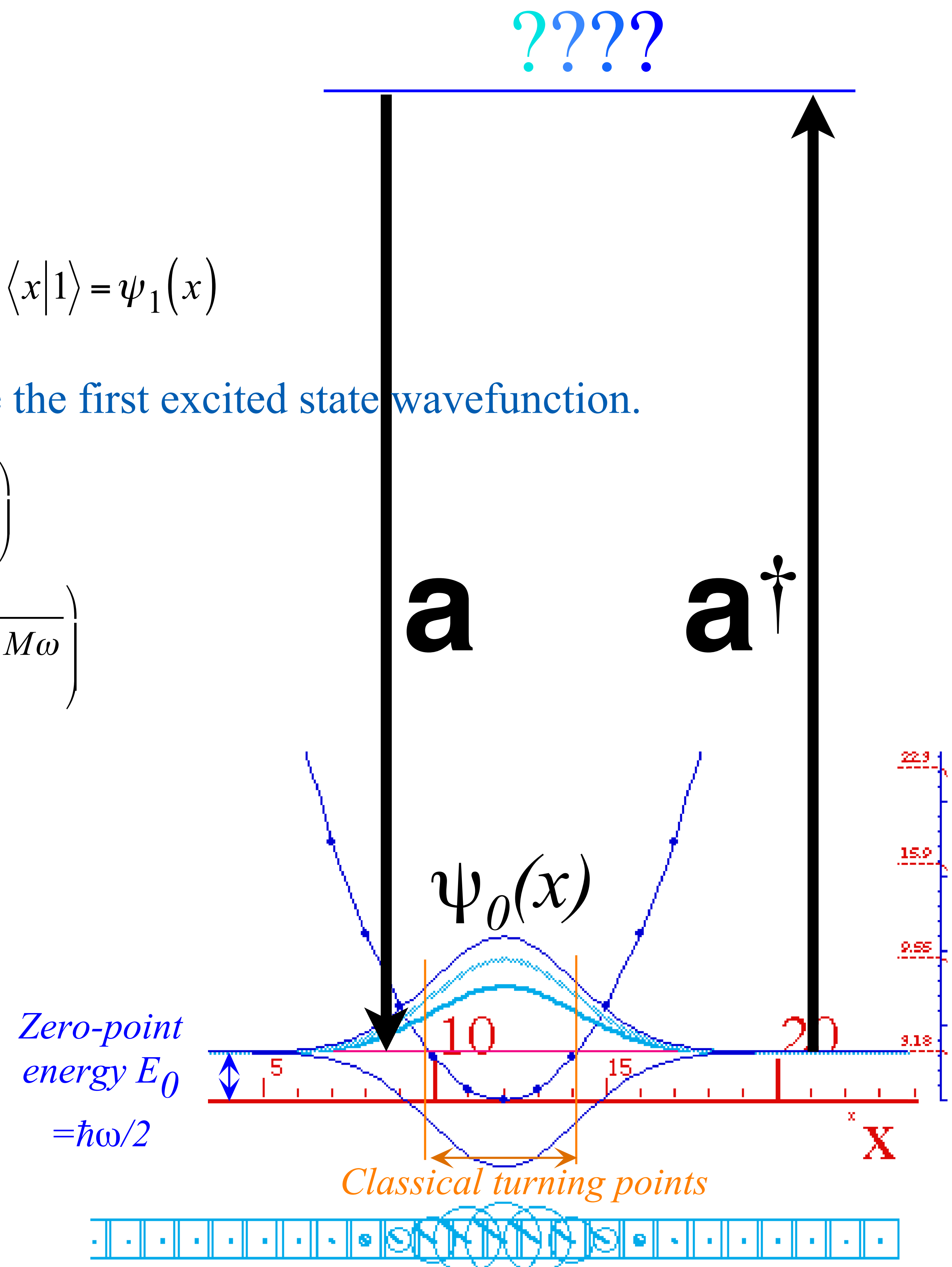

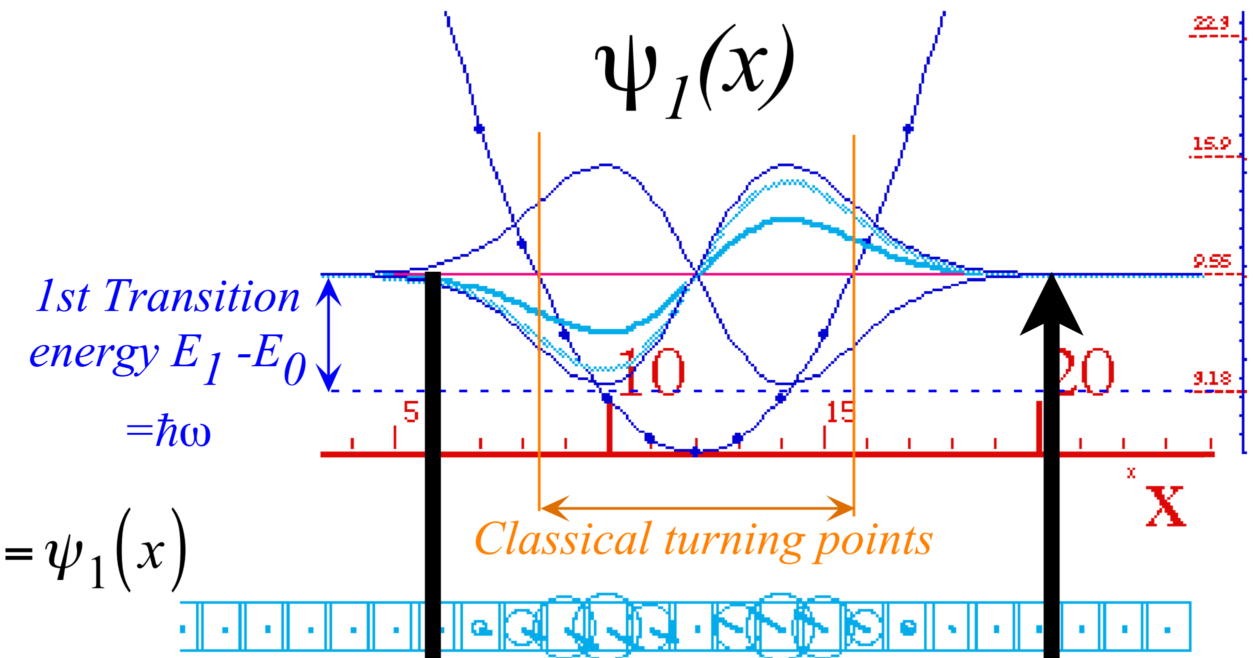

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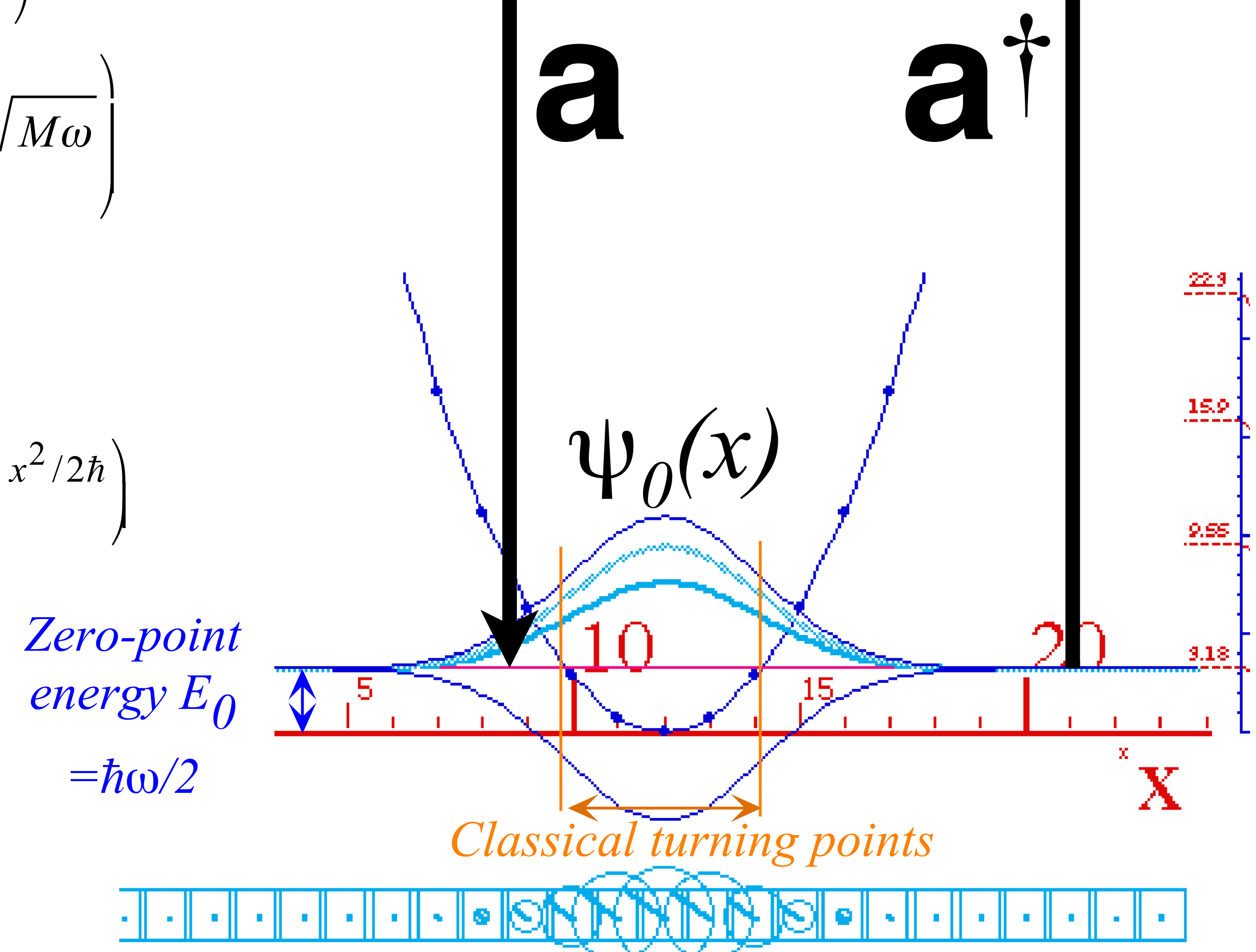
$$= \frac{\sqrt{M\omega}}{\sqrt{2\hbar}}\frac{e^{-M\omega x^2/2\hbar}}{const.}(2x) = \left(\frac{M\omega}{\pi\hbar}\right)^{3/4}\sqrt{2\pi}\left(x\,e^{-M\omega x^2/2\hbar}\right)$$



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$$[\mathbf{A}, \mathbf{BC}] = \mathbf{ABC} - \mathbf{BCA} = [\mathbf{A}, \mathbf{B}]\mathbf{C} + \mathbf{BAC} - \mathbf{BCA}$$
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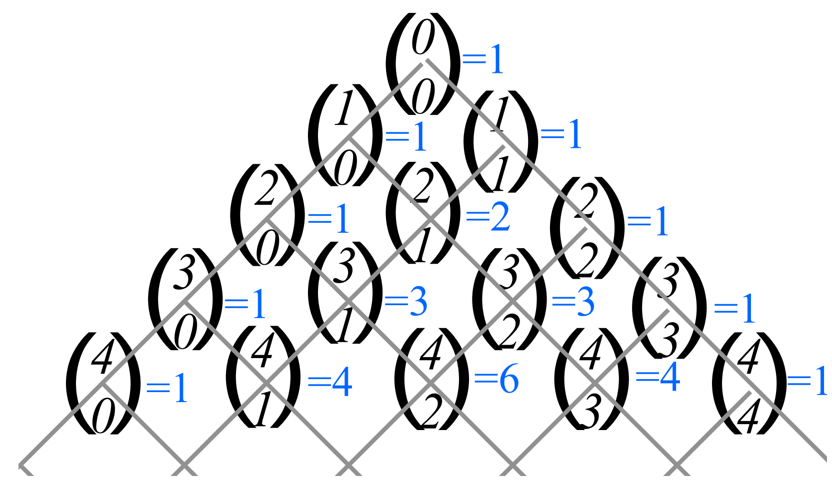

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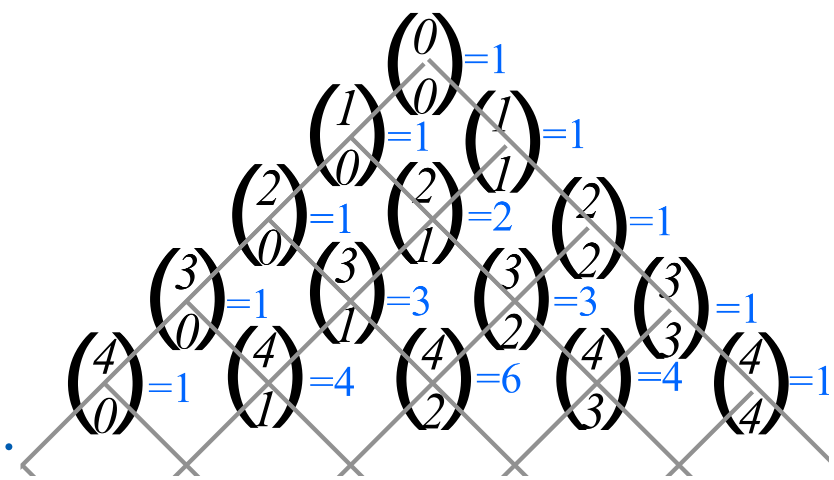

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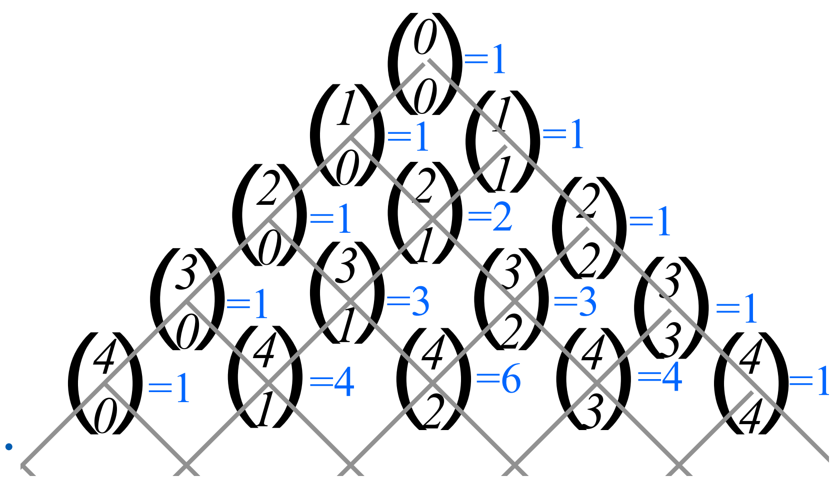

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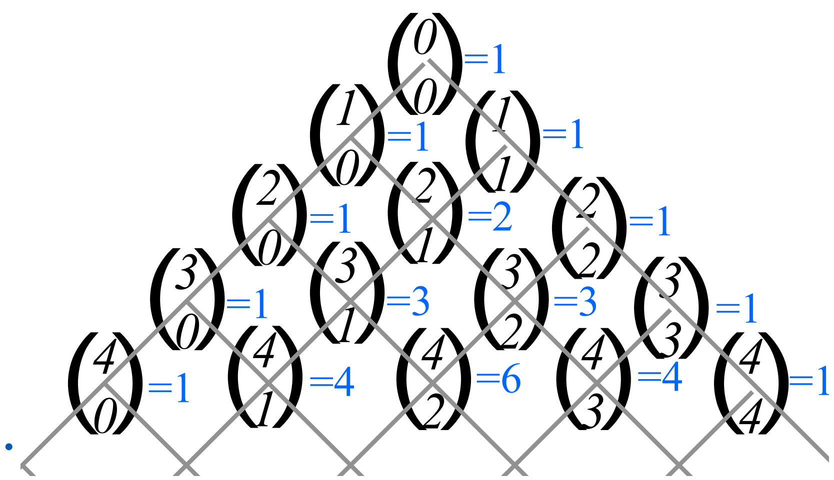

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So: $(const.)^2 = n!$

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Root-factorial normalization

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Apply destruction $\mathbf{a}$:

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$$|n\rangle = \frac{\mathbf{a}^{\dagger n}|0\rangle}{\sqrt{n!}}$$

Root-factorial normalization

Apply creation $\mathbf{a}^\dagger$:

$$\mathbf{a}^\dagger|n\rangle = \frac{\mathbf{a}^{\dagger n+1}|0\rangle}{\sqrt{n!}} = \sqrt{n+1}\,\frac{\mathbf{a}^{\dagger n+1}|0\rangle}{\sqrt{(n+1)!}}$$

Apply destruction $\mathbf{a}$:

$$\mathbf{a}|n\rangle = \frac{\mathbf{a}\mathbf{a}^{\dagger n}|0\rangle}{\sqrt{n!}}$$

Matrix $\langle \mathbf{a}^n \mathbf{a}^{\dagger n} \rangle$ *calculation*

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Feynman's mnemonic rule: Larger of two quanta goes in radical factor

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(They differ by exactly **1** !)

Many quantum operators* obey

$\mathbf{N}^\dagger\mathbf{N} = \mathbf{N}\mathbf{N}^\dagger$

Here is a case where $\mathbf{a}^\dagger\mathbf{a}$ does not quite equal $\mathbf{a}\mathbf{a}^\dagger$

$\mathbf{a}\mathbf{a}^\dagger - \mathbf{a}^\dagger\mathbf{a} = \mathbf{1}$

*Known as *Normal* operators.

Matrix $\langle \mathbf{a}^n \mathbf{a}^{\dagger n} \rangle$ *calculation*

Derive normalization for n^{th} state obtained by $(\mathbf{a}^\dagger)^n$ operator:

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Root-factorial normalization

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Feynman's mnemonic rule: Larger of two quanta goes in radical factor

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(Here is a case where $\mathbf{a}^\dagger \mathbf{a}$ does not quite equal $\mathbf{a}\,\mathbf{a}^\dagger$)

(Welcome to ∞-dimensional... quantum space!)

Symmetry group $\mathscr{G}$=U(1) representations, 1D HO Hamiltonian $\mathbf{H}=\hbar\omega\mathbf{a}^{\dagger}\mathbf{a}$ operators, 1D HO wave eigenfunctions Ψ_n, and coherent α-states

Factoring 1D-HO Hamiltonian $\mathbf{H}=\mathbf{p}^2+\mathbf{x}^2$

Creation-Destruction $\mathbf{a}^{\dagger}\mathbf{a}$ *algebra of U(1) operators*

Eigenstate creationism (and destructionism)

Vacuum state $|0\rangle$, *1*[st] *excited state* $|1\rangle$, $|2\rangle$, ...

Normal ordering for matrix calculation (creation $\mathbf{a}^{\dagger}$ *on left, destruction* $\mathbf{a}$ *on right)*

Commutator derivative identities, Binomial expansion identities

$\langle\mathbf{a}^n\mathbf{a}^{\dagger n}\rangle$ *operator calculations*

Number operator and Hamiltonian operator

Expectation values of position, momentum, and uncertainty for eigenstate $|n\rangle$

Harmonic oscillator beat dynamics of mixed states

Oscillator coherent states ("Shoved" and "kicked" states)

Translation operators vs. boost operators, boost-translation combinations

Time evolution of a coherent state $|\alpha\rangle$

Properties of coherent states and "squeezed" states

Matrix $\langle \mathbf{a}^{n}\mathbf{a}^{\dagger n}\rangle$ *calculation* *Number operator*

Derive normalization for n^{th} state obtained by $(\mathbf{a}^{\dagger})^n$ operator:

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Number operator and Hamiltonian operator

Number operator $\mathbf{N}=\mathbf{a}^{\dagger}\mathbf{a}$ counts quanta.

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$$|n\rangle = \frac{\mathbf{a}^{\dagger n}|0\rangle}{const.}, \quad \text{where:} \quad 1 = \langle n|n\rangle = \frac{\langle 0|\mathbf{a}^n \mathbf{a}^{\dagger n}|0\rangle}{(const.)^2} = n!\frac{\langle 0|\mathbf{1} + n\mathbf{a}^\dagger \mathbf{a} + ..|0\rangle}{(const.)^2} = \frac{n!}{(const.)^2}$$

$$|n\rangle = \frac{\mathbf{a}^{\dagger n}|0\rangle}{\sqrt{n!}}$$

Root-factorial normalization

Use: $\mathbf{a}\mathbf{a}^{\dagger n} = n\mathbf{a}^{\dagger n-1} + \mathbf{a}^{\dagger n}\mathbf{a}$

Apply creation $\mathbf{a}^\dagger$:

$$\mathbf{a}^\dagger|n\rangle = \frac{\mathbf{a}^{\dagger n+1}|0\rangle}{\sqrt{n!}} = \sqrt{n+1}\frac{\mathbf{a}^{\dagger n+1}|0\rangle}{\sqrt{(n+1)!}}$$

Apply destruction $\mathbf{a}$:

$$\mathbf{a}|n\rangle = \frac{\mathbf{a}\mathbf{a}^{\dagger n}|0\rangle}{\sqrt{n!}} = \frac{(n\mathbf{a}^{\dagger n-1} + \mathbf{a}^{\dagger n}\mathbf{a})|0\rangle}{\sqrt{n!}} = \sqrt{n}\frac{\mathbf{a}^{\dagger n-1}|0\rangle}{\sqrt{(n-1)!}}$$

$$\mathbf{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \qquad \mathbf{a}|n\rangle = \sqrt{n}|n-1\rangle$$

Feynman's mnemonic rule: Larger of two quanta goes in radical factor

$$\langle \mathbf{a}^\dagger \rangle = \begin{pmatrix} \cdot & & & & & \\ 1 & \cdot & & & & \\ & \sqrt{2} & \cdot & & & \\ & & \sqrt{3} & \cdot & & \\ & & & \sqrt{4} & \cdot & \\ & & & & \ddots & \cdot \end{pmatrix} \qquad \langle \mathbf{a} \rangle = \begin{pmatrix} \cdot & 1 & & & & \\ & \cdot & \sqrt{2} & & & \\ & & \cdot & \sqrt{3} & & \\ & & & \cdot & \sqrt{4} & \\ & & & & \cdot & \ddots \\ & & & & & \cdot \end{pmatrix}$$

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Number operator and Hamiltonian operator

Number operator $\mathbf{N}=\mathbf{a}^\dagger\mathbf{a}$ counts quanta.

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Matrix $\langle \mathbf{a}^n \mathbf{a}^{\dagger n} \rangle$ *calculation* *Number operator*

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$$\mathbf{H}\,|n\rangle = \hbar\omega\,\mathbf{a}^\dagger\mathbf{a}\,|n\rangle + \hbar\omega/2\mathbf{1}\,|n\rangle = \hbar\omega(n+1/2)|n\rangle$$

$$\langle \mathbf{H} \rangle = \hbar\omega\langle \mathbf{a}^\dagger\mathbf{a} + \tfrac{1}{2}\mathbf{1}\rangle = \hbar\omega\begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 2 & & \\ & & & 3 & \\ & & & & \ddots \end{pmatrix} + \hbar\omega\begin{pmatrix} 1/2 & & & & \\ & 1/2 & & & \\ & & 1/2 & & \\ & & & 1/2 & \\ & & & & \ddots \end{pmatrix}$$

Hamiltonian operator is $\hbar\omega$ $\mathbf{N}$ plus zero-point energy $\mathbf{1}\hbar\omega/2$.

Symmetry group $\mathscr{G}$=U(1) representations, 1D HO Hamiltonian $\mathbf{H}=\hbar\omega\mathbf{a}^\dagger\mathbf{a}$ operators, 1D HO wave eigenfunctions Ψ_n, and coherent α-states

Factoring 1D-HO Hamiltonian $\mathbf{H}=\mathbf{p}^2+\mathbf{x}^2$

Creation-Destruction $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(1) operators*

Eigenstate creationism (and destructionism)

Vacuum state $|0>$, *1st excited state* $|1>$, $|2>$, ...

Normal ordering for matrix calculation (creation $\mathbf{a}^\dagger$ *on left, destruction* $\mathbf{a}$ *on right)*

Commutator derivative identities, Binomial expansion identities

$\langle \mathbf{a}^n \mathbf{a}^{\dagger n} \rangle$ *operator calculations*

Number operator and Hamiltonian operator

Expectation values of position, momentum, and uncertainty for eigenstate $|n\rangle$

Harmonic oscillator beat dynamics of mixed states

Oscillator coherent states ("Shoved" and "kicked" states)

Translation operators vs. boost operators, boost-translation combinations

Time evolution of a coherent state $|\alpha>$

Properties of coherent states and "squeezed" states

Expectation values of position, momentum, and uncertainty for eigenstate $|n\rangle$

Operator for position **x**: $\sqrt{\frac{M\omega}{2\hbar}}\mathbf{x} = \frac{\mathbf{a}+\mathbf{a}^\dagger}{2}$

Operator for momentum **p**: $\sqrt{\frac{1}{2\hbar M\omega}}\,\mathbf{p} = \frac{\mathbf{a}-\mathbf{a}^\dagger}{2i}$

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Uncertainty or *standard deviation* Δq of a statistical quantity q is its root mean-square difference.

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expectation for (momentum)² $\langle \mathbf{p}^2 \rangle$:

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$$= \frac{\hbar M\omega}{2}\quad(2n+1)$$

Uncertainty or *standard deviation* Δq of a statistical quantity q is its root mean-square difference.

$$(\Delta q)^2 = \overline{(q-\bar{q})^2} \qquad \text{or:}\quad \Delta q = \sqrt{\overline{(q-\bar{q})^2}}$$

$$\Delta x|_n = \sqrt{\overline{\mathbf{x}^2}} = \sqrt{\frac{\hbar(2n+1)}{2M\omega}}$$

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Heisenberg uncertainty product for the n-quantum eigenstate $|n\rangle$

$$(\Delta x\cdot\Delta p)|_n = \sqrt{\overline{\mathbf{x}^2}}\sqrt{\overline{\mathbf{p}^2}} = \sqrt{\frac{\hbar(2n+1)}{2M\omega}}\sqrt{\frac{\hbar M\omega(2n+1)}{2}}$$

Expectation values of position, momentum, and uncertainty for eigenstate $|n\rangle$

Operator for position $\mathbf{x}$: $\sqrt{\frac{M\omega}{2\hbar}}\mathbf{x} = \frac{\mathbf{a}+\mathbf{a}^\dagger}{2}$

expectation for position $\langle \mathbf{x} \rangle$:

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$$= \frac{\hbar}{2M\omega}\quad(2n+1)$$

Use:
$\mathbf{a}\mathbf{a}^\dagger = \mathbf{1}+\mathbf{a}^\dagger\mathbf{a}$

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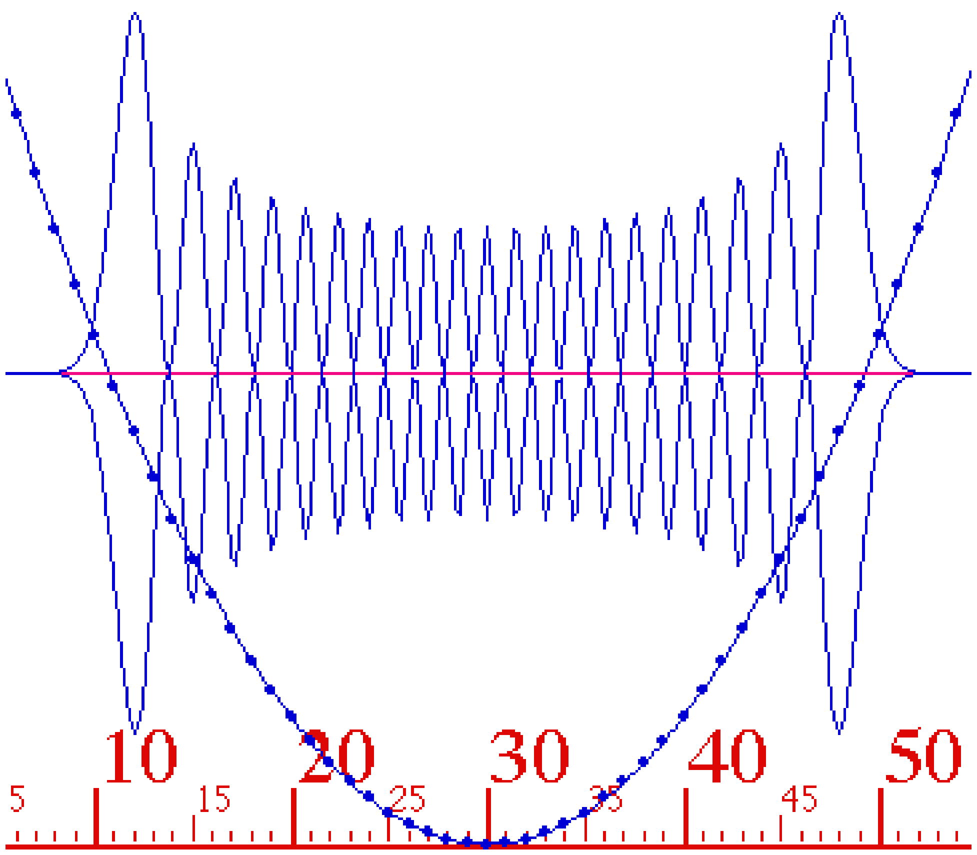
$$(\Delta x\cdot\Delta p)|_n = \hbar\left(n+\frac{1}{2}\right)$$

Heisenberg minimum uncertainty product occurs for the *0*-quantum (ground) eigenstate.

$$(\Delta x\cdot\Delta p)|_0 = \frac{\hbar}{2}$$

We pause for sobering considerations of the quantum world *vs.* the classical one.
Consider a "high"-quantum ($n=20$) eigenstate wavefunction:



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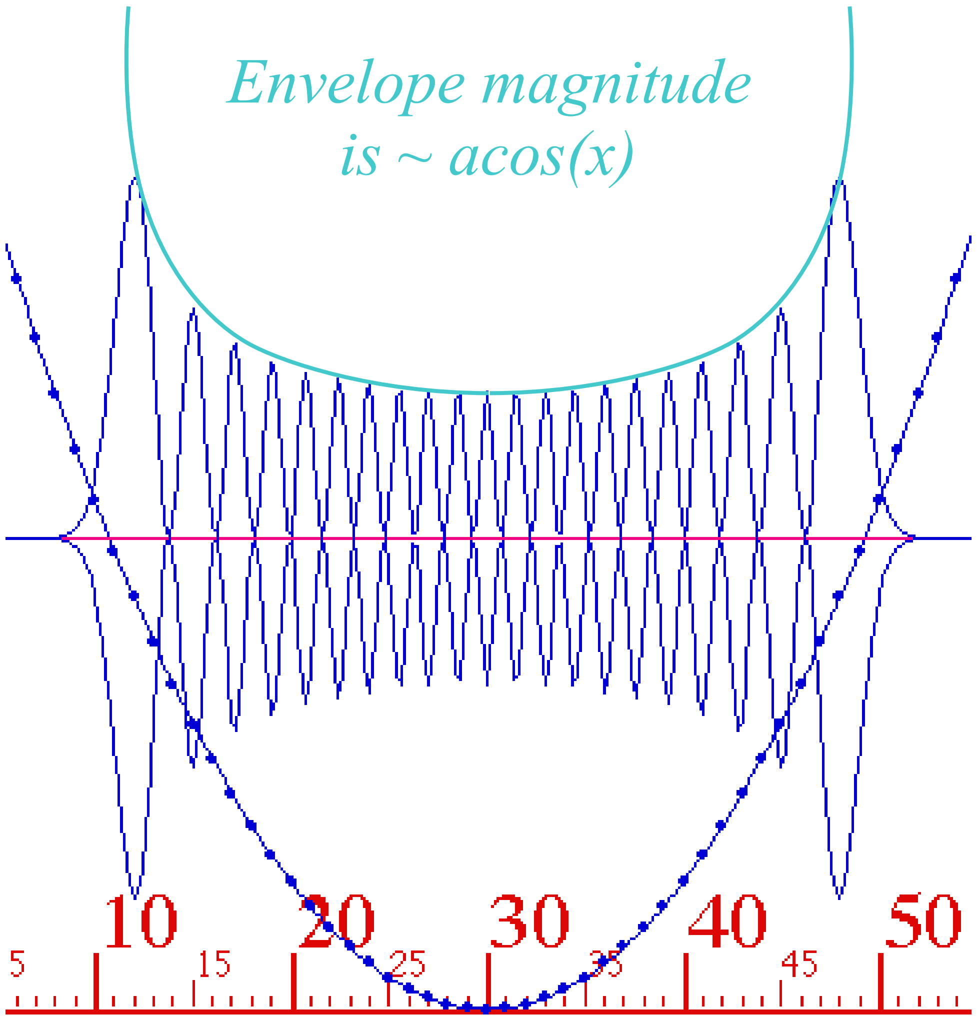

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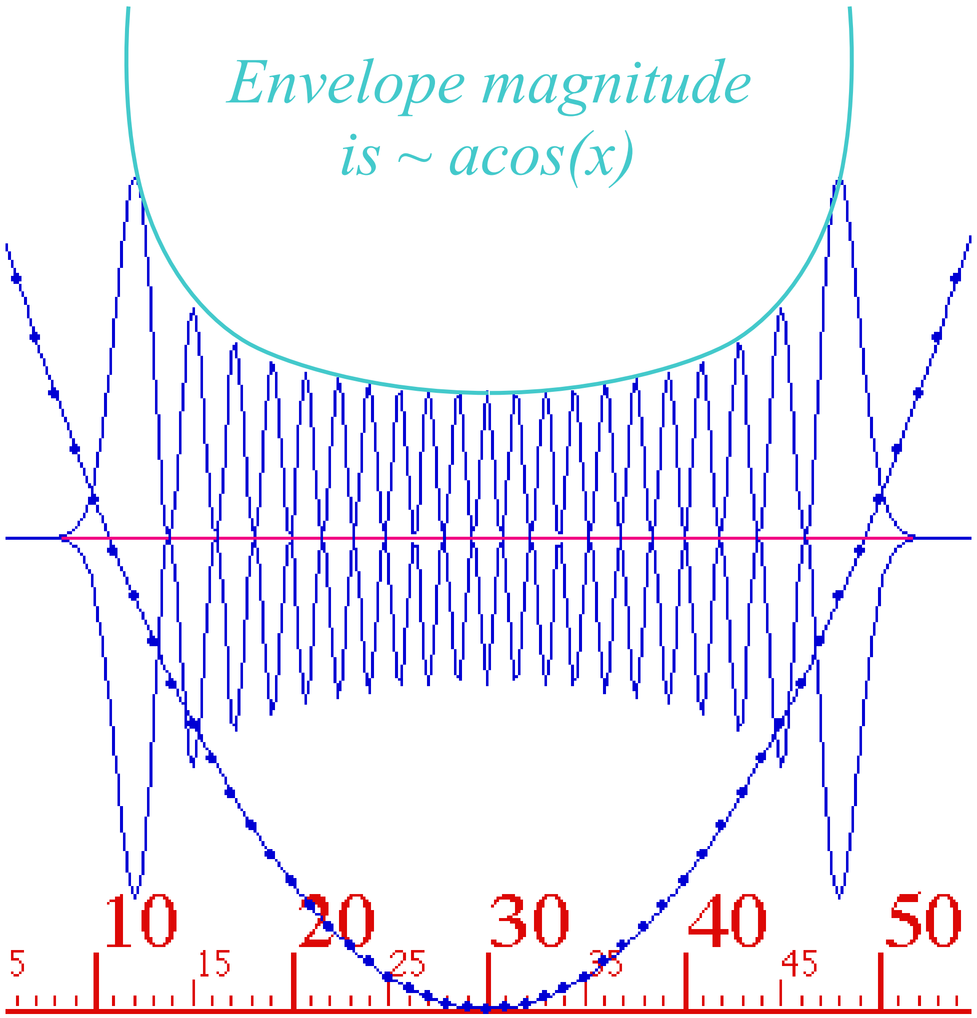

$n=20$ wave is still a long way from a classical energy value of *1 Joule*.
For a *1 Hz* oscillator, *1 Joule* would take a quantum number of roughly
$n = 100{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000 = 10^{35}$

Symmetry group $\mathscr{G}$=U(1) representations, 1D HO Hamiltonian $\mathbf{H}=\hbar\omega\mathbf{a}^\dagger\mathbf{a}$ operators, 1D HO wave eigenfunctions Ψ_n, and coherent α-states

Factoring 1D-HO Hamiltonian $\mathbf{H}=\mathbf{p}^2+\mathbf{x}^2$

Creation-Destruction $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(1) operators*

Eigenstate creationism (and destructionism)

Vacuum state $|0\rangle$, *1st excited state* $|1\rangle$, $|2\rangle$, ...

Normal ordering for matrix calculation (creation $\mathbf{a}^\dagger$ *on left, destruction* $\mathbf{a}$ *on right)*

Commutator derivative identities, Binomial expansion identities

$\langle\mathbf{a}^n\mathbf{a}^{\dagger n}\rangle$ *operator calculations*

Number operator and Hamiltonian operator

Expectation values of position, momentum, and uncertainty for eigenstate $|n\rangle$

Harmonic oscillator beat dynamics of mixed states

Oscillator coherent states ("Shoved" and "kicked" states)

Translation operators vs. boost operators, boost-translation combinations

Time evolution of a coherent state $|\alpha\rangle$

Properties of coherent states and "squeezed" states

Harmonic oscillator beat dynamics of mixed states

$$|\Psi\rangle = |0\rangle\langle 0|\Psi\rangle + |1\rangle\langle 1|\Psi\rangle = |0\rangle\Psi_0 + |1\rangle\Psi_1$$

$$\Psi(x) = \langle x|\Psi\rangle = \langle x|0\rangle\langle 0|\Psi\rangle + \langle x|1\rangle\langle 1|\Psi\rangle = \psi_0(x)\ \Psi 0 + \psi_1(x)\ \Psi 1$$

The time dependence $\Psi(x,t)$ of the mixed wave is then

$$\Psi(x,t) = \psi_0(x)\ e^{-i\omega_0 t}\ \Psi_0 + \psi_1(x)\ e^{-i\omega_1 t}\ \Psi_1 = (\psi_0(x)\ e^{-i\omega_0 t} + \psi_1(x)\ e^{-i\omega_1 t}\ )/\sqrt{2}$$

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Probability distribution "beats" back-and-forth

$$|\Psi(x,t)| = \sqrt{\Psi^*\Psi} = \sqrt{\left(e^{-i\omega_0 t}\psi_0(x) + e^{-i\omega_1 t}\psi_1(x)\right)^*\left(e^{-i\omega_0 t}\psi_0(x) + e^{-i\omega_1 t}\psi_1(x)\right)/2}$$

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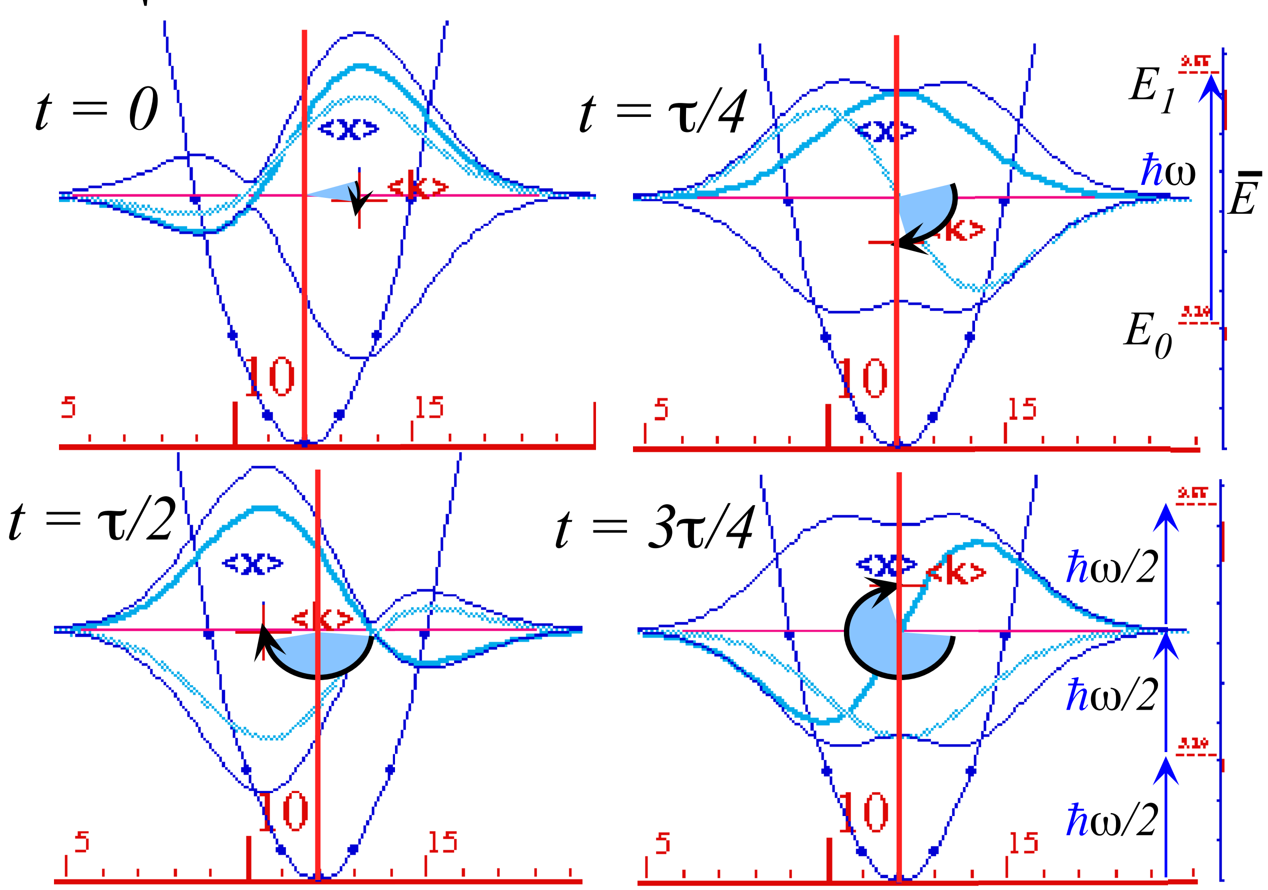

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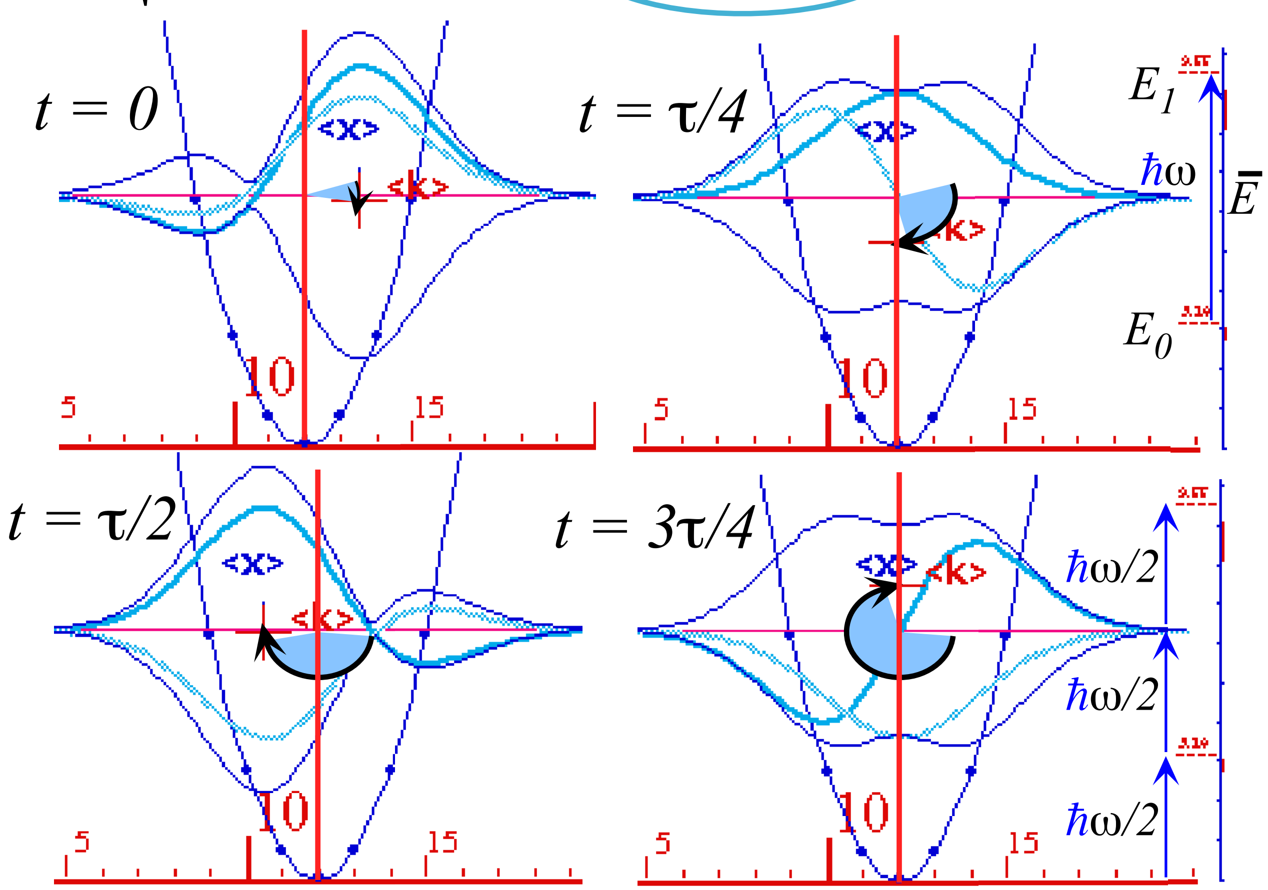
Need some *overlap*
somewhere
to get some *wiggle*



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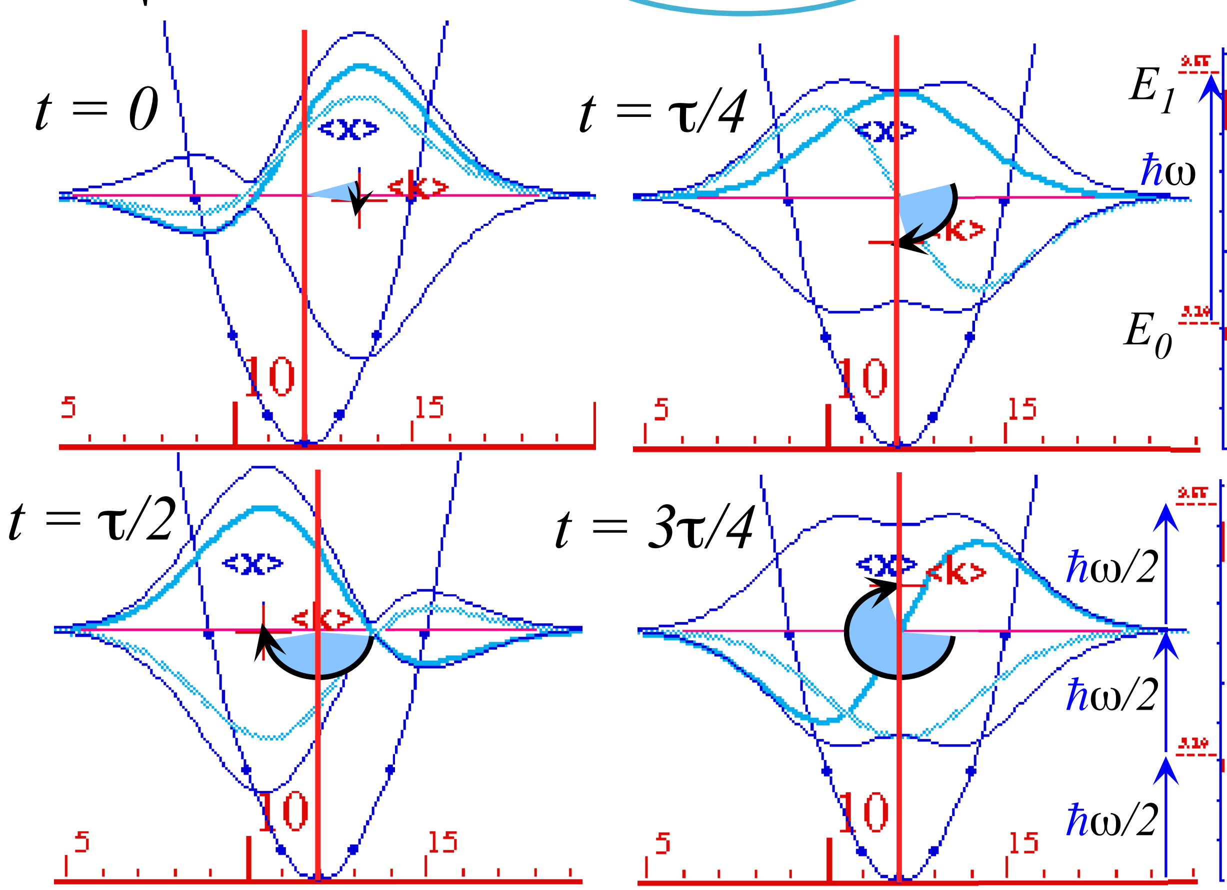

Beat frequency is eigenfrequency difference

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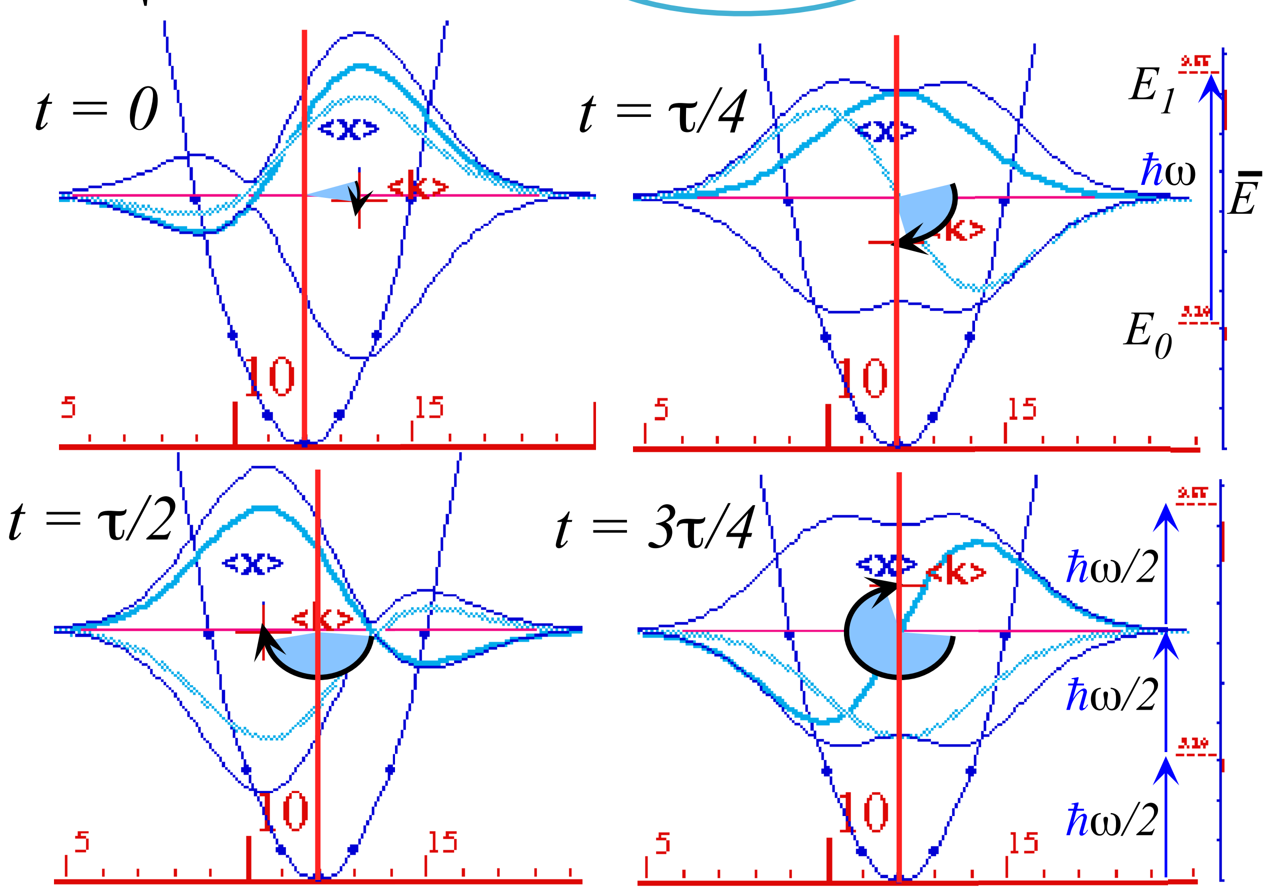

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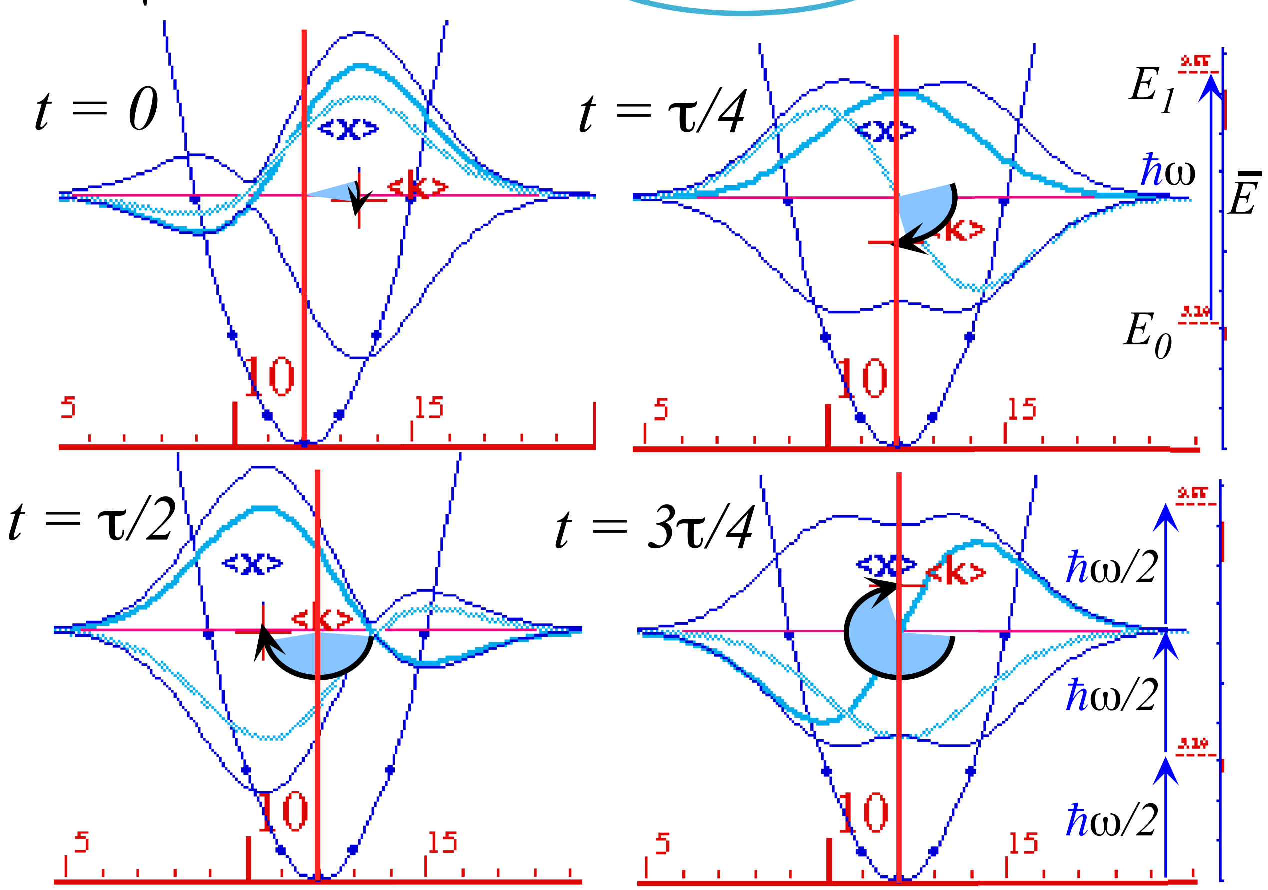

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Transition frequency is *transition energy*/$\hbar$

$$\Delta E = E_{1\leftarrow 0}\ transition = E_1 - E_0 = \hbar\omega$$

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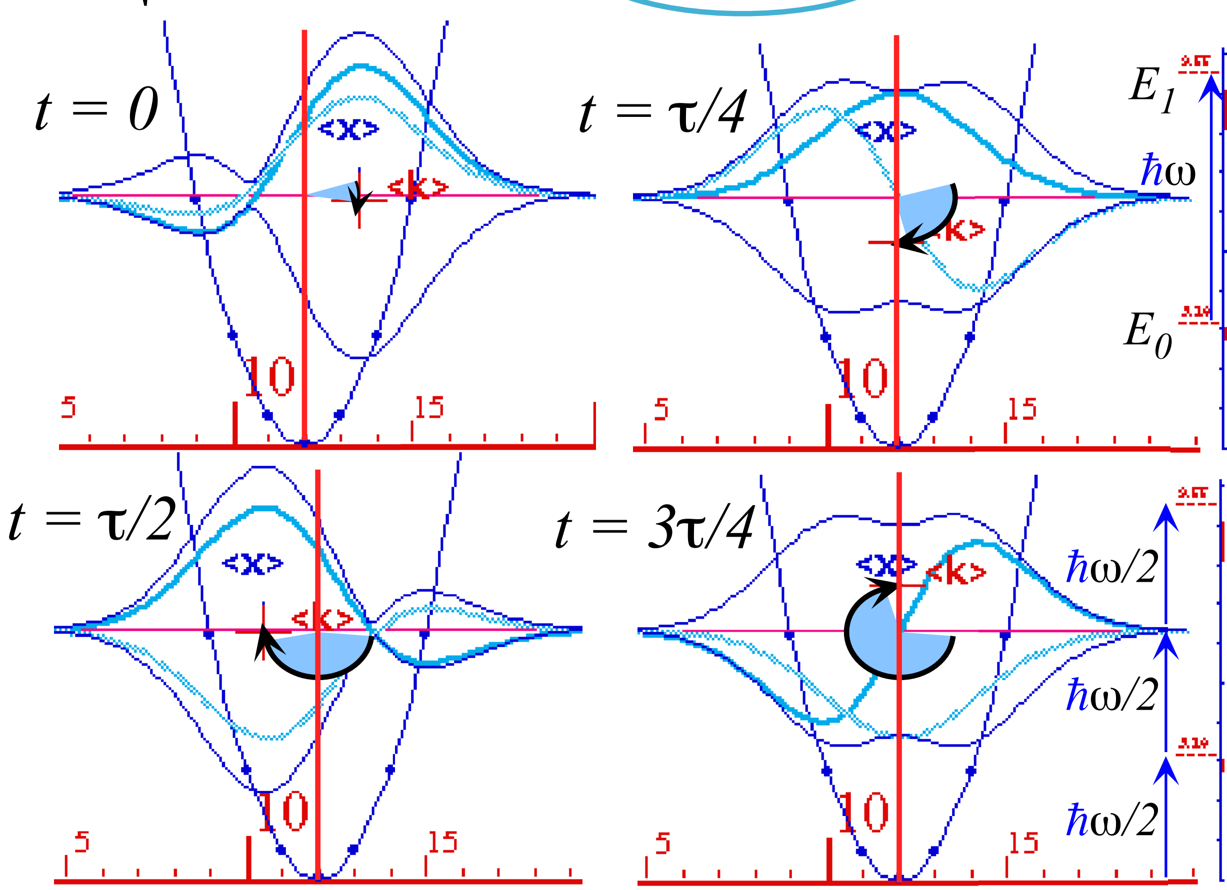

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Beat frequency ω = *Transition frequency* ω

Transition frequency is *transition energy*/ħ

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ω is frequency of radiating antenna of a transmitter or of a receiver, i.e., of an emitter or an absorber

(Usually of a dipole symmetry)

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Creation-Destruction $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(1) operators*

Eigenstate creationism (and destructionism)

Vacuum state $|0>$, *1st excited state* $|1>, |2>, \ldots$

Normal ordering for matrix calculation (creation $\mathbf{a}^\dagger$ *on left, destruction* $\mathbf{a}$ *on right)*

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Number operator and Hamiltonian operator

Expectation values of position, momentum, and uncertainty for eigenstate $|n\rangle$

Harmonic oscillator beat dynamics of mixed states

➡ *Oscillator coherent states ("Shoved" and "kicked" states)*

Translation operators vs. boost operators, ⬅ *boost-translation combinations*

Time evolution of a coherent state $|\alpha>$

Properties of coherent states and "squeezed" states

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Translation operators and generators: (A "shove")

Translation operator **T***(a)* shoves *x*-wavefunctions

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(Move up)

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Symmetry group $\mathscr{G}$=U(1) representations, 1D HO Hamiltonian $\mathbf{H}=\hbar\omega\mathbf{a}^\dagger\mathbf{a}$ operators, 1D HO wave eigenfunctions Ψ_n, and coherent α-states

Factoring 1D-HO Hamiltonian $\mathbf{H}=\mathbf{p}^2+\mathbf{x}^2$

Creation-Destruction $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(1) operators*

Eigenstate creationism (and destructionism)

Vacuum state $|0\rangle$, *1st excited state* $|1\rangle$, $|2\rangle$, ...

Normal ordering for matrix calculation (creation $\mathbf{a}^\dagger$ *on left, destruction* $\mathbf{a}$ *on right)*

Commutator derivative identities, Binomial expansion identities

$\langle\mathbf{a}^n\mathbf{a}^{\dagger n}\rangle$ *operator calculations*

Number operator and Hamiltonian operator

Expectation values of position, momentum, and uncertainty for eigenstate $|n\rangle$

Harmonic oscillator beat dynamics of mixed states

Oscillator coherent states ("Shoved" and "kicked" states)

Translation operators vs. boost operators, boost-translation combinations

→ *Time evolution of a coherent state* $|\alpha\rangle$ ←

Properties of coherent states and "squeezed" states

Time evolution of coherent state:

$$\left|\alpha_0(x_0, p_0)\right\rangle = e^{-|\alpha_0|^2/2} \sum_{n=0}^{\infty} \frac{(\alpha_0)^n}{\sqrt{n!}} |n\rangle$$

Time evolution operator for constant **H** has general form : $\mathbf{U}(t,0) = e^{-i\mathbf{H}t/\hbar}$

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Evolution simplifies to a variable-α_0 coherent state with a *time dependent phasor coordinate* α_t:

$$\mathbf{U}(t,0)\left|\alpha_0(x_0,p_0)\right\rangle = e^{-i\omega t/2}\left|\alpha_t(x_t,p_t)\right\rangle$$ where:

$$\alpha_t(x_t,p_t) = e^{-i\omega t}\alpha_0(x_0,p_0)$$

$$\left[x_t + i\frac{p_t}{M\omega}\right] = e^{-i\omega t}\left[x_0 + i\frac{p_0}{M\omega}\right]$$

Time evolution of coherent state:

$$\left|\alpha_0(x_0,p_0)\right\rangle = e^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\frac{(\alpha_0)^n}{\sqrt{n!}}|n\rangle$$

Time evolution operator for constant **H** has general form : $\mathbf{U}(t,0) = e^{-i\mathbf{H}t/\hbar}$

Oscillator eigenstate time evolution is simply determined by harmonic phases.

$$\mathbf{U}(t,0)|n\rangle = e^{-i\mathbf{H}t/\hbar}|n\rangle = e^{-i(n+1/2)\omega t}|n\rangle$$

Coherent state evolution results.

$$\mathbf{U}(t,0)\left|\alpha_0(x_0,p_0)\right\rangle = e^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\frac{(\alpha_0)^n}{\sqrt{n!}}\mathbf{U}(t,0)|n\rangle = e^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\frac{(\alpha_0)^n}{\sqrt{n!}}e^{-i(n+1/2)\omega t}|n\rangle$$

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$$\left[x_t + i\frac{p_t}{M\omega}\right] = e^{-i\omega t}\left[x_0 + i\frac{p_0}{M\omega}\right]$$

(x_t,p_t) mimics classical oscillator

$$x_t = x_0\cos\omega t + \frac{p_0}{M\omega}\sin\omega t$$

$$\frac{p_t}{M\omega} = -x_0\sin\omega t + \frac{p_0}{M\omega}\cos\omega t$$

Real and imaginary parts (x_t and $p_t/M\omega$) of α_t go clockwise on phasor circle

Symmetry group $\mathscr{G}$=U(1) representations, 1D HO Hamiltonian $\mathbf{H}=\hbar\omega\mathbf{a}^\dagger\mathbf{a}$ operators, 1D HO wave eigenfunctions Ψ_n, and coherent α-states

Factoring 1D-HO Hamiltonian $\mathbf{H}=\mathbf{p}^2+\mathbf{x}^2$

Creation-Destruction $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(1) operators*

Eigenstate creationism (and destructionism)

Vacuum state $|0\rangle$, *1st excited state* $|1\rangle$, $|2\rangle$, ...

Normal ordering for matrix calculation (creation $\mathbf{a}^\dagger$ *on left, destruction* $\mathbf{a}$ *on right)*

Commutator derivative identities, *Binomial expansion identities*

$\langle\mathbf{a}^n\mathbf{a}^{\dagger n}\rangle$ *operator calculations*

Number operator and Hamiltonian operator

Expectation values of position, momentum, and uncertainty for eigenstate $|n\rangle$

Harmonic oscillator beat dynamics of mixed states

Oscillator coherent states ("Shoved" and "kicked" states)

Translation operators vs. boost operators, *boost-translation combinations*

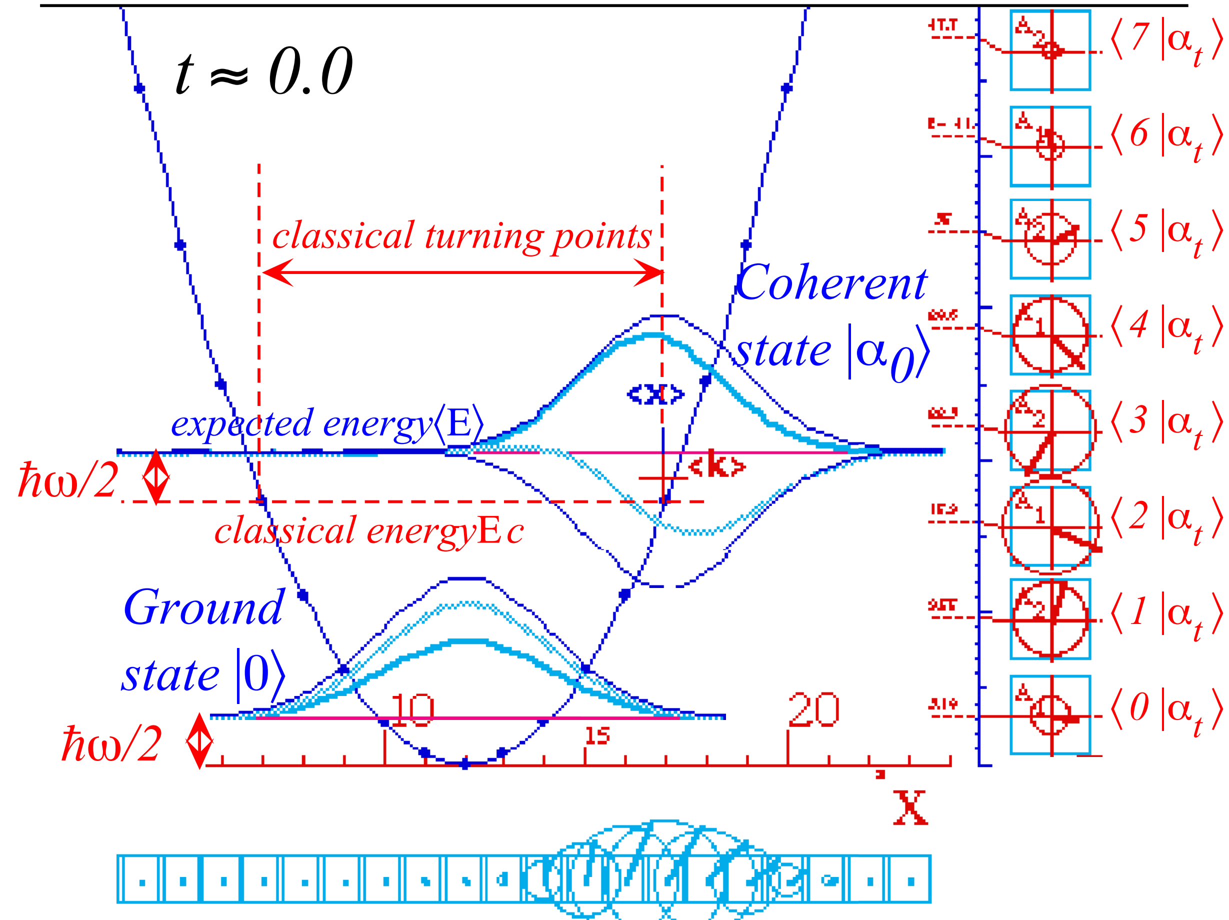
→ *Time evolution of a coherent state* $|\alpha\rangle$

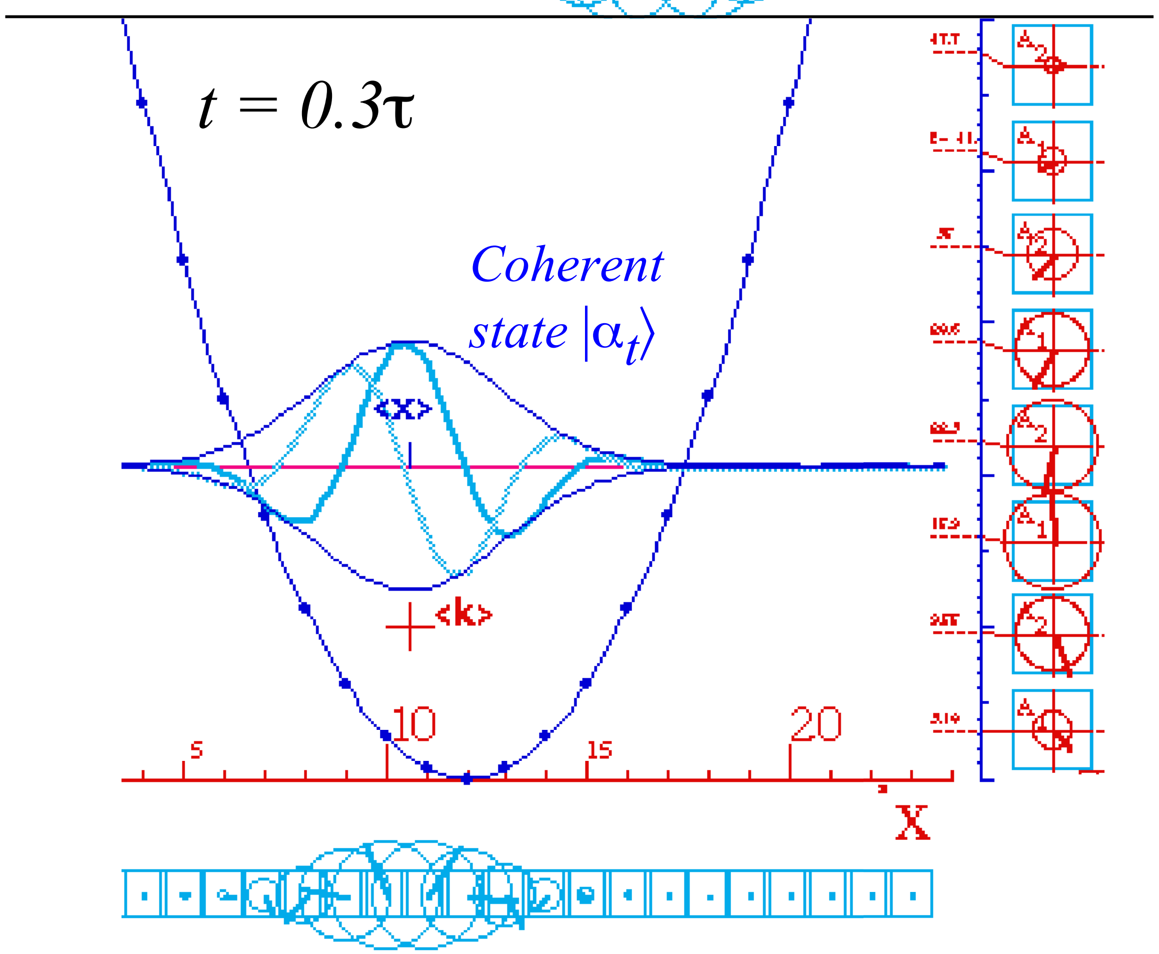
Properties of coherent states and "squeezed" states ←

Properties of coherent state

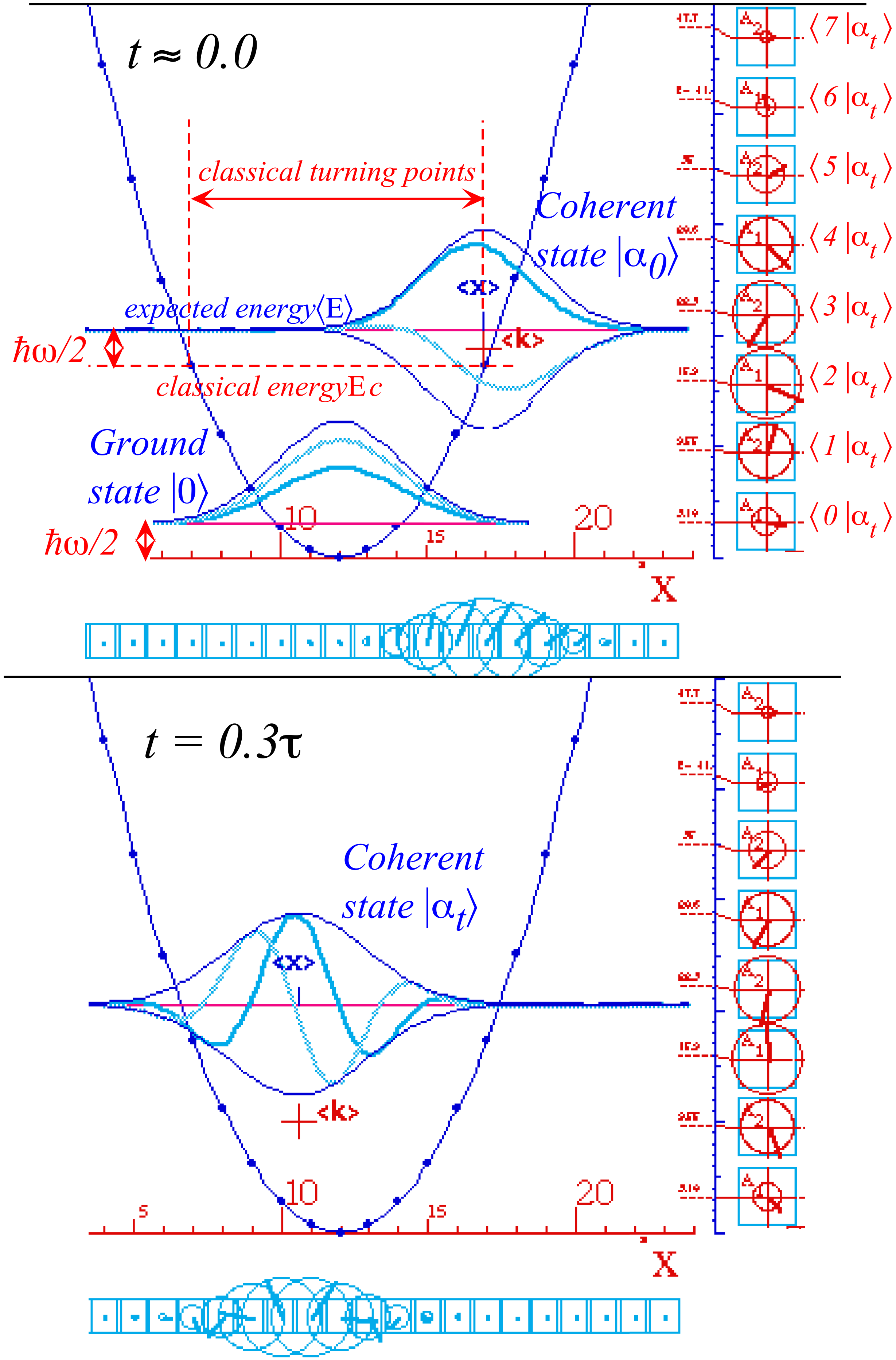
Coherent ket $|\alpha(x0,p0)\rangle$ is eigenvector of destruct-op. **a**.

$$\mathbf{a}|\alpha_0(x_0,p_0)\rangle = e^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\frac{(\alpha_0)^n}{\sqrt{n!}}\mathbf{a}|n\rangle$$





Properties of coherent state



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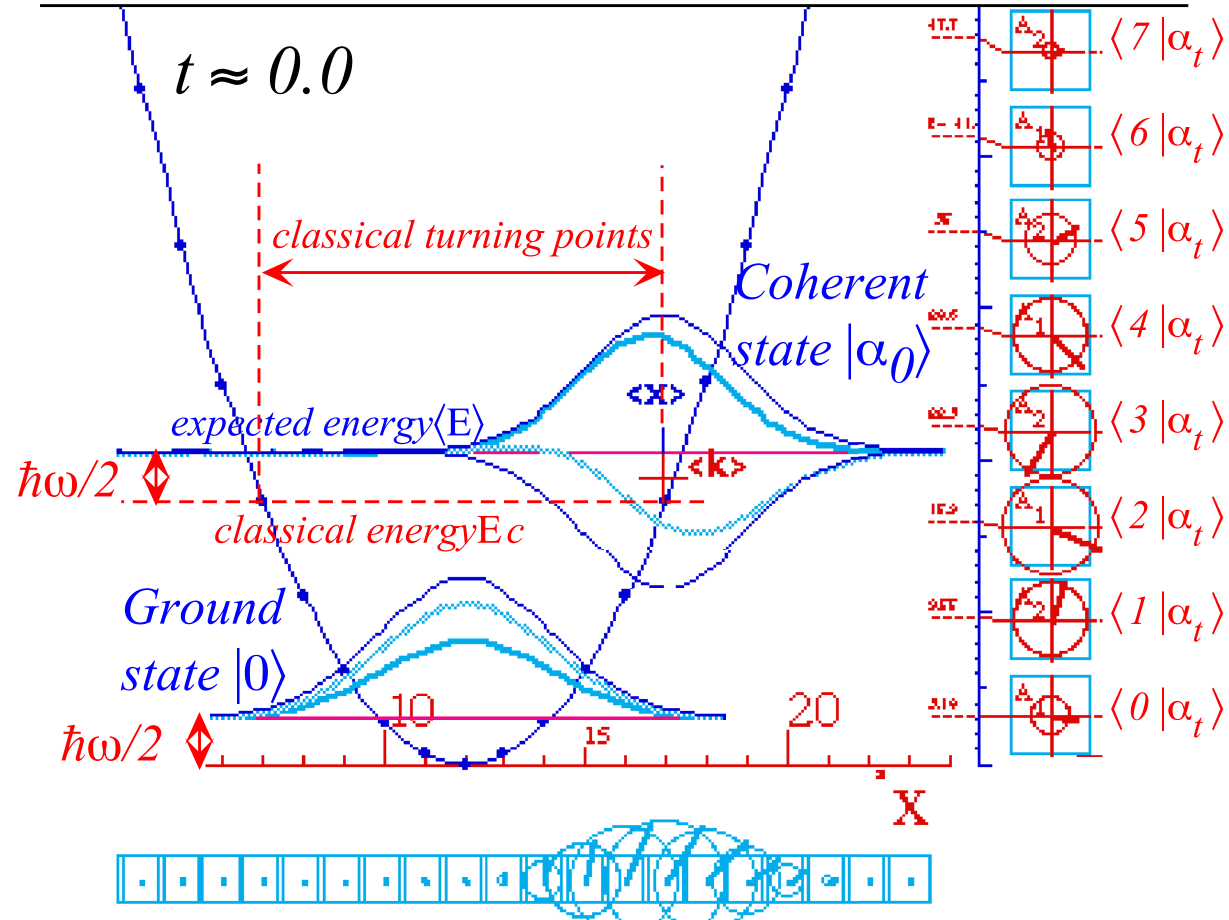
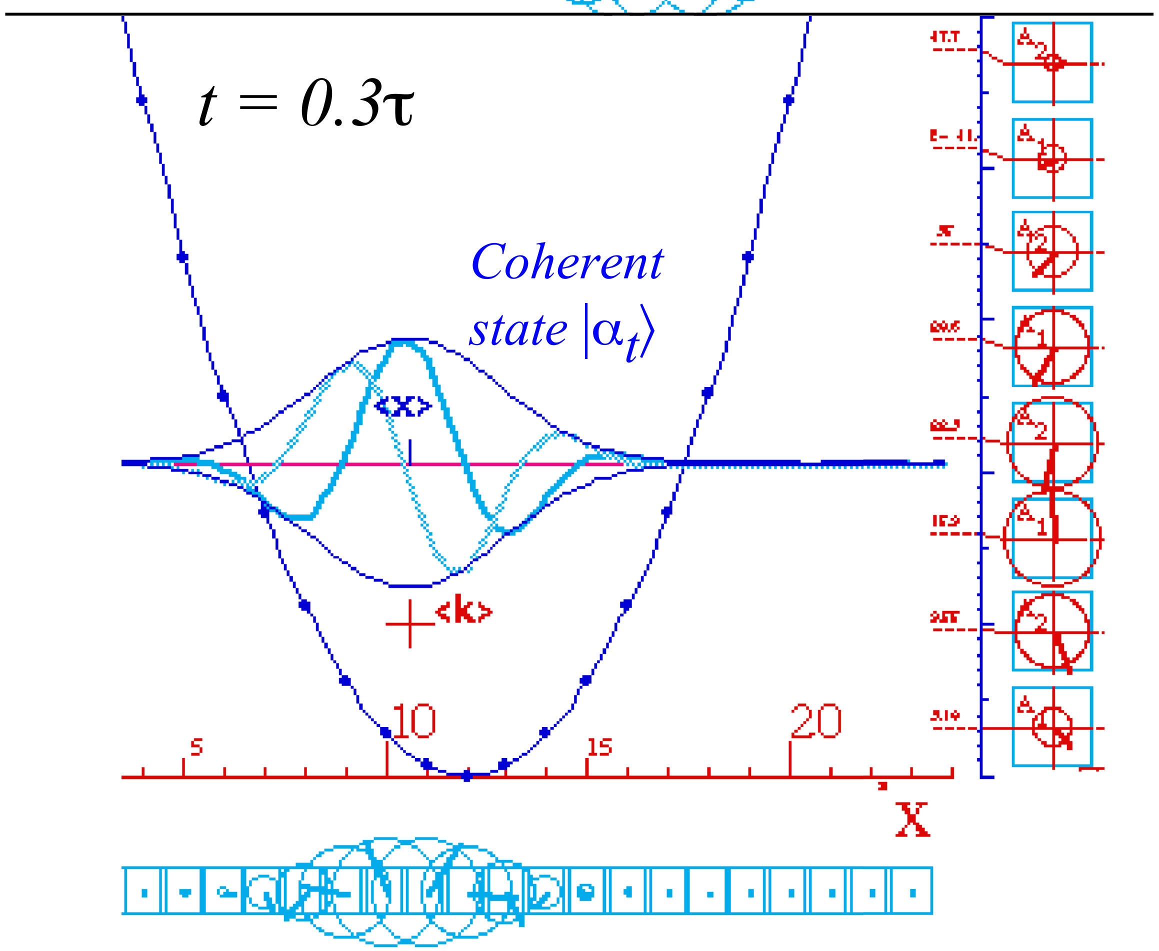
Properties of coherent state





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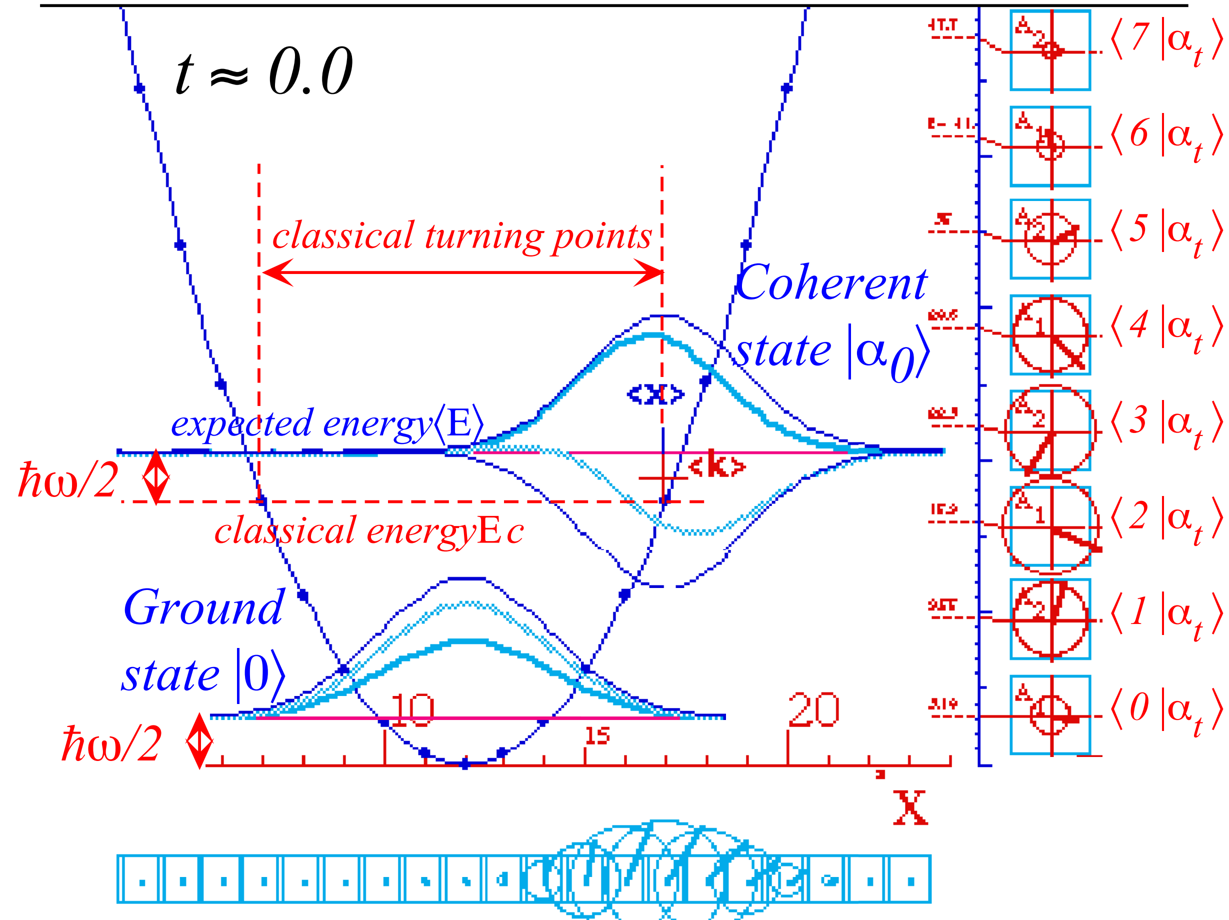
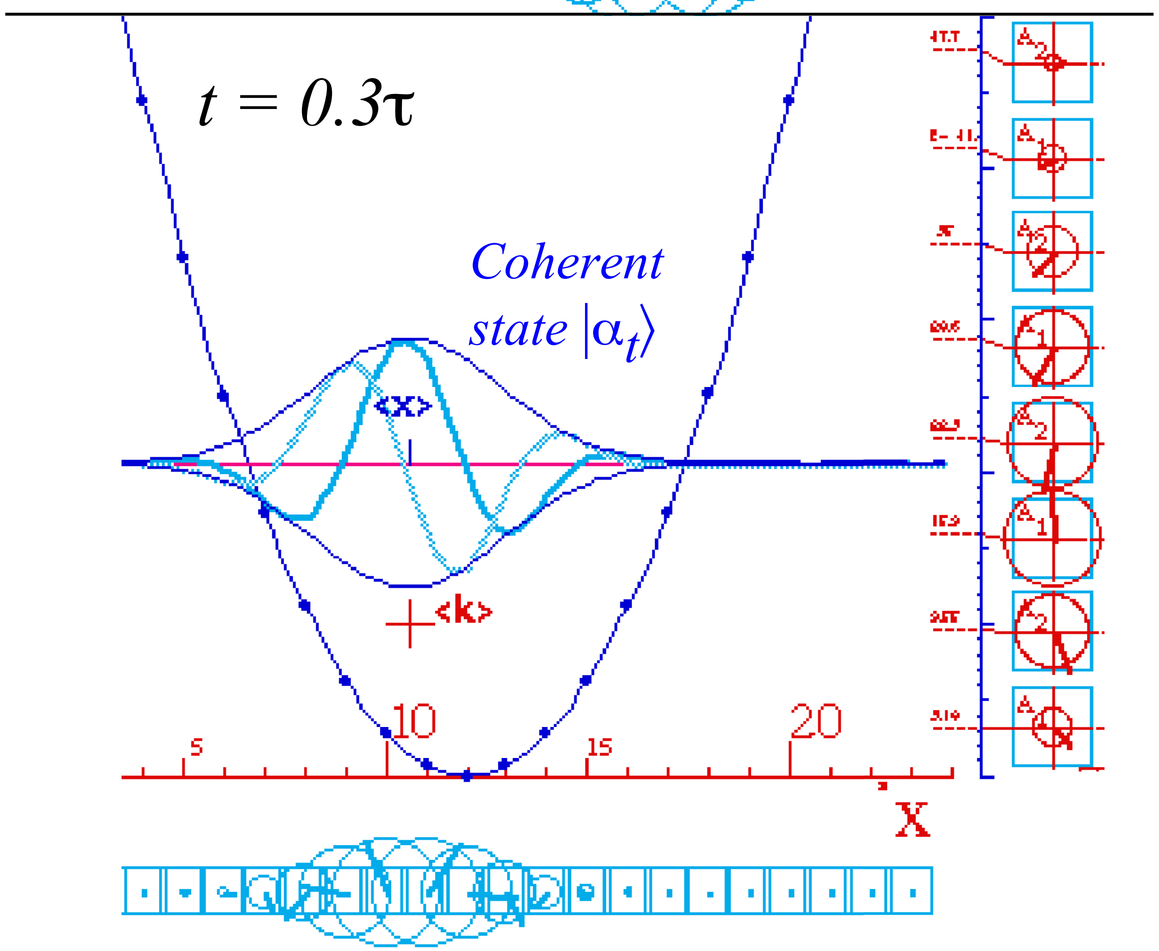
$$= \alpha_0|\alpha_0(x_0,p_0)\rangle$$

Properties of coherent state





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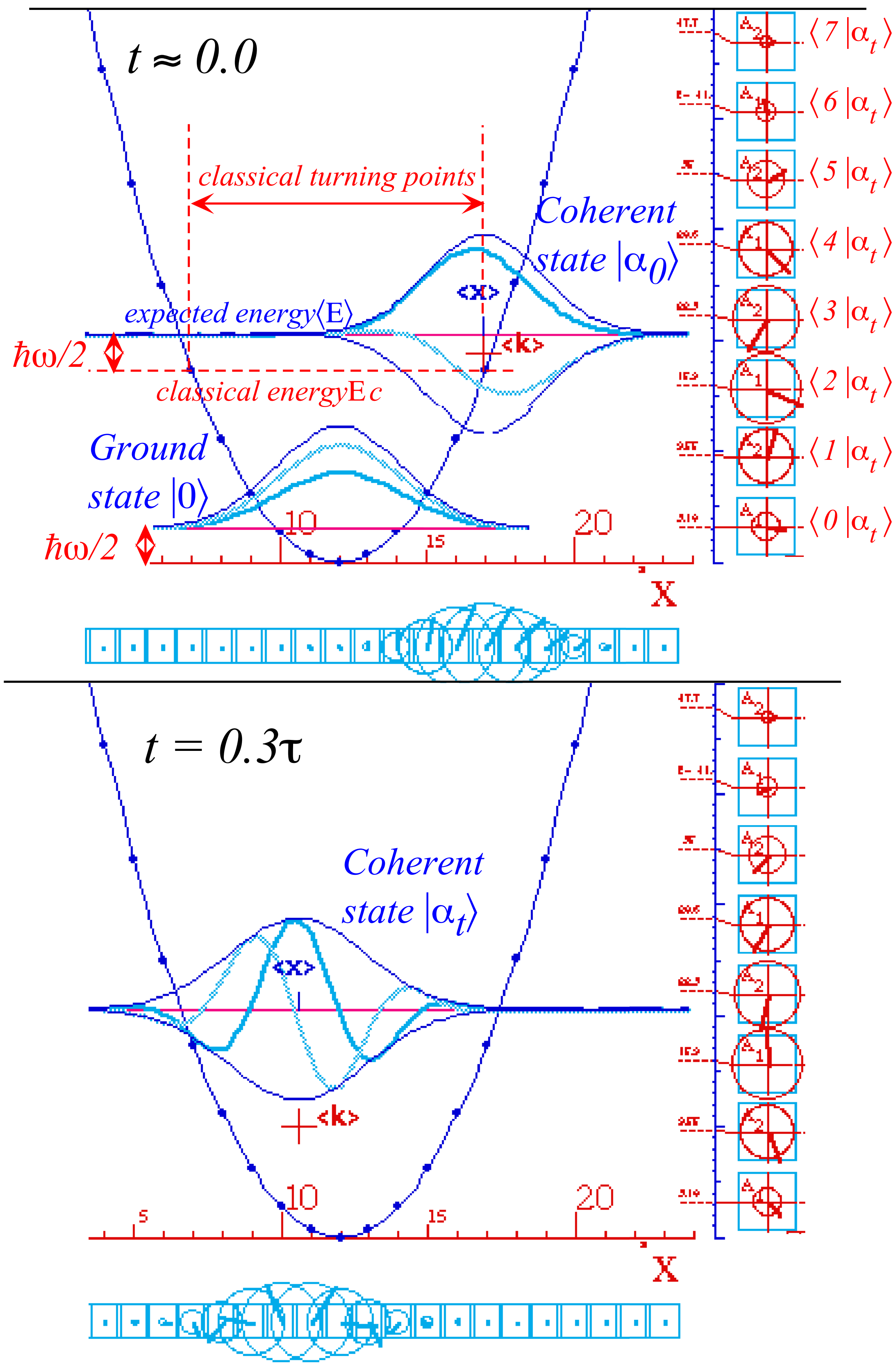
Properties of coherent state



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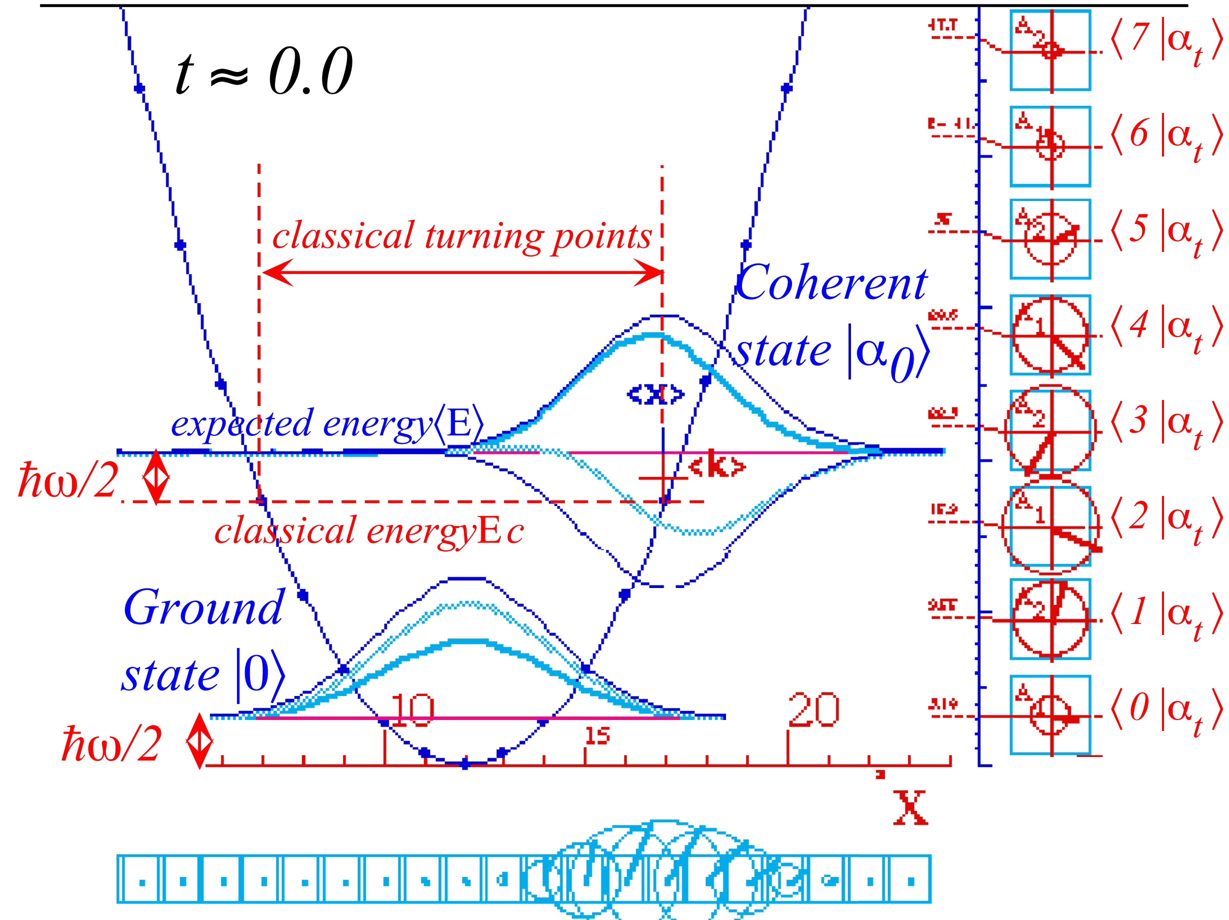
$$= \alpha_0|\alpha_0(x_0,p_0)\rangle$$ with eigenvalue α_0

Coherent bra $\langle\alpha(x0,p0)|$ is eigenvector of create-op. $\mathbf{a}^\dagger$.

$$\langle\alpha_0(x_0,p_0)|\mathbf{a}^\dagger = \langle\alpha_0(x_0,p_0)|\alpha_0^*$$

Properties of coherent state



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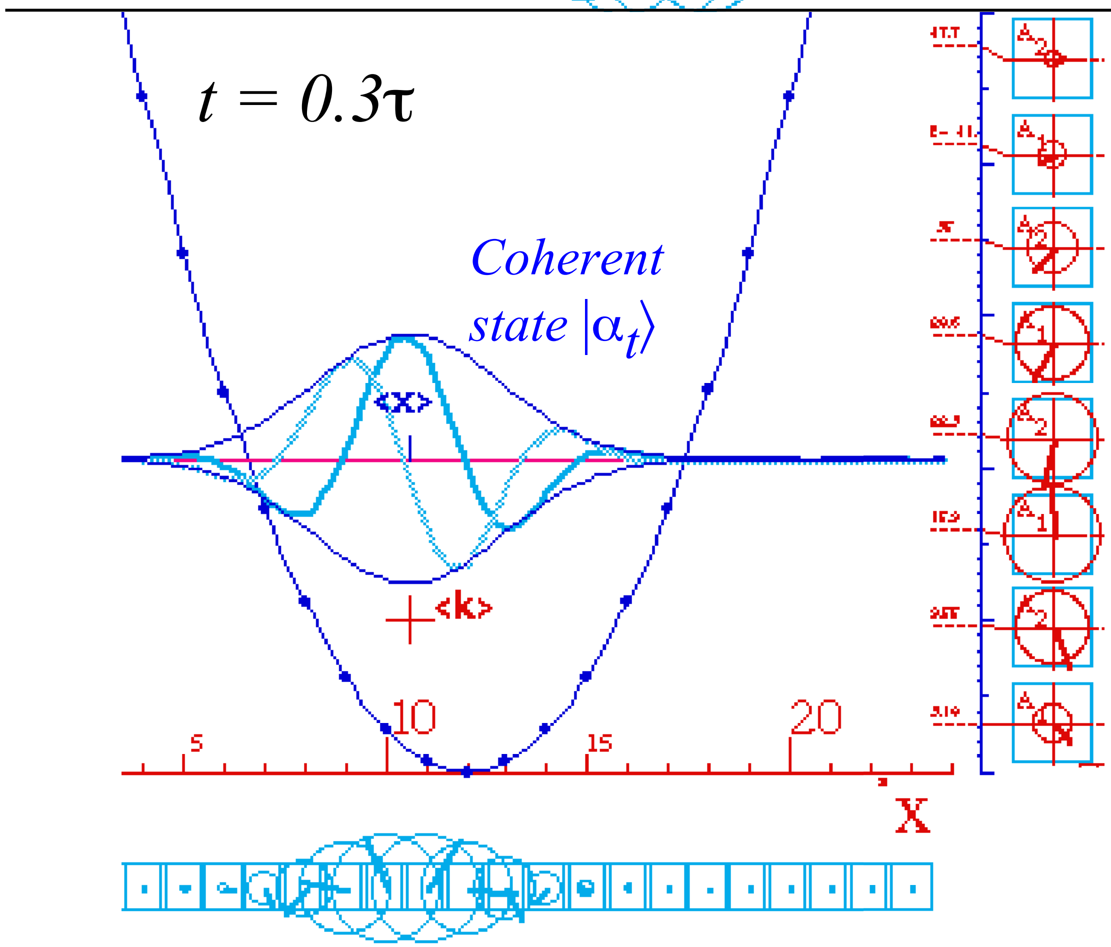
with eigenvalue $(\alpha_0)^*$



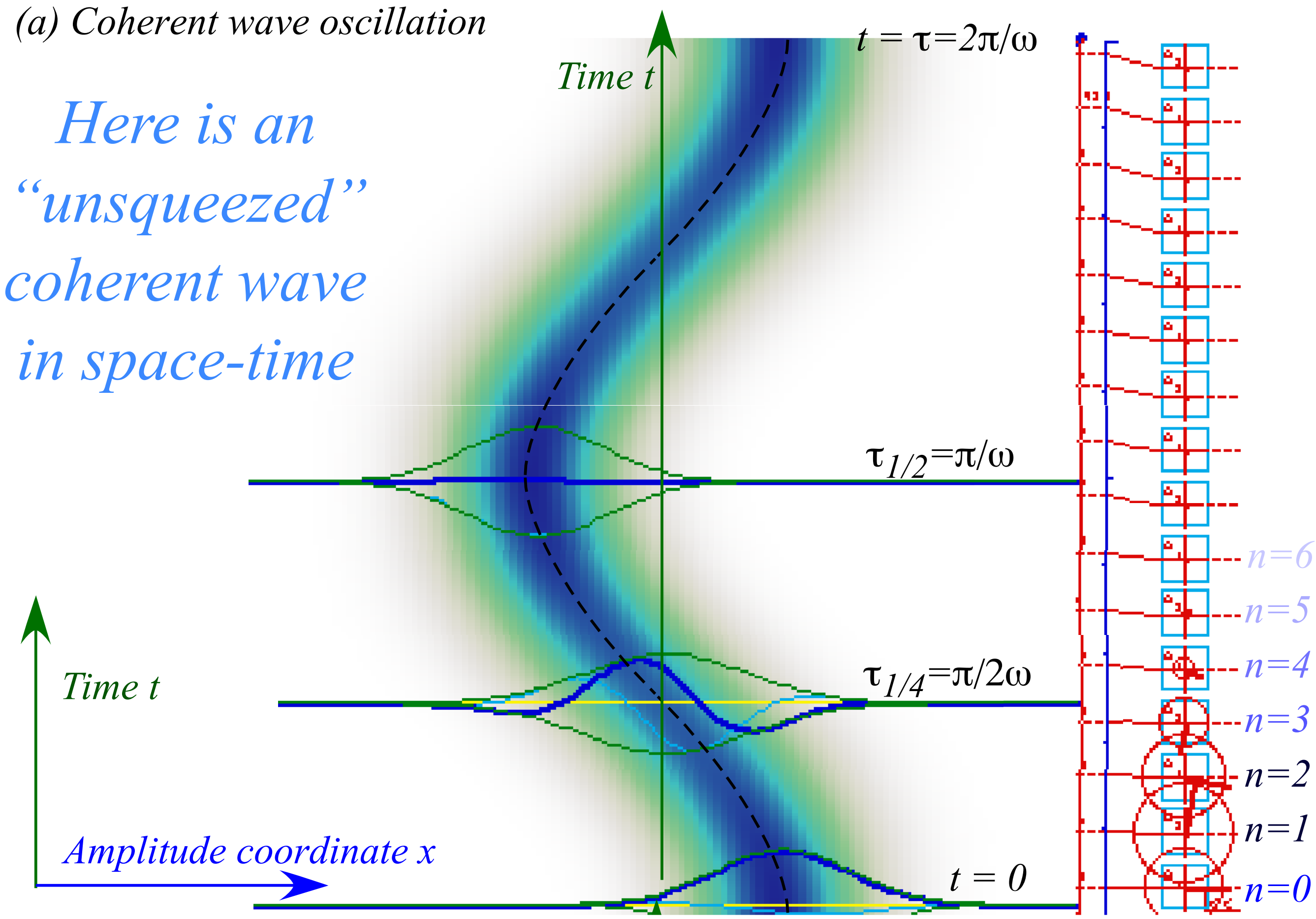
Expected quantum energy has simple time independent form.

$$\langle E\rangle\big|_{\alpha_0} = \langle\alpha_0(x_0,p_0)|\mathbf{H}|\alpha_0(x_0,p_0)\rangle$$

$$= \langle\alpha_0(x_0,p_0)|\left(\hbar\omega\mathbf{a}^\dagger\mathbf{a} + \frac{\hbar\omega}{2}\mathbf{1}\right)|\alpha_0(x_0,p_0)\rangle$$

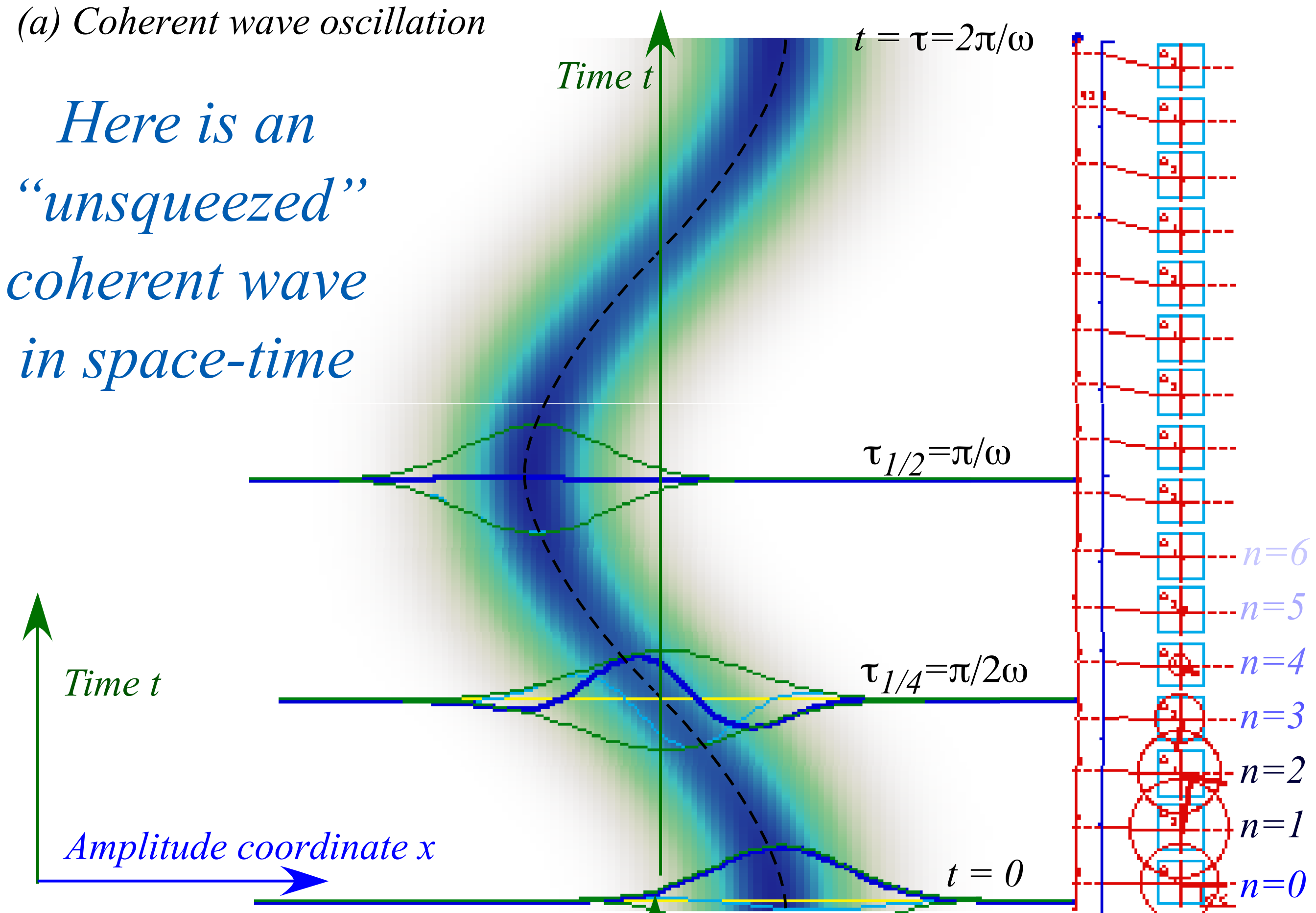
$$= \hbar\omega\alpha_0^*\alpha_0 + \frac{\hbar\omega}{2}$$

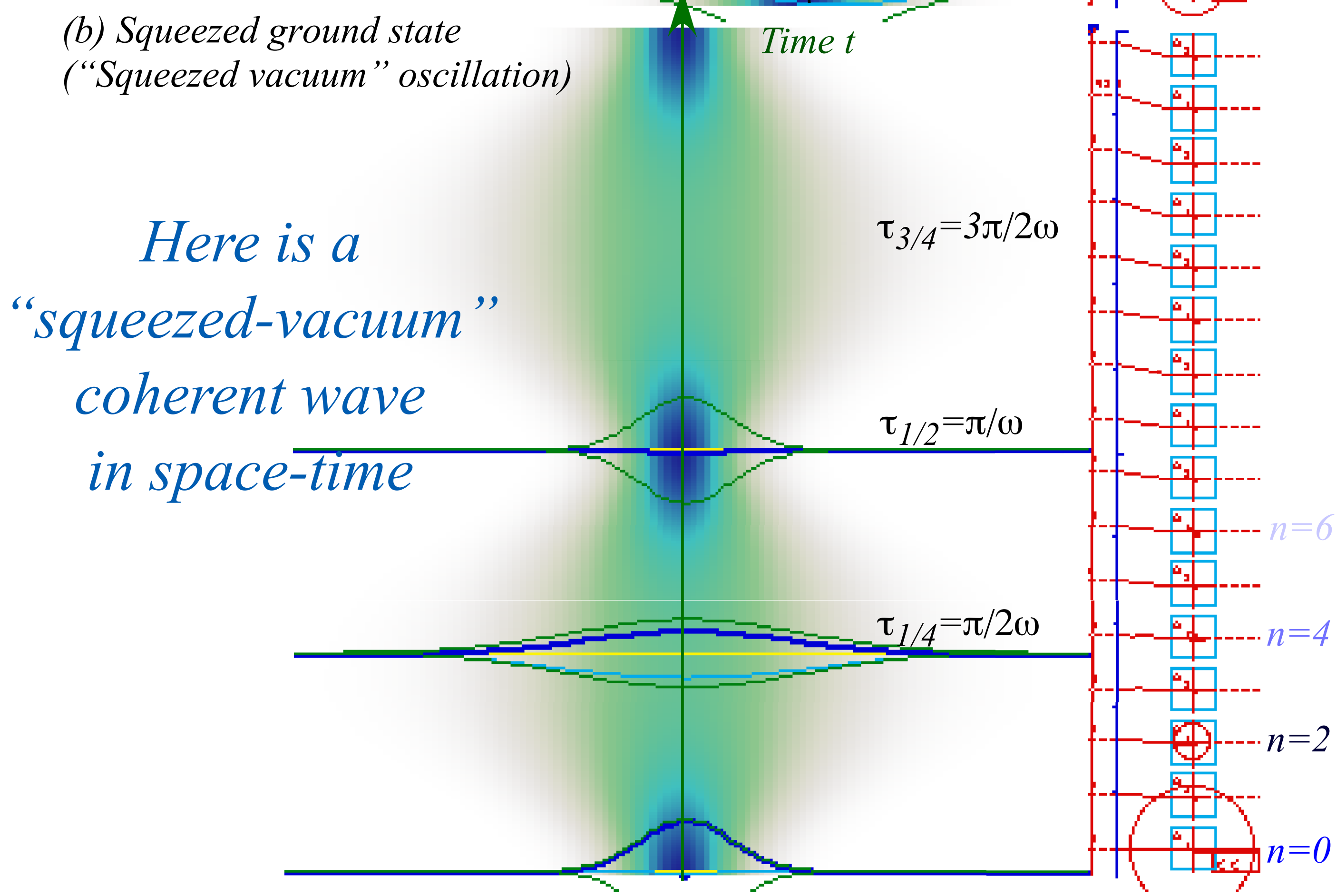
Properties of coherent state



Yay! Cosine trajectory! (That is "fat")

Properties of coherent state





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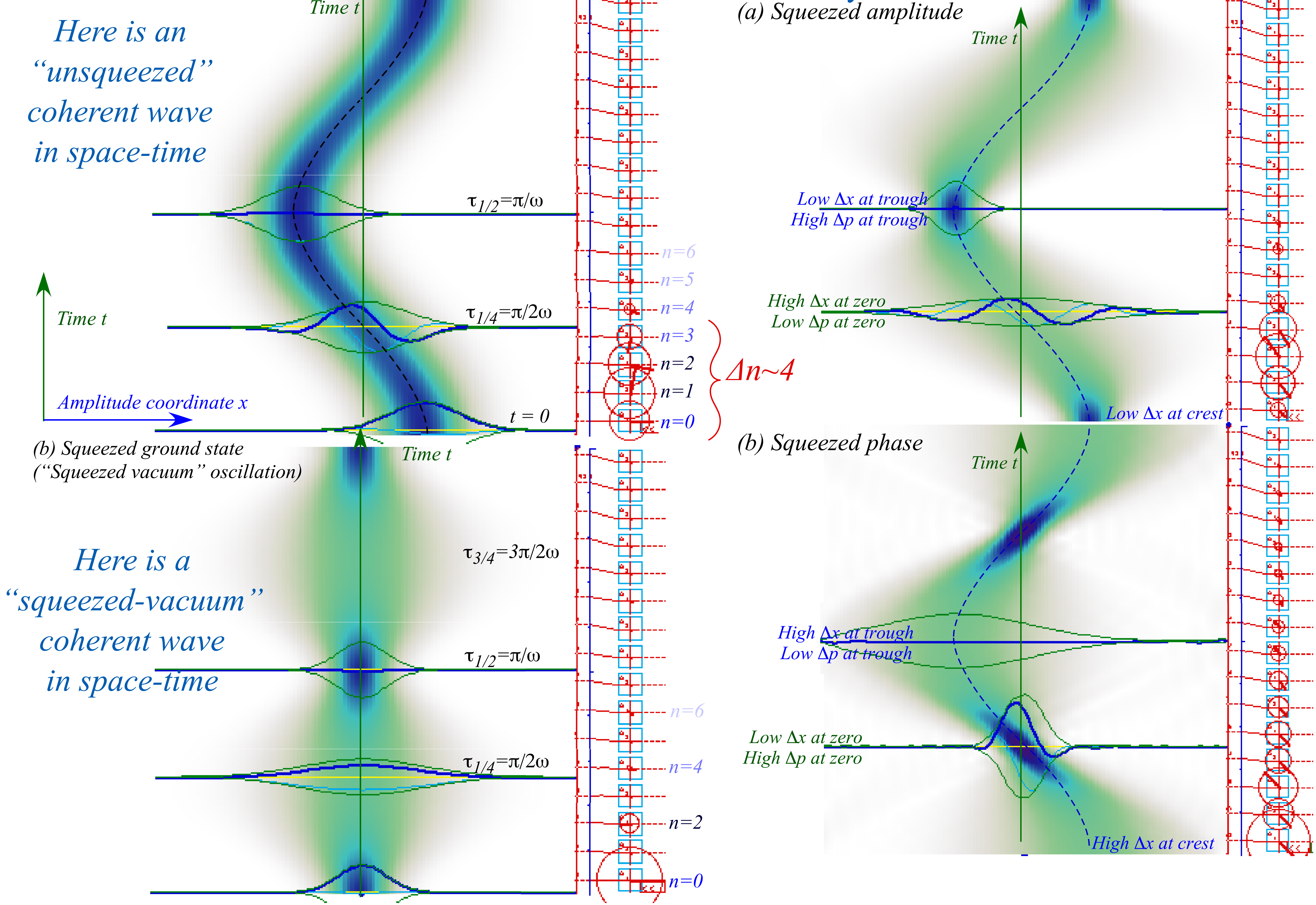
what happens if you apply operators with non-linear "tensor" exponents $exp(s\mathbf{x}^2)$, $exp(f\,\mathbf{p}^2)$, etc.

Properties of coherent state

Quantum-number *n* and phase *Φ* are conjugate variables with uncertainty relation $\Delta n\, \Delta\Phi \geq \pi/n$

(a) Coherent wave oscillation



Properties of 1D-HO coherent state

Coherent wave packet uncertainty relation: $\Delta n \cdot \Delta\phi > \pi / n$

???? Some uncertainty remains about this uncertainty ????