AMOP
reference links
on following page

1.31.18 class 6.0: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

Symmetry group $\mathscr{G}$ representations=>*AMOP* Hamiltonian **H** (or K) matrices, irreps $\mathscr{D}^{(\alpha)}$
=>*AMOP* wave functions $\Psi^{(\alpha)}$, eigensolutions

$\mathscr{G}$= U(2) product $\mathbf{R}[\boldsymbol{\Theta}]\mathbf{R}[\boldsymbol{\Theta}']=\mathbf{R}[\boldsymbol{\Theta}''']$ algebra (It's all done with σ_μ spinors)

Jordan-Pauli identity: U(2) product algebra of spinor σ_μ-operators

U(2) "Crazy-Thing" forms do products $\mathbf{R}[\boldsymbol{\Theta}]\mathbf{R}[\boldsymbol{\Theta}']=\mathbf{R}[\boldsymbol{\Theta}''']$ *algebraically*

$\mathscr{G}$= U(2) product $\mathbf{R}[\boldsymbol{\Theta}]\mathbf{R}[\boldsymbol{\Theta}']=\mathbf{R}[\boldsymbol{\Theta}''']$ by geometry (It's all done with σ_μ mirrors)

Mirror reflections by σ_μ-operators make rotations *The famous Clothing Store Mirror*

Hamilton-turns do products $\mathbf{R}[\boldsymbol{\Theta}]\mathbf{R}[\boldsymbol{\Theta}']=\mathbf{R}[\boldsymbol{\Theta}''']$ *geometrically*

Hamilton-turn slide rule and sundial *U(2) products and* (α,β,γ)-$[\varphi,\vartheta,\Theta]$ *conversions*

Finite group products by turns or by group link diagrams *D_3 example.* *O_h example*

$\mathscr{G}$= U(2) class transformation $\mathbf{R}[\boldsymbol{\Theta}]\mathbf{R}[\boldsymbol{\Theta}']\mathbf{R}[\boldsymbol{\Theta}]^{-1}=\mathbf{R}[\boldsymbol{\Theta}''']$ geometry

Group equivalence classes

U(2) density operator $\boldsymbol{\rho}$ and $[\boldsymbol{\rho},\mathbf{H}]$ mechanics

Density mechanics compared to spin vector **S** *rotated by crank vector* $\boldsymbol{\Theta}=\boldsymbol{\Omega}t$

Bloch equation $i\hbar\dot{\rho}=\left[\mathbf{H},\rho\right]$

AMOP reference links *(Updated list given on 2nd page of each class presentation)*

Web Resources - front page

UAF Physics UTube channel

2014 AMOP

2017 Group Theory for QM

2018 AMOP

Frame Transformation Relations And Multipole Transitions In Symmetric Polyatomic Molecules - RMP-1978 (Alt Scanned version)
Rotational energy surfaces and high- J eigenvalue structure of polyatomic molecules - Harter - Patterson - 1984
Galloping waves and their relativistic properties - ajp-1985-Harter
Asymptotic eigensolutions of fourth and sixth rank octahedral tensor operators - Harter-Patterson-JMP-1979
Nuclear spin weights and gas phase spectral structure of 12C60 and 13C60 buckminsterfullerene -Harter-Reimer-Cpl-1992 - (Alt1, Alt2 Erratum)

Double group theory on the half-shell and the two level system -Harter-Santos-1978-AJP.

- I) Rotation and half integral spin states (Alt scan)
- II) Optical polarization (Alt scan)

Theory of hyperfine and superfine levels in symmetric polyatomic molecules.

- I) Trigonal and tetrahedral molecules: Elementary spin-1/2 cases in vibronic ground states - PRA-1979-Harter-Patterson (Alt scan)
- II) Elementary cases in octahedral hexafluoride molecules - Harter-PRA-1981 (Alt scan)

Rotation-vibration scalar coupling zeta coefficients and spectroscopic band shapes of buckminsterfullerene - Weeks-Harter-CPL-1991 (Alt scan)
Fullerene symmetry reduction and rotational level fine structure/ the Buckyball isotopomer 12C 13C59 - jcp-Reimer-Harter-1997 (HiRez)
Molecular Eigensolution Symmetry Analysis and Fine Structure - IJMS-harter-mitchell-2013

Rotation–vibration spectra of icosahedral molecules.

- I) Icosahedral symmetry analysis and fine structure - harter-weeks-jcp-1989
- II) Icosahedral symmetry, vibrational eigenfrequencies, and normal modes of buckminsterfullerene - weeks-harter-jcp-1989
- III) Half-integral angular momentum - harter-reimer-jcp-1991

QTCA Unit 10 Ch 30 - 2013
AMOP Ch 32 Molecular Symmetry and Dynamics - 2019
AMOP Ch 0 Space-Time Symmetry - 2019

RESONANCE AND REVIVALS

- I) QUANTUM ROTOR AND INFINITE-WELL DYNAMICS - ISMSLi2012 (Talk) https://kb.osu.edu/dspace/handle/1811/52324
- II) Comparing Half-integer Spin and Integer Spin - Alva-ISMS-Ohio2013-R777 (Talks)
- III) Quantum Resonant Beats and Revivals in the Morse Oscillators and Rotors - (2013-Li-Diss)

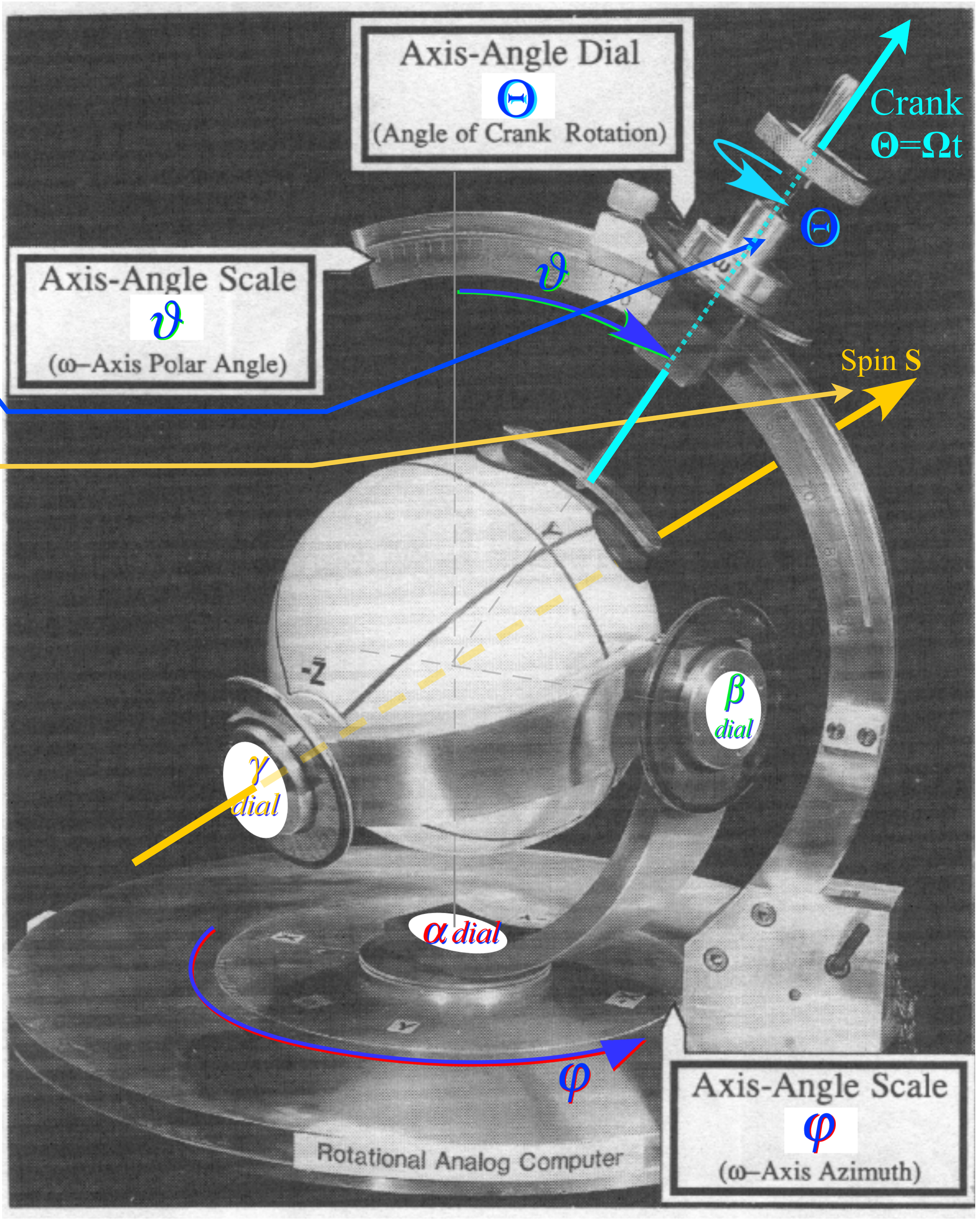
Rovibrational Spectral Fine Structure Of Icosahedral Molecules - Cpl 1986 (Alt Scan)
Gas Phase Level Structure of C60 Buckyball and Derivatives Exhibiting Broken Icosahedral Symmetry - reimer-diss-1996
Resonance and Revivals in Quantum Rotors - Comparing Half-integer Spin and Integer Spin - Alva-ISMS-Ohio2013-R777 (Talk)
Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-cpl-2013
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - harter - jms - 2001

Review of Class-5 showing that dynamics of $i\partial\Psi/\partial t = \mathbf{H}\Psi$ may be reduced to mechanics:

Crank $\boldsymbol{\Theta} = \boldsymbol{\Omega}t$ of Hamiltonian **H**

rotates

Spin vector $\mathbf{S} = \frac{1}{2}\boldsymbol{\sigma}$ of state Ψ.



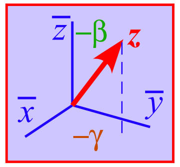
Darboux [φ,ϑ,Θ] *crank-axis angles*

Polar coordinates for unit axis vector $\hat{\boldsymbol{\Theta}}$

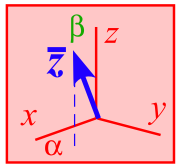
$$\hat{\Theta}_x = \cos\varphi \;\; \sin\vartheta$$
$$\hat{\Theta}_y = \sin\varphi \;\; \sin\vartheta$$
$$\hat{\Theta}_z = \qquad\quad \cos\vartheta$$

Here spin-rotor **S***-polar coordinates are Euler* (α,β,γ) *angles*

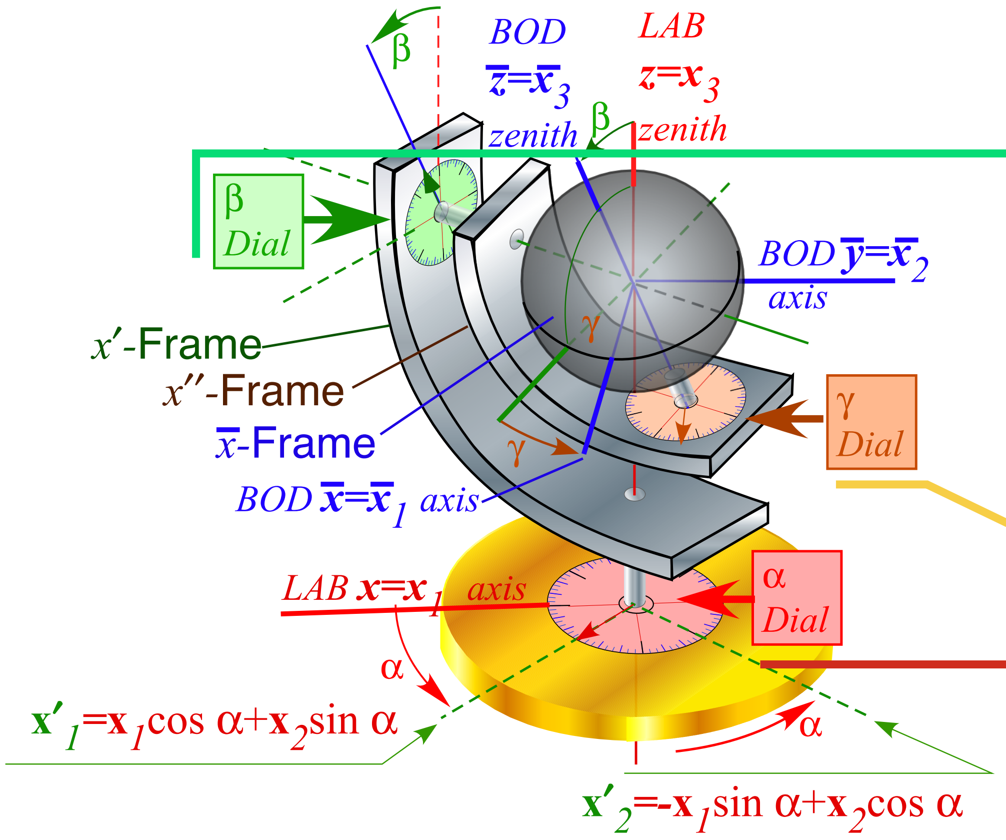
$$\hat{S}_X = \cos\alpha \ \sin\beta$$
$$\hat{S}_Y = \sin\alpha \ \sin\beta$$
$$\hat{S}_Z = \cos\beta$$



BOD frame view
Polar angles of LAB zenith $z=x_3$ *are*
(azimuth angle=$-\gamma$*,*
polar angle=$-\beta$ *)*



LAB frame view
Polar angles of BOD zenith $\bar{z}=\bar{x}_3$ *are*
(azimuth angle=α*,*
polar angle=β *)*

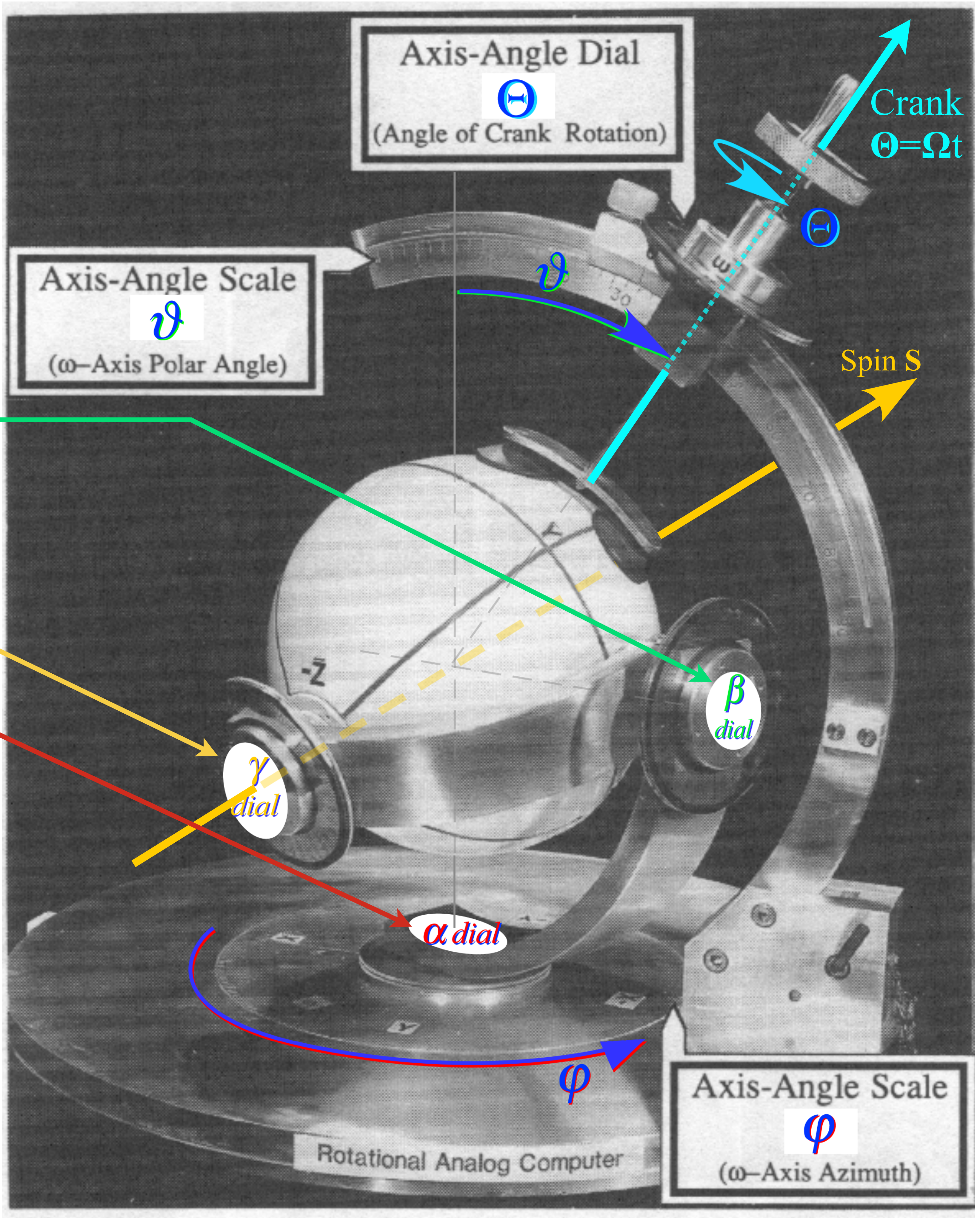


Versus

Darboux $[\varphi,\vartheta,\Theta]$ *crank-axis angles*

Polar coordinates
for unit axis vector $\hat{\Theta}$

$$\hat{\Theta}_x = \cos\varphi \ \sin\vartheta$$
$$\hat{\Theta}_y = \sin\varphi \ \sin\vartheta$$
$$\hat{\Theta}_z = \cos\vartheta$$



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Finite group products by turns or by group link diagrams

D_3 example.

O_h example

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Operator-on-Operator transformations

Product algebra Multiplication rules[†] for Pauli's "σ_μ-quaternions" and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$.

•	**1**	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
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$\mathbf{q}_X$	$\mathbf{q}_X$	$-\mathbf{1}$	$\mathbf{q}_Z$	$-\mathbf{q}_Y$
$\mathbf{q}_Y$	$\mathbf{q}_Y$	$-\mathbf{q}_Z$	$-\mathbf{1}$	$\mathbf{q}_X$
$\mathbf{q}_Z$	$\mathbf{q}_Z$	$\mathbf{q}_Y$	$-\mathbf{q}_X$	$-\mathbf{1}$

,

•	**1**	σ_X	σ_Y	σ_Z
1	**1**	σ_X	σ_Y	σ_Z
σ_X	σ_X	**1**	$i\sigma_Z$	$-i\sigma_Y$
σ_Y	σ_Y	$-i\sigma_Z$	**1**	$i\sigma_X$
σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	**1**

[†] Class 4 p. 28 link

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$\mathbf{1}$	$\mathbf{1}$	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{q}_X$	$\mathbf{q}_X$	$-\mathbf{1}$	$\mathbf{q}_Z$	$-\mathbf{q}_Y$
$\mathbf{q}_Y$	$\mathbf{q}_Y$	$-\mathbf{q}_Z$	$-\mathbf{1}$	$\mathbf{q}_X$
$\mathbf{q}_Z$	$\mathbf{q}_Z$	$\mathbf{q}_Y$	$-\mathbf{q}_X$	$-\mathbf{1}$

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•	$\mathbf{1}$	σ_X	σ_Y	σ_Z
$\mathbf{1}$	$\mathbf{1}$	σ_X	σ_Y	σ_Z
σ_X	σ_X	$\mathbf{1}$	$i\sigma_Z$	$-i\sigma_Y$
σ_Y	σ_Y	$-i\sigma_Z$	$\mathbf{1}$	$i\sigma_X$
σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	$\mathbf{1}$

$$\mathbf{q}_\mu \, \mathbf{q}_\nu = -\delta_{\mu\nu} \, \mathbf{1} + \varepsilon_{\mu\nu\lambda} \, \mathbf{q}_\lambda$$

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$\mathbf{q}_Y$	$\mathbf{q}_Y$	$-\mathbf{q}_Z$	$-\mathbf{1}$	$\mathbf{q}_X$
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Commutation rules for Pauli ops: σ_μ

$$\sigma_\mu\sigma_\nu - \sigma_\nu\sigma_\mu = [\sigma_\mu, \sigma_\nu] = 2i\,\varepsilon_{\mu\nu\lambda}\,\sigma_\lambda$$

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Jordan's spin-ops: $\mathbf{J}_\mu = \mathbf{S}_\mu = \sigma_\mu/2$.

$$\mathbf{S}_\mu\mathbf{S}_\nu - \mathbf{S}_\nu\mathbf{S}_\mu = [\mathbf{S}_\mu, \mathbf{S}_\nu] = i \, \varepsilon_{\mu\nu\lambda} \, \mathbf{S}_\lambda$$

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$\mathbf{q}_Z$	$\mathbf{q}_Z$	$\mathbf{q}_Y$	$-\mathbf{q}_X$	$-\mathbf{1}$

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$\mathbf{1}$	$\mathbf{1}$	σ_X	σ_Y	σ_Z
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Group products: $\mathbf{R}_{ab}(\vec{\Theta}_{ab}) \equiv \mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b) = e^{-i(\vec{\sigma}\bullet\vec{\Theta}_a)/2} e^{-i(\vec{\sigma}\bullet\vec{\Theta}_b)/2}$

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$\mathbf{q}_Y$	$\mathbf{q}_Y$	$-\mathbf{q}_Z$	$-\mathbf{1}$	$\mathbf{q}_X$
$\mathbf{q}_Z$	$\mathbf{q}_Z$	$\mathbf{q}_Y$	$-\mathbf{q}_X$	$-\mathbf{1}$

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This NOT just $e^{i\mathbf{a}}e^{i\mathbf{b}}=e^{i(\mathbf{a}+\mathbf{b})}$ *!*

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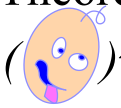
$$\mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b) = \left(\mathbf{1}\cos\frac{\Theta_a}{2} - i(\vec{\sigma}\bullet\hat{\Theta}_a)\sin\frac{\Theta_a}{2}\right)\left(\mathbf{1}\cos\frac{\Theta_b}{2} - i(\vec{\sigma}\bullet\hat{\Theta}_b)\sin\frac{\Theta_b}{2}\right) = \mathbf{R}_{ab}(\vec{\Theta}_{ab})$$

†Class 4 p. 52 link

The 
Crazy Thing
Theorem:
If ()^2= -**1**
Then:
e()θ =**1**cosθ+()sinθ

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σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	**1**

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Commutation rules for Pauli ops: σ_μ

$$\sigma_\mu\sigma_\nu - \sigma_\nu\sigma_\mu = [\sigma_\mu, \sigma_\nu] = 2i\ \varepsilon_{\mu\nu\lambda}\ \sigma_\lambda$$

Group products: $\mathbf{R}_{ab}(\vec{\Theta}_{ab}) \equiv \mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b) = e^{-i(\vec{\sigma}\bullet\vec{\Theta}_a)/2}e^{-i(\vec{\sigma}\bullet\vec{\Theta}_b)/2}$

This <u>NOT</u> just $e^{i\mathbf{a}}e^{i\mathbf{b}}=e^{i(\mathbf{a}+\mathbf{b})}$ *!*

$$\mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b) = \left(\mathbf{1}\cos\frac{\Theta_a}{2} - i(\vec{\sigma}\bullet\hat{\Theta}_a)\sin\frac{\Theta_a}{2}\right)\left(\mathbf{1}\cos\frac{\Theta_b}{2} - i(\vec{\sigma}\bullet\hat{\Theta}_b)\sin\frac{\Theta_b}{2}\right) = \mathbf{R}_{ab}(\vec{\Theta}_{ab})$$

$$= \cos\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2}\mathbf{1} - i\left[\hat{\Theta}_a\sin\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} + \hat{\Theta}_b\cos\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\right]\bullet\vec{\sigma} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}(\vec{\sigma}\bullet\hat{\Theta}_a)(\vec{\sigma}\bullet\hat{\Theta}_b)$$

Operator-on-Operator transformations

Product algebra Multiplication rules for Pauli's "σ_μ-quaternions" and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$.

•	$\mathbf{1}$	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{q}_X$	$\mathbf{q}_X$	$-\mathbf{1}$	$\mathbf{q}_Z$	$-\mathbf{q}_Y$
$\mathbf{q}_Y$	$\mathbf{q}_Y$	$-\mathbf{q}_Z$	$-\mathbf{1}$	$\mathbf{q}_X$
$\mathbf{q}_Z$	$\mathbf{q}_Z$	$\mathbf{q}_Y$	$-\mathbf{q}_X$	$-\mathbf{1}$

,

•	$\mathbf{1}$	σ_X	σ_Y	σ_Z
$\mathbf{1}$	$\mathbf{1}$	σ_X	σ_Y	σ_Z
σ_X	σ_X	$\mathbf{1}$	$i\sigma_Z$	$-i\sigma_Y$
σ_Y	σ_Y	$-i\sigma_Z$	$\mathbf{1}$	$i\sigma_X$
σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	$\mathbf{1}$

$$\mathbf{q}_\mu \mathbf{q}_\nu = -\delta_{\mu\nu}\,\mathbf{1} + \varepsilon_{\mu\nu\lambda}\,\mathbf{q}_\lambda$$

$$\sigma_\mu \sigma_\nu = \delta_{\mu\nu}\,\mathbf{1} + i\,\varepsilon_{\mu\nu\lambda}\,\sigma_\lambda$$

Commutation rules for Pauli ops: σ_μ

$$\sigma_\mu\sigma_\nu - \sigma_\nu\sigma_\mu = [\sigma_\mu, \sigma_\nu] = 2i\,\varepsilon_{\mu\nu\lambda}\,\sigma_\lambda$$

Group products: $\mathbf{R}_{ab}(\vec{\Theta}_{ab}) \equiv \mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b) = e^{-i(\vec{\sigma}\bullet\vec{\Theta}_a)/2}e^{-i(\vec{\sigma}\bullet\vec{\Theta}_b)/2}$ *This NOT just* $e^{i\mathbf{a}}e^{i\mathbf{b}}=e^{i(\mathbf{a}+\mathbf{b})}$ *!*

†Class 4 p. 28 link

$$\mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b) = \left(\mathbf{1}\cos\frac{\Theta_a}{2} - i(\vec{\sigma}\bullet\hat{\Theta}_a)\sin\frac{\Theta_a}{2}\right)\left(\mathbf{1}\cos\frac{\Theta_b}{2} - i(\vec{\sigma}\bullet\hat{\Theta}_b)\sin\frac{\Theta_b}{2}\right) = \mathbf{R}_{ab}(\vec{\Theta}_{ab})$$

$$=\cos\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2}\mathbf{1} - i\left[\hat{\Theta}_a\sin\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} + \hat{\Theta}_b\cos\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\right]\bullet\vec{\sigma} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}(\vec{\sigma}\bullet\hat{\Theta}_a)(\vec{\sigma}\bullet\hat{\Theta}_b)$$

†Class 4 p. 45 link

Jordan-Pauli† identity $(\vec{\sigma}\bullet\mathbf{a})(\vec{\sigma}\bullet\mathbf{b})=(\mathbf{a}\bullet\mathbf{b})\mathbf{1}+i(\mathbf{a}\times\mathbf{b})\bullet\vec{\sigma}$ reduces $(\vec{\sigma}\bullet\hat{\Theta}_a)(\vec{\sigma}\bullet\hat{\Theta}_b)$ to: $\left(\hat{\Theta}_a\bullet\hat{\Theta}_b\right)\mathbf{1}+\left(\hat{\Theta}_a\times\hat{\Theta}_b\right)\bullet\vec{\sigma}$

Operator-on-Operator transformations

Product algebra Multiplication rules for Pauli's "σ_μ-quaternions" and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$.

•	1	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
1	**1**	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{q}_X$	$\mathbf{q}_X$	$-\mathbf{1}$	$\mathbf{q}_Z$	$-\mathbf{q}_Y$
$\mathbf{q}_Y$	$\mathbf{q}_Y$	$-\mathbf{q}_Z$	$-\mathbf{1}$	$\mathbf{q}_X$
$\mathbf{q}_Z$	$\mathbf{q}_Z$	$\mathbf{q}_Y$	$-\mathbf{q}_X$	$-\mathbf{1}$

,

•	1	σ_X	σ_Y	σ_Z
1	**1**	σ_X	σ_Y	σ_Z
σ_X	σ_X	**1**	$i\sigma_Z$	$-i\sigma_Y$
σ_Y	σ_Y	$-i\sigma_Z$	**1**	$i\sigma_X$
σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	**1**

$$\mathbf{q}_\mu \mathbf{q}_\nu = -\delta_{\mu\nu}\,\mathbf{1} + \varepsilon_{\mu\nu\lambda}\,\mathbf{q}_\lambda$$

$$\sigma_\mu \sigma_\nu = \delta_{\mu\nu}\,\mathbf{1} + i\,\varepsilon_{\mu\nu\lambda}\,\sigma_\lambda$$

Commutation rules for Pauli ops: σ_μ

$$\sigma_\mu\sigma_\nu - \sigma_\nu\sigma_\mu = [\sigma_\mu, \sigma_\nu] = 2i\,\varepsilon_{\mu\nu\lambda}\,\sigma_\lambda$$

Group products: $\mathbf{R}_{ab}(\vec{\Theta}_{ab}) \equiv \mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b) = e^{-i(\vec{\sigma}\bullet\vec{\Theta}_a)/2}e^{-i(\vec{\sigma}\bullet\vec{\Theta}_b)/2}$ *This <u>NOT</u> just $e^{i\mathbf{a}}e^{i\mathbf{b}}=e^{i(\mathbf{a}+\mathbf{b})}$!*

$$\mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b) = \left(\mathbf{1}\cos\frac{\Theta_a}{2} - i(\vec{\boldsymbol{\sigma}}\bullet\hat{\boldsymbol{\Theta}}_a)\sin\frac{\Theta_a}{2}\right)\left(\mathbf{1}\cos\frac{\Theta_b}{2} - i(\vec{\boldsymbol{\sigma}}\bullet\hat{\boldsymbol{\Theta}}_b)\sin\frac{\Theta_b}{2}\right) = \mathbf{R}_{ab}(\vec{\Theta}_{ab})$$

$$= \cos\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2}\mathbf{1} - i\left[\hat{\boldsymbol{\Theta}}_a\sin\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} + \hat{\boldsymbol{\Theta}}_b\cos\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\right]\bullet\vec{\boldsymbol{\sigma}} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}(\vec{\boldsymbol{\sigma}}\bullet\hat{\boldsymbol{\Theta}}_a)(\vec{\boldsymbol{\sigma}}\bullet\hat{\boldsymbol{\Theta}}_b)$$

†Class 4 p. 45 link

Jordan-Pauli† identity $(\vec{\sigma}\bullet\mathbf{a})(\vec{\sigma}\bullet\mathbf{b}) = (\mathbf{a}\bullet\mathbf{b})\mathbf{1} + i(\mathbf{a}\times\mathbf{b})\bullet\vec{\sigma}$ reduces $(\vec{\boldsymbol{\sigma}}\bullet\hat{\boldsymbol{\Theta}}_a)(\vec{\boldsymbol{\sigma}}\bullet\hat{\boldsymbol{\Theta}}_b)$ to: $\left(\hat{\boldsymbol{\Theta}}_a\bullet\hat{\boldsymbol{\Theta}}_b\right)\mathbf{1} + \left(\hat{\boldsymbol{\Theta}}_a\times\hat{\boldsymbol{\Theta}}_b\right)\bullet\vec{\boldsymbol{\sigma}}$

$$= \left(\cos\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\left(\hat{\boldsymbol{\Theta}}_a\bullet\hat{\boldsymbol{\Theta}}_b\right)\right)\mathbf{1}$$

$$- i\left\{\hat{\boldsymbol{\Theta}}_a\sin\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} + \hat{\boldsymbol{\Theta}}_b\cos\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\left(\hat{\boldsymbol{\Theta}}_a\times\hat{\boldsymbol{\Theta}}_b\right)\right\}\bullet\vec{\boldsymbol{\sigma}}$$

Operator-on-Operator transformations

Product algebra Multiplication rules for Pauli's "σ_μ-quaternions" and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$.

•	$\mathbf{1}$	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{q}_X$	$\mathbf{q}_X$	$-\mathbf{1}$	$\mathbf{q}_Z$	$-\mathbf{q}_Y$
$\mathbf{q}_Y$	$\mathbf{q}_Y$	$-\mathbf{q}_Z$	$-\mathbf{1}$	$\mathbf{q}_X$
$\mathbf{q}_Z$	$\mathbf{q}_Z$	$\mathbf{q}_Y$	$-\mathbf{q}_X$	$-\mathbf{1}$

,

•	$\mathbf{1}$	σ_X	σ_Y	σ_Z
$\mathbf{1}$	$\mathbf{1}$	σ_X	σ_Y	σ_Z
σ_X	σ_X	$\mathbf{1}$	$i\sigma_Z$	$-i\sigma_Y$
σ_Y	σ_Y	$-i\sigma_Z$	$\mathbf{1}$	$i\sigma_X$
σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	$\mathbf{1}$

$$\mathbf{q}_\mu\,\mathbf{q}_\nu = -\delta_{\mu\nu}\,\mathbf{1} + \varepsilon_{\mu\nu\lambda}\,\mathbf{q}_\lambda$$
$$\sigma_\mu\,\sigma_\nu = \delta_{\mu\nu}\,\mathbf{1} + i\,\varepsilon_{\mu\nu\lambda}\,\sigma_\lambda$$

Commutation rules for Pauli ops: σ_μ

$$\sigma_\mu\sigma_\nu - \sigma_\nu\sigma_\mu = [\sigma_\mu, \sigma_\nu] = 2i\,\varepsilon_{\mu\nu\lambda}\,\sigma_\lambda$$

Group products: $\mathbf{R}_{ab}(\vec{\Theta}_{ab}) \equiv \mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b) = e^{-i(\vec{\sigma}\bullet\vec{\Theta}_a)/2}e^{-i(\vec{\sigma}\bullet\vec{\Theta}_b)/2}$ *This NOT just* $e^{i\mathbf{a}}e^{i\mathbf{b}}=e^{i(\mathbf{a}+\mathbf{b})}$ *!*

$$\mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b) = \left(\mathbf{1}\cos\frac{\Theta_a}{2} - i(\vec{\sigma}\bullet\hat{\Theta}_a)\sin\frac{\Theta_a}{2}\right)\left(\mathbf{1}\cos\frac{\Theta_b}{2} - i(\vec{\sigma}\bullet\hat{\Theta}_b)\sin\frac{\Theta_b}{2}\right) = \mathbf{R}_{ab}(\vec{\Theta}_{ab})$$

$$=\cos\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2}\mathbf{1} - i\left[\hat{\Theta}_a\sin\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} + \hat{\Theta}_b\cos\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\right]\bullet\vec{\sigma} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}(\vec{\sigma}\bullet\hat{\Theta}_a)(\vec{\sigma}\bullet\hat{\Theta}_b)$$

†Class 4 p. 45 link

Jordan-Pauli† identity $(\vec{\sigma}\bullet\mathbf{a})(\vec{\sigma}\bullet\mathbf{b})=(\mathbf{a}\bullet\mathbf{b})\mathbf{1}+i(\mathbf{a}\times\mathbf{b})\bullet\vec{\sigma}$ reduces $(\vec{\sigma}\bullet\hat{\Theta}_a)(\vec{\sigma}\bullet\hat{\Theta}_b)$ to: $\left(\hat{\Theta}_a\bullet\hat{\Theta}_b\right)\mathbf{1}+\left(\hat{\Theta}_a\times\hat{\Theta}_b\right)\bullet\vec{\sigma}$

$$=\left(\cos\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\left(\hat{\Theta}_a\bullet\hat{\Theta}_b\right)\right)\mathbf{1}$$

$$-i\left\{\hat{\Theta}_a\sin\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} + \hat{\Theta}_b\cos\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\left(\hat{\Theta}_a\times\hat{\Theta}_b\right)\right\}\bullet\vec{\sigma}$$

$$=\left(\cos\frac{\Theta_{ab}}{2}\right)\mathbf{1} - i\left\{\hat{\Theta}_{ab}\sin\frac{\Theta_{ab}}{2}\right\}\bullet\vec{\sigma} = \mathbf{R}_{ab}(\vec{\Theta}_{ab}) = \mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b)$$

Match with “Crazy-Thing” form of product $\mathbf{R}_{ab}(\vec{\Theta}_{ab})$

†Class 4 p. 52 link

Operator-on-Operator transformations

Product algebra Multiplication rules for Pauli's "σ_μ-quaternions" and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$.

•	1	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
1	1	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{q}_X$	$\mathbf{q}_X$	−1	$\mathbf{q}_Z$	$-\mathbf{q}_Y$
$\mathbf{q}_Y$	$\mathbf{q}_Y$	$-\mathbf{q}_Z$	−1	$\mathbf{q}_X$
$\mathbf{q}_Z$	$\mathbf{q}_Z$	$\mathbf{q}_Y$	$-\mathbf{q}_X$	−1

,

•	1	σ_X	σ_Y	σ_Z
1	1	σ_X	σ_Y	σ_Z
σ_X	σ_X	1	$i\sigma_Z$	$-i\sigma_Y$
σ_Y	σ_Y	$-i\sigma_Z$	1	$i\sigma_X$
σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	1

$$\mathbf{q}_\mu\,\mathbf{q}_\nu = -\delta_{\mu\nu}\,\mathbf{1} + \varepsilon_{\mu\nu\lambda}\,\mathbf{q}_\lambda$$
$$\sigma_\mu\,\sigma_\nu = \delta_{\mu\nu}\,\mathbf{1} + i\,\varepsilon_{\mu\nu\lambda}\,\sigma_\lambda$$

Commutation rules for Pauli ops: σ_μ

$$\sigma_\mu\sigma_\nu - \sigma_\nu\sigma_\mu = [\sigma_\mu, \sigma_\nu] = 2i\,\varepsilon_{\mu\nu\lambda}\,\sigma_\lambda$$

Group products: $\mathbf{R}_{ab}(\vec{\Theta}_{ab}) \equiv \mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b) = e^{-i(\vec{\sigma}\bullet\vec{\Theta}_a)/2}e^{-i(\vec{\sigma}\bullet\vec{\Theta}_b)/2}$ *This NOT just* $e^{i\mathbf{a}}e^{i\mathbf{b}}=e^{i(\mathbf{a}+\mathbf{b})}$ *!*

$$\mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b) = \left(\mathbf{1}\cos\frac{\Theta_a}{2} - i(\vec{\sigma}\bullet\hat{\Theta}_a)\sin\frac{\Theta_a}{2}\right)\left(\mathbf{1}\cos\frac{\Theta_b}{2} - i(\vec{\sigma}\bullet\hat{\Theta}_b)\sin\frac{\Theta_b}{2}\right) = \mathbf{R}_{ab}(\vec{\Theta}_{ab})$$

$$=\cos\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2}\mathbf{1} - i\left[\hat{\Theta}_a\sin\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} + \hat{\Theta}_b\cos\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\right]\bullet\vec{\sigma} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}(\vec{\sigma}\bullet\hat{\Theta}_a)(\vec{\sigma}\bullet\hat{\Theta}_b)$$

†Class 4 p. 45 link

Jordan-Pauli† identity $(\vec{\sigma}\bullet\mathbf{a})(\vec{\sigma}\bullet\mathbf{b})=(\mathbf{a}\bullet\mathbf{b})\mathbf{1}+i(\mathbf{a}\times\mathbf{b})\bullet\vec{\sigma}$ reduces $(\vec{\sigma}\bullet\hat{\Theta}_a)(\vec{\sigma}\bullet\hat{\Theta}_b)$ to: $\left(\hat{\Theta}_a\bullet\hat{\Theta}_b\right)\mathbf{1}+\left(\hat{\Theta}_a\times\hat{\Theta}_b\right)\bullet\vec{\sigma}$

$$=\left(\cos\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\left(\hat{\Theta}_a\bullet\hat{\Theta}_b\right)\right)\mathbf{1}$$

$$-i\left\{\hat{\Theta}_a\sin\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} + \hat{\Theta}_b\cos\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\left(\hat{\Theta}_a\times\hat{\Theta}_b\right)\right\}\bullet\vec{\sigma}$$

$$=\left(\cos\frac{\Theta_{ab}}{2}\right)\mathbf{1} - i\left\{\hat{\Theta}_{ab}\sin\frac{\Theta_{ab}}{2}\right\}\bullet\vec{\sigma} = \mathbf{R}_{ab}(\vec{\Theta}_{ab}) = \mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b)$$

Match with "Crazy-Thing" form of product $\mathbf{R}_{ab}(\vec{\Theta}_{ab})$

†Class 4 p. 52 link

1st Step: Coefficient ( ) of **unit 1**
derives ***angle of rotation***: Θ_{ab}

Operator-on-Operator transformations

Product algebra Multiplication rules for Pauli's "σ_μ-quaternions" and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$.

•	1	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
1	**1**	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{q}_X$	$\mathbf{q}_X$	$-\mathbf{1}$	$\mathbf{q}_Z$	$-\mathbf{q}_Y$
$\mathbf{q}_Y$	$\mathbf{q}_Y$	$-\mathbf{q}_Z$	$-\mathbf{1}$	$\mathbf{q}_X$
$\mathbf{q}_Z$	$\mathbf{q}_Z$	$\mathbf{q}_Y$	$-\mathbf{q}_X$	$-\mathbf{1}$

,

•	1	σ_X	σ_Y	σ_Z
1	**1**	σ_X	σ_Y	σ_Z
σ_X	σ_X	**1**	$i\sigma_Z$	$-i\sigma_Y$
σ_Y	σ_Y	$-i\sigma_Z$	**1**	$i\sigma_X$
σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	**1**

$$\mathbf{q}_\mu\, \mathbf{q}_\nu = -\delta_{\mu\nu}\,\mathbf{1} + \varepsilon_{\mu\nu\lambda}\,\mathbf{q}_\lambda$$
$$\sigma_\mu\, \sigma_\nu = \delta_{\mu\nu}\,\mathbf{1} + i\,\varepsilon_{\mu\nu\lambda}\,\sigma_\lambda$$

Commutation rules for Pauli ops: σ_μ

$$\sigma_\mu\sigma_\nu - \sigma_\nu\sigma_\mu = [\sigma_\mu, \sigma_\nu] = 2i\,\varepsilon_{\mu\nu\lambda}\,\sigma_\lambda$$

Group products: $\mathbf{R}_{ab}(\vec{\Theta}_{ab}) \equiv \mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b) = e^{-i(\vec{\sigma}\bullet\vec{\Theta}_a)/2}e^{-i(\vec{\sigma}\bullet\vec{\Theta}_b)/2}$ *This NOT just* $e^{ia}e^{ib}=e^{i(a+b)}$ *!*

$$\mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b) = \left(\mathbf{1}\cos\frac{\Theta_a}{2} - i(\vec{\sigma}\bullet\hat{\Theta}_a)\sin\frac{\Theta_a}{2}\right)\left(\mathbf{1}\cos\frac{\Theta_b}{2} - i(\vec{\sigma}\bullet\hat{\Theta}_b)\sin\frac{\Theta_b}{2}\right) = \mathbf{R}_{ab}(\vec{\Theta}_{ab})$$

$$=\cos\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2}\mathbf{1} - i\left[\hat{\Theta}_a\sin\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} + \hat{\Theta}_b\cos\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\right]\bullet\vec{\sigma} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}(\vec{\sigma}\bullet\hat{\Theta}_a)(\vec{\sigma}\bullet\hat{\Theta}_b)$$

†Class 4 p. 45 link

Jordan-Pauli† identity $(\vec{\sigma}\bullet\mathbf{a})(\vec{\sigma}\bullet\mathbf{b}) = (\mathbf{a}\bullet\mathbf{b})\mathbf{1} + i(\mathbf{a}\times\mathbf{b})\bullet\vec{\sigma}$ reduces $(\vec{\sigma}\bullet\hat{\Theta}_a)(\vec{\sigma}\bullet\hat{\Theta}_b)$ to: $\left(\hat{\Theta}_a\bullet\hat{\Theta}_b\right)\mathbf{1} + \left(\hat{\Theta}_a\times\hat{\Theta}_b\right)\bullet\vec{\sigma}$

$$=\left(\cos\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\left(\hat{\Theta}_a\bullet\hat{\Theta}_b\right)\right)\mathbf{1}$$

$$-i\left\{\hat{\Theta}_a\sin\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} + \hat{\Theta}_b\cos\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\left(\hat{\Theta}_a\times\hat{\Theta}_b\right)\right\}\bullet\vec{\sigma}$$

$$=\left(\cos\frac{\Theta_{ab}}{2}\right)\mathbf{1} - i\left\{\hat{\Theta}_{ab}\sin\frac{\Theta_{ab}}{2}\right\}\bullet\vec{\sigma} = \mathbf{R}_{ab}(\vec{\Theta}_{ab}) = \mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b)$$

Match with "Crazy-Thing" form of product $\mathbf{R}_{ab}(\vec{\Theta}_{ab})$

†Class 4 p. 52 link

1st Step: Coefficient ( ) of **unit 1**
derives ***angle of rotation***: Θ_{ab}

2nd Step: Coefficient { } of $-i\bullet\vec{\sigma}$
derives ***unit-vector*** $\hat{\Theta}_{ab}$ ***of rotation***:

Operator-on-Operator transformations

Product algebra Multiplication rules for Pauli's "σ_μ-quaternions" and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$.

•	1	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
1	1	$\mathbf{q}_X$	$\mathbf{q}_Y$	$\mathbf{q}_Z$
$\mathbf{q}_X$	$\mathbf{q}_X$	-1	$\mathbf{q}_Z$	$-\mathbf{q}_Y$
$\mathbf{q}_Y$	$\mathbf{q}_Y$	$-\mathbf{q}_Z$	-1	$\mathbf{q}_X$
$\mathbf{q}_Z$	$\mathbf{q}_Z$	$\mathbf{q}_Y$	$-\mathbf{q}_X$	-1

,

•	1	σ_X	σ_Y	σ_Z
1	1	σ_X	σ_Y	σ_Z
σ_X	σ_X	1	$i\sigma_Z$	$-i\sigma_Y$
σ_Y	σ_Y	$-i\sigma_Z$	1	$i\sigma_X$
σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	1

$$\mathbf{q}_\mu\,\mathbf{q}_\nu = -\delta_{\mu\nu}\,\mathbf{1} + \varepsilon_{\mu\nu\lambda}\,\mathbf{q}_\lambda$$
$$\sigma_\mu\,\sigma_\nu = \delta_{\mu\nu}\,\mathbf{1} + i\,\varepsilon_{\mu\nu\lambda}\,\sigma_\lambda$$

Commutation rules for Pauli ops: σ_μ

$$\sigma_\mu\sigma_\nu - \sigma_\nu\sigma_\mu = [\sigma_\mu, \sigma_\nu] = 2i\,\varepsilon_{\mu\nu\lambda}\,\sigma_\lambda$$

Group products: $\mathbf{R}_{ab}(\vec{\Theta}_{ab}) \equiv \mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b) = e^{-i(\vec{\sigma}\bullet\vec{\Theta}_a)/2}e^{-i(\vec{\sigma}\bullet\vec{\Theta}_b)/2}$ *This NOT just* $e^{ia}e^{ib}=e^{i(a+b)}$ *!*

$$\mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b) = \left(\mathbf{1}\cos\frac{\Theta_a}{2} - i(\vec{\sigma}\bullet\hat{\Theta}_a)\sin\frac{\Theta_a}{2}\right)\left(\mathbf{1}\cos\frac{\Theta_b}{2} - i(\vec{\sigma}\bullet\hat{\Theta}_b)\sin\frac{\Theta_b}{2}\right) = \mathbf{R}_{ab}(\vec{\Theta}_{ab})$$

$$=\cos\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2}\mathbf{1} - i\left[\hat{\Theta}_a\sin\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} + \hat{\Theta}_b\cos\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\right]\bullet\vec{\sigma} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}(\vec{\sigma}\bullet\hat{\Theta}_a)(\vec{\sigma}\bullet\hat{\Theta}_b)$$

†Class 4 p. 45 link

Jordan-Pauli[†] identity $(\vec{\sigma}\bullet\mathbf{a})(\vec{\sigma}\bullet\mathbf{b}) = (\mathbf{a}\bullet\mathbf{b})\mathbf{1} + i(\mathbf{a}\times\mathbf{b})\bullet\vec{\sigma}$ reduces $(\vec{\sigma}\bullet\hat{\Theta}_a)(\vec{\sigma}\bullet\hat{\Theta}_b)$ to: $\left(\hat{\Theta}_a\bullet\hat{\Theta}_b\right)\mathbf{1} + \left(\hat{\Theta}_a\times\hat{\Theta}_b\right)\bullet\vec{\sigma}$

$$=\left(\cos\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\left(\hat{\Theta}_a\bullet\hat{\Theta}_b\right)\right)\mathbf{1}$$

$$-i\left\{\hat{\Theta}_a\sin\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} + \hat{\Theta}_b\cos\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2} - \sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\left(\hat{\Theta}_a\times\hat{\Theta}_b\right)\right\}\bullet\vec{\sigma}$$

$$=\left(\cos\frac{\Theta_{ab}}{2}\right)\mathbf{1} - i\left\{\hat{\Theta}_{ab}\sin\frac{\Theta_{ab}}{2}\right\}\bullet\vec{\sigma} = \mathbf{R}_{ab}(\vec{\Theta}_{ab}) = \mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b)$$

Match with "Crazy-Thing" form of product $\mathbf{R}_{ab}(\vec{\Theta}_{ab})$

†Class 4 p. 52 link

1st Step: Coefficient ( ) of **unit 1** derives ***angle of rotation***: Θ_{ab}

2nd Step: Coefficient { } of $-i\bullet\vec{\sigma}$ derives ***unit-vector*** $\hat{\Theta}_{ab}$ ***of rotation***:

Operator-on-Operator transformations

Product algebra $\mathbf{R}_{ab}(\vec{\Theta}_{ab}) \equiv \mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b) = e^{-i(\vec{\sigma}\bullet\vec{\Theta}_a)/2}e^{-i(\vec{\sigma}\bullet\vec{\Theta}_b)/2}$

This NOT just $e^{ia}e^{ib}=e^{i(a+b)}$ *!*

(...except when $\hat{\Theta}_a=\hat{\Theta}_b$ *!)*

$$=\left(\cos\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2}-\sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\left(\hat{\Theta}_a\bullet\hat{\Theta}_b\right)\right)\mathbf{1}$$

$$-i\left\{\hat{\Theta}_a\sin\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2}+\hat{\Theta}_b\cos\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}-\sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\left(\hat{\Theta}_a\times\hat{\Theta}_b\right)\right\}\bullet\vec{\sigma}$$

$$=\left(\cos\frac{\Theta_{ab}}{2}\right)\mathbf{1}-i\left\{\hat{\Theta}_{ab}\sin\frac{\Theta_{ab}}{2}\right\}\bullet\vec{\sigma}=\mathbf{R}_{ab}(\vec{\Theta}_{ab})=\mathbf{R}_a(\vec{\Theta}_a)\cdot\mathbf{R}_b(\vec{\Theta}_b)$$

Match with "Crazy-Thing" form of product $\mathbf{R}_{ab}(\vec{\Theta}_{ab})$

1st Step: Coefficient ( ) of **unit 1** derives ***angle of rotation****:* Θ_{ab}

2nd Step: Coefficient { } of *-i*•$\vec{\sigma}$ derives ***unit-vector*** $\hat{\Theta}_{ab}$ ***of rotation****:*

Now easy to find the *product angle* Θ_{ab} and *crank unit vector* $\hat{\Theta}_{ab}$.

U(2) and R(3) Group Product Formulae

$$\frac{\Theta_{ab}}{2}=\cos^{-1}\left(\cos\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2}-\sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\hat{\Theta}_a\bullet\hat{\Theta}_b\right)$$

$$\vec{\Theta}_{ab}=\left[\sin\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2}\ \hat{\Theta}_a+\cos\frac{\Theta}{2}\sin\frac{\Theta'}{2}\ \hat{\Theta}_b+\sin\frac{\Theta'}{2}\sin\frac{\Theta}{2}\hat{\Theta}_a\times\hat{\Theta}_b\right]/\sin\frac{\Theta_{ab}}{2}$$

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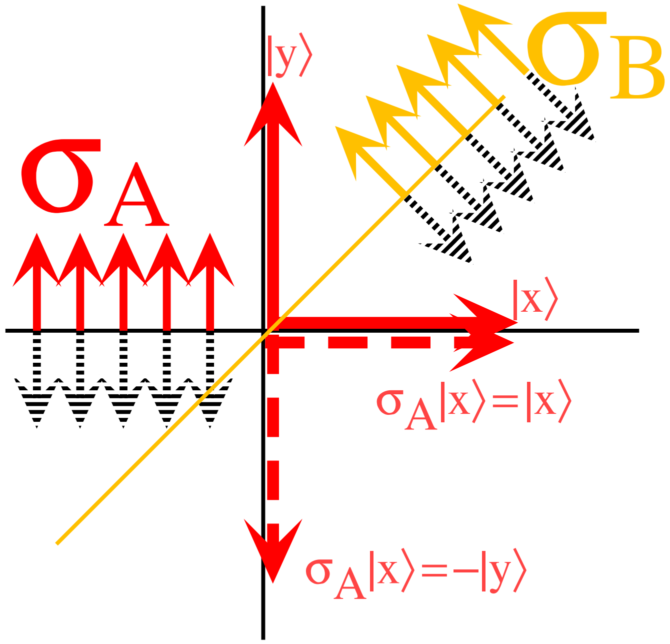
Density mechanics compared to spin vector **S** *rotated by crank vector* $\boldsymbol{\Theta}=\boldsymbol{\Omega}t$

Bloch equation $i\hbar\dot{\rho}=\left[\mathbf{H},\rho\right]$

Geometry of U(2) transformations. It's all done with mirrors!

$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

+σ_A is an x-plane mirror



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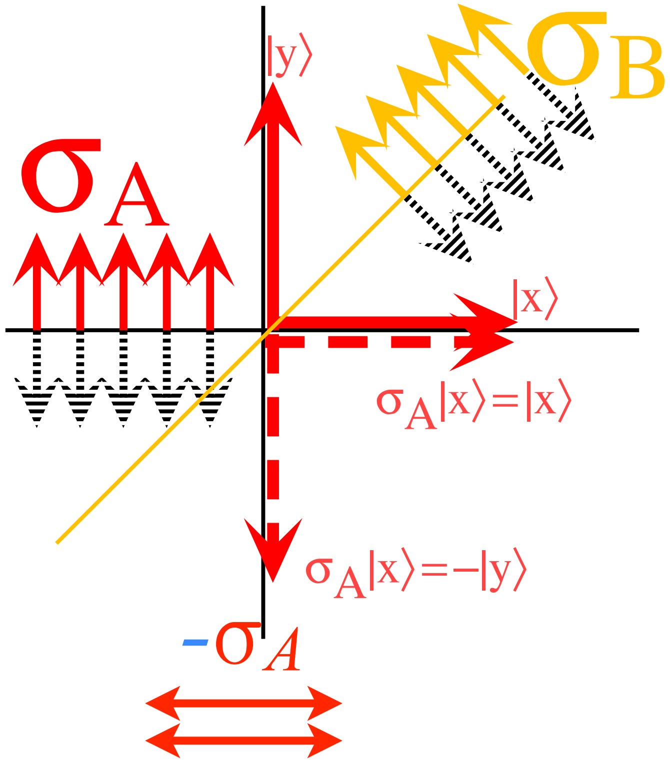
$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$+\sigma_A$ is an *x*-plane mirror



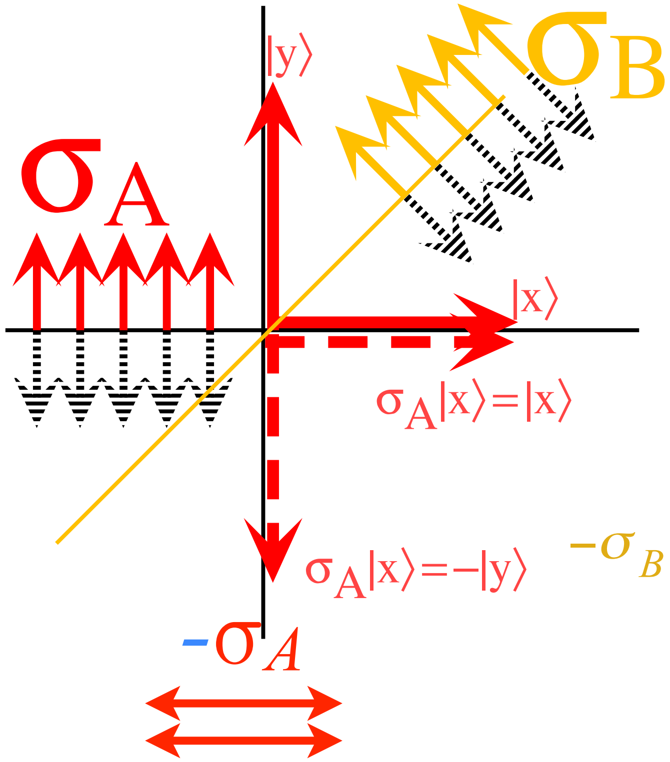
Note that $-\sigma_A$ is a *y*-plane mirror

$$-\sigma_A = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$$

Geometry of U(2) transformations. It's all done with mirrors!

$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$+\sigma_A$ *is an x-plane mirror*



$$-\sigma_B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = ?$$

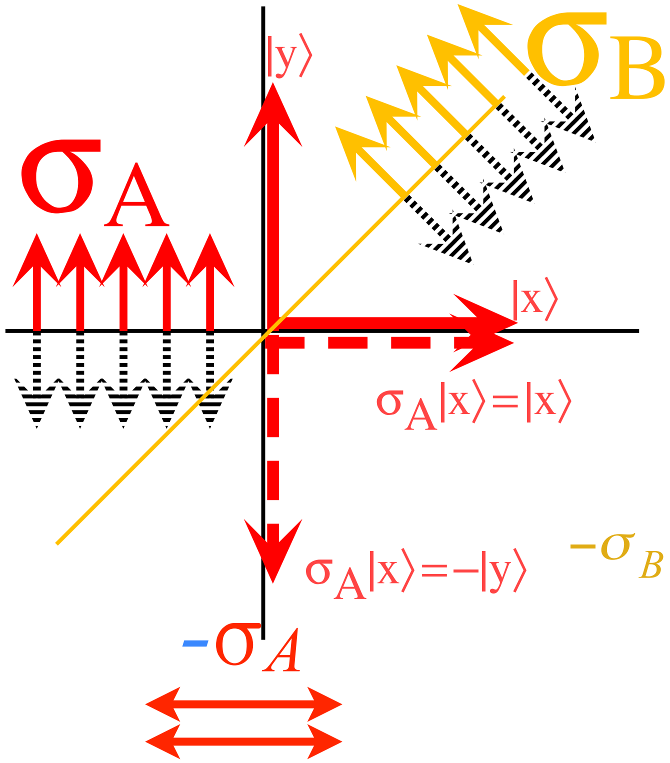
Note that $-\sigma_A$ *is a y-plane mirror*

$$-\sigma_A = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$$

Geometry of U(2) transformations. It's all done with mirrors!

$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

+σ_B is an 45°-plane mirror



+σ_A is an x-plane mirror

$$-\sigma_B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = ?$$

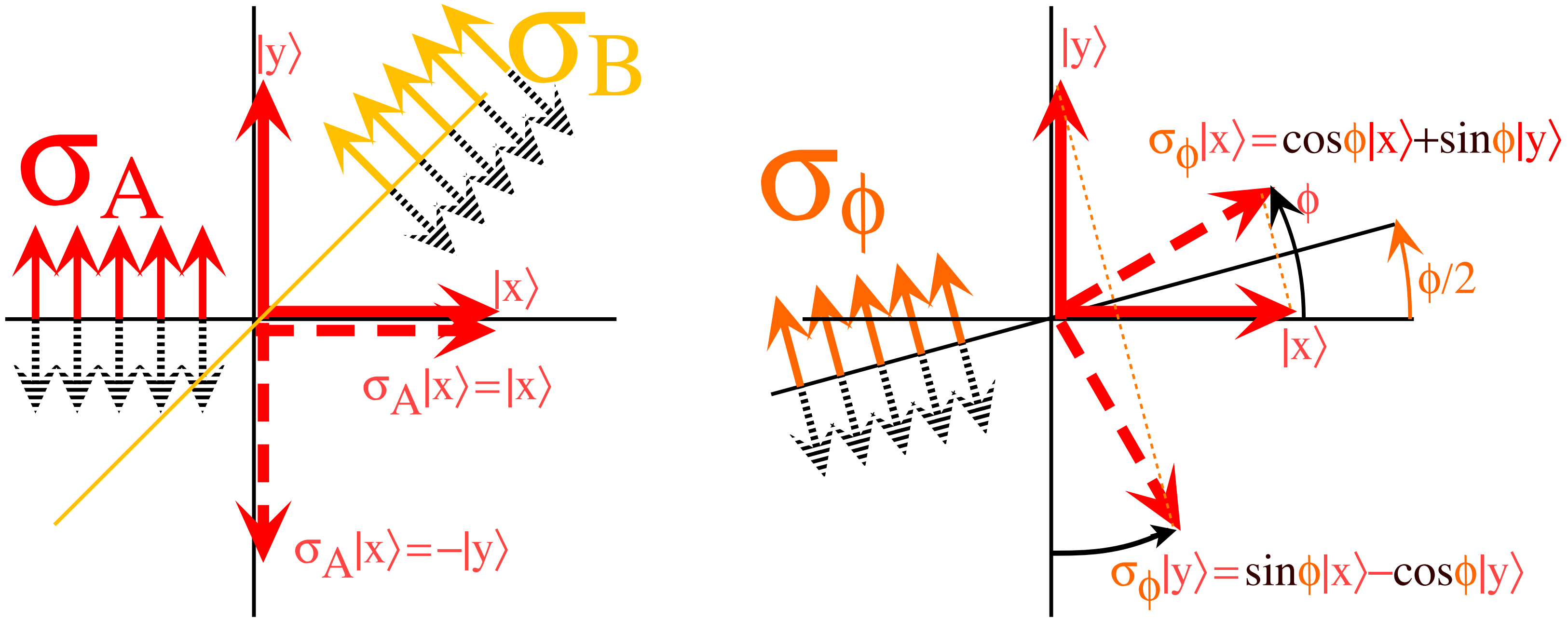
-σ_B is an -45°-plane mirror

Note that -σ_A is a y-plane mirror

$$-\sigma_A = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$$

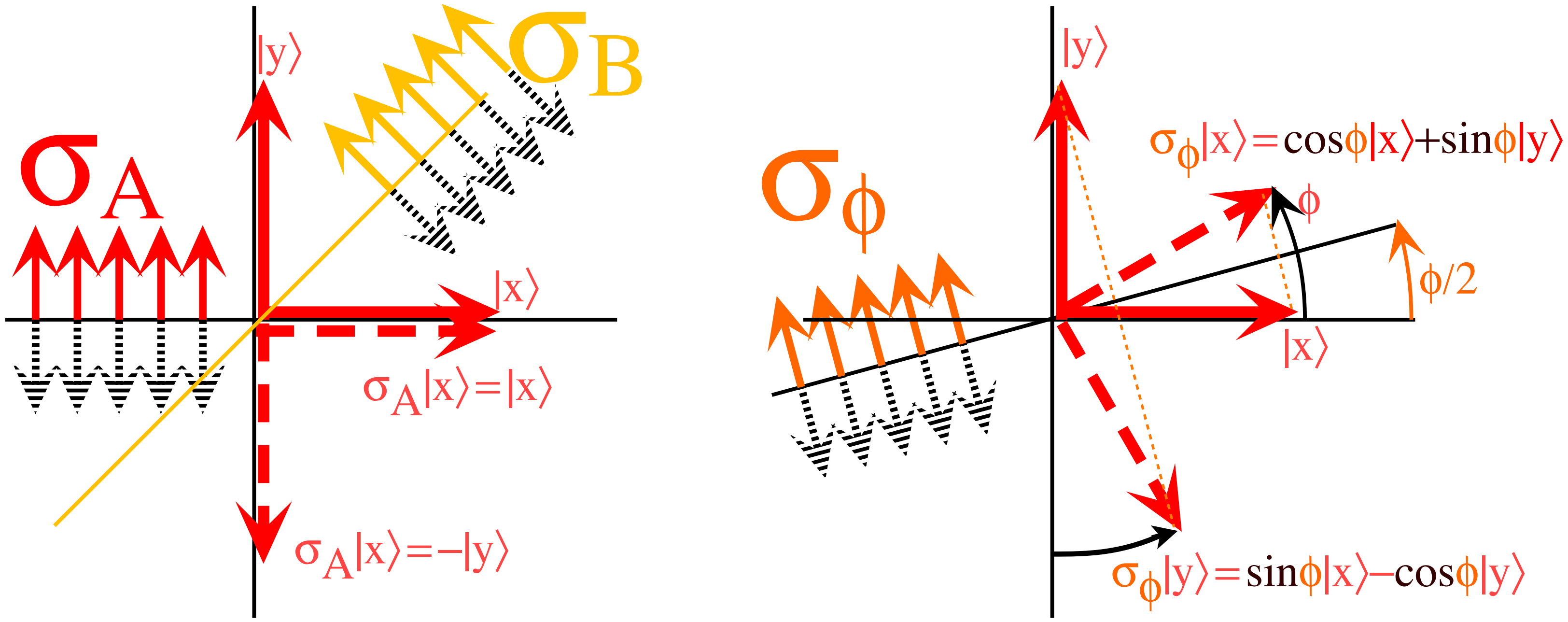
Geometry of U(2) transformations. It's all done with mirrors!

$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;,\; \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;,\; \sigma_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = \sigma_A \cos\phi + \sigma_B \sin\phi$$



Geometry of U(2) transformations. It's all done with mirrors!

$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = \sigma_A \cos\phi + \sigma_B \sin\phi$$

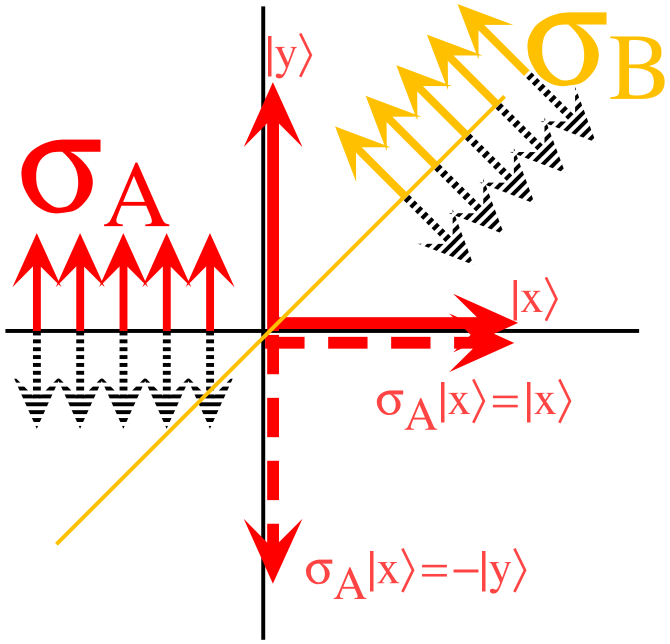


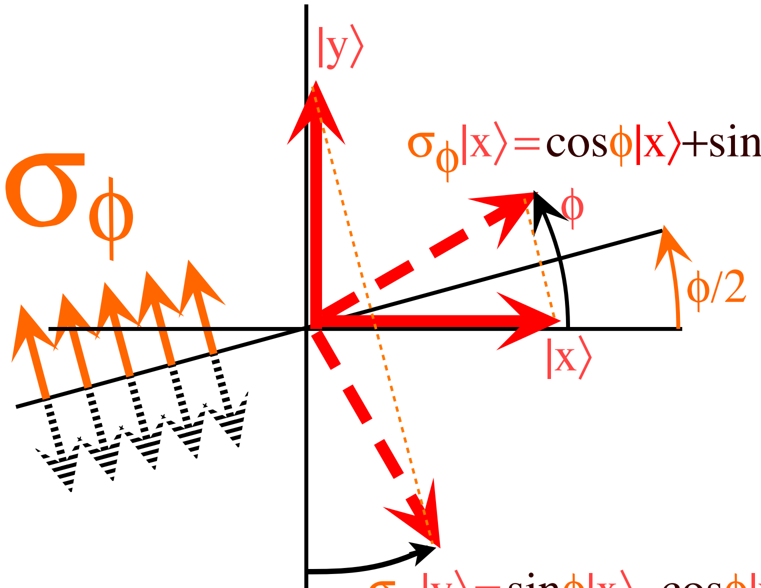
implies : $\langle x|\sigma_\phi|x\rangle = \cos\phi$

and : $\langle y|\sigma_\phi|x\rangle = \sin\phi$

Geometry of U(2) transformations. It's all done with mirrors!

$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = \sigma_A \cos\phi + \sigma_B \sin\phi$$





$\sigma_\phi|x\rangle = \cos\phi|x\rangle + \sin\phi|y\rangle$ *implies* : $\langle x|\sigma_\phi|x\rangle = \cos\phi$

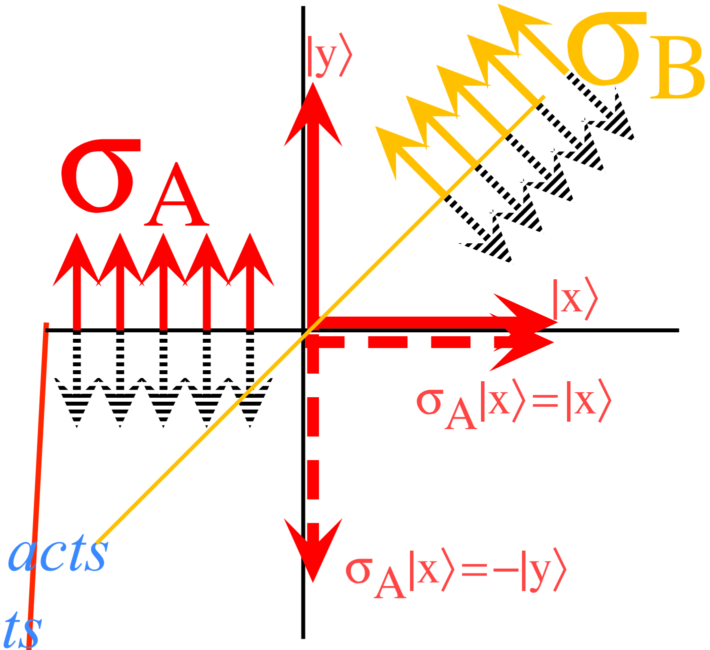
and : $\langle y|\sigma_\phi|x\rangle = \sin\phi$

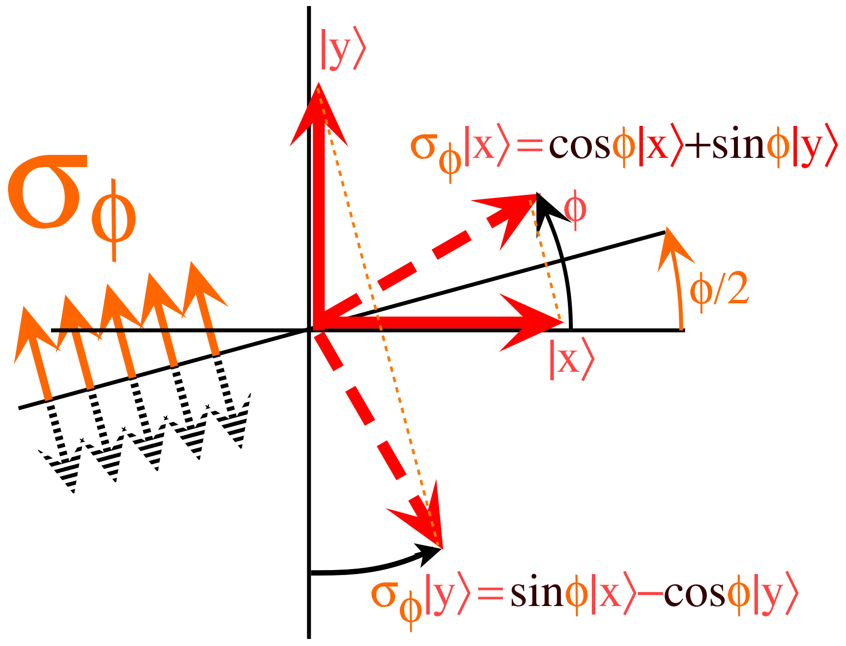
$\sigma_\phi|y\rangle = \sin\phi|x\rangle - \cos\phi|y\rangle$ *implies* : $\langle x|\sigma_\phi|y\rangle = \sin\phi$

and : $\langle y|\sigma_\phi|y\rangle = -\cos\phi$

Geometry of U(2) transformations. It's all done with mirrors!

$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = \sigma_A \cos\phi + \sigma_B \sin\phi$$





mirror σ

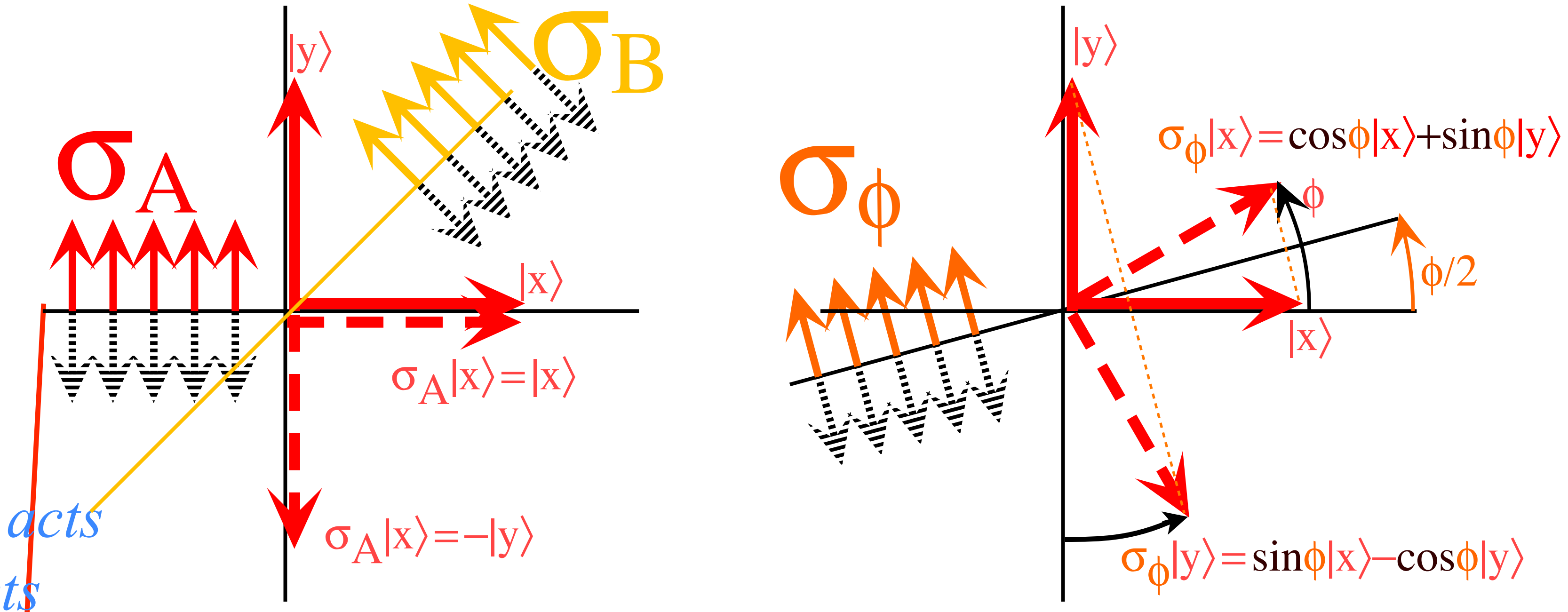
mirror σ_A acts first on kets

$$\sigma_\phi \sigma_A = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

mirror σ_ϕ goes 2nd

Geometry of U(2) transformations. It's all done with mirrors!

$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = \sigma_A \cos\phi + \sigma_B \sin\phi$$



mirror σ_A acts first on kets

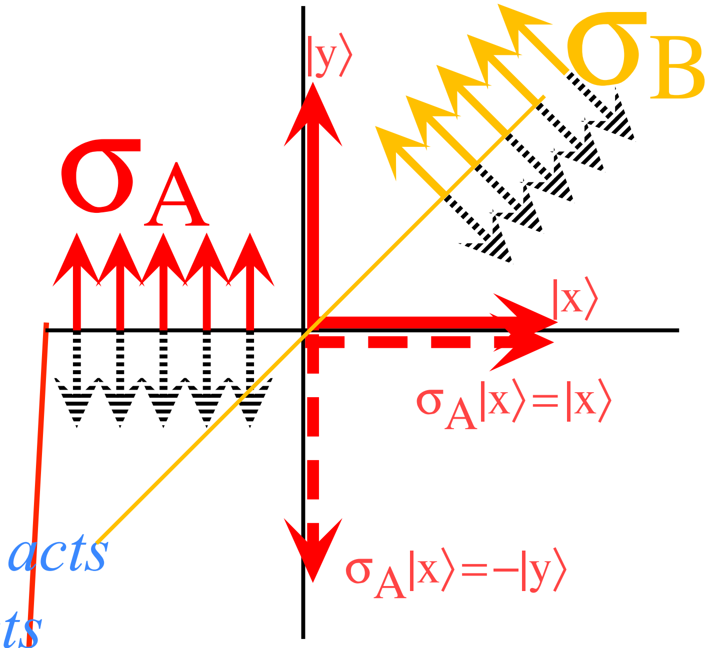
mirror σ_ϕ goes 2nd

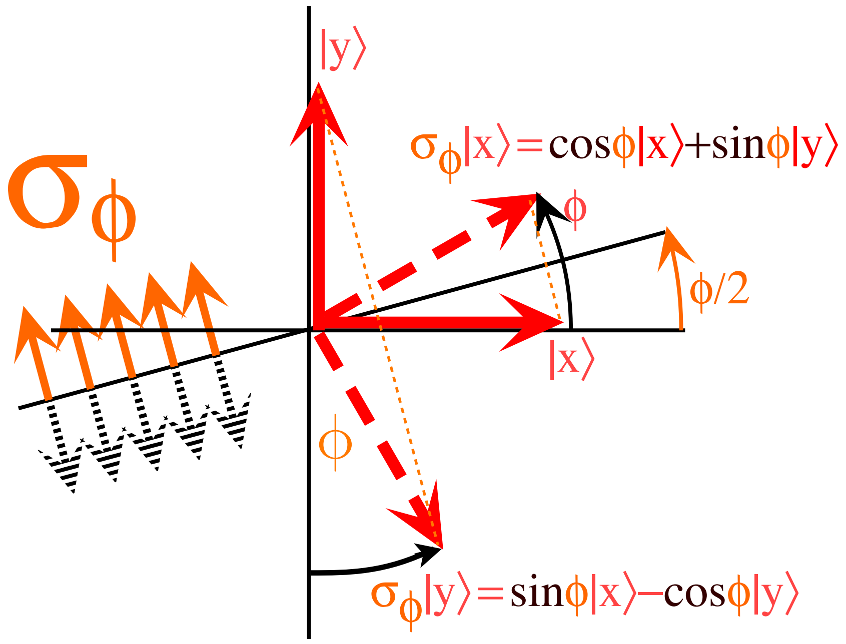
$$\sigma_\phi \sigma_A = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = R[\phi] \; ,$$

Geometry of U(2) transformations. It's all done with mirrors!

$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;,\; \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;,\; \sigma_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = \sigma_A \cos\phi + \sigma_B \sin\phi$$





mirror σ_A acts
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$$\sigma_\phi \sigma_A = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

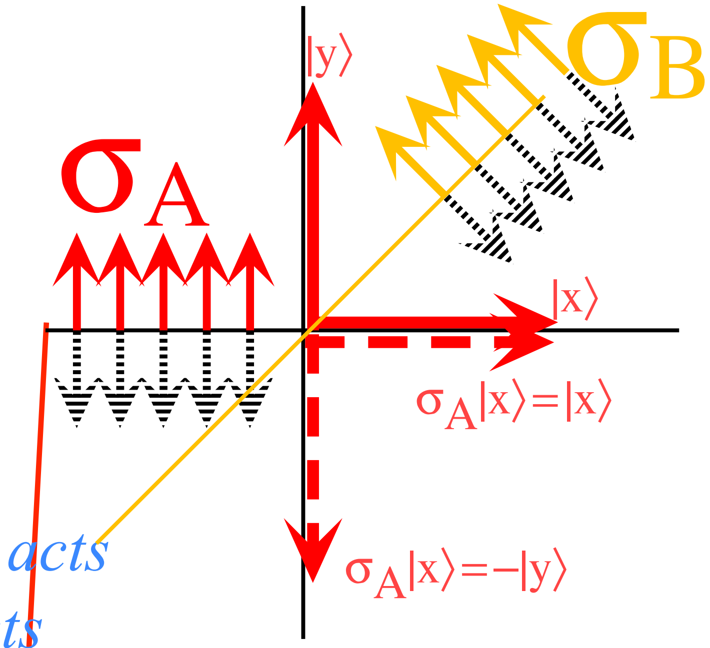
mirror σ_ϕ
goes 2nd

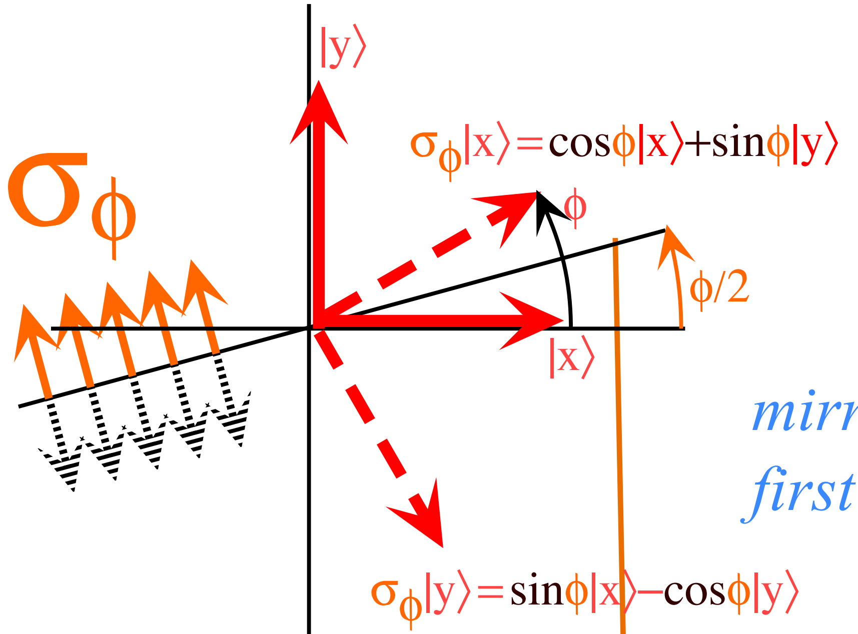
$$= \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = R[\phi] \;,$$

Rotation angle ϕ is TWICE the angle $\phi/2$ between mirror σ_A and mirror σ_ϕ

Geometry of U(2) transformations. It's all done with mirrors!

$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \ \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \sigma_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = \sigma_A \cos\phi + \sigma_B \sin\phi$$





mirror σ$_A$ acts first on kets

mirror σ$_\phi$ acts first on kets

mirror σ$_\phi$ goes 2nd

$$\sigma_\phi \sigma_A = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = \mathrm{R}[\phi] \ ,$$

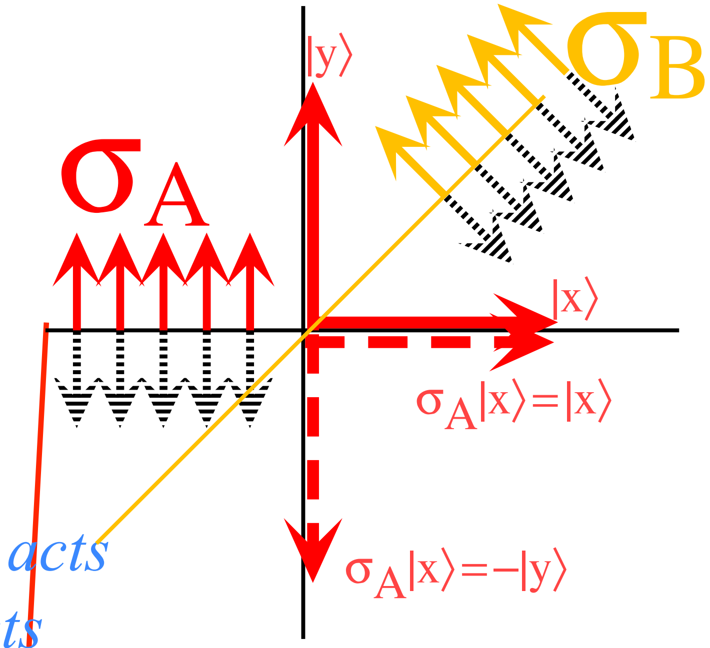
mirror σ$_A$ goes 2nd

$$\sigma_A \sigma_\phi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = \mathrm{R}[-\phi]$$

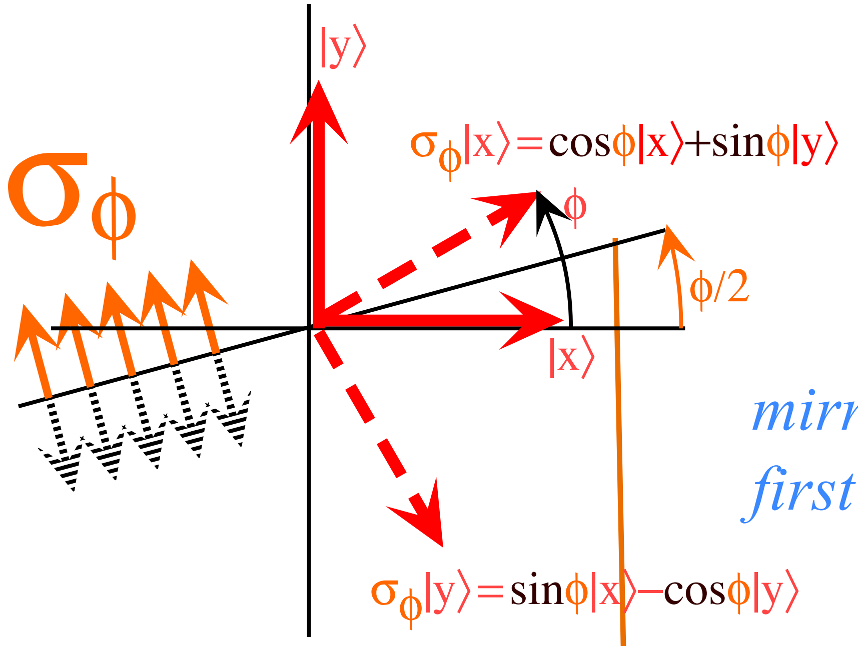
Rotation angle ϕ is TWICE the angle $\phi/2$ between mirror σ$_A$ and mirror σ$_\phi$

Geometry of U(2) transformations. It's all done with mirrors!

$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;,\; \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;,\; \sigma_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = \sigma_A \cos\phi + \sigma_B \sin\phi$$



mirror σ$_A$ acts first on kets



mirror σ$_\phi$ acts first on kets

mirror σ$_\phi$ goes 2nd

$$\sigma_\phi \sigma_A = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = R[\phi] \;,$$

$$\sigma_A \sigma_\phi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = R[-\phi]$$

mirror σ$_A$ goes 2nd

Rotation angle ϕ is TWICE the angle $\phi/2$ between mirror σ$_A$ and mirror σ$_\phi$

$$\sigma_A \sigma_B = \overset{\substack{\text{determinant} \\ \text{det = -1}}}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \overset{\substack{\text{determinant} \\ \text{det = -1}}}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} = \underset{\substack{\text{xy-rotation} \\ \text{by -90°}}}{\overset{\substack{\text{determinant} \\ \text{det = +1}}}{\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}}} = i \underset{\substack{\text{imaginary} \\ \text{reflection?}}}{\overset{\substack{\text{determinant} \\ \text{det = -1}}}{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}} = i\sigma_C$$

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reference links
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Density mechanics compared to spin vector **S** *rotated by crank vector* $\boldsymbol{\Theta}=\boldsymbol{\Omega}t$

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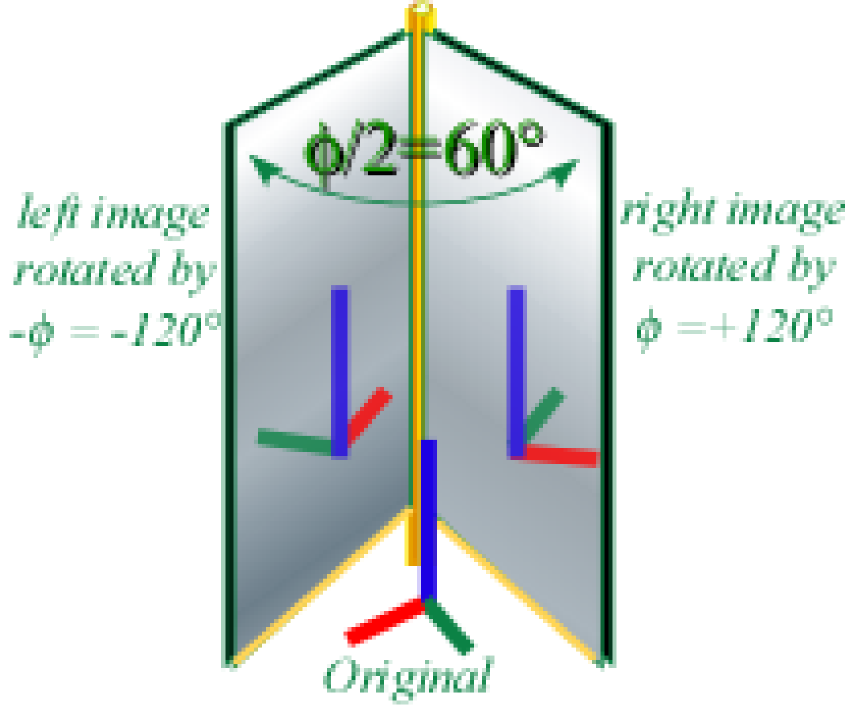
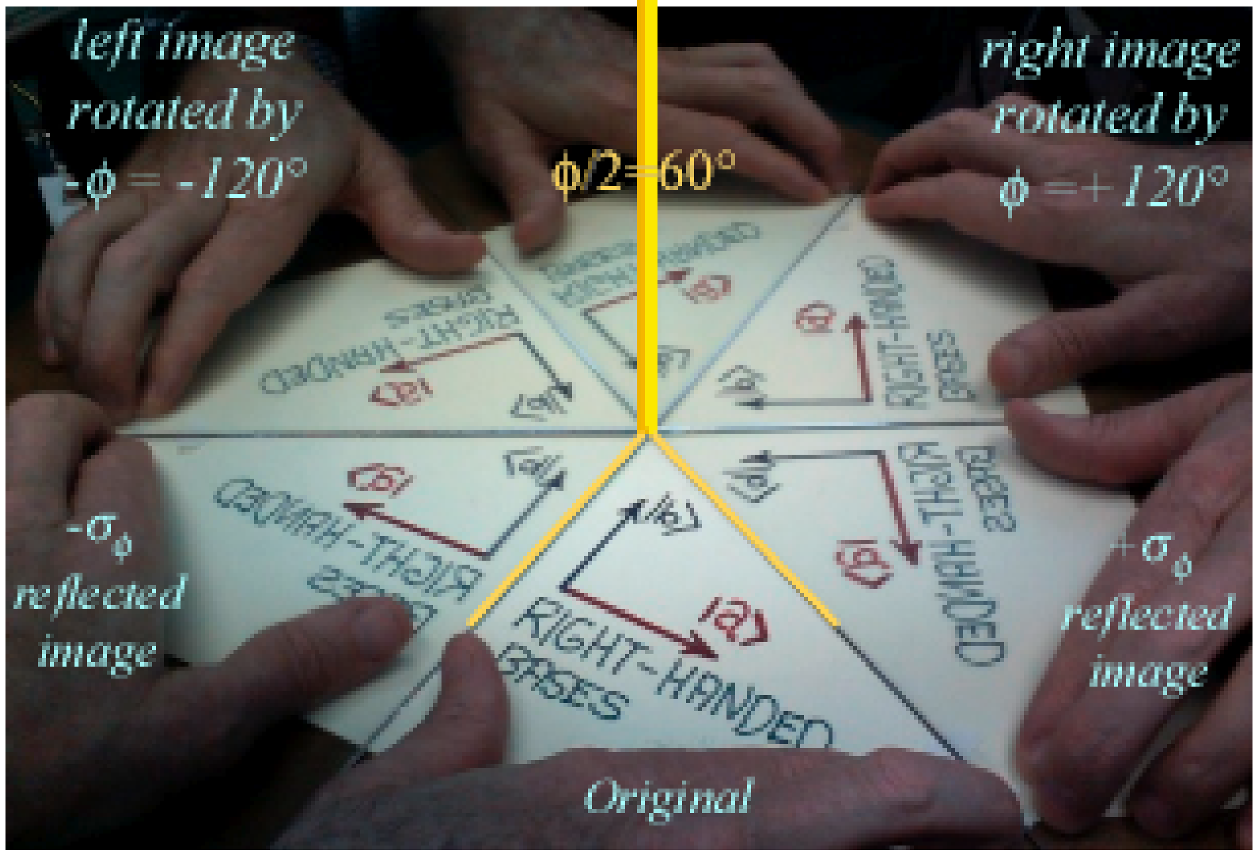
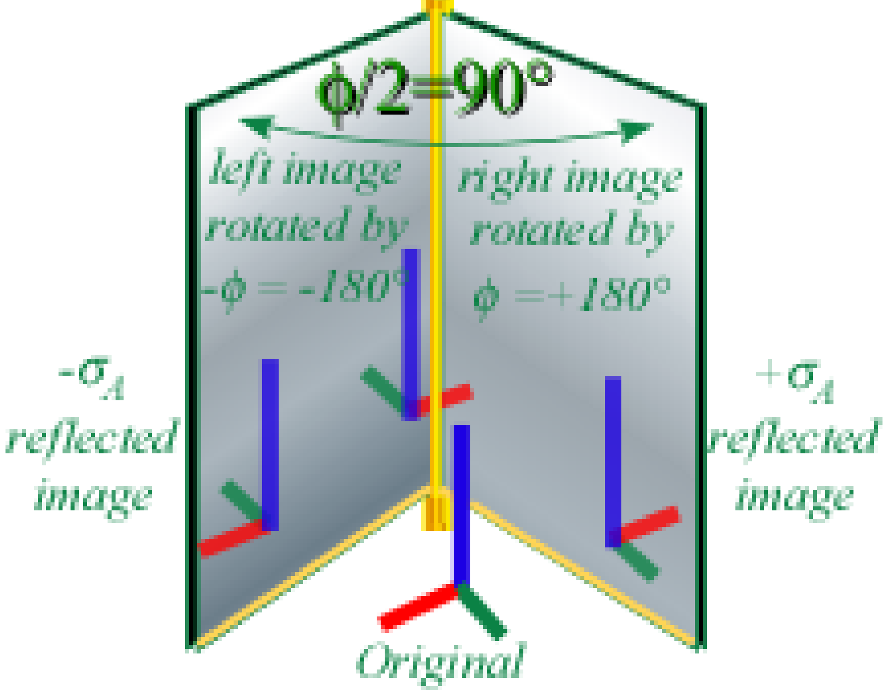
Reflections in clothing store mirrors

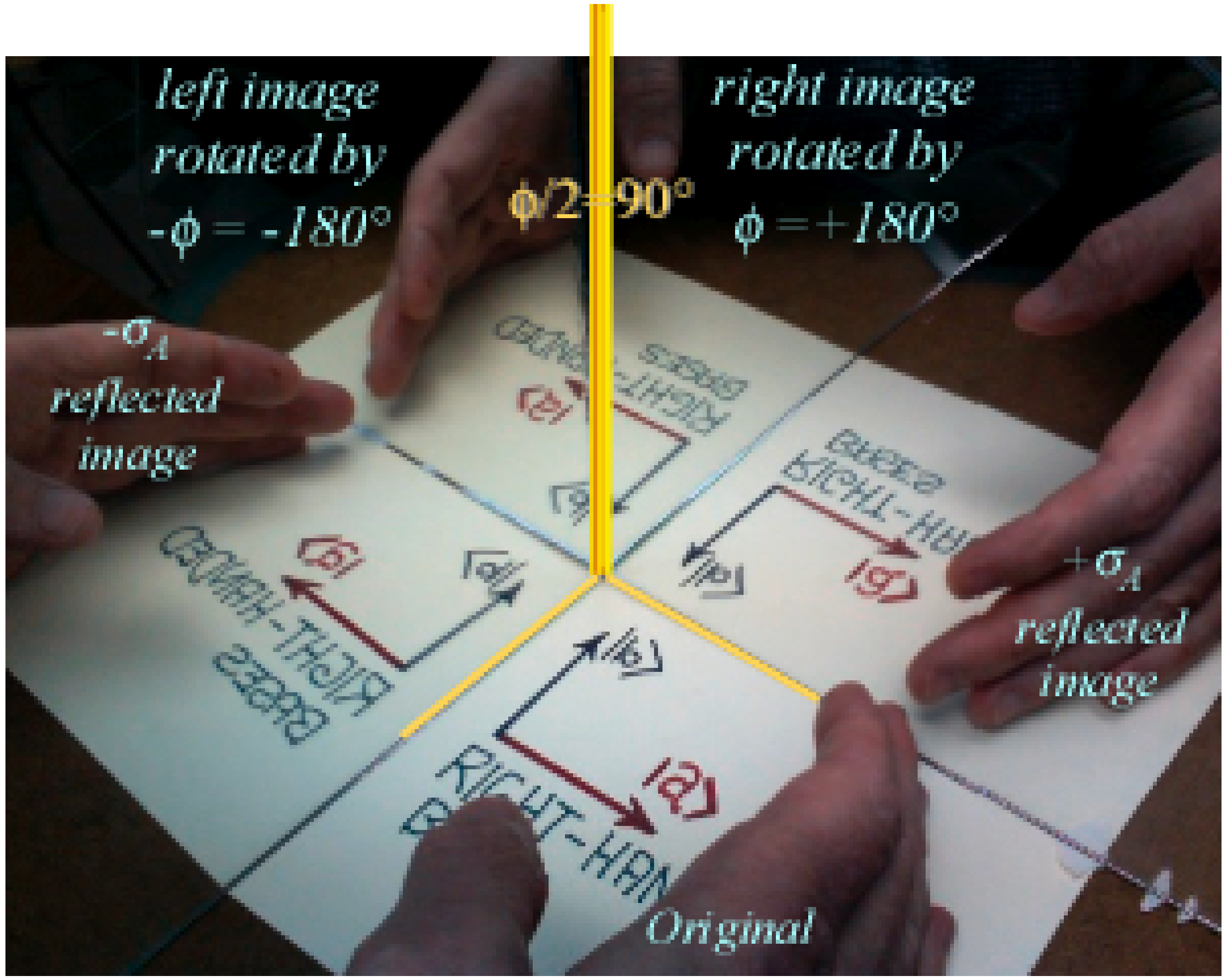
(a) $\phi=\pm 120°$ rotations





(b) $\phi=\pm 180°$ rotations





CMech with
a Bang!
p27-43

Fig.
5.4a-b

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Mirror reflections by σ_μ*-operators make rotations* *The famous Clothing Store Mirror*

➡ *Hamilton-turns do products* $\mathbf{R}[\boldsymbol{\Theta}]\mathbf{R}[\boldsymbol{\Theta}']=\mathbf{R}[\boldsymbol{\Theta}''']$ *geometrically*

Hamilton-turn slide rule and sundial *U(2) products and* (α,β,γ)-$[\varphi,\vartheta,\Theta]$ *conversions*

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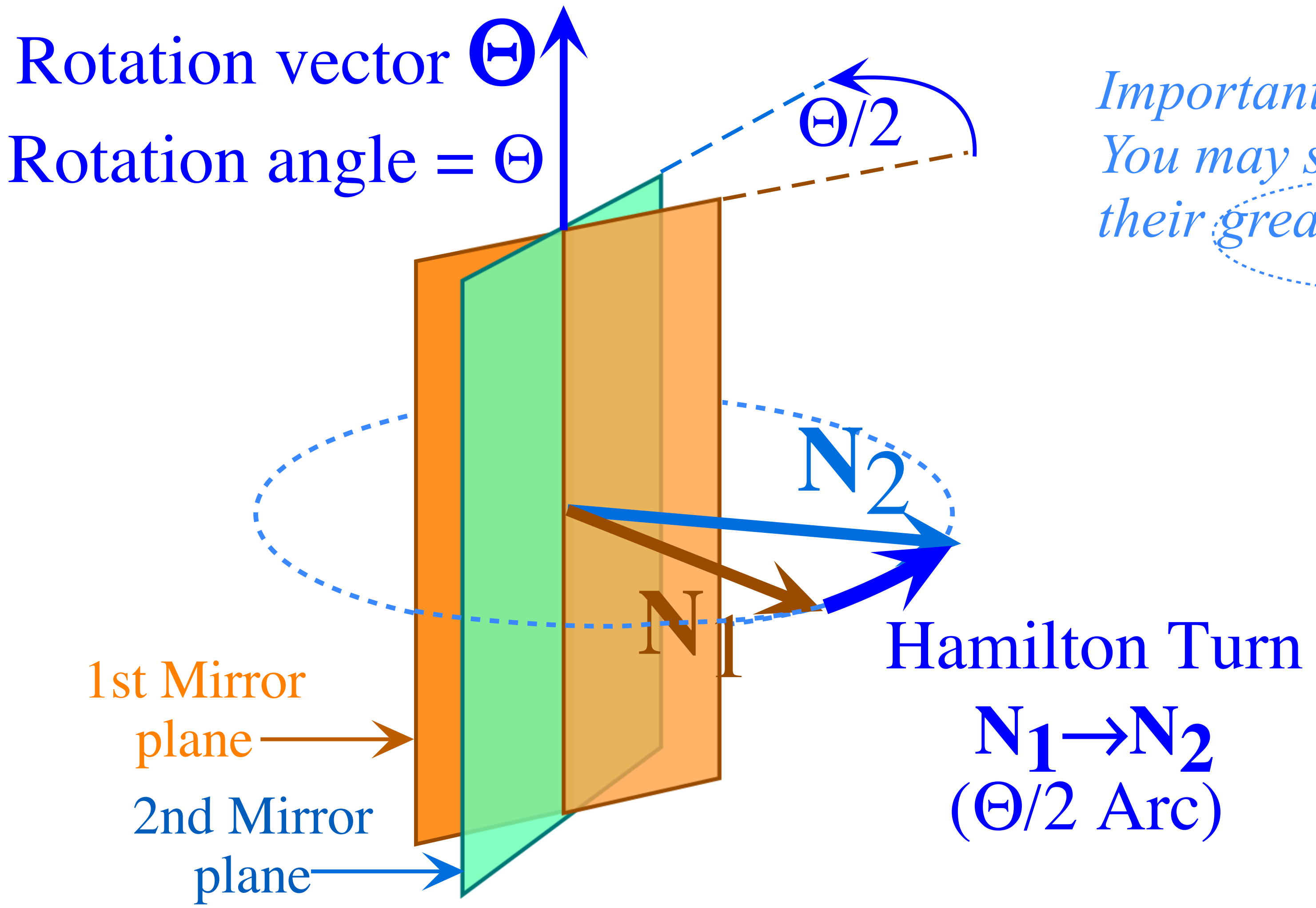
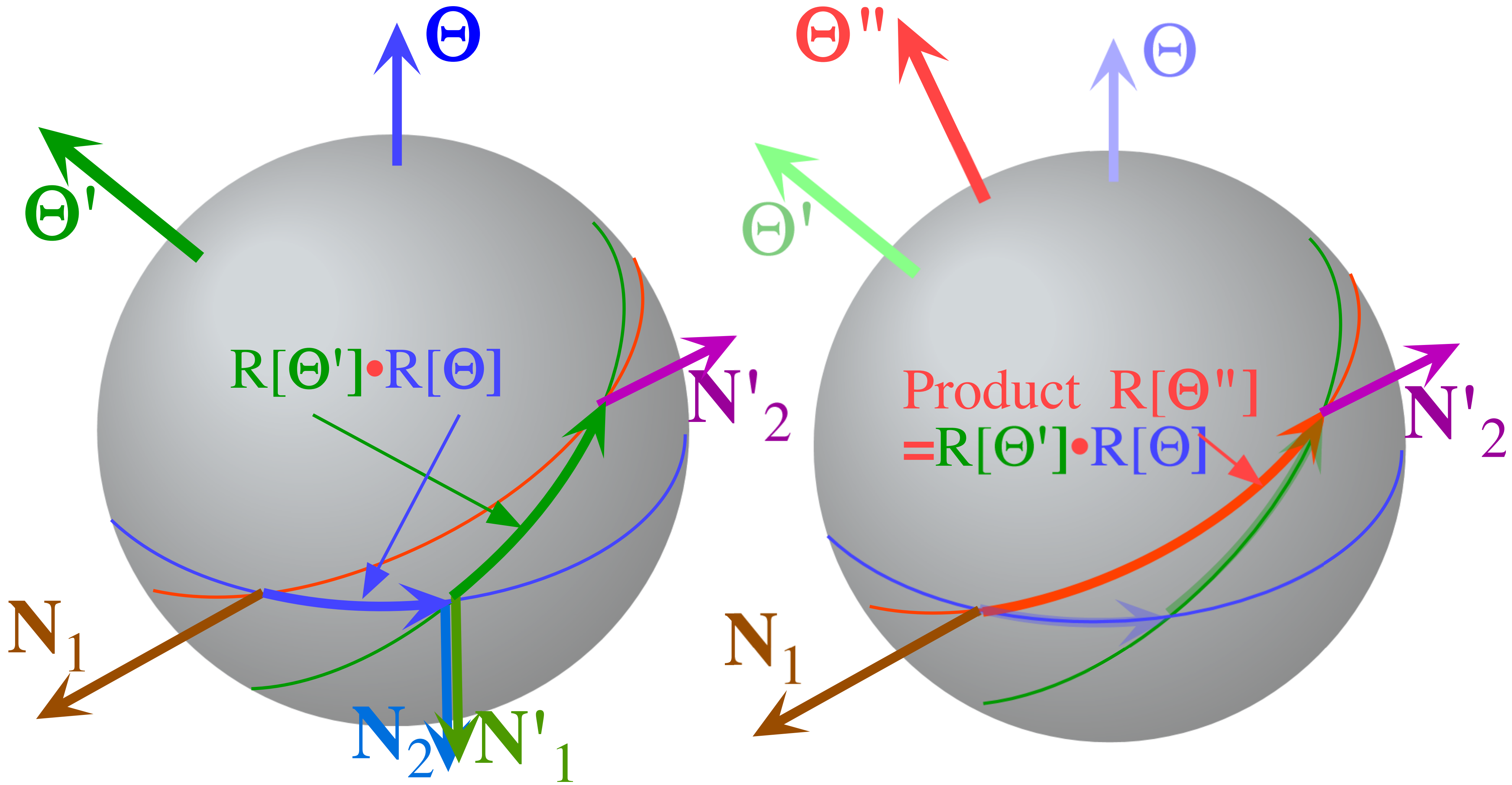
Hamilton-turns do U(2) products $\mathbf{R}[\mathbf{\Theta}]\mathbf{R}[\mathbf{\Theta}']=\mathbf{R}[\mathbf{\Theta}'']$ *geometrically*



*Important point about **Turn arcs**:*
You may slide them anywhere on
their great-circle arc

Fig. 10.A.7 Mirror reflection planes, normals, and Hamilton-turn arc vector.

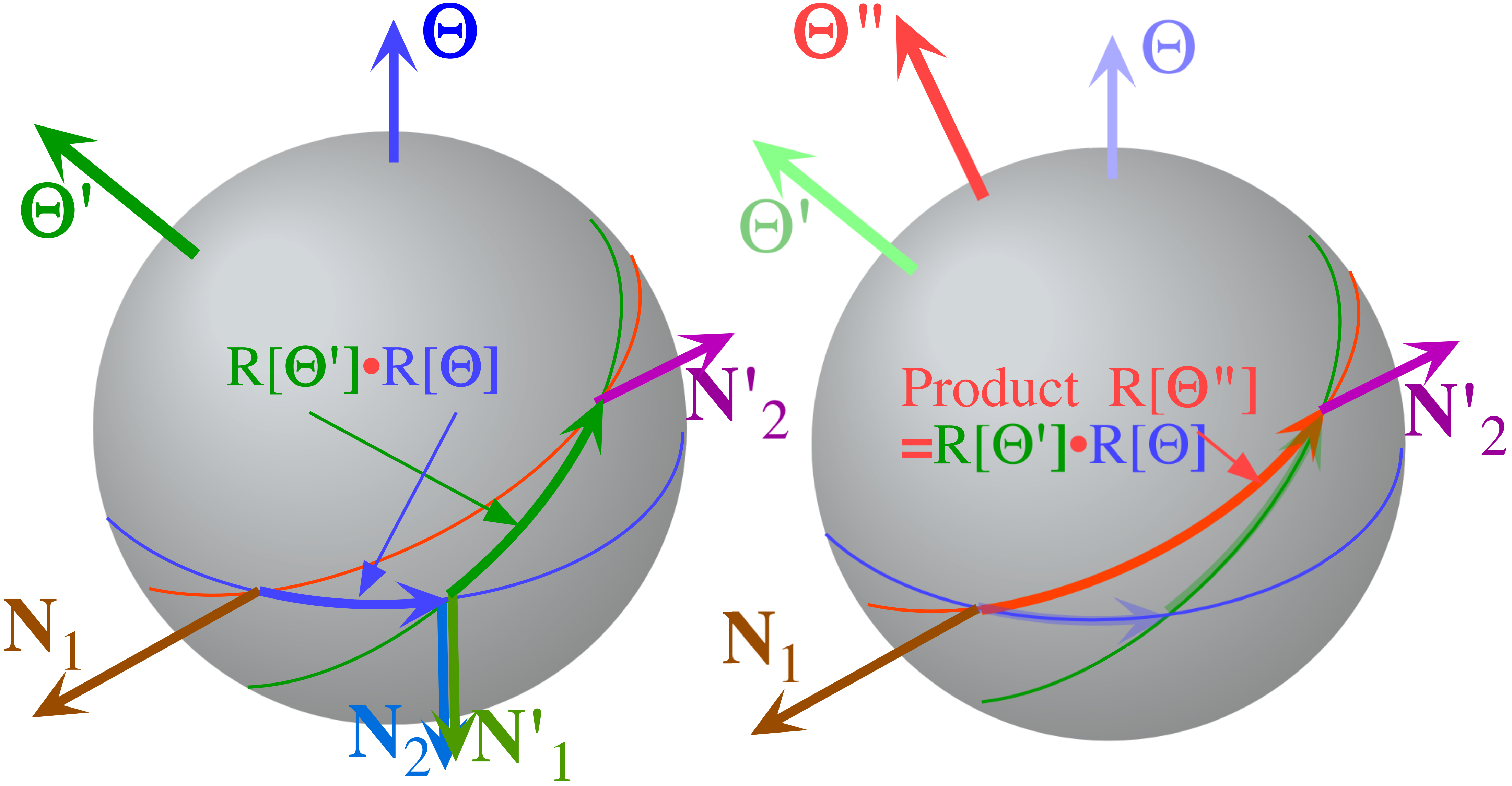
Hamilton-turns do U(2) products **R[Θ]R[Θ′]=R[Θ′′]** *geometrically*



QTforCA Fig. 10.A.8 Adding Hamilton-turn arcs to compute a U(2) product **R**[Θ"]=**R**[Θ']**R**[Θ].

Each arc Θ/2, Θ'/2, or Θ"/2 is 1/2 actual angle Θ, Θ', or Θ" of rotation **R**[Θ], **R**[Θ'], or **R**[Θ"].

Hamilton-turns do U(2) products **R[Θ]R[Θ′]=R[Θ′′]** *geometrically*



QTforCA Fig. 10.A.8 Adding Hamilton-turn arcs to compute a U(2) product **R[Θ"]=R[Θ']R[Θ]**.

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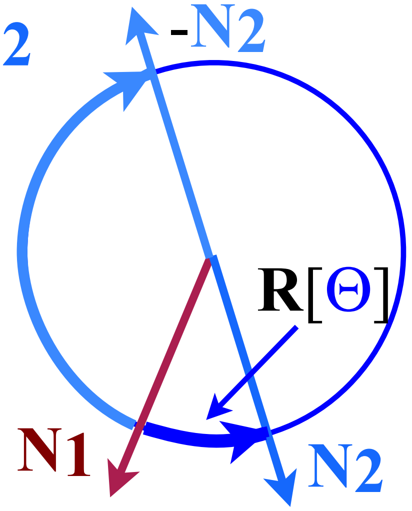
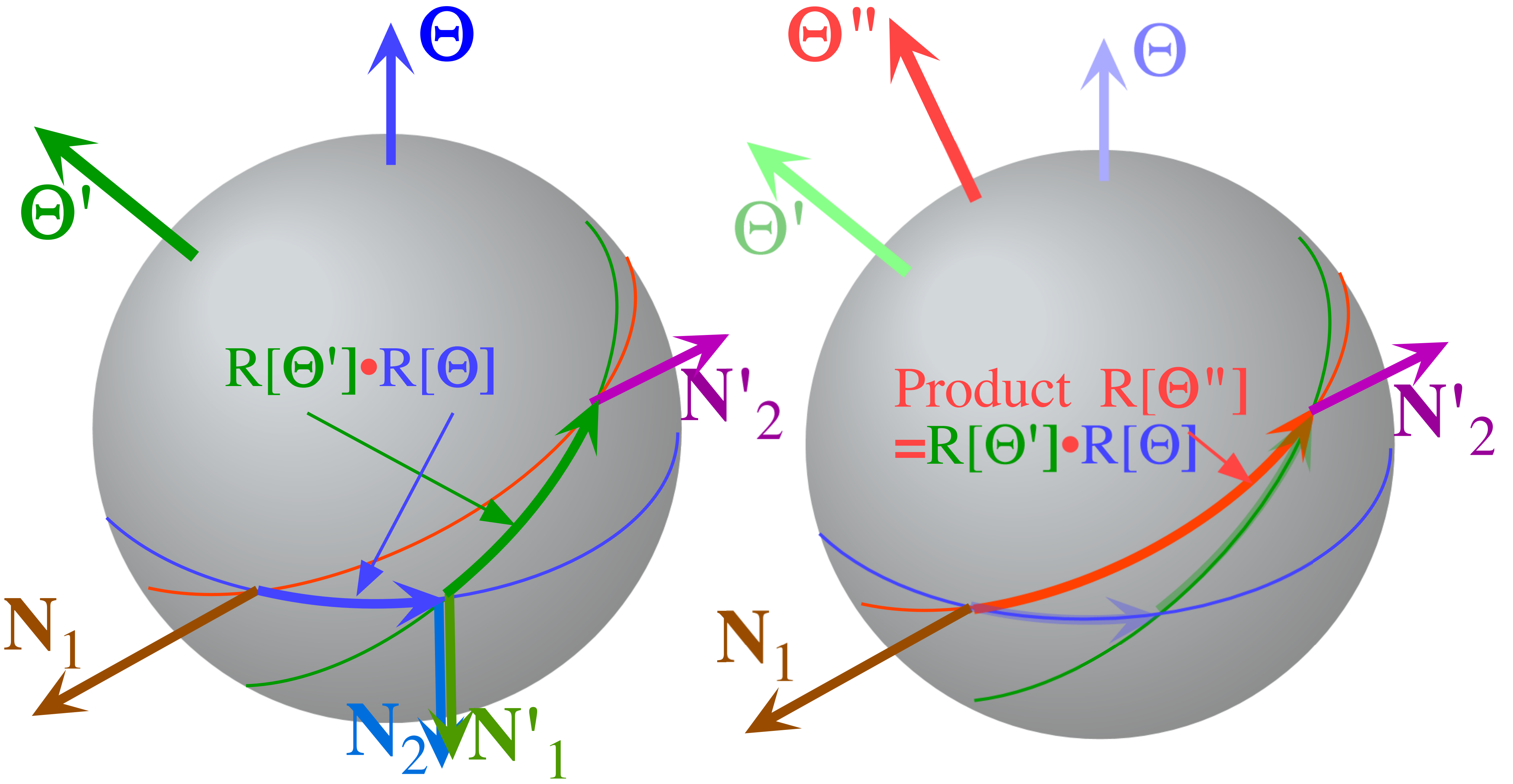
Arc Θ/2 between **N1** and **N2** and its supplement (Θ±2π)/2=Θ/2±π between **N1** and -**N2** represent the same classical rotation by Θ.



Hamilton-turns do U(2) products **R[Θ]R[Θ′]=R[Θ′′′]** *geometrically*



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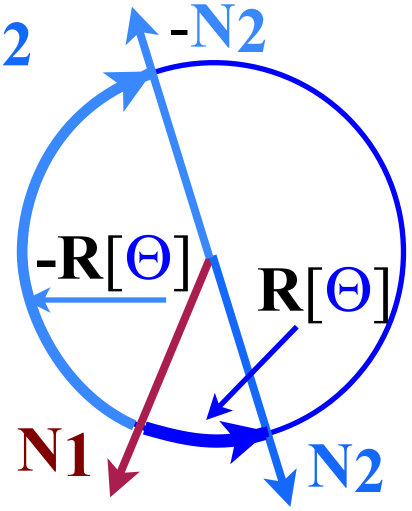
For quantum spin-1/2 object, the arc pointing from **N1** to the antipodal normal -**N2** represents a Θ-rotation with an extra π-phase factor $e^{\pm i\pi} = -1$, that is, -**R**[Θ] .



AMOP
reference links
on following page

1.31.18 class 6.0: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

Symmetry group $\mathscr{G}$ representations=>*AMOP* Hamiltonian **H** (or K) matrices, irreps $\mathscr{D}^{(\alpha)}$
=>*AMOP* wave functions $\Psi^{(\alpha)}$, eigensolutions

$\mathscr{G}$= U(2) product **R[Θ]R[Θ′]=R[Θ′′′]** algebra (It's all done with σ_μ spinors)

Jordan-Pauli identity: U(2) product algebra of spinor σ_μ-operators

U(2) "Crazy-Thing" forms do products **R[Θ]R[Θ′]=R[Θ′′′]** *algebraically*

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Mirror reflections by σ_μ-operators make rotations

Hamilton-turns do products **R[Θ]R[Θ′]=R[Θ′′′]** *geometrically*

➡ *Hamilton-turn slide rule and sundial* *U(2) products and* (α,β,γ)-[φ,ϑ,Θ] *conversions*

Finite group products by turns or by group link diagrams *D_3 example.* *O_h example*

$\mathscr{G}$= U(2) class transformation **R[Θ]R[Θ′]R[Θ]$^{-1}$=R[Θ′′′]** geometry

Group equivalence classes

U(2) density operator **ρ** and [**ρ**,**H**] mechanics

Density mechanics compared to spin vector **S** *rotated by crank vector* **Θ**=**Ω***t*

Bloch equation $i\hbar\dot{\rho} = [\mathbf{H},\rho]$

R(3)-U(2) slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

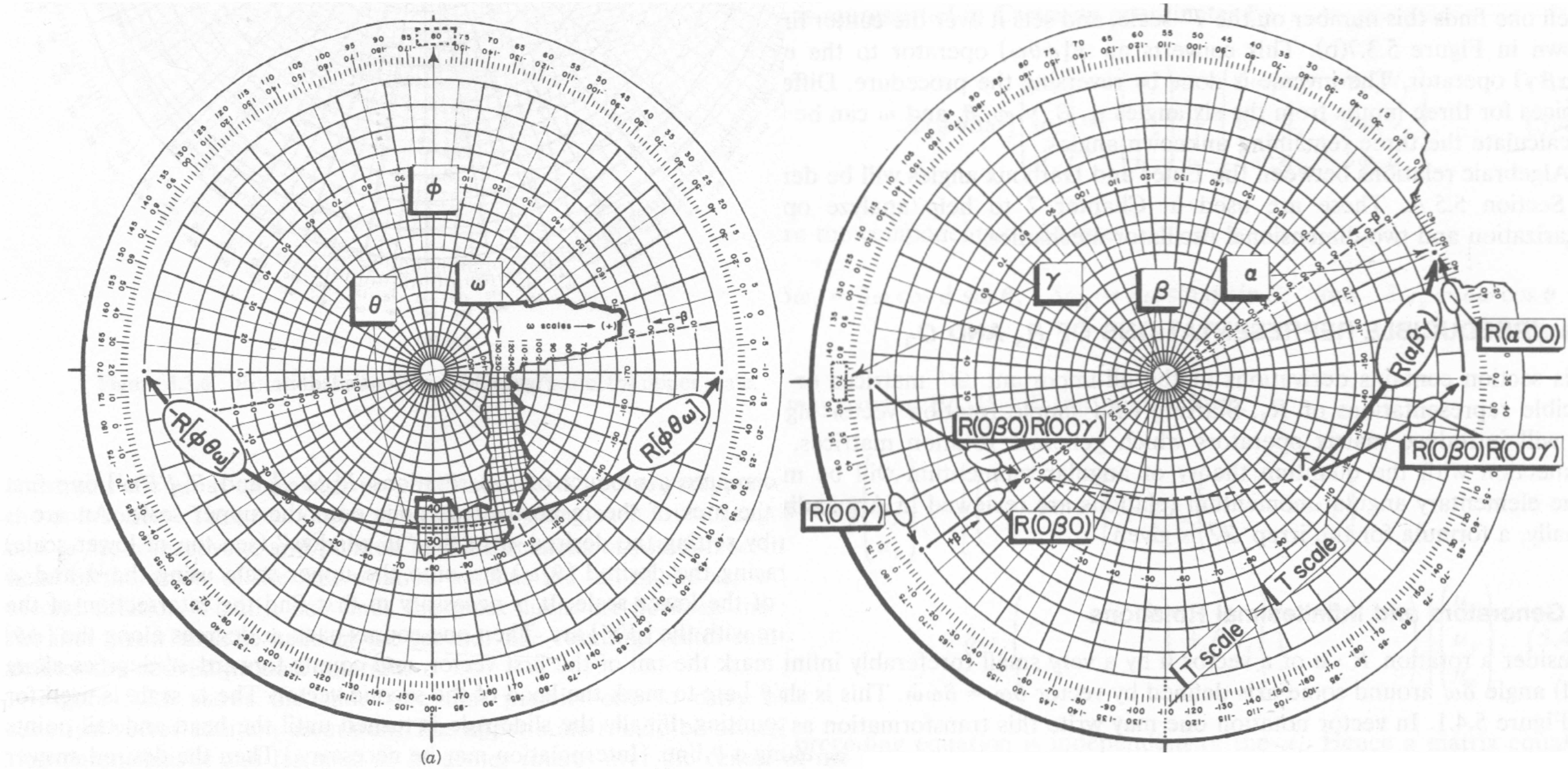
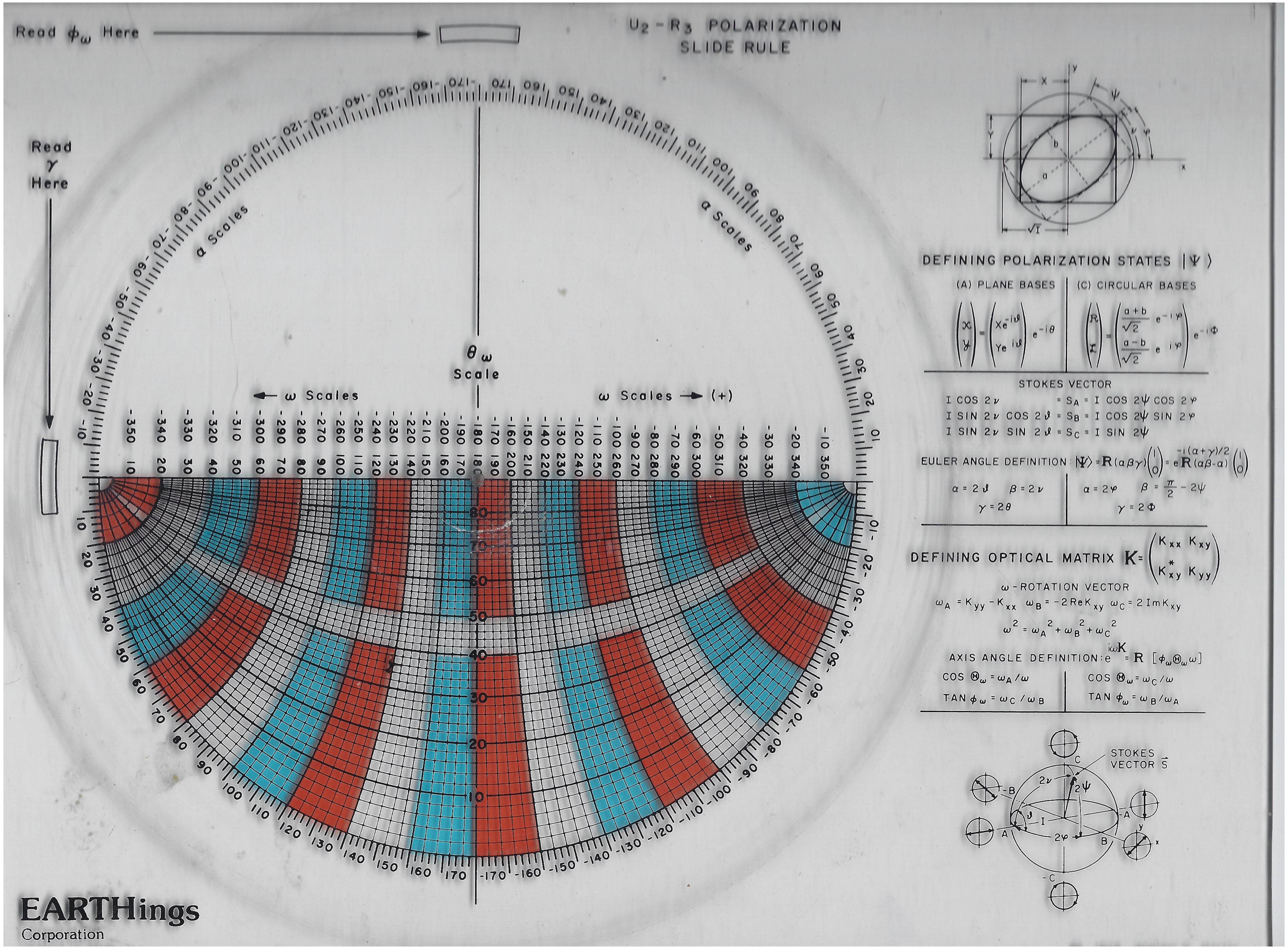


Figure 5.3.7 Setting the rotational slide rule. (a) Darboux or axis angles. (b) Euler angles.

Harter and Dos Santos
Double-Group Theory on the Half-Shell I and II
Am. J. Phys. ***46*** *251 (1978)*

Double group theory on the half-shell and the two level system -Harter-Santos-1978-AJP.
I) Rotation and half integral spin states (Alt scan)
II) Optical polarization (Alt scan)



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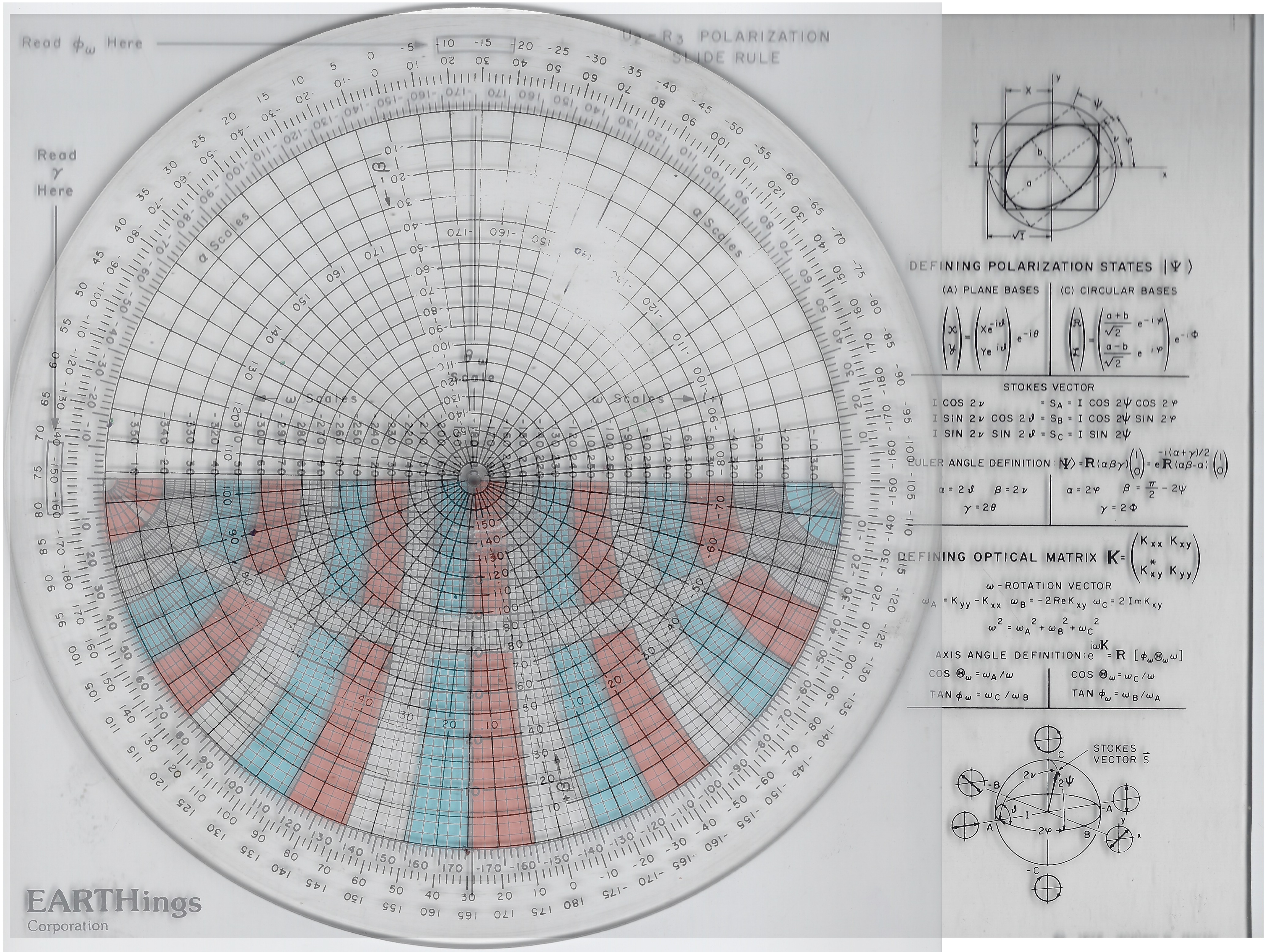

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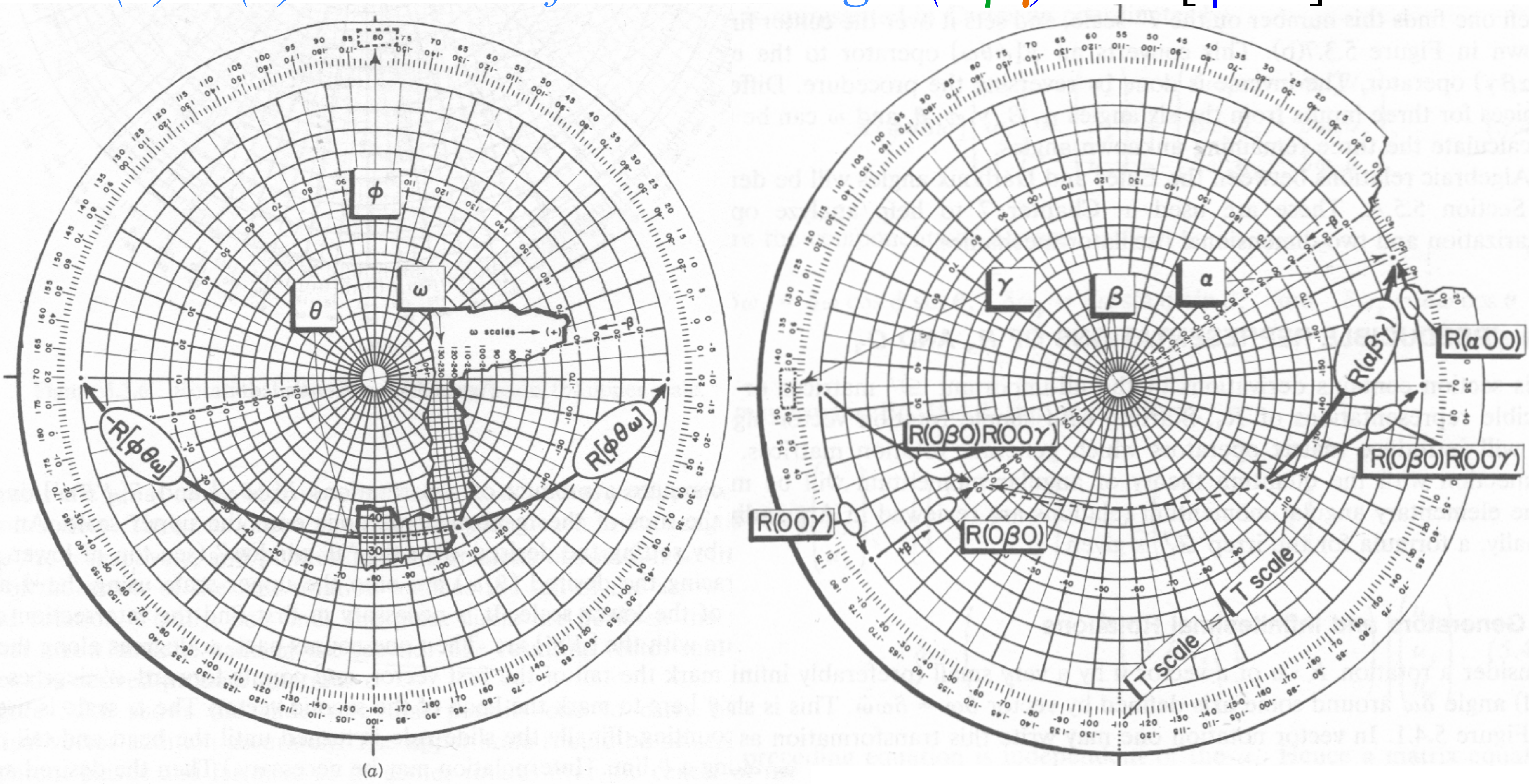

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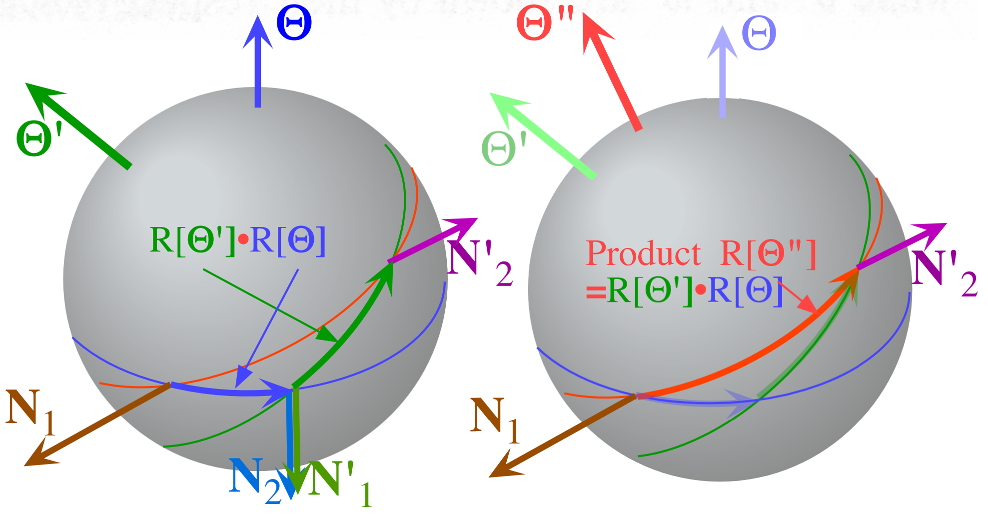
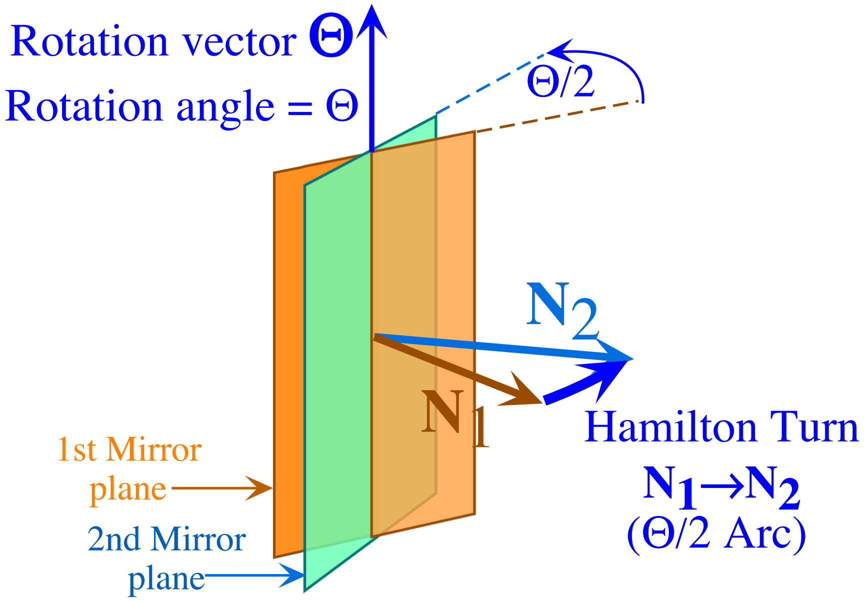
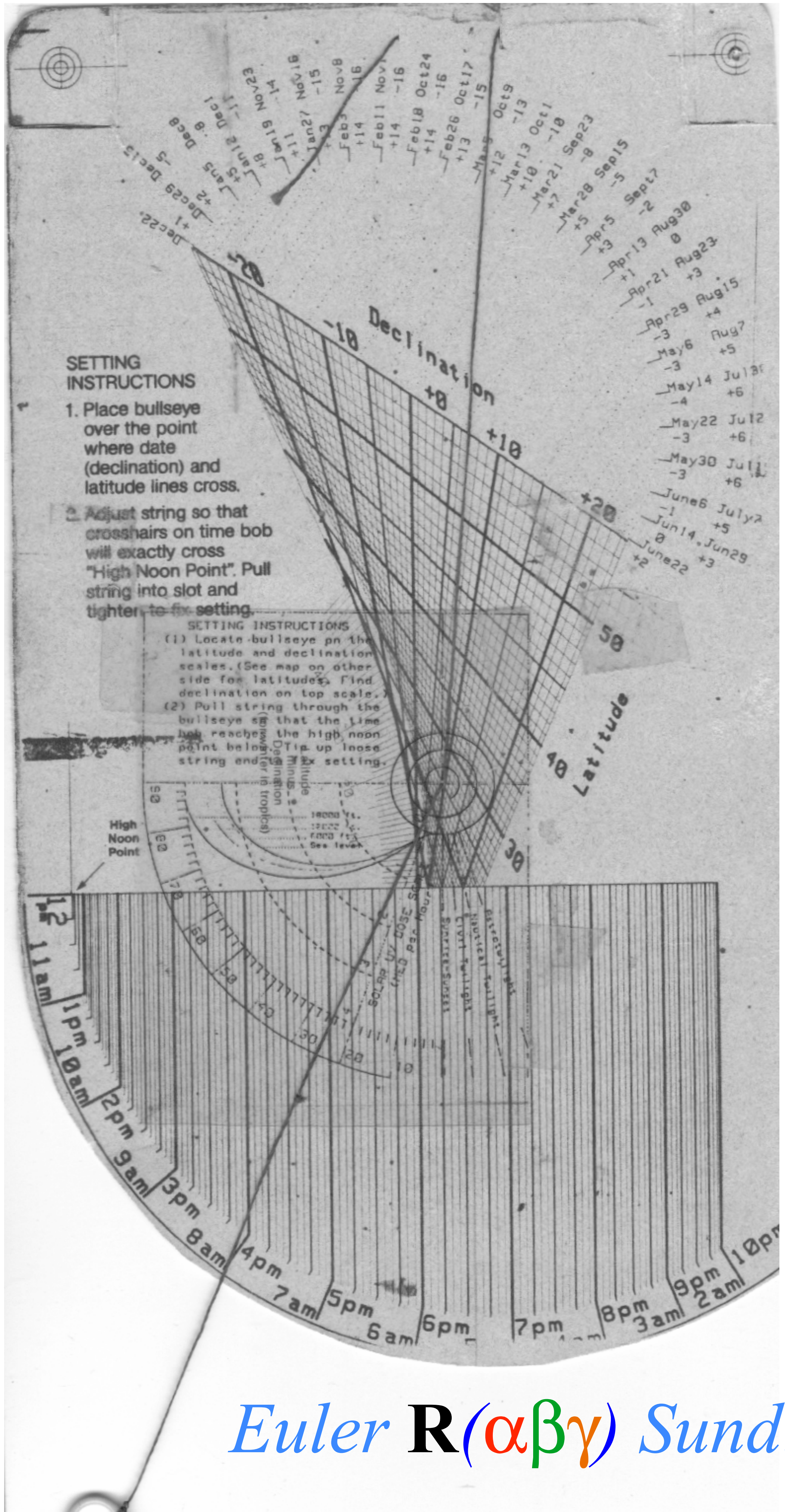






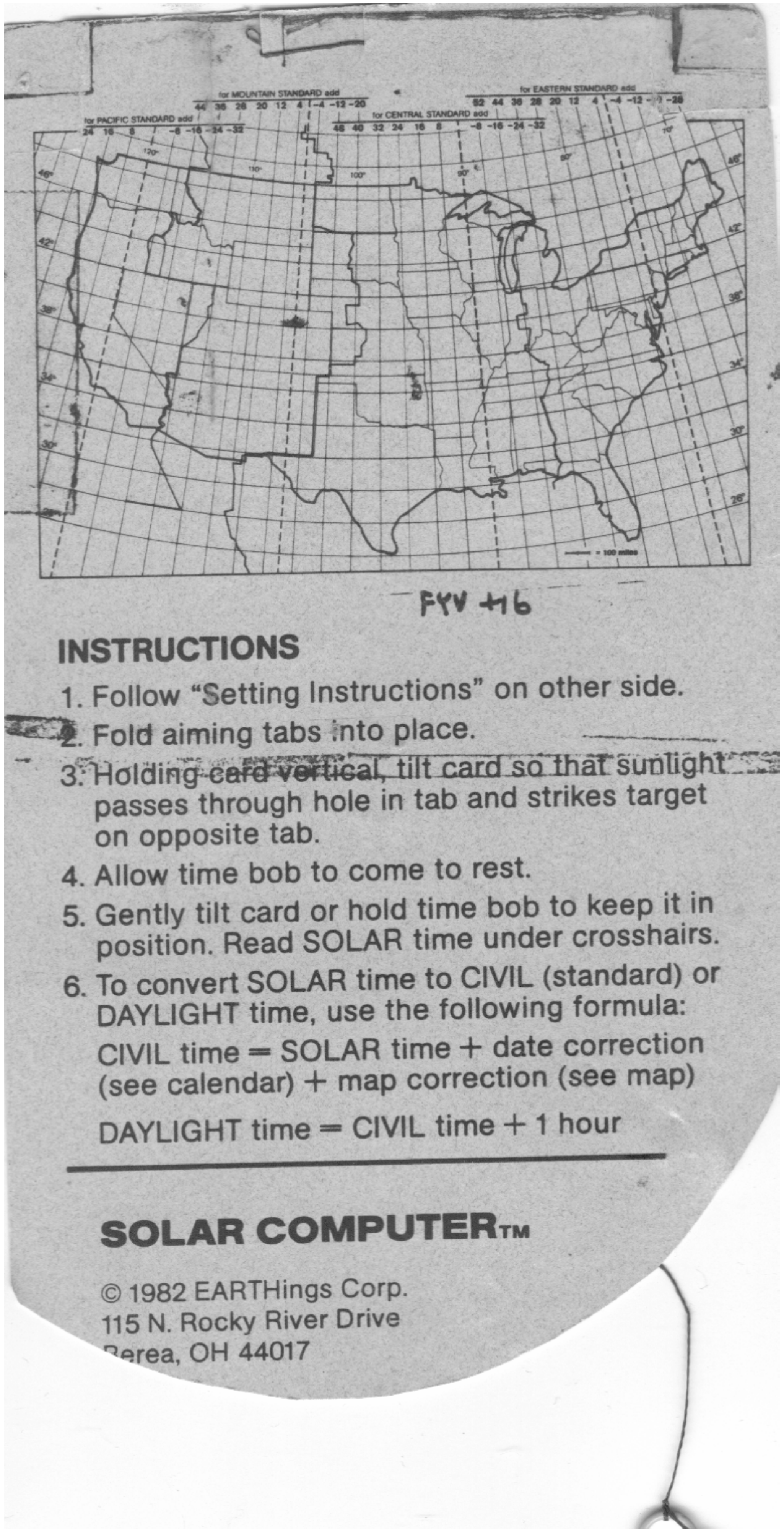
SETTING INSTRUCTIONS

1. Place bullseye over the point where date (declination) and latitude lines cross.
2. Adjust string so that crosshairs on time bob will exactly cross "High Noon Point". Pull string into slot and tighten to fix setting.

SETTING INSTRUCTIONS

(1) Locate bullseye on the latitude and declination scales. (See map on other side for latitudes. Find declination on top scale.)

(2) Pull string through the bullseye so that the time bob reaches the high noon point below. Tip up loose string end to fix setting.



FYV +16

INSTRUCTIONS

1. Follow "Setting Instructions" on other side.
2. Fold aiming tabs into place.
3. Holding card vertical, tilt card so that sunlight passes through hole in tab and strikes target on opposite tab.
4. Allow time bob to come to rest.
5. Gently tilt card or hold time bob to keep it in position. Read SOLAR time under crosshairs.
6. To convert SOLAR time to CIVIL (standard) or DAYLIGHT time, use the following formula:
   CIVIL time = SOLAR time + date correction (see calendar) + map correction (see map)
   DAYLIGHT time = CIVIL time + 1 hour

SOLAR COMPUTERTM

115 N. Rocky River Drive
Berea, OH 44017

Euler R(αβγ) Sundial

AMOP
reference links
on following page

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Mirror reflections by σ_μ-operators make rotations *The famous Clothing Store Mirror*

Hamilton-turns do products $\mathbf{R}[\boldsymbol{\Theta}]\mathbf{R}[\boldsymbol{\Theta}']=\mathbf{R}[\boldsymbol{\Theta}''']$ geometrically

Hamilton-turn slide rule and sundial *U(2) products and (α,β,γ)-$[\varphi,\vartheta,\Theta]$ conversions*

➡ *Finite group products by turns or by group link diagrams* *D_3 example.* *O_h example*

$\mathscr{G}$= U(2) class transformation $\mathbf{R}[\boldsymbol{\Theta}]\mathbf{R}[\boldsymbol{\Theta}']\mathbf{R}[\boldsymbol{\Theta}]^{-1}=\mathbf{R}[\boldsymbol{\Theta}''']$ geometry

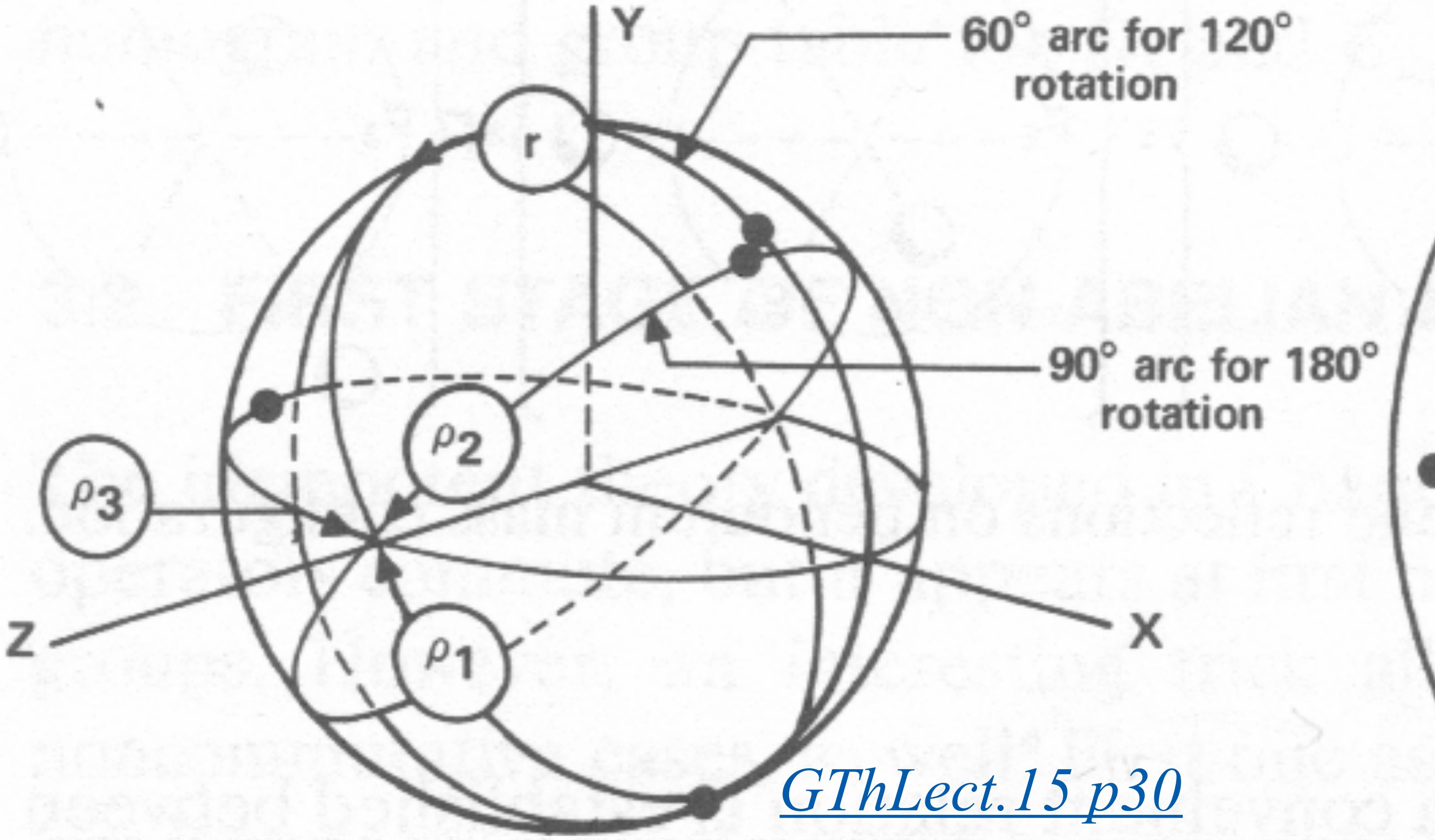
Group equivalence classes

U(2) density operator $\boldsymbol{\rho}$ and $[\boldsymbol{\rho},\mathbf{H}]$ mechanics

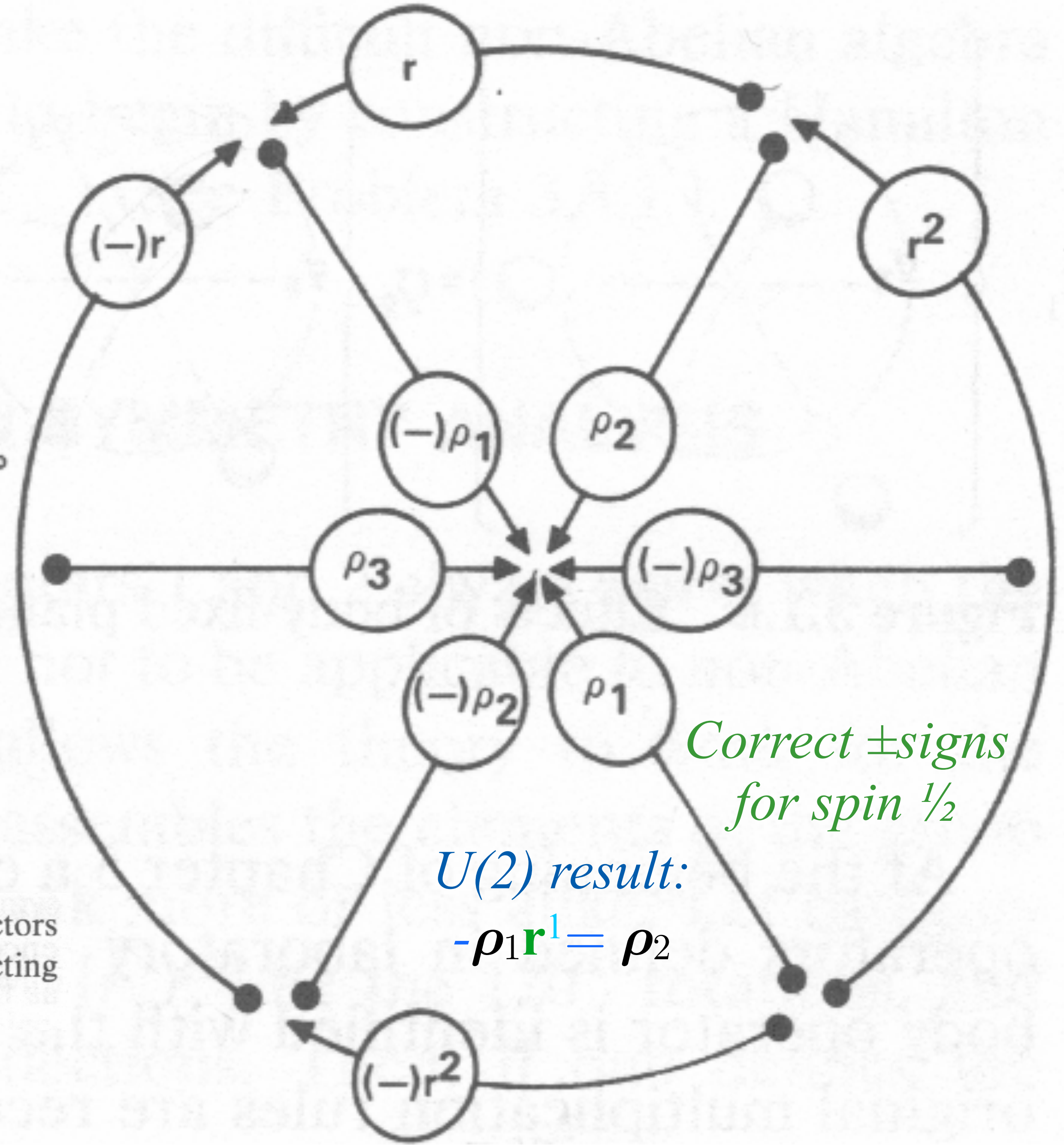
*Density mechanics compared to spin vector **S** rotated by crank vector $\boldsymbol{\Theta}=\boldsymbol{\Omega}t$*

Bloch equation $i\hbar\dot{\rho}=\left[\mathbf{H},\rho\right]$

D_3 products by U(2) Hamilton-turns



GThLect.15 p30



Correct ±signs for spin ½

U(2) result:
$-\boldsymbol{\rho}_1 \mathbf{r}^1 = \boldsymbol{\rho}_2$

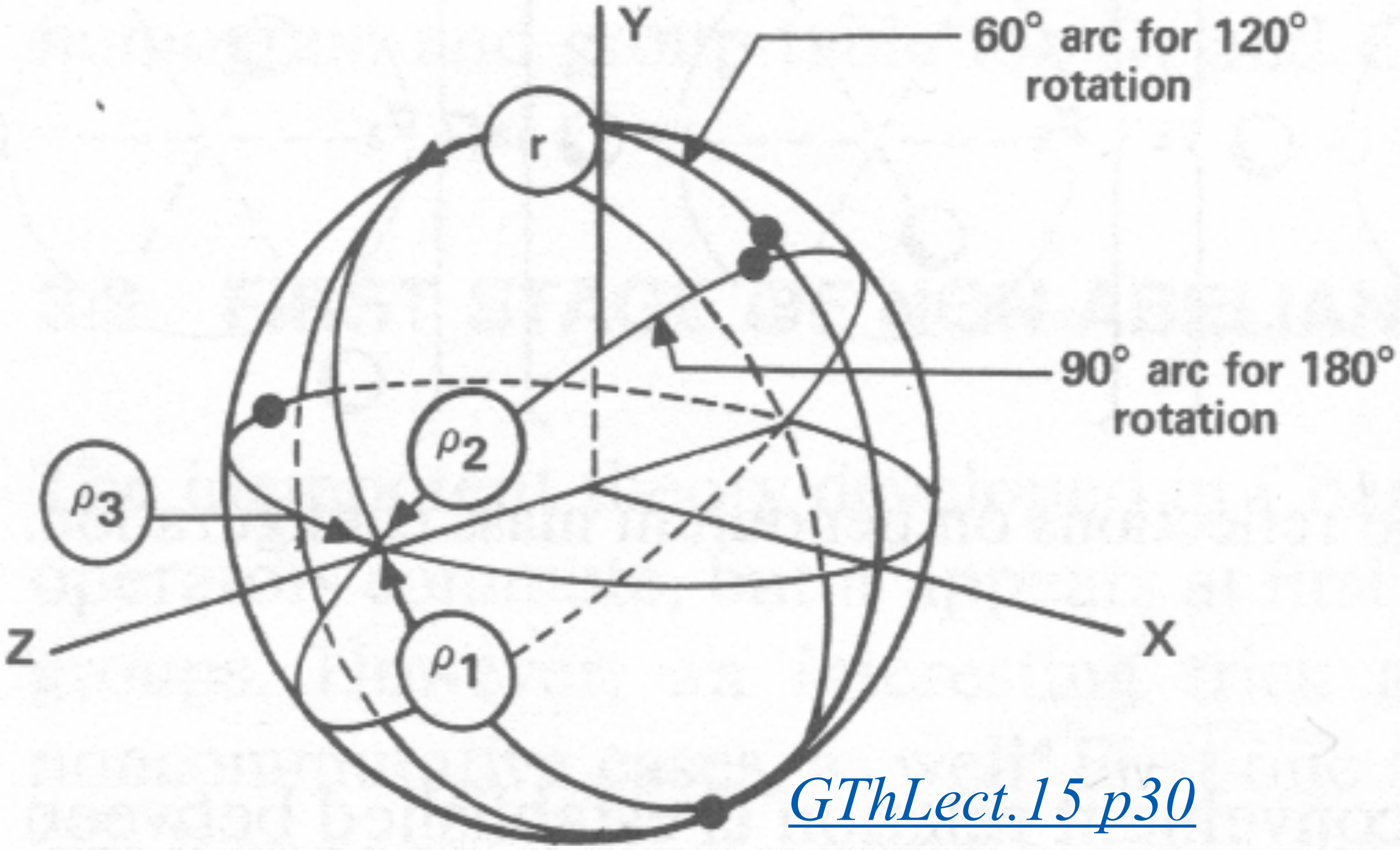
Figure 3.1.7 Geometrical definition of symmetry group D_3. (a) Hamilton arc vectors are drawn for rotations r, i_1, and i_3. (b) Group nomogram is obtained by projecting (a) onto the xy plane.

1	$\mathbf{r}^1$	$\mathbf{r}^2$	ρ_1	ρ_2	ρ_3
h^4	-1	$-h^2$	$-\rho_2$	$-\rho_3$	ρ_1
h^2	h^4	-1	$-\rho_3$	ρ_1	ρ_2
ρ_1	ρ_2	ρ_3	-1	$-h^2$	$-h^4$
ρ_2	ρ_3	$-\rho_1$	h^4	-1	$-h^2$
ρ_3	$-\rho_1$	$-\rho_2$	h^2	h^4	-1

Note $\boldsymbol{h}^2=\mathbf{r}^1$ and $\boldsymbol{h}^4=\mathbf{r}^2$ for D_6 notation

D_3 products by U(2) Hamilton-turns

GrpThLect 15 p.30.



GThLect.15 p30

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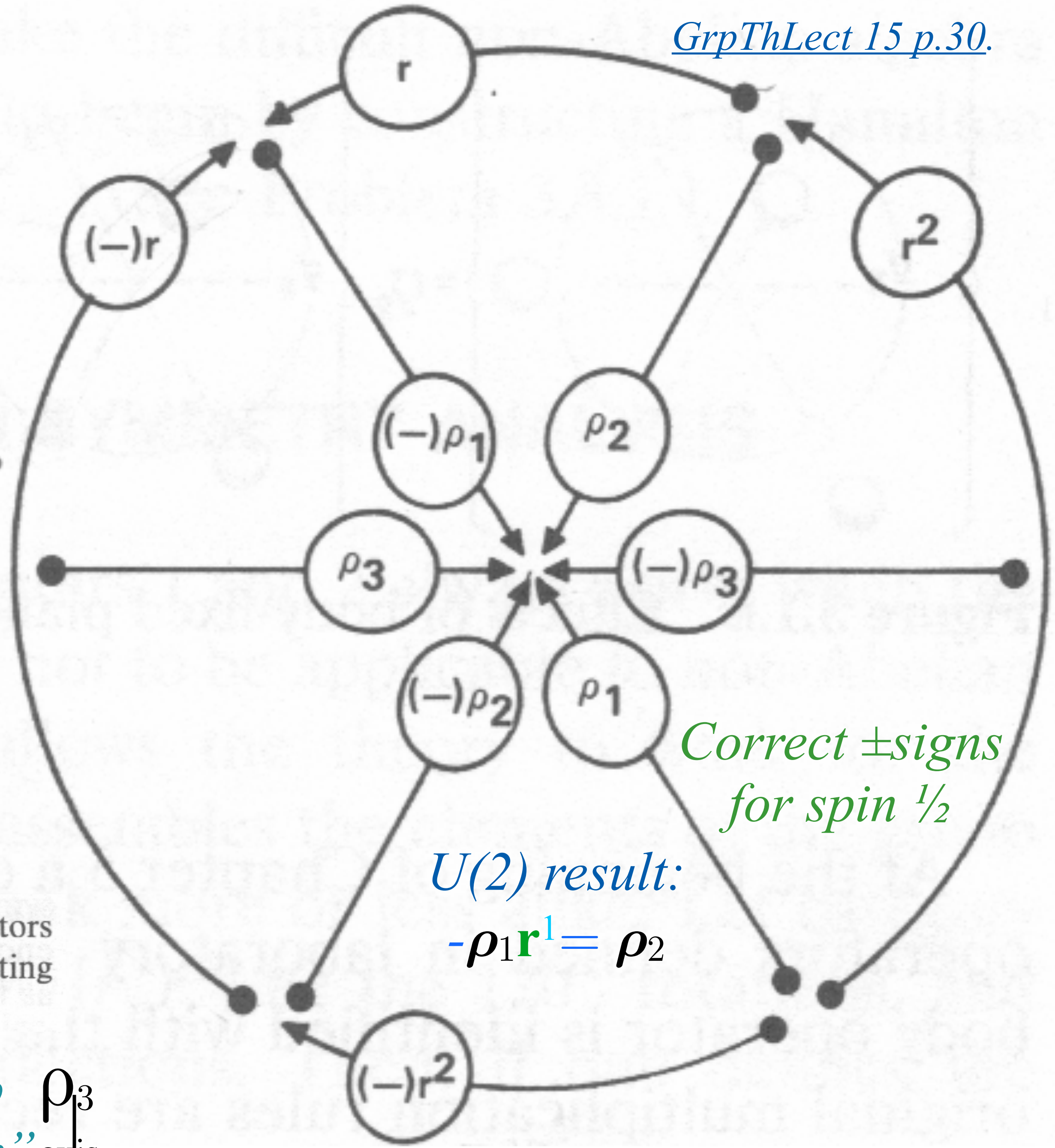
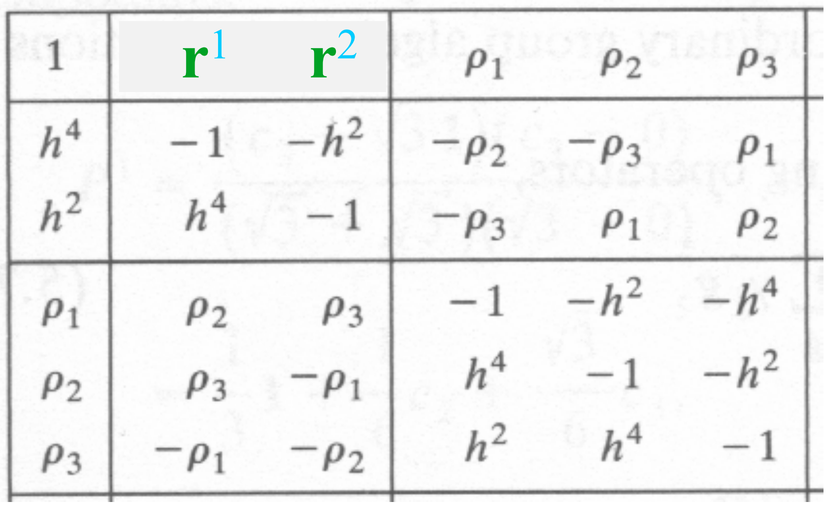


Correct ±signs
for spin ½

U(2) result:
$-\rho_1\mathbf{r}^1 = \rho_2$

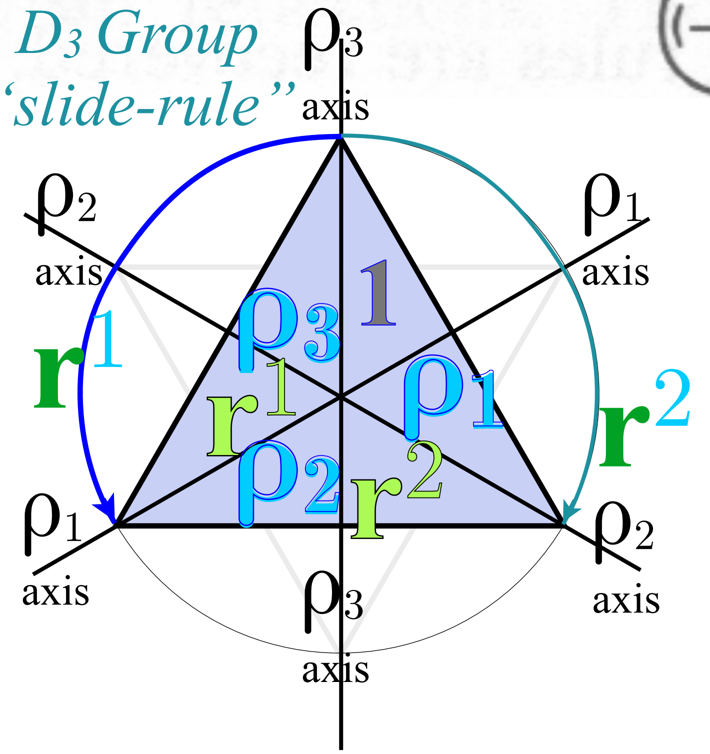
D_3 products by link diagram



1	$\mathbf{r}^1$	$\mathbf{r}^2$	ρ_1	ρ_2	ρ_3
h^4	-1	$-h^2$	$-\rho_2$	$-\rho_3$	ρ_1
h^2	h^4	-1	$-\rho_3$	ρ_1	ρ_2
ρ_1	ρ_2	ρ_3	-1	$-h^2$	$-h^4$
ρ_2	ρ_3	$-\rho_1$	h^4	-1	$-h^2$
ρ_3	$-\rho_1$	$-\rho_2$	h^2	h^4	-1

Note $\mathbf{h}^2=\mathbf{r}^1$ and $\mathbf{h}^4=\mathbf{r}^2$ for D_6 notation

D_3 Group
"slide-rule"



R(3) result:
$\rho_1\mathbf{r}^1 = \rho_2$

Ignores ±signs
for spin ½

GrpThLect 15 p.6.

Octahedral⊃Tetrahedral symmetry $O_h \supset O \sim T_d \supset T$, $O_h \supset O \sim T_d$, and $O_h \supset T_h \supset T$

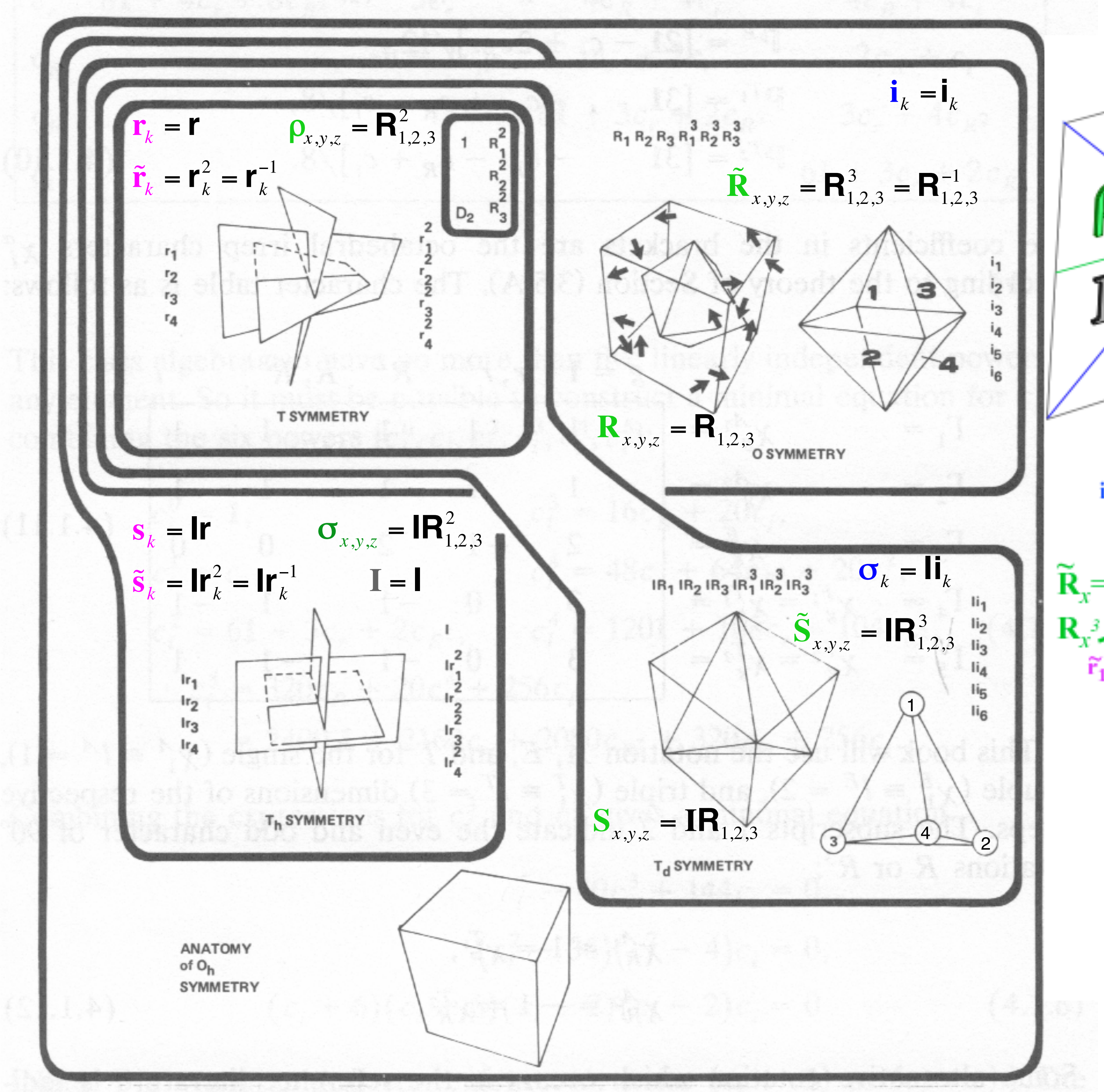
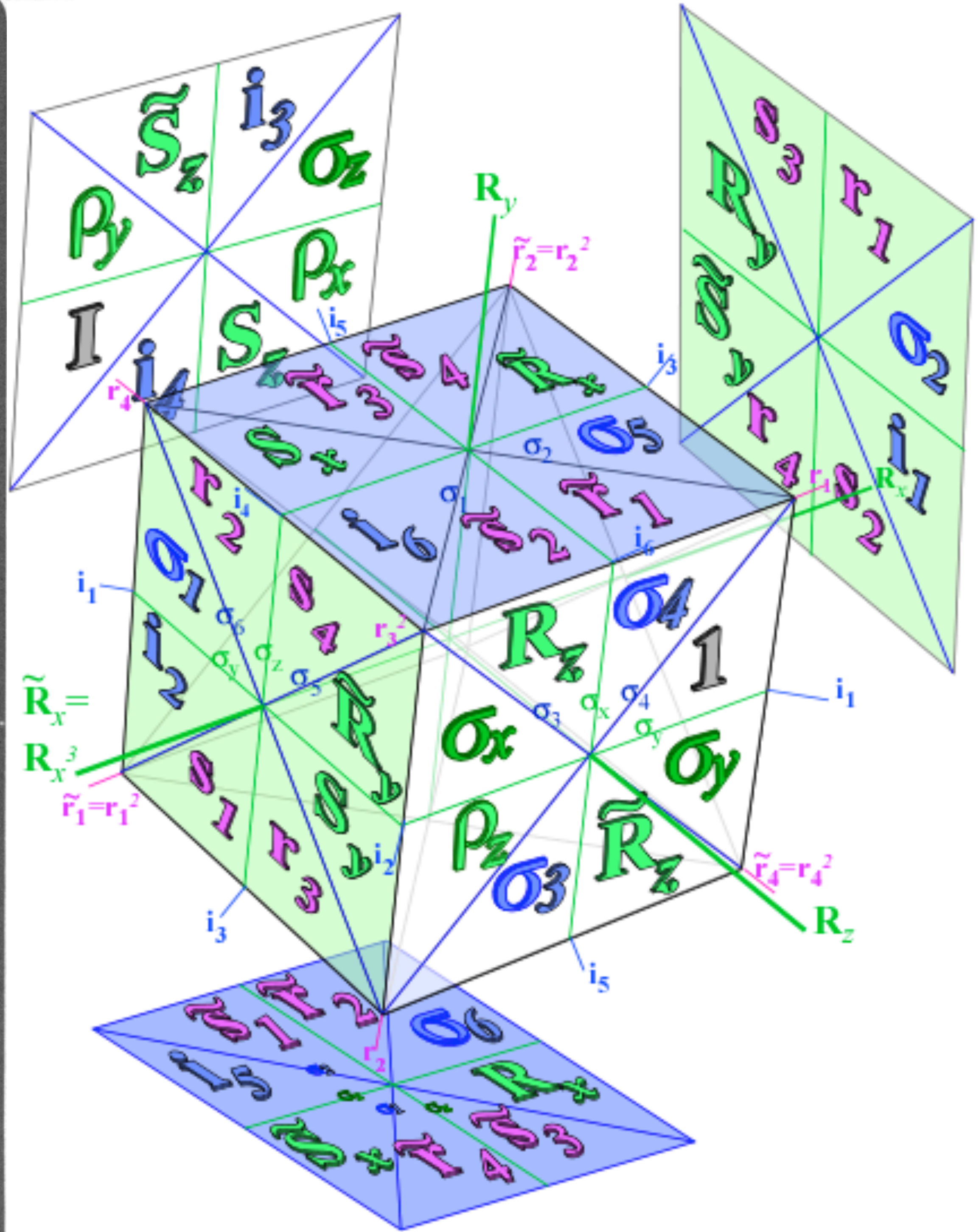


Figure 4.1.5 The full octahedral group (O_h) and four non-Abelian subgroups T, T_h, T_d, and O. The Abelian D_2 subgroup of T is indicated also.

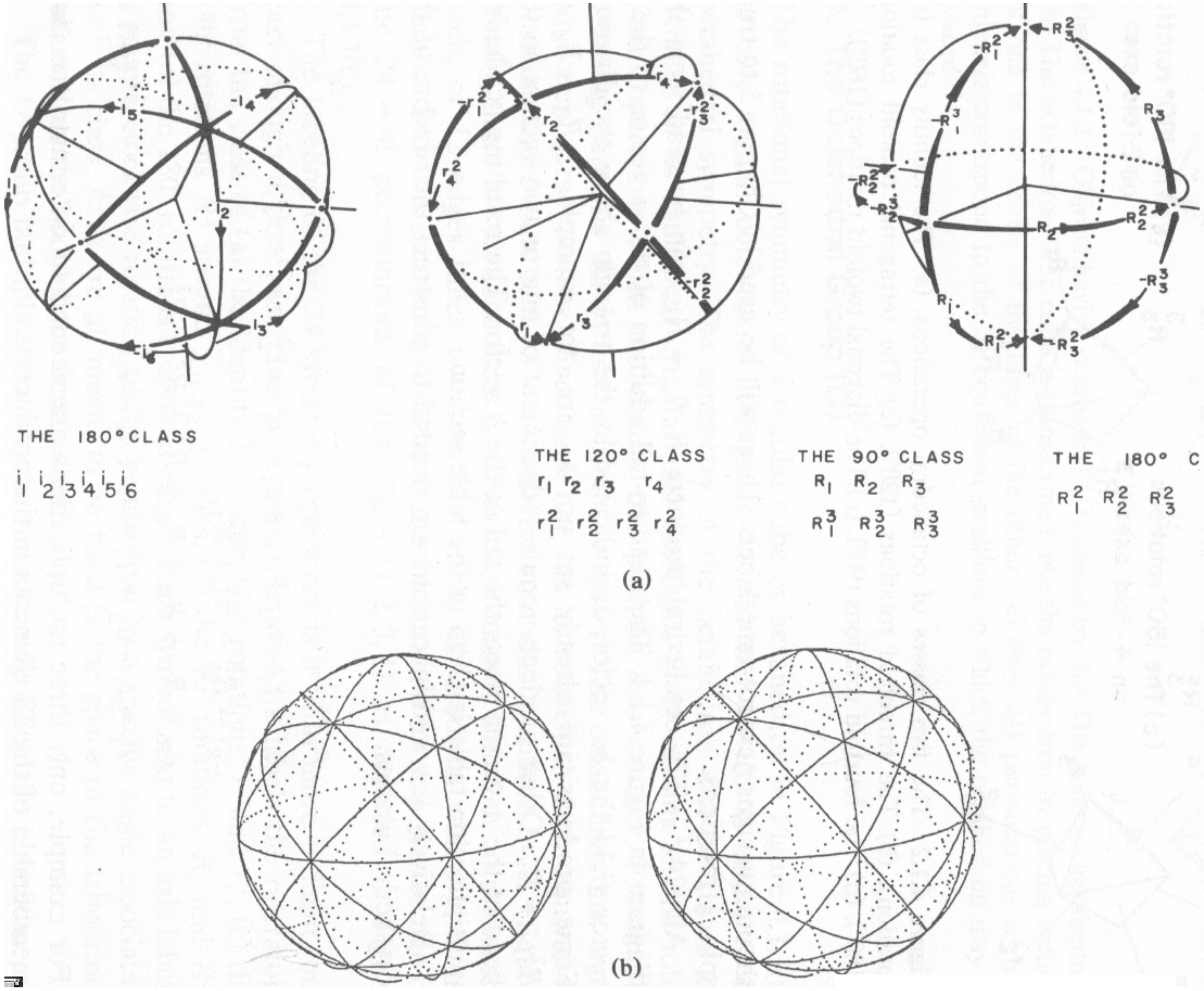


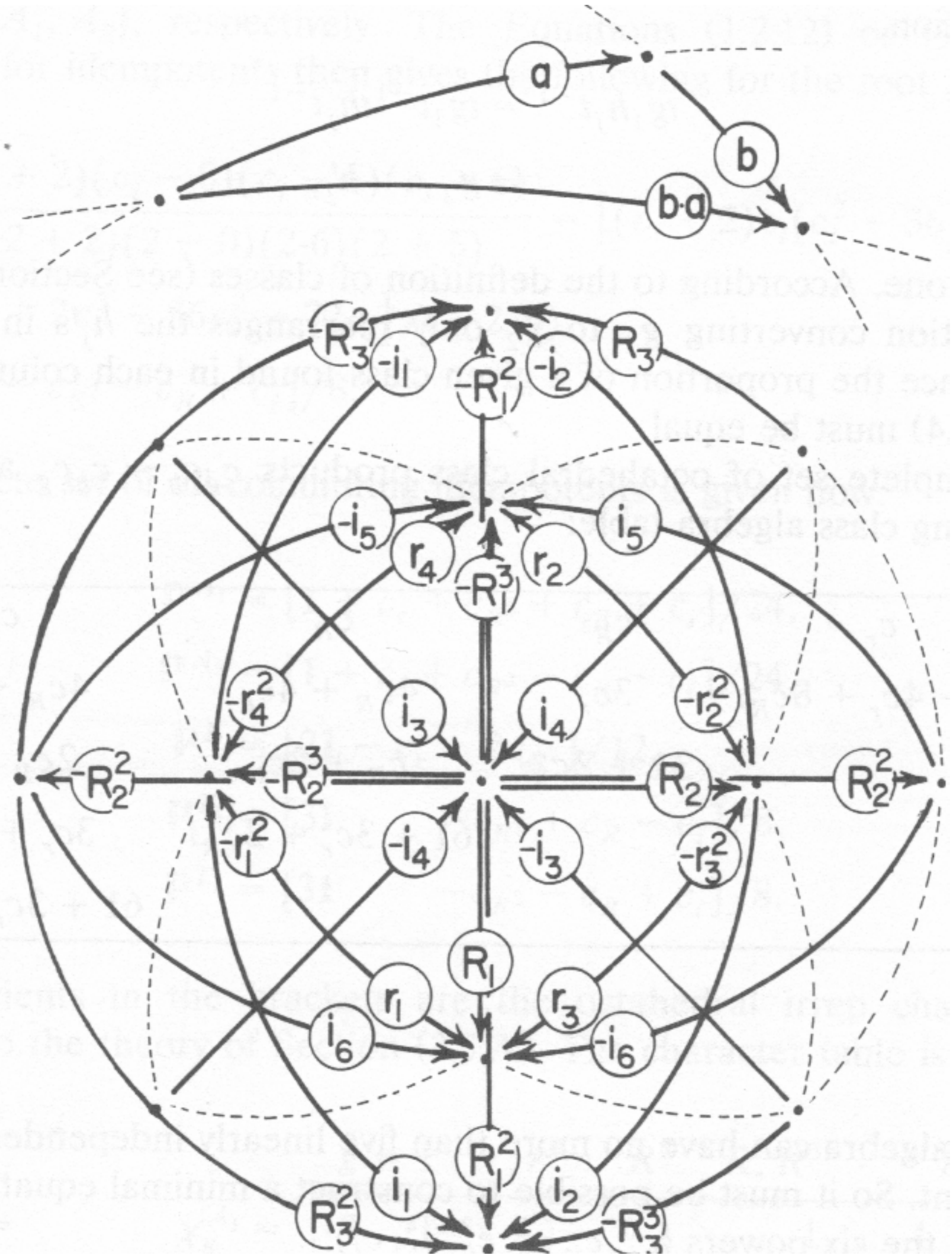
GThLect.19 p14-20

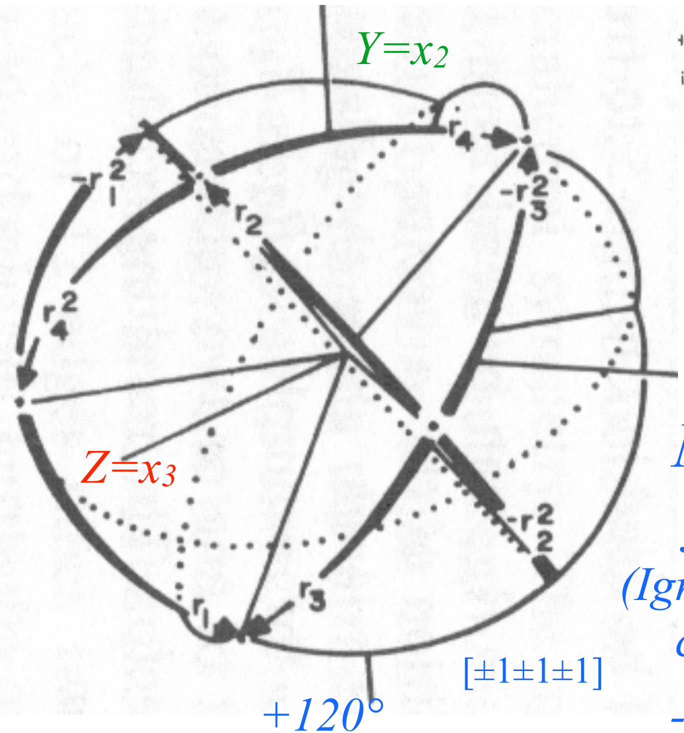
Fig. 4.1.5 from Principles of Symmetry, Dynamics and Spectroscopy

The symmetry group of Sulfur Hexafluoride SF_6

Octahedral O and spin-O⊂U(2) rotation nomogram from Fig. 4.1.3-4 Principles of Symmetry, Dynamics and Spectroscopy



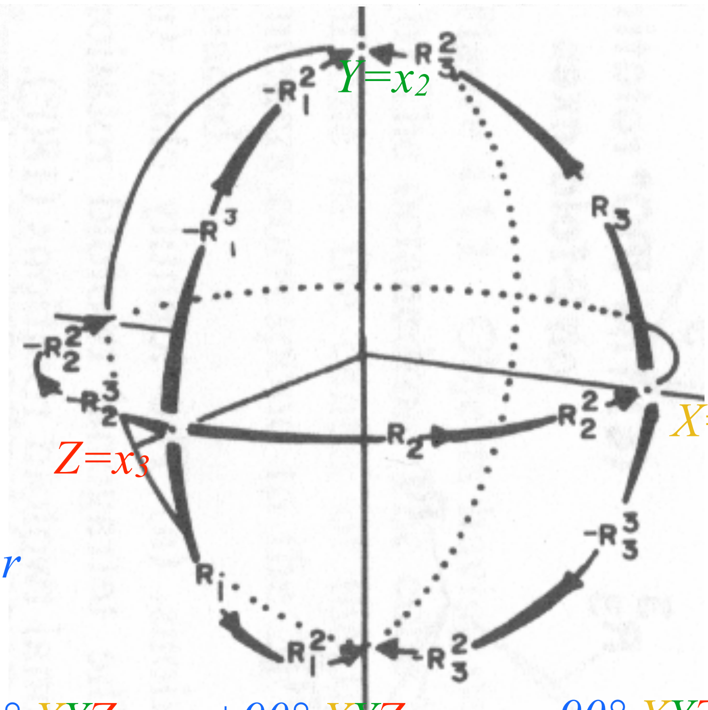


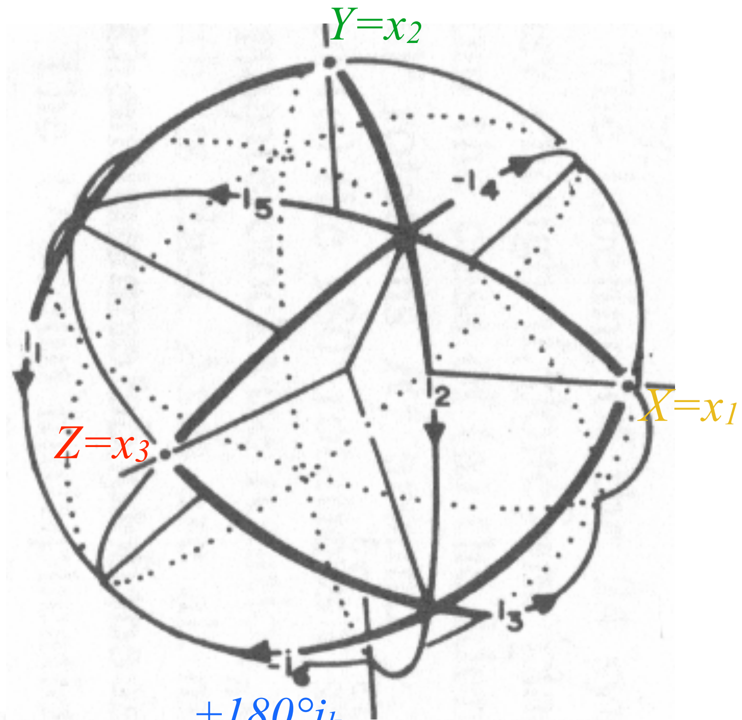


HE 180°CLASS
$i_2\ i_3\ i_4\ i_5\ i_6$

Minus (-)signs for Fermions

(Ignore (-) for Bosons or classical particles)





[±1±1±1]

+120° -120° ±180° XYZ +90° XYZ -90° XYZ ±180°i_k

	$[111]$	$[\bar{1}\bar{1}1]$	$[1\bar{1}\bar{1}]$	$[\bar{1}1\bar{1}]$	$[\bar{1}\bar{1}\bar{1}]$	$[11\bar{1}]$	$[\bar{1}11]$	$[1\bar{1}1]$	$[100]$	$[010]$	$[001]$	$[100]$	$[010]$	$[001]$	$[\bar{1}00]$	$[0\bar{1}0]$	$[00\bar{1}]$	$[101]$	$[10\bar{1}]$	$[110]$	$[1\bar{1}0]$	$[01\bar{1}]$	$[011]$
1	r_1	r_2	r_3	r_4	r_1^2	r_2^2	r_3^2	r_4^2	R_1^2	R_2^2	R_3^2	R_1	R_2	R_3	R_1^3	R_2^3	R_3^3	i_1	i_2	i_3	i_4	i_5	i_6
r_1	r_1^2	$-r_4^2$	$-r_2^2$	$-r_3^2$	-1	$-R_2^2$	$-R_3^2$	$-R_1^2$	$-r_2$	$-r_3$	$-r_4$	i_3	i_6	i_1	$-R_3$	$-R_1$	$-R_2$	R_1^3	i_5	R_2^3	i_2	$-i_4$	R_3^3
r_2	$-r_3^2$	r_2^2	$-r_4^2$	$-r_1^2$	R_2^2	-1	R_1^2	$-R_3^2$	r_1	r_4	$-r_3$	R_3	$-R_1^3$	i_2	i_3	$-i_5$	R_2^3	i_6	$-R_1$	R_2	$-i_1$	R_3^3	i_4
r_3	$-r_4^2$	$-r_1^2$	r_3^2	$-r_2^2$	R_3^2	$-R_1^2$	-1	R_2^2	$-r_4$	r_1	r_2	$-i_4$	R_1	$-R_2^3$	R_3^3	i_6	i_2	i_5	$-R_1^3$	i_1	R_2	$-i_3$	R_3
r_4	$-r_2^2$	$-r_3^2$	$-r_1^2$	r_4^2	R_1^2	R_3^2	$-R_2^2$	-1	r_3	$-r_2$	r_1	$-R_3^3$	$-i_5$	R_2	$-i_4$	R_1^3	i_1	R_1	i_6	$-i_2$	R_2^3	R_3	i_3
r_1^2	-1	R_1^2	R_2^2	R_3^2	$-r_1$	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2	R_2^3	R_3^3	R_1^3	$-i_1$	$-i_3$	$-i_6$	$-R_3$	$-i_4$	$-R_1$	i_5	$-i_2$	$-R_2$
r_2^2	$-R_1^2$	-1	R_3^2	$-R_2^2$	r_4	$-r_2$	r_1	r_3	$-r_3^2$	$-r_1^2$	r_4^2	i_2	$-i_3$	$-R_1$	R_2	$-R_3^3$	$-i_5$	i_4	$-R_3$	$-R_1^3$	$-i_6$	R_2^3	$-i_1$
r_3^2	$-R_2^2$	$-R_3^2$	-1	R_1^2	r_2	r_4	$-r_3$	r_1	r_2^2	$-r_4^2$	$-r_1^2$	$-R_2$	$-i_4$	$-i_6$	i_2	R_3	$-R_1^3$	$-i_3$	$-R_3^3$	i_5	R_1	$-i_1$	$-R_2^3$
r_4^2	$-R_3^2$	R_2^2	$-R_1^2$	-1	r_3	r_1	r_2	$-r_4$	$-r_1^2$	r_3^2	$-r_2^2$	$-i_1$	$-R_3$	$-i_5$	$-R_2^3$	$-i_4$	R_1	$-R_3^3$	i_3	$-i_6$	R_1^3	R_2	$-i_2$
R_1^2	$-r_4$	r_3	$-r_2$	r_1	r_2^2	$-r_1^2$	r_4^2	$-r_3^2$	-1	R_3^2	$-R_2^2$	R_1^3	i_1	$-i_4$	$-R_1$	i_2	$-i_3$	$-R_2$	$-R_2^3$	R_3^3	R_3	$-i_6$	i_5
R_2^2	$-r_2$	r_1	r_4	$-r_3$	r_3^2	$-r_4^2$	$-r_1^2$	r_2^2	$-R_3^2$	-1	R_1^2	$-i_5$	R_2^3	i_3	$-i_6$	$-R_2$	$-i_4$	$-i_2$	i_1	$-R_3$	R_3^3	R_1	R_1^3
R_3^2	$-r_3$	$-r_4$	r_1	r_2	r_4^2	r_3^2	$-r_2^2$	$-r_1^2$	R_2^2	$-R_1^2$	-1	i_6	i_2	R_3^3	$-i_5$	$-i_1$	$-R_3$	R_2^3	$-R_2$	i_4	$-i_3$	R_1^3	$-R_1$
R_1	i_1	$-R_2^3$	$-i_2$	R_2	R_3^3	$-i_3$	$-R_3$	i_4	R_1^3	i_6	i_5	R_1^2	r_1	$-r_4^2$	-1	$-r_3$	r_2^2	$-r_4$	r_2	r_1^2	$-r_3^2$	$-R_2^2$	R_3^2
R_2	i_3	R_3	$-R_3^3$	i_4	R_1^3	i_5	$-i_6$	$-R_1$	$-i_2$	R_2^3	i_1	$-r_2^2$	R_2^2	r_1	r_3^2	-1	$-r_4$	R_1^2	R_3^2	$-r_2$	$-r_3$	$-r_4^2$	r_1^2
R_3	i_6	i_5	R_1	$-R_1^3$	R_2^3	$-R_2$	$-i_2$	$-i_1$	i_3	i_4	R_3^3	r_1	$-r_3^2$	R_3^2	$-r_2$	r_4^2	-1	r_1^2	r_2^2	R_2^2	$-R_1^2$	$-r_4$	$-r_3$
R_1^3	$-R_2$	$-i_2$	R_2^3	i_1	$-i_3$	$-R_3^3$	i_4	R_3	$-R_1$	i_5	$-i_6$	-1	$-r_4$	r_3^2	$-R_1^2$	r_2	$-r_1^2$	$-r_1$	r_3	r_2^2	$-r_4^2$	$-R_3^2$	$-R_2^2$
R_2^3	$-R_3$	i_3	i_4	R_3^3	$-i_6$	R_1	$-R_1^3$	i_5	$-i_1$	$-R_2$	$-i_2$	r_4^2	-1	$-r_2$	$-r_1^2$	$-R_2^2$	r_3	$-R_3^2$	R_1^2	$-r_1$	$-r_4$	$-r_2^2$	r_3^2
R_3^3	$-R_1$	R_1^3	i_6	i_5	$-i_1$	$-i_2$	R_2	$-R_2^3$	i_4	$-i_3$	$-R_3$	$-r_3$	r_2^2	-1	r_4	$-r_1^2$	$-R_3^2$	r_4^2	r_3^2	$-R_1^2$	$-R_2^2$	$-r_2$	$-r_1$
i_1	R_3^3	$-i_4$	i_3	R_3	$-R_1$	$-i_6$	$-i_5$	$-R_1^3$	R_2^3	i_2	$-R_2$	r_1^2	R_3^2	$-r_4$	r_4^2	$-R_1^2$	$-r_1$	-1	$-R_2^2$	$-r_3$	r_2	r_3^2	r_2^2
i_2	i_4	R_3^3	R_3	$-i_3$	$-i_5$	R_1^3	R_1	$-i_6$	R_2	$-i_1$	R_2^3	$-r_3^2$	$-R_1^2$	$-r_3$	$-r_2^2$	$-R_3^2$	$-r_2$	R_2^2	-1	r_4	$-r_1$	r_1^2	r_4^2
i_3	R_1^3	R_1	$-i_5$	i_6	$-R_2$	$-R_2^3$	$-i_1$	i_2	$-R_3$	R_3^3	$-i_4$	$-r_2$	r_1^2	R_1^2	$-r_1$	r_2^2	$-R_2^2$	r_3^2	$-r_4^2$	-1	R_3^2	r_3	$-r_4$
i_4	$-i_5$	i_6	$-R_1^3$	$-R_1$	$-i_2$	i_1	$-R_2^3$	$-R_2$	$-R_3^3$	$-R_3$	i_3	r_4	r_4^2	R_2^2	r_3	r_3^2	R_1^2	$-r_2^2$	r_1^2	$-R_3^3$	-1	r_1	$-r_2$
i_5	i_2	$-R_2$	i_1	$-R_2^3$	i_4	$-R_3$	i_3	$-R_3^3$	i_6	$-R_1^3$	$-R_1$	R_3^2	r_2	r_2^2	R_2^2	r_4	r_4^2	$-r_3$	$-r_1$	$-r_3^2$	$-r_1^2$	-1	$-R_1^2$
i_6	R_2^3	i_1	R_2	i_2	$-R_3$	$-i_4$	$-R_3^3$	$-i_3$	$-i_5$	$-R_1$	R_1^3	R_2^2	$-r_3$	r_1^2	$-R_3^2$	$-r_1$	r_3^2	$-r_2$	$-r_4$	r_4^2	r_2^2	R_1^2	-1

Octahedral O and spin-O⊂U(2) rotation product Table F.2.1 from Principles of Symmetry, Dynamics and Spectroscopy

The symmetry group of “Buckyball” C_{60}

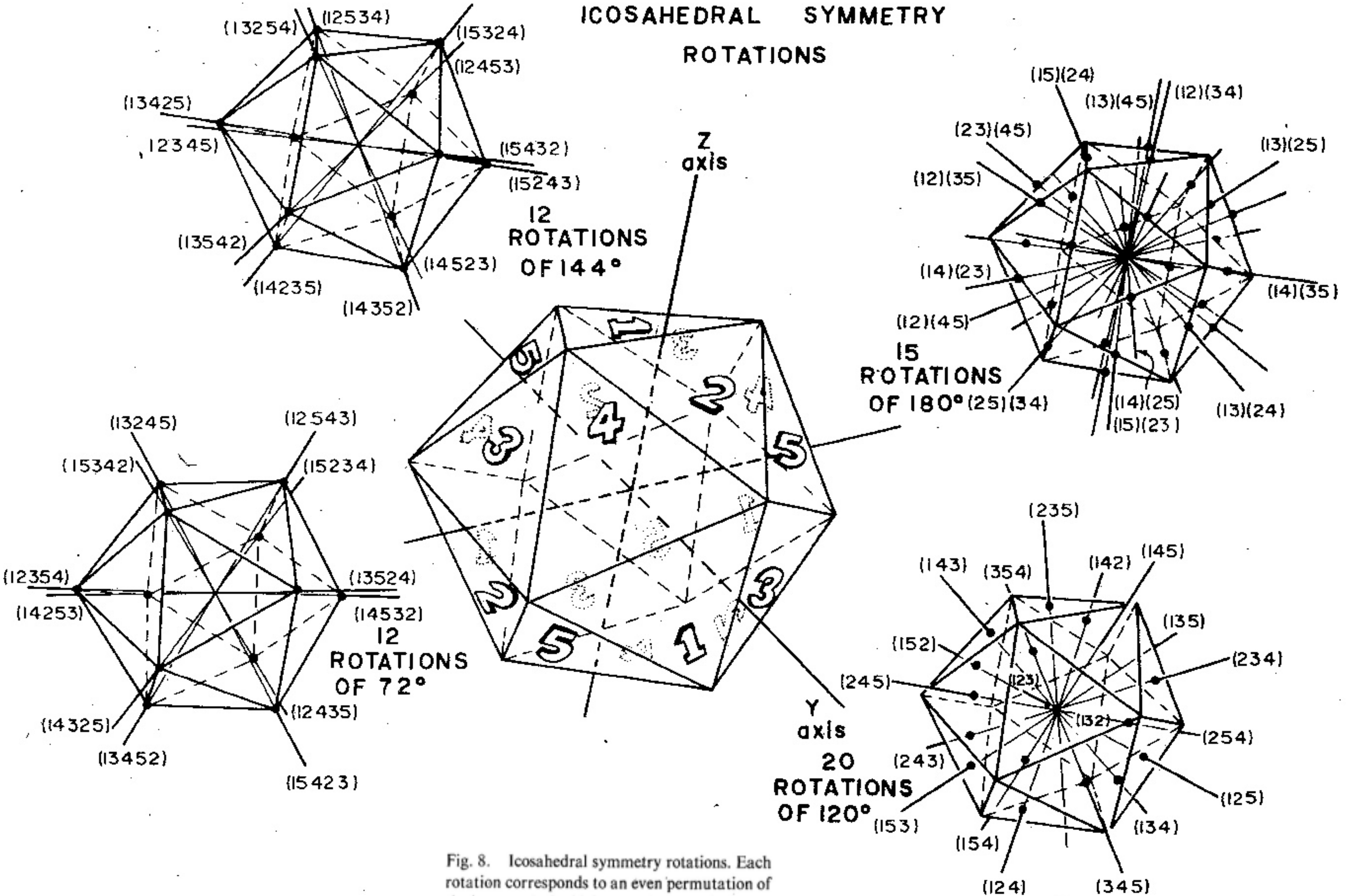


Fig. 8. Icosahedral symmetry rotations. Each rotation corresponds to an even permutation of the integers 1–5 which are written on the icosahedron.

The symmetry group of "Buckyball" C_{60}

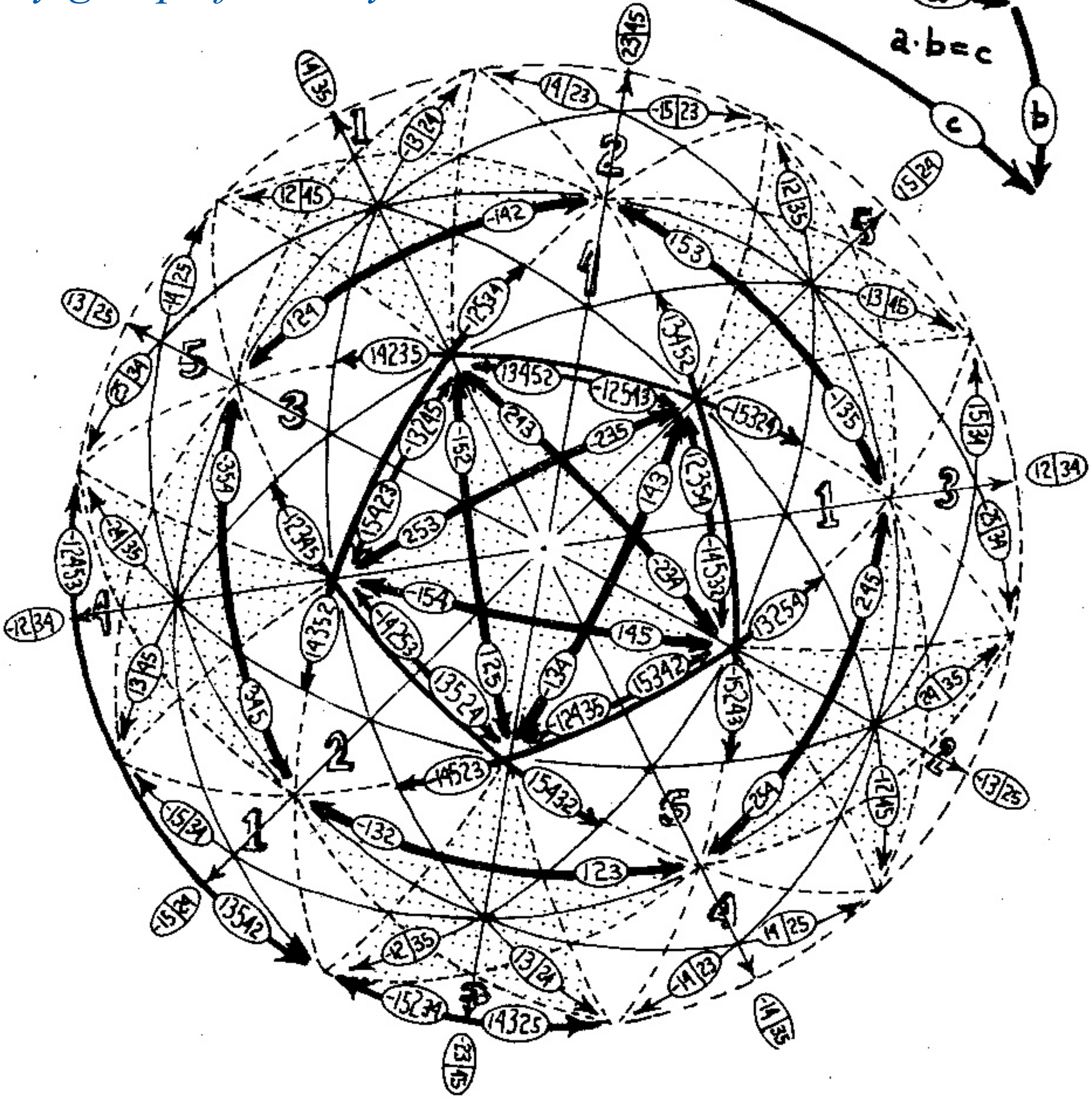


Fig. 10. Icosahedral vector addition nomogram.

AMOP reference links on following page

1.31.18 class 6.0: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

Symmetry group $\mathscr{G}$ representations=>*AMOP* Hamiltonian **H** (or K) matrices, irreps $\mathscr{D}^{(\alpha)}$
=>*AMOP* wave functions $\Psi^{(\alpha)}$, eigensolutions

$\mathscr{G}$= U(2) product $\mathbf{R}[\mathbf{\Theta}]\mathbf{R}[\mathbf{\Theta}']=\mathbf{R}[\mathbf{\Theta}'']$ algebra (It's all done with σ_μ spinors)

Jordan-Pauli identity: U(2) product algebra of spinor σ_μ-operators

U(2) "Crazy-Thing" forms do products $\mathbf{R}[\mathbf{\Theta}]\mathbf{R}[\mathbf{\Theta}']=\mathbf{R}[\mathbf{\Theta}'']$ algebraically

$\mathscr{G}$= U(2) product $\mathbf{R}[\mathbf{\Theta}]\mathbf{R}[\mathbf{\Theta}']=\mathbf{R}[\mathbf{\Theta}'']$ by geometry (It's all done with σ_μ mirrors)

Mirror reflections by σ_μ-operators make rotations *The famous Clothing Store Mirror*

Hamilton-turns do products $\mathbf{R}[\mathbf{\Theta}]\mathbf{R}[\mathbf{\Theta}']=\mathbf{R}[\mathbf{\Theta}'']$ geometrically

Hamilton-turn slide rule and sundial *U(2) products and (α,β,γ)-$[\varphi,\vartheta,\Theta]$ conversions*

Finite group products by turns or by group link diagrams *D_3 example.* *O_h example*

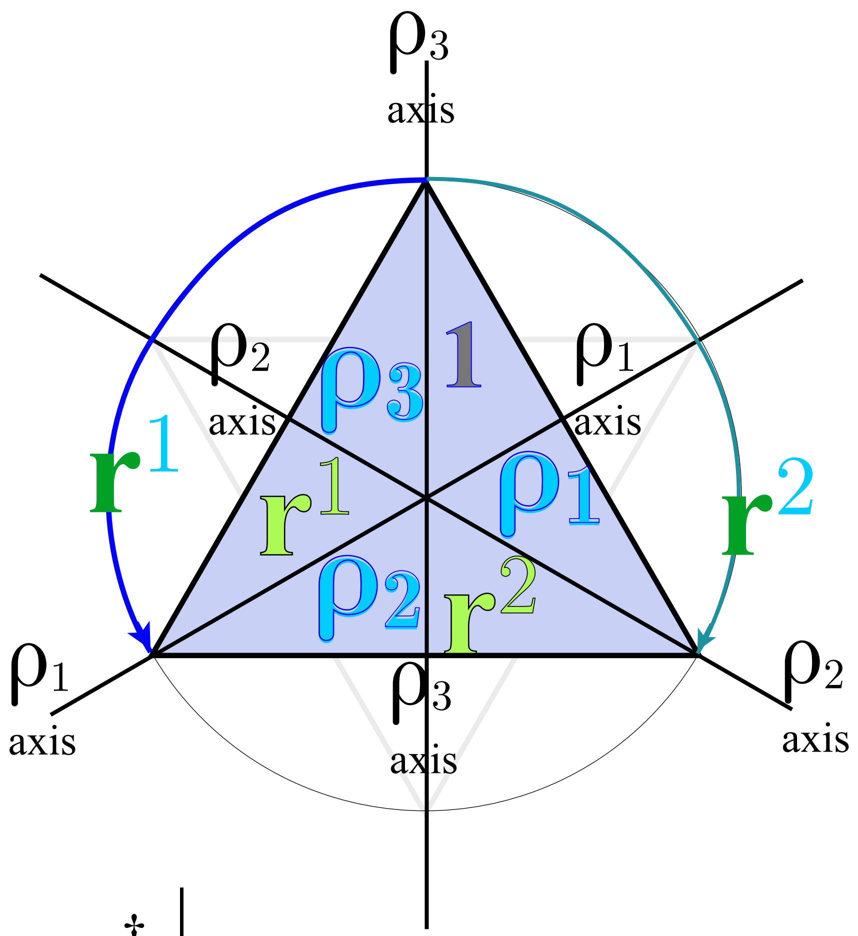
$\mathscr{G}$= U(2) class transformation $\mathbf{R}[\mathbf{\Theta}]\mathbf{R}[\mathbf{\Theta}']\mathbf{R}[\mathbf{\Theta}]^{-1}=\mathbf{R}[\mathbf{\Theta}'']$ geometry

➡ *Group equivalence classes*

U(2) density operator $\boldsymbol{\rho}$ and $[\boldsymbol{\rho},\mathbf{H}]$ mechanics

*Density mechanics compared to spin vector **S** rotated by crank vector $\mathbf{\Theta}=\mathbf{\Omega}t$*

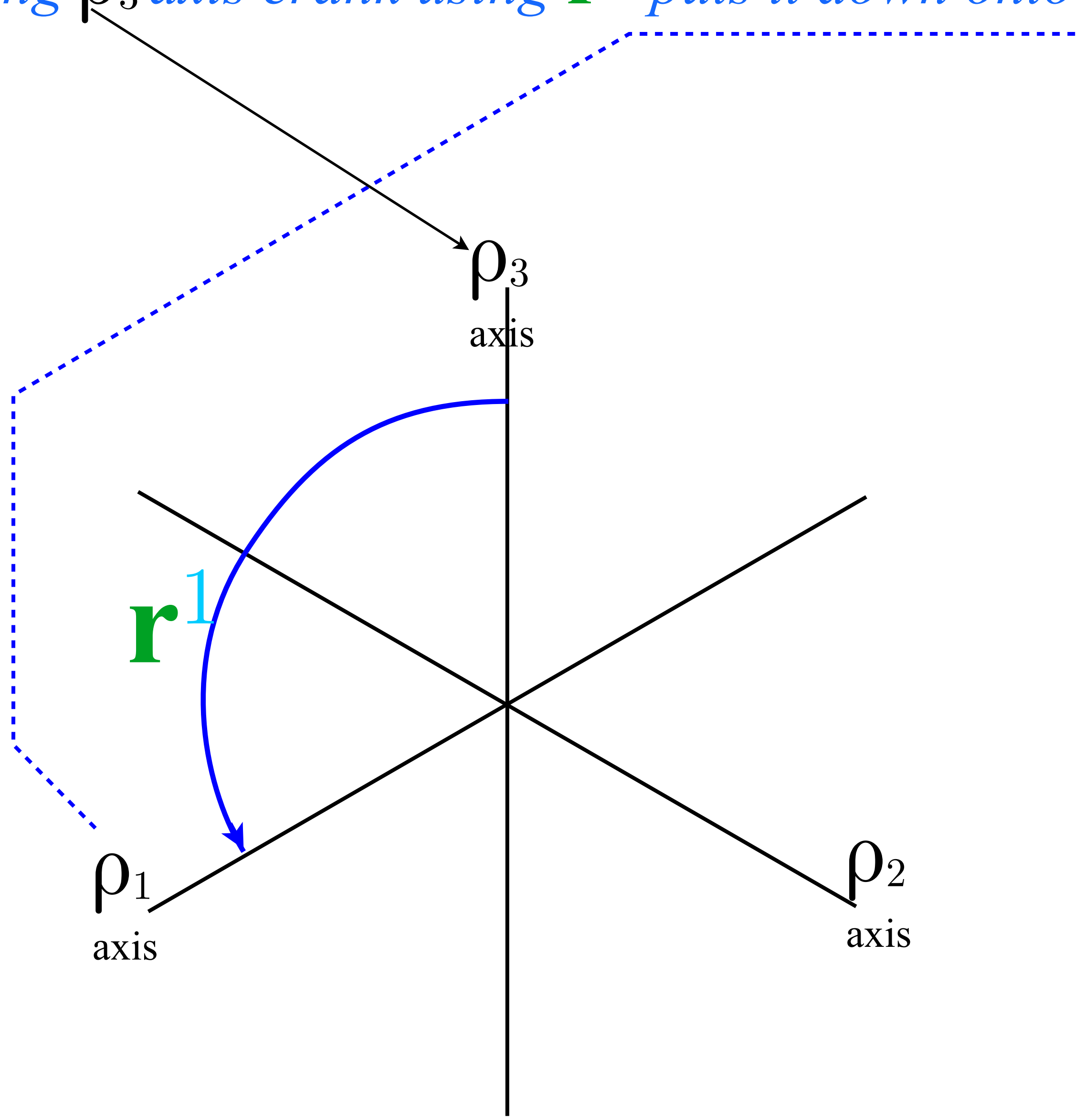
Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes



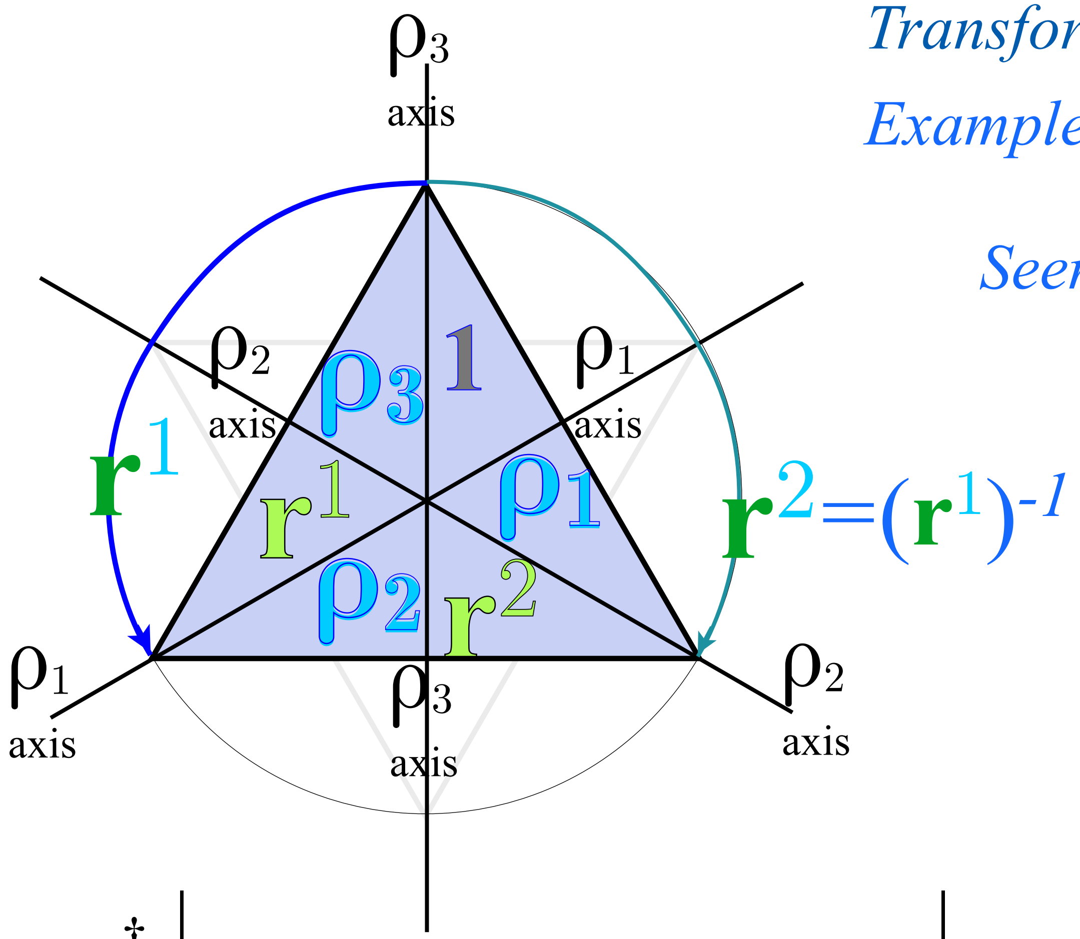
D_3 $gg^\dagger$ *form*	1	r^2	r^1	ρ_1	ρ_2	ρ_3
1	1	r^2	r^1	ρ_1	ρ_2	ρ_3
r^1	r^1	1	r^2	ρ_3	ρ_1	ρ_2
r^2	r^2	r^1	1	ρ_2	ρ_3	ρ_1
ρ_1	ρ_1	ρ_3	ρ_2	1	r^1	r^2
ρ_2	ρ_2	ρ_1	ρ_3	r^2	1	r^1
ρ_3	ρ_3	ρ_2	ρ_1	r^1	r^2	1

Transforming D_3 operators using D_3 operators

Example 1: Rotating ρ_3 *axis crank using* $\mathbf{r}^1$ *puts it down onto* ρ_1



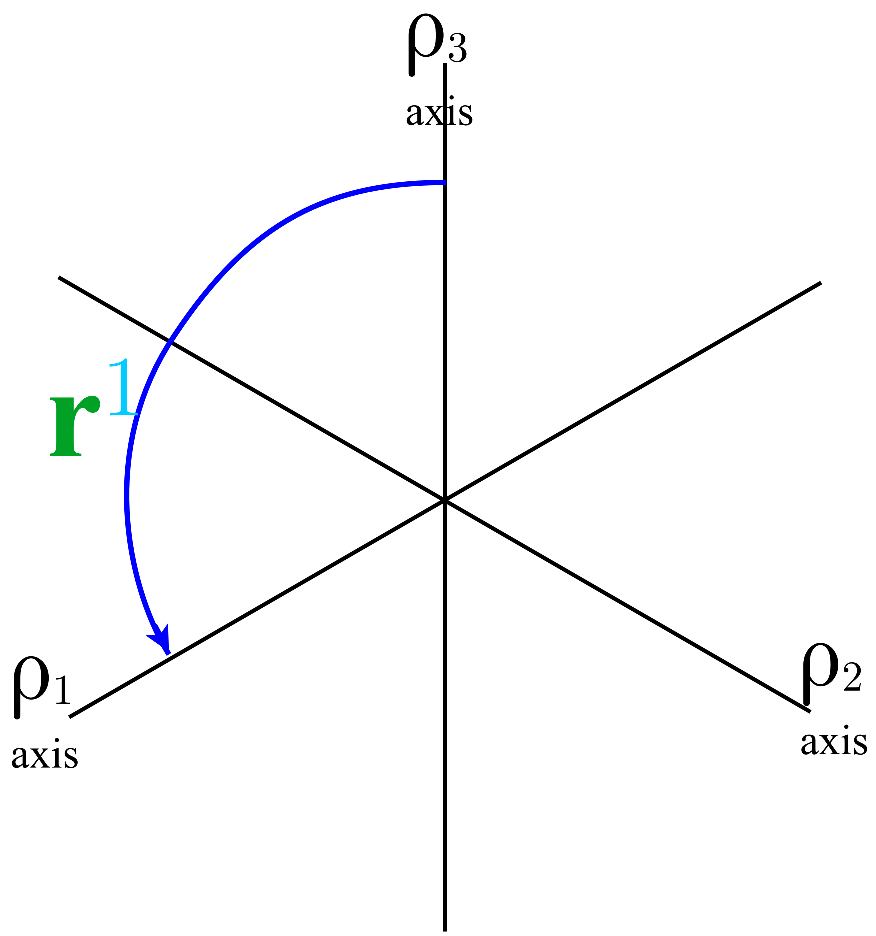
Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes



Transforming D_3 operators using D_3 operators

Example 1: Rotating ρ_3 axis crank using $\mathbf{r}^1$ puts it down onto ρ_1

Seems to imply: $\mathbf{r}^1\rho_3(\mathbf{r}^1)^{-1}=\mathbf{r}^1\rho_3\mathbf{r}^2=\rho_1$



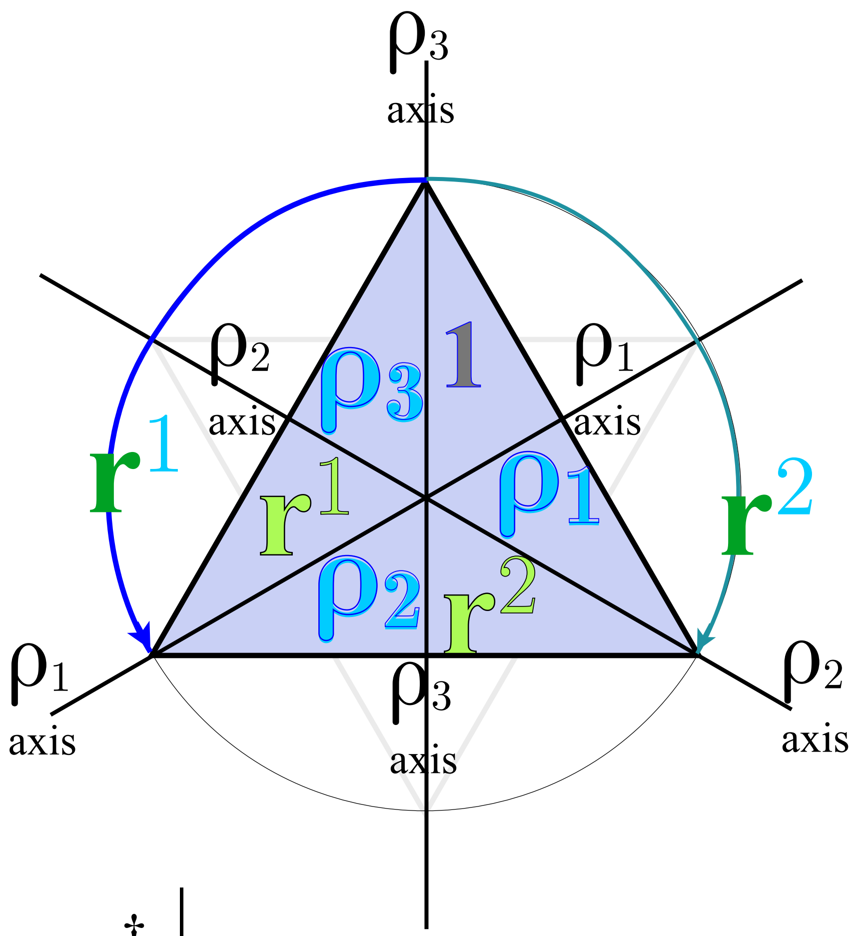
D_3 **gg**† *form*	**1**	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
1	**1**	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{r}^1$	$\mathbf{r}^1$	**1**	$\mathbf{r}^2$	ρ_3	ρ_1	ρ_2
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	**1**	ρ_2	ρ_3	ρ_1
ρ_1	ρ_1	ρ_3	ρ_2	**1**	$\mathbf{r}^1$	$\mathbf{r}^2$
ρ_2	ρ_2	ρ_1	ρ_3	$\mathbf{r}^2$	**1**	$\mathbf{r}^1$
ρ_3	ρ_3	ρ_2	ρ_1	$\mathbf{r}^1$	$\mathbf{r}^2$	**1**

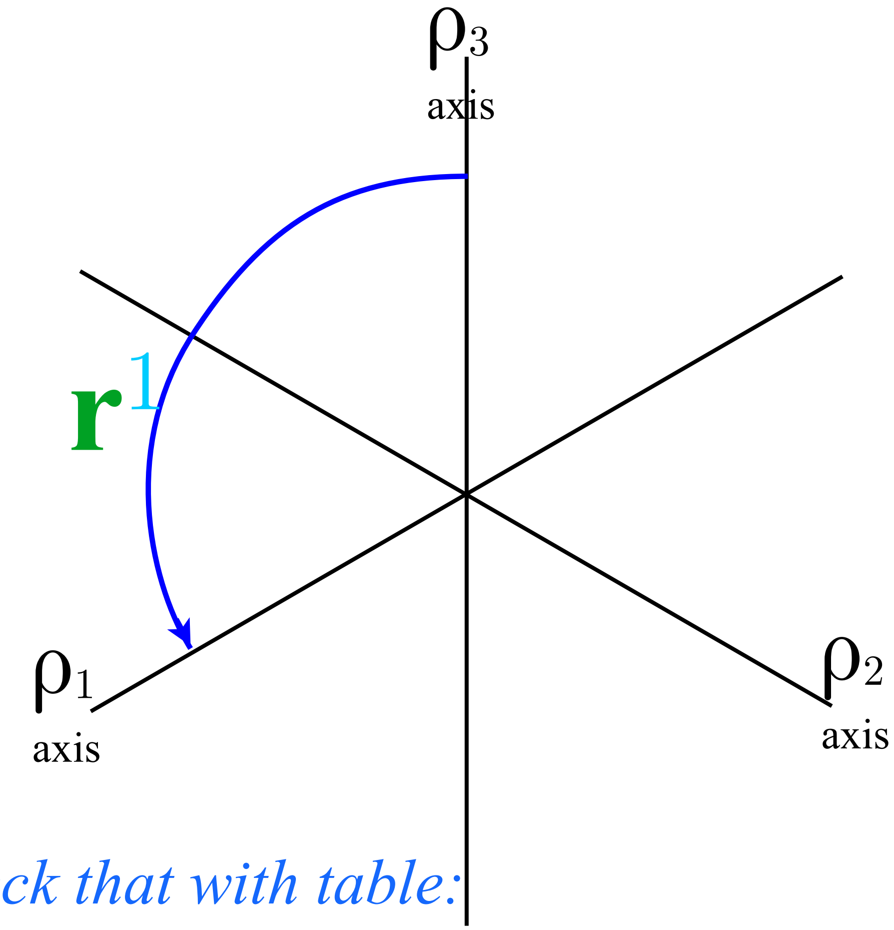
Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes

Transforming D_3 operators using D_3 operators

Example 1: Rotating ρ_3 *axis crank using* $\mathbf{r}^1$ *puts it down onto* ρ_1

Seems to imply: $\mathbf{r}^1\rho_3(\mathbf{r}^1)^{-1}=\mathbf{r}^1\rho_3\mathbf{r}^2=\rho_1$





Need to check that with table:

$$\underbrace{\mathbf{r}^1\rho_3}\mathbf{r}^2=\underbrace{\rho_2}\mathbf{r}^2$$

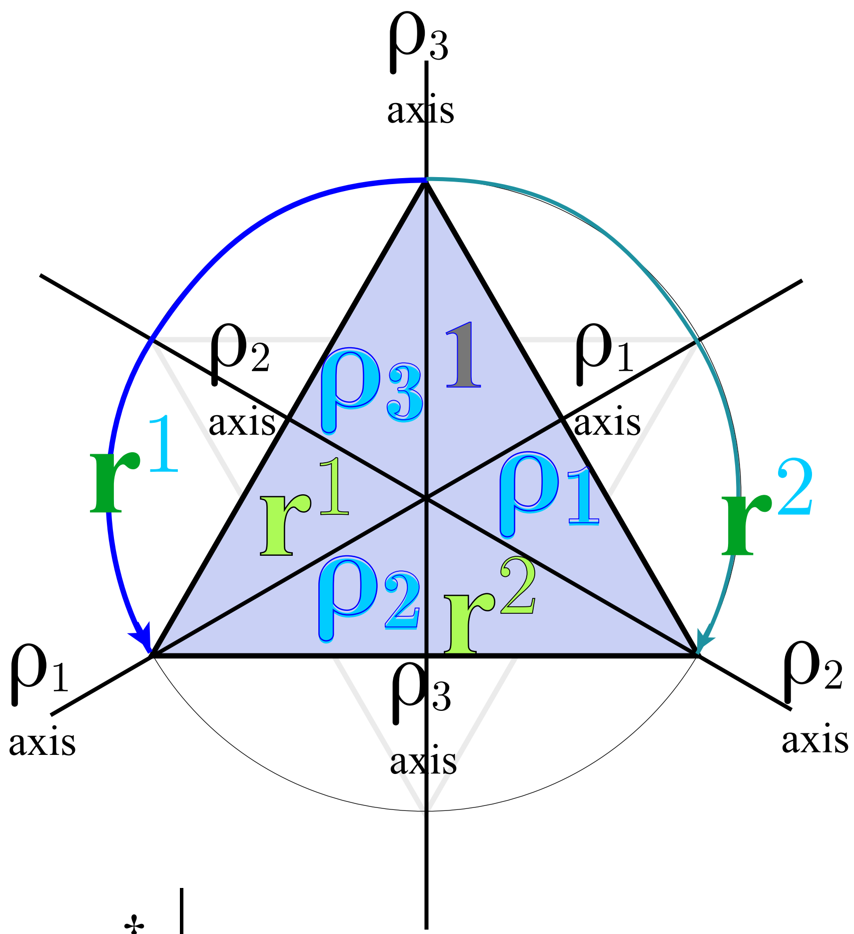
D_3 $\mathbf{gg}^\dagger$ *form*	**1**	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
1	**1**	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{r}^1$	$\mathbf{r}^1$	**1**	$\mathbf{r}^2$	ρ_3	ρ_1	ρ_2
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	**1**	ρ_2	ρ_3	ρ_1
ρ_1	ρ_1	ρ_3	ρ_2	**1**	$\mathbf{r}^1$	$\mathbf{r}^2$
ρ_2	ρ_2	ρ_1	ρ_3	$\mathbf{r}^2$	**1**	$\mathbf{r}^1$
ρ_3	ρ_3	ρ_2	ρ_1	$\mathbf{r}^1$	$\mathbf{r}^2$	**1**

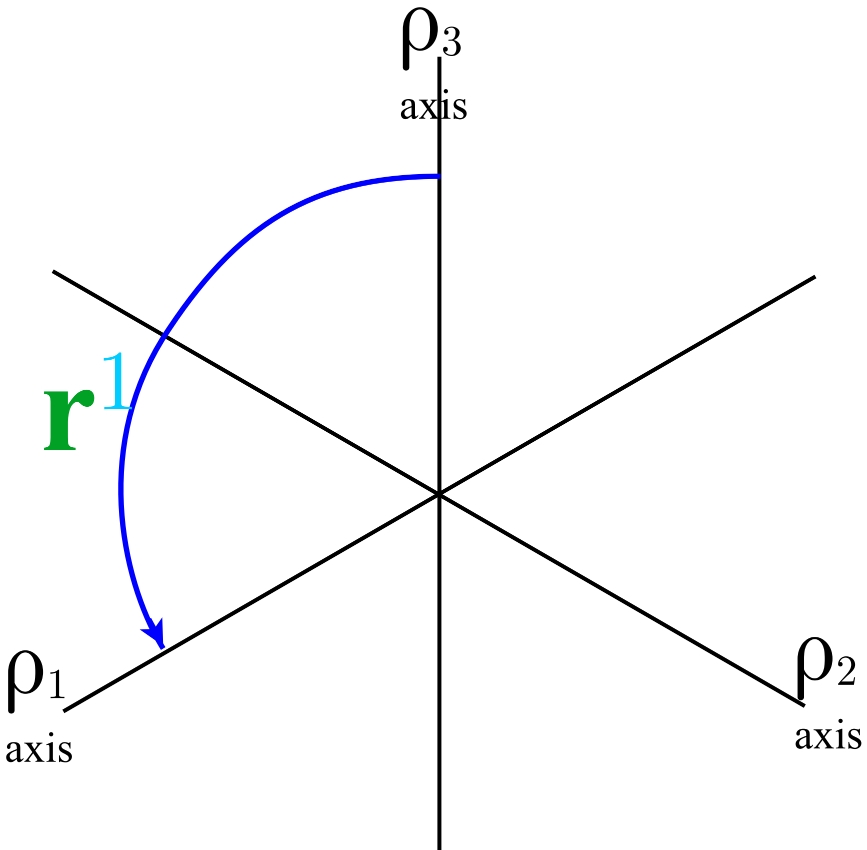
Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes

Transforming D_3 operators using D_3 operators

Example 1: Rotating ρ_3 *axis crank using* $\mathbf{r}^1$ *puts it down onto* ρ_1

Seems to imply: $\mathbf{r}^1\rho_3(\mathbf{r}^1)^{-1}=\mathbf{r}^1\rho_3\mathbf{r}^2=\rho_1$





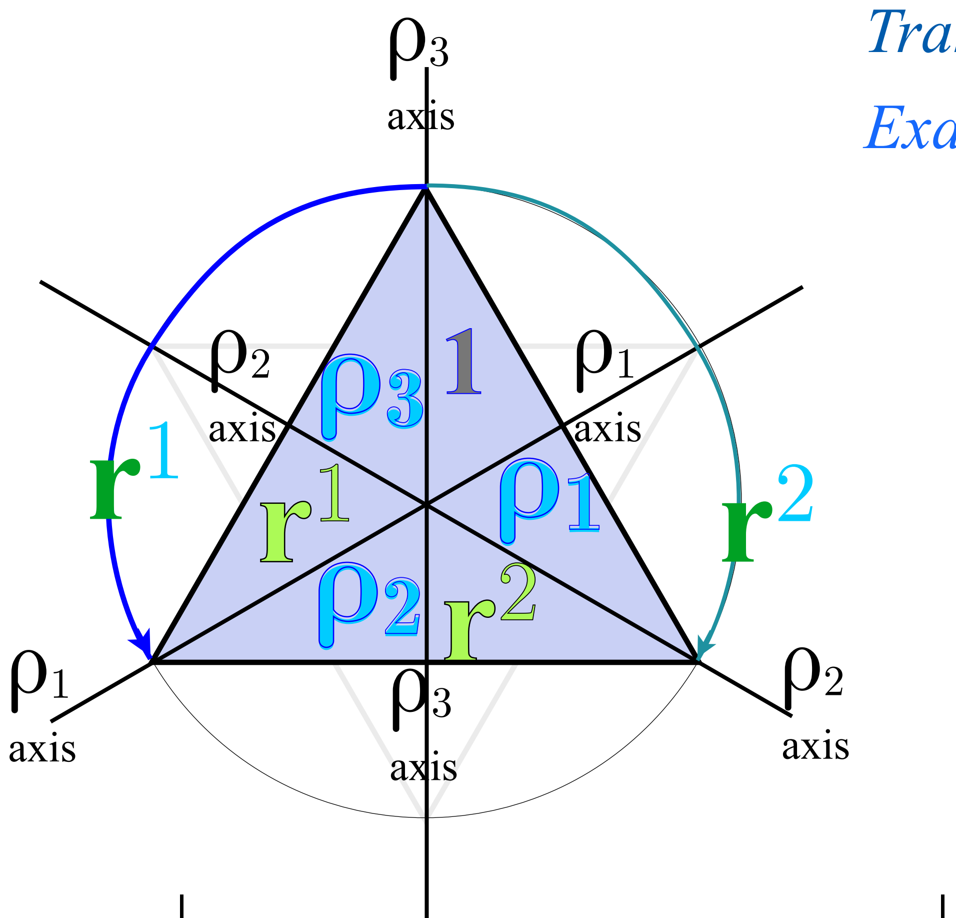
D_3 $\mathbf{gg}^\dagger$ *form*	1	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
1	1	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{r}^1$	$\mathbf{r}^1$	1	$\mathbf{r}^2$	ρ_3	ρ_1	ρ_2
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	1	ρ_2	ρ_3	ρ_1
ρ_1	ρ_1	ρ_3	ρ_2	1	$\mathbf{r}^1$	$\mathbf{r}^2$
ρ_2	ρ_2	$\boxed{\rho_1}$	ρ_3	$\mathbf{r}^2$	1	$\mathbf{r}^1$
ρ_3	ρ_3	ρ_2	ρ_1	$\mathbf{r}^1$	$\mathbf{r}^2$	1

Need to check that with table:

$$\mathbf{r}^1\rho_3\mathbf{r}^2=\rho_2\mathbf{r}^2=\boxed{\rho_1}$$

Checks out!

Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes

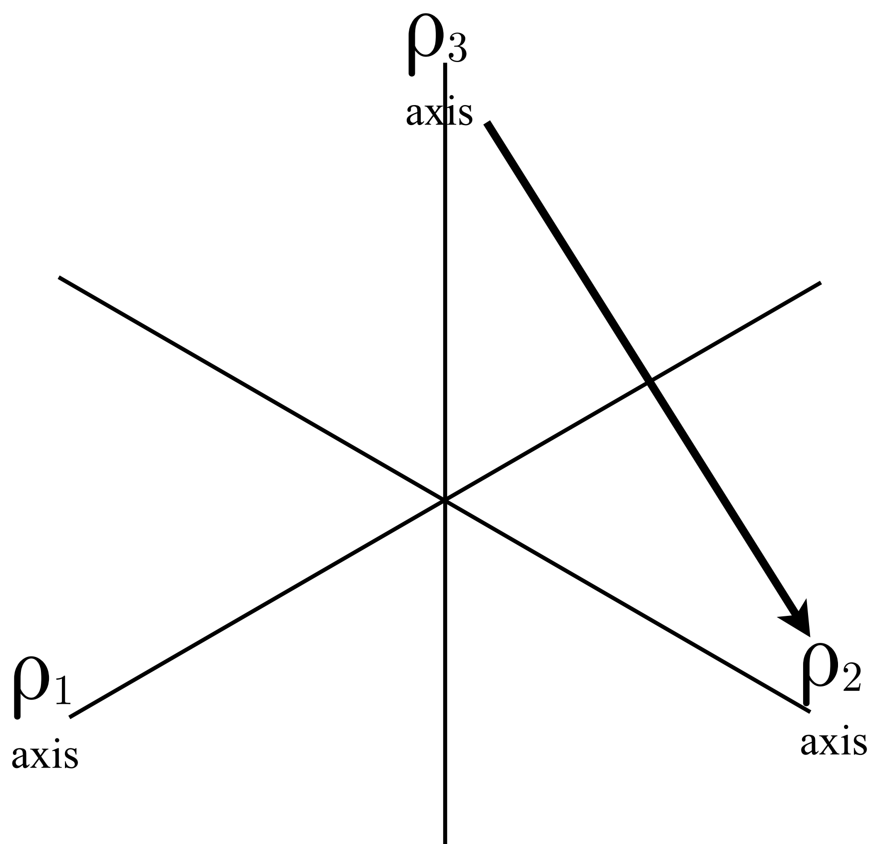


D_3 **gg**† *form*	**1**	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
1	**1**	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{r}^1$	$\mathbf{r}^1$	**1**	$\mathbf{r}^2$	ρ_3	ρ_1	ρ_2
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	**1**	ρ_2	ρ_3	ρ_1
ρ_1	ρ_1	ρ_3	ρ_2	**1**	$\mathbf{r}^1$	$\mathbf{r}^2$
ρ_2	ρ_2	ρ_1	ρ_3	$\mathbf{r}^2$	**1**	$\mathbf{r}^1$
ρ_3	ρ_3	ρ_2	ρ_1	$\mathbf{r}^1$	$\mathbf{r}^2$	**1**

Transforming D_3 operators using D_3 operators

Example 2: Rotating ρ_3 axis crank using ρ_1 puts it down onto ρ_2

Seems to imply: $\rho_1 \rho_3 (\rho_1)^{-1} = \rho_1 \rho_3\ \rho_1 = \rho_2$



Need to check that with table:

$$\rho_1 \rho_3\ \rho_1 = \mathbf{r}^2\ \rho_1 = \rho_2$$

Checks out!

Also:

$$\mathbf{r}^2 \rho_3\ (\mathbf{r}^2)^{-1} = \mathbf{r}^2\ \rho_1 = \rho_2$$

Equivalence transformations and equivalence classes

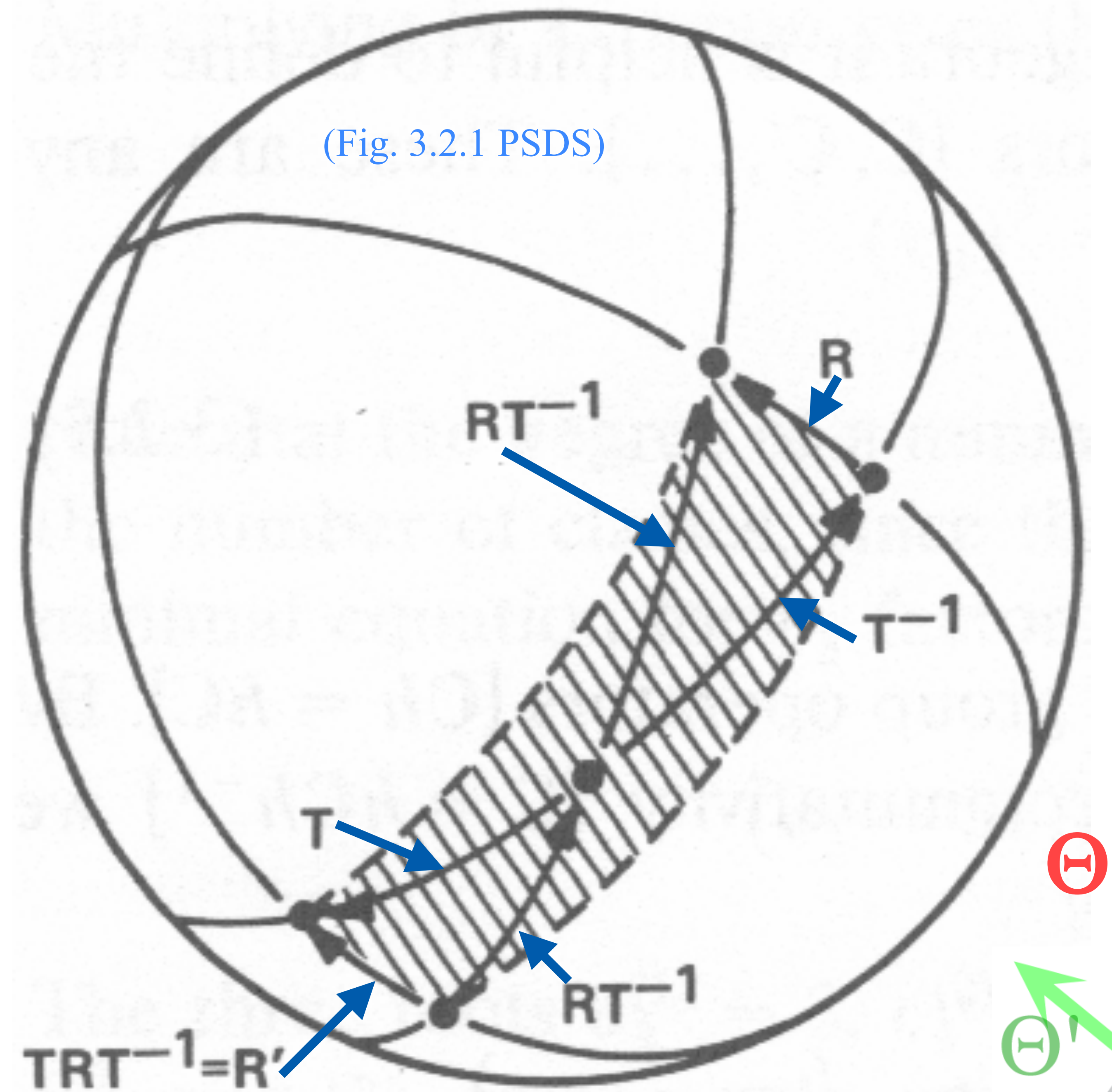
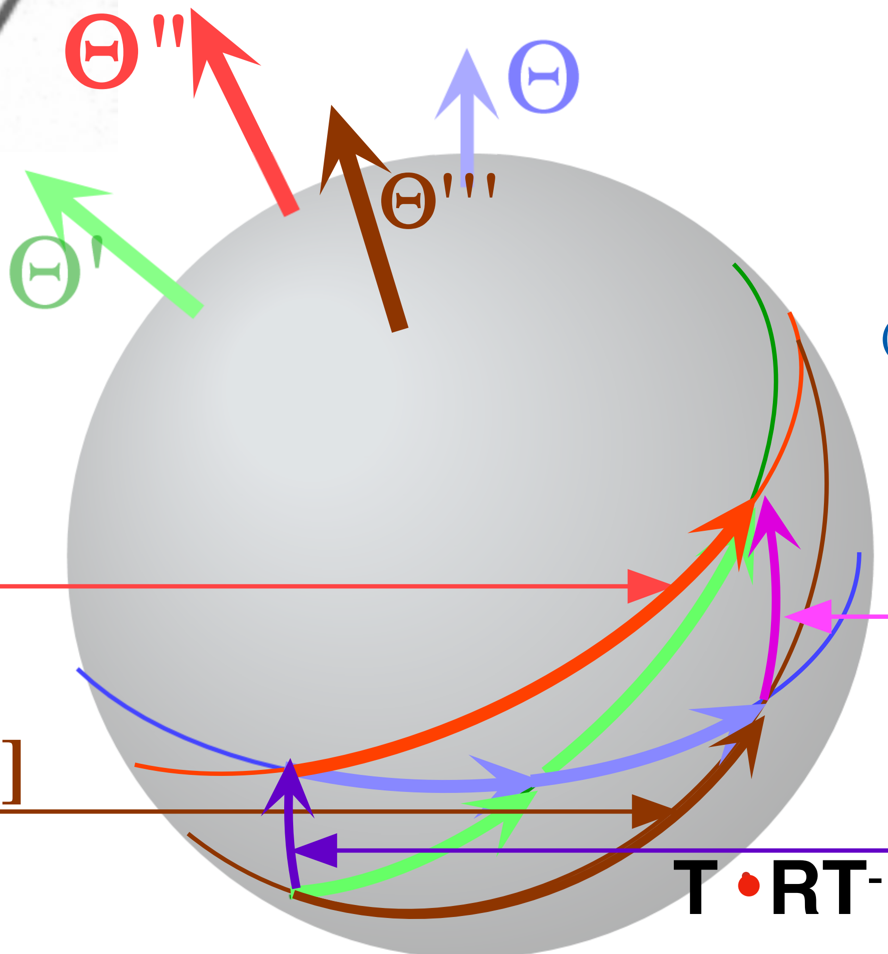


Figure 3.2.1 Showing class equivalence using Hamilton's vectors. Operation R is equivalent to $R' = TRT^{-1}$.

(From GThLect.15 p.36-41)



(Fig. 10.A.9 QTCA)

Product R[Θ'']

$\mathbf{RT^{-1} \bullet T^{-1}} = R[\Theta'] \bullet R[\Theta]$

Product R[Θ''']

$\mathbf{T^{-1} \bullet RT^{-1}} = R[\Theta] \bullet R[\Theta']$

Product

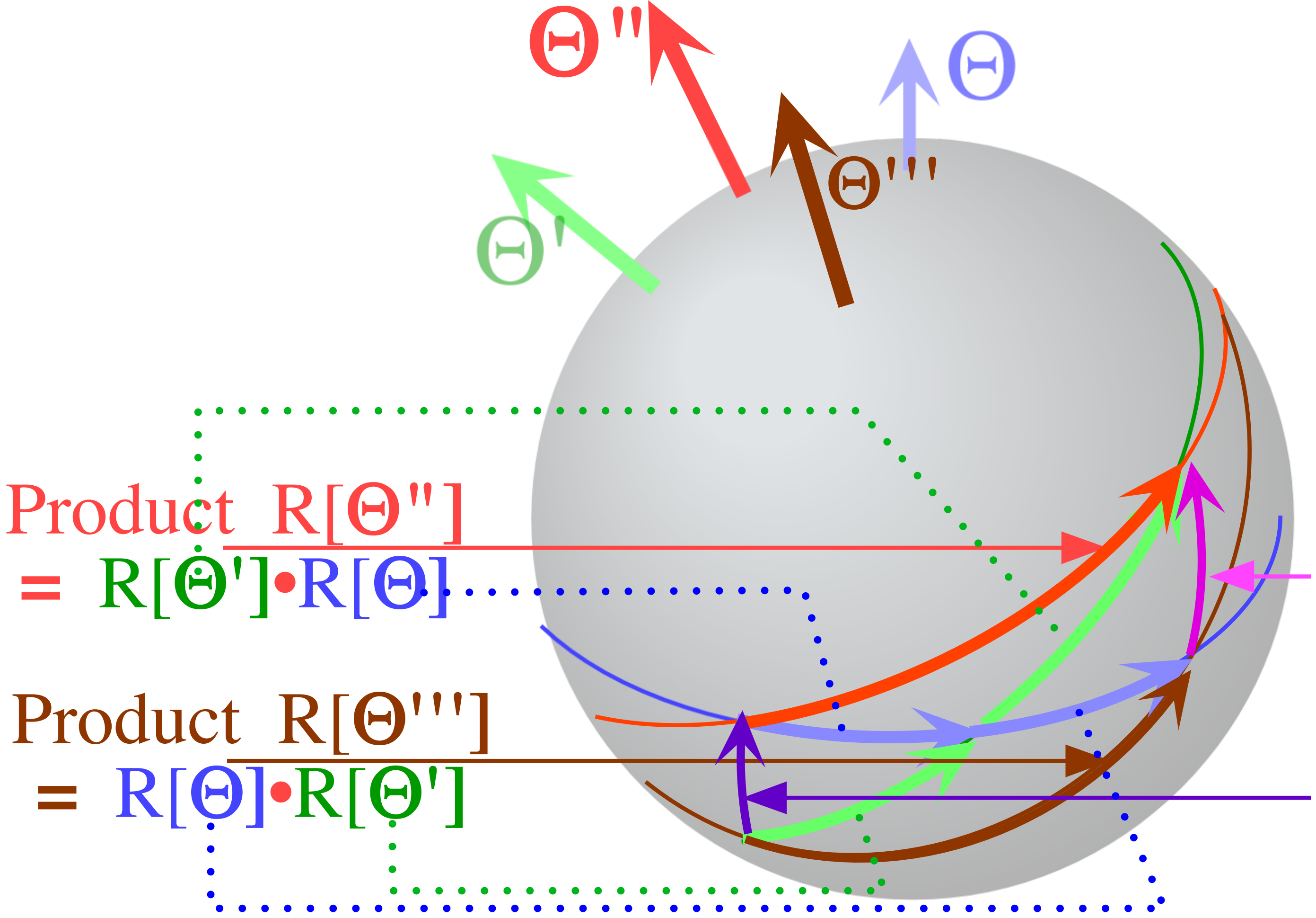
$R[\Theta'] \bullet R^{-1}[\Theta]$

$\mathbf{RT^{-1} \bullet T=R}$

Product

$\mathbf{T \bullet RT^{-1}} = R^{-1}[\Theta] \bullet R[\Theta']$

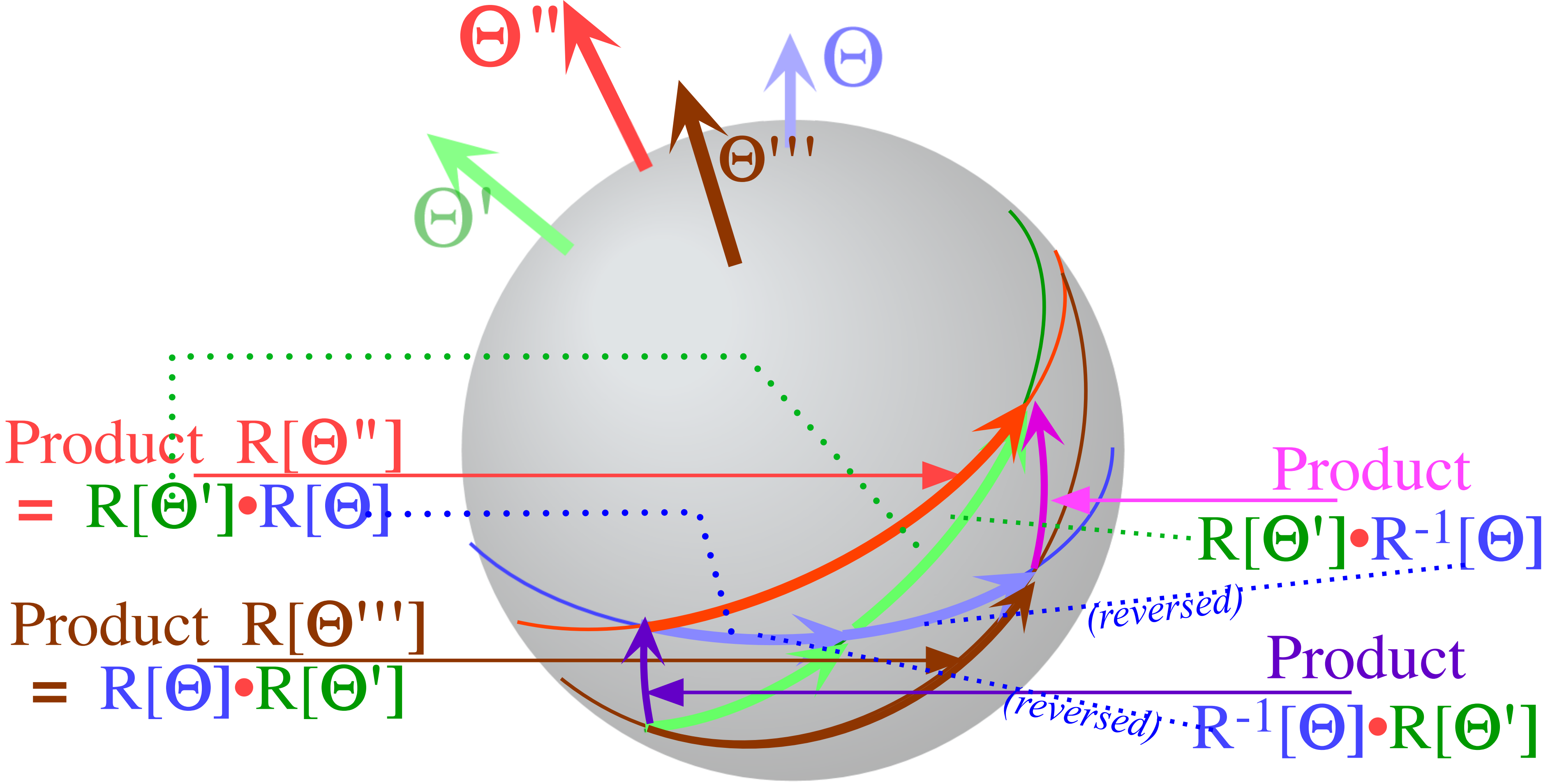
Geometry of transformation of rotational **R[Θ]***-operators*



QTforCA

Fig. 10.A.9 Hamilton-turn arc parallelogram with **R**[Θ"]=**R**[Θ']**R**[Θ] *and* **R**[Θ''']=**R**[Θ]**R**[Θ']

Geometry of transformation of rotational $\mathbf{R}[\Theta]$*-operators*



Product $R[\Theta'']$
$= R[\Theta']\bullet R[\Theta]$

Product $R[\Theta''']$
$= R[\Theta]\bullet R[\Theta']$

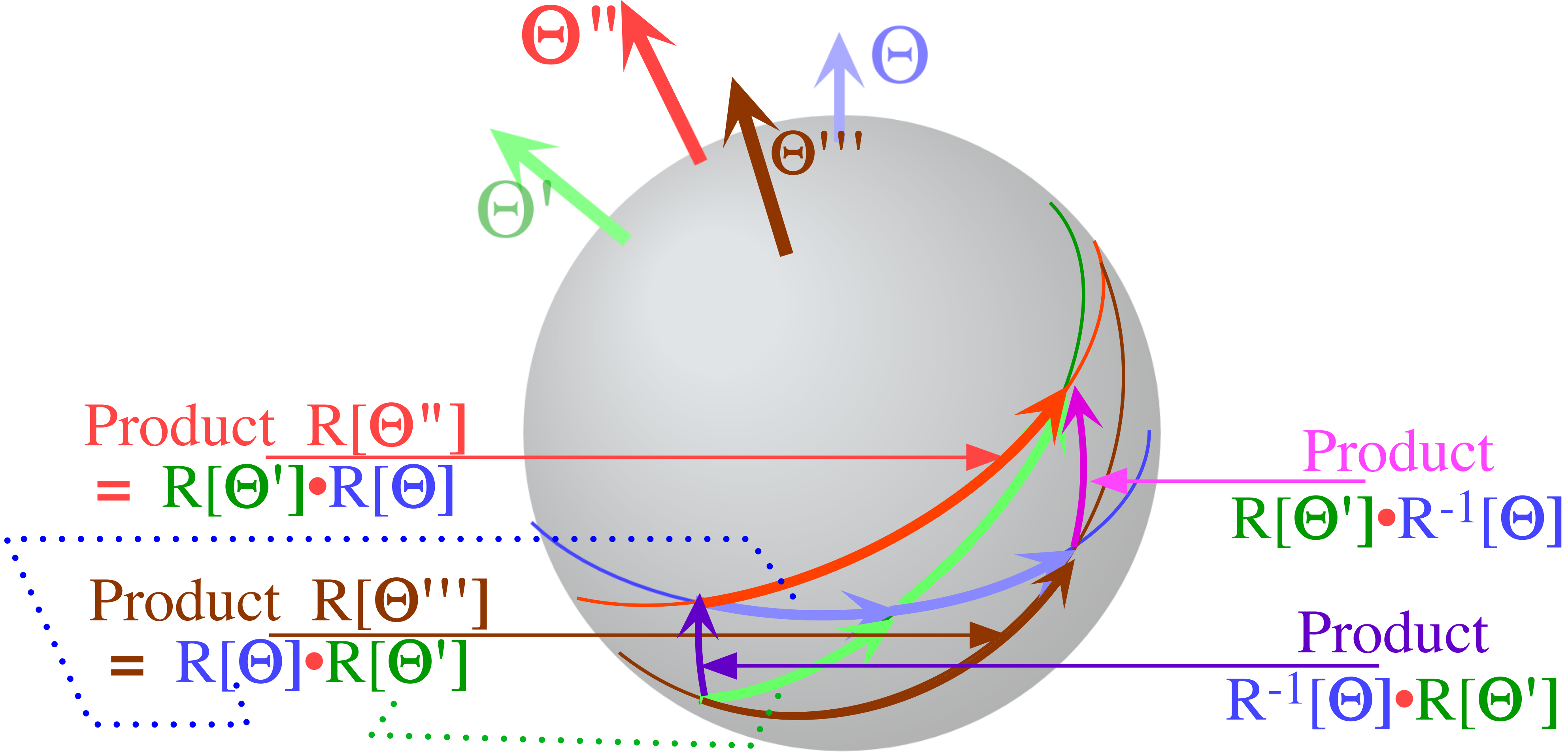
Product
$R[\Theta']\bullet R^{-1}[\Theta]$

Product
$R^{-1}[\Theta]\bullet R[\Theta']$

QTforCA

Fig. 10.A.9 Hamilton-turn arc parallelogram with $\mathbf{R}[\Theta'']=\mathbf{R}[\Theta']\mathbf{R}[\Theta]$ *and* $\mathbf{R}[\Theta''']=\mathbf{R}[\Theta]\mathbf{R}[\Theta']$

Geometry of transformation of rotational **R[Θ]***-operators*



Product $R[\Theta'']$
$= R[\Theta']\bullet R[\Theta]$

Product
$R[\Theta']\bullet R^{-1}[\Theta]$

Product $R[\Theta''']$
$= R[\Theta]\bullet R[\Theta']$

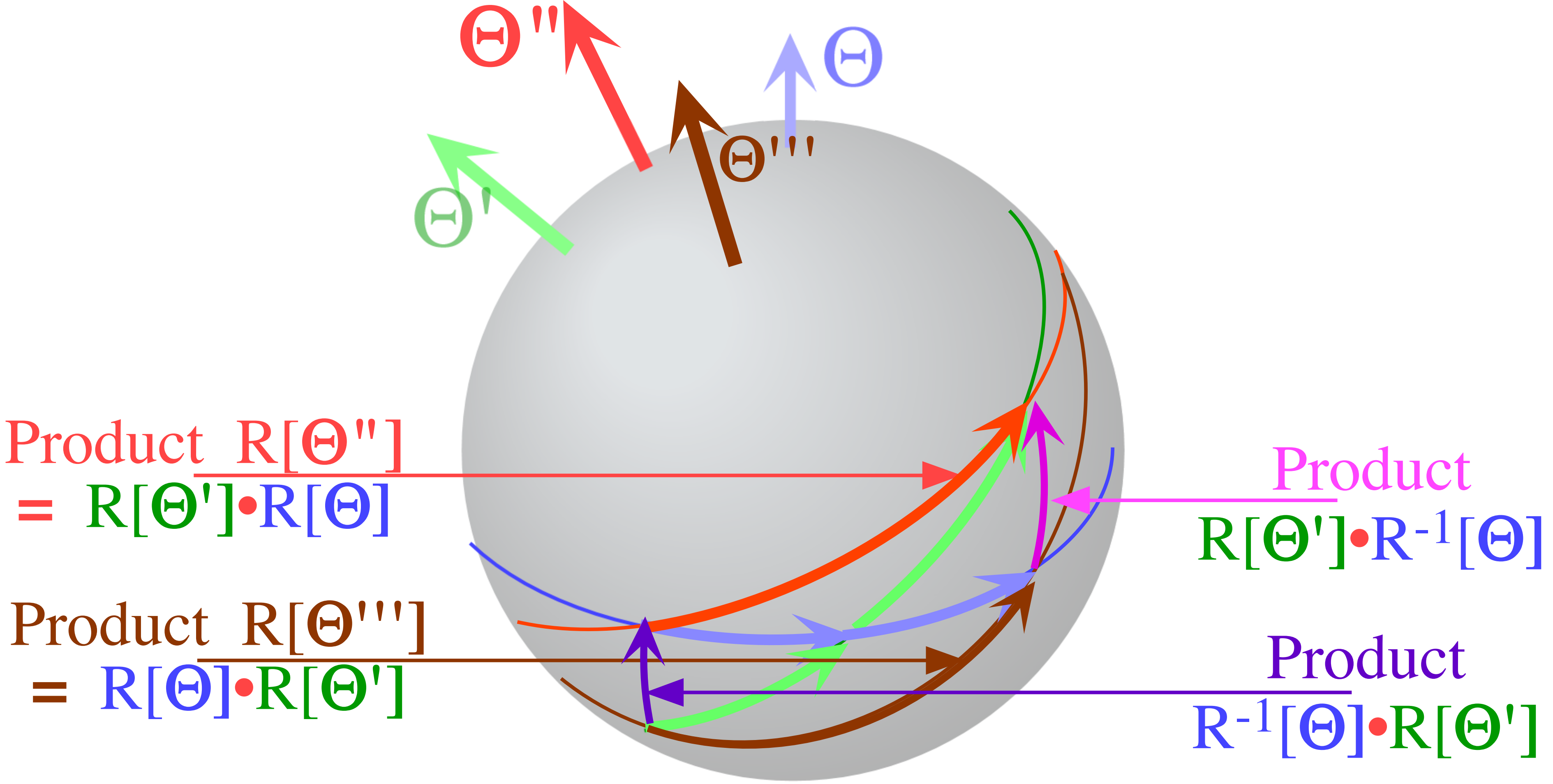
Product
$R^{-1}[\Theta]\bullet R[\Theta']$

QTforCA

Fig. 10.A.9 Hamilton-turn arc parallelogram with $\mathbf{R}[\Theta'']=\mathbf{R}[\Theta']\mathbf{R}[\Theta]$ *and* $\mathbf{R}[\Theta''']=\mathbf{R}[\Theta]\mathbf{R}[\Theta']$

Vectors added in the reverse order give $\mathbf{R}[\Theta''']=\mathbf{R}[\Theta]\mathbf{R}[\Theta']$ instead of $\mathbf{R}[\Theta'']=\mathbf{R}[\Theta']\mathbf{R}[\Theta]$.

Geometry of transformation of rotational **R[Θ]***-operators*



Product R[Θ'']
= R[Θ']•R[Θ]

Product R[Θ''']
= R[Θ]•R[Θ']

Product
R[Θ']•R^{-1}[Θ]

Product
R^{-1}[Θ]•R[Θ']

QTforCA

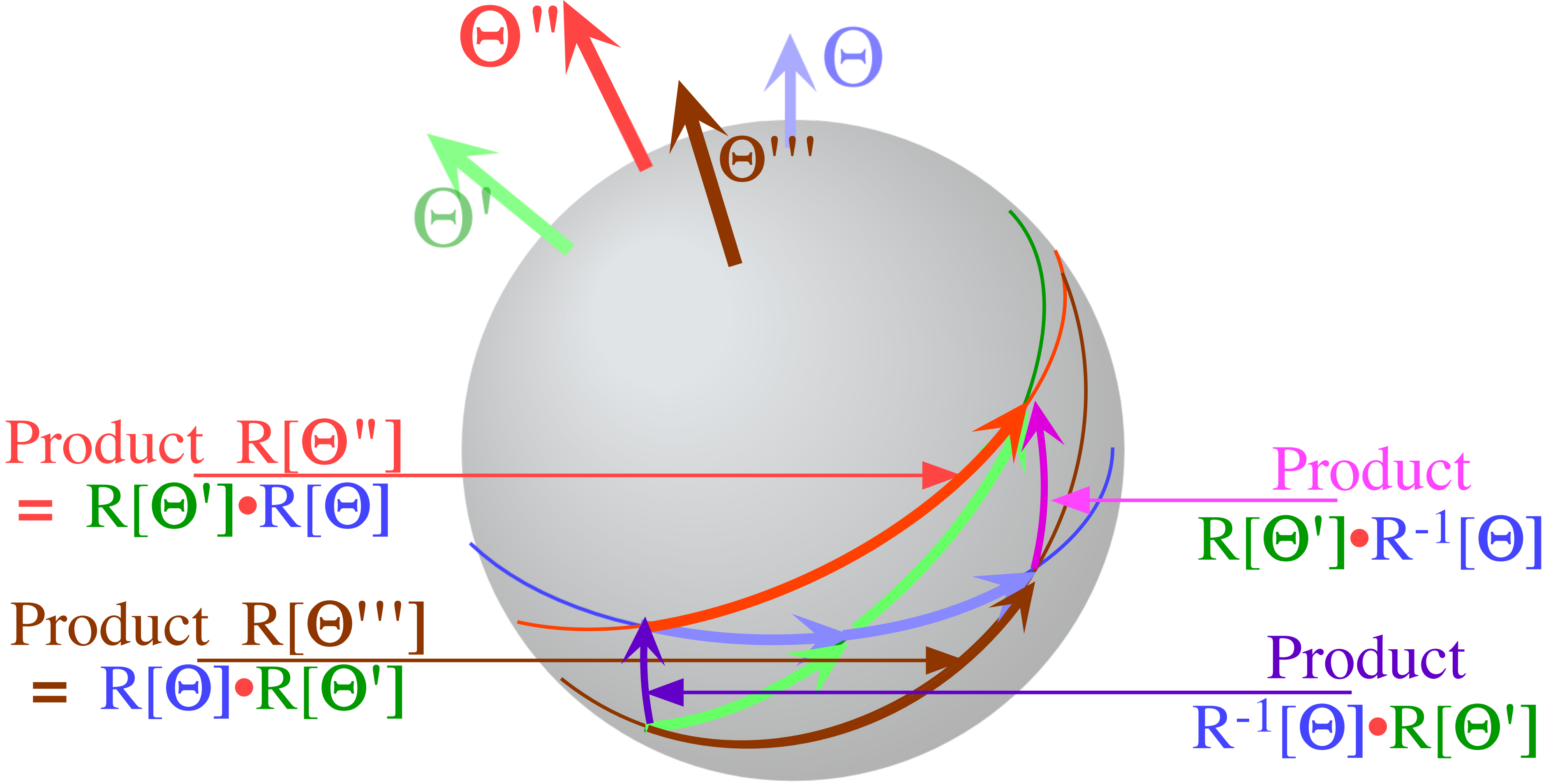
Fig. 10.A.9 Hamilton-turn arc parallelogram with **R**[Θ'']=**R**[Θ']**R**[Θ] *and* **R**[Θ''']=**R**[Θ]**R**[Θ']

Vectors added in the reverse order give **R**[Θ''']=**R**[Θ]**R**[Θ'] instead of **R**[Θ'']=**R**[Θ']**R**[Θ].

A similarity transformation of rotation **R**[Θ''] by rotation **R**[Θ] gives rotation **R**[Θ''']

$$\mathbf{R}[\Theta]\ \underbrace{\mathbf{R}[\Theta'']\ \mathbf{R}[-\Theta]}_{\mathbf{R}[\Theta']} = \mathbf{R}[\Theta''']$$

Geometry of transformation of rotational **R[Θ]***-operators*



Product $R[\Theta'']$
$= R[\Theta']\bullet R[\Theta]$

Product $R[\Theta''']$
$= R[\Theta]\bullet R[\Theta']$

Product
$R[\Theta']\bullet R^{-1}[\Theta]$

Product
$R^{-1}[\Theta]\bullet R[\Theta']$

QTforCA

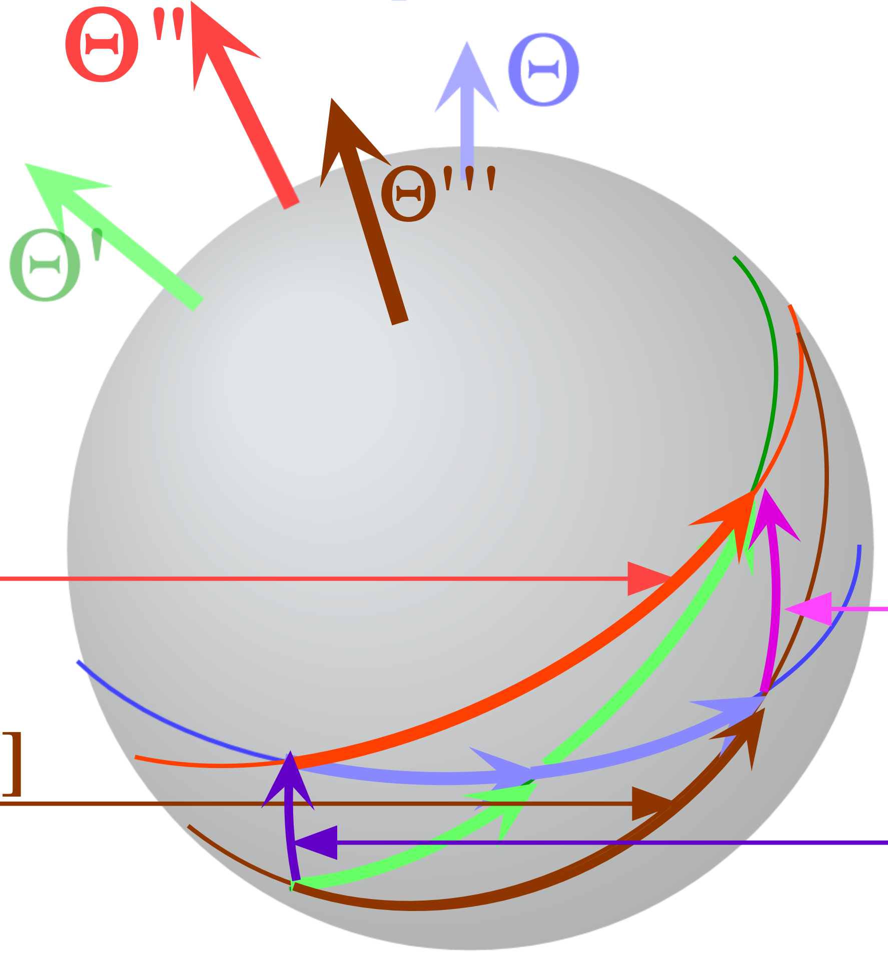
Fig. 10.A.9 Hamilton-turn arc parallelogram with $\mathbf{R}[\Theta'']=\mathbf{R}[\Theta']\mathbf{R}[\Theta]$ *and* $\mathbf{R}[\Theta''']=\mathbf{R}[\Theta]\mathbf{R}[\Theta']$

Vectors added in the reverse order give $\mathbf{R}[\Theta''']=\mathbf{R}[\Theta]\mathbf{R}[\Theta']$ instead of $\mathbf{R}[\Theta'']=\mathbf{R}[\Theta']\mathbf{R}[\Theta]$.

A similarity transformation of rotation $\mathbf{R}[\Theta'']$ by rotation $\mathbf{R}[\Theta]$ gives rotation $\mathbf{R}[\Theta''']$ and *vice-versa*:

$$\mathbf{R}[\Theta]\,\underbrace{\mathbf{R}[\Theta'']\,\mathbf{R}[-\Theta]}_{\mathbf{R}[\Theta']} = \mathbf{R}[\Theta'''] \qquad\qquad \underbrace{\mathbf{R}[-\Theta]\,\mathbf{R}[\Theta''']}_{\mathbf{R}[\Theta']}\,\mathbf{R}[\Theta] = \mathbf{R}[\Theta'']$$

Geometry of transformation of rotational **R[Θ]**-*operators*



Product R[Θ'']
= R[Θ']•R[Θ]

Product
R[Θ']•R^{-1}[Θ]

Product R[Θ''']
= R[Θ]•R[Θ']

Product
R^{-1}[Θ]•R[Θ']

QTforCA

Fig. 10.A.9 Hamilton-turn arc parallelogram with **R**[Θ'']=**R**[Θ']**R**[Θ] *and* **R**[Θ''']=**R**[Θ]**R**[Θ']

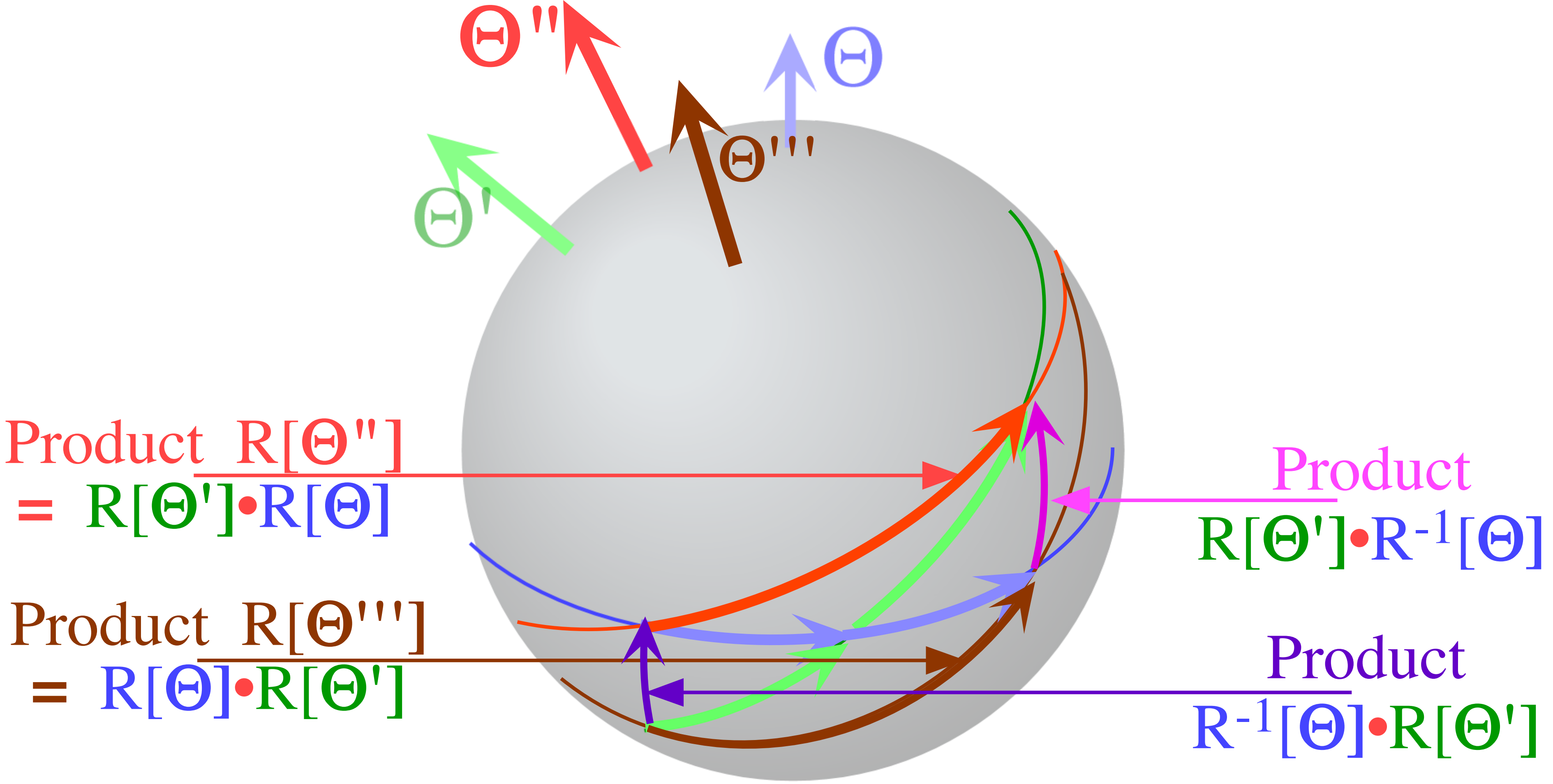
Vectors added in the reverse order give **R**[Θ''']=**R**[Θ]**R**[Θ'] instead of **R**[Θ'']=**R**[Θ']**R**[Θ].

A similarity transformation of rotation **R**[Θ''] by rotation **R**[Θ] gives rotation **R**[Θ'''] and *vice-versa*:

$$\mathbf{R}[\Theta]\,\underbrace{\mathbf{R}[\Theta'']\,\mathbf{R}[-\Theta]}_{\mathbf{R}[\Theta']} = \mathbf{R}[\Theta'''] \qquad \underbrace{\mathbf{R}[-\Theta]\,\mathbf{R}[\Theta''']}_{\mathbf{R}[\Theta']}\,\mathbf{R}[\Theta] = \mathbf{R}[\Theta'']$$

Everything associated with rotation **R**[Θ''] is rotated by full angle Θ around axis Θ.

Geometry of transformation of rotational $\mathbf{R}[\Theta]$*-operators*



Product $\mathrm{R}[\Theta'']$
$= \mathrm{R}[\Theta']\bullet\mathrm{R}[\Theta]$

Product
$\mathrm{R}[\Theta']\bullet\mathrm{R}^{-1}[\Theta]$

Product $\mathrm{R}[\Theta''']$
$= \mathrm{R}[\Theta]\bullet\mathrm{R}[\Theta']$

Product
$\mathrm{R}^{-1}[\Theta]\bullet\mathrm{R}[\Theta']$

QTforCA

Fig. 10.A.9 Hamilton-turn arc parallelogram with $\mathbf{R}[\Theta'']=\mathbf{R}[\Theta']\mathbf{R}[\Theta]$ *and* $\mathbf{R}[\Theta''']=\mathbf{R}[\Theta]\mathbf{R}[\Theta']$

Vectors added in the reverse order give $\mathbf{R}[\Theta''']=\mathbf{R}[\Theta]\mathbf{R}[\Theta']$ instead of $\mathbf{R}[\Theta'']=\mathbf{R}[\Theta']\mathbf{R}[\Theta]$.

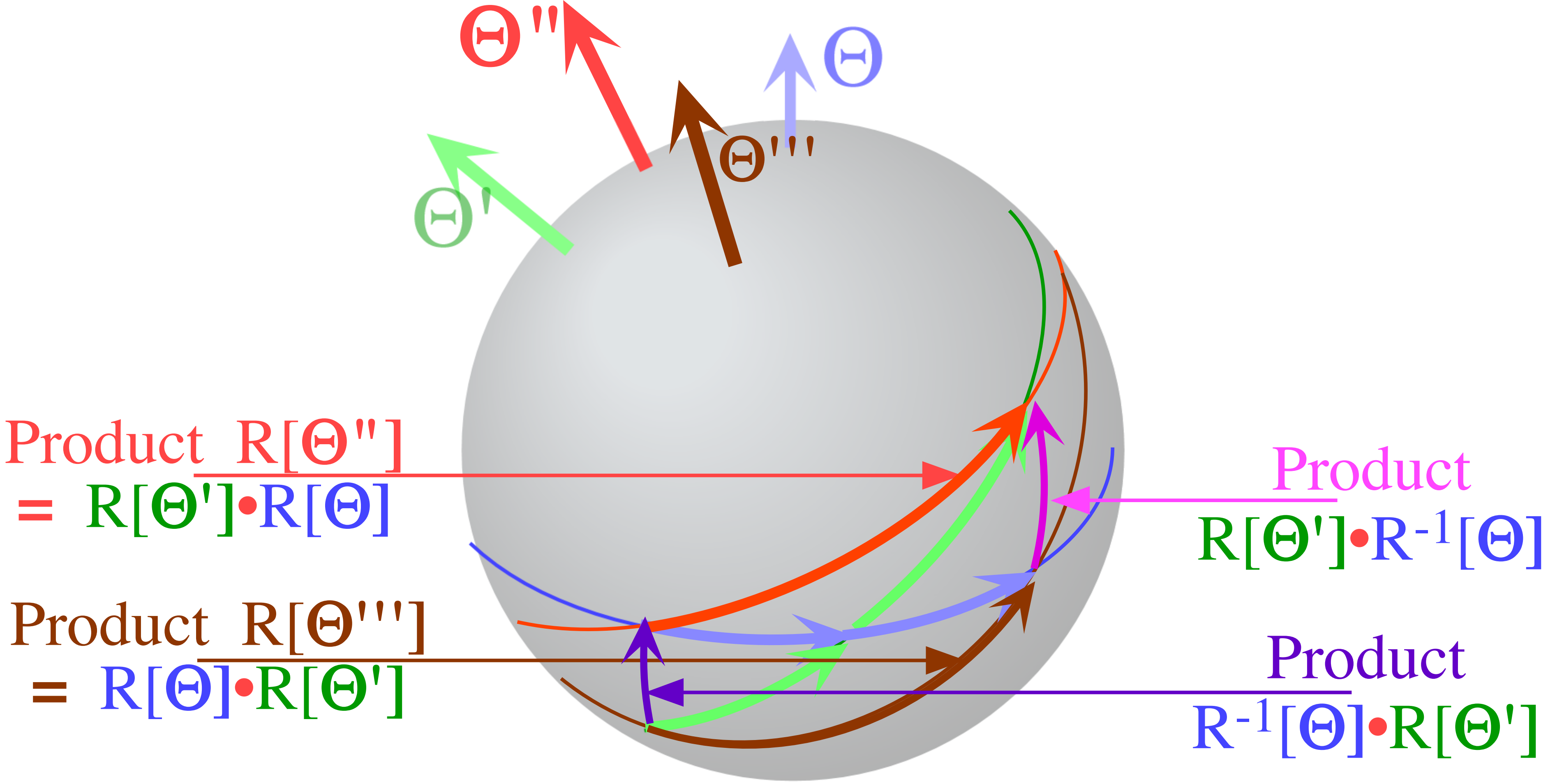
A similarity transformation of rotation $\mathbf{R}[\Theta'']$ by rotation $\mathbf{R}[\Theta]$ gives rotation $\mathbf{R}[\Theta''']$ and *vice-versa*:

$$\mathbf{R}[\Theta]\,\underbrace{\mathbf{R}[\Theta'']\,\mathbf{R}[-\Theta]}_{\mathbf{R}[\Theta']} = \mathbf{R}[\Theta'''] \qquad \underbrace{\mathbf{R}[-\Theta]\,\mathbf{R}[\Theta''']}_{\mathbf{R}[\Theta']}\,\mathbf{R}[\Theta] = \mathbf{R}[\Theta'']$$

Everything associated with rotation $\mathbf{R}[\Theta'']$ is rotated by full angle Θ around axis Θ.

Crank vector Θ and its turn arc moved by <u>two</u> $\mathbf{R}[\Theta]$ turn arcs into turn arc of $\mathbf{R}[\Theta''']$ below it.

Geometry of transformation of rotational **R[Θ]***-operators*



QTforCA

Fig. 10.A.9 Hamilton-turn arc parallelogram with **R**[Θ'']=**R**[Θ']**R**[Θ] *and* **R**[Θ''']=**R**[Θ]**R**[Θ']

Vectors added in the reverse order give **R**[Θ''']=**R**[Θ]**R**[Θ'] instead of **R**[Θ'']=**R**[Θ']**R**[Θ].

A similarity transformation of rotation **R**[Θ''] by rotation **R**[Θ] gives rotation **R**[Θ'''] and *vice-versa*:

$$\mathbf{R}[\Theta]\,\underbrace{\mathbf{R}[\Theta'']\,\mathbf{R}[-\Theta]}_{\mathbf{R}[\Theta']} = \mathbf{R}[\Theta'''] \qquad \underbrace{\mathbf{R}[-\Theta]\,\mathbf{R}[\Theta''']}_{\mathbf{R}[\Theta']}\,\mathbf{R}[\Theta] = \mathbf{R}[\Theta'']$$

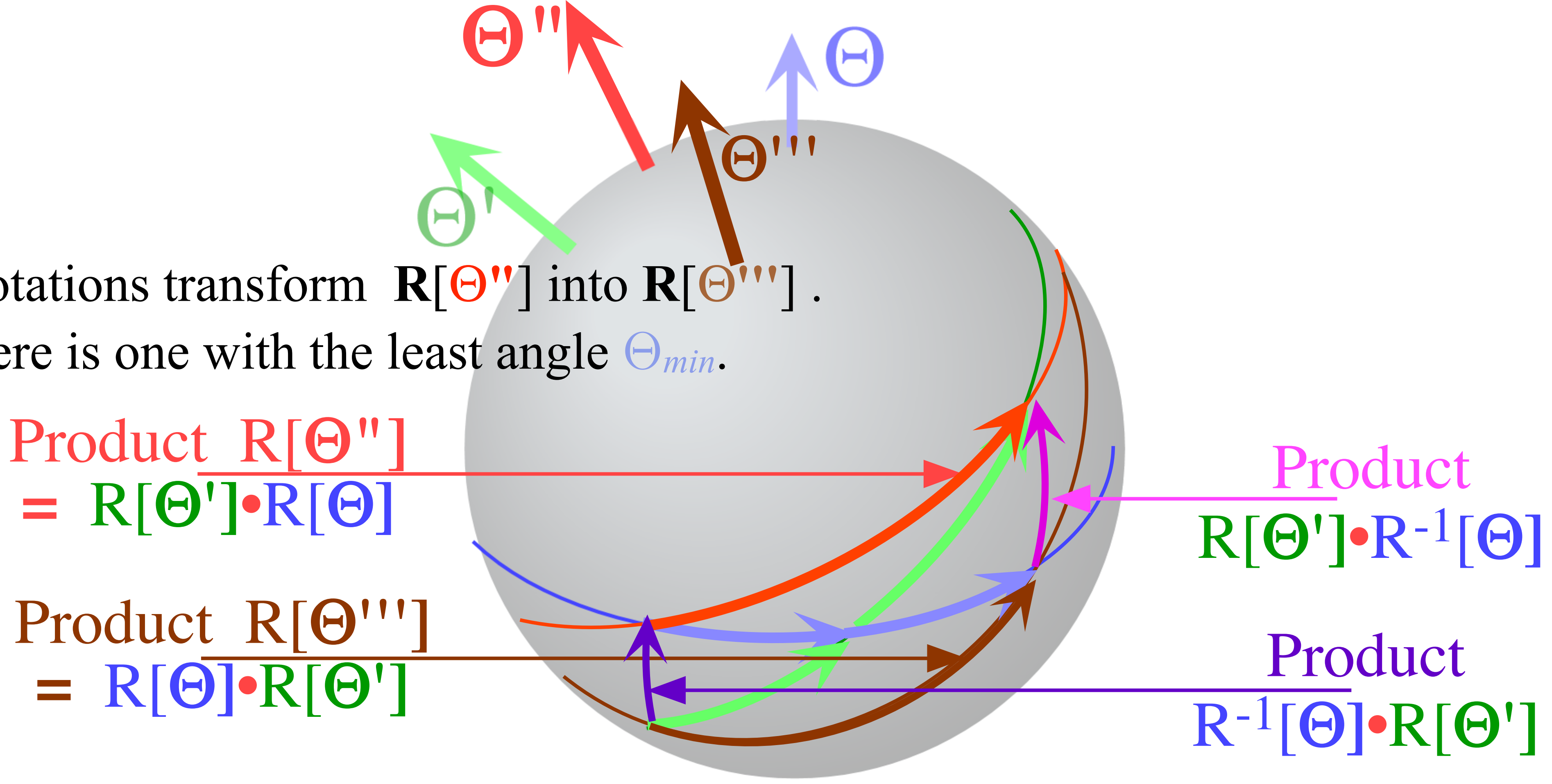
Everything associated with rotation **R**[Θ''] is rotated by full angle Θ around axis Θ.

Crank vector Θ and its turn arc moved by two **R**[Θ] turn arcs into turn arc of **R**[Θ'''] below it.

Another similarity transformation of rotation **R**[Θ'''] by rotation **R**[Θ'] to **R**[Θ'']

$$\mathbf{R}[\Theta']\,\underbrace{\mathbf{R}[\Theta''']\,\mathbf{R}[-\Theta']}_{\mathbf{R}[\Theta]} = \mathbf{R}[\Theta'']$$

Geometry of transformation of rotational **R[Θ]***-operators*



Many (∞) rotations transform **R**[Θ''] into **R**[Θ'''] .
Of these, there is one with the least angle Θ_{min}.

Product R[Θ'']
= R[Θ']•R[Θ]

Product R[Θ''']
= R[Θ]•R[Θ']

Product
R[Θ']•R^{-1}[Θ]

Product
R^{-1}[Θ]•R[Θ']

QTforCA

Fig. 10.A.9 Hamilton-turn arc parallelogram with **R**[Θ'']=**R**[Θ']**R**[Θ] *and* **R**[Θ''']=**R**[Θ]**R**[Θ']

Vectors added in the reverse order give **R**[Θ''']=**R**[Θ]**R**[Θ'] instead of **R**[Θ'']=**R**[Θ']**R**[Θ].

A similarity transformation of rotation **R**[Θ''] by rotation **R**[Θ] gives rotation **R**[Θ'''] and *vice-versa*:

$$\mathbf{R}[\Theta]\,\underbrace{\mathbf{R}[\Theta'']\,\mathbf{R}[-\Theta]}_{\mathbf{R}[\Theta']} = \mathbf{R}[\Theta'''] \qquad \underbrace{\mathbf{R}[-\Theta]\,\mathbf{R}[\Theta''']}_{\mathbf{R}[\Theta']}\,\mathbf{R}[\Theta] = \mathbf{R}[\Theta'']$$

Everything associated with rotation **R**[Θ''] is rotated by full angle Θ around axis Θ.

Crank vector Θ and its turn arc moved by <u>two</u> **R**[Θ] turn arcs into turn arc of **R**[Θ'''] below it.

Another similarity transformation of rotation **R**[Θ'''] by rotation **R**[Θ'] to **R**[Θ'']

$$\mathbf{R}[\Theta']\,\underbrace{\mathbf{R}[\Theta''']\,\mathbf{R}[-\Theta']}_{\mathbf{R}[\Theta]} = \mathbf{R}[\Theta''] \qquad \underbrace{\mathbf{R}[-\Theta']\,\mathbf{R}[\Theta'']}_{\mathbf{R}[\Theta]}\,\mathbf{R}[\Theta'] = \mathbf{R}[\Theta''']$$

AMOP
reference links
on following page

1.31.18 class 6.0: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

Symmetry group $\mathscr{G}$ representations=>*AMOP* Hamiltonian **H** (or K) matrices, irreps $\mathscr{D}^{(\alpha)}$
=>*AMOP* wave functions $\Psi^{(\alpha)}$, eigensolutions

$\mathscr{G}$= U(2) product $\mathbf{R}[\Theta]\mathbf{R}[\Theta']=\mathbf{R}[\Theta''']$ algebra (It's all done with σ_μ spinors)

Jordan-Pauli identity: U(2) product algebra of spinor σ_μ-operators

U(2) "Crazy-Thing" forms do products $\mathbf{R}[\Theta]\mathbf{R}[\Theta']=\mathbf{R}[\Theta''']$ *algebraically*

$\mathscr{G}$= U(2) product $\mathbf{R}[\Theta]\mathbf{R}[\Theta']=\mathbf{R}[\Theta''']$ by geometry (It's all done with σ_μ mirrors)

Mirror reflections by σ_μ-operators make rotations *The famous Clothing Store Mirror*

Hamilton-turns do products $\mathbf{R}[\Theta]\mathbf{R}[\Theta']=\mathbf{R}[\Theta''']$ *geometrically*

Hamilton-turn slide rule and sundial *U(2) products and* (α,β,γ)-$[\varphi,\vartheta,\Theta]$ *conversions*

Finite group products by turns or by group link diagrams *D_3 example.* *O_h example*

$\mathscr{G}$= U(2) class equivalence transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^{-1}=\mathbf{R}[\Theta''']$ geometry

Group equivalence classes

U(2) density operator $\boldsymbol{\rho}$ and $[\boldsymbol{\rho},\mathbf{H}]$ mechanics

Density mechanics compared to spin vector **S** *rotated by crank vector* $\boldsymbol{\Theta}=\boldsymbol{\Omega}t$

Bloch equation $i\hbar\dot{\rho}=\left[\mathbf{H},\rho\right]$

U(2) density operator approach to symmetry dynamics

Euler phase-angle coordinates (α, β, γ)
and ***norm*** *N of quantum state* $|\Psi\rangle$

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \sqrt{N}\begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix} = \sqrt{N}\begin{pmatrix} e^{-i\alpha/2}\cos\beta/2 \\ e^{i\alpha/2}\sin\beta/2 \end{pmatrix} e^{-i\gamma/2}$$

$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1=-\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2=\cos[(\gamma-\alpha)/2]\sin\beta/2$
$p_2=-\sin[(\gamma-\alpha)/2]\sin\beta/2$

1/2 times $\boldsymbol{\sigma}$*-operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle = N = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N\left(\underbrace{p_1^2+x_1^2+p_2^2+x_2^2}_{4D\text{-}norm=1}\right)$$

scaled by $\frac{1}{2}$:

$$\frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right) = \frac{N}{2}$$

$$\langle\Psi|\sigma_Z|\Psi\rangle = 2S_A = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N\left(p_1^2+x_1^2-p_2^2-x_2^2\right)$$

scaled by $\frac{1}{2}$:

$$S_Z = S_A = \frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right) = \frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right) = \frac{N}{2}\cos\beta$$

U(2) density operator approach to symmetry dynamics

$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1=-\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2=\cos[(\gamma-\alpha)/2]\sin\beta/2$
$p_2=-\sin[(\gamma-\alpha)/2]\sin\beta/2$

Euler phase-angle coordinates (α,β,γ)
and ***norm*** *N of quantum state $|\Psi\rangle$*

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\beta/2\\ e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$

1/2 times **σ***-operator expectation values* $\langle\Psi|\boldsymbol{\sigma}_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle = N = \begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = N\underbrace{\left(p_1{}^2+x_1{}^2+p_2{}^2+x_2{}^2\right)}_{4D\text{-}norm=1}$$

scaled by $\frac{1}{2}$:

$$\frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right)=\frac{N}{2}$$

$$\langle\Psi|\sigma_Z|\Psi\rangle = 2S_A = \begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = N\left(p_1{}^2+x_1{}^2-p_2{}^2-x_2{}^2\right)$$

scaled by $\frac{1}{2}$:

$$S_Z=S_A=\frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right)=\frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)=\frac{N}{2}\cos\beta$$

$$\langle\Psi|\sigma_X|\Psi\rangle = 2S_B = \begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = 2N\left(x_1x_2+p_1p_2\right)$$

scaled by $\frac{1}{2}$:

$$S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2 = N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{N}{2}\cos\alpha\,\sin\beta$$

U(2) density operator approach to symmetry dynamics

x_1=cos[(γ+α)/2]cosβ/2
p_1=-sin[(γ+α)/2]cosβ/2

x_2=cos[(γ–α)/2]sinβ/2
p_2=-sin[(γ–α)/2]sinβ/2

Euler phase-angle coordinates (α, β, γ)
and norm N of quantum state |Ψ⟩

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \sqrt{N}\begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix} = \sqrt{N}\begin{pmatrix} e^{-i\alpha/2}\cos\beta/2 \\ e^{i\alpha/2}\sin\beta/2 \end{pmatrix} e^{-i\gamma/2}$$

1/2 times **σ**-*operator expectation values* ⟨Ψ|σ$_\mu$|Ψ⟩ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle = N = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N\left(\underbrace{p_1^2+x_1^2+p_2^2+x_2^2}_{4D\text{-}norm=1}\right)$$

scaled by $\frac{1}{2}$:

$$\frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right) = \frac{N}{2}$$

$$\langle\Psi|\sigma_Z|\Psi\rangle = 2S_A = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N\left(p_1^2+x_1^2-p_2^2-x_2^2\right)$$

scaled by $\frac{1}{2}$:

$$S_Z = S_A = \frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right) = \frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right) = \frac{N}{2}\cos\beta$$

$$\langle\Psi|\sigma_X|\Psi\rangle = 2S_B = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = 2N\left(x_1x_2+p_1p_2\right)$$

scaled by $\frac{1}{2}$:

$$S_X = S_B = \mathrm{Re}\,\Psi_1^*\Psi_2 = N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{N}{2}\cos\alpha\,\sin\beta$$

$$\langle\Psi|\sigma_Y|\Psi\rangle = 2S_C = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = 2N\left(x_1p_2-x_2p_1\right)$$

scaled by $\frac{1}{2}$:

$$S_Y = S_C = \mathrm{Im}\,\Psi_1^*\Psi_2 = N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{N}{2}\sin\alpha\,\sin\beta$$

U(2) density operator approach to symmetry dynamics

x_1=cos[(γ+α)/2]cosβ/2
p_1=-sin[(γ+α)/2]cosβ/2

x_2=cos[(γ–α)/2]sinβ/2
p_2=-sin[(γ–α)/2]sinβ/2

Euler phase-angle coordinates (α, β, γ) *and norm N of quantum state* $|\Psi\rangle$

$$|\Psi\rangle = \begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = \sqrt{N}\begin{pmatrix} x_1+ip_1\\ x_2+ip_2\end{pmatrix} = \sqrt{N}\begin{pmatrix} e^{-i\alpha/2}\cos\beta/2 \\ e^{i\alpha/2}\sin\beta/2\end{pmatrix} e^{-i\gamma/2}$$

1/2 times **σ**-*operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle = N = \begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = N\underbrace{\left(p_1{}^2+x_1{}^2+p_2{}^2+x_2{}^2\right)}_{4D\text{-}norm=1} \quad \textit{scaled by } \tfrac{1}{2}: \quad \frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right) = \frac{N}{2}$$

$$\langle\Psi|\sigma_Z|\Psi\rangle = 2S_A = \begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = N\left(p_1{}^2+x_1{}^2-p_2{}^2-x_2{}^2\right) \quad \textit{scaled by } \tfrac{1}{2}: \quad S_Z = S_A = \frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right) = \frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right) = \frac{N}{2}\cos\beta$$

$$\langle\Psi|\sigma_X|\Psi\rangle = 2S_B = \begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = 2N\left(x_1x_2+p_1p_2\right) \quad \textit{scaled by } \tfrac{1}{2}: \quad S_X = S_B = \mathrm{Re}\,\Psi_1^*\Psi_2 = N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{N}{2}\cos\alpha\,\sin\beta$$

$$\langle\Psi|\sigma_Y|\Psi\rangle = 2S_C = \begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = 2N\left(x_1p_2-x_2p_1\right) \quad \textit{scaled by } \tfrac{1}{2}: \quad S_Y = S_C = \mathrm{Im}\,\Psi_1^*\Psi_2 = N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{N}{2}\sin\alpha\,\sin\beta$$

The *density operator* $\rho = |\Psi\rangle\langle\Psi| =$

$$\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}\otimes\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix} = \begin{pmatrix}\Psi_1\Psi_1^* & \Psi_1\Psi_2^*\\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^*\end{pmatrix} = \begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix} = \begin{pmatrix}\Psi_1^*\Psi_1 & \Psi_2^*\Psi_1\\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2\end{pmatrix}$$

U(2) density operator approach to symmetry dynamics

$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1=-\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2=\cos[(\gamma-\alpha)/2]\sin\beta/2$
$p_2=-\sin[(\gamma-\alpha)/2]\sin\beta/2$

Euler phase-angle coordinates (α, β, γ)
and norm N of quantum state $|\Psi\rangle$

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\beta/2\\ e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$

1/2 times **σ***-operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle= N =\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\underbrace{\left(p_1{}^2+x_1{}^2+p_2{}^2+x_2{}^2\right)}_{4D\text{-}norm=1}$$

scaled by $\frac{1}{2}$: $\frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right)=\frac{N}{2}$

$$\langle\Psi|\sigma_Z|\Psi\rangle=2S_A=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\left(p_1{}^2+x_1{}^2-p_2{}^2-x_2{}^2\right)$$

scaled by $\frac{1}{2}$: $S_Z=S_A=\frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right)=\frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)=\frac{N}{2}\cos\beta$

$$\langle\Psi|\sigma_X|\Psi\rangle=2S_B=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1x_2+p_1p_2\right)$$

scaled by $\frac{1}{2}$: $S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2\quad=N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}\quad=\frac{N}{2}\cos\alpha\,\sin\beta$

$$\langle\Psi|\sigma_Y|\Psi\rangle=2S_C=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1p_2-x_2p_1\right)$$

scaled by $\frac{1}{2}$: $S_Y=S_C=\mathrm{Im}\,\Psi_1^*\Psi_2\quad=N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}\quad=\frac{N}{2}\sin\alpha\,\sin\beta$

The *density operator* ρ = |Ψ⟩⟨Ψ|= $\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}\otimes\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}=\begin{pmatrix}\Psi_1\Psi_1^* & \Psi_1\Psi_2^*\\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^*\end{pmatrix}=\begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix}=\begin{pmatrix}\Psi_1^*\Psi_1 & \Psi_2^*\Psi_1\\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2\end{pmatrix}$

$\rho_{11}=\Psi_1^*\Psi_1$ $=\frac{1}{2}N+S_Z$	$\rho_{12}=\Psi_2^*\Psi_1$ $=S_X-iS_Y,$
$\rho_{21}=\Psi_1^*\Psi_2$ $=S_X+iS_Y$	$\rho_{22}=\Psi_2^*\Psi_2$ $=\frac{1}{2}N-S_Z$

$$=\begin{pmatrix}\frac{1}{2}N+S_Z & S_X-iS_Y\\ S_X+iS_Y & \frac{1}{2}N-S_Z\end{pmatrix}$$

Norm: $N=\Psi1^*\Psi1+\Psi2^*\Psi2$

...2-by-2 *density operator* ρ

U(2) density operator approach to symmetry dynamics

$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1=-\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2=\cos[(\gamma-\alpha)/2]\sin\beta/2$
$p_2=-\sin[(\gamma-\alpha)/2]\sin\beta/2$

Euler phase-angle coordinates (α,β,γ)
and norm N *of quantum state* $|\Psi\rangle$

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\beta/2\\ e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$

1/2 times **σ**-*operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle=N=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\underbrace{\left(p_1{}^2+x_1{}^2+p_2{}^2+x_2{}^2\right)}_{4D\text{-}norm=1}$$

scaled by $\frac{1}{2}$: $\frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right)=\frac{N}{2}$

$$\langle\Psi|\sigma_Z|\Psi\rangle=2S_A=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\left(p_1{}^2+x_1{}^2-p_2{}^2-x_2{}^2\right)$$

scaled by $\frac{1}{2}$: $S_Z=S_A=\frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right)=\frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)=\frac{N}{2}\cos\beta$

$$\langle\Psi|\sigma_X|\Psi\rangle=2S_B=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1x_2+p_1p_2\right)$$

scaled by $\frac{1}{2}$: $S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2=N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\cos\alpha\,\sin\beta$

$$\langle\Psi|\sigma_Y|\Psi\rangle=2S_C=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1p_2-x_2p_1\right)$$

scaled by $\frac{1}{2}$: $S_Y=S_C=\mathrm{Im}\,\Psi_1^*\Psi_2=N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\sin\alpha\,\sin\beta$

The *density operator* ρ = |Ψ⟩⟨Ψ|=

$$\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}\otimes\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}=\begin{pmatrix}\Psi_1\Psi_1^* & \Psi_1\Psi_2^*\\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^*\end{pmatrix}=\begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix}=\begin{pmatrix}\Psi_1^*\Psi_1 & \Psi_2^*\Psi_1\\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2\end{pmatrix}$$

$\rho_{11}=\Psi_1^*\Psi_1$ $=\frac{1}{2}N+S_Z$	$\rho_{12}=\Psi_2^*\Psi_1$ $=S_X-iS_Y,$
$\rho_{21}=\Psi_1^*\Psi_2$ $=S_X+iS_Y$	$\rho_{22}=\Psi_2^*\Psi_2$ $=\frac{1}{2}N-S_Z$

Norm: $N=\Psi_1^*\Psi_1+\Psi_2^*\Psi_2$ ←

$$=\underset{\substack{\uparrow\\ \rho}}{\begin{pmatrix}\frac{1}{2}N+S_Z & S_X-iS_Y\\ S_X+iS_Y & \frac{1}{2}N-S_Z\end{pmatrix}}=\frac{1}{2}N\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+S_X\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}+S_Y\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}+S_Z\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

...so state *density operator* **ρ** has **σ**-expansion

U(2) density operator approach to symmetry dynamics

$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1=-\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2=\cos[(\gamma-\alpha)/2]\sin\beta/2$
$p_2=-\sin[(\gamma-\alpha)/2]\sin\beta/2$

Euler phase-angle coordinates (α,β,γ)
and norm N of quantum state $|\Psi\rangle$

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\beta/2\\ e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$

1/2 times $\boldsymbol{\sigma}$*-operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle=N=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\underbrace{\left(p_1^2+x_1^2+p_2^2+x_2^2\right)}_{4D\text{-}norm=1}$$

scaled by $\frac{1}{2}$: $\frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right)=\frac{N}{2}$

$$\langle\Psi|\sigma_Z|\Psi\rangle=2S_A=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\left(p_1^2+x_1^2-p_2^2-x_2^2\right)$$

scaled by $\frac{1}{2}$: $S_Z=S_A=\frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right)=\frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)=\frac{N}{2}\cos\beta$

$$\langle\Psi|\sigma_X|\Psi\rangle=2S_B=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1x_2+p_1p_2\right)$$

scaled by $\frac{1}{2}$: $S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2=N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\cos\alpha\sin\beta$

$$\langle\Psi|\sigma_Y|\Psi\rangle=2S_C=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1p_2-x_2p_1\right)$$

scaled by $\frac{1}{2}$: $S_Y=S_C=\mathrm{Im}\,\Psi_1^*\Psi_2=N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\sin\alpha\sin\beta$

The *density operator* $\rho=|\Psi\rangle\langle\Psi|=$

$$\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}\otimes\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}=\begin{pmatrix}\Psi_1\Psi_1^* & \Psi_1\Psi_2^*\\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^*\end{pmatrix}=\begin{pmatrix}\rho_{11}&\rho_{12}\\ \rho_{21}&\rho_{22}\end{pmatrix}=\begin{pmatrix}\Psi_1^*\Psi_1 & \Psi_2^*\Psi_1\\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2\end{pmatrix}$$

$\rho_{11}=\Psi_1^*\Psi_1$ $=\frac{1}{2}N+S_Z$	$\rho_{12}=\Psi_2^*\Psi_1$ $=S_X-iS_Y$,
$\rho_{21}=\Psi_1^*\Psi_2$ $=S_X+iS_Y$	$\rho_{22}=\Psi_2^*\Psi_2$ $=\frac{1}{2}N-S_Z$

Norm: $N=\Psi_1^*\Psi_1+\Psi_2^*\Psi_2$ ←

$$=\begin{pmatrix}\frac{1}{2}N+S_Z & S_X-iS_Y\\ S_X+iS_Y & \frac{1}{2}N-S_Z\end{pmatrix}=\frac{1}{2}N\begin{pmatrix}1&0\\0&1\end{pmatrix}+S_X\underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}}+S_Y\underbrace{\begin{pmatrix}0&-i\\i&0\end{pmatrix}}+S_Z\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}$$

$$\uparrow\ \rho=\frac{1}{2}N\ \mathbf{1}+S_X\ \sigma_X+S_Y\ \sigma_Y+S_Z\ \sigma_Z=\frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

...so state *density operator* $\boldsymbol{\rho}$ has $\boldsymbol{\sigma}$-expansion

U(2) density operator approach to symmetry dynamics

x_1=cos[(γ+α)/2]cosβ/2
p_1=-sin[(γ+α)/2]cosβ/2

x_2=cos[(γ−α)/2]sinβ/2
p_2=-sin[(γ−α)/2]sinβ/2

Euler phase-angle coordinates (α, β, γ) *and norm N of quantum state* $|\Psi\rangle$

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\beta/2\\ e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$

1/2 times **σ***-operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle = N = \begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = N\underbrace{\left(p_1^2+x_1^2+p_2^2+x_2^2\right)}_{4D\text{-}norm=1}$$

scaled by $\frac{1}{2}$:

$$\frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right)=\frac{N}{2}$$

$$\langle\Psi|\sigma_Z|\Psi\rangle = 2S_A = \begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = N\left(p_1^2+x_1^2-p_2^2-x_2^2\right)$$

scaled by $\frac{1}{2}$:

$$S_Z=S_A=\frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right)=\frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)=\frac{N}{2}\cos\beta$$

$$\langle\Psi|\sigma_X|\Psi\rangle = 2S_B = \begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = 2N\left(x_1x_2+p_1p_2\right)$$

scaled by $\frac{1}{2}$:

$$S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2 \quad = N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} \quad =\frac{N}{2}\cos\alpha\,\sin\beta$$

$$\langle\Psi|\sigma_Y|\Psi\rangle = 2S_C = \begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = 2N\left(x_1p_2-x_2p_1\right)$$

scaled by $\frac{1}{2}$:

$$S_Y=S_C=\mathrm{Im}\,\Psi_1^*\Psi_2 \quad = N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} \quad =\frac{N}{2}\sin\alpha\,\sin\beta$$

The *density operator* $\rho=|\Psi\rangle\langle\Psi|=$

$$\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}\otimes\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}=\begin{pmatrix}\Psi_1\Psi_1^* & \Psi_1\Psi_2^*\\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^*\end{pmatrix}=\begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix}=\begin{pmatrix}\Psi_1^*\Psi_1 & \Psi_2^*\Psi_1\\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2\end{pmatrix}$$

$\rho_{11}=\Psi_1^*\Psi_1$ $=\frac{1}{2}N+S_Z$	$\rho_{12}=\Psi_2^*\Psi_1$ $=S_X-iS_Y$,
$\rho_{21}=\Psi_1^*\Psi_2$ $=S_X+iS_Y$	$\rho_{22}=\Psi_2^*\Psi_2$ $=\frac{1}{2}N-S_Z$

Norm: $N=\Psi1^*\Psi1+\Psi2^*\Psi2$

$$=\begin{pmatrix}\frac{1}{2}N+S_Z & S_X-iS_Y\\ S_X+iS_Y & \frac{1}{2}N-S_Z\end{pmatrix}=\frac{1}{2}N\begin{pmatrix}1&0\\0&1\end{pmatrix}+S_X\underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}}+S_Y\underbrace{\begin{pmatrix}0&-i\\i&0\end{pmatrix}}+S_Z\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}$$

$$\uparrow\ \rho \quad = \frac{1}{2}N\ \mathbf{1}\ +S_X\ \sigma_X\ +S_Y\ \sigma_Y\ +S_Z\ \sigma_Z\ =\frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

...so state *density operator* **ρ** has **σ**-expansion like *Hamiltonian operator* **H**

$$\begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix}=\mathbf{H}=\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{A-D}{2}\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}+B\underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}}+C\underbrace{\begin{pmatrix}0&-i\\i&0\end{pmatrix}}$$

$$\mathbf{H}=\ \omega_0\ \sigma_0\ +\frac{\Omega_A}{2}\ \sigma_A\ +\frac{\Omega_B}{2}\ \sigma_B\ +\frac{\Omega_C}{2}\ \sigma_C\ =\omega_0\sigma_0+\frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

U(2) density operator approach to symmetry dynamics

$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1=-\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2=\cos[(\gamma-\alpha)/2]\sin\beta/2$
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$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\beta/2\\ e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$

1/2 times $\boldsymbol{\sigma}$*-operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle=N=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\underbrace{\left(p_1^2+x_1^2+p_2^2+x_2^2\right)}_{4D\text{-}norm=1}\quad \textit{scaled by } \tfrac{1}{2}:\quad \frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right)=\frac{N}{2}$$

$$\langle\Psi|\sigma_Z|\Psi\rangle=2S_A=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\left(p_1^2+x_1^2-p_2^2-x_2^2\right)\quad \textit{scaled by } \tfrac{1}{2}:\quad S_Z=S_A=\frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right)=\frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)=\frac{N}{2}\cos\beta$$

$$\langle\Psi|\sigma_X|\Psi\rangle=2S_B=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1x_2+p_1p_2\right)\quad \textit{scaled by } \tfrac{1}{2}:\quad S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2=N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\cos\alpha\sin\beta$$

$$\langle\Psi|\sigma_Y|\Psi\rangle=2S_C=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1p_2-x_2p_1\right)\quad \textit{scaled by } \tfrac{1}{2}:\quad S_Y=S_C=\mathrm{Im}\,\Psi_1^*\Psi_2=N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\sin\alpha\sin\beta$$

The *density operator* $\rho=|\Psi\rangle\langle\Psi|=$

$$\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}\otimes\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}=\begin{pmatrix}\Psi_1\Psi_1^* & \Psi_1\Psi_2^*\\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^*\end{pmatrix}=\begin{pmatrix}\rho_{11}&\rho_{12}\\ \rho_{21}&\rho_{22}\end{pmatrix}=\begin{pmatrix}\Psi_1^*\Psi_1 & \Psi_2^*\Psi_1\\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2\end{pmatrix}$$

$\rho_{11}=\Psi_1^*\Psi_1$ $=\frac{1}{2}N+S_Z$	$\rho_{12}=\Psi_2^*\Psi_1$ $=S_X-iS_Y,$
$\rho_{21}=\Psi_1^*\Psi_2$ $=S_X+iS_Y$	$\rho_{22}=\Psi_2^*\Psi_2$ $=\frac{1}{2}N-S_Z$

Norm: $N=\Psi 1^*\Psi 1+\Psi 2^*\Psi 2$

$$=\begin{pmatrix}\frac{1}{2}N+S_Z & S_X-iS_Y\\ S_X+iS_Y & \frac{1}{2}N-S_Z\end{pmatrix}=\frac{1}{2}N\begin{pmatrix}1&0\\0&1\end{pmatrix}+S_X\underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}}+S_Y\underbrace{\begin{pmatrix}0&-i\\i&0\end{pmatrix}}+S_Z\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}$$

$$\uparrow\ \boldsymbol{\rho}\quad=\frac{1}{2}N\ \mathbf{1}\ +S_X\ \boldsymbol{\sigma}_X\ +S_Y\ \boldsymbol{\sigma}_Y\ +S_Z\ \boldsymbol{\sigma}_Z\ =\frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

...so state *density operator* $\boldsymbol{\rho}$ has $\boldsymbol{\sigma}$-expansion like *Hamiltonian operator* **H**

$$\begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix}=\mathbf{H}=\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{A-D}{2}\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}+B\underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}}+C\underbrace{\begin{pmatrix}0&-i\\i&0\end{pmatrix}}$$

$$\mathbf{H}=\ \omega_0\ \sigma_0\ +\frac{\Omega_A}{2}\sigma_A\ +\frac{\Omega_B}{2}\sigma_B\ +\frac{\Omega_C}{2}\sigma_C\ =\omega_0\sigma_0+\frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H}=\ \Omega_0\ \mathbf{1}\ +\Omega_A\ \mathbf{S}_A\ +\Omega_B\ \mathbf{S}_B\ +\Omega_C\mathbf{S}_C\ =\Omega_0\mathbf{1}+\vec{\Omega}\bullet\mathbf{S}$$

$$\boxed{\begin{aligned}\boldsymbol{\rho}&=\frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\\ \mathbf{H}&=\Omega_0\mathbf{1}+\frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}\end{aligned}}$$

AMOP
reference links
on following page

1.31.18 class 6.0: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

Symmetry group $\mathscr{G}$ representations=>*AMOP* Hamiltonian **H** (or K) matrices, irreps $\mathscr{D}^{(\alpha)}$
=>*AMOP* wave functions $\Psi^{(\alpha)}$, eigensolutions

$\mathscr{G}$= U(2) product $\mathbf{R}[\boldsymbol{\Theta}]\mathbf{R}[\boldsymbol{\Theta}']=\mathbf{R}[\boldsymbol{\Theta}''']$ algebra (It's all done with σ_μ spinors)

Jordan-Pauli identity: U(2) product algebra of spinor σ_μ-operators

U(2) "Crazy-Thing" forms do products $\mathbf{R}[\boldsymbol{\Theta}]\mathbf{R}[\boldsymbol{\Theta}']=\mathbf{R}[\boldsymbol{\Theta}''']$ *algebraically*

$\mathscr{G}$= U(2) product $\mathbf{R}[\boldsymbol{\Theta}]\mathbf{R}[\boldsymbol{\Theta}']=\mathbf{R}[\boldsymbol{\Theta}''']$ by geometry (It's all done with σ_μ mirrors)

Mirror reflections by σ_μ-operators make rotations *The famous Clothing Store Mirror*

Hamilton-turns do products $\mathbf{R}[\boldsymbol{\Theta}]\mathbf{R}[\boldsymbol{\Theta}']=\mathbf{R}[\boldsymbol{\Theta}''']$ *geometrically*

Hamilton-turn slide rule and sundial *U(2) products and* (α,β,γ)-$[\varphi,\vartheta,\Theta]$ *conversions*

Finite group products by turns or by group link diagrams *D_3 example.* *O_h example*

$\mathscr{G}$= U(2) class transformation $\mathbf{R}[\boldsymbol{\Theta}]\mathbf{R}[\boldsymbol{\Theta}']\mathbf{R}[\boldsymbol{\Theta}]^{-1}=\mathbf{R}[\boldsymbol{\Theta}''']$ geometry

Group classes and subgroup cosets

U(2) density operator $\boldsymbol{\rho}$ and $[\boldsymbol{\rho},\mathbf{H}]$ mechanics

➡ *Density mechanics compared to spin vector* **S** *rotated by crank vector* $\boldsymbol{\Theta}=\boldsymbol{\Omega}t$

Bloch equation $i\hbar\dot{\rho}=[\mathbf{H},\rho]$

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\boldsymbol{\rho} = \frac{1}{2} N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

Ket equation (time *forward*) and "daggered" bra-equation (time *reversed*).

$$i\hbar\left|\dot{\Psi}\right\rangle = \mathbf{H}\left|\Psi\right\rangle, \qquad \Leftarrow \mathit{Daggar}^{\dagger} \Rightarrow \qquad -i\hbar\left\langle\dot{\Psi}\right| = \left\langle\Psi\right|\mathbf{H}$$

Note: $\mathbf{H}^{\dagger} = \mathbf{H}$.

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\sigma$$

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$$i\hbar\left|\dot{\Psi}\right\rangle = \mathbf{H}\left|\Psi\right\rangle, \quad \Leftarrow Daggar^{\dagger} \Rightarrow \quad -i\hbar\left\langle\dot{\Psi}\right| = \left\langle\Psi\right|\mathbf{H}$$

Note: $\mathbf{H}^{\dagger} = \mathbf{H}$.
$\rho^{\dagger} = \rho$

Combining these gives a time derivative of the density operator $\rho = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right| + i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right| = \mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\boldsymbol{\rho} = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

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Note: $\mathbf{H}^{\dagger} = \mathbf{H}$.
$\boldsymbol{\rho}^{\dagger} = \boldsymbol{\rho}$

Combining these gives a time derivative of the density operator $\boldsymbol{\rho} = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho} = i\hbar\dot{\boldsymbol{\rho}} = i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right| + i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right| = \mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

The result is called a *Bloch equation*.

$$i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho} = i\hbar\dot{\boldsymbol{\rho}} = \mathbf{H}\boldsymbol{\rho} - \boldsymbol{\rho}\mathbf{H} = \left[\mathbf{H},\boldsymbol{\rho}\right]$$

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\rho = \frac{1}{2} N\mathbf{1} + \vec{\mathbf{S}}\bullet\sigma$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\sigma$$

Ket equation (time *forward*) and "daggered" bra-equation (time *reversed*).

$$i\hbar\left|\dot{\Psi}\right\rangle = \mathbf{H}\left|\Psi\right\rangle, \qquad \Leftarrow Daggar^{\dagger} \Rightarrow \qquad -i\hbar\left\langle\dot{\Psi}\right| = \left\langle\Psi\right|\mathbf{H}$$

Note: $\mathbf{H}^{\dagger} = \mathbf{H}$.
$\rho^{\dagger} = \rho$

Combining these gives a time derivative of the density operator $\rho = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right| + i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right| = \mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

The result is called a *Bloch equation*.

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = \mathbf{H}\rho - \rho\mathbf{H} = \left[\mathbf{H},\rho\right]$$

Given ρ and **H** in terms *spin* **S**-vector and *crank* **Ω**-vector:

$$\mathbf{H}\rho = \left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\Omega}\bullet\sigma\right)\left(\frac{N}{2}\mathbf{1} + \vec{S}\bullet\sigma\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\Omega}\bullet\sigma + \hbar\Omega_0\vec{S}\bullet\sigma + \frac{\hbar}{2}\left(\vec{\Omega}\bullet\sigma\right)\left(\vec{S}\bullet\sigma\right)$$

$$-\rho\mathbf{H} = \left(\frac{N}{2}\mathbf{1} + \vec{S}\bullet\sigma\right)\left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\Omega}\bullet\sigma\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\Omega}\bullet\sigma + \hbar\Omega_0\vec{S}\bullet\sigma + \frac{\hbar}{2}\left(\vec{S}\bullet\sigma\right)\left(\vec{\Omega}\bullet\sigma\right)$$

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\rho = \frac{1}{2} N\mathbf{1} + \vec{\mathbf{S}} \bullet \sigma$$

$$\mathbf{H} = \Omega_0 \mathbf{1} + \frac{\vec{\mathbf{\Omega}}}{2} \bullet \sigma$$

Ket equation (time *forward*) and "daggered" bra-equation (time *reversed*).

$$i\hbar\left|\dot{\Psi}\right\rangle = \mathbf{H}\left|\Psi\right\rangle, \qquad \Leftarrow \textit{Daggar}^{\dagger} \Rightarrow \qquad -i\hbar\left\langle\dot{\Psi}\right| = \left\langle\Psi\right|\mathbf{H}$$

Note: $\mathbf{H}^{\dagger} = \mathbf{H}$.
$\rho^{\dagger} = \rho$

Combining these gives a time derivative of the density operator $\rho = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right| + i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right| = \mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

The result is called a *Bloch equation*.

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = \mathbf{H}\rho - \rho\mathbf{H} = \left[\mathbf{H},\rho\right]$$

Given ρ and **H** in terms *spin* **S**-vector and *crank* **Ω**-vector:

$$\mathbf{H}\rho = \left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\mathbf{\Omega}}\bullet\sigma\right)\left(\frac{N}{2}\mathbf{1} + \vec{\mathbf{S}}\bullet\sigma\right) = \cancel{\hbar\Omega_0\frac{N}{2}\mathbf{1}} + \cancel{\frac{N}{4}\hbar\vec{\mathbf{\Omega}}\bullet\sigma} + \cancel{\hbar\Omega_0\vec{\mathbf{S}}\bullet\sigma} + \frac{\hbar}{2}\left(\vec{\mathbf{\Omega}}\bullet\sigma\right)\left(\vec{\mathbf{S}}\bullet\sigma\right)$$

$$-\rho\mathbf{H} = \left(\frac{N}{2}\mathbf{1} + \vec{\mathbf{S}}\bullet\sigma\right)\left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\mathbf{\Omega}}\bullet\sigma\right) = \cancel{\hbar\Omega_0\frac{N}{2}\mathbf{1}} + \cancel{\frac{N}{4}\hbar\vec{\mathbf{\Omega}}\bullet\sigma} + \cancel{\hbar\Omega_0\vec{\mathbf{S}}\bullet\sigma} + \frac{\hbar}{2}\left(\vec{\mathbf{S}}\bullet\sigma\right)\left(\vec{\mathbf{\Omega}}\bullet\sigma\right)$$

Last terms don't cancel if the *spin* **S** and *crank* **Ω** *point in different directions*.

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\boldsymbol{\rho} = \frac{1}{2} N\mathbf{1} + \vec{\mathbf{S}} \bullet \boldsymbol{\sigma}$$

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Note: $\mathbf{H}^{\dagger} = \mathbf{H}$.
$\boldsymbol{\rho}^{\dagger} = \boldsymbol{\rho}$

Combining these gives a time derivative of the density operator $\boldsymbol{\rho} = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho} = i\hbar\dot{\boldsymbol{\rho}} = i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right| + i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right| = \mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

The result is called a *Bloch equation*.

$$\boxed{i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho} = i\hbar\dot{\boldsymbol{\rho}} = \mathbf{H}\boldsymbol{\rho} - \boldsymbol{\rho}\mathbf{H} = \left[\mathbf{H}, \boldsymbol{\rho}\right]}$$

$$(\mathbf{A}\bullet\boldsymbol{\sigma})(\mathbf{B}\bullet\boldsymbol{\sigma}) = A_\alpha B_\beta \sigma_\alpha \sigma_\beta = A_\alpha B_\beta\left(\delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma\right)$$

$$= A_\alpha B_\alpha + i\varepsilon_{\alpha\beta\gamma} A_\alpha B_\beta \sigma_\gamma$$

$$= \mathbf{A}\bullet\mathbf{B} + i(\mathbf{A}\times\mathbf{B})\cdot\boldsymbol{\sigma}$$

This cancels *This remains*

Given $\boldsymbol{\rho}$ and $\mathbf{H}$ in terms *spin* $\mathbf{S}$-vector and *crank* $\boldsymbol{\Omega}$-vector:

$$\mathbf{H}\boldsymbol{\rho} = \left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\frac{N}{2}\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\cancel{\frac{N}{2}\mathbf{1}} + \cancel{\frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}} + \cancel{\hbar\Omega_0\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}} + \frac{\hbar}{2}\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)$$

$$-\boldsymbol{\rho}\mathbf{H} = \left(\frac{N}{2}\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)\left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\cancel{\frac{N}{2}\mathbf{1}} + \cancel{\frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}} + \cancel{\hbar\Omega_0\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}} + \frac{\hbar}{2}\left(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)$$

Last terms don't cancel if the *spin* $\mathbf{S}$ and *crank* $\boldsymbol{\Omega}$ *point in different directions*.

$$\mathbf{H}\boldsymbol{\rho} - \boldsymbol{\rho}\mathbf{H} = \frac{\hbar}{2}\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right) - \frac{\hbar}{2}\left(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)$$

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\rho = \frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1}+\frac{\vec{\boldsymbol{\Omega}}}{2}\bullet\boldsymbol{\sigma}$$

Ket equation (time forward) and "daggered" bra-equation (time reversed).

$$i\hbar\left|\dot{\Psi}\right\rangle = \mathbf{H}\left|\Psi\right\rangle, \qquad \Leftarrow \textit{Daggar}^{\dagger} \Rightarrow \qquad -i\hbar\left\langle\dot{\Psi}\right| = \left\langle\Psi\right|\mathbf{H}$$

Combining these gives a time derivative of the density operator ρ = |Ψ⟩⟨Ψ|

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right| + i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right| = \mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

The result is called a *Bloch equation*.

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = \mathbf{H}\rho - \rho\mathbf{H} = \left[\mathbf{H},\rho\right]$$

$$\left(\mathbf{A}\bullet\boldsymbol{\sigma}\right)\left(\mathbf{B}\bullet\boldsymbol{\sigma}\right) = A_\alpha B_\beta\sigma_\alpha\sigma_\beta = A_\alpha B_\beta\left(\delta_{\alpha\beta}+i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma\right)$$

$$=A_\alpha B_\alpha + i\varepsilon_{\alpha\beta\gamma}A_\alpha B_\beta\sigma_\gamma$$

$$=\mathbf{A}\bullet\mathbf{B} + i\left(\mathbf{A}\times\mathbf{B}\right)\cdot\boldsymbol{\sigma}$$

This cancels

This remains

Given ρ and **H** in terms *spin* **S**-vector and *crank* **Ω**-vector:

$$\mathbf{H}\rho = \left(\hbar\Omega_0\mathbf{1}+\frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\frac{N}{2}\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1}+\frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}+\hbar\Omega_0\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}+\frac{\hbar}{2}\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)$$

$$\rho\mathbf{H} = \left(\frac{N}{2}\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)\left(\hbar\Omega_0\mathbf{1}+\frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1}+\frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}+\hbar\Omega_0\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}+\frac{\hbar}{2}\left(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)$$

Last terms don't cancel if the *spin* **S** and *crank* **Ω** *point in different directions*.

$$\mathbf{H}\rho - \rho\mathbf{H} = \frac{\hbar}{2}\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right) - \frac{\hbar}{2}\left(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)$$

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = \frac{i\hbar}{2}\left(\vec{\boldsymbol{\Omega}}\times\vec{\mathbf{S}}\right)\bullet\boldsymbol{\sigma} - \frac{i\hbar}{2}\left(\vec{\mathbf{S}}\times\vec{\boldsymbol{\Omega}}\right)\bullet\boldsymbol{\sigma}$$

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathrm{S}}\bullet\sigma$$
$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\sigma$$

Ket equation (time forward) and "daggered" bra-equation (time reversed).

$$i\hbar\left|\dot{\Psi}\right\rangle = \mathbf{H}\left|\Psi\right\rangle, \qquad \Leftarrow Daggar^{\dagger} \Rightarrow \qquad -i\hbar\left\langle\dot{\Psi}\right| = \left\langle\Psi\right|\mathbf{H}$$

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The result is called a *Bloch equation*.

$$\boxed{i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = \mathbf{H}\rho - \rho\mathbf{H} = \left[\mathbf{H},\rho\right]}$$

$$(\mathbf{A}\bullet\sigma)(\mathbf{B}\bullet\sigma) = A_\alpha B_\beta \sigma_\alpha\sigma_\beta = A_\alpha B_\beta\left(\delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma\right)$$
$$= A_\alpha B_\alpha + i\varepsilon_{\alpha\beta\gamma}A_\alpha B_\beta\sigma_\gamma$$
$$= \mathbf{A}\bullet\mathbf{B} + i\left(\mathbf{A}\times\mathbf{B}\right)\cdot\sigma$$

This cancels | *This remains*

Given ρ and $\mathbf{H}$ in terms *spin* S-vector and *crank* Ω-vector:

$$\mathbf{H}\rho = \left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\Omega}\bullet\sigma\right)\left(\frac{N}{2}\mathbf{1} + \vec{\mathrm{S}}\bullet\sigma\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\Omega}\bullet\sigma + \hbar\Omega_0\vec{\mathrm{S}}\bullet\sigma + \frac{\hbar}{2}\left(\vec{\Omega}\bullet\sigma\right)\left(\vec{\mathrm{S}}\bullet\sigma\right)$$

$$\rho\mathbf{H} = \left(\frac{N}{2}\mathbf{1} + \vec{\mathrm{S}}\bullet\sigma\right)\left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\Omega}\bullet\sigma\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\Omega}\bullet\sigma + \hbar\Omega_0\vec{\mathrm{S}}\bullet\sigma + \frac{\hbar}{2}\left(\vec{\mathrm{S}}\bullet\sigma\right)\left(\vec{\Omega}\bullet\sigma\right)$$

Last terms don't cancel if the *spin* S and *crank* Ω *point in different directions*.

$$\mathbf{H}\rho - \rho\mathbf{H} = \frac{\hbar}{2}\left(\vec{\Omega}\bullet\sigma\right)\left(\vec{\mathrm{S}}\bullet\sigma\right) - \frac{\hbar}{2}\left(\vec{\mathrm{S}}\bullet\sigma\right)\left(\vec{\Omega}\bullet\sigma\right)$$

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = \frac{i\hbar}{2}\left(\vec{\Omega}\times\vec{\mathrm{S}}\right)\bullet\sigma - \frac{i\hbar}{2}\left(\vec{\mathrm{S}}\times\vec{\Omega}\right)\bullet\sigma$$

$$i\hbar\frac{\partial}{\partial t}\left(\frac{N}{2}\mathbf{1} + \vec{\mathrm{S}}\bullet\sigma\right) = i\hbar\dot{\vec{\mathrm{S}}}\bullet\sigma = i\hbar\left(\vec{\Omega}\times\mathrm{S}\right)\bullet\sigma$$

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\boxed{\begin{array}{l}\boldsymbol{\rho} = \frac{1}{2}N\mathbf{1} + \vec{\mathrm{S}}\bullet\boldsymbol{\sigma} \\ \mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\boldsymbol{\Omega}}}{2}\bullet\boldsymbol{\sigma}\end{array}}$$

Ket equation (time forward) and "daggered" bra-equation (time reversed).

$$i\hbar\left|\dot{\Psi}\right\rangle = \mathbf{H}\left|\Psi\right\rangle, \quad \Leftarrow \textit{Daggar}^{\dagger} \Rightarrow \quad -i\hbar\left\langle\dot{\Psi}\right| = \left\langle\Psi\right|\mathbf{H}$$

Note: $\mathbf{H}^{\dagger} = \mathbf{H}$.
$\boldsymbol{\rho}^{\dagger} = \boldsymbol{\rho}$

Combining these gives a time derivative of the density operator $\boldsymbol{\rho} = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho} = i\hbar\dot{\boldsymbol{\rho}} = i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right| + i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right| = \mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

The result is called a *Bloch equation*.

$$\boxed{i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho} = i\hbar\dot{\boldsymbol{\rho}} = \mathbf{H}\boldsymbol{\rho} - \boldsymbol{\rho}\mathbf{H} = \left[\mathbf{H},\boldsymbol{\rho}\right]}$$

$$\left(\mathbf{A}\bullet\boldsymbol{\sigma}\right)\left(\mathbf{B}\bullet\boldsymbol{\sigma}\right) = A_\alpha B_\beta \sigma_\alpha \sigma_\beta = A_\alpha B_\beta\left(\delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma\right)$$
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Given $\boldsymbol{\rho}$ and $\mathbf{H}$ in terms *spin* S-vector and *crank* $\boldsymbol{\Omega}$-vector:

$$\mathbf{H}\boldsymbol{\rho} = \left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\frac{N}{2}\mathbf{1} + \vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma} + \hbar\Omega_0\vec{\mathrm{S}}\bullet\boldsymbol{\sigma} + \frac{\hbar}{2}\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right)$$

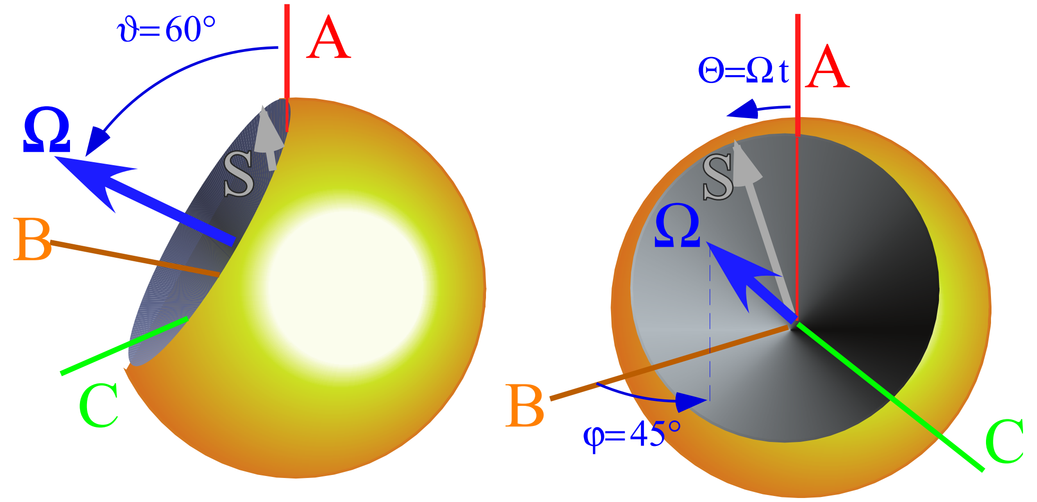
$$\boldsymbol{\rho}\mathbf{H} = \left(\frac{N}{2}\mathbf{1} + \vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right)\left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma} + \hbar\Omega_0\vec{\mathrm{S}}\bullet\boldsymbol{\sigma} + \frac{\hbar}{2}\left(\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)$$

Last terms don't cancel if the *spin* S and *crank* $\boldsymbol{\Omega}$ *point in different directions*.

$$\mathbf{H}\boldsymbol{\rho} - \boldsymbol{\rho}\mathbf{H} = \frac{\hbar}{2}\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right) - \frac{\hbar}{2}\left(\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)$$

$$i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho} = i\hbar\dot{\boldsymbol{\rho}} = \frac{i\hbar}{2}\left(\vec{\boldsymbol{\Omega}}\times\vec{\mathrm{S}}\right)\bullet\boldsymbol{\sigma} - \frac{i\hbar}{2}\left(\vec{\mathrm{S}}\times\vec{\boldsymbol{\Omega}}\right)\bullet\boldsymbol{\sigma}$$

$$i\hbar\frac{\partial}{\partial t}\left(\frac{N}{2}\mathbf{1} + \vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right) = i\hbar\dot{\vec{\mathrm{S}}}\bullet\boldsymbol{\sigma} = i\hbar\left(\vec{\boldsymbol{\Omega}}\times\mathbf{S}\right)\bullet\boldsymbol{\sigma}$$



Factoring out •σ gives a classical/quantum *gyro-precession equation*. $\boxed{\frac{\partial\vec{\mathrm{S}}}{\partial t} = \dot{\vec{\mathrm{S}}} = \vec{\boldsymbol{\Omega}}\times\vec{\mathrm{S}}}$