AMOP reference links on following page

1.29.18 class 5.0: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

Symmetry group $\mathscr{G}$ representations=>*AMOP* Hamiltonian **H** (or K) matrices, irreps $\mathscr{D}^{(\alpha)}$ =>*AMOP* wave functions $\Psi^{(\alpha)}$, eigensolutions

$\mathscr{G}$= U(2) spin-½ irreps: Euler **R**(αβγ) *vs* Darboux **R**[φϑΘ] rotations and applications

Relating Euler and Darboux angles to U(2) phasor coordinates x_1+ip_1 *and* x_2+ip_2 .

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Conventional amplitude-phase-$(A_1,A_2,,\omega t,\rho_1)$ *labeling of optical polarization*

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Euler angle labeling of optical polarization *C-Type* $(\alpha_C,\beta_C,\gamma_C)$ *vs* *A-Type* $(\alpha_A,\beta_A,\gamma_A)$,

AMOP reference links *(Updated list given on 2nd page of each class presentation)*

Web Resources - front page

UAF Physics UTube channel

2014 AMOP

2017 Group Theory for QM

2018 AMOP

Frame Transformation Relations And Multipole Transitions In Symmetric Polyatomic Molecules - RMP-1978 (Alt Scanned version)
Rotational energy surfaces and high- J eigenvalue structure of polyatomic molecules - Harter - Patterson - 1984
Galloping waves and their relativistic properties - ajp-1985-Harter
Asymptotic eigensolutions of fourth and sixth rank octahedral tensor operators - Harter-Patterson-JMP-1979
Nuclear spin weights and gas phase spectral structure of 12C60 and 13C60 buckminsterfullerene -Harter-Reimer-Cpl-1992 - (Alt1, Alt2 Erratum)

Theory of hyperfine and superfine levels in symmetric polyatomic molecules.

I) Trigonal and tetrahedral molecules: Elementary spin-1/2 cases in vibronic ground states - PRA-1979-Harter-Patterson (Alt scan)
II) Elementary cases in octahedral hexafluoride molecules - Harter-PRA-1981 (Alt scan)

Rotation-vibration scalar coupling zeta coefficients and spectroscopic band shapes of buckminsterfullerene - Weeks-Harter-CPL-1991 (Alt scan)
Fullerene symmetry reduction and rotational level fine structure/ the Buckyball isotopomer 12C 13C59 - jcp-Reimer-Harter-1997 (HiRez)
Molecular Eigensolution Symmetry Analysis and Fine Structure - IJMS-harter-mitchell-2013

Rotation–vibration spectra of icosahedral molecules.

I) Icosahedral symmetry analysis and fine structure - harter-weeks-jcp-1989
II) Icosahedral symmetry, vibrational eigenfrequencies, and normal modes of buckminsterfullerene - weeks-harter-jcp-1989
III) Half-integral angular momentum - harter-reimer-jcp-1991

QTCA Unit 10 Ch 30 - 2013
AMOP Ch 32 Molecular Symmetry and Dynamics - 2019
AMOP Ch 0 Space-Time Symmetry - 2019

RESONANCE AND REVIVALS

I) QUANTUM ROTOR AND INFINITE-WELL DYNAMICS - ISMSLi2012 (Talk) https://kb.osu.edu/dspace/handle/1811/52324
II) Comparing Half-integer Spin and Integer Spin - Alva-ISMS-Ohio2013-R777 (Talks)
III) Quantum Resonant Beats and Revivals in the Morse Oscillators and Rotors - (2013-Li-Diss)

Rovibrational Spectral Fine Structure Of Icosahedral Molecules - Cpl 1986 (Alt Scan)
Gas Phase Level Structure of C60 Buckyball and Derivatives Exhibiting Broken Icosahedral Symmetry - reimer-diss-1996
Resonance and Revivals in Quantum Rotors - Comparing Half-integer Spin and Integer Spin - Alva-ISMS-Ohio2013-R777 (Talk)
Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-cpl-2013
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - harter - jms - 2001

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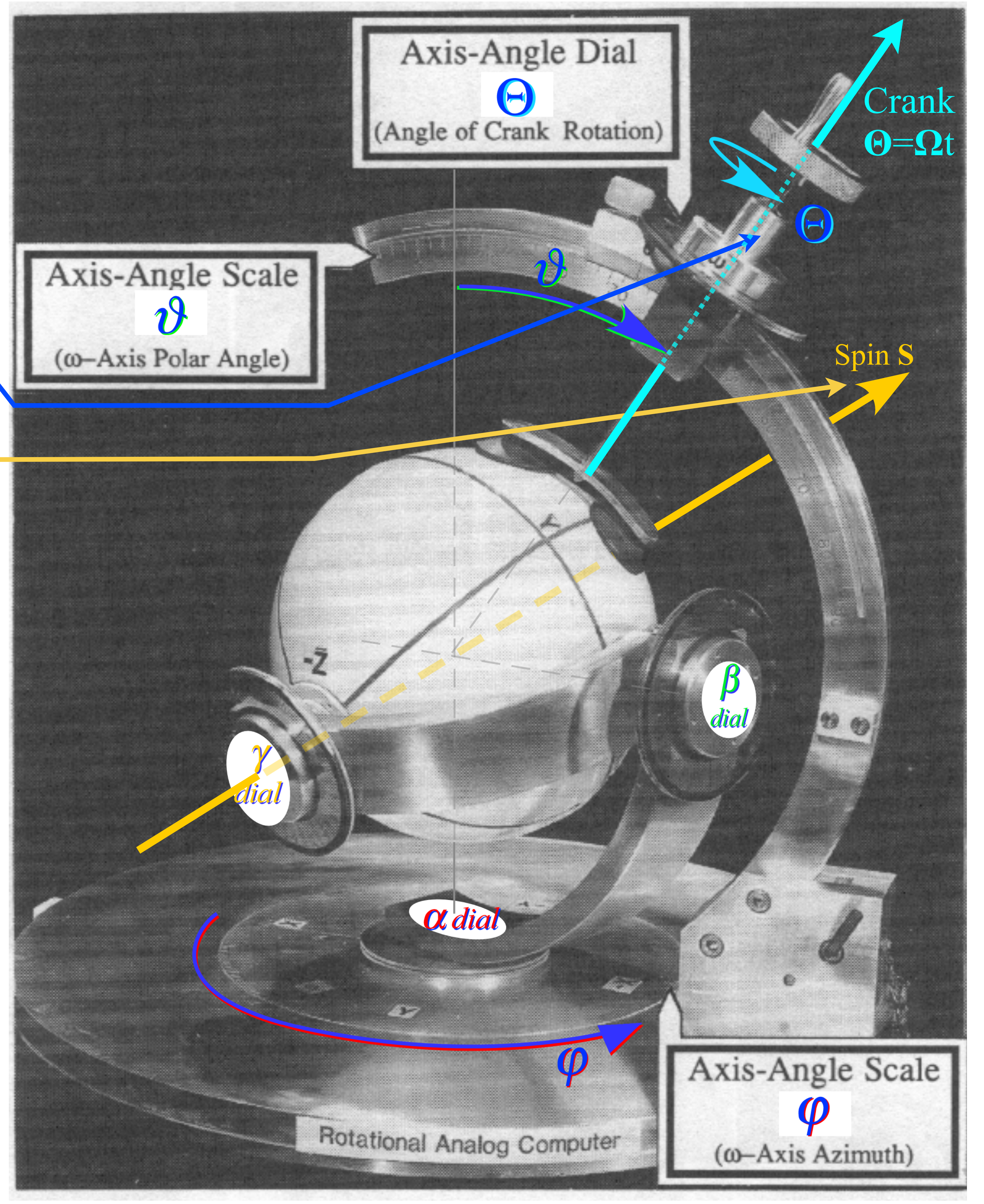
Preceding Class-4 showed that dynamics of $i\partial\Psi/\partial t = \mathbf{H}\Psi$ may be reduced to mechanics:

Crank $\boldsymbol{\Theta} = \boldsymbol{\Omega}t$ of Hamiltonian **H**

rotates

Spin vector $\mathbf{S}=\frac{1}{2}\boldsymbol{\sigma}$ of state Ψ.



Darboux $[\varphi,\vartheta,\Theta]$ *crank-axis angles*

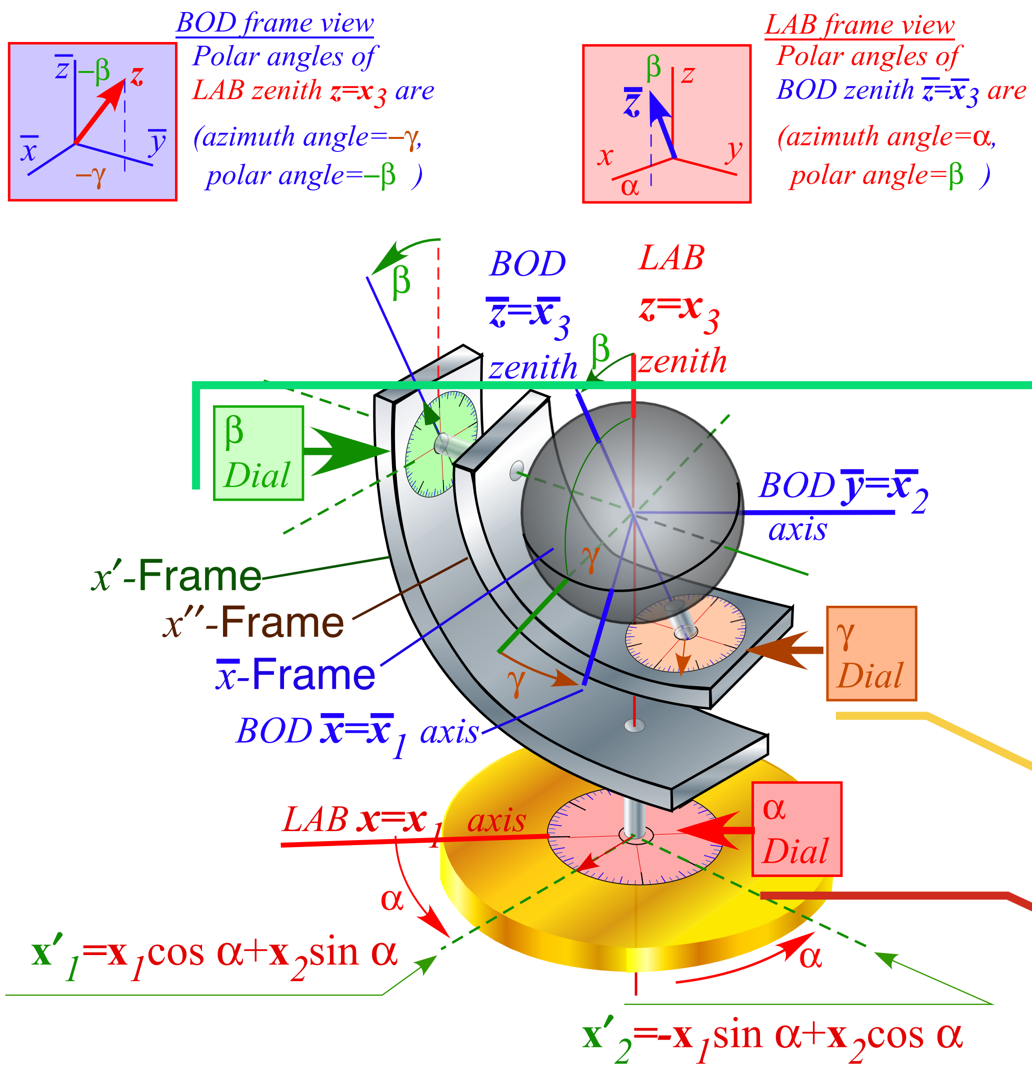
Polar coordinates for unit axis vector $\hat{\boldsymbol{\Theta}}$

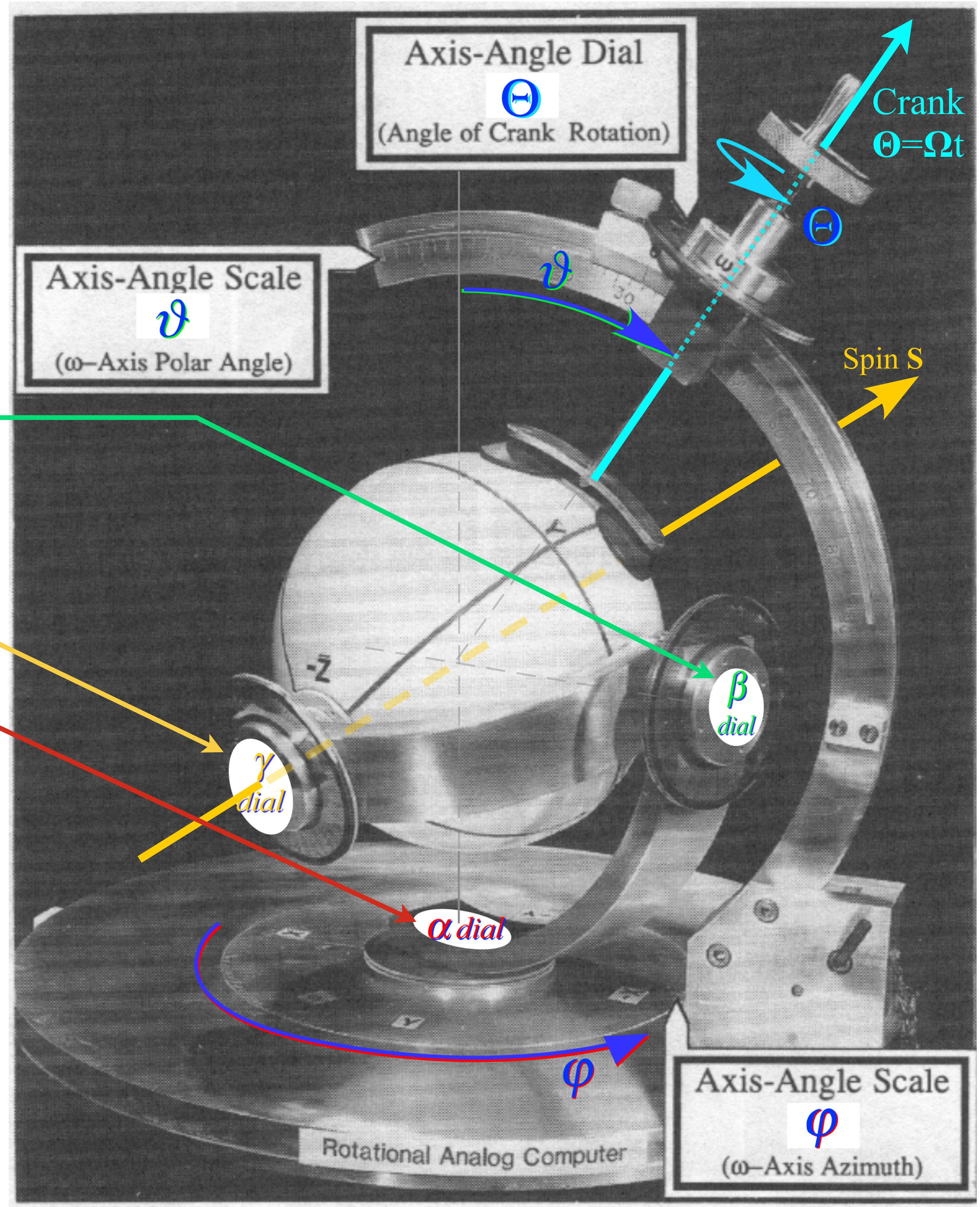
$\hat{\Theta}_x = \cos\varphi \ \sin\vartheta$

$\hat{\Theta}_y = \sin\varphi \ \sin\vartheta$

$\hat{\Theta}_z = \cos\vartheta$

*Here spin-rotor **S**-polar coordinates are Euler* (α,β,γ) *angles*





Versus

Darboux $[\varphi,\vartheta,\Theta]$ *crank-axis angles*

Polar coordinates for unit axis vector $\hat{\Theta}$

$$\hat{\Theta}_x = \cos\varphi \;\sin\vartheta$$
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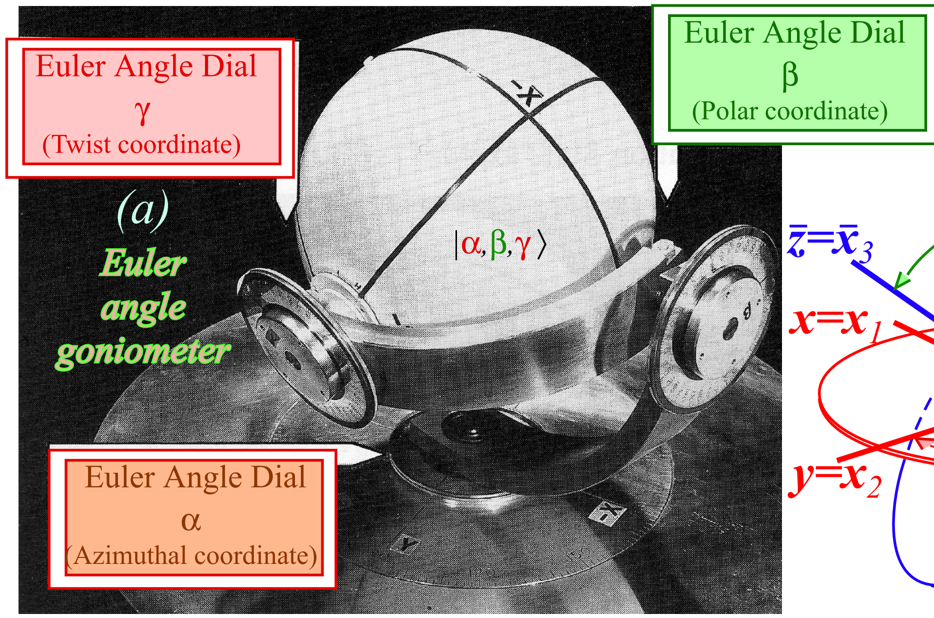
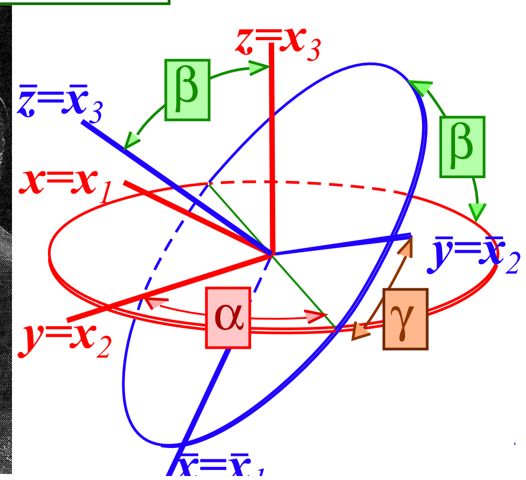
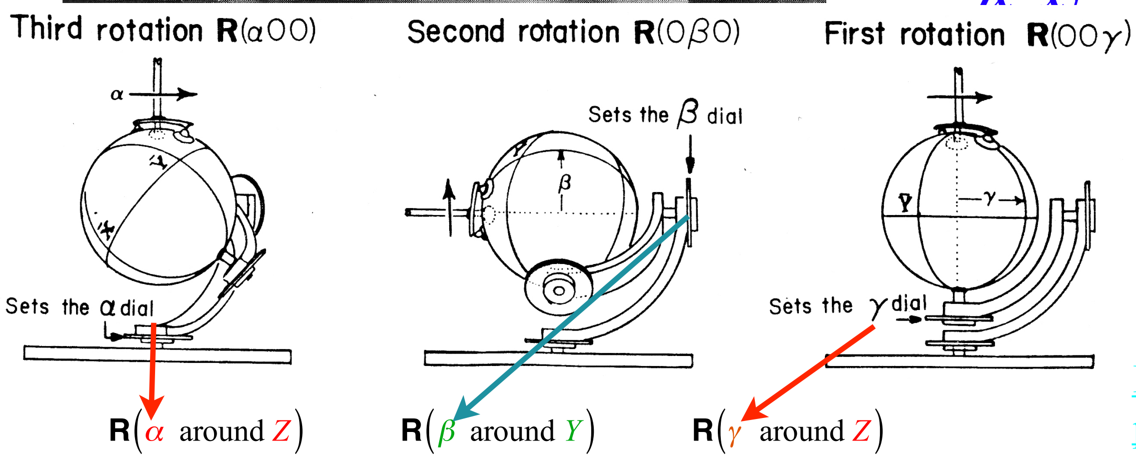


An astronomer's diagram

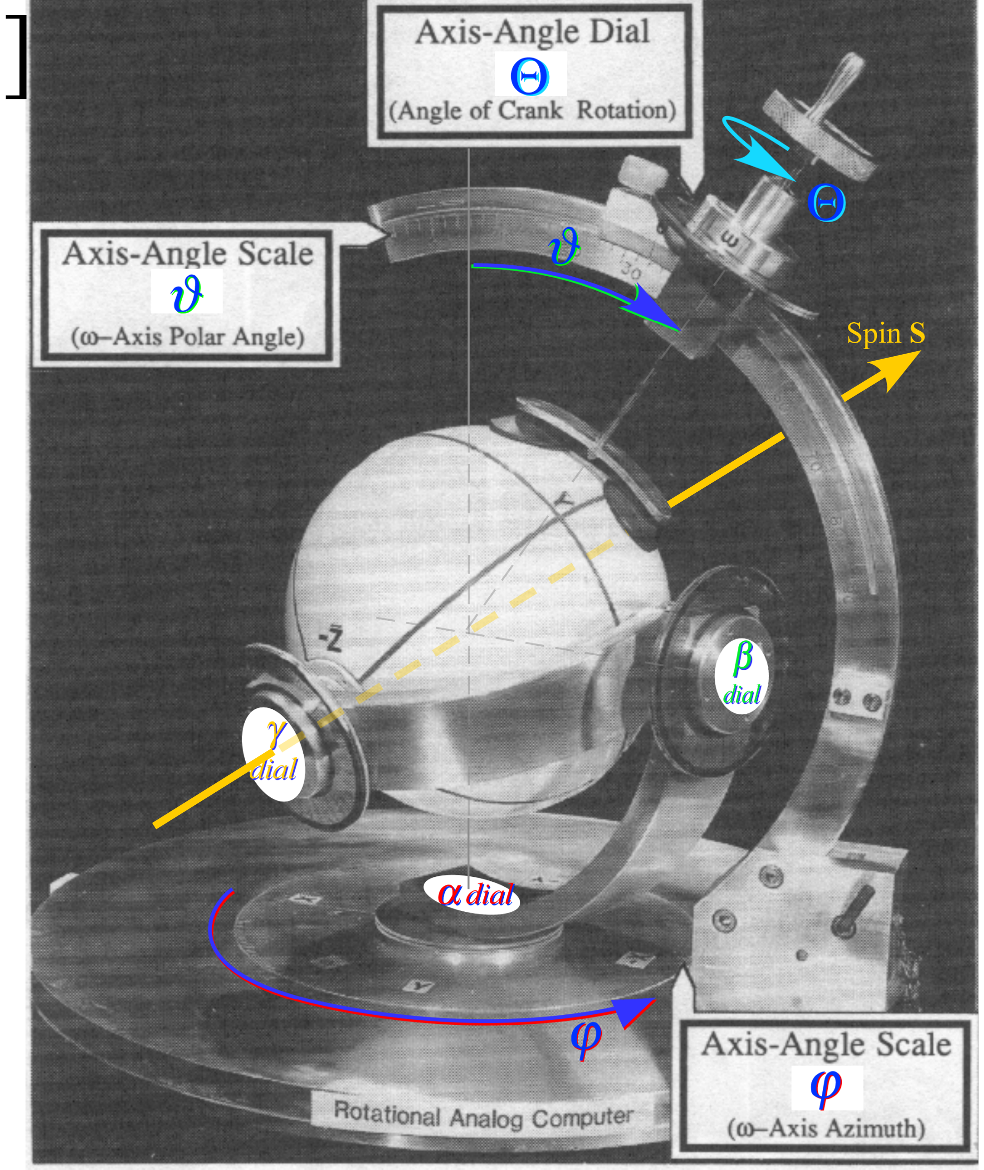




From Class 4 page 86 to 92

GThLect.8 p17-24

From Class 4 page 57 to 58



$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

$$\mathbf{R}[\vec{\Theta}]=\begin{pmatrix} \cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X-i\hat{\Theta}_Y\right) \\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X+i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2} \end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

$$=\cos\frac{\Theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$=\begin{pmatrix} \cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta\right) \\ \sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta\right) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2} \end{pmatrix}$$

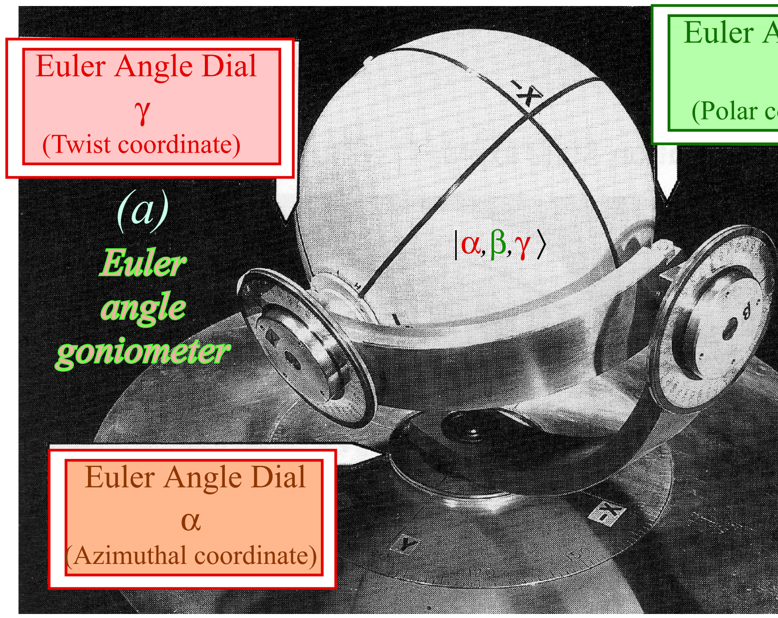
Euler **R**(αβγ) is simpler to form than Θ-axis *Darboux* **R**[φϑΘ].

Euler *state definition* lets us relate **R**(αβγ) to **R**[φϑΘ] ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ *(αβγ make better coordinates)*

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}=\begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$



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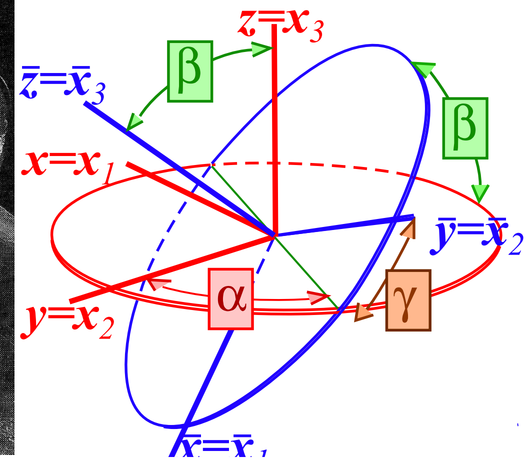
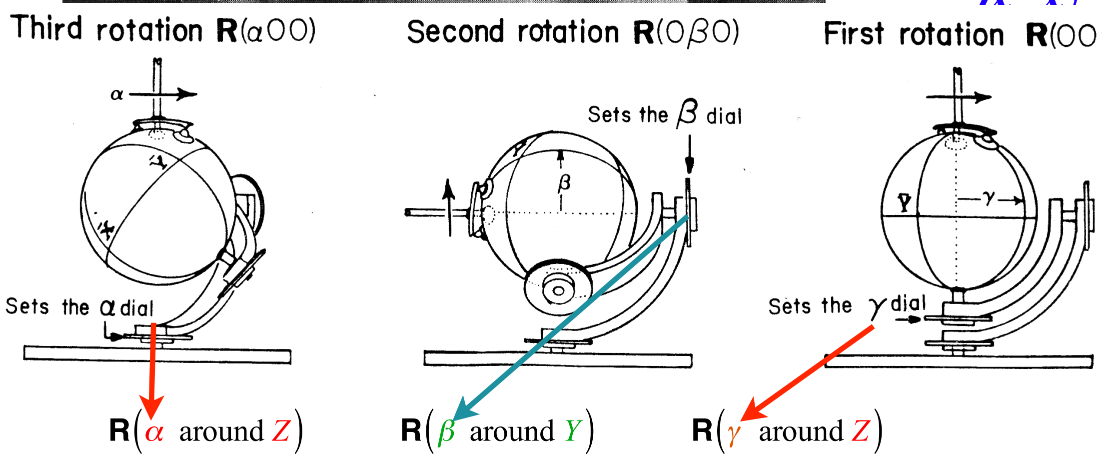
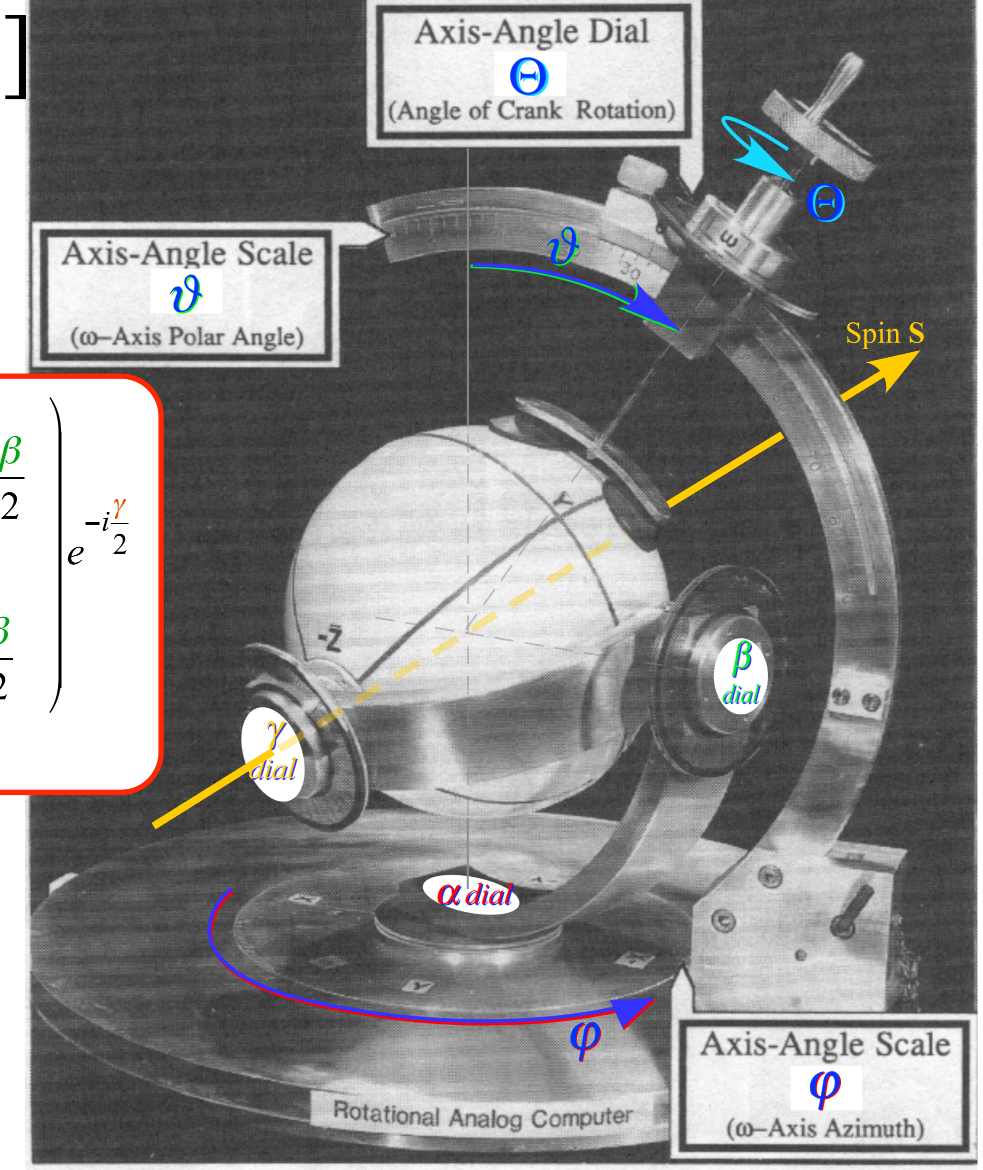




$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$

From Class 4 page 96 to 97



$$\mathbf{R}(\alpha\beta\gamma) = \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix} =$$

$$= \cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2} - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2} - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

$$\mathbf{R}[\vec{\Theta}] = \begin{pmatrix} \cos\frac{\Theta}{2} - i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X - i\hat{\Theta}_Y\right) \\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X + i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2} + i\hat{\Theta}_Z\sin\frac{\Theta}{2} \end{pmatrix} = \mathbf{R}[\varphi\vartheta\Theta] = e^{-i\mathbf{H}t}$$

$$= \cos\frac{\Theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2} - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2} - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$= \begin{pmatrix} \cos\frac{\Theta}{2} - i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta + i\cos\varphi\sin\vartheta\right) \\ \sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta - i\cos\varphi\sin\vartheta\right) & \cos\frac{\Theta}{2} + i\cos\vartheta\sin\frac{\Theta}{2} \end{pmatrix}$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ is simpler to form than Θ-axis *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$.

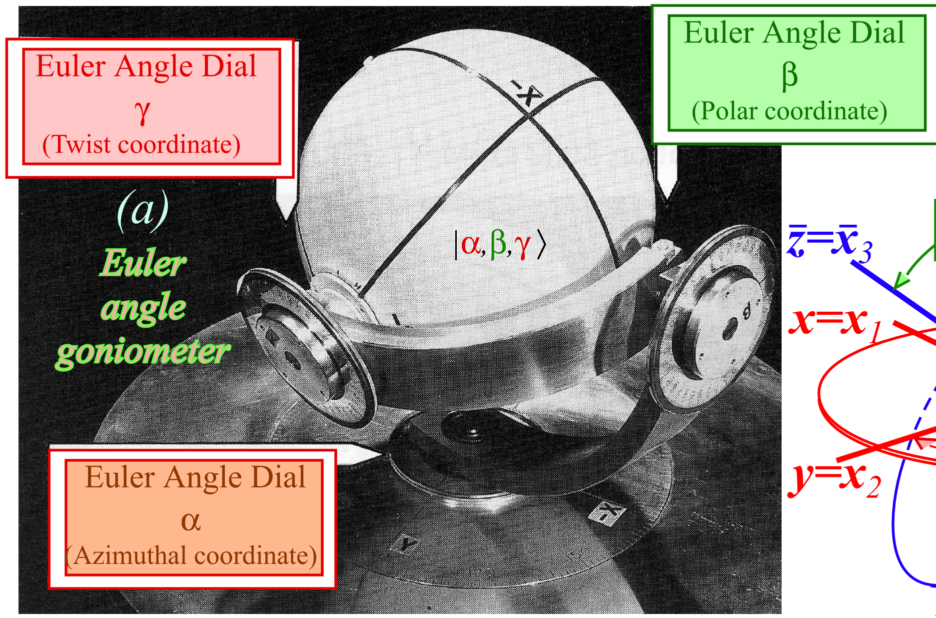
Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$ ...

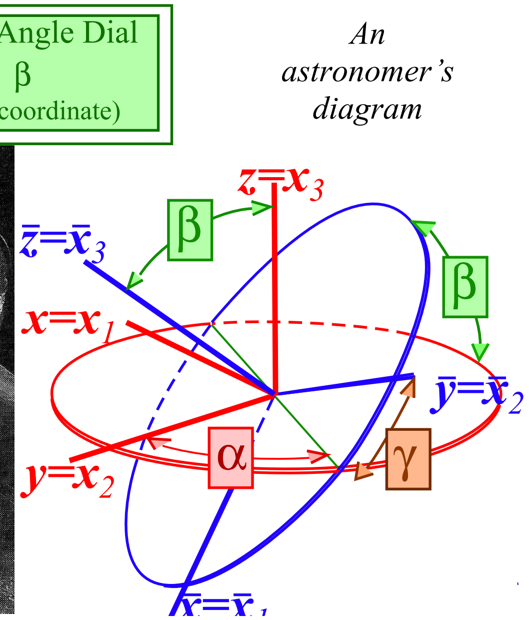
$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ *(αβγ make better coordinates)*

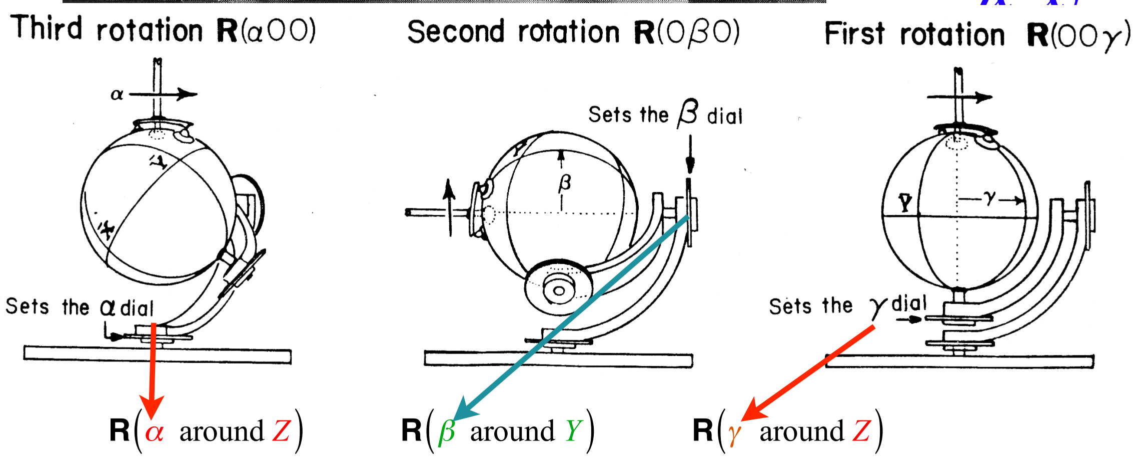
$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

Euler R(αβγ) versus *Darboux* R[φϑΘ]

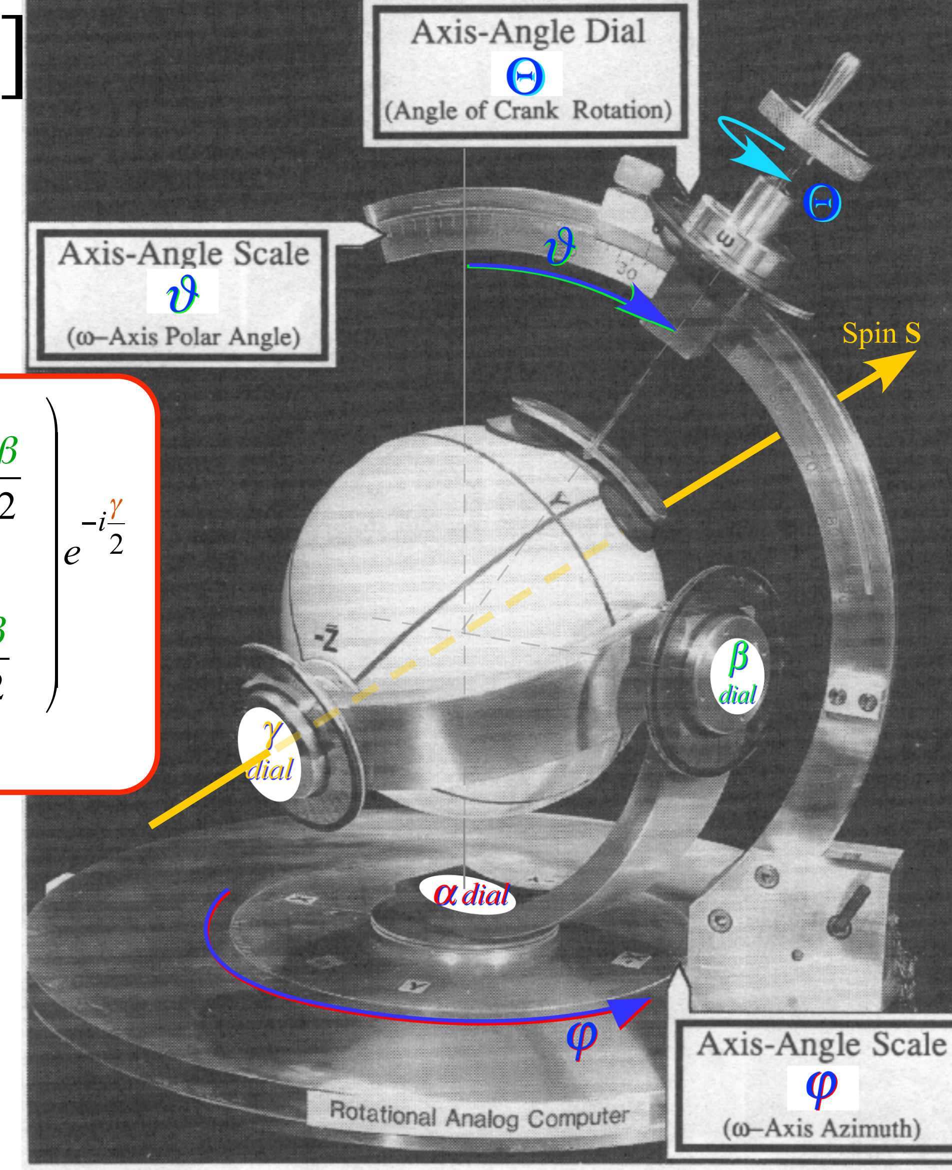






$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle=\begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2}\\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2}\end{pmatrix}e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$

From Class 4 page 96 to 97



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$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

$$\mathbf{R}[\vec{\Theta}]=\begin{pmatrix}\cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}(\hat{\Theta}_X-i\hat{\Theta}_Y)\\ -i\sin\frac{\Theta}{2}(\hat{\Theta}_X+i\hat{\Theta}_Y) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2}\end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

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$$-p_2=\sin[(\gamma-\alpha)/2]\sin\beta/2=\hat{\Theta}_X\sin\Theta/2=\cos\varphi\ \sin\vartheta\ \sin\Theta/2$$

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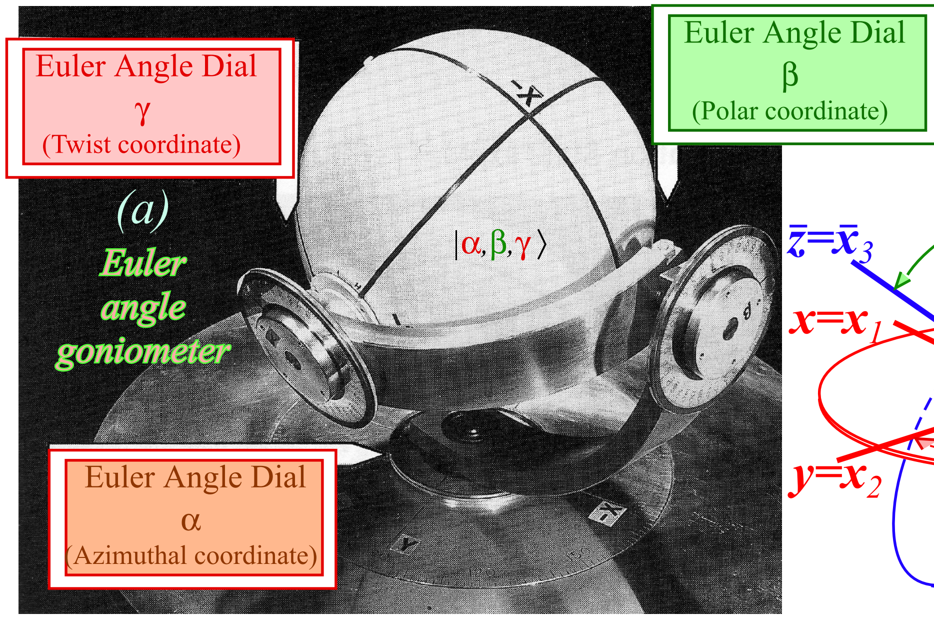
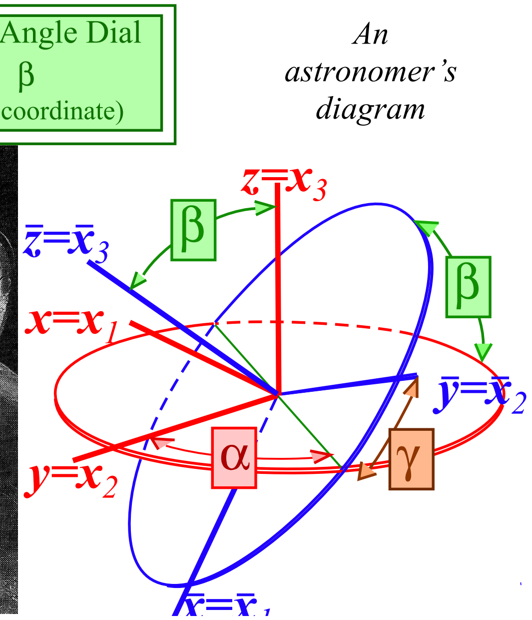
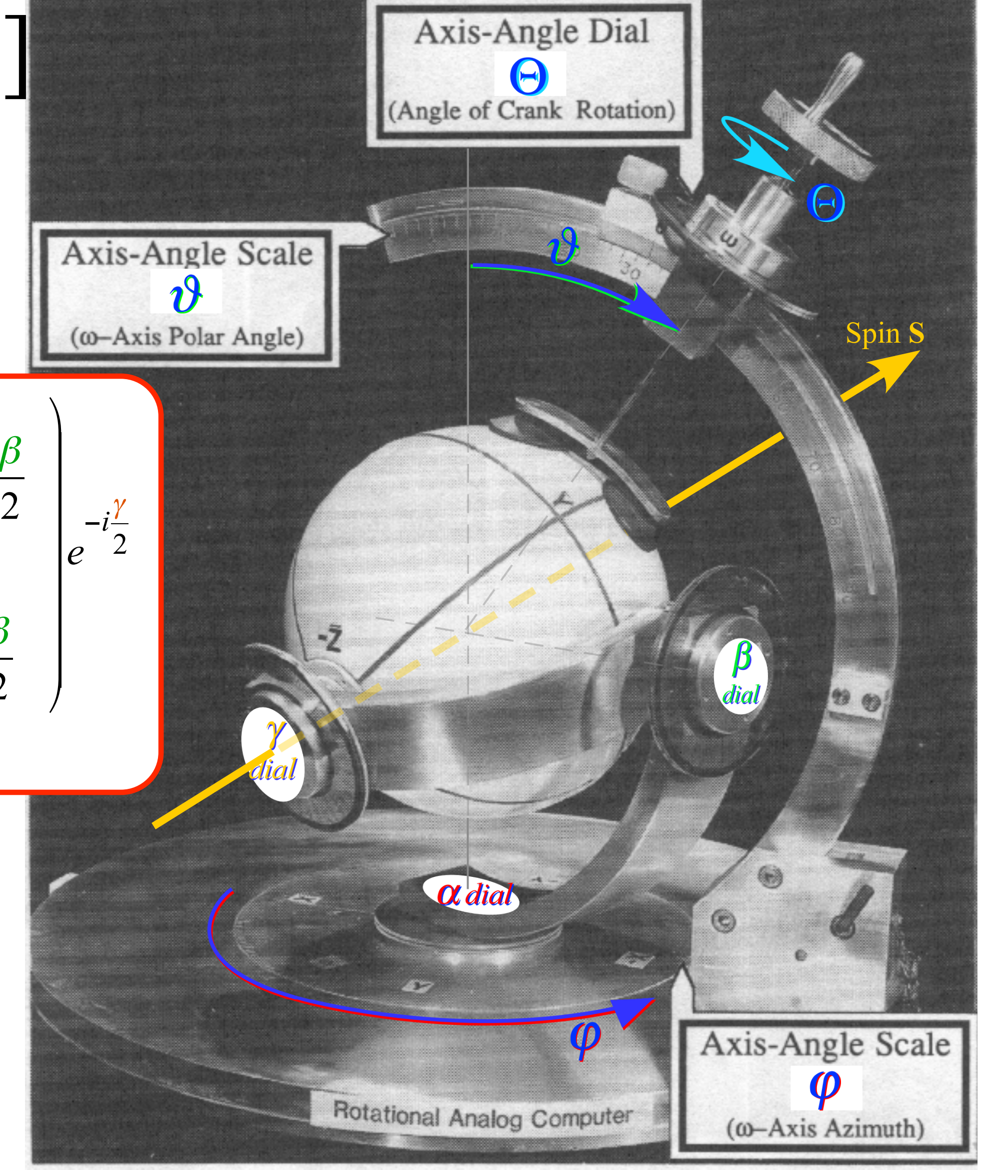
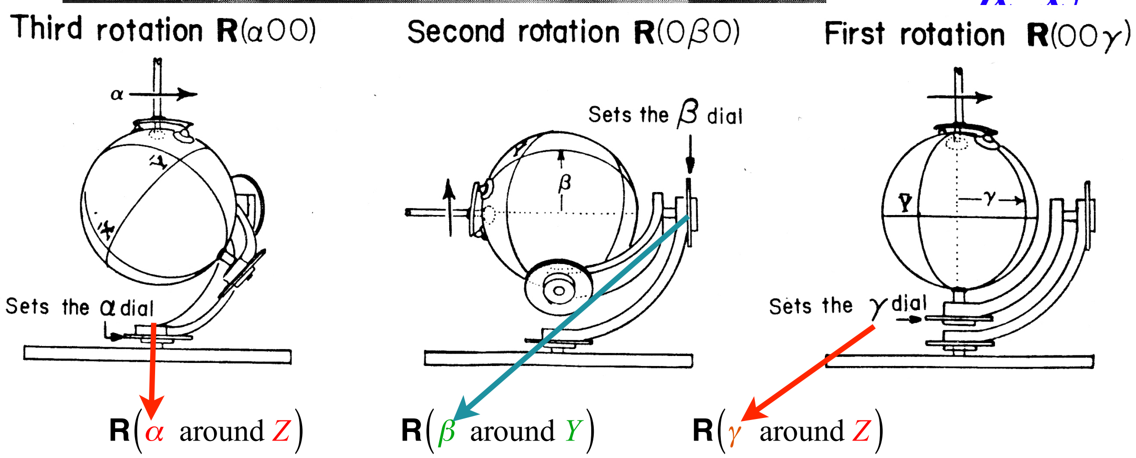




$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$

From Class 4 page 96 to 97





$$\mathbf{R}(\alpha\beta\gamma) = \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix} =$$

$$= \cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2} - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2} - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

$$\mathbf{R}[\vec{\Theta}] = \begin{pmatrix} \cos\frac{\Theta}{2} - i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X - i\hat{\Theta}_Y\right) \\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X + i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2} + i\hat{\Theta}_Z\sin\frac{\Theta}{2} \end{pmatrix} = \mathbf{R}[\varphi\vartheta\Theta] = e^{-i\mathbf{H}t}$$

$$= \cos\frac{\Theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\hat{\Theta}_X\sin\frac{\Theta}{2} - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\hat{\Theta}_Y\sin\frac{\Theta}{2} - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\hat{\Theta}_Z\sin\frac{\Theta}{2}$$

$\hat{\Theta}_X$: cosφ sinϑ $\hat{\Theta}_Y$: sinφ sinϑ $\hat{\Theta}_Z$: cosϑ

$$= \begin{pmatrix} \cos\frac{\Theta}{2} - i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta + i\cos\varphi\sin\vartheta\right) \\ \sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta - i\cos\varphi\sin\vartheta\right) & \cos\frac{\Theta}{2} + i\cos\vartheta\sin\frac{\Theta}{2} \end{pmatrix}$$

Euler R(αβγ) is simpler to form than Θ-axis *Darboux* R[φϑΘ].

Euler *state definition* lets us relate R(αβγ) to R[φϑΘ] ...

|αβγ⟩= R(αβγ)|000⟩   *(αβγ make better coordinates)*

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

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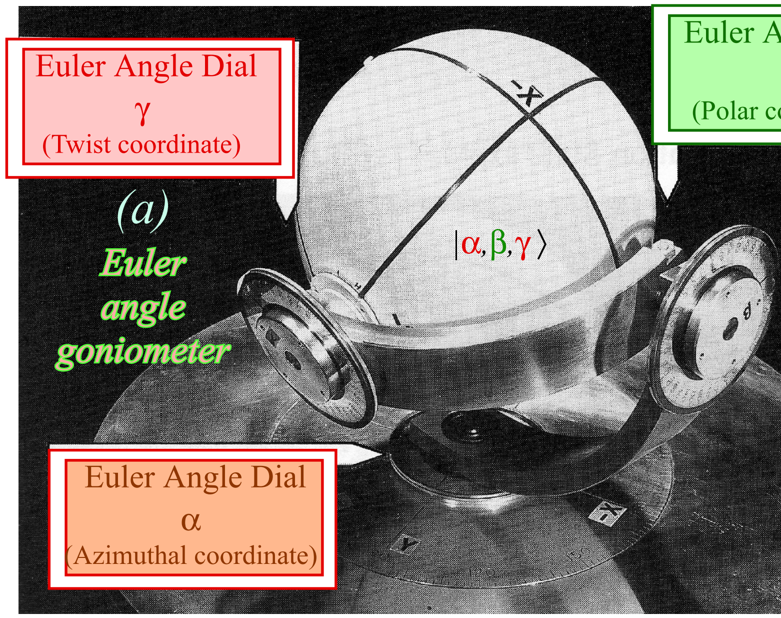
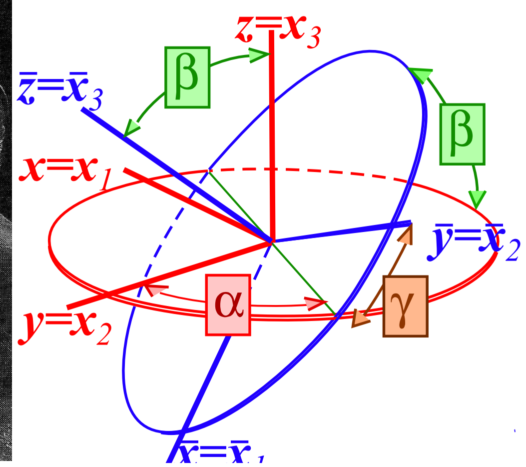
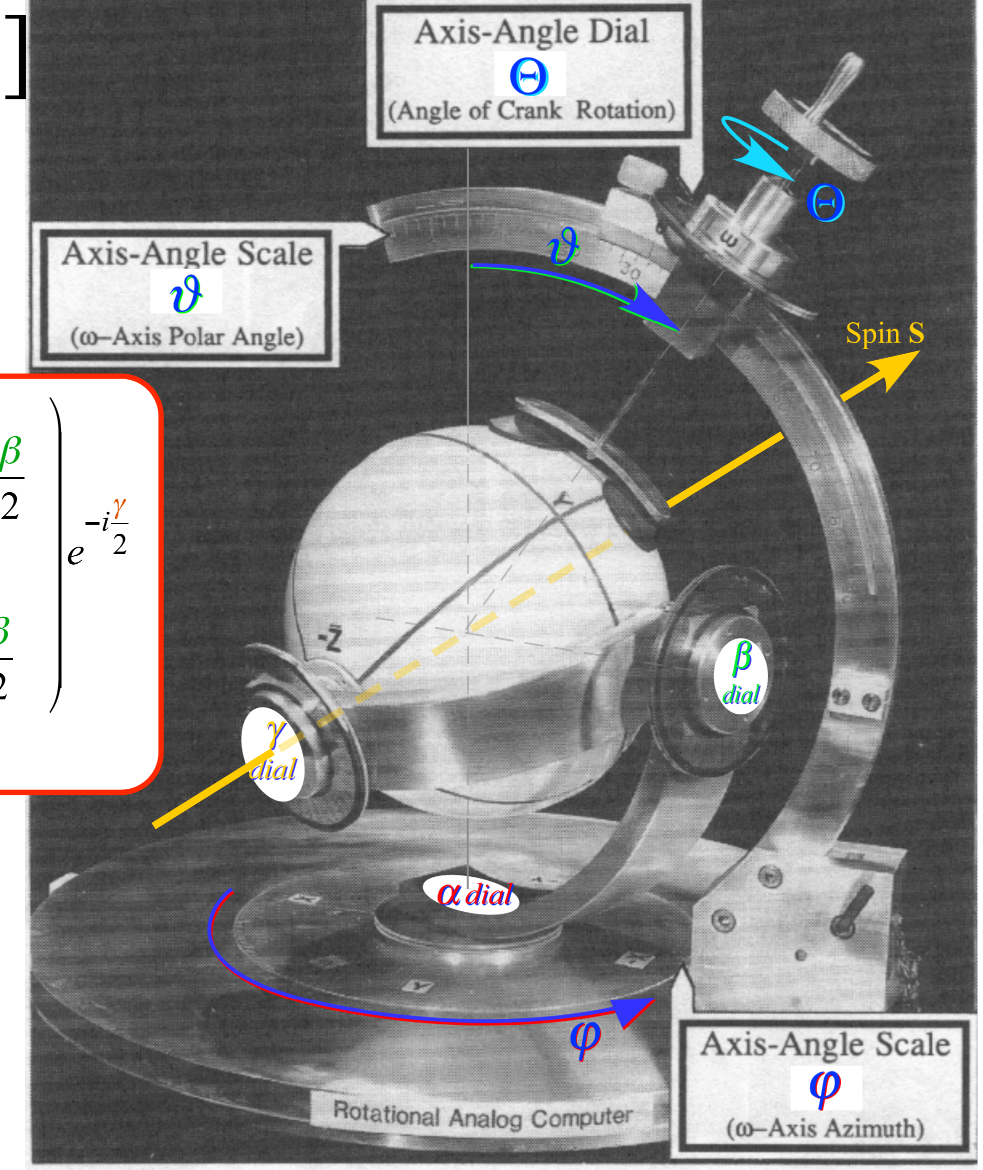
Euler R(αβγ) versus *Darboux* R[φϑΘ]

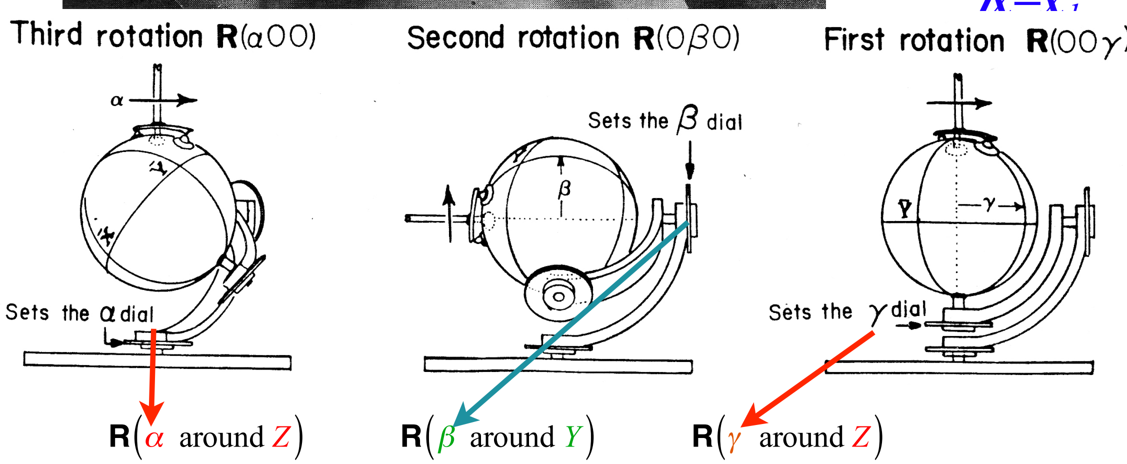






$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$

From Class 4 page 96 to 97



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Euler $\mathbf{R}(\alpha\beta\gamma)$ related to Darboux $\mathbf{R}[\varphi\vartheta\Theta]$

Step-by-Step Development GThLect.8 p89-99

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ $\alpha\beta\gamma$ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2\end{pmatrix}$$

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$$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z\sin\Theta/2 = \cos\vartheta\,\sin\Theta/2$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2$$

$$\tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2}-\varphi]$$

$$(\gamma+\alpha)/2 = \tan^{-1}[\cos\vartheta\tan\Theta/2]$$

$$(\gamma-\alpha)/2 = \frac{\pi}{2}-\varphi$$

$$\sin[(\gamma-\alpha)/2] = \sin[\frac{\pi}{2}-\varphi] = \cos\varphi$$

$$\sin\beta/2 = \sin\vartheta\,\sin\Theta/2$$

This gives *Euler angles* $(\alpha\beta\gamma)$ in terms of *Darboux angles* $[\varphi\vartheta\Theta]$

$$\alpha = \varphi - \pi/2 + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

$$\beta = 2\sin^{-1}(\sin\Theta/2\,\sin\vartheta)$$

$$\gamma = \pi/2 - \varphi + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

Inverse relations have *Darboux axis angles* $[\varphi\vartheta\Theta]$ in terms of *Euler angles* $(\alpha\beta\gamma)$

$$\varphi = (\alpha-\gamma+\pi)/2$$

$$\vartheta = \tan^{-1}[\tan\beta/2/\sin(\alpha+\gamma)/2]$$

$$\Theta = 2\cos^{-1}[\cos\beta/2\cos(\alpha+\gamma)/2]$$

$$\cos[(\gamma-\alpha)/2] = \cos[\frac{\pi}{2}-\varphi] = \sin\varphi$$

$$\frac{\cos[(\gamma-\alpha)/2]\sin\beta/2}{\sin[(\gamma+\alpha)/2]\cos\beta/2} = \sin\varphi\tan\vartheta \Rightarrow \frac{\tan\beta/2}{\sin[(\gamma+\alpha)/2]} = \tan\vartheta$$

Example: *Euler angles* $(\alpha=50^\circ\ \beta=60^\circ\ \gamma=70^\circ)$

$$\varphi = (50^\circ - 70^\circ + 180^\circ)/2 = 80^\circ$$

$$\vartheta = \tan^{-1}[\tan 60^\circ/2/\sin(50^\circ+\gamma)/2] = 33.7^\circ$$

$$\Theta = 2\cos^{-1}[\cos 60^\circ/2\cos(50^\circ+\gamma)/2] = 128.7^\circ$$

Reverse check: $(\alpha\beta\gamma)$ in terms of $[\varphi\vartheta\Theta]$

$$\alpha = 80^\circ - 90^\circ + \tan^{-1}(\tan(128.7^\circ/2)\cos 33.7^\circ) = 50.007^\circ$$

$$\beta = 2\sin^{-1}(\sin 128.7^\circ/2\,\sin 33.7^\circ) = 60.022^\circ$$

$$\gamma = \pi/2 - 128.7^\circ + \tan^{-1}(\tan(128.7^\circ/2)) = 70.007^\circ$$

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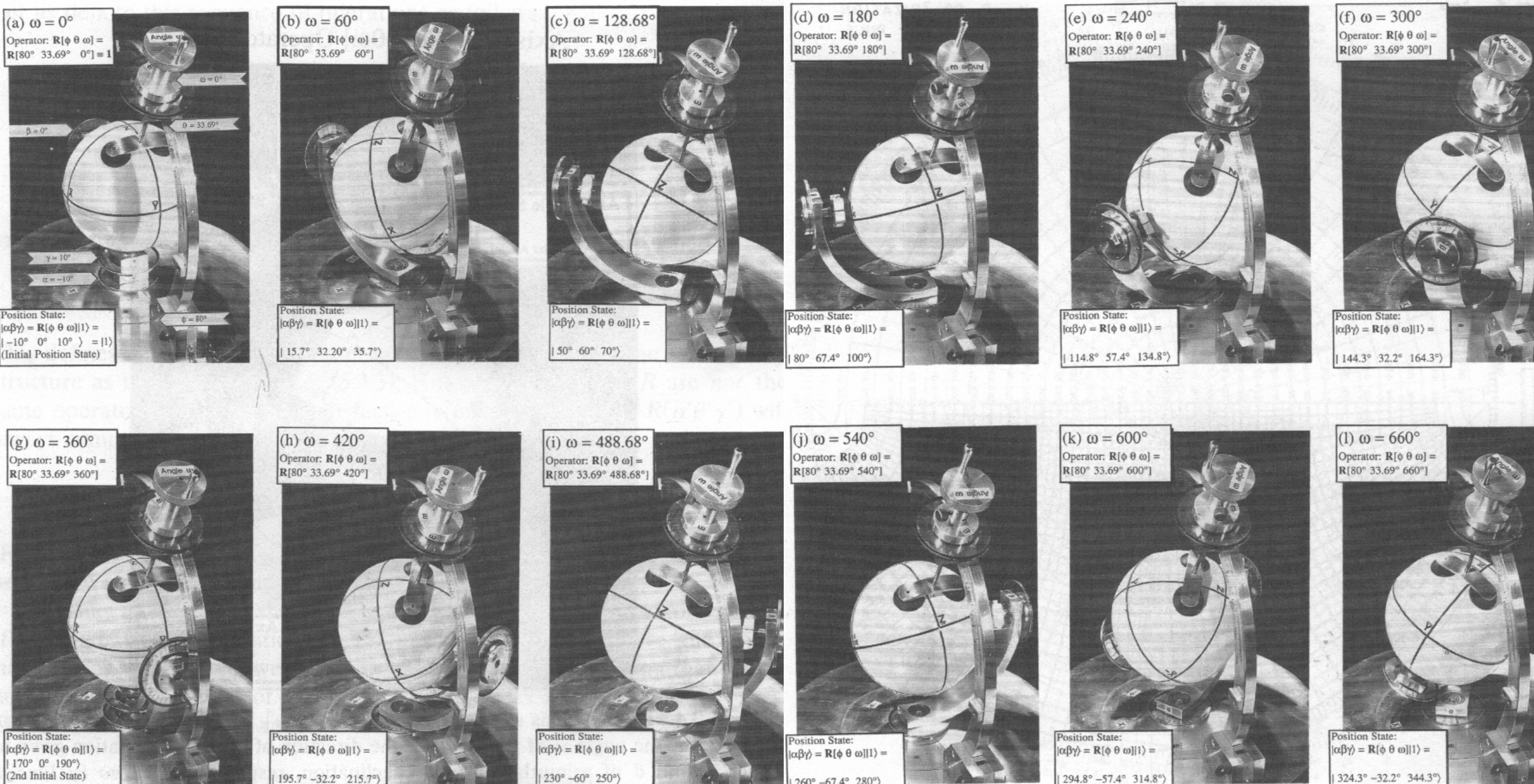
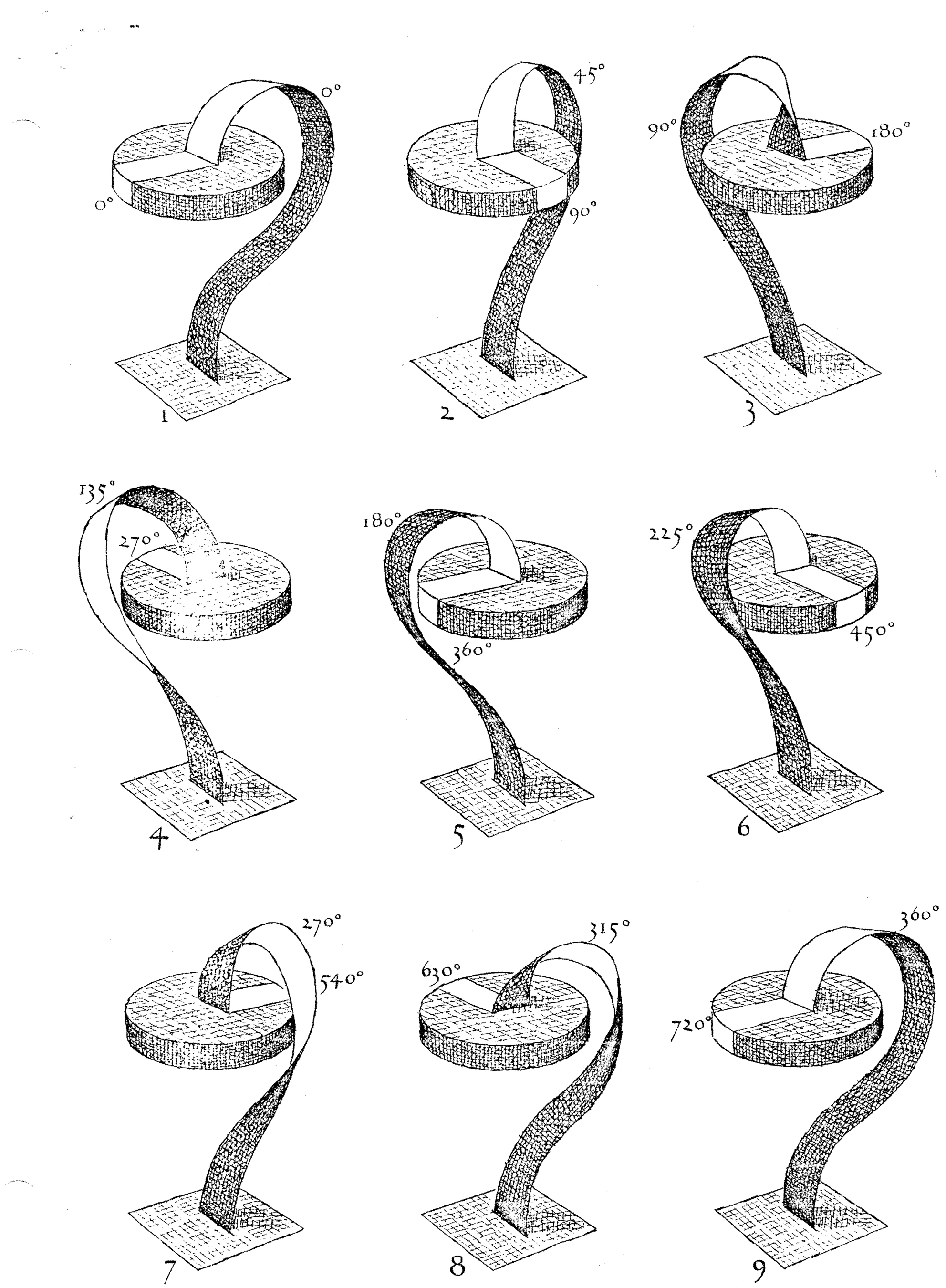
Euler **R**(αβγ) *rotation* Θ=0−4π-*sequence* [φϑ] *fixed*

Θ=0° Θ=60° Θ=128.7° Θ=180° Θ=240° Θ=300°



Θ=360° Θ=420° Θ=488.7° Θ=540° Θ=600° Θ=660°

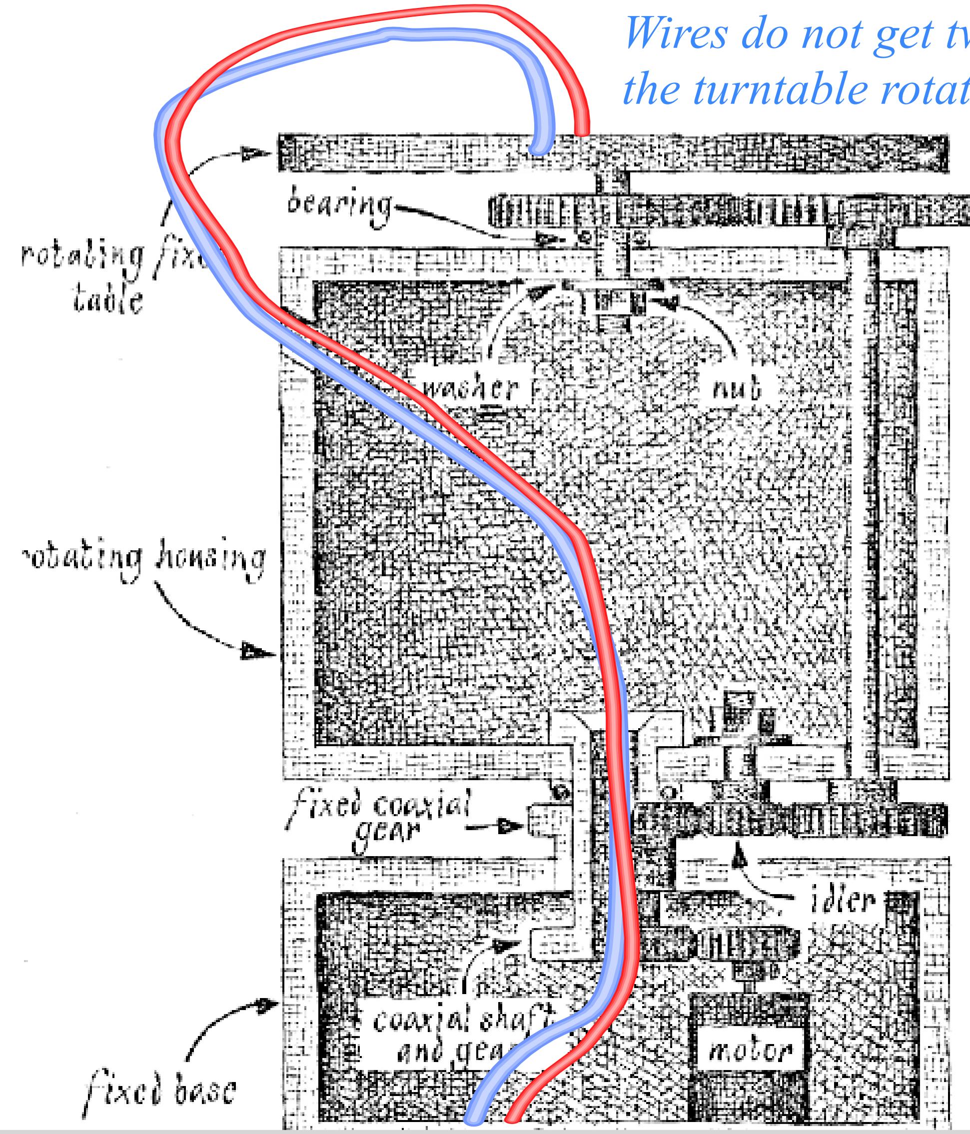
Some "real-world" applications of the U(2)-R(3) spinor-vector topology

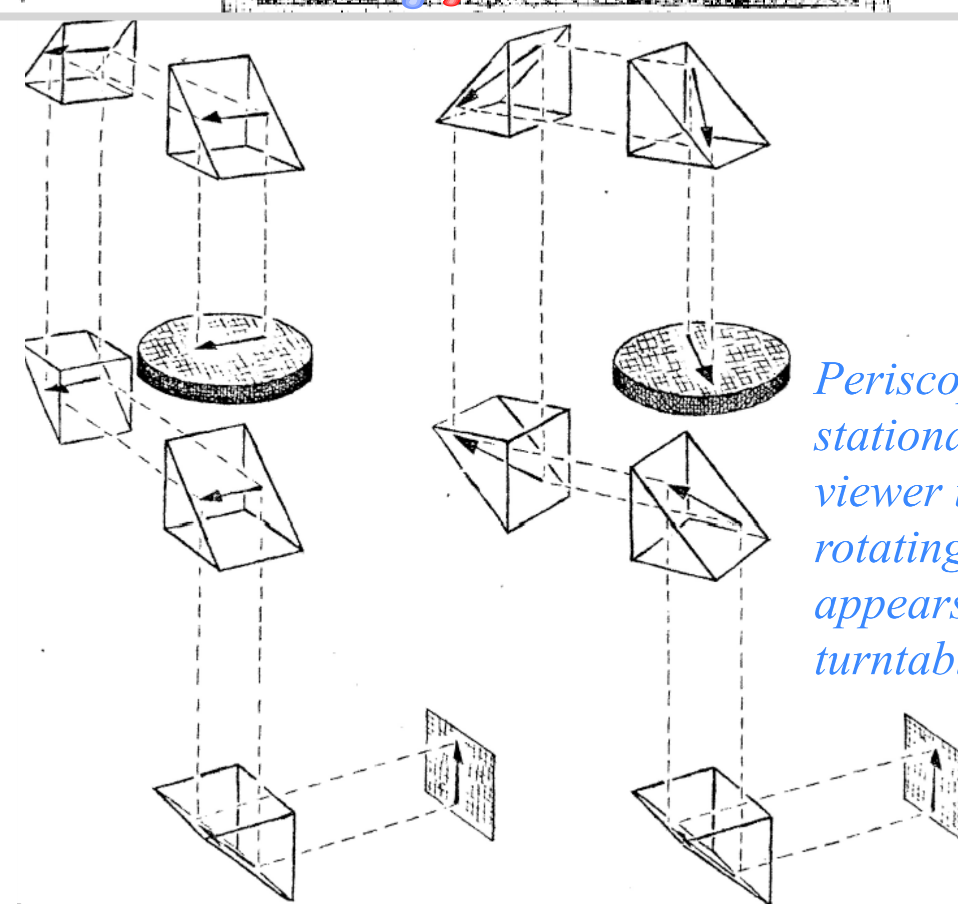


Sequential models of D. A. Adams' antitwister mechanism

From Scientific American December 1975-p.120-125

Wires do not get twisted up as the turntable rotates





Periscope allows stationary outside viewer to see into a rotating frame that appears fixed as the turntable rotates

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reference links
on following page

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The ABC's of U(2) dynamics

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

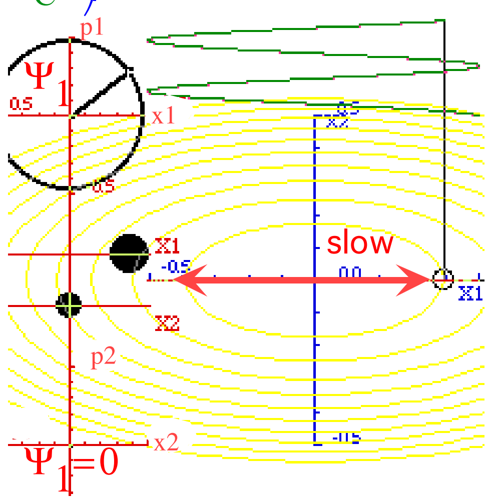
$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

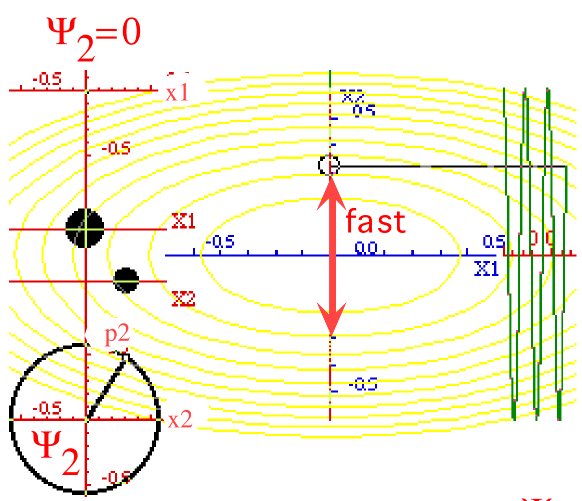
$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

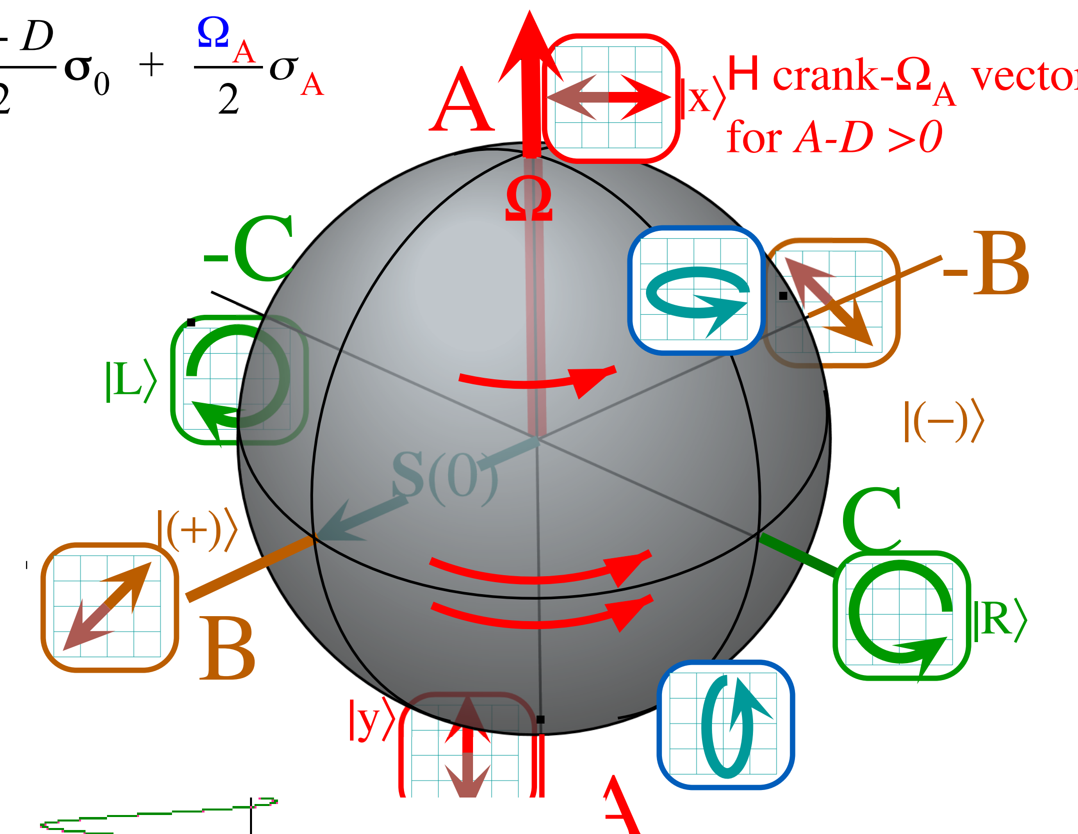
Asymmetric Diagonal A-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^A|1\rangle & \langle 1|\mathbf{H}^A|2\rangle \\ \langle 2|\mathbf{H}^A|1\rangle & \langle 2|\mathbf{H}^A|2\rangle \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_A}{2}\sigma_A$$

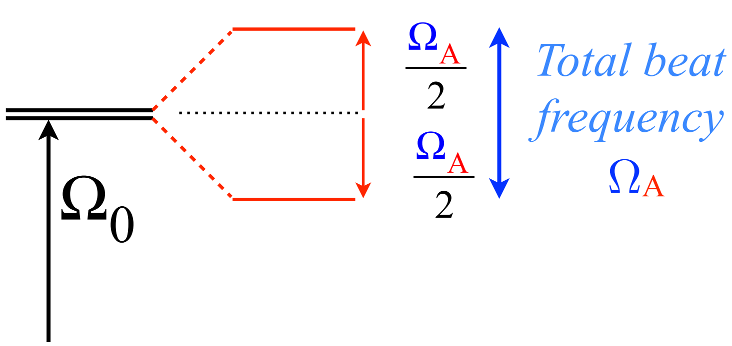
$$Crank: \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 0 \\ 0 \end{pmatrix} \qquad Eigen-Spin: \vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} \pm S \\ 0 \\ 0 \end{pmatrix}$$

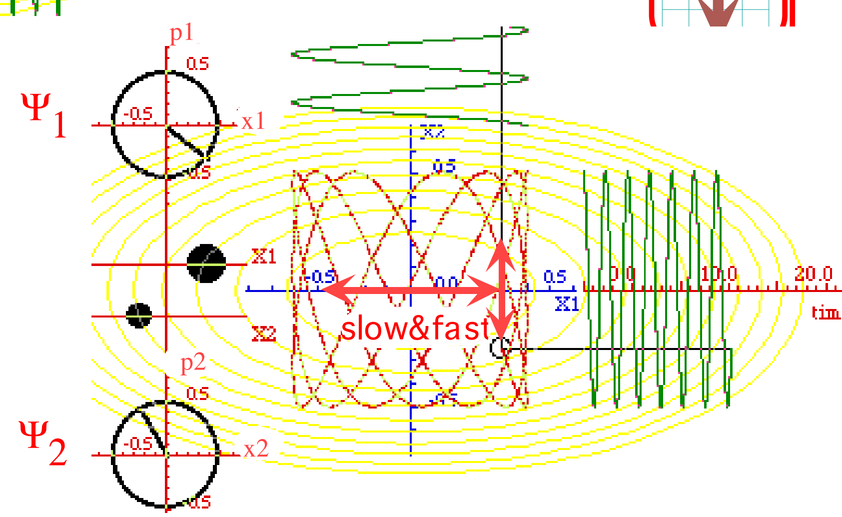






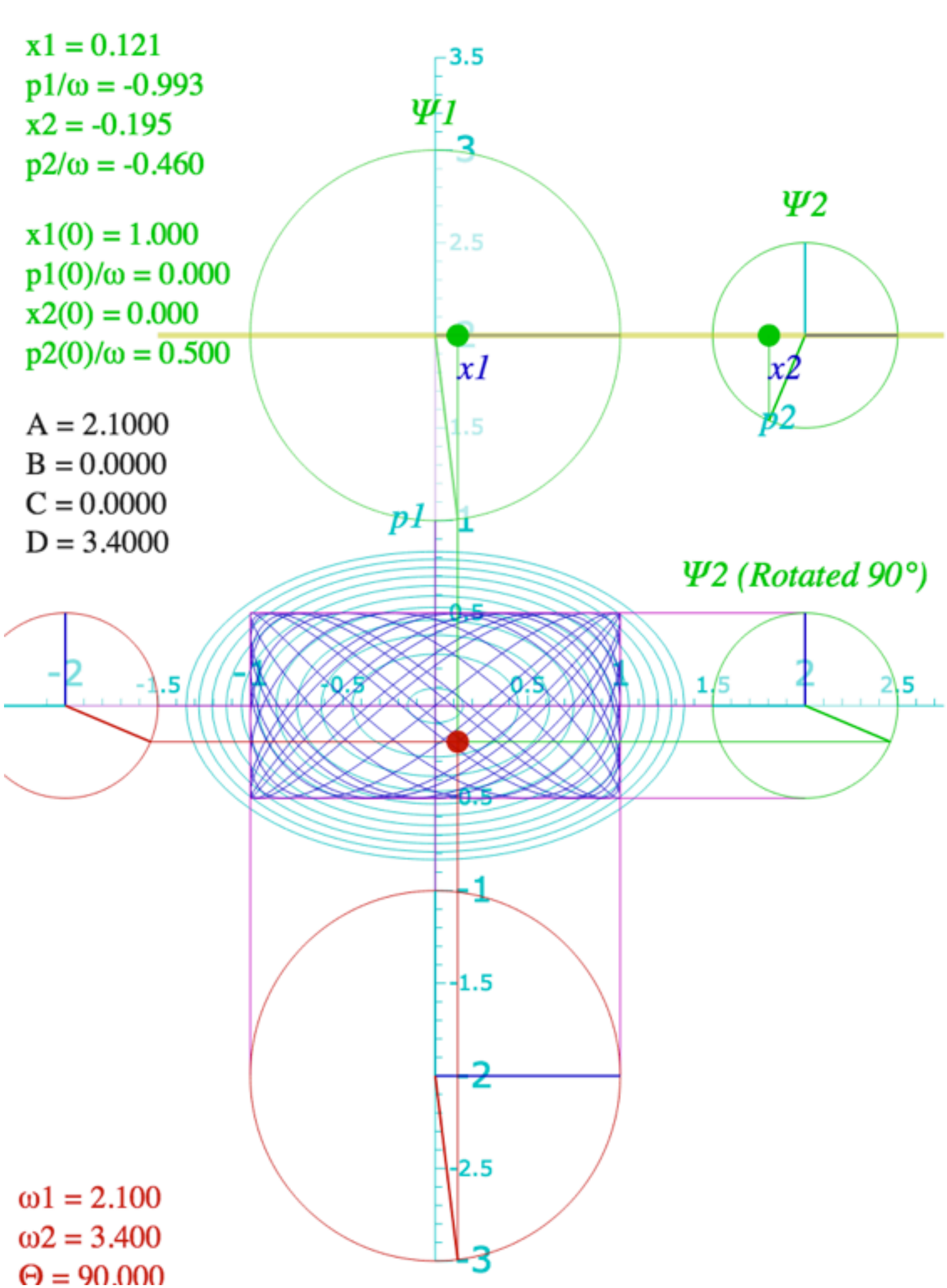
Beat dynamics:

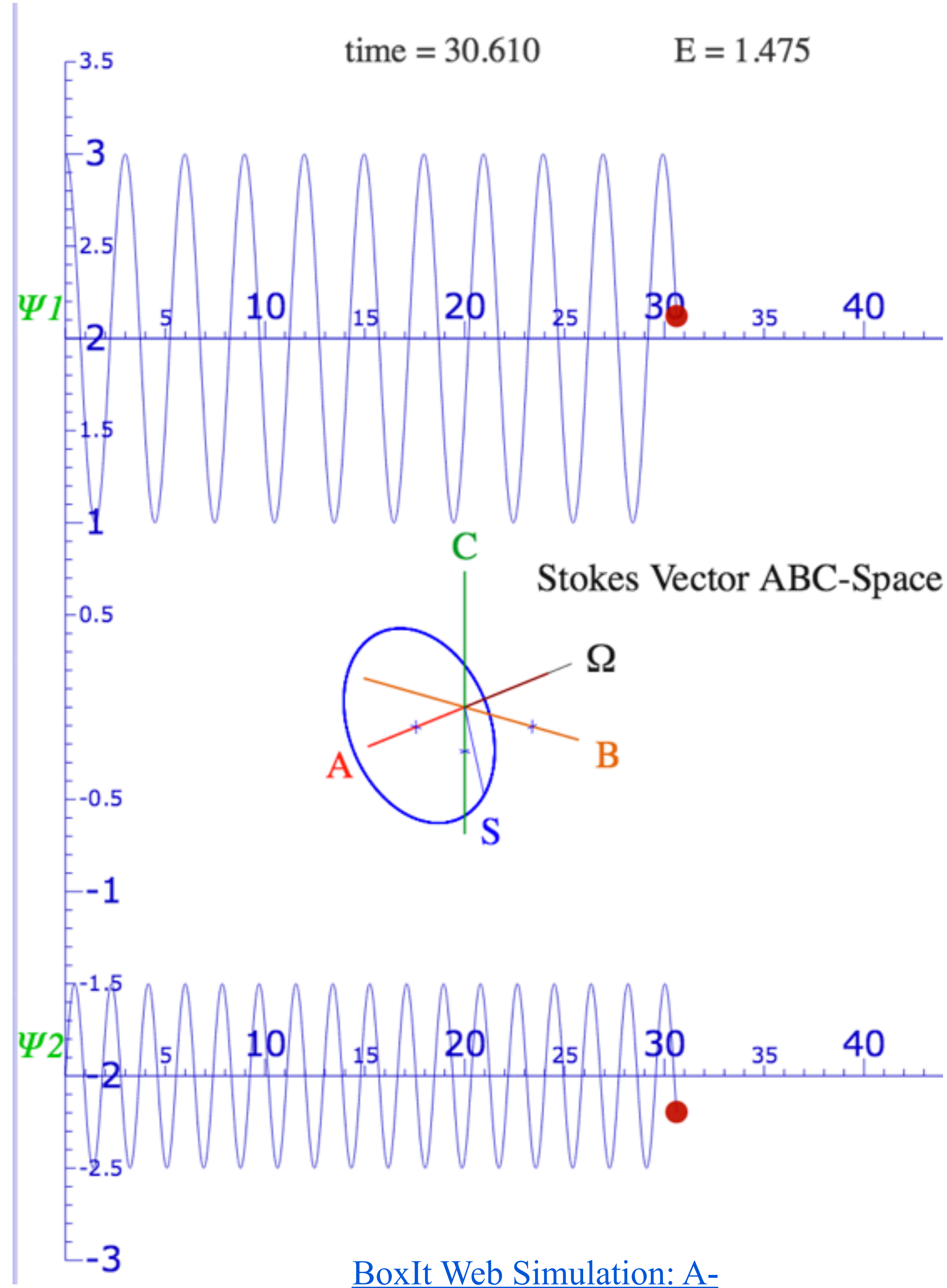




[BoxIt (A-Type) Web Simulation]

A-Type elliptical polarized motion





[BoxIt Web Simulation: A-Type with A=2.1, D=3.4]

The ABC's of U(2) dynamics

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\boldsymbol{\Omega}}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

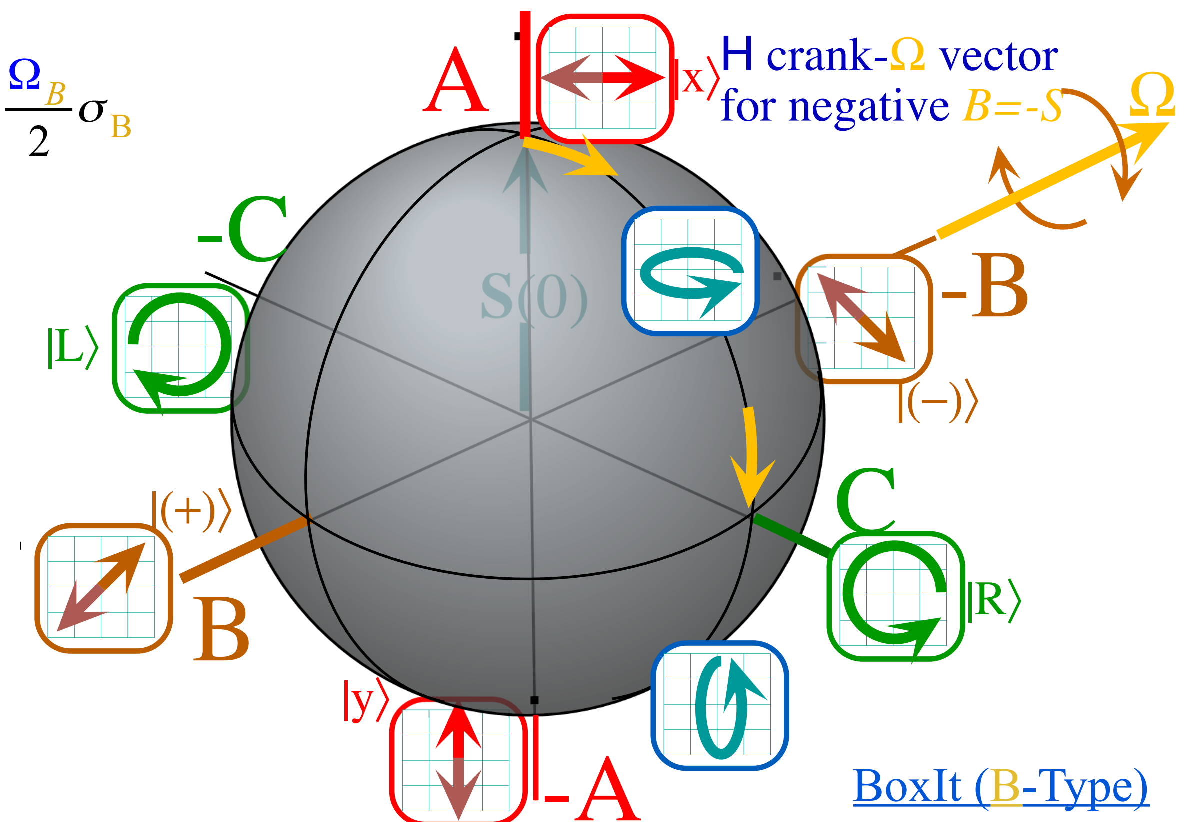
$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

$$\vec{\boldsymbol{\Omega}} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Bilateral-Balanced B-Type motion

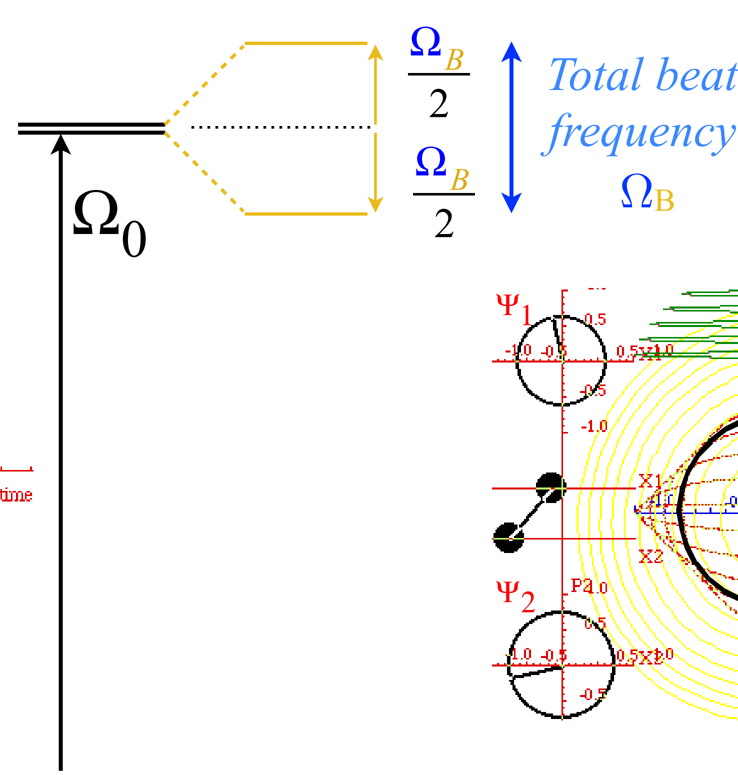
$$\begin{pmatrix} \langle 1|\mathbf{H}^B|1\rangle & \langle 1|\mathbf{H}^B|2\rangle \\ \langle 2|\mathbf{H}^B|1\rangle & \langle 2|\mathbf{H}^B|2\rangle \end{pmatrix} = \begin{pmatrix} \Omega_0 & B \\ B & \Omega_0 \end{pmatrix} = \Omega_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B$$

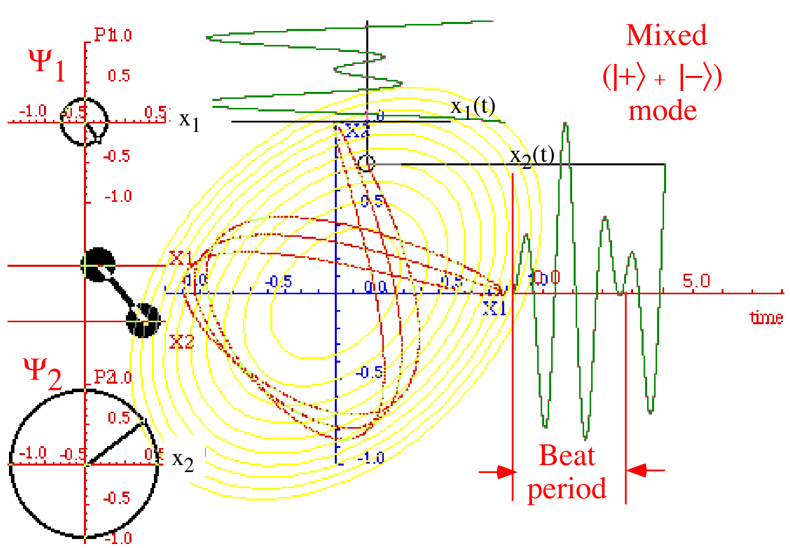
$$\textit{Crank}:\ \vec{\boldsymbol{\Omega}} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} 0 \\ 2B \\ 0 \end{pmatrix} \qquad \textit{Eigen-Spin}:\ \vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} 0 \\ \pm S \\ 0 \end{pmatrix}$$

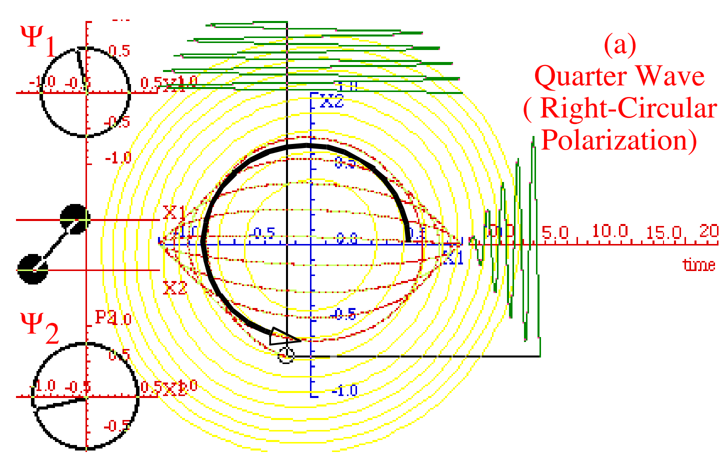


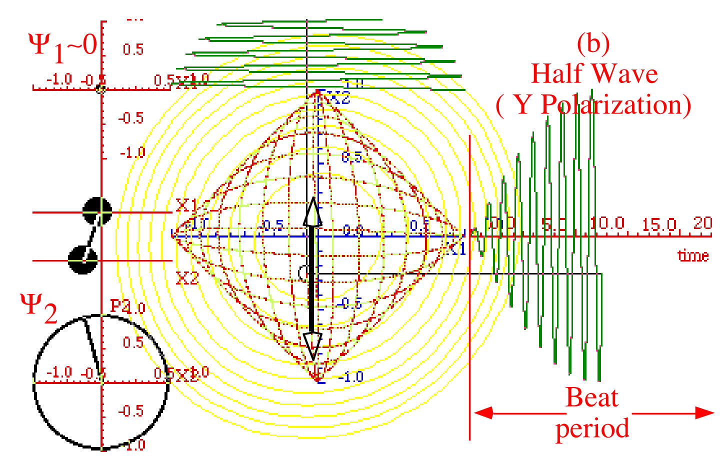
[BoxIt (B-Type) Web Simulation]

Beat dynamics:

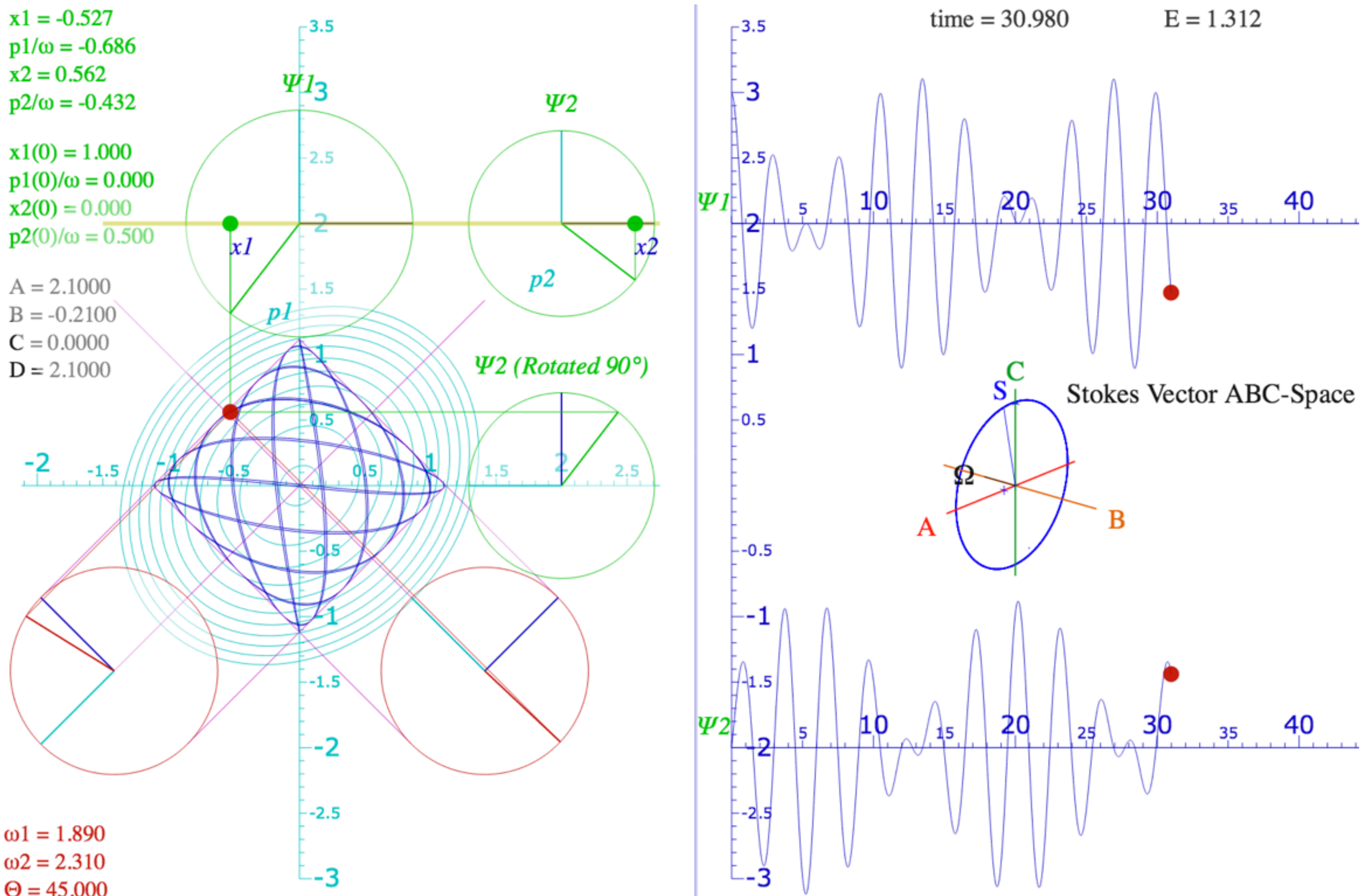








B-Type elliptical polarized motion



BoxIt Web Simulation:
B-Type with A, D=2.1; B=-0.21

AMOP reference links on following page

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$\mathbf{H}=A\boldsymbol{\sigma}_Z+B\boldsymbol{\sigma}_X+C\boldsymbol{\sigma}_Y$ (*ABC-Type*), *Invariant Tori in* (x_1,p_1,x_2,p_2)*-space.*

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$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\sigma$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\sigma$$

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$$= \frac{A+D}{2}\;\mathbf{1} \;+\; B\;\sigma_B \;+\; C\;\sigma_C \;+\; \frac{A-D}{2}\;\sigma_A$$

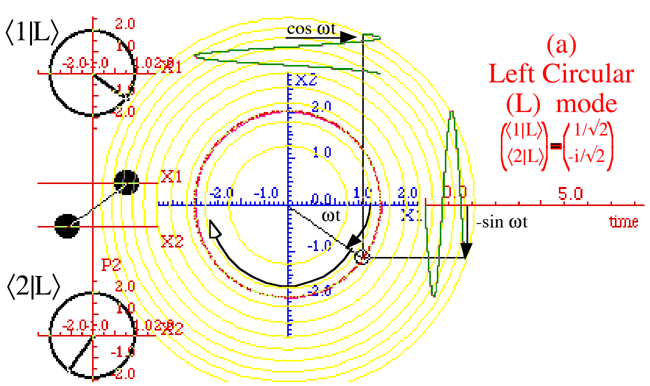
$$= \frac{A+D}{2}\;\sigma_0 \;+\; \frac{\Omega_B}{2}\;\sigma_B \;+\; \frac{\Omega_C}{2}\;\sigma_C \;+\; \frac{\Omega_A}{2}\;\sigma_A$$

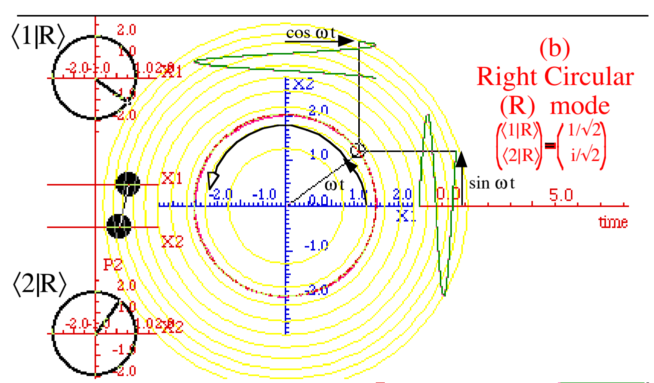
$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

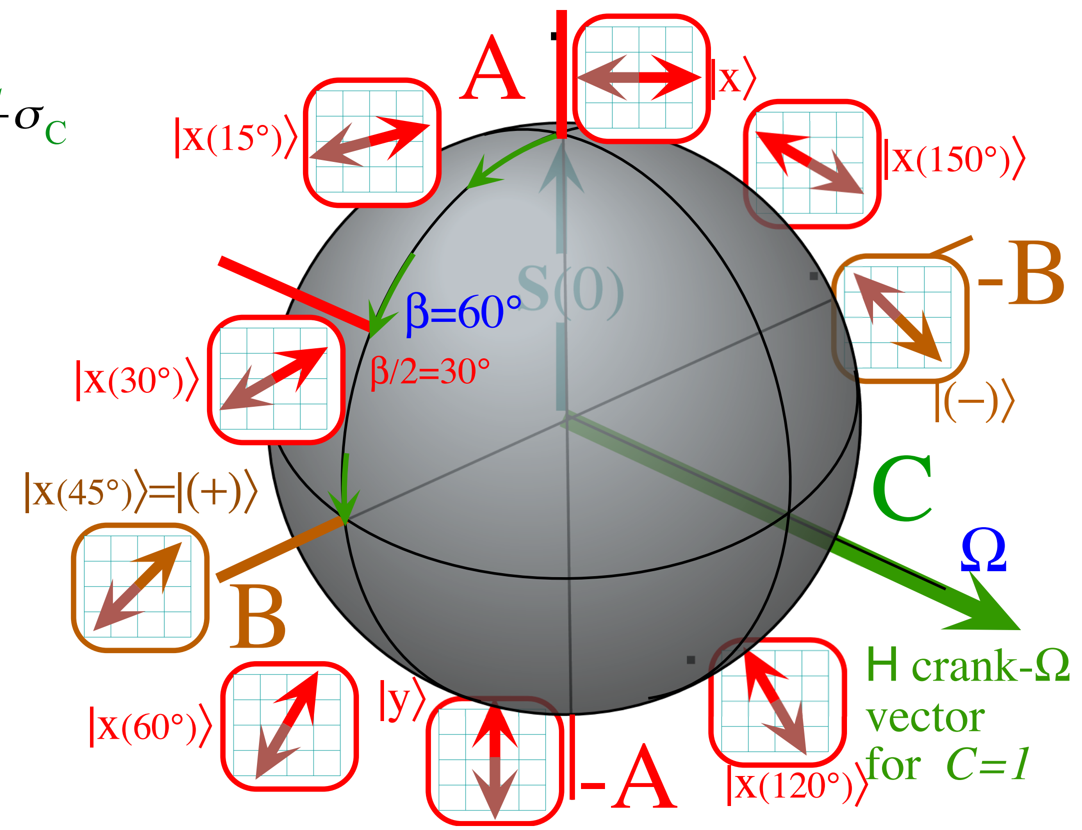
Circular-Coriolis... C-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^C|1\rangle & \langle 1|\mathbf{H}^C|2\rangle \\ \langle 2|\mathbf{H}^C|1\rangle & \langle 2|\mathbf{H}^C|2\rangle \end{pmatrix} = \begin{pmatrix} \Omega_0 & -iC \\ iC & \Omega_0 \end{pmatrix} = \Omega_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \Omega_0\sigma_0 + \frac{\Omega_C}{2}\sigma_C$$

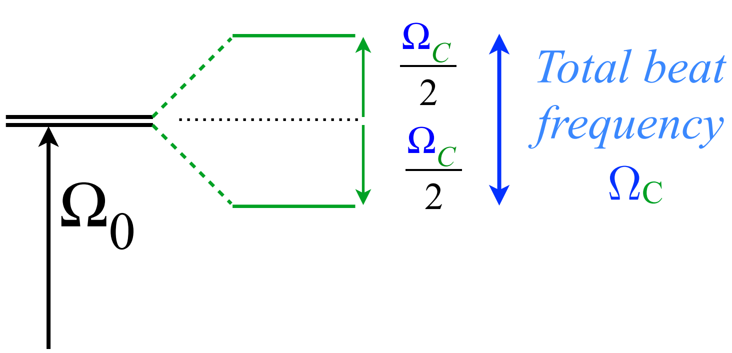
$$Crank:\ \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2C \end{pmatrix} \qquad Eigen-Spin:\vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \pm S \end{pmatrix}$$

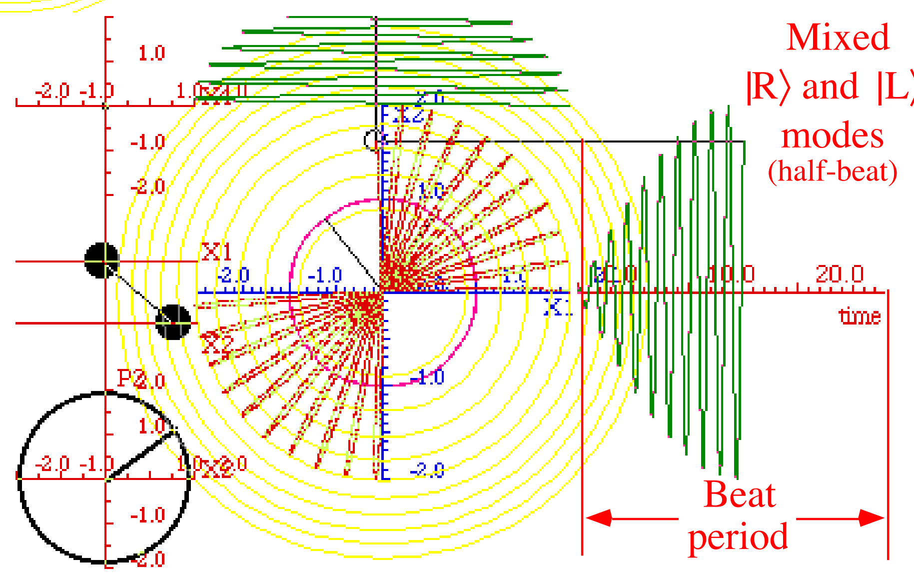






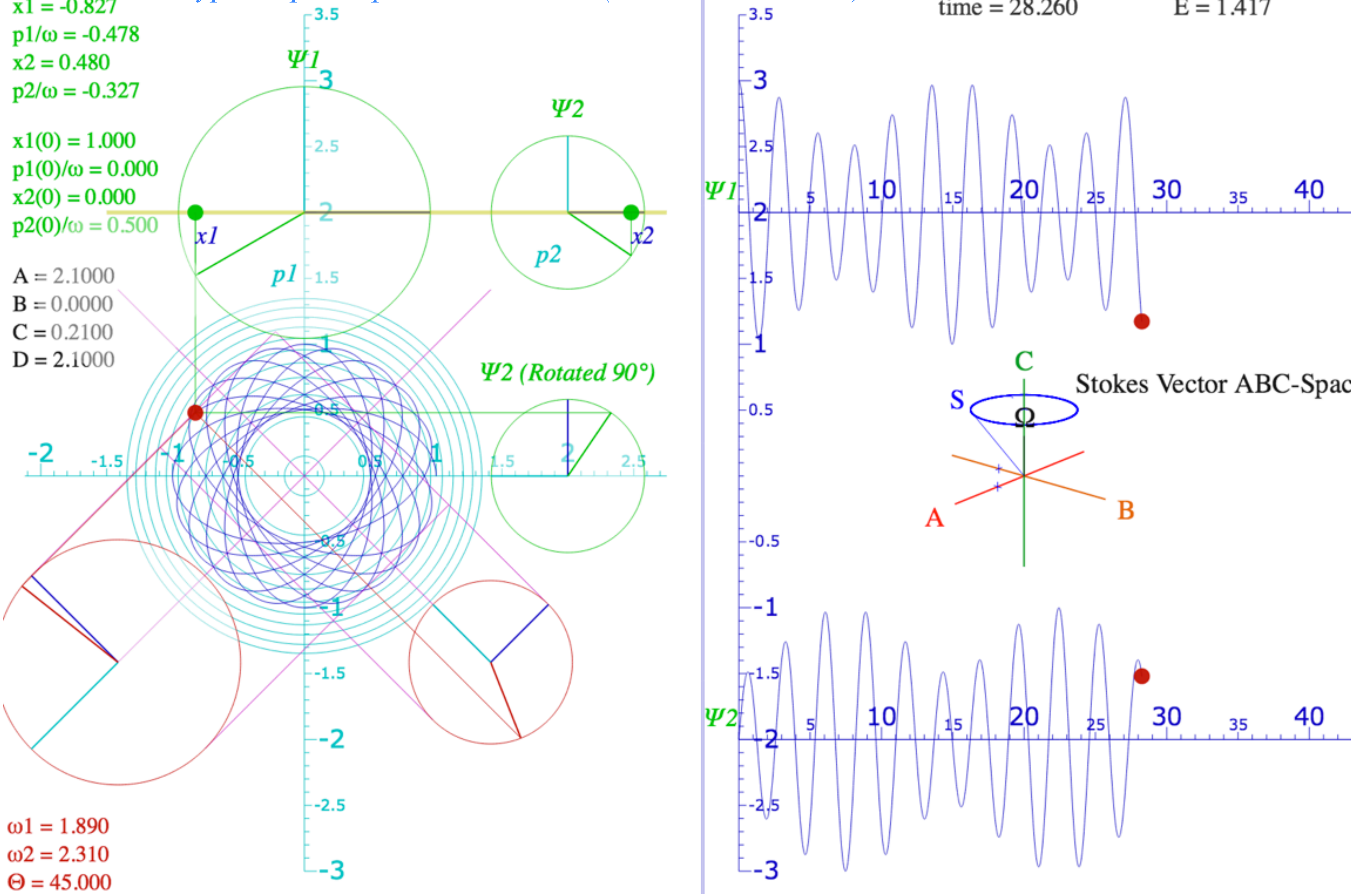
Beat dynamics:





BoxIt (Web Simulation

C-Type elliptical polarized motion (BoxIt Web Simulation)



BoxIt Web Simulation:
C-Type with A, D=2.1; C=-0.21

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Euler angle labeling of optical polarization *C-Type* $(\alpha_C,\beta_C,\gamma_C)$ *vs A-Type* $(\alpha_A,\beta_A,\gamma_A)$,

$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} E_1 & V \\ V & E_2 \end{pmatrix}$$

2nd order perturbation terms

$$\lambda_1 = E_1 + \frac{V^2}{E_1 - E_2} \,,$$

$$\lambda_2 = E_2 + \frac{V^2}{E_2 - E_1} \,.$$

Shows how hi-order polynomial perturbation algebra is hard to match to hyperbolic energy levels.

$$\lambda^2 - (Trace\mathbf{H})\lambda + \det|\mathbf{H}| = 0 = \lambda^2 - (E_1 + E_2)\lambda + (E_1E_2 - V^2)$$

$$\lambda_{1,2} = \frac{E_1 + E_2 \pm \sqrt{(E_1 + E_2)^2 - 4E_1E_2 + 4V^2}}{2} = \frac{E_1 + E_2 \pm \sqrt{(E_1 - E_2)^2 + 4V^2}}{2} \,,$$

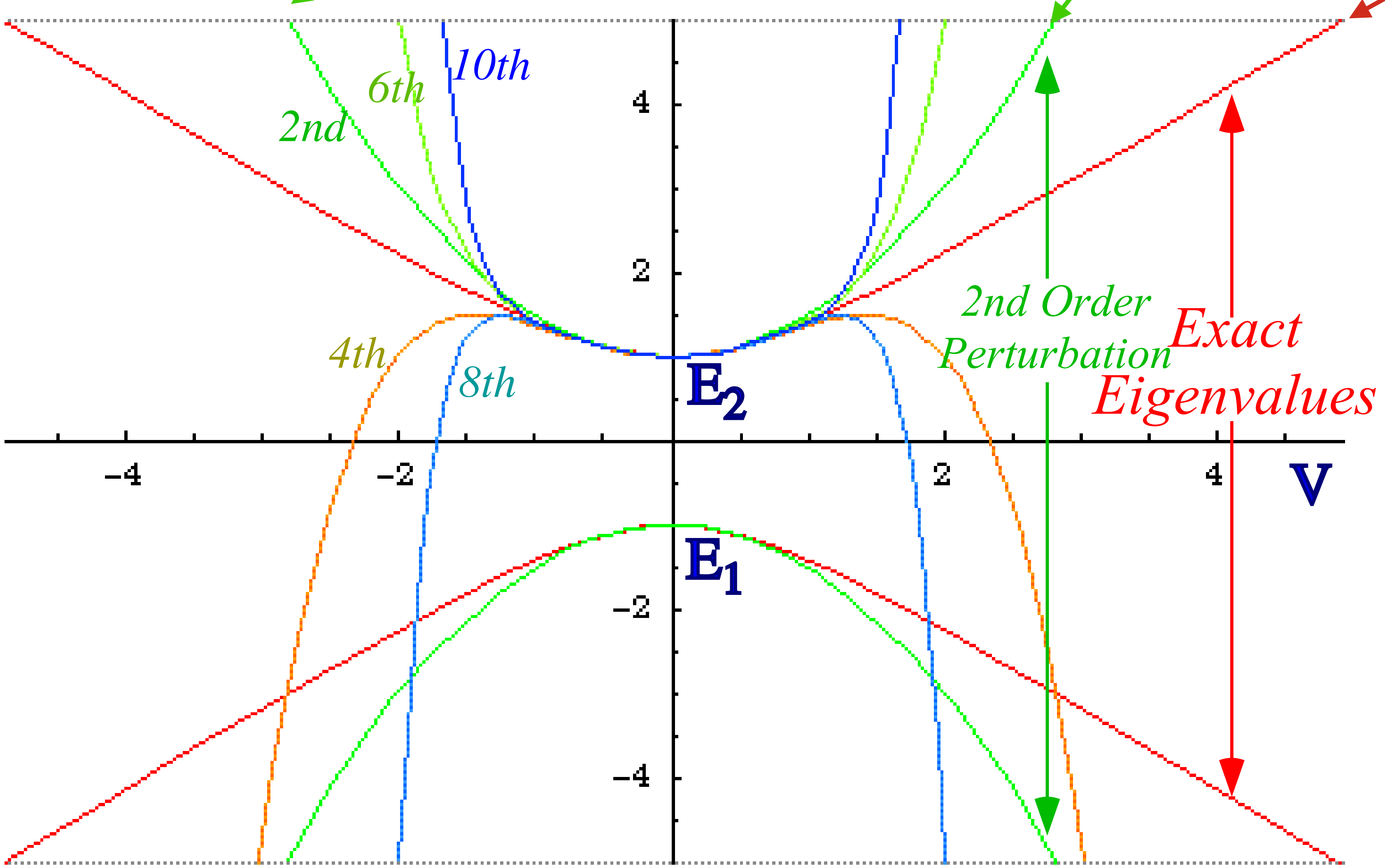


Fig. 3.2.2 Comparison of exact vs. 2nd-order thru 10th-order perturbation approximations

$$E_2 = \frac{\Delta}{2} + \frac{V^2}{\Delta} - \frac{V^4}{\Delta^3} + \frac{V^6}{\Delta^5} - \frac{V^8}{\Delta^7} + \frac{V^{10}}{\Delta^9} \cdots \text{, where: } \Delta = |E_1 - E_2|$$

The ABC's of U(2) dynamics-Mixed modes

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$
$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

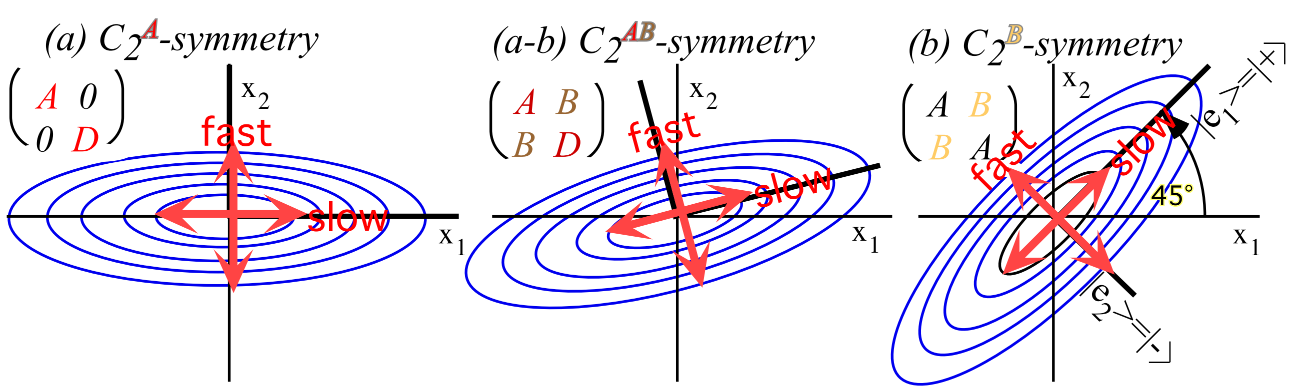
$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

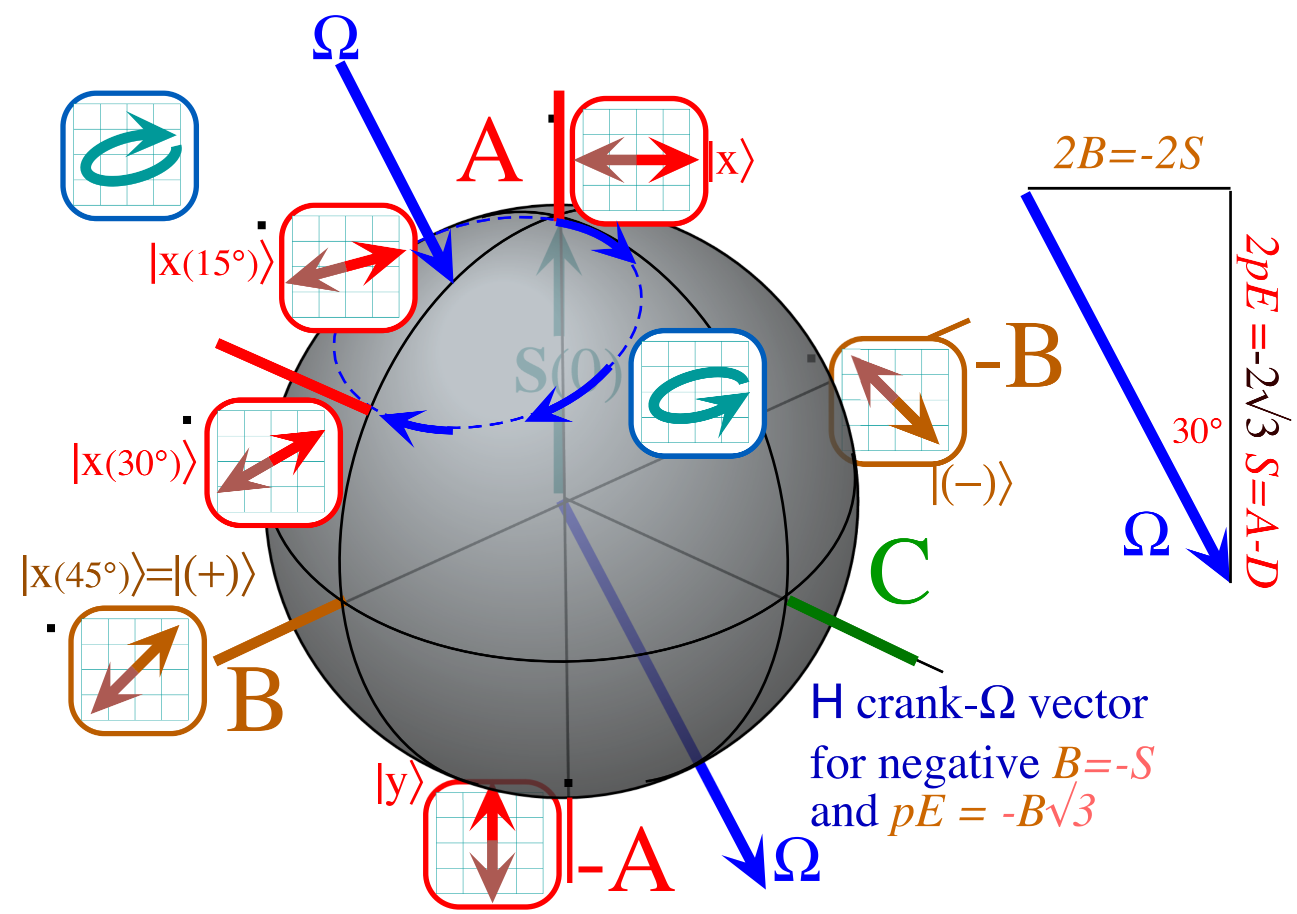
Tilted-plane polarization AB-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^{AB}|1\rangle & \langle 1|\mathbf{H}^{AB}|2\rangle \\ \langle 2|\mathbf{H}^{AB}|1\rangle & \langle 2|\mathbf{H}^{AB}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B \\ B & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_A}{2}\sigma_A + \frac{\Omega_B}{2}\sigma_B$$

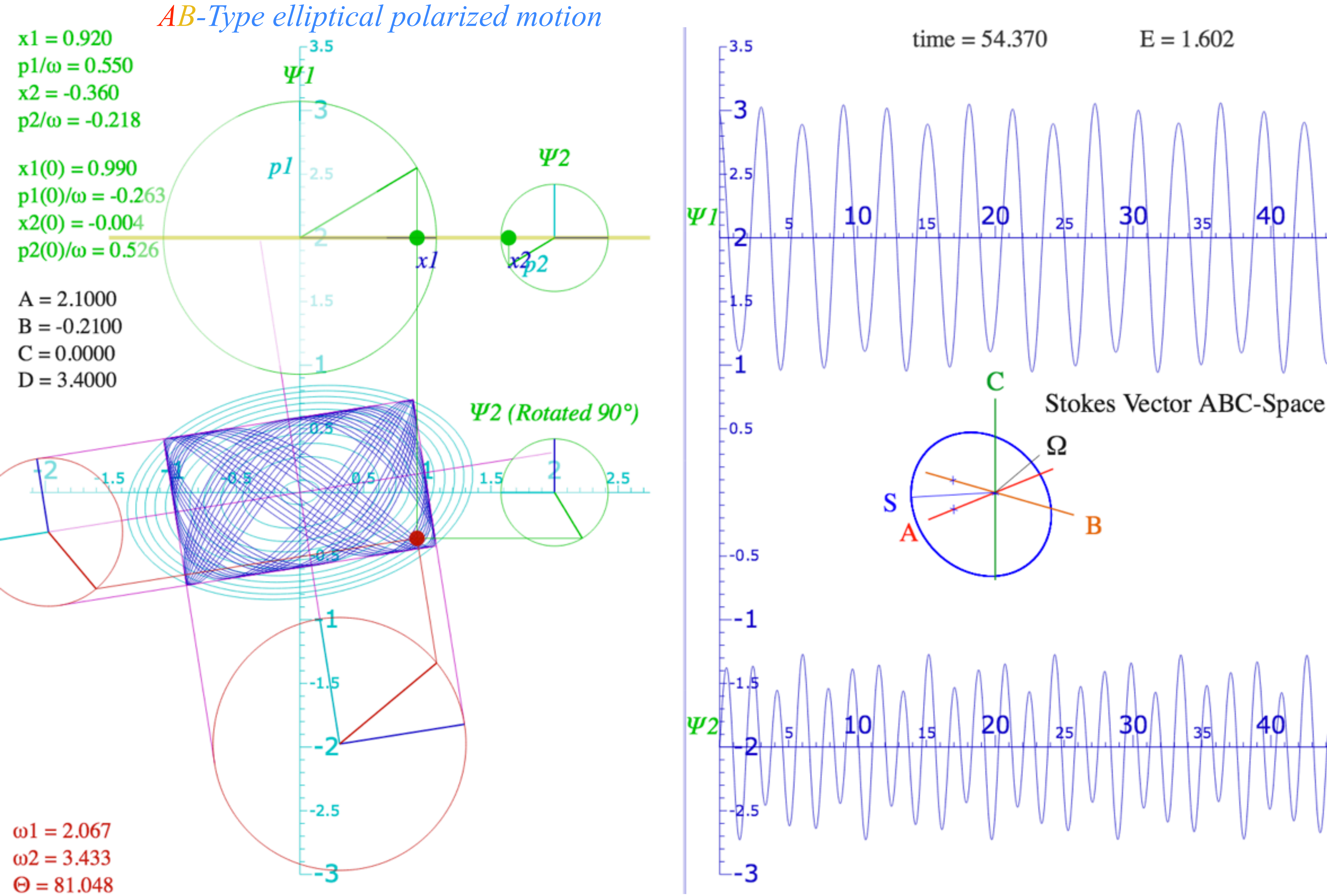
$$Crank: \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 0 \end{pmatrix} \qquad Eigen-Spin: \vec{\mathbf{S}} = \pm S\cdot\hat{\Omega}$$



Beat dynamics:



BoxIt (AB-Type Motion)
Web Simulation



[BoxIt Web Simulation:](#)
[AB-Type with A=2.1; B=-0.21; D=3.4](#)

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A to B to A Symmetry breaking described by hyperbolic eigenvalues of $A\sigma_A + B\sigma_B = \mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$

$$\mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \quad \text{Secular equation: } \varepsilon^2 - 0 \cdot \varepsilon - (A^2 + B^2) \quad \text{gives } \textit{hyperbolic} \text{ energy levels: } \varepsilon = \pm\sqrt{A^2 + B^2}$$

Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa and with varying C, too.

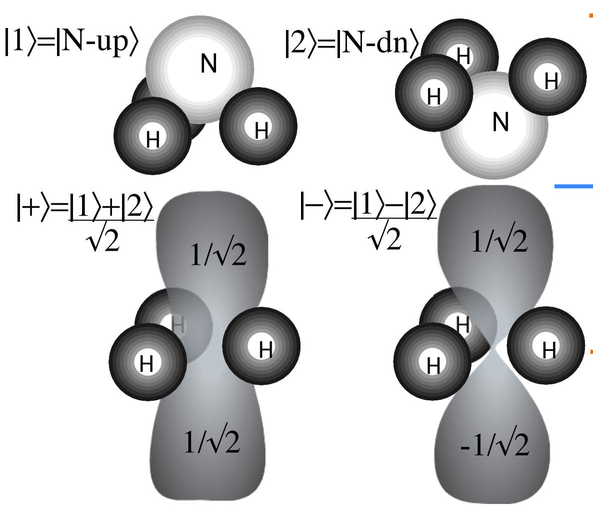


Fig. 10.3.2 Ammonia (NH₃) inversion states
(a) Base states (b) C₂-Eigenstates

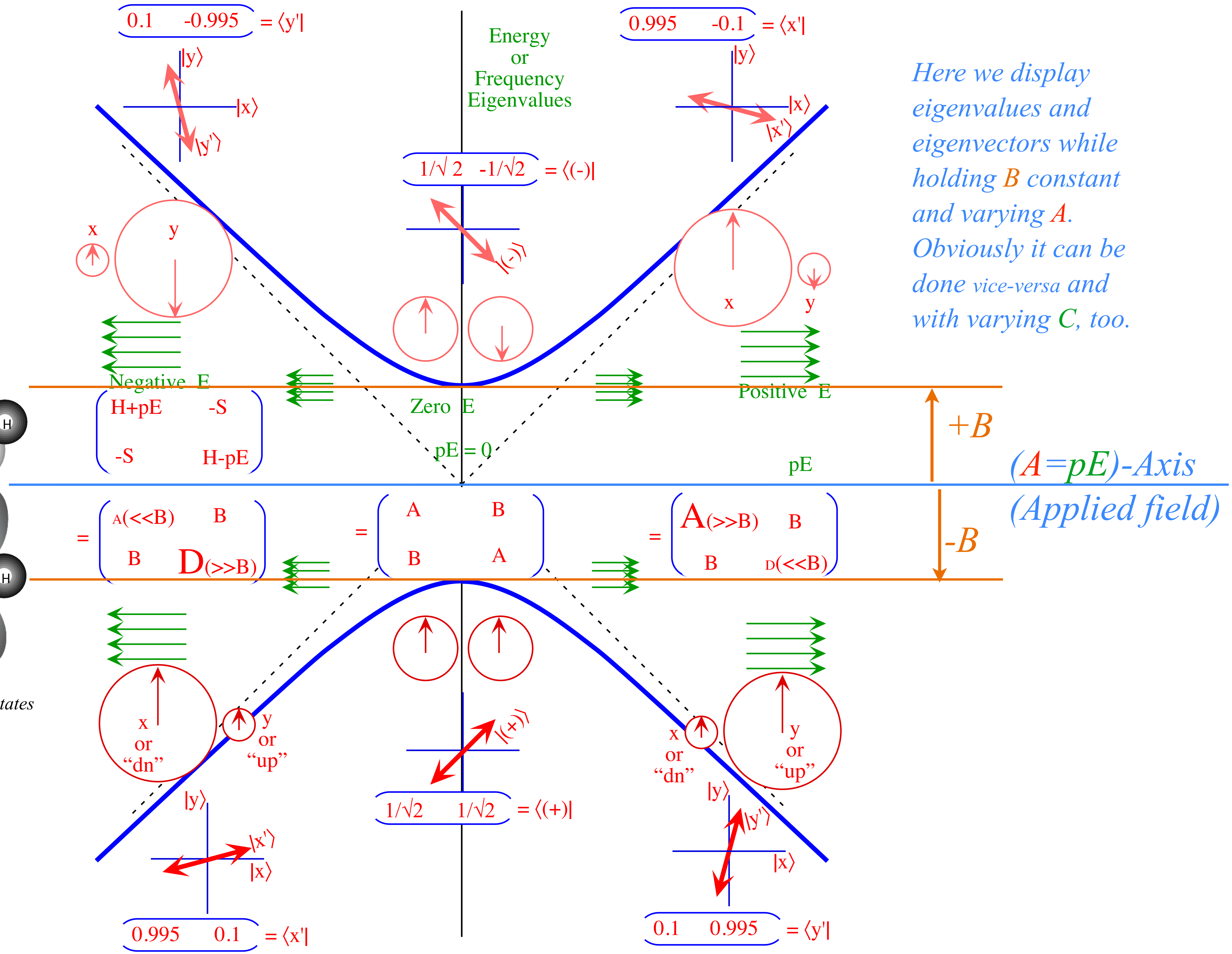


Fig. 10.3.1 (b) Wigner avoided level crossing. (Fixed tunneling B=-S and variable A-D=pE field.)

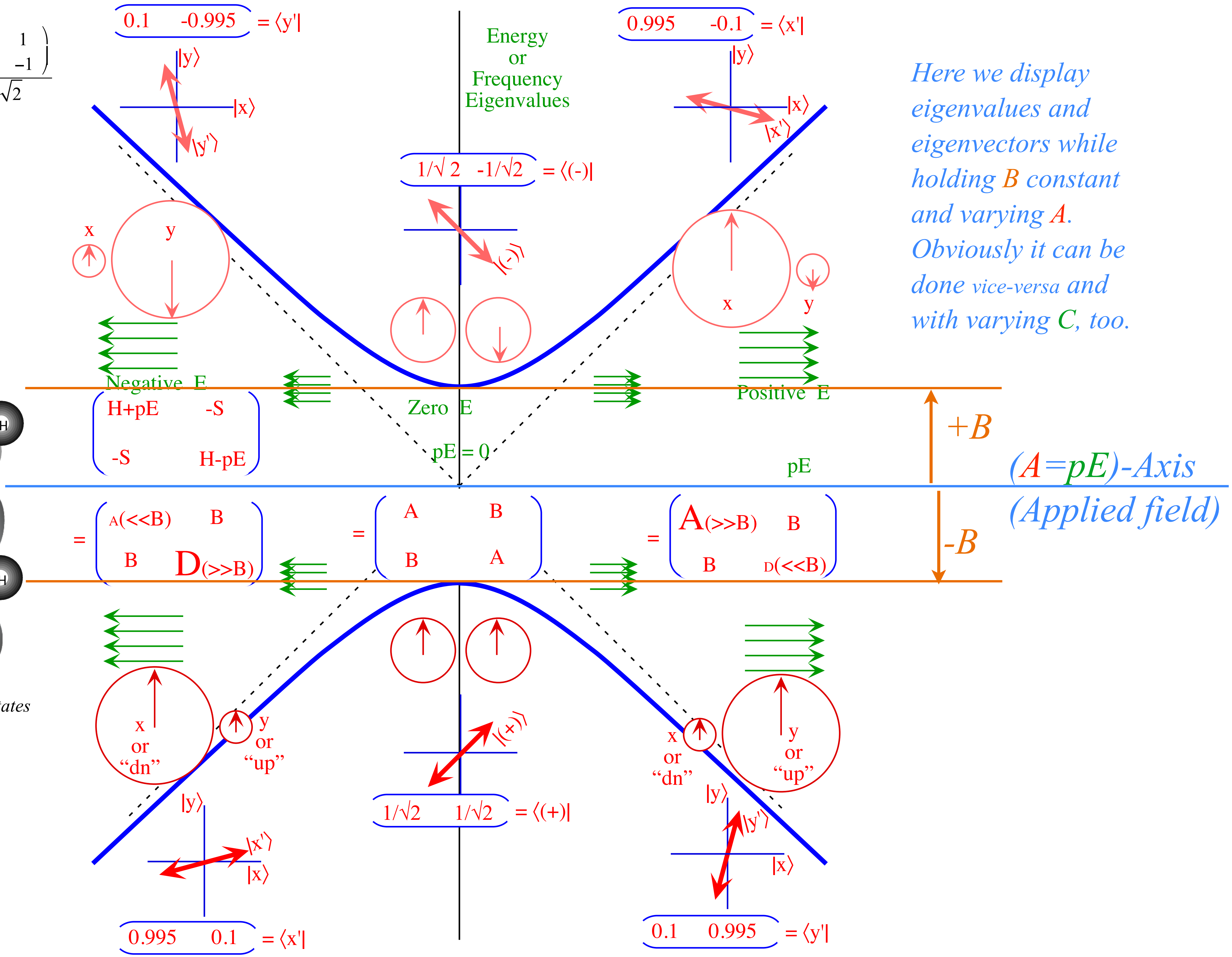
A to B to A Symmetry breaking described by hyperbolic eigenvalues of $A\boldsymbol{\sigma}_A + B\boldsymbol{\sigma}_B = \mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$

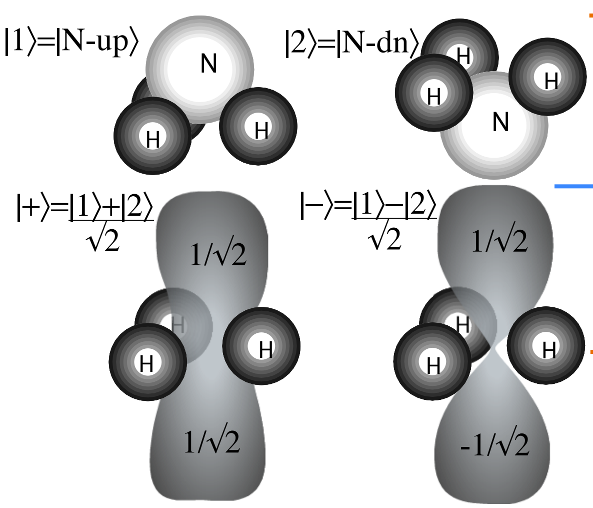
$\mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$ Secular equation: $\varepsilon^2 - 0 \cdot \varepsilon - (A^2 + B^2)$ gives *hyperbolic* energy levels: $\varepsilon = \pm\sqrt{A^2 + B^2}$

H(*B – basis*) **H**(*A – basis*)

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}} \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}}$$

Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa and with varying C, too.





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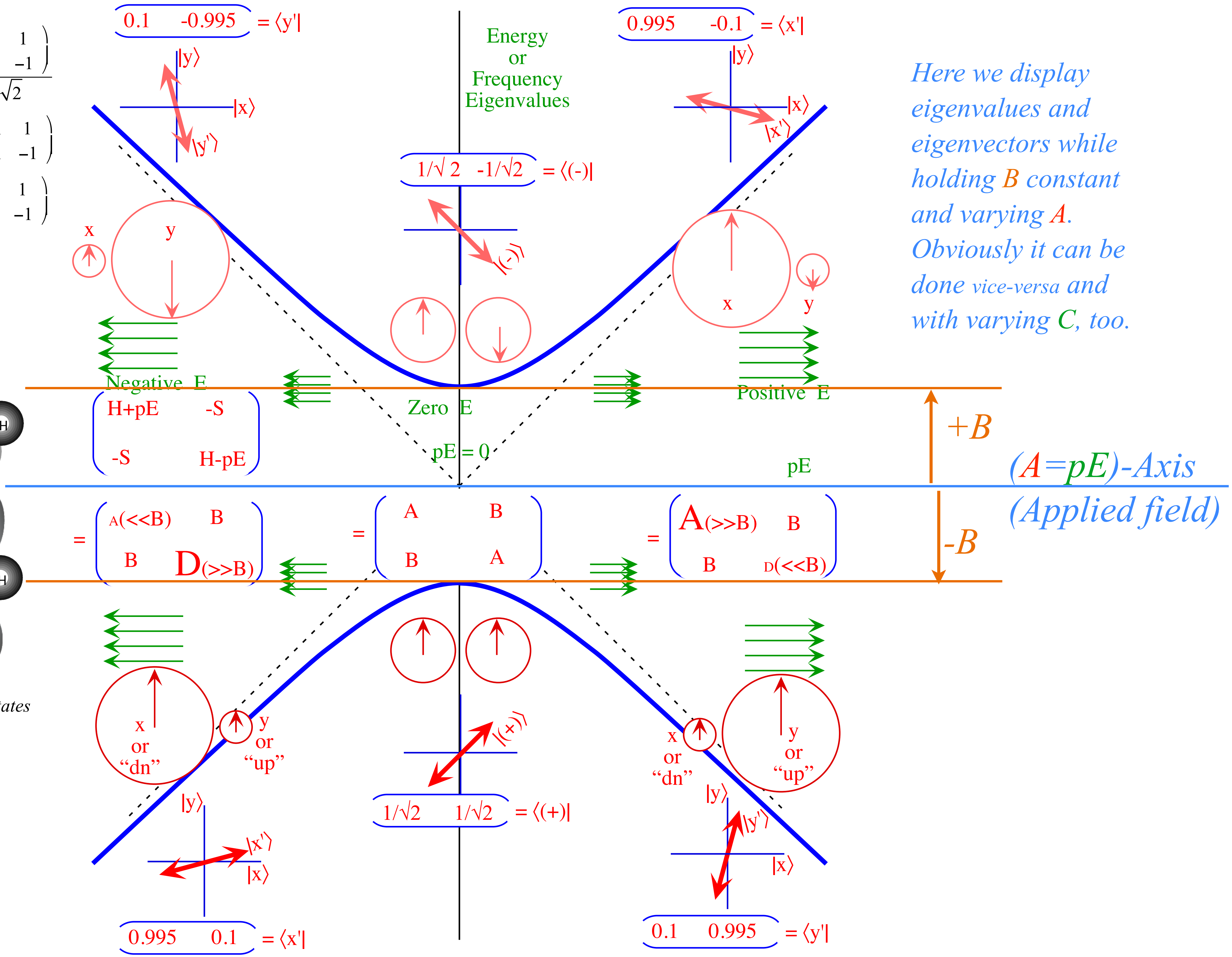
$\mathbf{H}(B-basis)$ $\mathbf{H}(A-basis)$

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}} \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} +A & B \\ B & -A \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} +A+B & B-A \\ +A-B & B+A \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$



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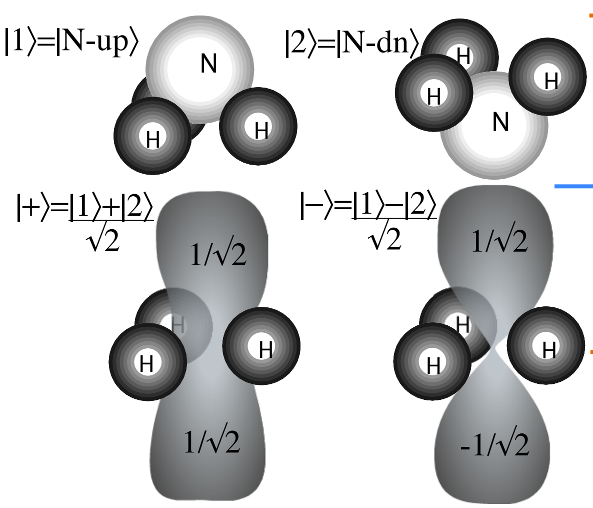


Fig. 10.3.2 Ammonia (NH_3) inversion states
(a) Base states (b) C_2-Eigenstates

Fig. 10.3.1 (b) Wigner avoided level crossing. (Fixed tunneling B=-S and variable A-D=pE field.)

A to B to A Symmetry breaking described by hyperbolic eigenvalues of $A\boldsymbol{\sigma}_A + B\boldsymbol{\sigma}_B = \mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$

$$\mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \quad \text{Secular equation: } \varepsilon^2 - 0\cdot\varepsilon - (A^2 + B^2) \quad \text{gives } \textit{hyperbolic} \text{ energy levels: } \varepsilon = \pm\sqrt{A^2 + B^2}$$

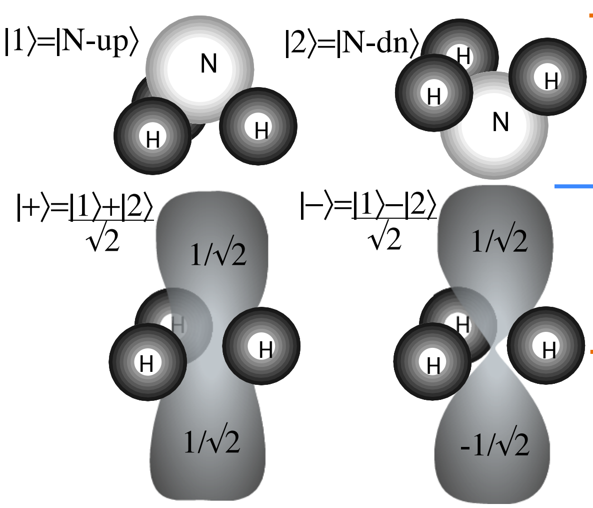
H(*B – basis*) **H**(*A – basis*)

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}} \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} +A & B \\ B & -A \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} +A+B & B-A \\ +A-B & B+A \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2B & 2A \\ 2A & -2B \end{pmatrix}$$



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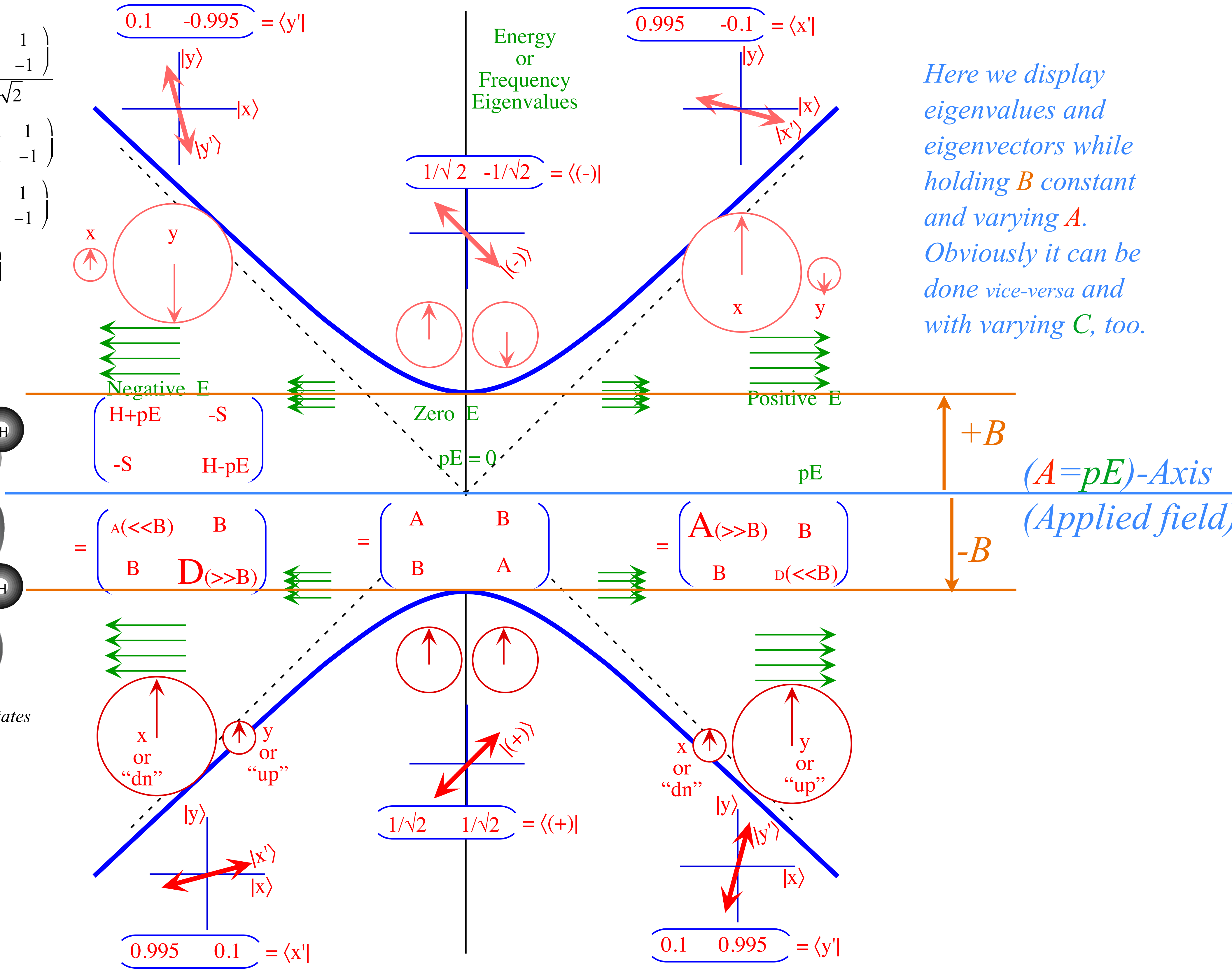

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$\mathbf{H}(B-basis)$ $\quad$ $\mathbf{H}(A-basis)$

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}} \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}}$$

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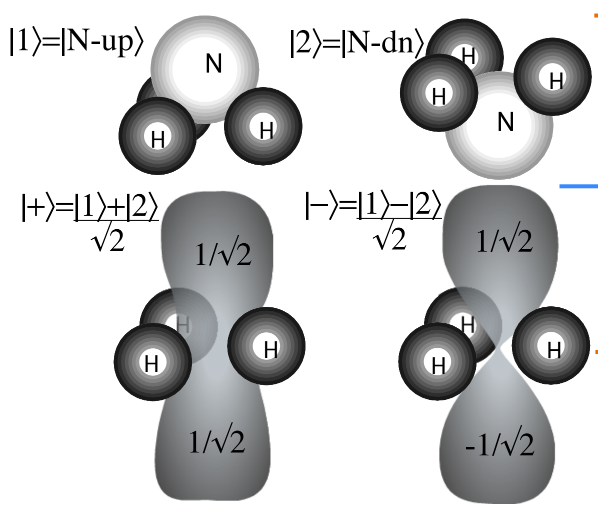

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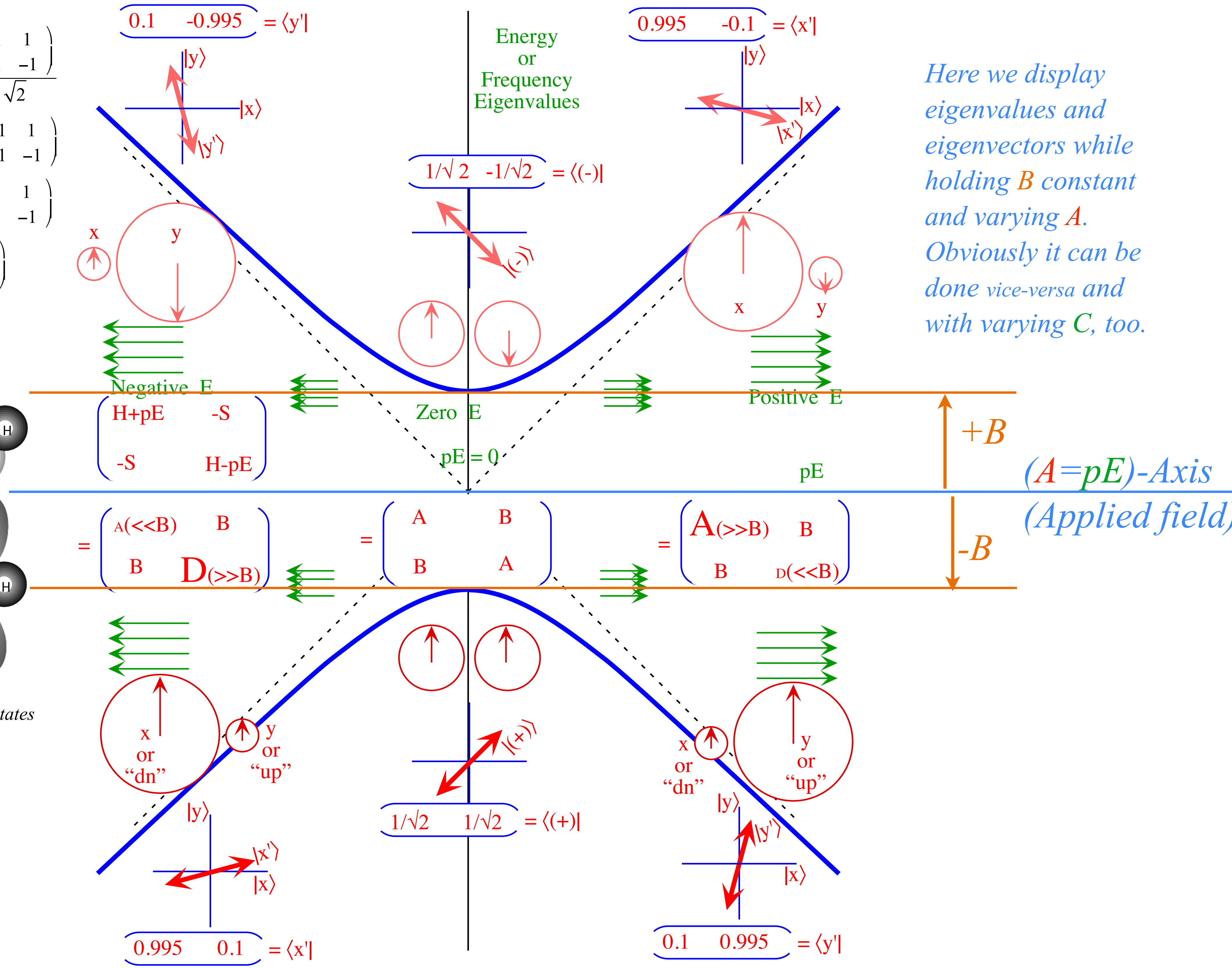

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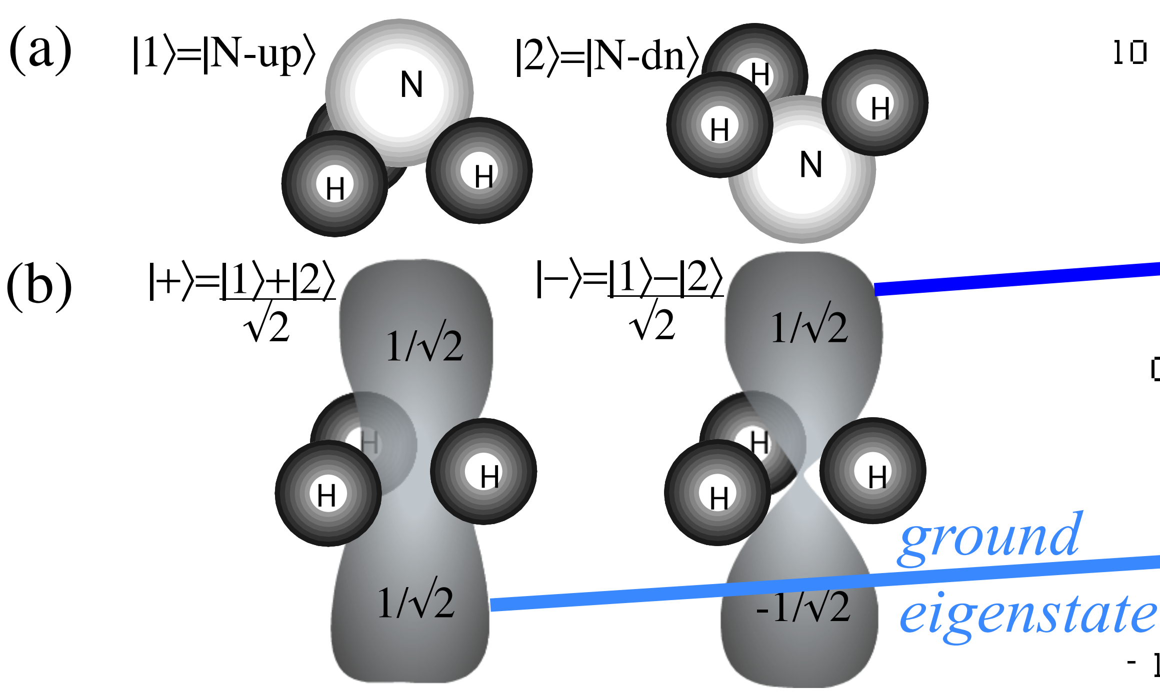
A view of a conical intersection:



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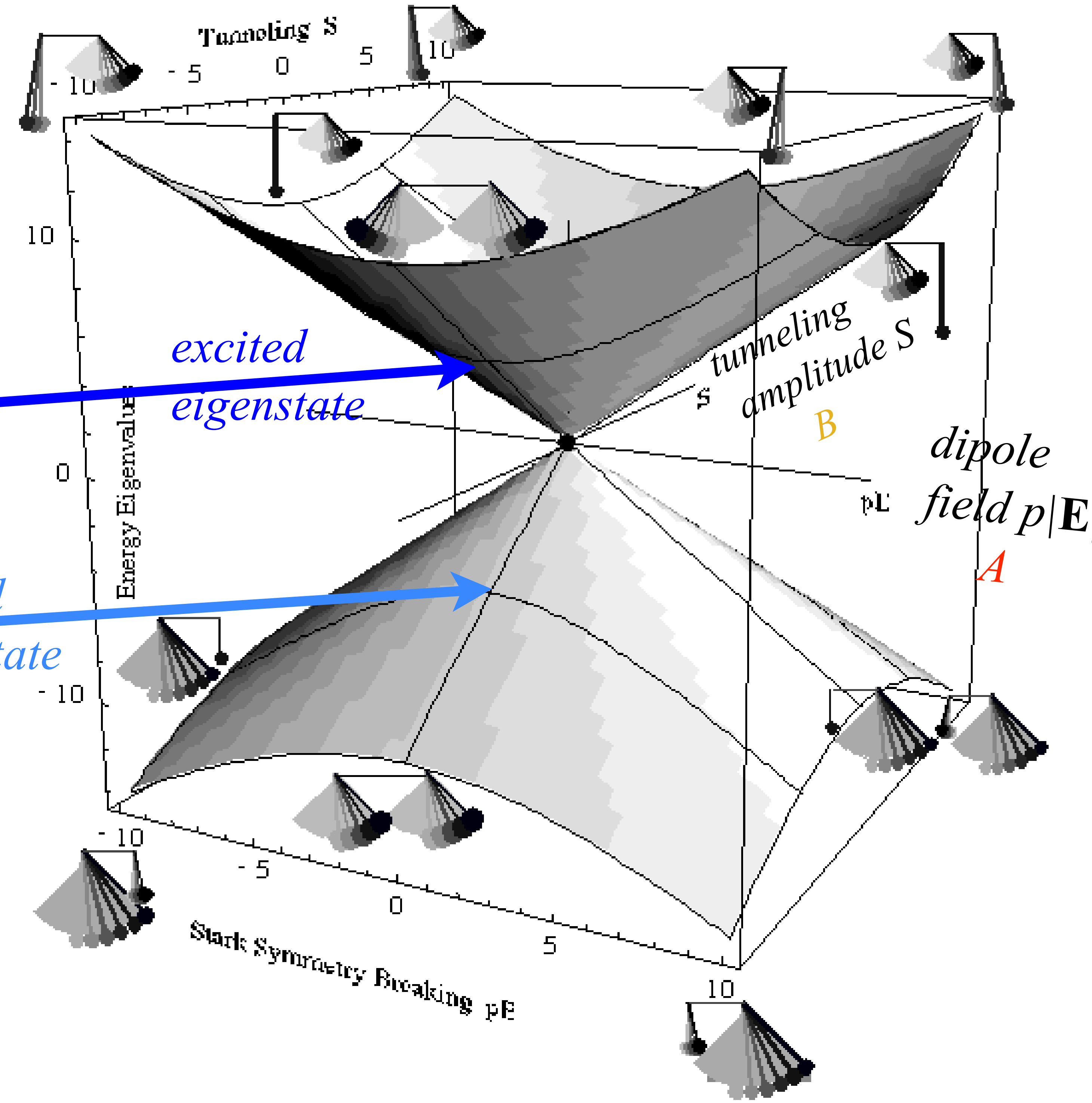
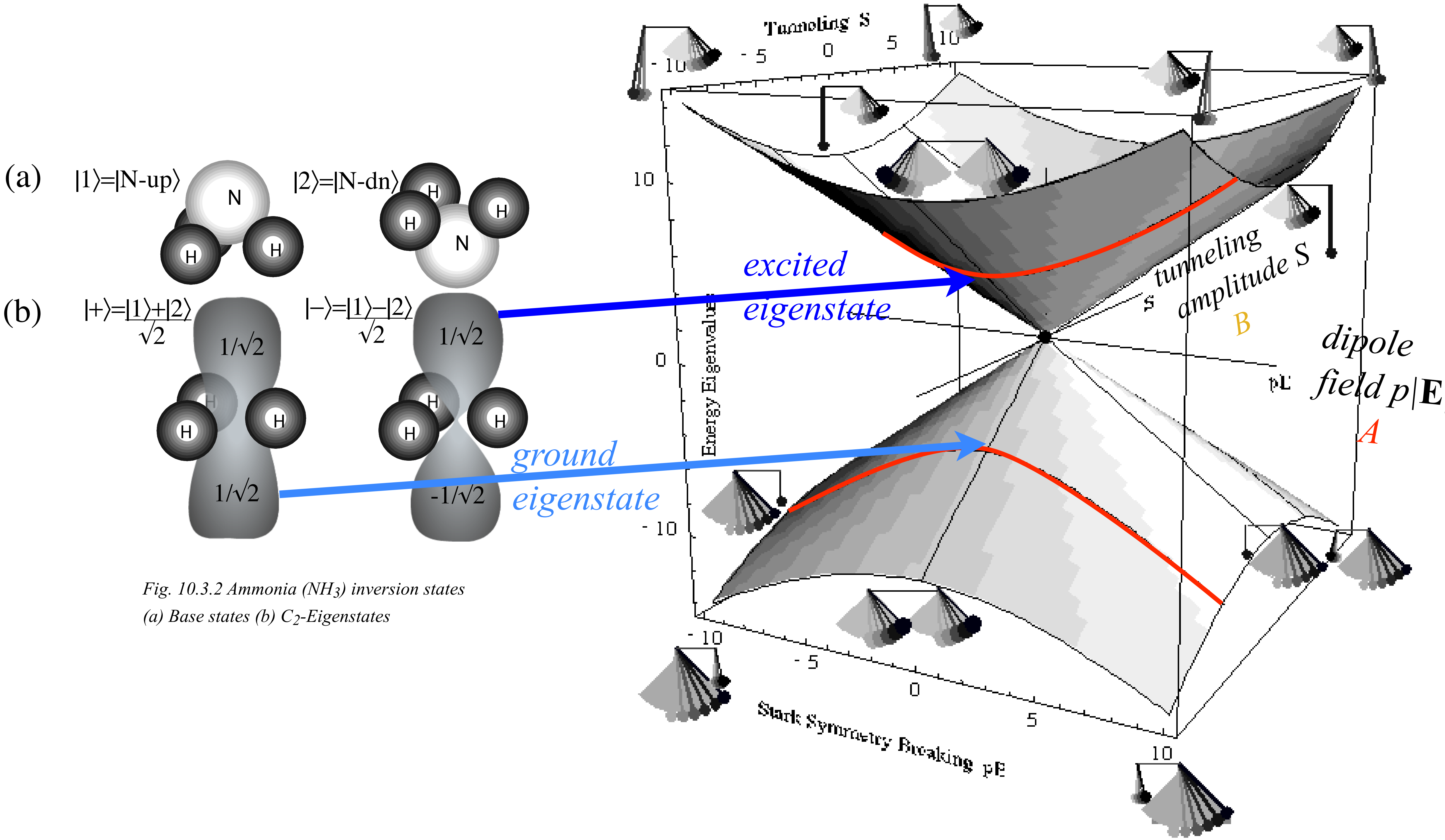
(a) Base states (b) C_2-Eigenstates



10.3.1 (a) Two state eigenvalue "diablo" surfaces and conical intersection and pendulum eigenstates.

A view of a conical intersection: Any vertical cross-section is hyperbolic avoided-crossing



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ABC-Type elliptical polarized motion

(from Principles of Symmetry, Dynamics, and Spectroscopy)

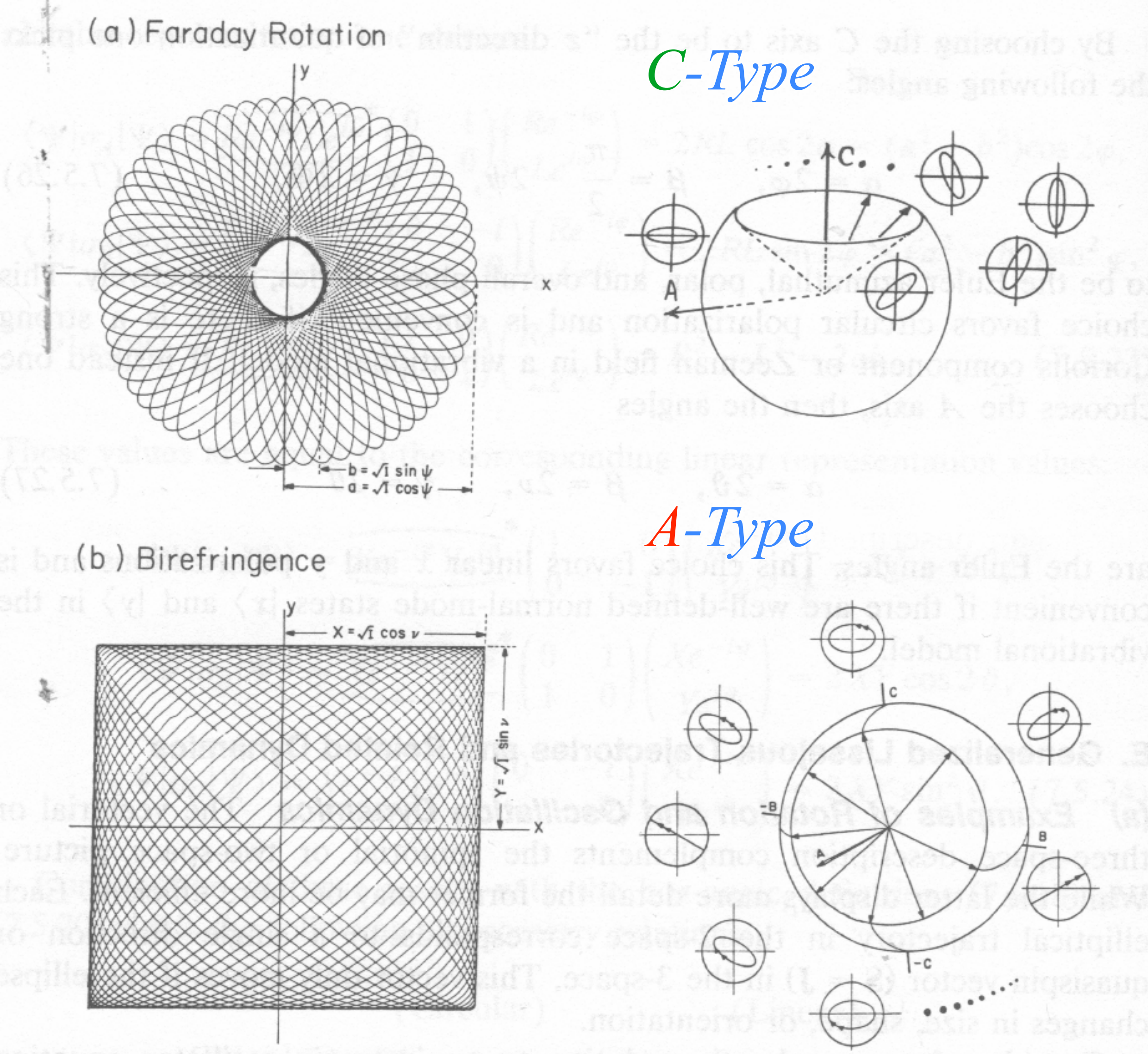
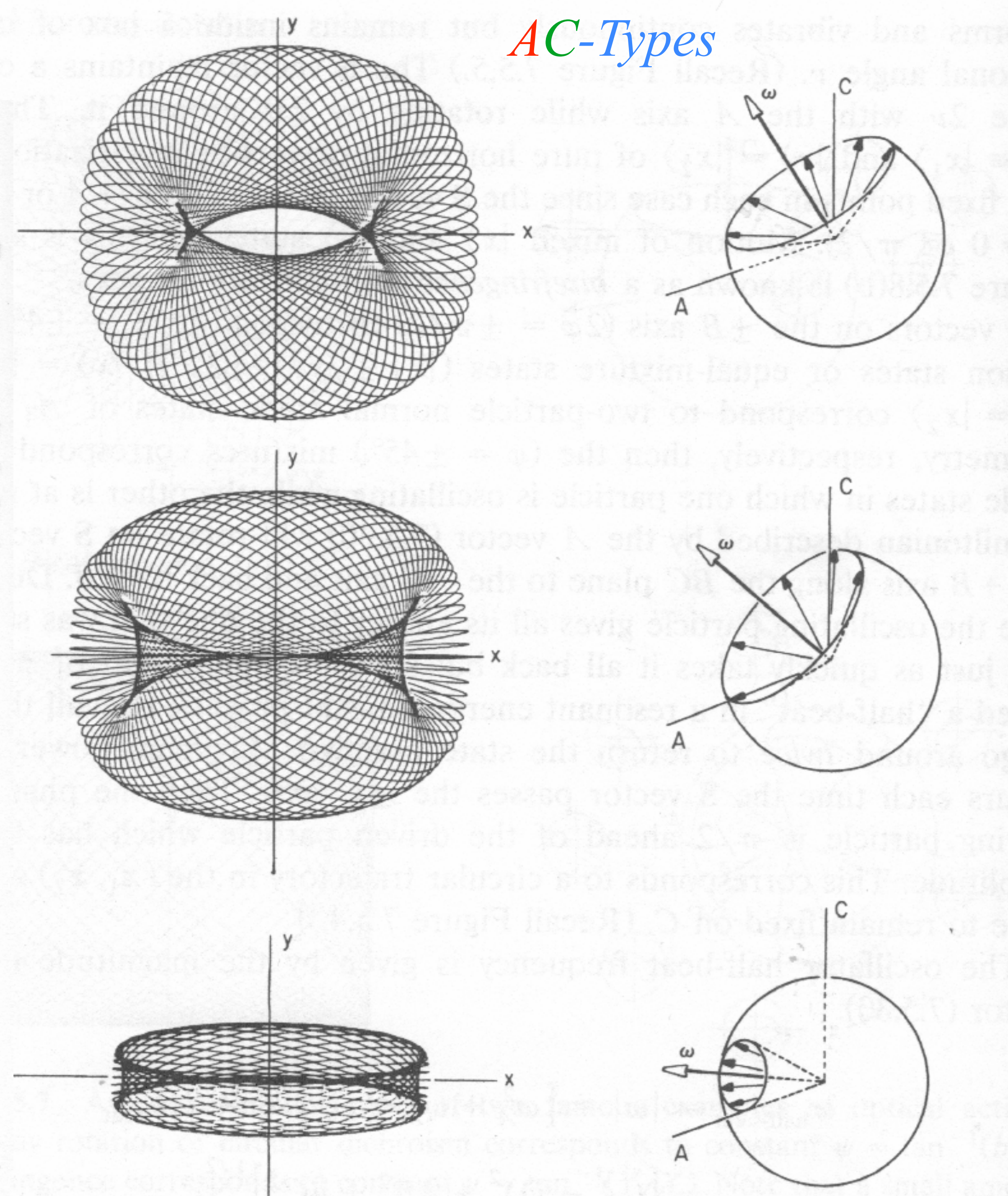
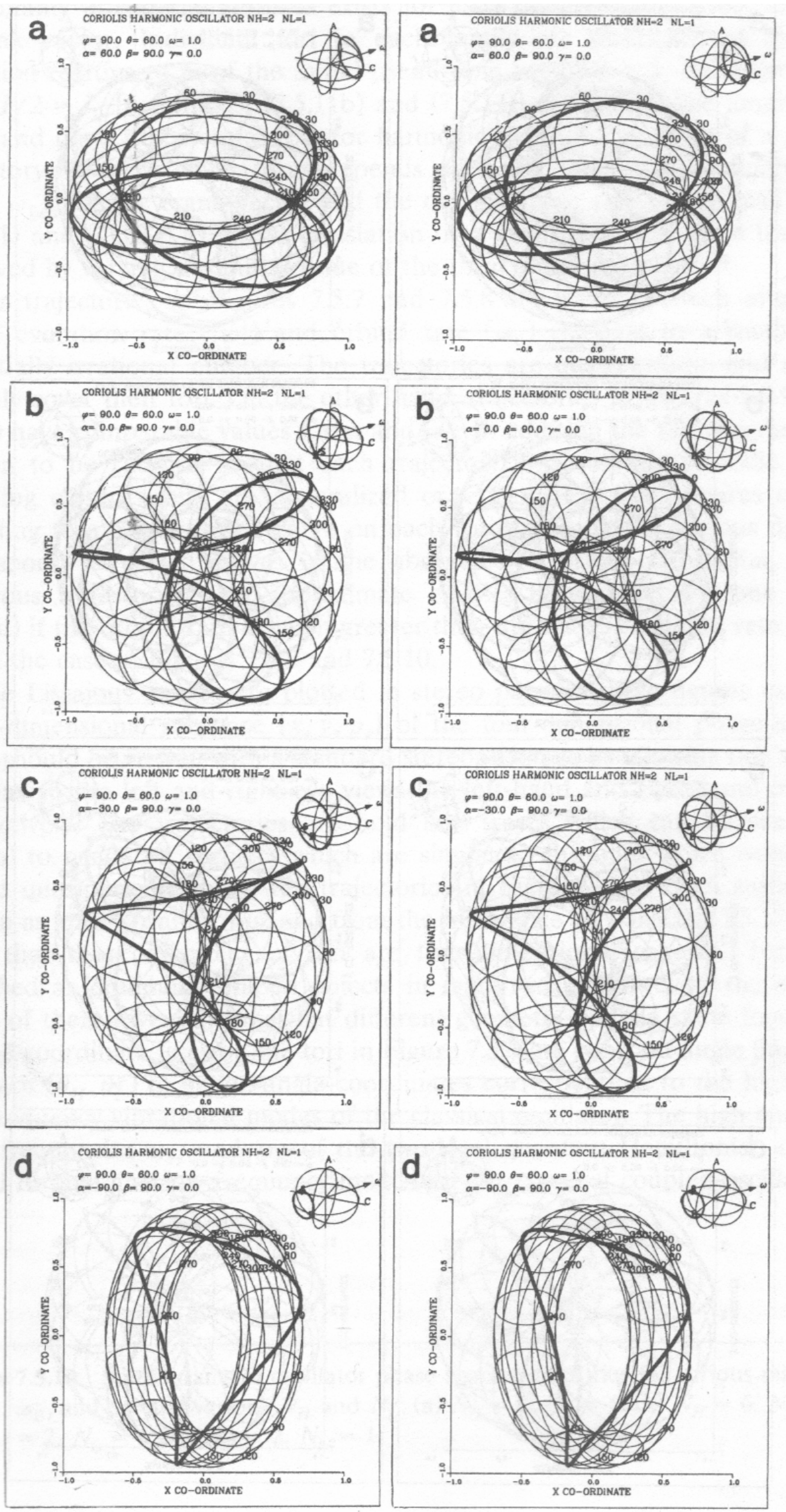


Figure 7.5.7 Analog computer plots of two famous examples of optical activity. (a) Faraday rotation or circular dichroism corresponds to constant $\psi = \tan^{-1}(b/a)$. (b) Birefringence corresponds to constant $\nu = \tan^{-1}(Y/X)$. Note that a small amount of birefringence is present in Figure 7.11(a); i.e., ψ oscillates slightly. Pure Faraday rotation is difficult to achieve on an analog computer.

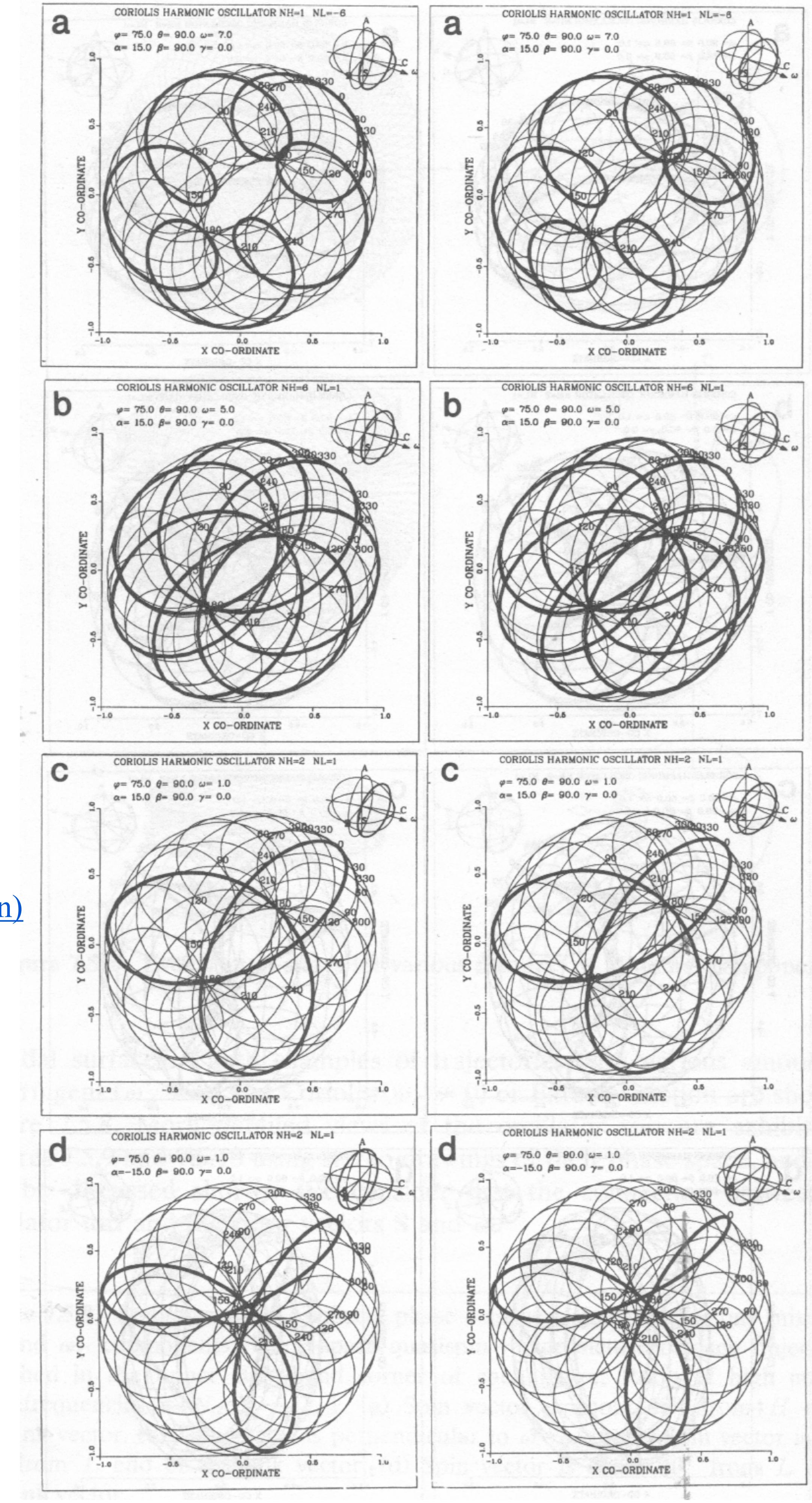


7.5.8 Evolution of states for various mixtures of A and C components.

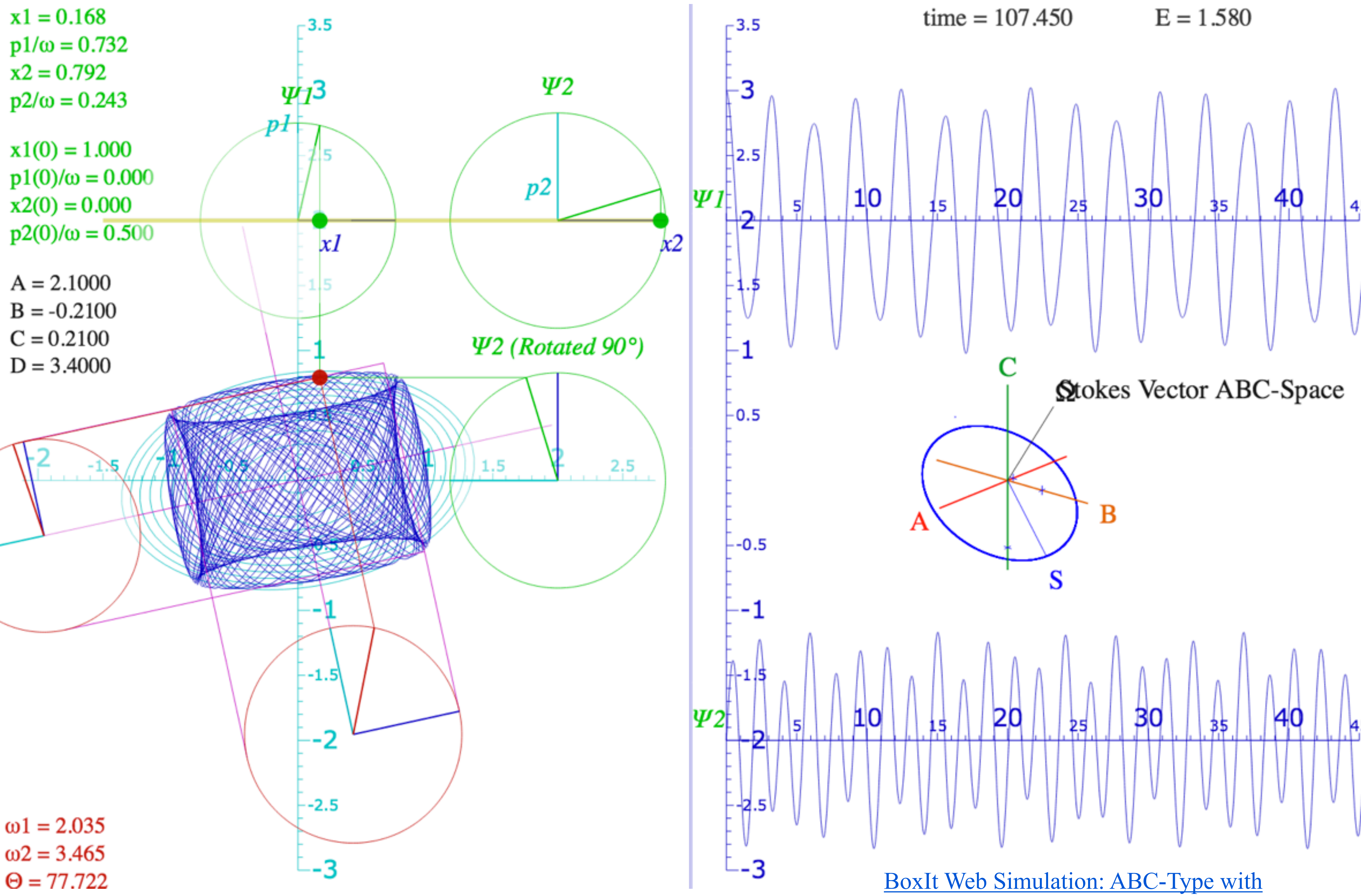


ABC-Type elliptical polarized dynamics

BoxIt (ABC-Motion) Web Simulation



ABC-Type elliptical polarized motion



[BoxIt Web Simulation: ABC-Type with A=2.1; B=-0.21; C=0.21; D=3.4]

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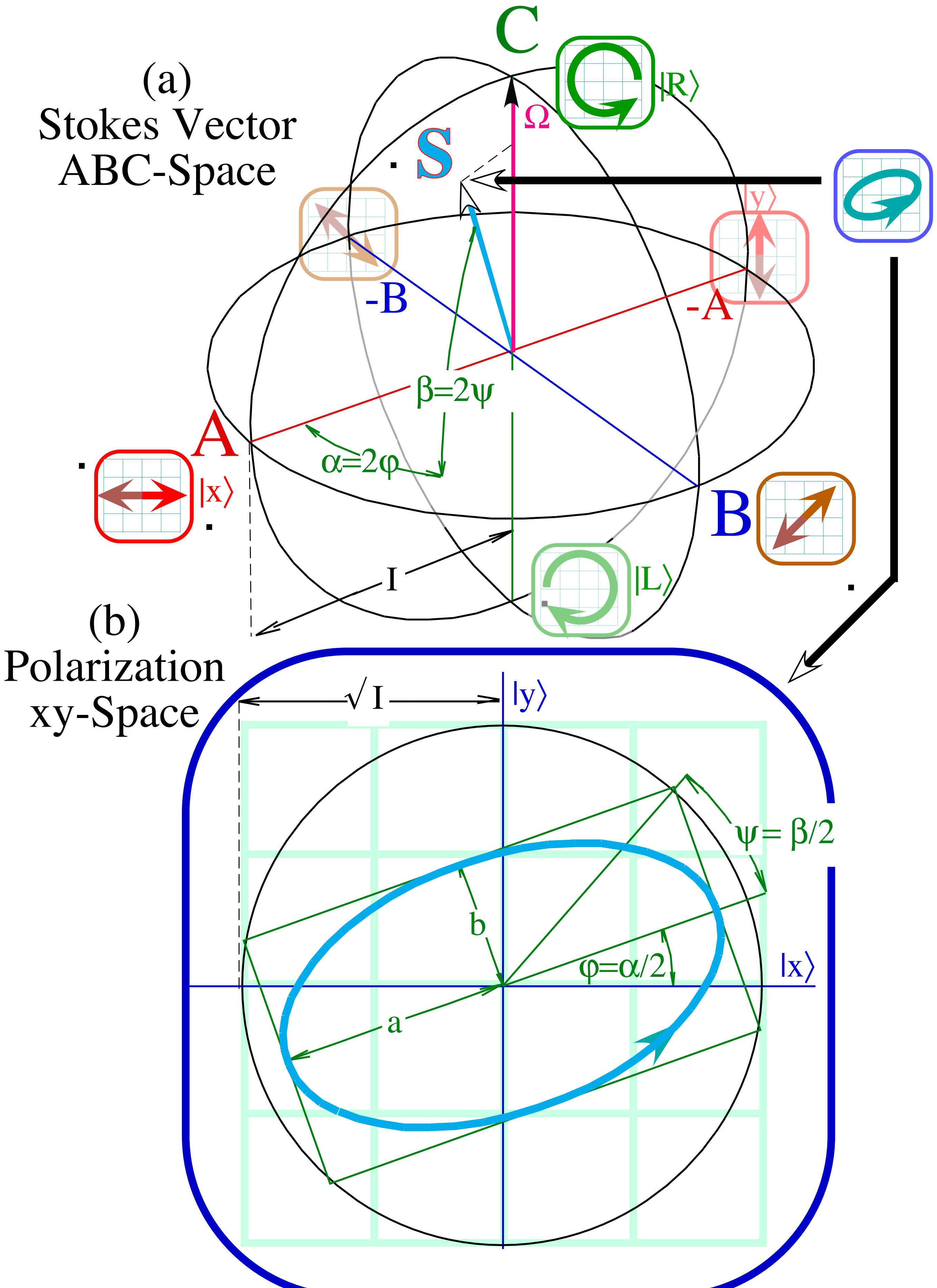
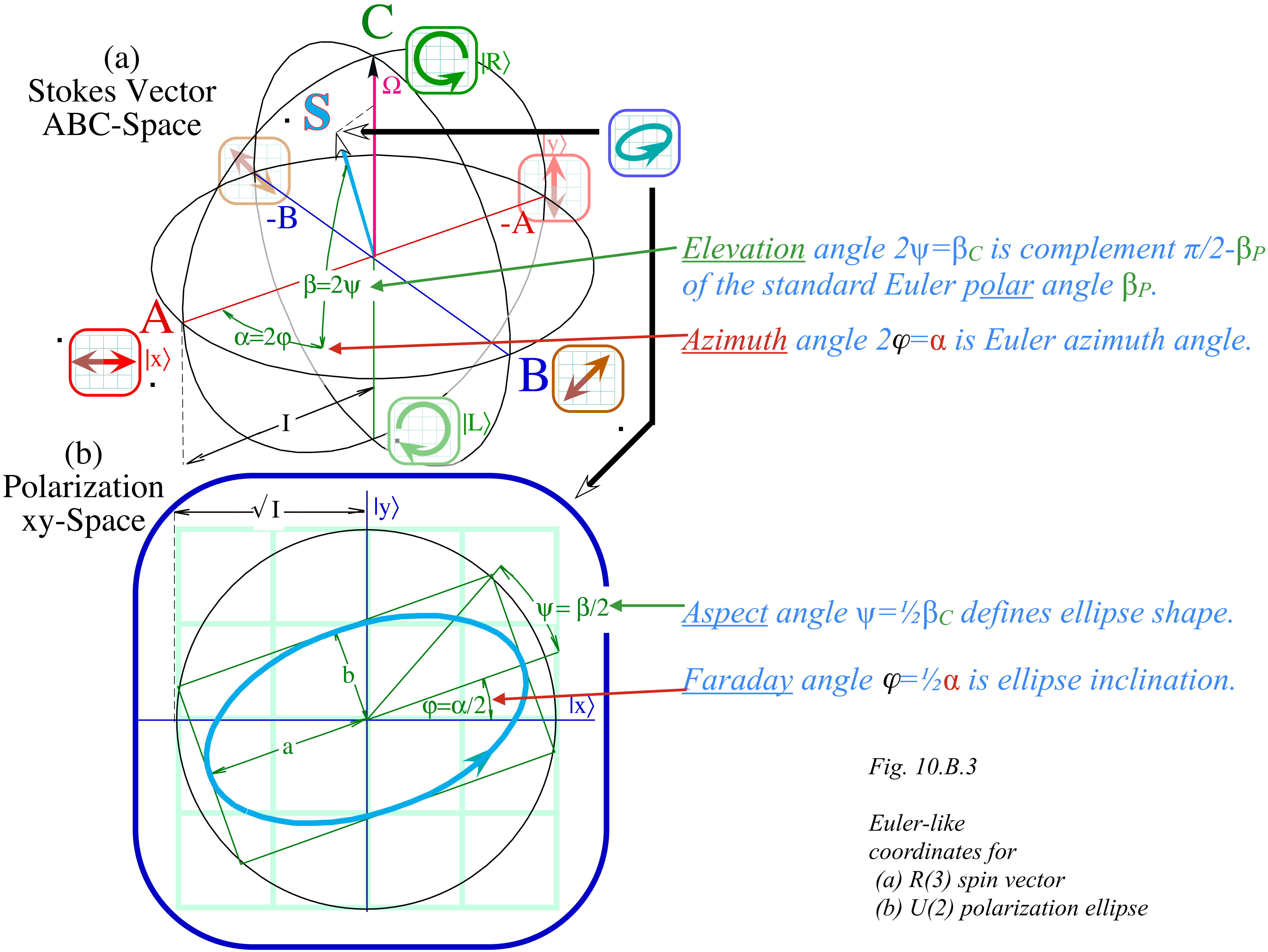


Fig. 10.B.3

Euler-like coordinates for
(a) R(3) spin vector
(b) U(2) polarization ellipse

ABC-Type elliptical polarization



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➡ *Conventional amplitude-phase-*$(A_1,A_2,\omega t,\rho_1)$ *labeling of optical polarization Relation Euler angle*

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Polarization ellipsometry and U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates and related to Euler Angles ($\alpha\beta\gamma$)

2D elliptic frequency ω orbit has amplitudes

A_1 and A_2, and phase shifts ρ_1 and $\rho_2=-\rho_1$.

$$x_1= A_1 cos(\omega t+\rho_1)$$

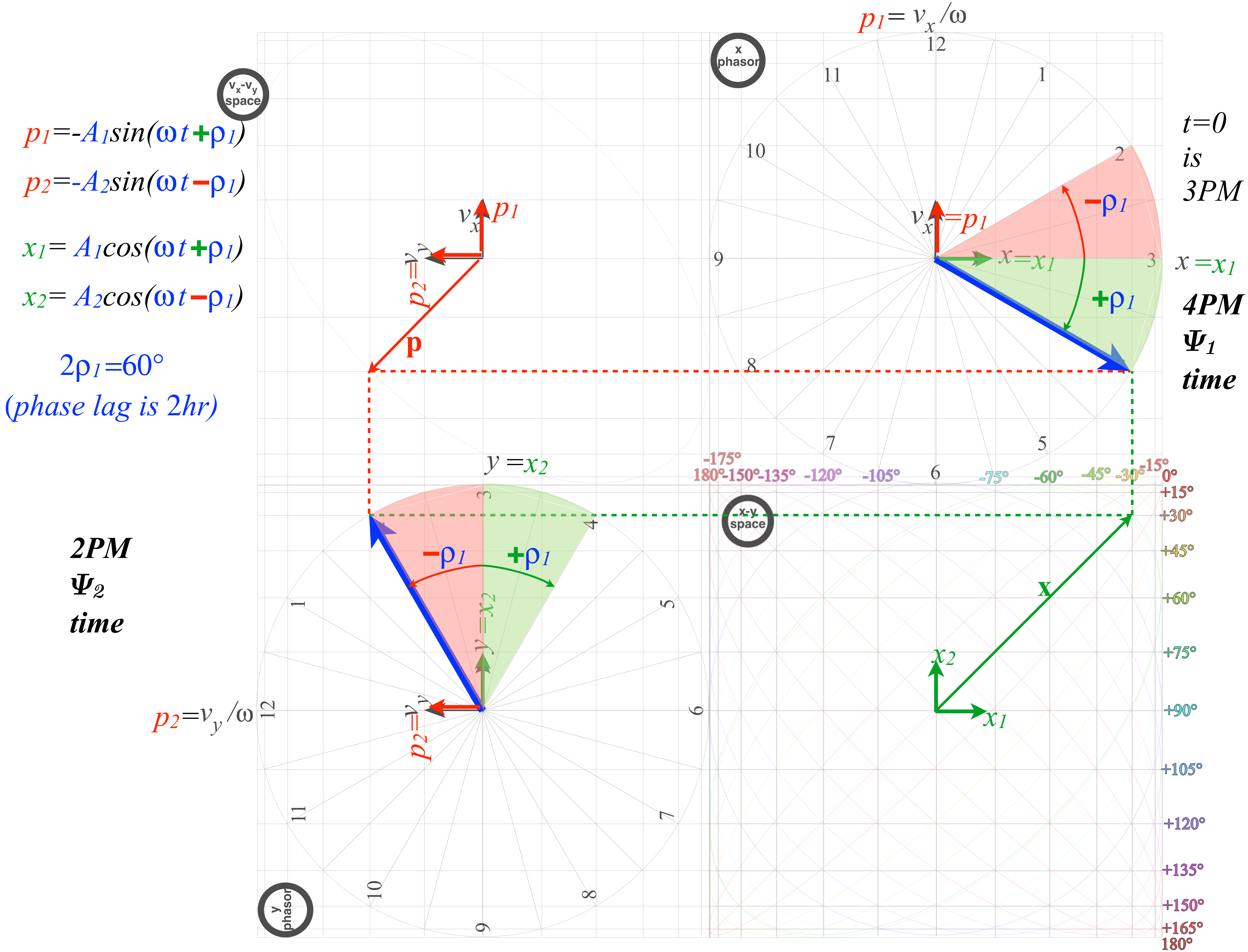
$$-p_1=A_1 sin(\omega t+\rho_1)$$

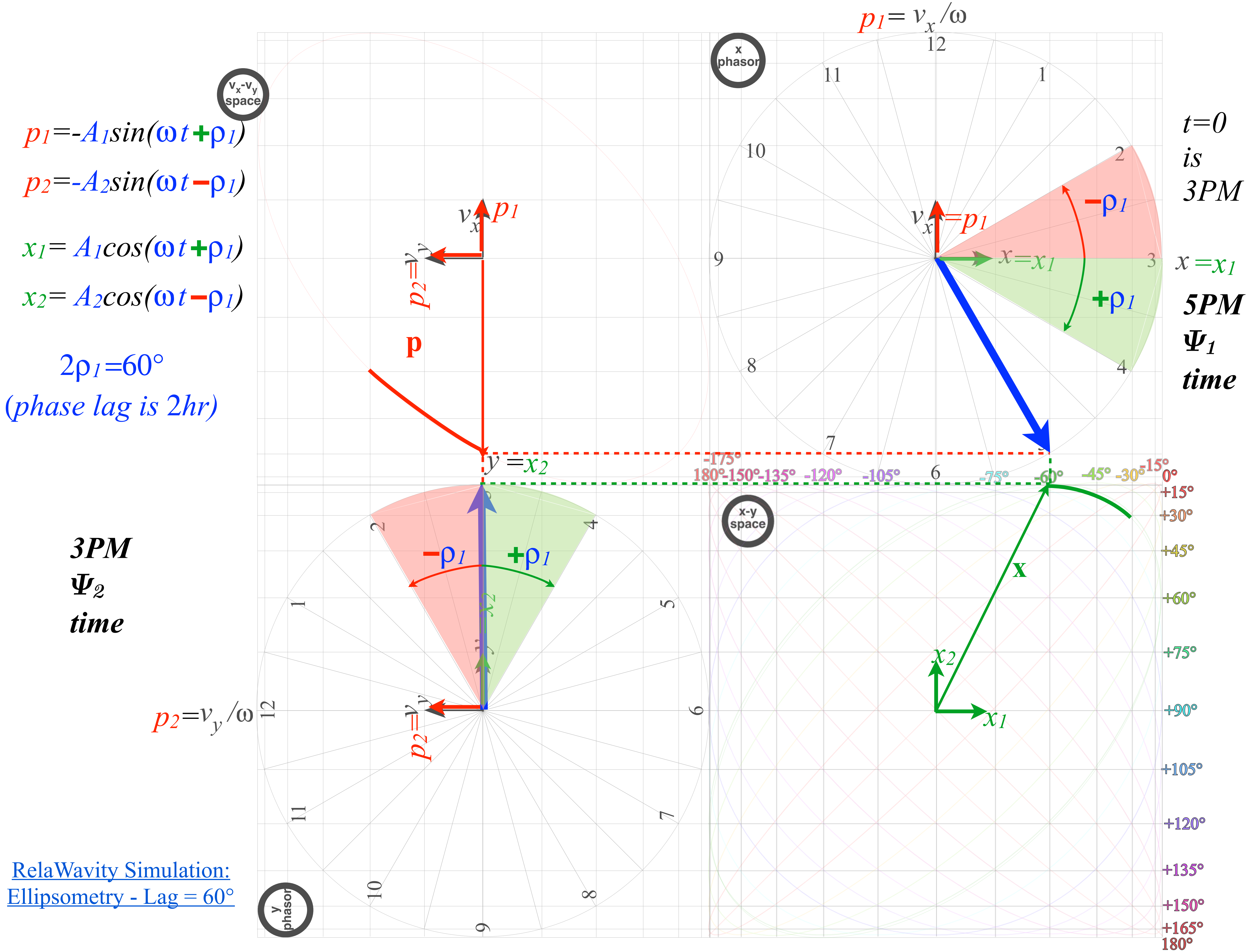
$$x_2= A_2 cos(\omega t-\rho_1)$$

$$-p_2=A_2 sin(\omega t-\rho_1)$$

Amp-phase parameters $(A_1,A_2, \omega t,\rho_1)$

$$\begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$


$p_1 = -A_1 \sin(\omega t + \rho_1)$
$p_2 = -A_2 \sin(\omega t - \rho_1)$
$x_1 = A_1 \cos(\omega t + \rho_1)$
$x_2 = A_2 \cos(\omega t - \rho_1)$
$2\rho_1 = 60°$
(phase lag is 2hr)
v_x-v_y space
x phasor
$p_1 = v_x/\omega$
$v_x = p_1$
$x = x_1$
$-\rho_1$
$+\rho_1$
t=0 is 3PM
4PM Ψ_1 time
p
$p_2 = v_y$
$y = x_2$
2PM Ψ_2 time
x-y space
x
x_1
x_2
$p_2 = v_y/\omega$
y phasor
-175° 180° -150° -135° -120° -105° -75° -60° -45° -30° -15° 0°
+15° +30° +45° +60° +75° +90° +105° +120° +135° +150° +165° 180°



[RelaWavity Simulation: Ellipsometry - Lag = 60°]

$p_1 = -A_1 \sin(\omega t + \rho_1)$

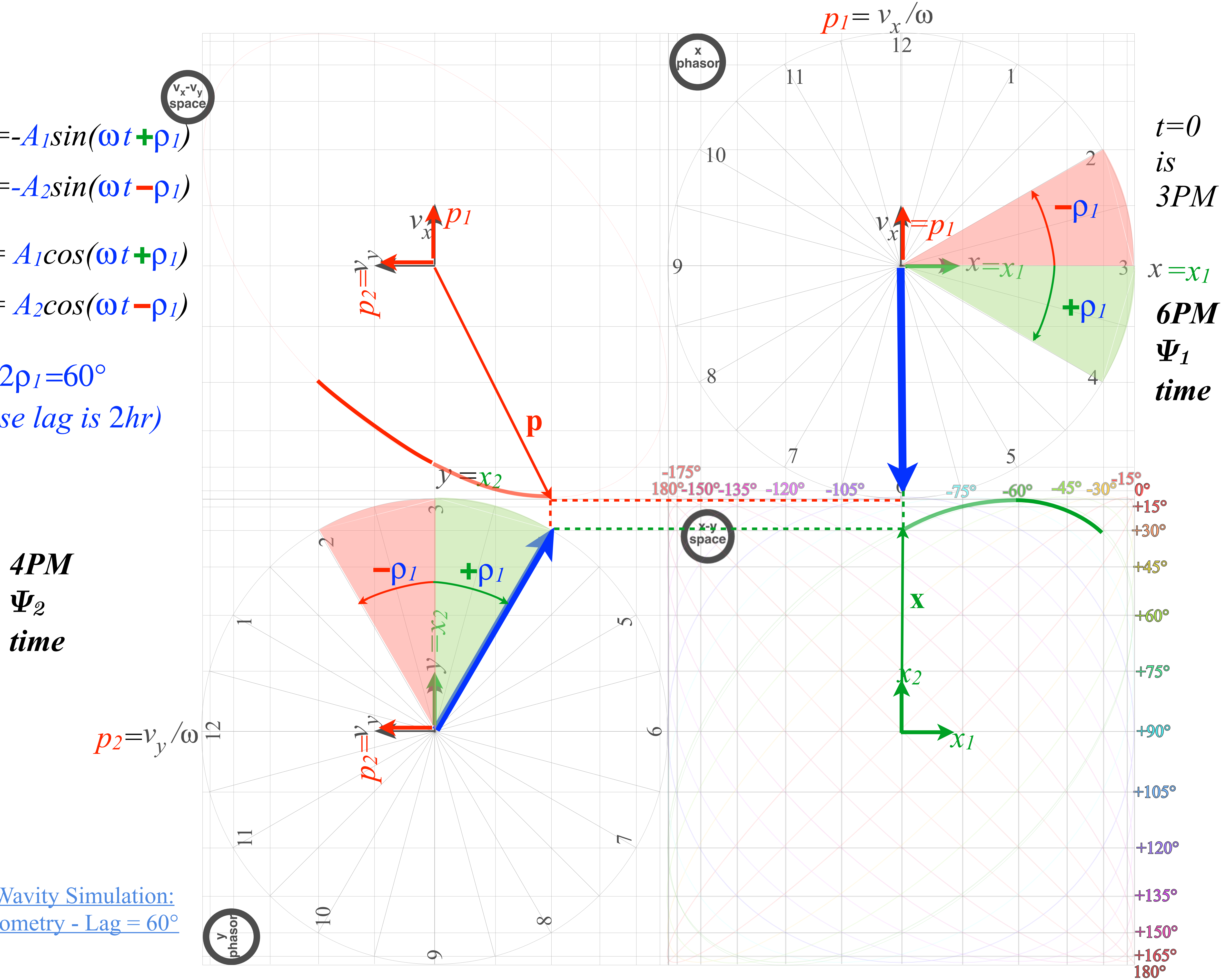
$p_2 = -A_2 \sin(\omega t - \rho_1)$

$x_1 = A_1 \cos(\omega t + \rho_1)$

$x_2 = A_2 \cos(\omega t - \rho_1)$

$2\rho_1 = 60°$

(phase lag is 2hr)



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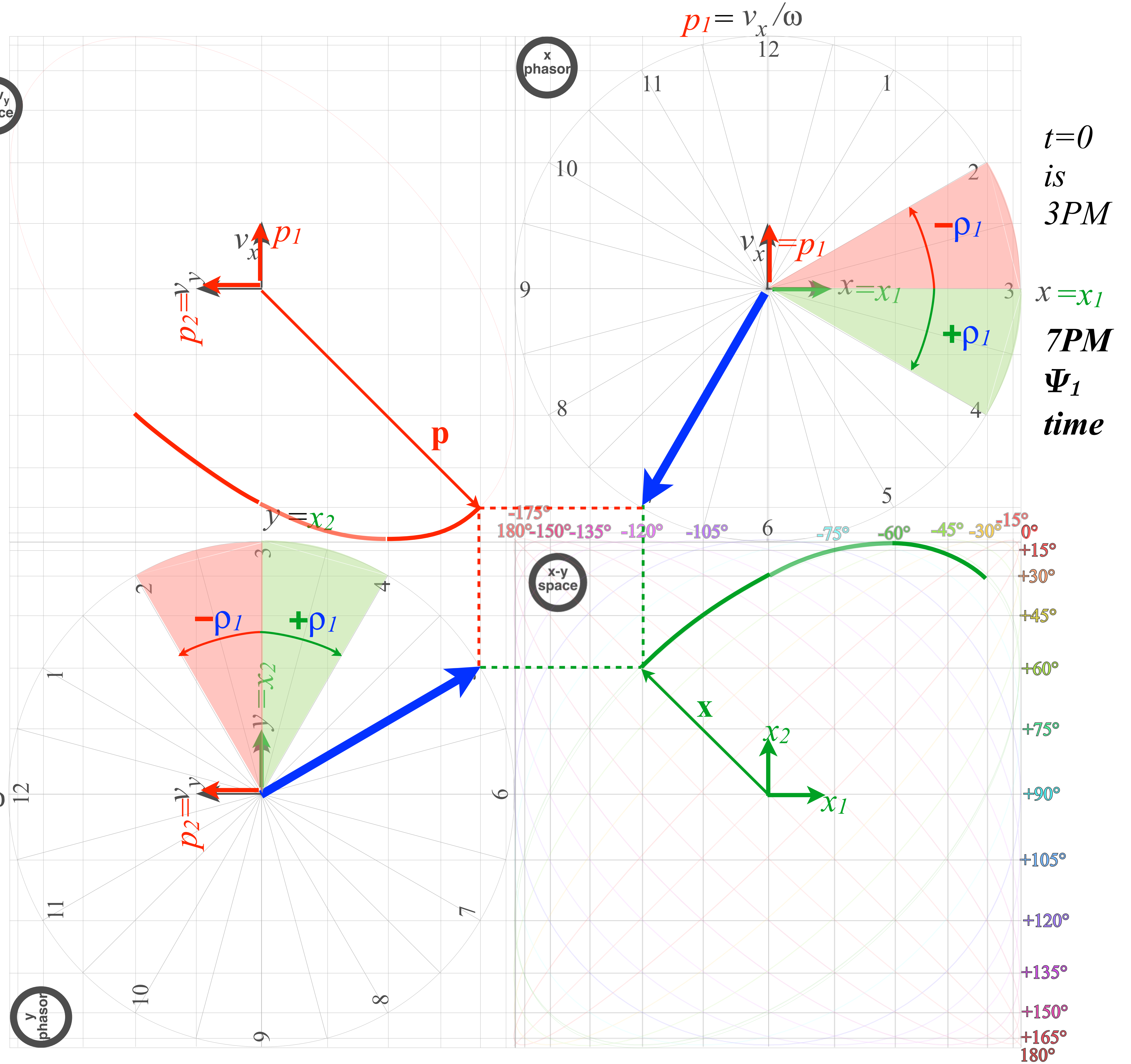
$p_2 = -A_2 \sin(\omega t - \rho_1)$

$x_1 = A_1 \cos(\omega t + \rho_1)$

$x_2 = A_2 \cos(\omega t - \rho_1)$

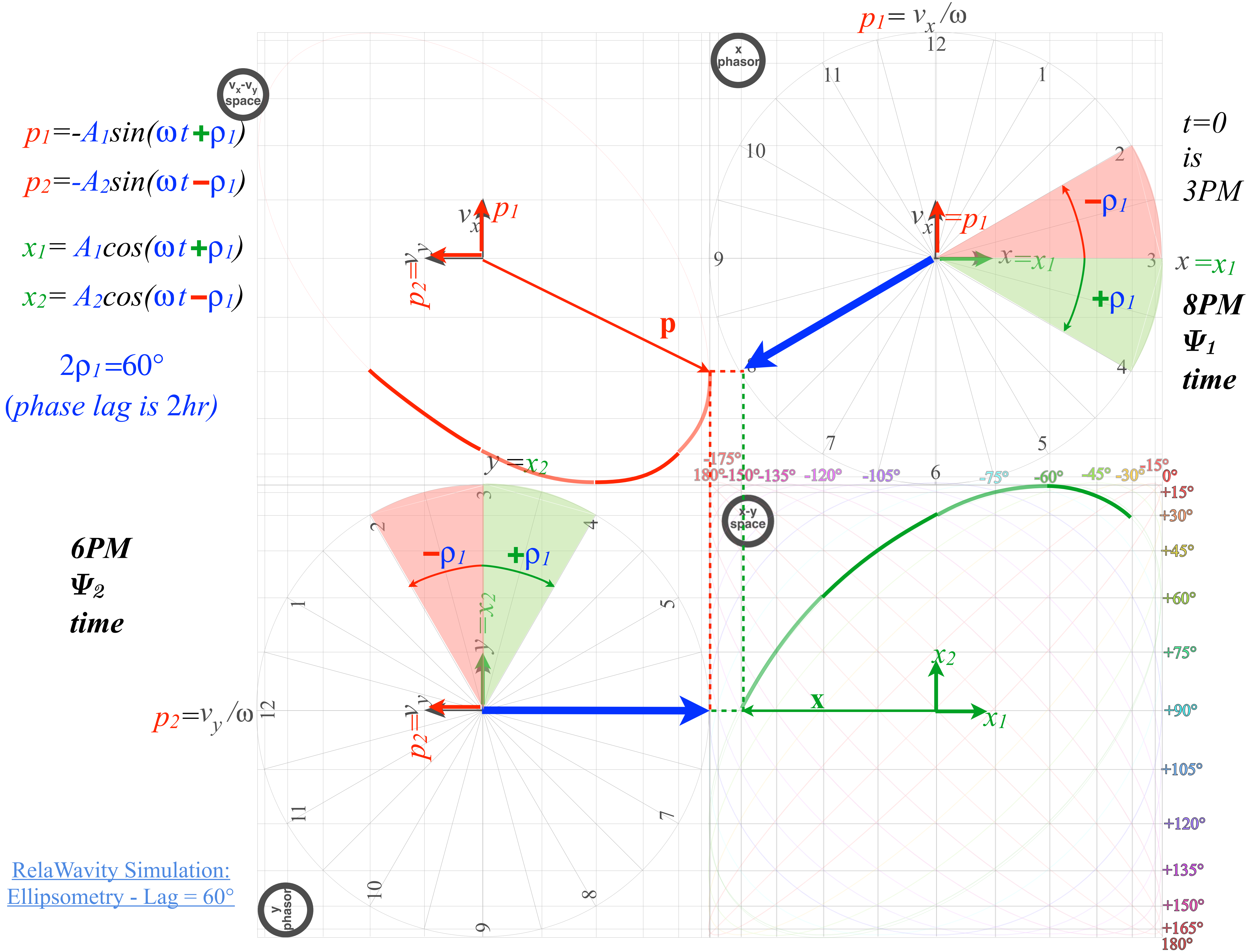
$2\rho_1 = 60°$

(phase lag is 2hr)

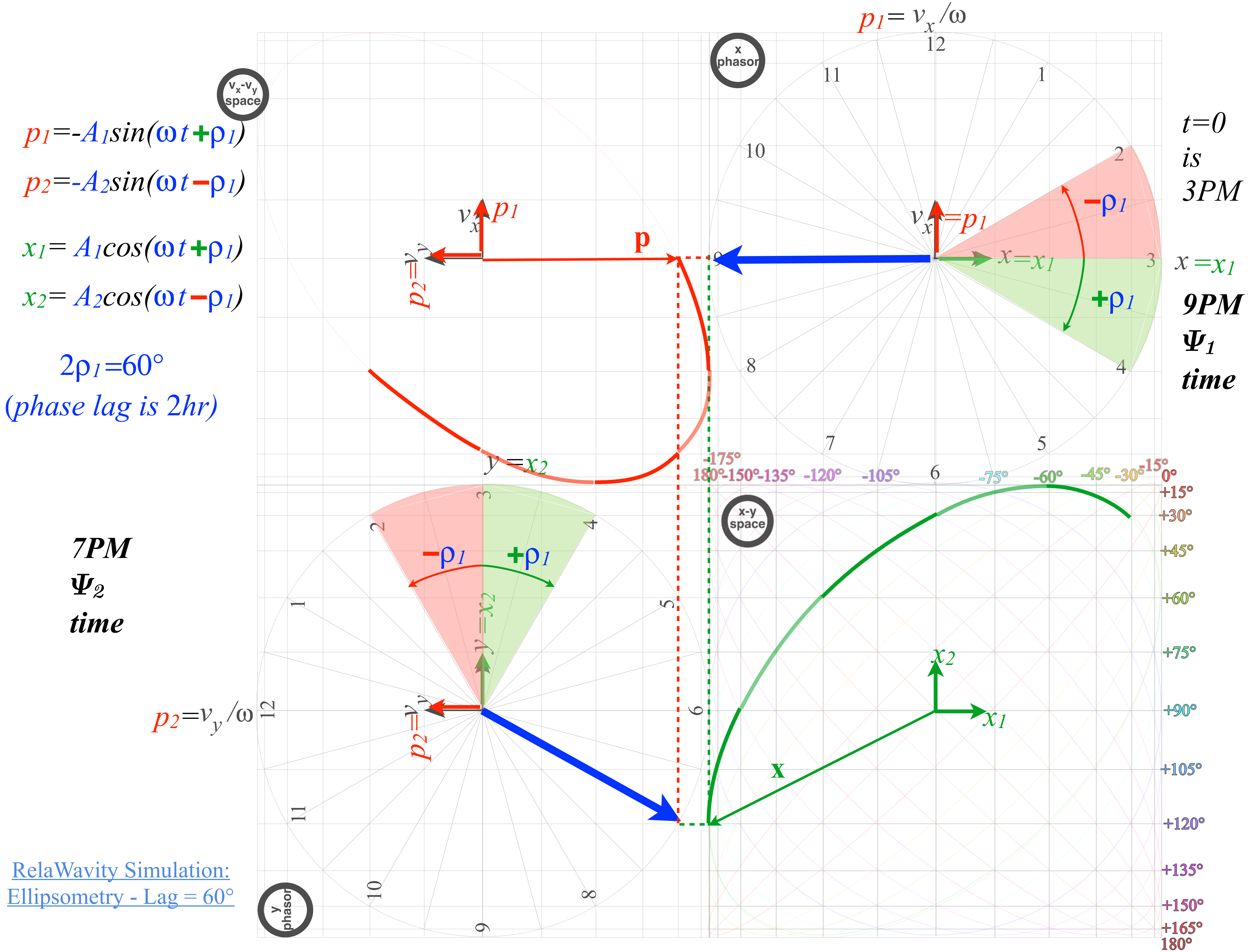


5PM
Ψ_2
time

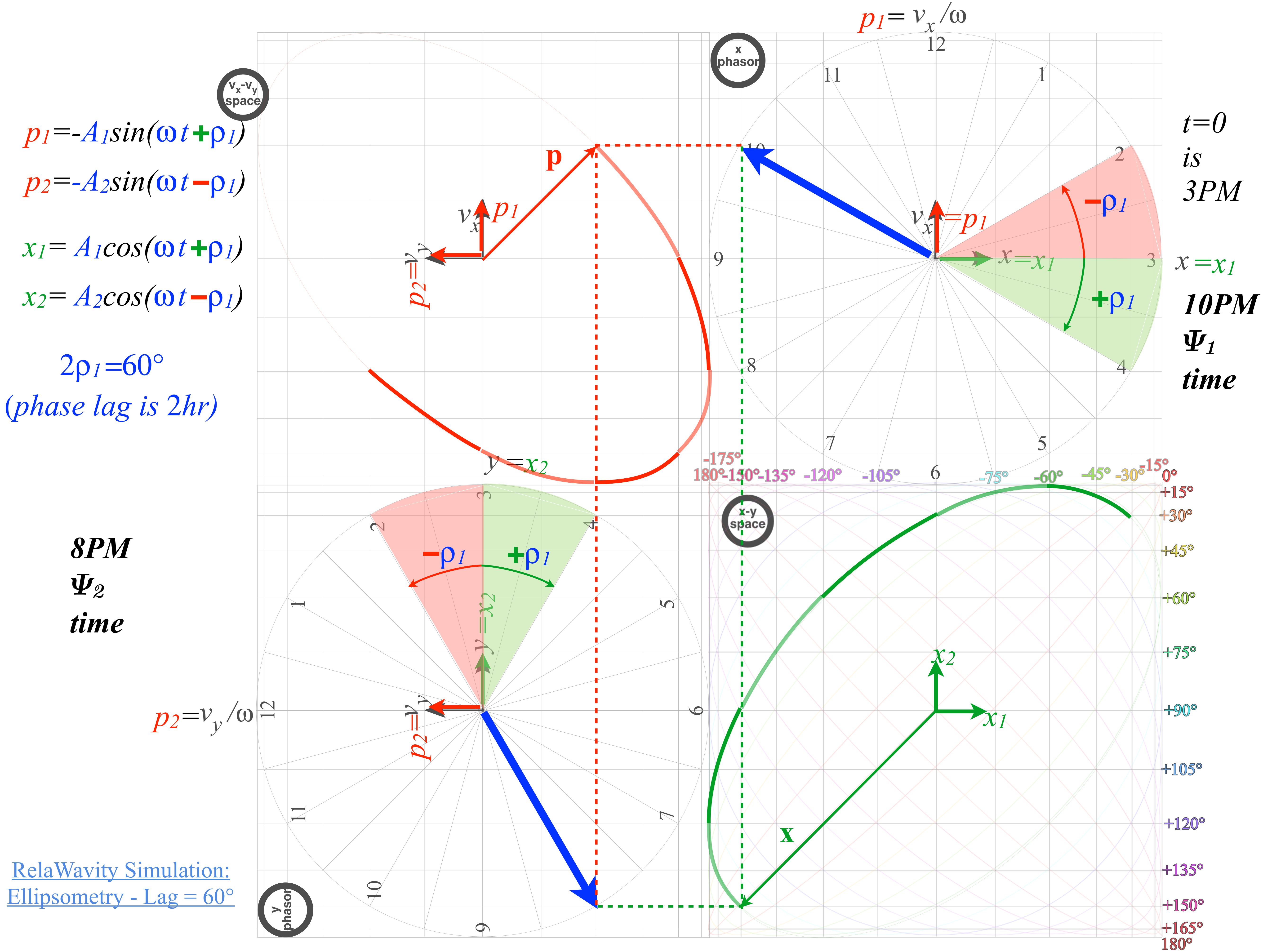
[RelaWavity Simulation: Ellipsometry - Lag = 60°]



RelaWavity Simulation: Ellipsometry - Lag = 60°



p1=-A1sin(ωt+ρ1)
p2=-A2sin(ωt−ρ1)
x1= A1cos(ωt+ρ1)
x2= A2cos(ωt−ρ1)
2ρ1=60°
(phase lag is 2hr)
vx-vy space
x phasor
y phasor
x-y space
p1= vx /ω
p2=vy /ω
t=0
is
3PM
9PM
Ψ1
time
7PM
Ψ2
time
−ρ1
+ρ1
x =x1
y =x2
p
x
x1
x2
0°
+15°
+30°
+45°
+60°
+75°
+90°
+105°
+120°
+135°
+150°
+165°
180°
-15°
-30°
-45°
-60°
-75°
-105°
-120°
-135°
-150°
-175°
180°
RelaWavity Simulation:
Ellipsometry - Lag = 60°



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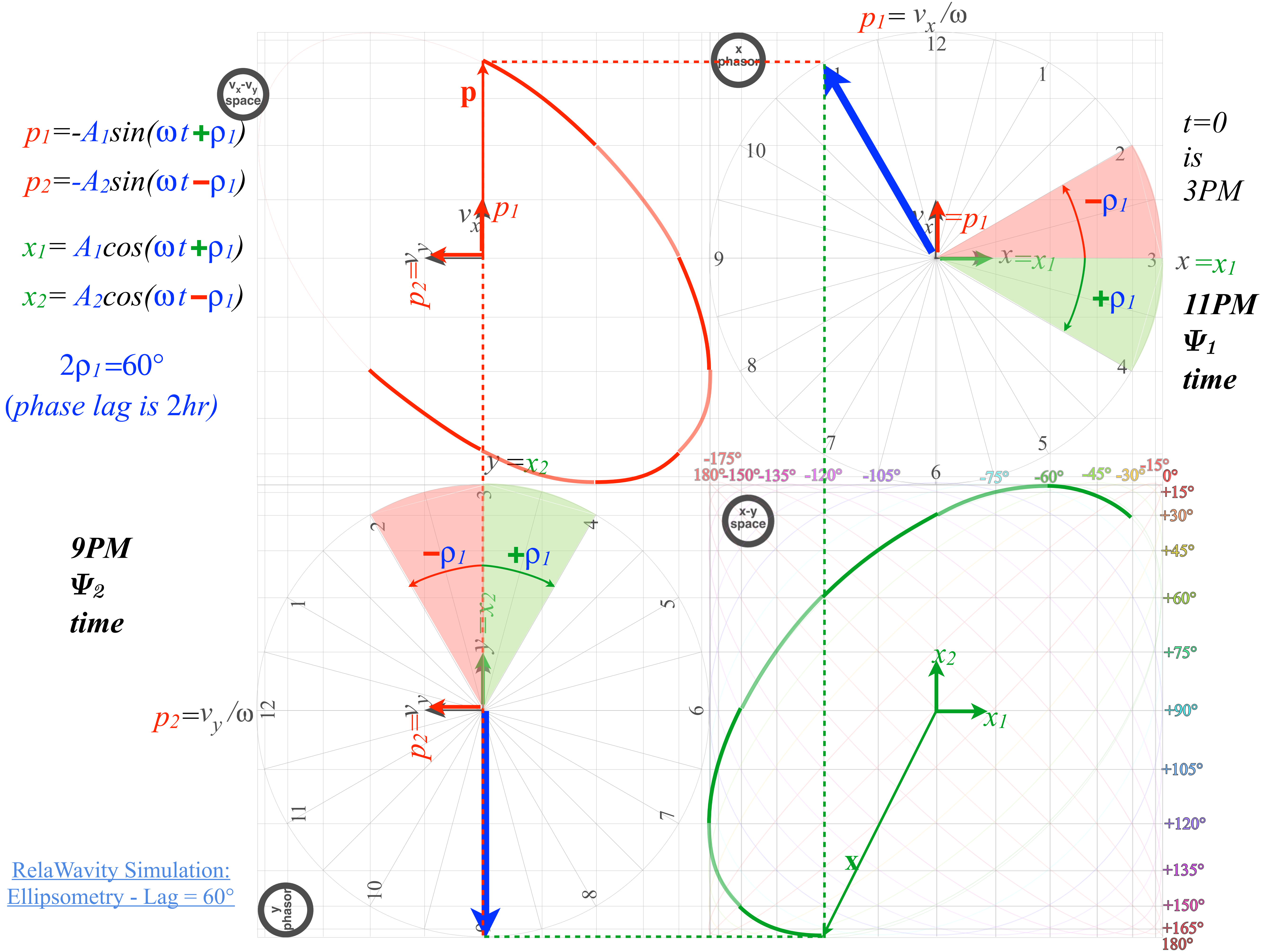
x2= A2cos(ωt−ρ1)
2ρ1=60°
(phase lag is 2hr)
vx-vy space
x phasor
x-y space
y phasor
p1= vx/ω
p2=vy/ω
p
x=x1
y=x2
x1
x2
x
−ρ1
+ρ1
t=0
is
3PM
10PM
Ψ1
time
8PM
Ψ2
time
180° -150° -135° -120° -105° -75° -60° -45° -30° -15° 0°
-175°
+15° +30° +45° +60° +75° +90° +105° +120° +135° +150° +165° 180°
RelaWavity Simulation:
Ellipsometry - Lag = 60°



$p_1=-A_1\sin(\omega t+\rho_1)$
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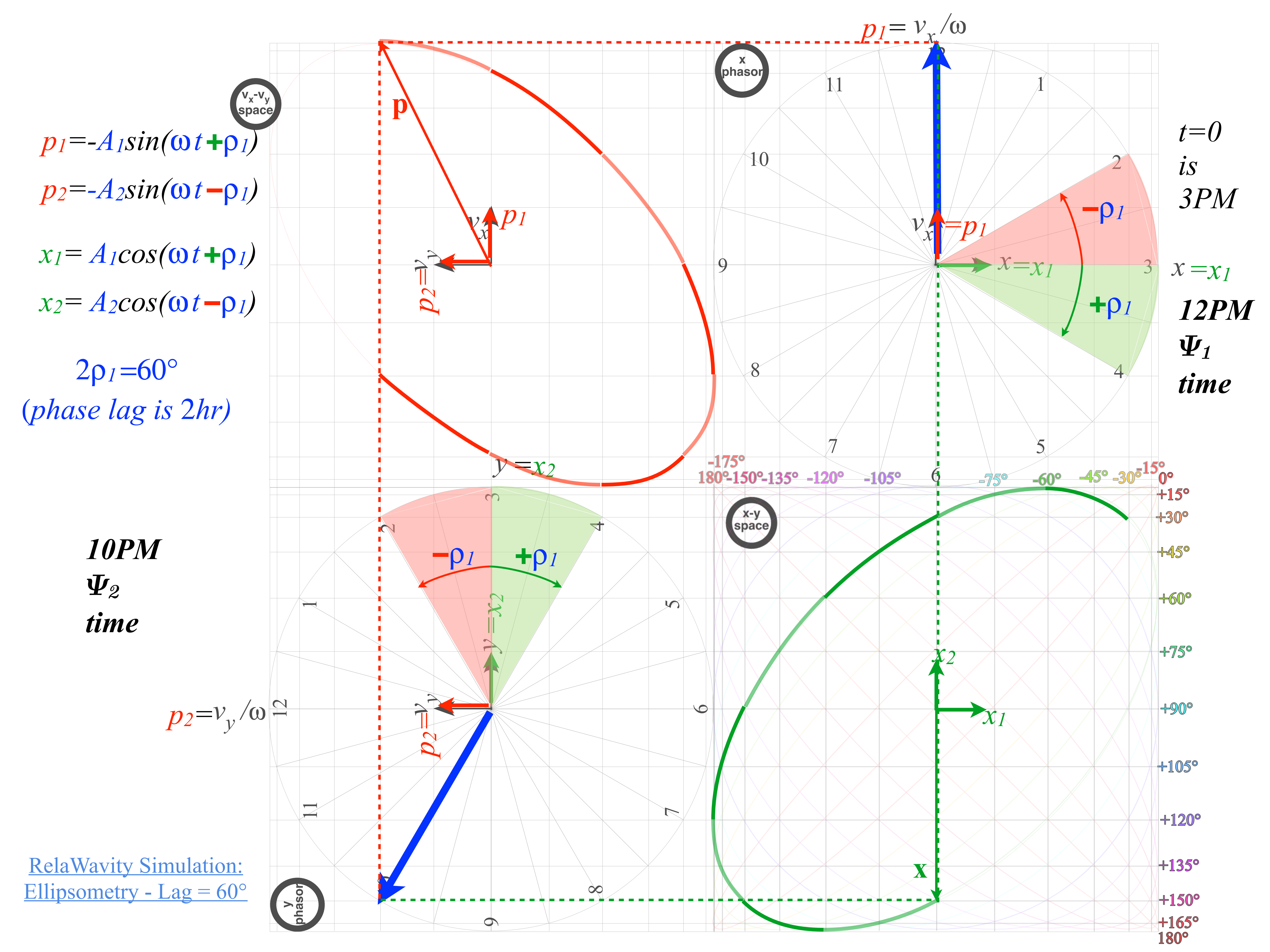
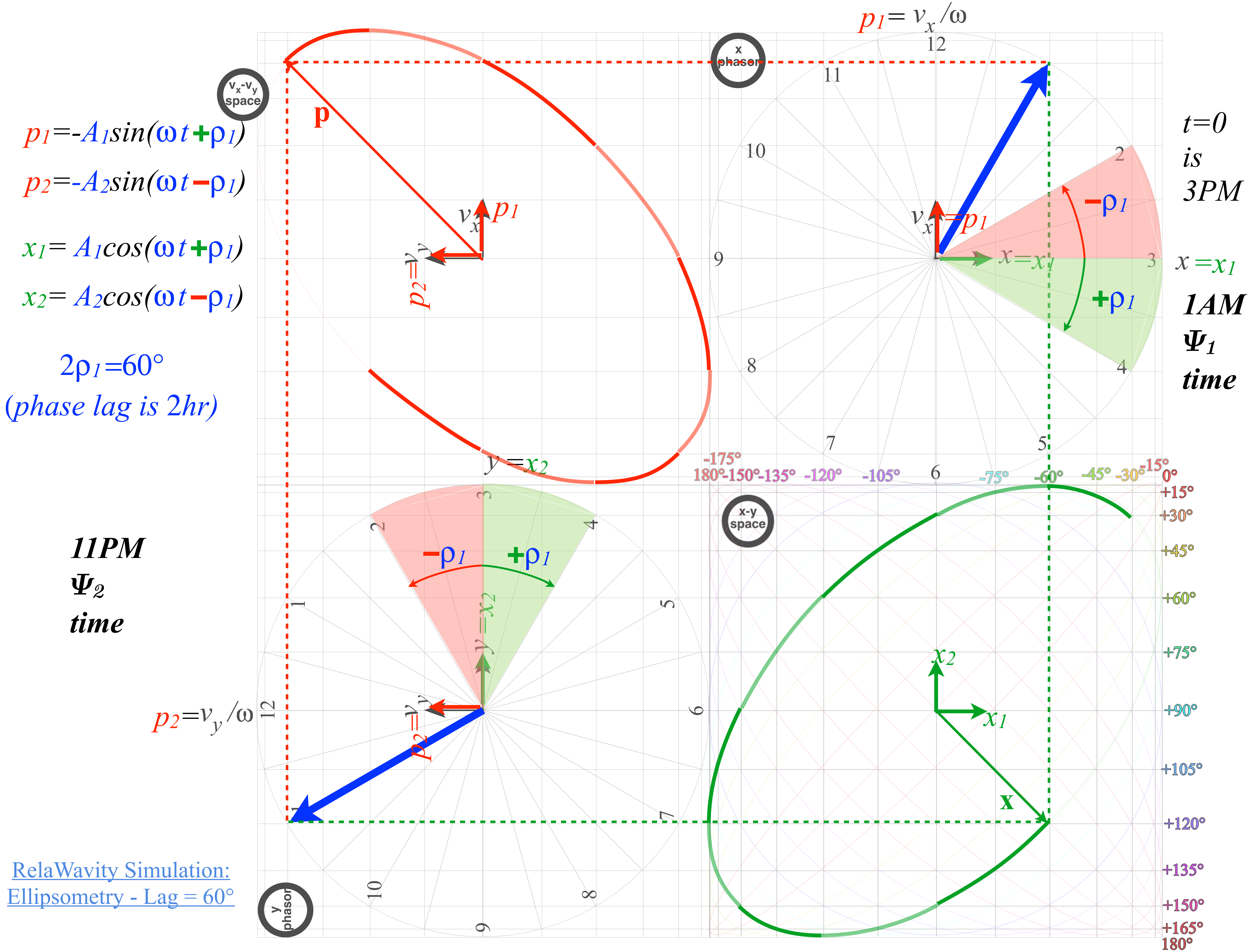
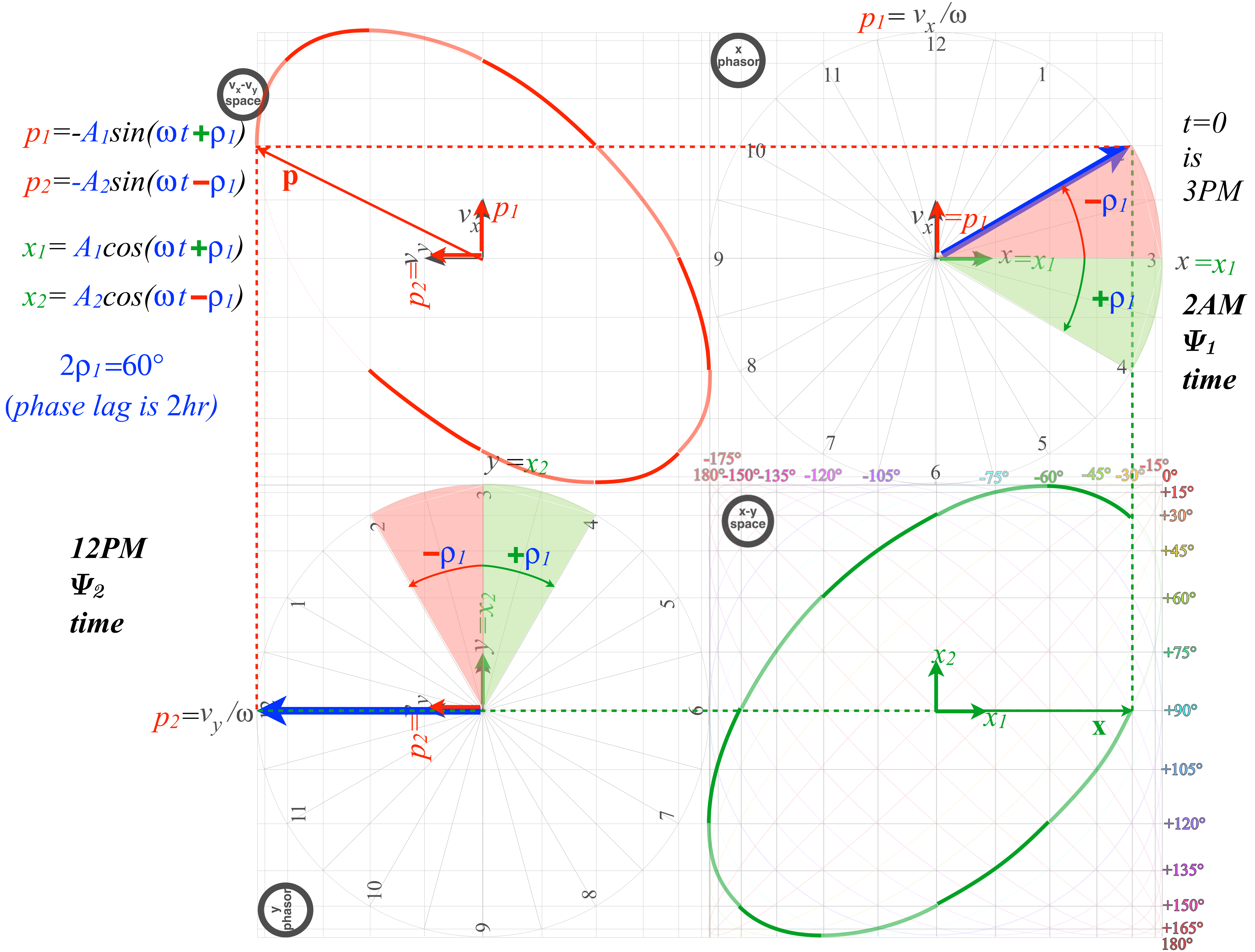
$x_1=A_1\cos(\omega t+\rho_1)$
$x_2=A_2\cos(\omega t-\rho_1)$
$2\rho_1=60°$
(phase lag is 2hr)
v_x-v_y space
x phasor
y phasor
x-y space
p
$p_1=v_x/\omega$
$p_2=v_y/\omega$
$v_x=p_1$
$p_2=v_y$
$y=x_2$
$x=x_1$
$-\rho_1$
$+\rho_1$
t=0 is 3PM
11PM Ψ_1 time
9PM Ψ_2 time
x_1
x_2
x
-175° 180° -150° -135° -120° -105° -75° -60° -45° -30° -15° 0°
+15° +30° +45° +60° +75° +90° +105° +120° +135° +150° +165° 180°
RelaWavity Simulation: Ellipsometry - Lag = 60°



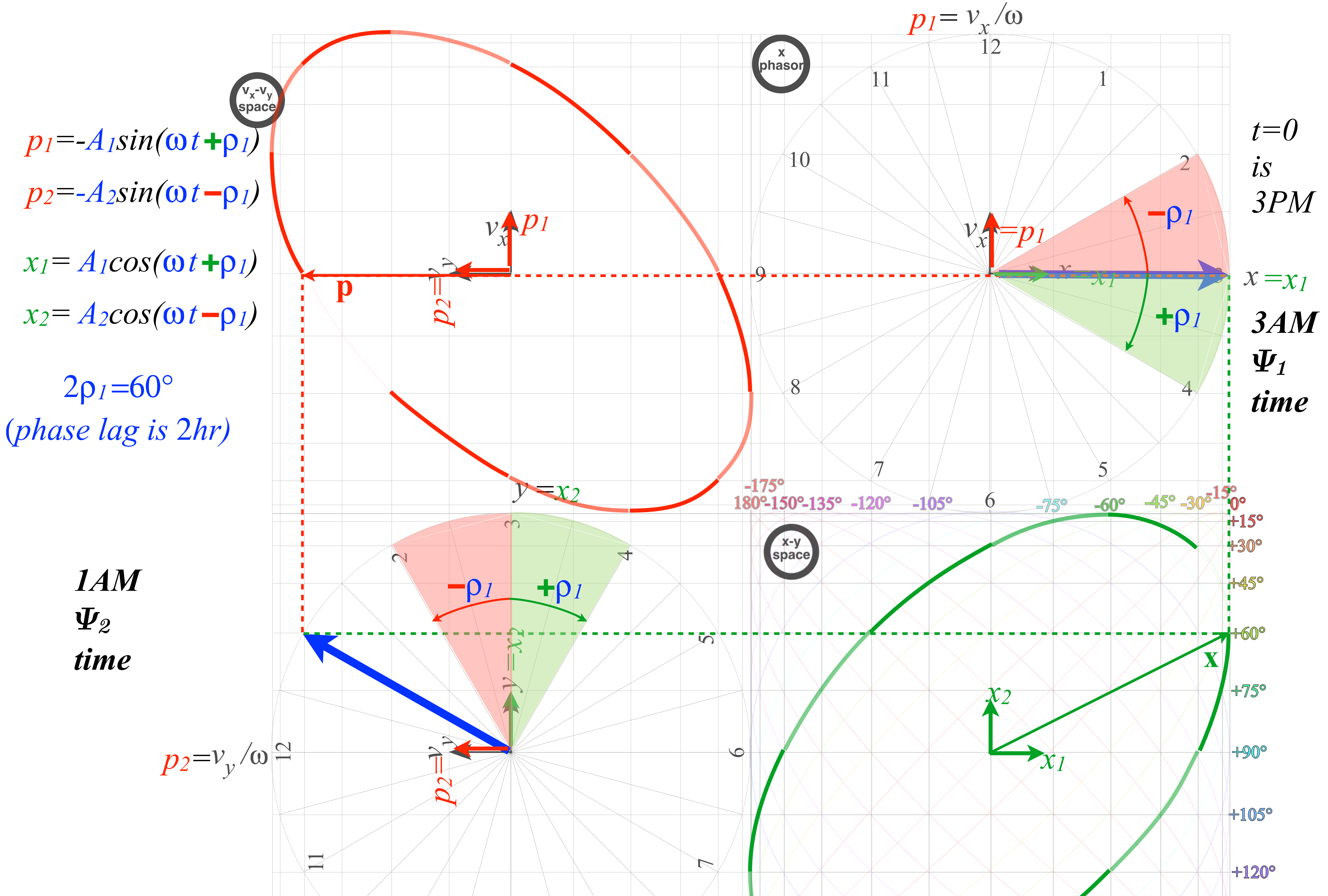
p_1=-A_1sin(ωt+ρ$_1$)
p_2=-A_2sin(ωt−ρ$_1$)
x_1= A_1cos(ωt+ρ$_1$)
x_2= A_2cos(ωt−ρ$_1$)
2ρ$_1$=60°
(phase lag is 2hr)
v_x-v_y space
x phasor
x-y space
y phasor
p
p_1= v_x/ω
p_2=v_y/ω
x =x_1
y =x_2
−ρ$_1$
+ρ$_1$
t=0
is
3PM
12PM
Ψ$_1$
time
10PM
Ψ$_2$
time
x
RelaWavity Simulation:
Ellipsometry - Lag = 60°



RelaWavity Simulation:
Ellipsometry - Lag = 60°



$p_1 = v_x/\omega$
x phasor
v_x-v_y space
$p_1 = -A_1 \sin(\omega t + \rho_1)$
$p_2 = -A_2 \sin(\omega t - \rho_1)$
$x_1 = A_1 \cos(\omega t + \rho_1)$
$x_2 = A_2 \cos(\omega t - \rho_1)$
$2\rho_1 = 60°$
(phase lag is 2hr)
p
$v_x = p_1$
$v_y = p_2$
$x = x_1$
$-\rho_1$
$+\rho_1$
t=0
is
3PM
2AM
Ψ_1
time
$y = x_2$
12PM
Ψ_2
time
x-y space
y phasor
$p_2 = v_y/\omega$
x_1
x_2
x



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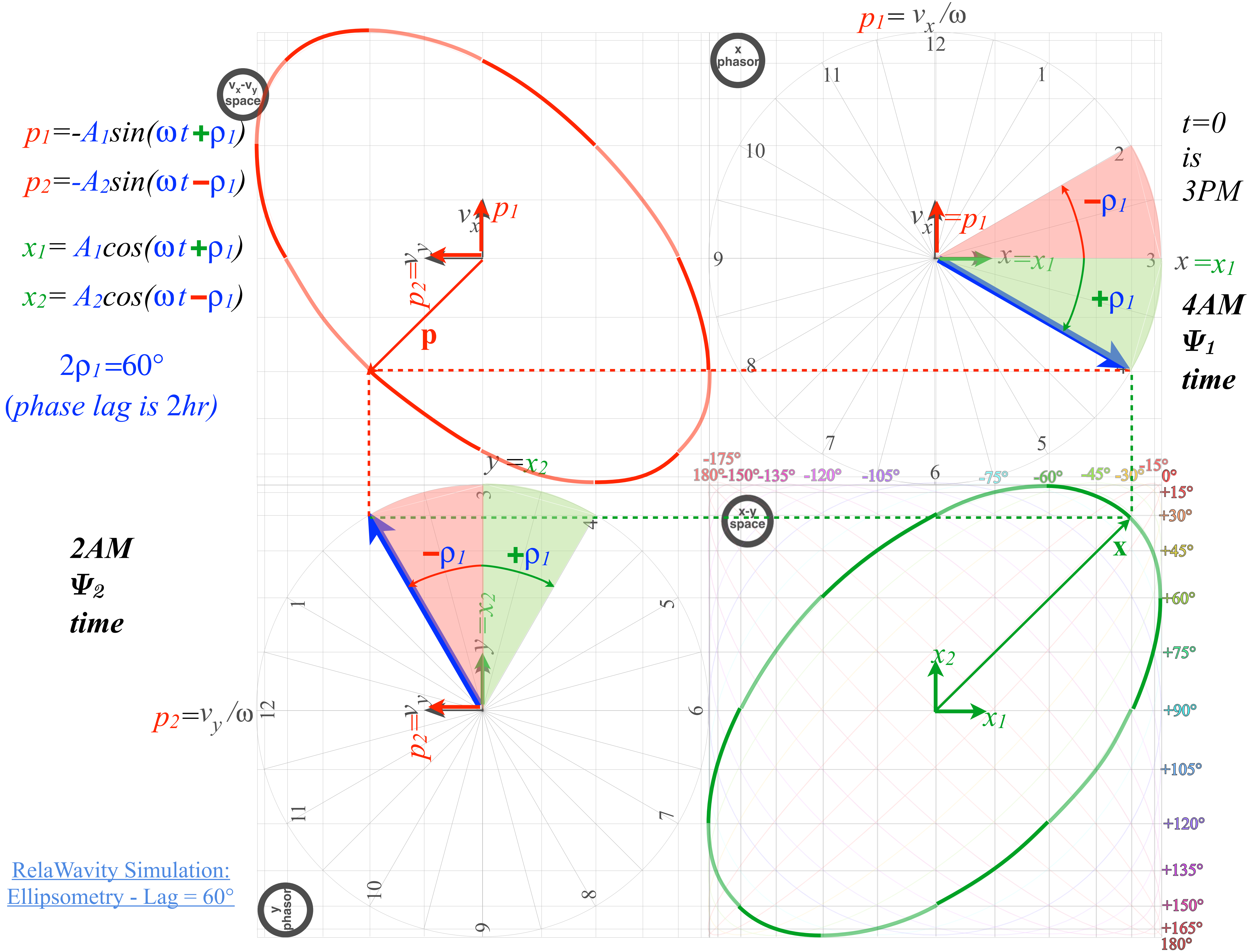

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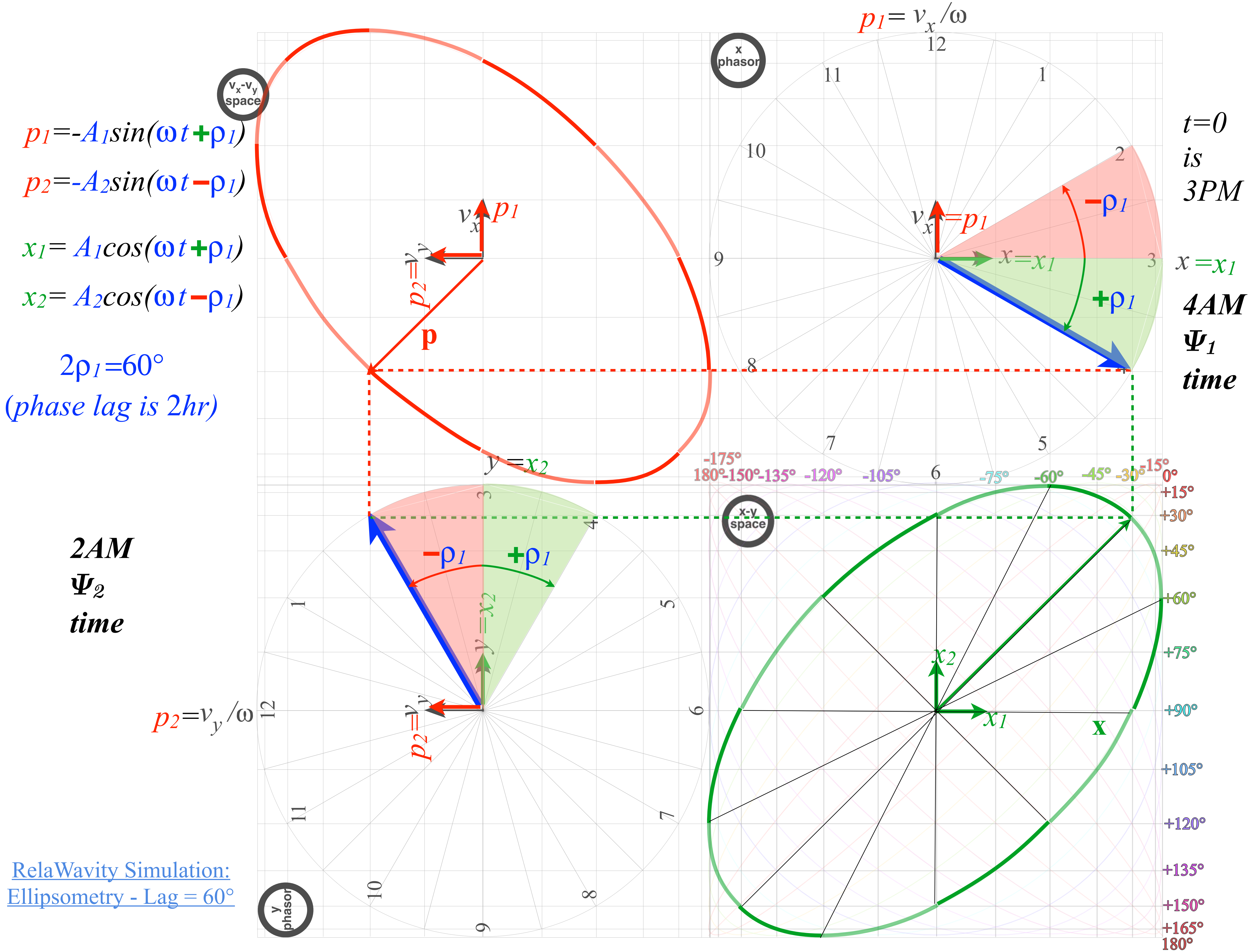

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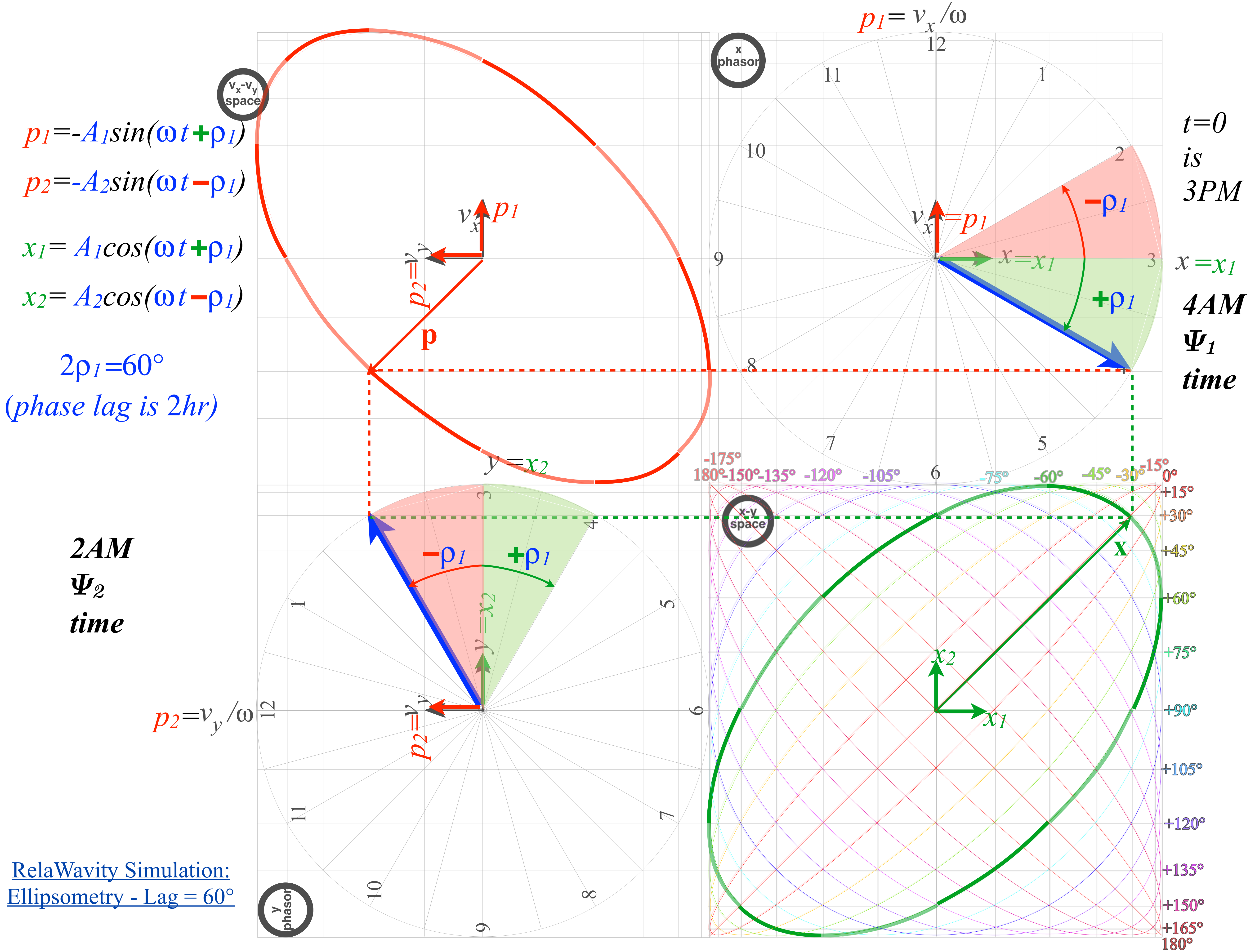

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1.29.18 class 5.0: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

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Symmetry group $\mathscr{G}$ representations=>*AMOP* Hamiltonian **H** (or K) matrices, irreps $\mathscr{D}^{(\alpha)}$
=>*AMOP* wave functions $\Psi^{(\alpha)}$, eigensolutions

$\mathscr{G}$= U(2) spin-½ irreps: Euler **R**(αβγ) *vs* Darboux **R**[φϑΘ] rotations and applications

Relating Euler and Darboux angles to U(2) phasor coordinates x_1+ip_1 *and* x_2+ip_2 .

Derivation of Euler-to-Darboux and Darboux-to-Euler conversion formulae, *Test of formulae.*

Darboux **R**[φϑΘ] *spin-½ rotation* Θ=0 *to* 4π *for fixed* [φϑ] *"Real-world" 4π spin-½ behavior.*

Review of U(2) dynamics: $\mathbf{H}=A\boldsymbol{\sigma}_Z$ (*A-Type*), $\mathbf{H}=B\boldsymbol{\sigma}_X$ (*B-Type*), $\mathbf{H}=C\boldsymbol{\sigma}_Y$ (*C-Type*).

U(2) dynamics of mixed-Types: $\mathbf{H}=A\boldsymbol{\sigma}_Z+B\boldsymbol{\sigma}_X$ (*AB-Type*), *Avoided crossing around Dirac-point.*
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Conventional amplitude-phase-$(A_1,A_2,\omega t,\rho_1)$ *labeling of optical polarizatio* *Relation to Euler angle*

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2D elliptic frequency ω orbit has amplitudes A_1 and A_2, and phase shifts ρ_1 and $\rho_2=-\rho_1$.

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Let: $\omega t + \rho_1 = (\gamma + \alpha)/2$

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$\omega t-\rho_1 = (\gamma-\alpha)/2$

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$$\tan\beta/2 = A_2/A_1 \qquad A^2 = A_1^2 + A_2^2$$

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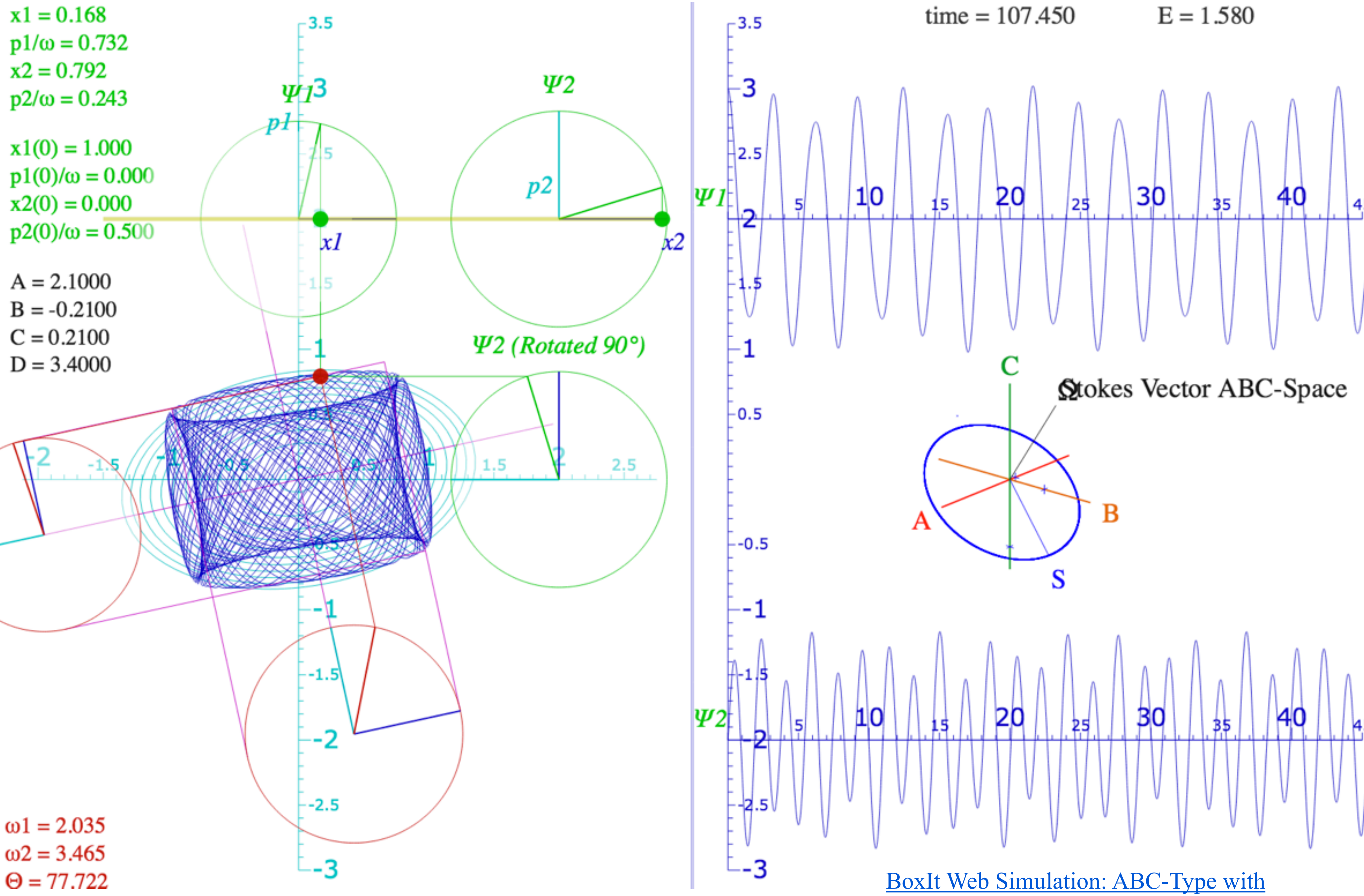
$\omega t-\rho_1 = (\gamma-\alpha)/2$

$$\alpha = 2\rho_1 \qquad \gamma = 2\omega\cdot t$$

Euler parameters (α,β,γ, A) in terms of *amp-phase parameters* $(A_1, A_2, \omega t, \rho_1)$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$



[BoxIt Web Simulation: ABC-Type with A=2.1; B=-0.21; C=0.21; D=3.4]

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To find U(2) *eigenstates: Match* **H** *axis-angles* $[\varphi,\vartheta,\Theta]$ *to* **S** *Euler angles* (α,β,γ) *A-Type* $(\alpha_A,\beta_A,\gamma_A)$,

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To find U(2) eigenstates: Match **H** axis-angles $[\varphi,\vartheta,\Theta]$ to **S** Euler angles (α,β,γ)

Given Hamiltonian:

$$\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}=\begin{pmatrix} \frac{3}{4} & \frac{\sqrt{3}}{8}+i\frac{3}{8} \\ \frac{\sqrt{3}}{8}-i\frac{3}{8} & \frac{1}{4} \end{pmatrix}$$

Find its **Ω**-vector components and axis-angles $[\varphi,\vartheta,\Theta]$:

$$\vec{\Omega}=\begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix}=\begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}=\begin{pmatrix} \frac{3}{4}-\frac{1}{4} \\ 2\frac{\sqrt{3}}{8} \\ 2(\frac{-3}{8}) \end{pmatrix}=\begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{4} \\ \frac{-3}{4} \end{pmatrix}=\begin{pmatrix} \cos\vartheta_A \\ \sin\vartheta_A\cos\varphi_A \\ \sin\vartheta_A\sin\varphi_A \end{pmatrix}$$

H eigenstates have their **S**-vector along (or opposite) to **Ω**-vector. This derives their Euler angles:

$$\vec{\mathbf{S}}=\begin{pmatrix} \cos\beta_A \\ \sin\beta_A\cos\alpha_A \\ \sin\beta_A\sin\alpha_A \end{pmatrix}=\begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{4} \\ \frac{-3}{4} \end{pmatrix},\quad \vec{\mathbf{S}}\bullet\vec{\mathbf{S}}=\frac{1}{2}^2+\frac{\sqrt{3}}{4}^2+\left(\frac{-3}{4}\right)^2=\frac{1}{4}+\frac{3}{16}+\frac{9}{16}=1=\Omega^2$$

…beat frequency Ω :

and Euler angles: $\beta_A=\vartheta_A=\cos^{-1}\frac{1}{2}=60°$ and: $\alpha_A=\varphi_A=\tan^{-1}\frac{\frac{-3}{4}}{\frac{\sqrt{3}}{4}}=\tan^{-1}-\sqrt{3}=-60°$

Then spin-½ *up* U(2) *A*-basis fast eigenstate is:

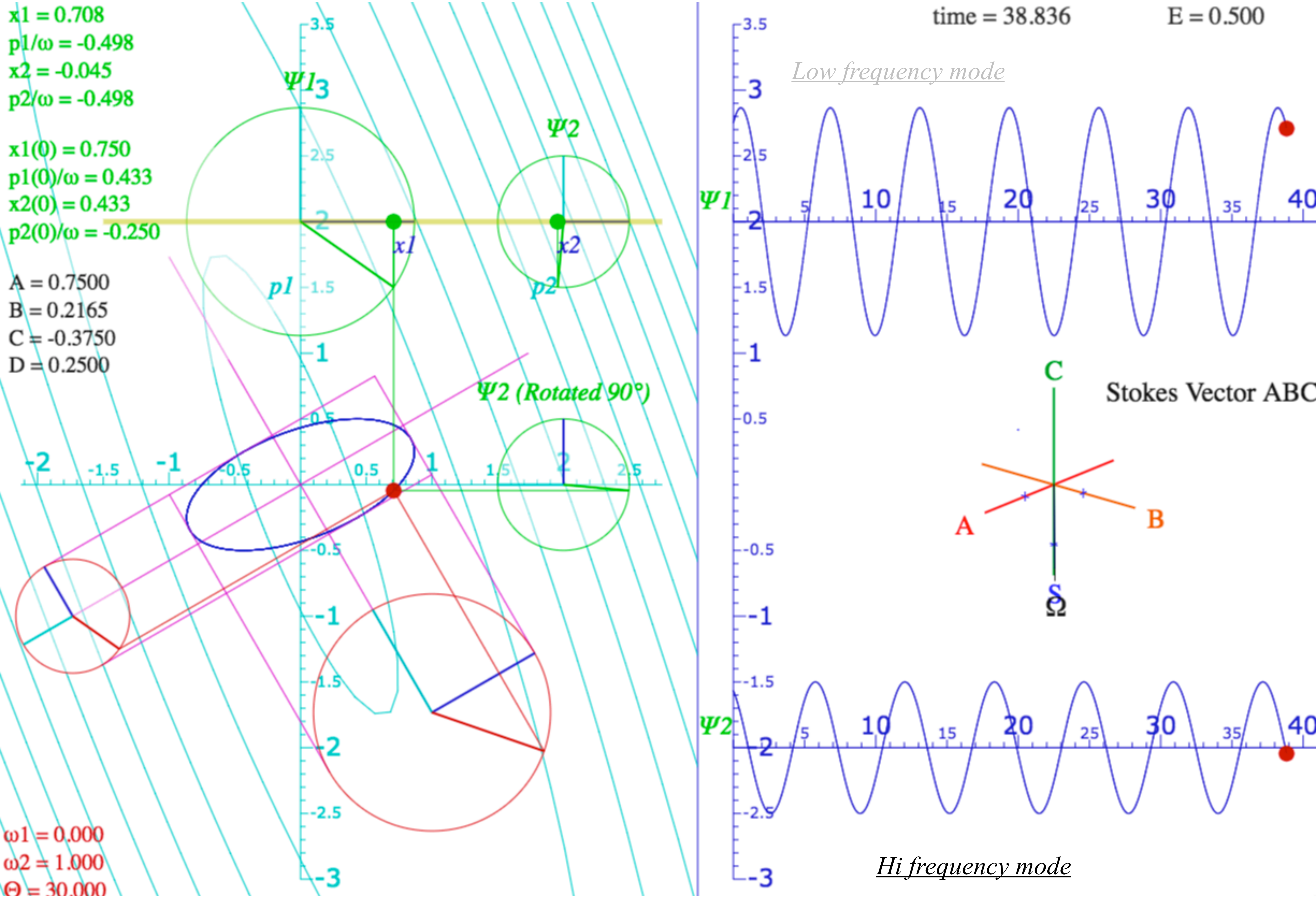
$$|\uparrow_A\rangle=\begin{pmatrix} x_1^{(\uparrow_A)}+ip_1 \\ x_2^{(\uparrow_A)}+ip_2 \end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha_A}{2}}\cos\frac{\beta_A}{2} \\ e^{+i\frac{\alpha_A}{2}}\sin\frac{\beta_A}{2} \end{pmatrix}=\begin{pmatrix} \cos\frac{-\alpha_A}{2}\cos\frac{\beta_A}{2}+i\sin\frac{-\alpha_A}{2}\cos\frac{\beta_A}{2} \\ \cos\frac{\alpha_A}{2}\sin\frac{\beta_A}{2}+i\sin\frac{\alpha_A}{2}\sin\frac{\beta_A}{2} \end{pmatrix}$$

Fast-mode eigenfrequency:
$\omega_\uparrow=\omega_0+\frac{\Omega}{2}=\frac{A+D}{2}+\frac{\Omega}{2}=\frac{1}{2}+\frac{1}{2}=1$

$$=\begin{pmatrix} \cos 30°\cos 30°+i\sin 30°\cos 30° \\ \cos\text{-}30°\sin 30°+i\sin\text{-}30°\sin 30° \end{pmatrix}=\begin{pmatrix} \frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2}+i\frac{1}{2}\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2}\frac{1}{2}+i\frac{-1}{2}\frac{1}{2} \end{pmatrix}=\begin{pmatrix} \frac{3}{4}+i\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4}-i\frac{1}{4} \end{pmatrix}=\begin{pmatrix} 0.75+i0.433 \\ 0.433-i0.25 \end{pmatrix}=\begin{pmatrix} x_1^{(\uparrow_A)}+ip_1 \\ x_2^{(\uparrow_A)}+ip_2 \end{pmatrix}$$

Slow-mode eigenfrequency is: $\omega_\uparrow=\omega_0-\frac{\Omega}{2}=\frac{A+D}{2}-\frac{\Omega}{2}=\frac{1}{2}-\frac{1}{2}=0$

Example of High frequency mode



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Conventional amplitude-phase-$(A_1,A_2,\omega t,\rho_1)$ *labeling of optical polarization Relation to Euler angle*

To find U(2) *eigenstates: Match* **H** *axis-angles* $[\varphi,\vartheta,\Theta]$ *to* **S** *Euler angles* (α,β,γ) *A-Type* $(\alpha_A,\beta_A,\gamma_A)$,

➡ *Fast mode of elliptic polarization vs Slow mode (or no-mode) of orthogonal elliptic orbit*

Euler angle labeling of optical polarization C-Type $(\alpha_C,\beta_C,\gamma_C)$ *vs A-Type* $(\alpha_A,\beta_A,\gamma_A)$,

H eigenstates have their **S**-vector along (or opposite) to **Ω**-vector. This derives their Euler angles:

$$\vec{\mathbf{S}} = \begin{pmatrix} \cos\beta_A \\ \sin\beta_A \cos\alpha_A \\ \sin\beta_A \sin\alpha_A \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{4} \\ \frac{-3}{4} \end{pmatrix}, \quad \vec{\mathbf{S}} \bullet \vec{\mathbf{S}} = \frac{1}{2}^2 + \frac{\sqrt{3}}{4}^2 + \left(\frac{-3}{4}\right)^2 = \frac{1}{4} + \frac{3}{16} + \frac{9}{16} = 1 = \Omega^2$$

...beat frequency Ω :

and Euler angles: $\beta_A = \vartheta_A = \cos^{-1}\frac{1}{2} = 60°$ and: $\alpha_A = \varphi_A = \tan^{-1}\frac{\frac{-3}{4}}{\frac{\sqrt{3}}{4}} = \tan^{-1} -\sqrt{3} = -60°$

spin-½ *up* U(2) *A*-basis fast eigenstate is:

$$|\uparrow_A\rangle = \begin{pmatrix} x_1^{(\uparrow_A)} + ip_1 \\ x_2^{(\uparrow_A)} + ip_2 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha_A}{2}} \cos\frac{\beta_A}{2} \\ e^{+i\frac{\alpha_A}{2}} \sin\frac{\beta_A}{2} \end{pmatrix} = \begin{pmatrix} \cos\frac{-\alpha_A}{2}\cos\frac{\beta_A}{2} + i\sin\frac{-\alpha_A}{2}\cos\frac{\beta_A}{2} \\ \cos\frac{\alpha_A}{2}\sin\frac{\beta_A}{2} + i\sin\frac{\alpha_A}{2}\sin\frac{\beta_A}{2} \end{pmatrix}$$

Fast-mode eigenfrequency:

$$\omega_{\uparrow} = \omega_0 + \frac{\Omega}{2} = \frac{A+D}{2} + \frac{\Omega}{2} = \frac{1}{2} + \frac{1}{2} = 1$$

$$= \begin{pmatrix} \cos 30°\cos 30° + i\sin 30°\cos 30° \\ \cos -30°\sin 30° + i\sin -30°\sin 30° \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2} + i\frac{1}{2}\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2}\frac{1}{2} + i\frac{-1}{2}\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{3}{4} + i\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} - i\frac{1}{4} \end{pmatrix} = \begin{pmatrix} 0.75 + i0.433 \\ 0.433 - i0.25 \end{pmatrix} = \begin{pmatrix} x_1^{(\uparrow_A)} + ip_1 \\ x_2^{(\uparrow_A)} + ip_2 \end{pmatrix}$$

spin-½ *down* U(2) *A*-basis slow eigenstate is: (*very slow*)

$$|\downarrow_A\rangle = \begin{pmatrix} x_1^{(\downarrow_A)} + ip_1 \\ x_2^{(\downarrow_A)} + ip_2 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha_A}{2}} \sin\frac{\beta_A}{2} \\ -e^{+i\frac{\alpha_A}{2}} \cos\frac{\beta_A}{2} \end{pmatrix} = \begin{pmatrix} \cos\frac{-\alpha_A}{2}\sin\frac{\beta_A}{2} + i\sin\frac{-\alpha_A}{2}\sin\frac{\beta_A}{2} \\ -\cos\frac{\alpha_A}{2}\cos\frac{\beta_A}{2} - i\sin\frac{\alpha_A}{2}\cos\frac{\beta_A}{2} \end{pmatrix}$$

Slow-mode eigenfrequency:

$$\omega_{\downarrow} = \omega_0 - \frac{\Omega}{2} = \frac{A+D}{2} - \frac{\Omega}{2} = \frac{1}{2} - \frac{1}{2} = 0$$

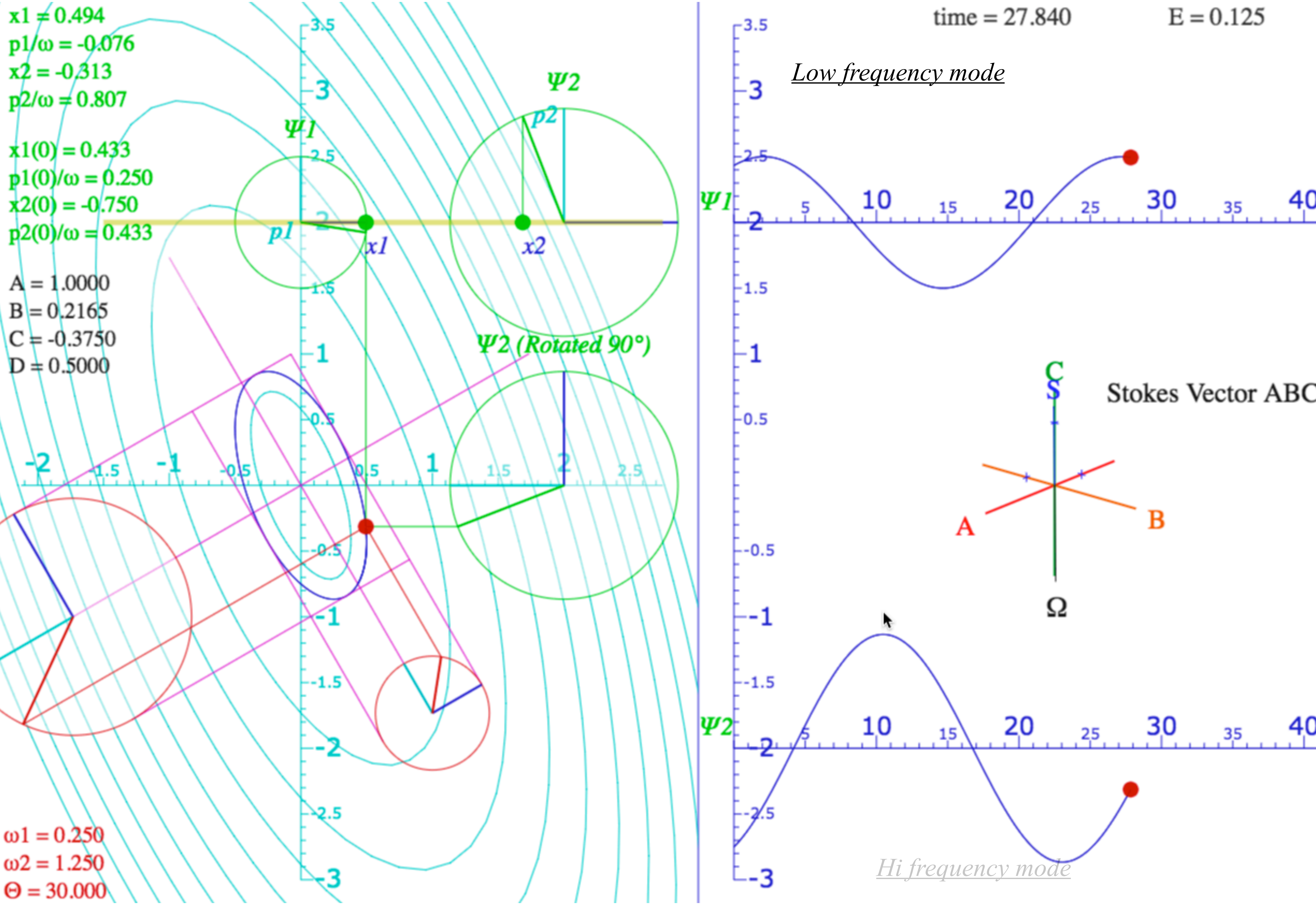
Get ↓down ↓spin: replace β_A by β_A-180°

$$= \begin{pmatrix} \cos 30°\sin 30° + i\sin 30°\sin 30° \\ -\cos -30°\cos 30° - i\sin -30°\cos 30° \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2}\frac{1}{2} + i\frac{1}{2}\frac{1}{2} \\ -\frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2} - i\frac{-1}{2}\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{4} + i\frac{1}{4} \\ -\frac{3}{4} + i\frac{\sqrt{3}}{4} \end{pmatrix} = \begin{pmatrix} 0.433 + i0.25 \\ -0.75 + i0.433 \end{pmatrix}$$

To make slow-mode move replace $(A,D) = (\frac{3}{4},\frac{1}{4})$ by $(A,D) = (1,\frac{1}{2})$ so $\omega_0 = \frac{A+D}{2} = \frac{3}{4}$

$$\omega_{\downarrow} = \frac{A+D}{2} - \frac{\Omega}{2} = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$$

Example of Low frequency mode



AMOP reference links on following page

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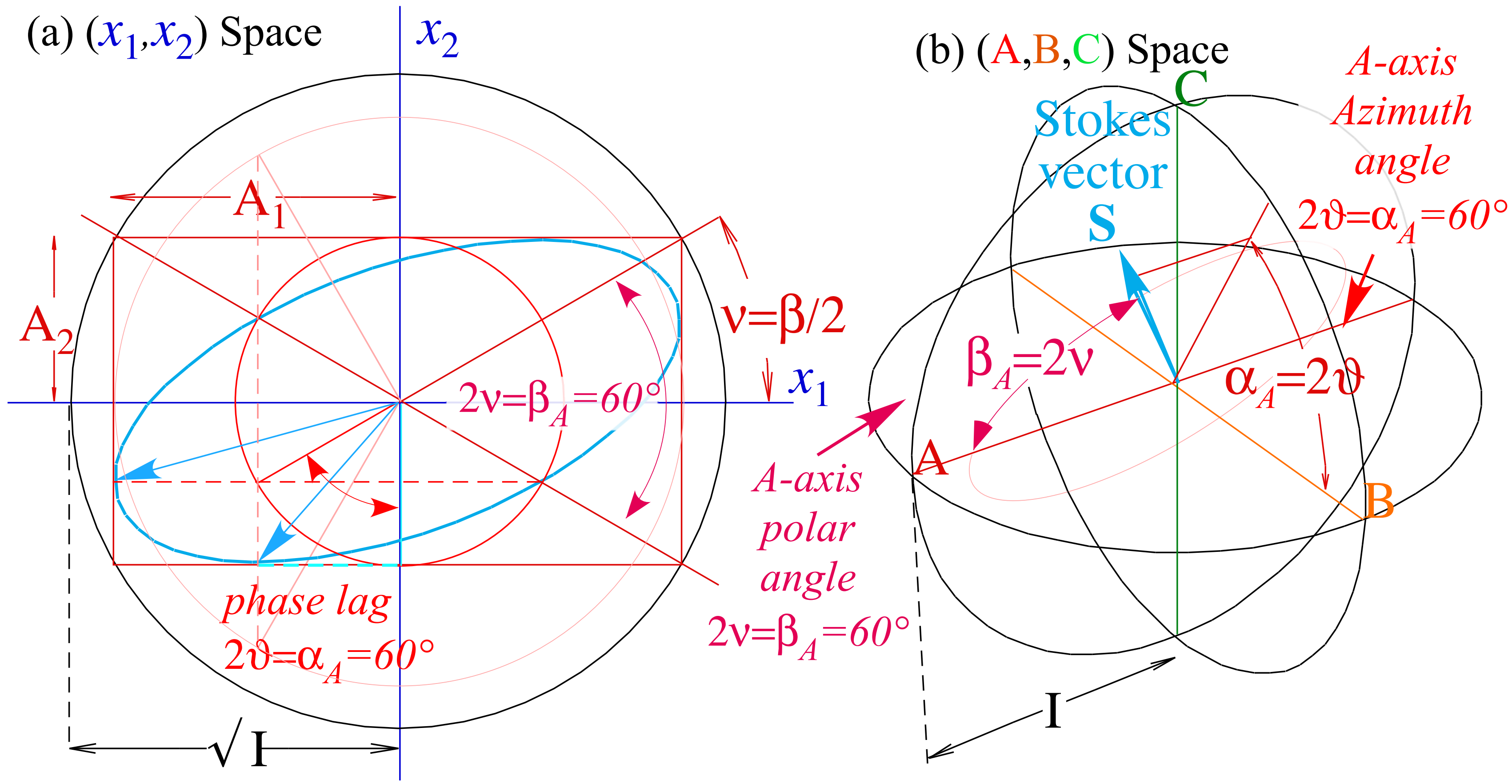
➡ *Euler angle labeling of optical polarization* *C-Type* $(\alpha_C,\beta_C,\gamma_C)$ *vs A-Type* $(\alpha_A,\beta_A,\gamma_A)$,

The A-view in {x1,x2}-basis

Angles $\alpha_A = \rho_1 - \rho_2 = 2\rho_1$, $\beta_A = 2\tan^{-1} A_2/A_1$, $\gamma_A = 2\omega\cdot t$ define ellipses with intensity $I = A^2 = A_1^2 + A_2^2$.

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = A \begin{pmatrix} e^{-i\alpha_A/2}\cos\frac{\beta_A}{2} \\ e^{+i\alpha_A/2}\sin\frac{\beta_A}{2} \end{pmatrix} e^{-i\omega t} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$



A or *Z*-axis Euler angles

$2\vartheta = \alpha = \alpha_A = \rho_1 - \rho_2 = 2\rho_1 = 60°$

$2\nu = \beta = \beta_A = 2\tan^{-1} A_2/A_1 = 60°$

$\gamma_A = 2\omega\cdot t$

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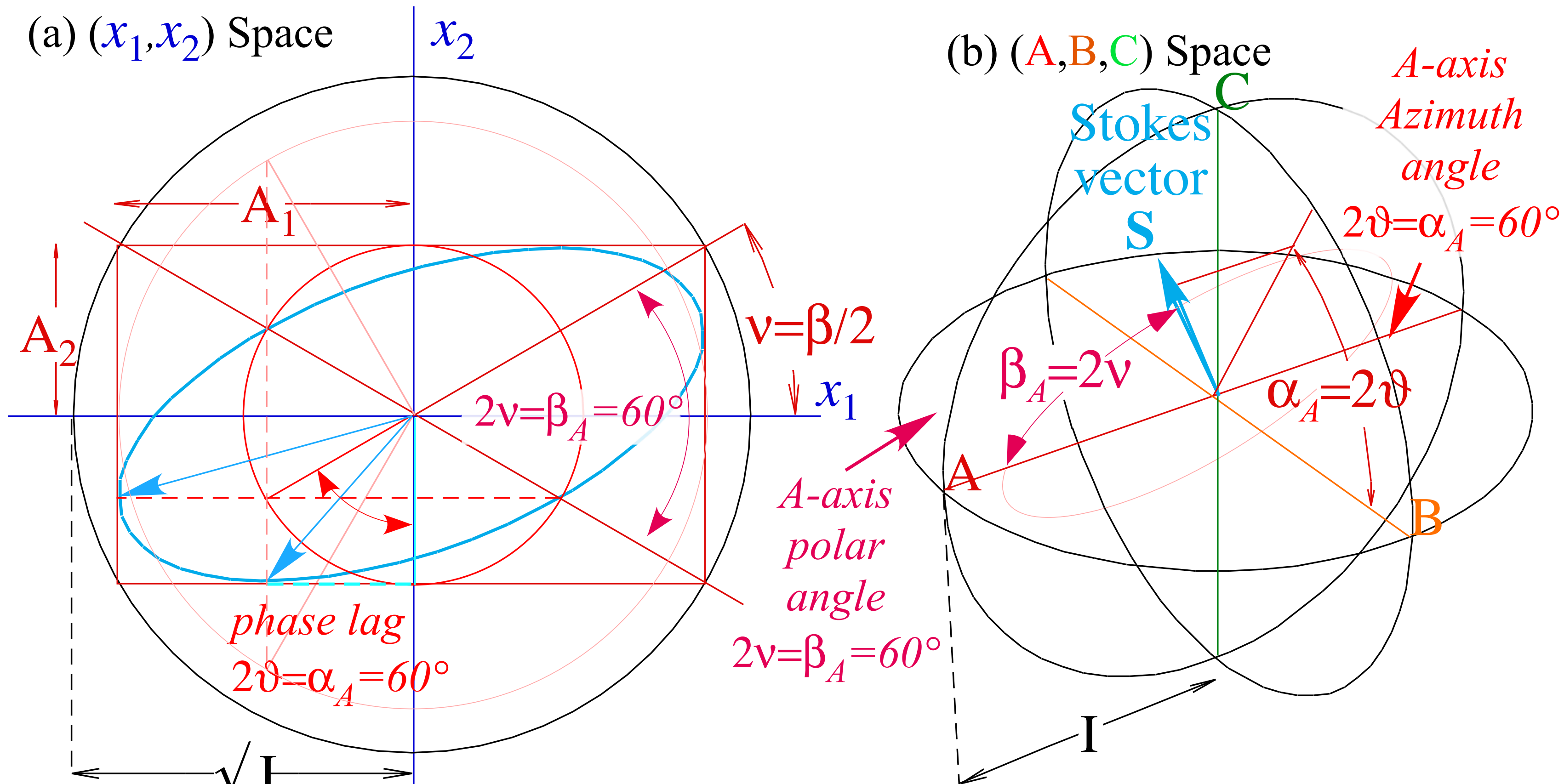


A or *Z*-axis Euler angles

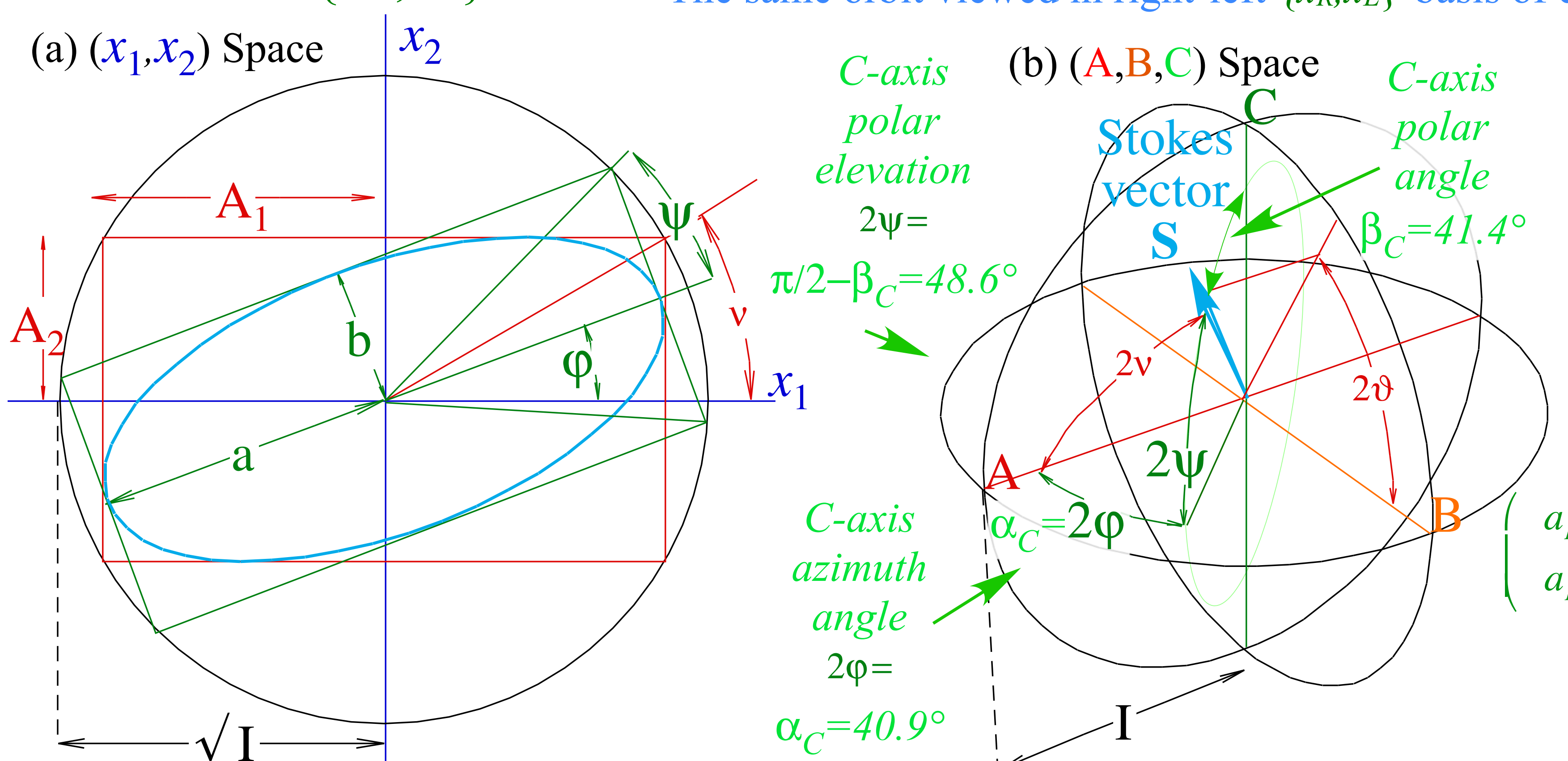
$\alpha=\alpha_A=\rho_1-\rho_2=2\rho_1=60°$

$\beta=\beta_A=2\tan^{-1}A_2/A_1=60°$

$\gamma_A=2\omega\cdot t$

The C-view in {xR,xL}-basis

The same orbit viewed in right-left $\{x_R,x_L\}$-basis of circular polarization with angles $(\alpha_C,\beta_C,\gamma_C)$.



$$\begin{pmatrix} a_R \\ a_L \end{pmatrix} = A \begin{pmatrix} e^{-i\alpha_C/2}\cos\frac{\beta_C}{2} \\ e^{+i\alpha_C/2}\sin\frac{\beta_C}{2} \end{pmatrix} e^{-i\frac{\gamma_C}{2}} = \begin{pmatrix} x_R+ip_R \\ x_R+ip_R \end{pmatrix}$$

Converting an A-based set of Stokes parameters into a C-based set or a B-based set involves cyclic permutation of A, B, and C polar formulas

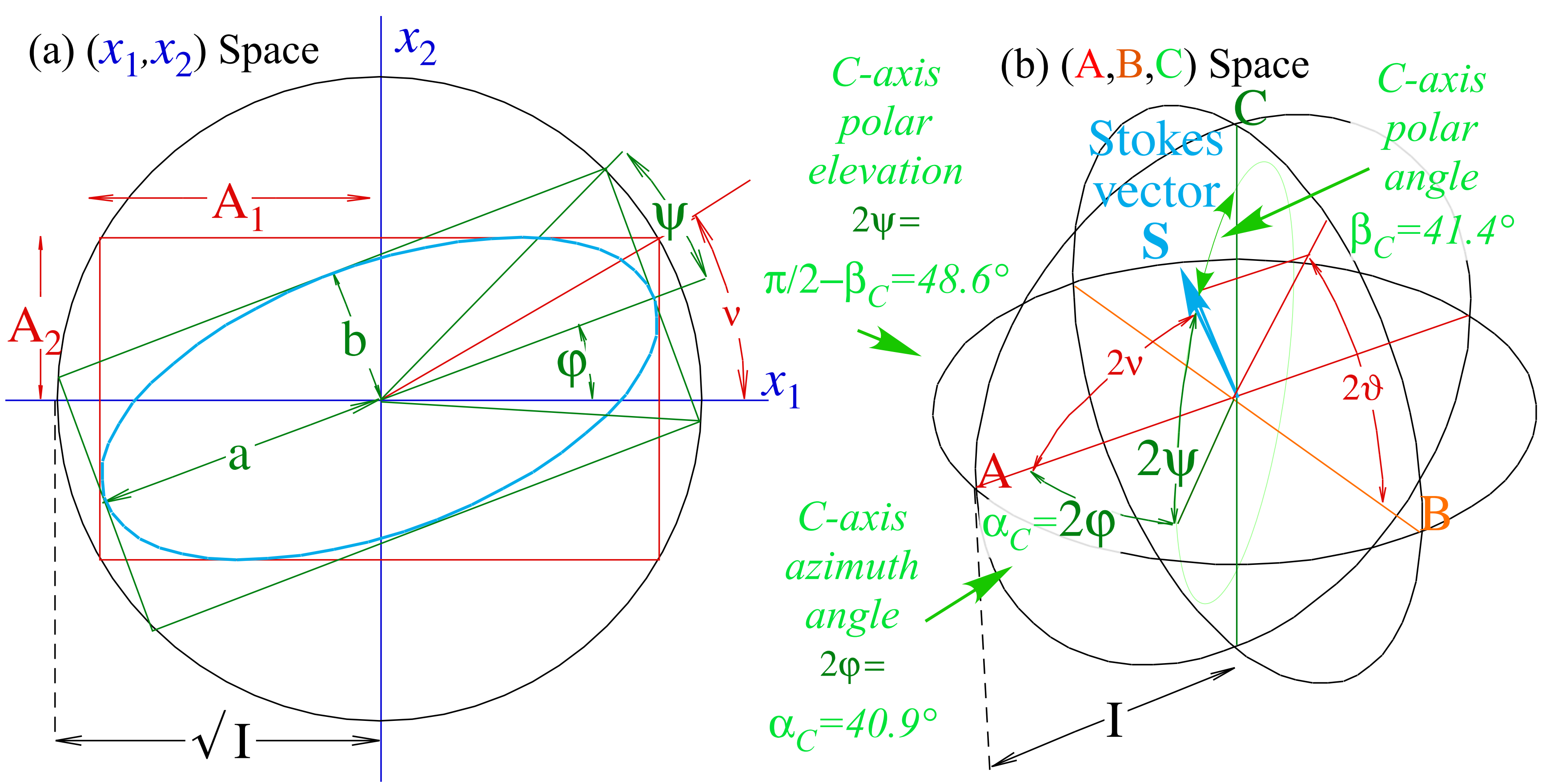
$$\text{Asymmetry } S_A = \frac{I}{2}\cos\beta_A = \frac{I}{2}\sin\alpha_B\sin\beta_B = \frac{I}{2}\cos\alpha_C\sin\beta_C$$

$$\text{Balance } S_B = \frac{I}{2}\cos\alpha_A\sin\beta_A = \frac{I}{2}\cos\beta_B = \frac{I}{2}\sin\alpha_C\sin\beta_C$$

$$\text{Chirality } S_C = \frac{I}{2}\sin\alpha_A\sin\beta_A = \frac{I}{2}\cos\alpha_B\sin\beta_B = \frac{I}{2}\cos\beta_C$$

The C-view in $\{x_R,x_L\}$-basis

The same orbit viewed in right and left circular polarization $\{x_R,x_L\}$-bases using angles $(\alpha_C,\beta_C,\gamma_C)$.



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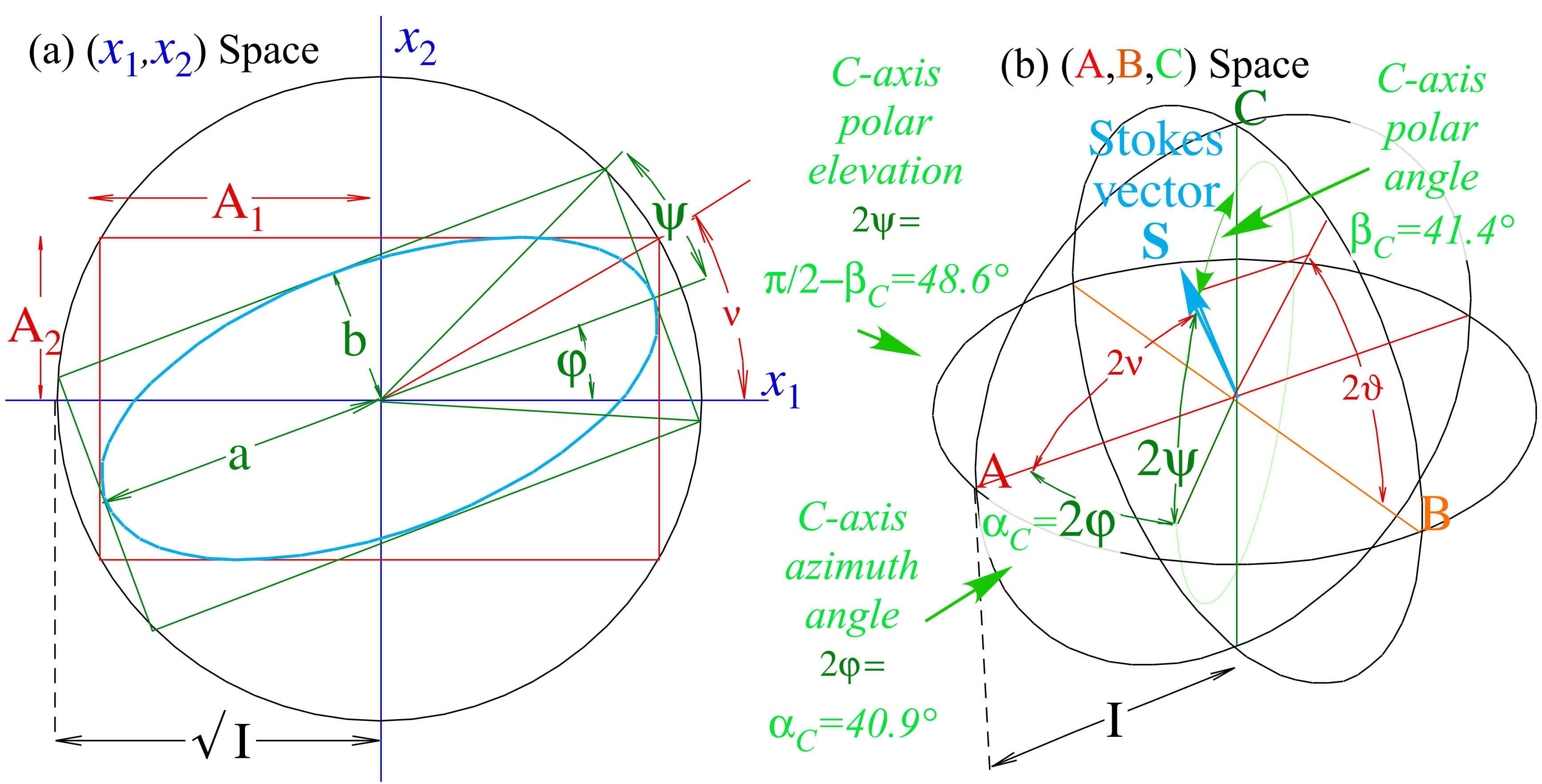
The same orbit viewed in right and left circular polarization $\{x_R, x_L\}$-bases using angles $(\alpha_C, \beta_C, \gamma_C)$.

Angles (α_C, β_C): C-axial polar angle β_C from above.

$\sin\alpha_A \sin\beta_A = \cos\beta_C$ or: $\beta_C = \cos^{-1}(\sin\alpha_A \sin\beta_A) = \cos^{-1}(\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}) = 41.4°$



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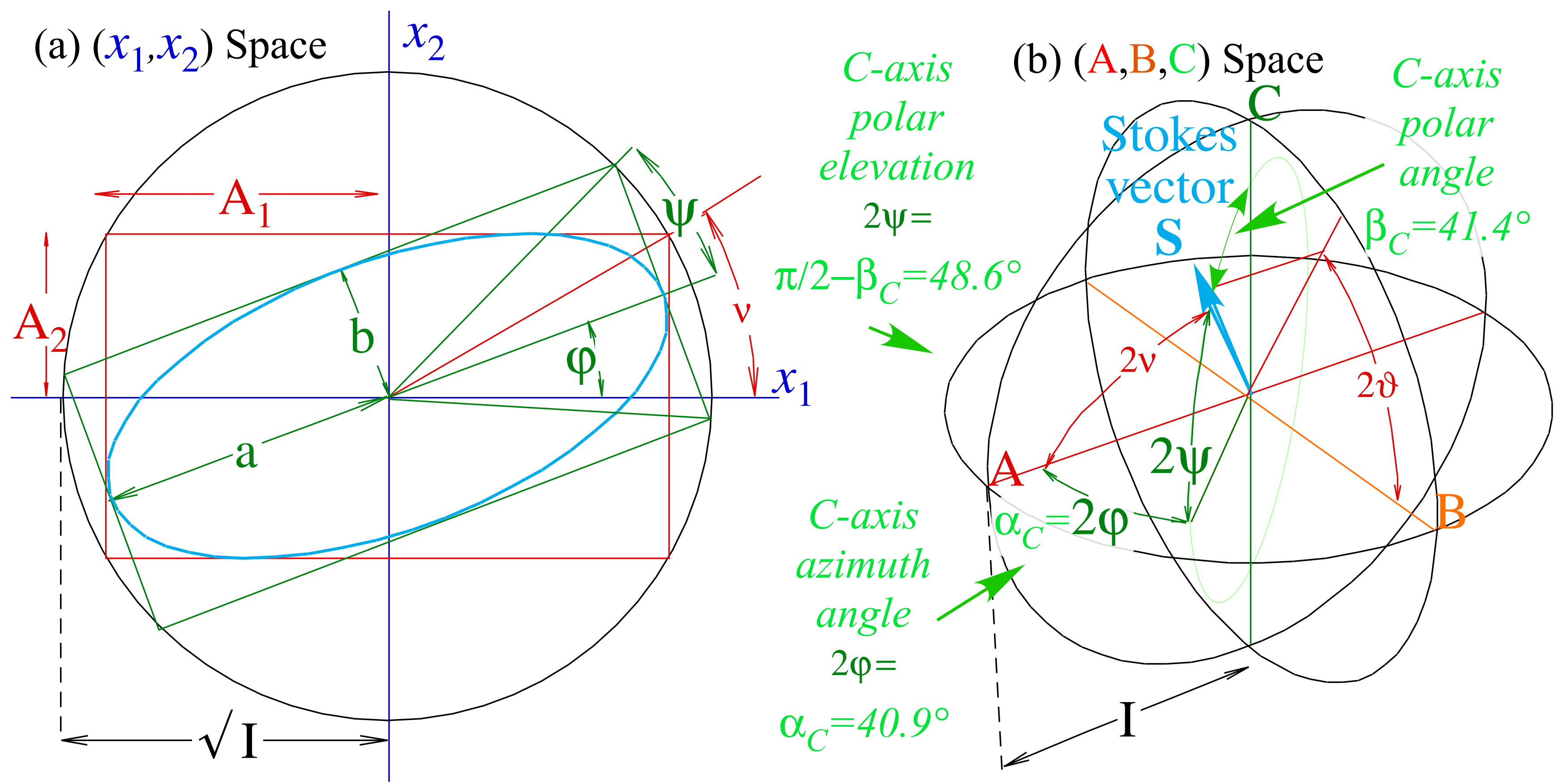
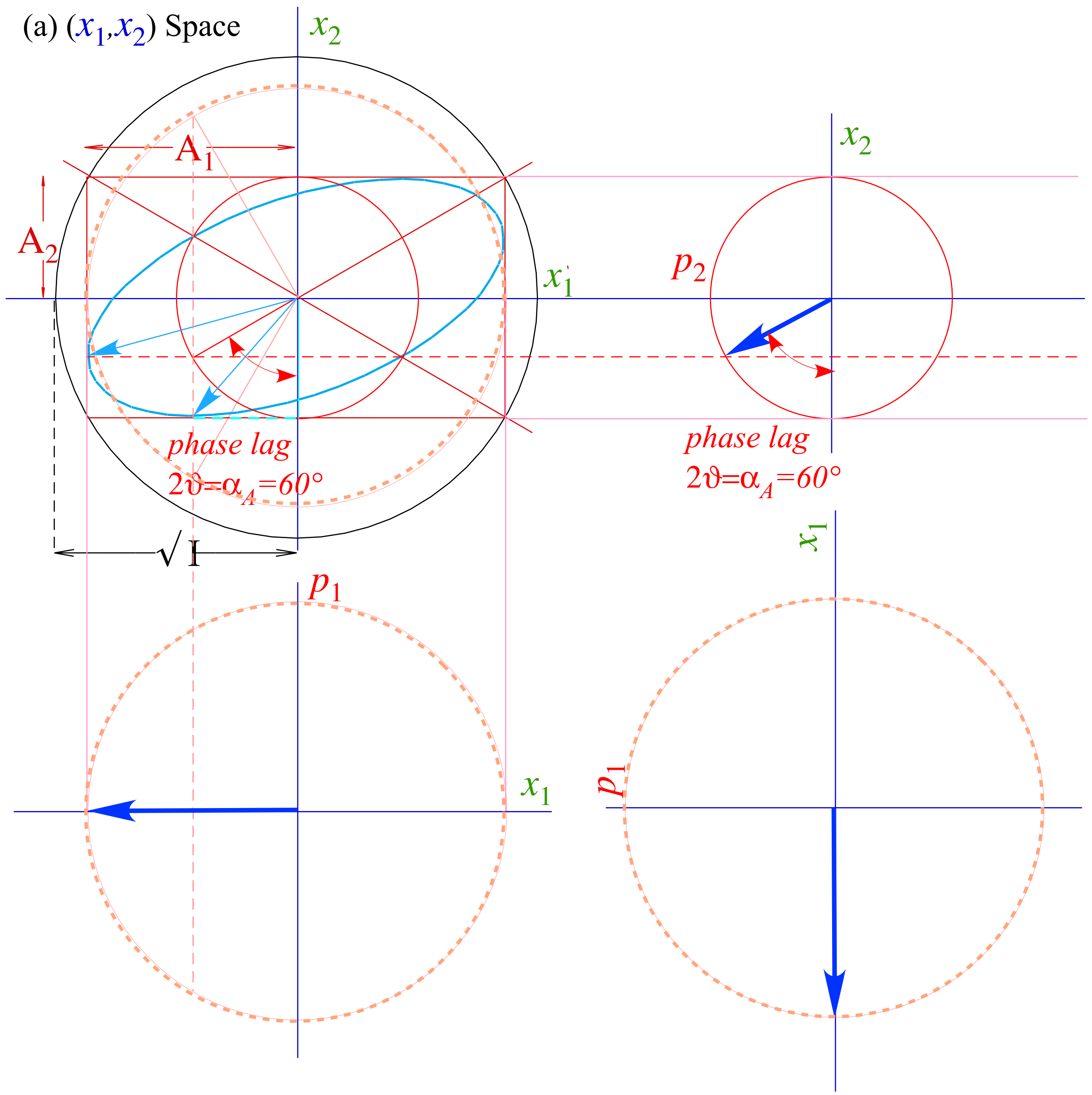
Angles (α_C,β_C): C-axial polar angle β_C from above.

$$\sin\alpha_A \sin\beta_A = \cos\beta_C \qquad \text{or:} \quad \beta_C = \cos^{-1}(\sin\alpha_A \sin\beta_A) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\right) = 41.4°$$

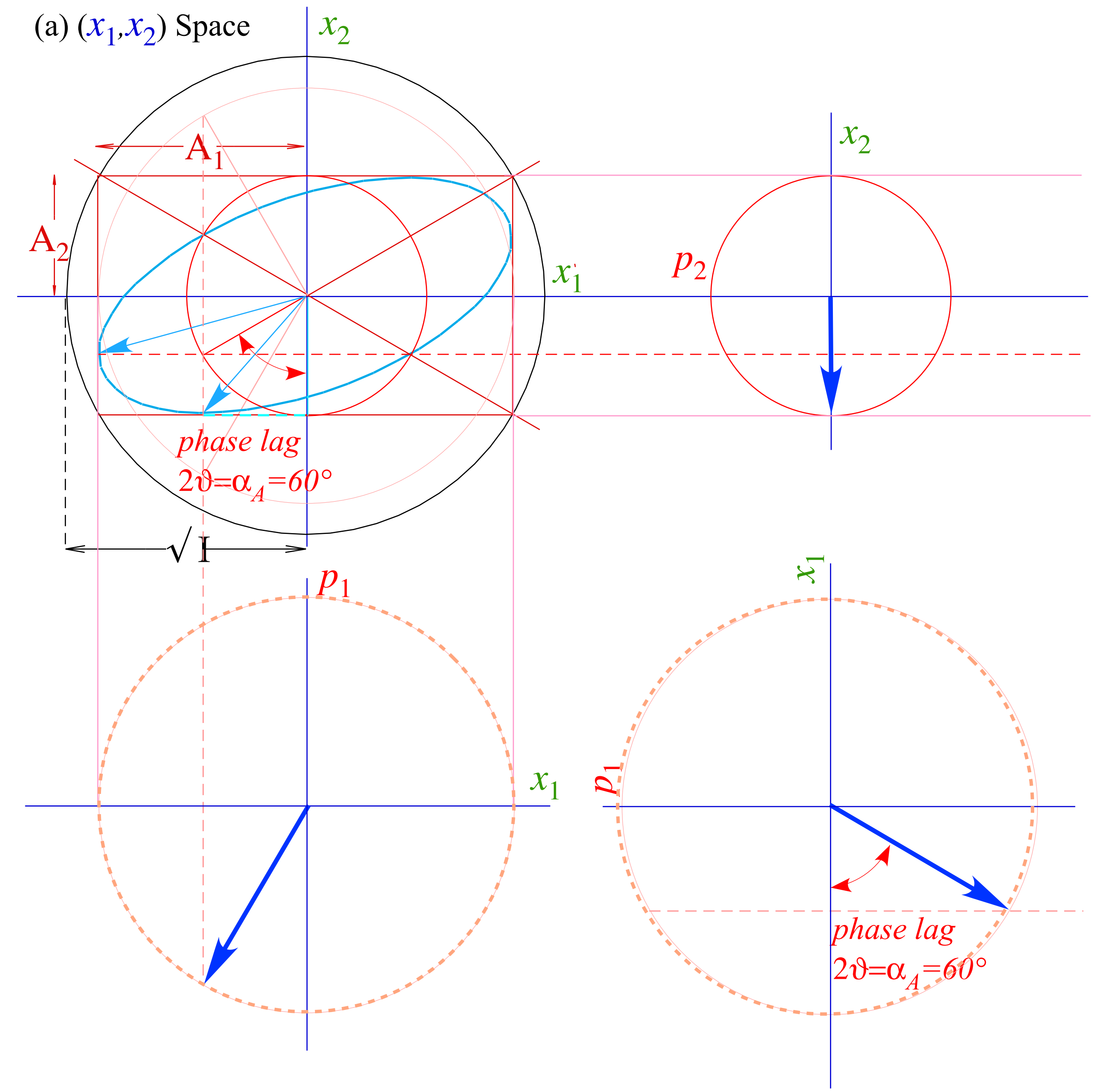
C-axis azimuth angle α_C relates to A-axis angles α_A and β_A . See $\alpha_C = 2\varphi$ below.

$$\frac{\cos\alpha_A \sin\beta_A}{\cos\beta_A} = \tan\alpha_C \quad \text{or:} \quad \alpha_C = ATN2(\cos\alpha_A \sin\beta_A / \cos\beta_A) = ATN2\left(\frac{1}{2}\cdot\frac{\sqrt{3}}{2}/\frac{1}{2}\right) = 40.9°$$





(a) (x_1,x_2) Space
x_2
A_1
A_2
x_1
p_2
phase lag
$2\vartheta=\alpha_A=60°$
phase lag
$2\vartheta=\alpha_A=60°$
$\sqrt{I}$
p_1
x_1
p_1
x_1



(a) (x_1,x_2) Space
x_2
A_1
A_2
x_1
p_2
phase lag
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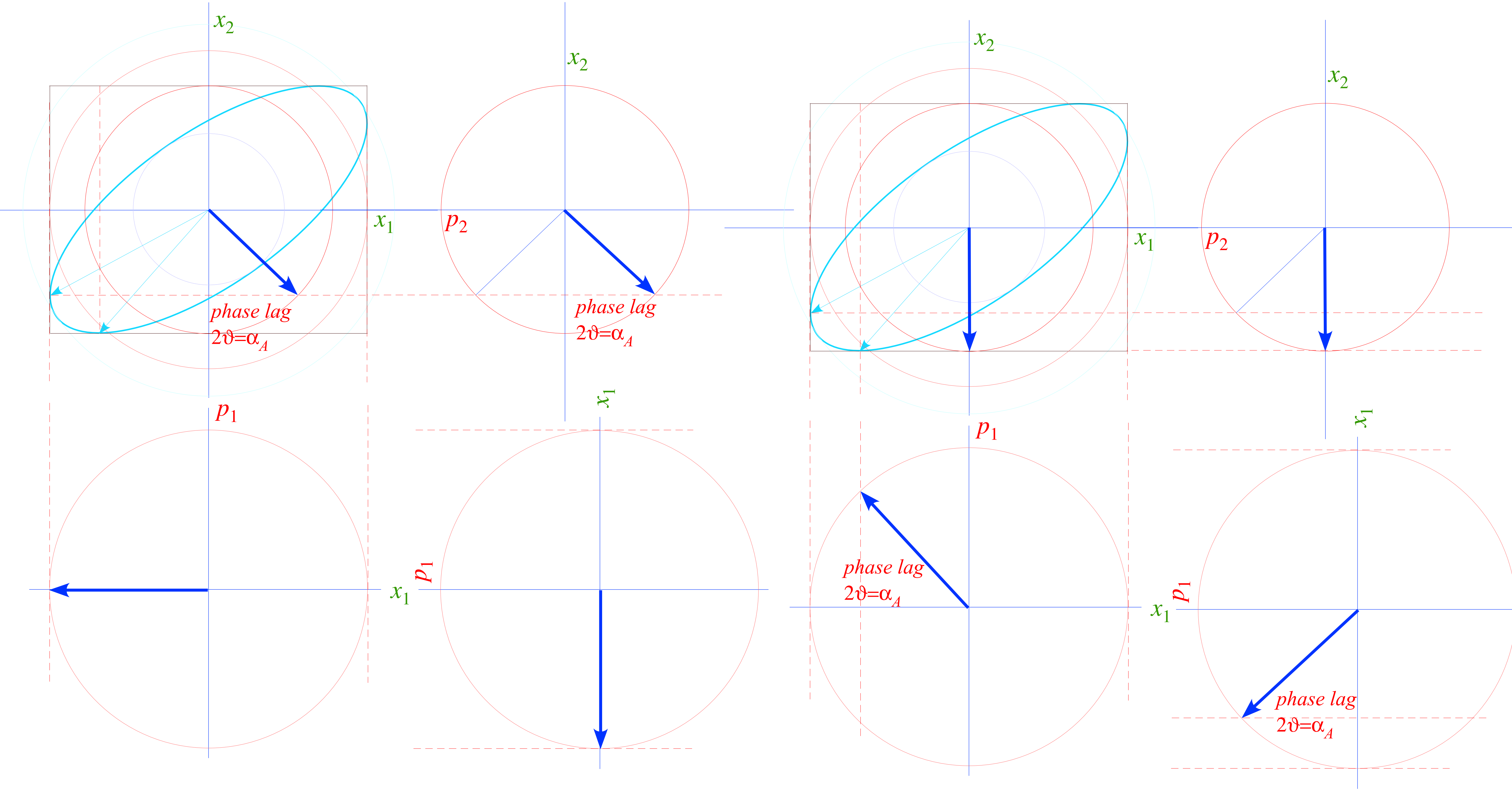
$\sqrt{I}$
p_1
x_1
p_1
x_1
phase lag
$2\vartheta=\alpha_A=60°$

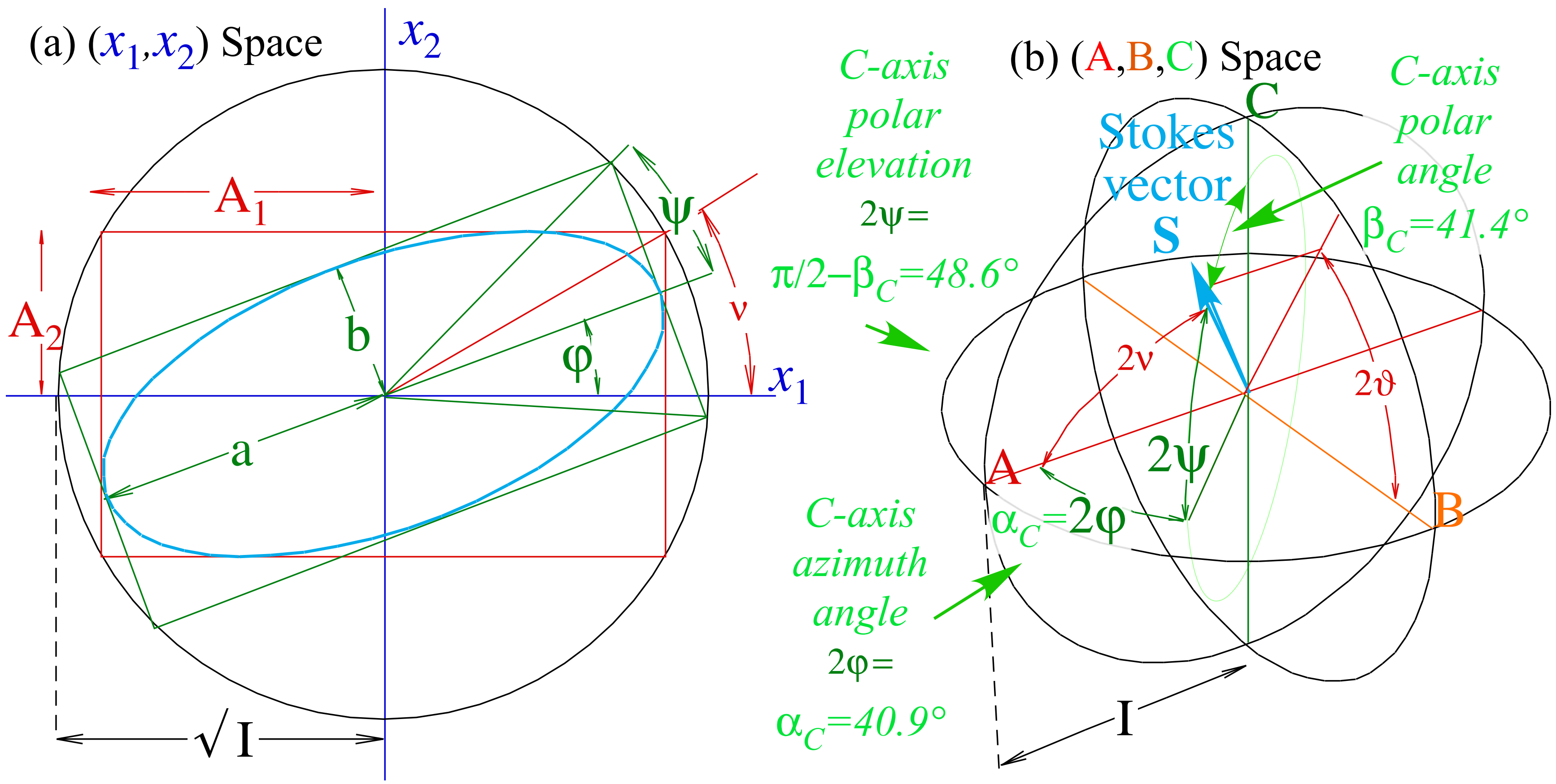


x2
x1
p2
x2
phase lag
2ϑ=αA
phase lag
2ϑ=αA
p1
x1
x1
p1
x2
x1
p2
x2
p1
phase lag
2ϑ=αA
x1
x1
p1
phase lag
2ϑ=αA

The C*-view in* $\{x_R, x_L\}$*-basis*

The same orbit viewed in right and left circular polarization $\{x_R, x_L\}$-bases using angles $(\alpha_C, \beta_C, \gamma_C)$.

$$\begin{pmatrix} a_R \\ a_L \end{pmatrix} = A \begin{pmatrix} e^{-i\alpha_C/2}\cos\frac{\beta_C}{2} \\ e^{+i\alpha_C/2}\sin\frac{\beta_C}{2} \end{pmatrix} e^{-i\frac{\gamma_C}{2}} = \begin{pmatrix} x_R + ip_R \\ x_R + ip_R \end{pmatrix}$$



90° B –rotation $\mathbf{R}(\pi/4)|x_1\rangle = |x_R\rangle$ of axis A into C gets $(\alpha_C, \beta_C, \gamma_C)$ from $(\alpha_A, \beta_A, \gamma_A)$ all at once.

$$\begin{pmatrix} \cos\frac{\pi}{4} & i\sin\frac{\pi}{4} \\ i\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} Ae^{-i\alpha_A/2}\cos\frac{\beta_A}{2} \\ Ae^{+i\alpha_A/2}\sin\frac{\beta_A}{2} \end{pmatrix} e^{-i\frac{\gamma_A}{2}} = \begin{pmatrix} Ae^{-i\alpha_C/2}\cos\frac{\beta_C}{2} \\ Ae^{+i\alpha_C/2}\sin\frac{\beta_C}{2} \end{pmatrix} e^{-i\frac{\gamma_C}{2}} = \begin{pmatrix} x_R + ip_R \\ x_R + ip_R \end{pmatrix}$$

Polarization ellipse and spinor state dynamics

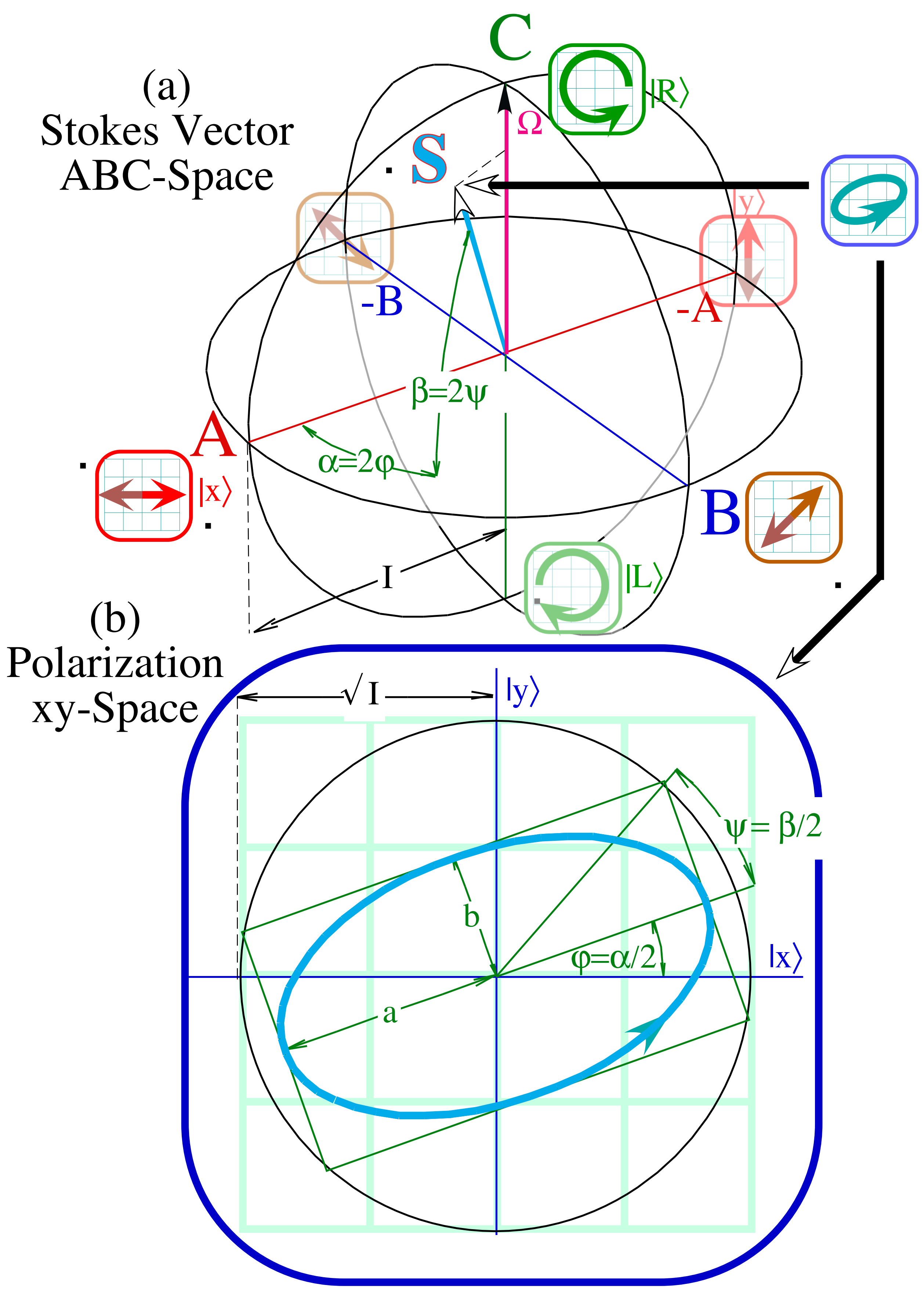


Fig. 3.4.5 Polarization variables (a) Stokes real-vector space (ABC) (b) Complex xy-spinor-space (x_1,x_2).

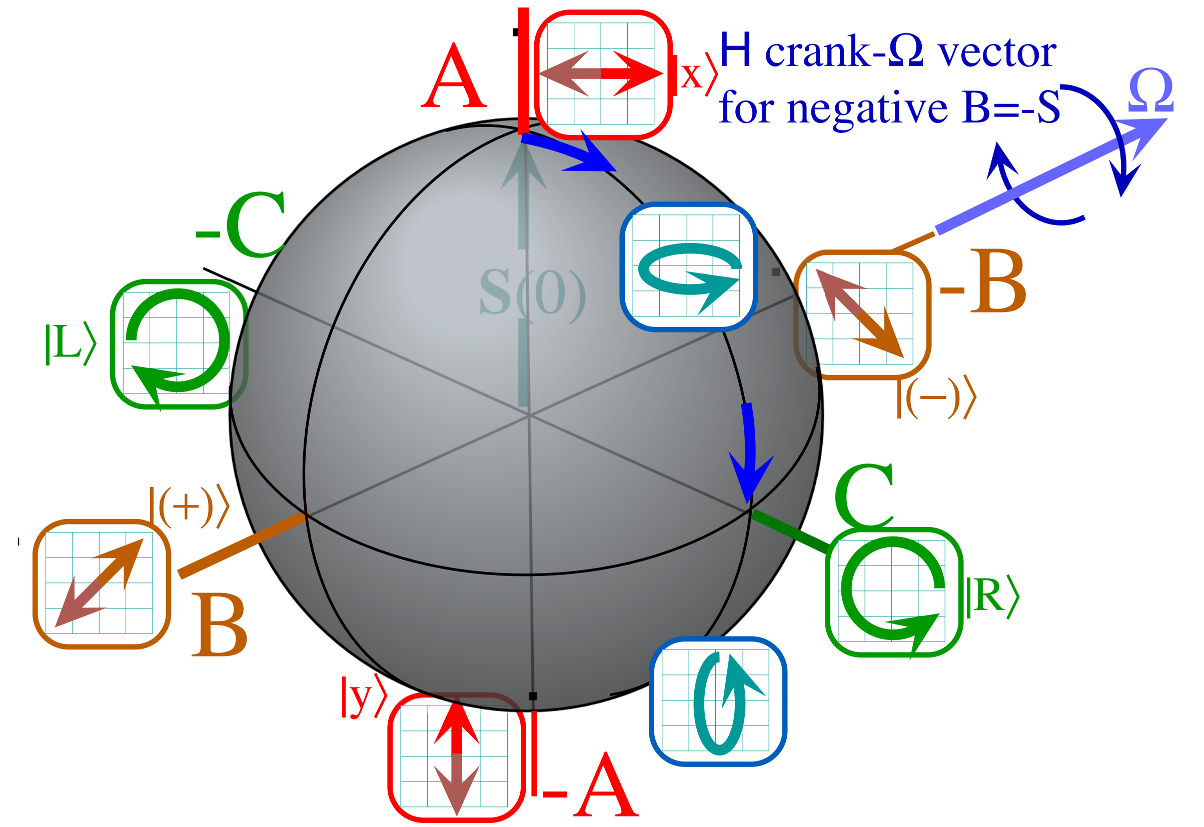
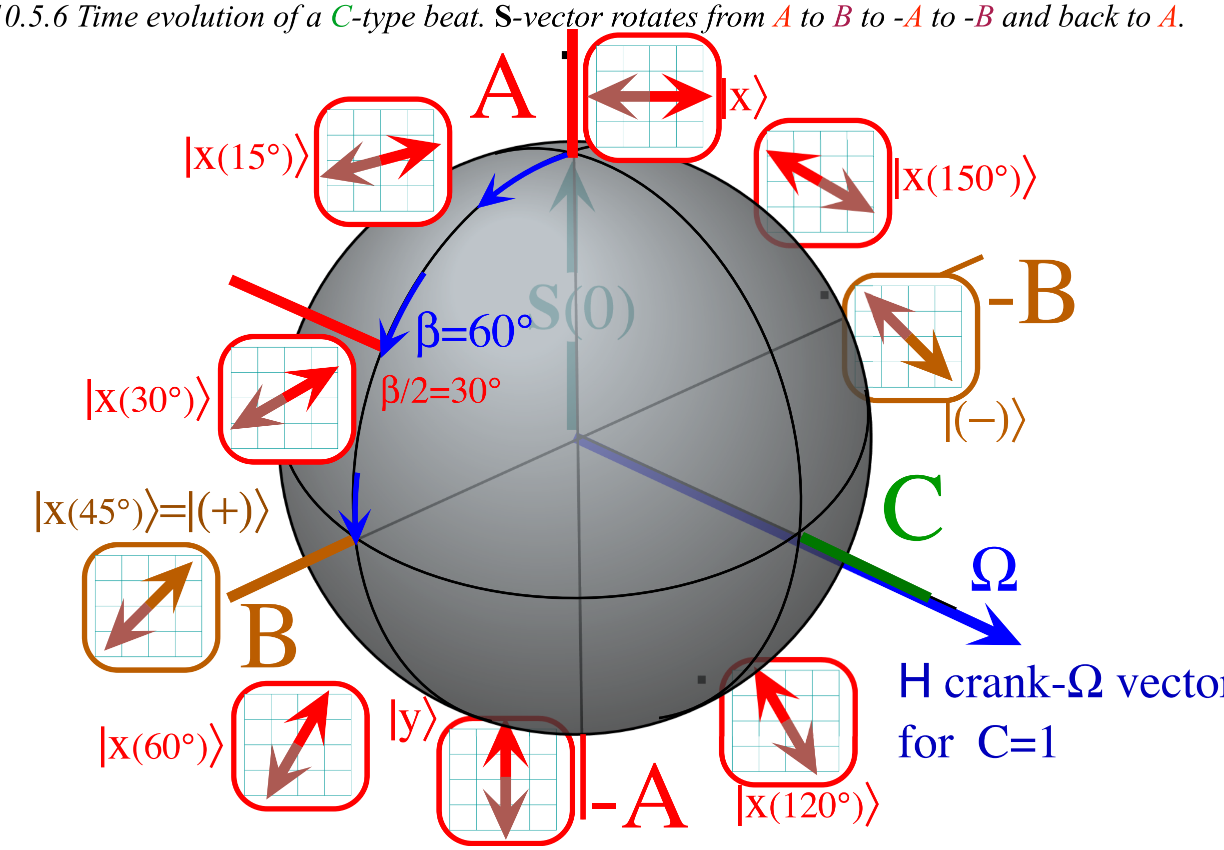


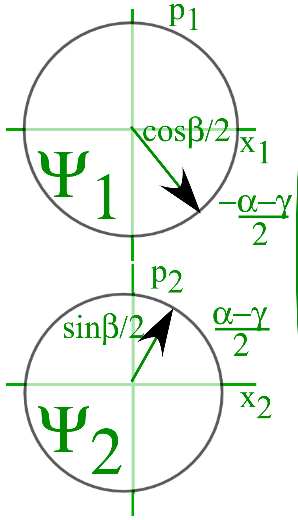
Fig. 10.5.5 Time evolution of a B-type beat. **S***-vector rotates from A to C to -A to -C and back to A.*

Fig. 10.5.6 Time evolution of a C-type beat. **S***-vector rotates from A to B to -A to -B and back to A.*

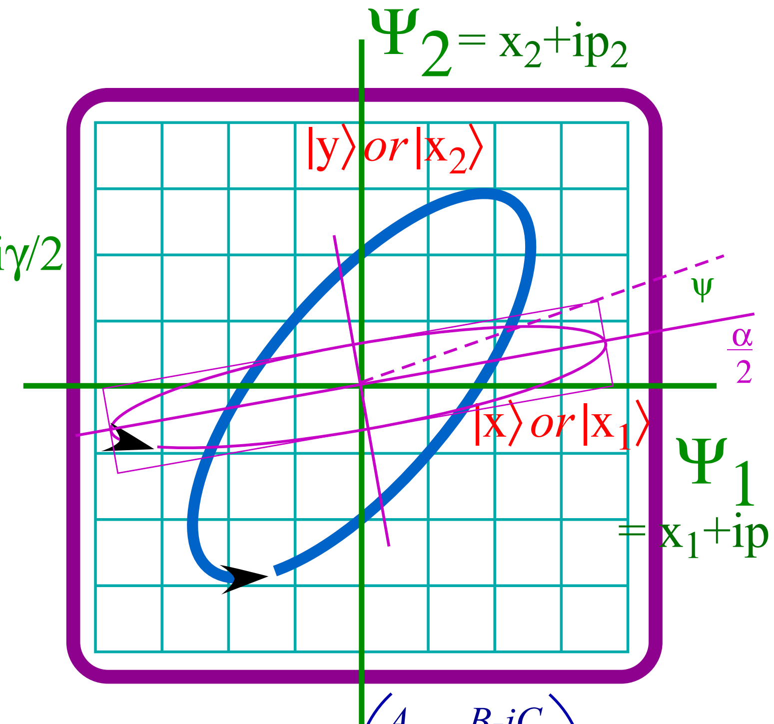


U(2) World : Complex 2D Spinors

2-State ket $|\Psi\rangle=$



$$\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\begin{pmatrix}\sqrt{N}e^{-i\alpha/2}\cos\beta/2\\ \sqrt{N}e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$



R(3) World : Real 3D Vectors

$|\Psi\rangle$ State

Spin Vector S

$$\begin{pmatrix}S_B\\ S_C\\ S_A\end{pmatrix}=\begin{pmatrix}N\sin\beta\cos\alpha\\ N\sin\beta\sin\alpha\\ N\cos\beta\end{pmatrix}\frac{1}{2}$$

(for α=15° β=45°)

H-Operator $\begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix}$

Angular velocity

$$\Omega=\begin{pmatrix}\Omega_B\\ \Omega_C\\ \Omega_A\end{pmatrix}=\begin{pmatrix}2B\\ 2C\\ A-D\end{pmatrix}=\begin{pmatrix}\Omega\sin\vartheta\cos\varphi\\ \Omega\sin\vartheta\sin\varphi\\ \Omega\cos\vartheta\end{pmatrix}$$