AMOP reference links on page 2

4.02.18 class 20: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

Interwining ($S_1 \subset S_2 \subset S_3 \subset S_4 \subset S_5$ …)*(U(1)⊂U(2)⊂U(3)⊂U(4)⊂U(5) …) algebras and tensor operator applications to spinor-rotor or orbital correlations

U(2) tensor product states and S_n permutation symmetry
Rank-1 tensor (or spinor)

Rank-2 tensor (2 particles each with U(2) state space)
2-particle U(2) transform and permutation operation
S_2 symmetry of U(2): Trust but verify

Applying S_2 projection to build DTran
Applying DTran for S_2
Applying DTran for U(2)

S_3 permutations related to C_{3v}~D_3 geometry
S_3 permutation matrices
Hooklength formula for S_n reps
S_3 symmetry of U(2): Applying S_3 projection (Note Pauli-exclusion principle basis)
Building S_3 DTran T from projectors
Effect of S_3 DTran T: Introducing intertwining S_3 - U(2) irep matrices
Multi-spin $(1/2)^N$ product state (Comparison to previous cases)

AMOP reference links *(Updated list given on 2nd page of each class presentation)*

Web Resources - front page
UAF Physics UTube channel

Quantum Theory for the Computer Age
Principles of Symmetry, Dynamics, and Spectroscopy
Classical Mechanics with a Bang!
Modern Physics and its Classical Foundations

2014 AMOP
2017 Group Theory for QM
2018 AMOP

Frame Transformation Relations And Multipole Transitions In Symmetric Polyatomic Molecules - RMP-1978
Rotational energy surfaces and high- J eigenvalue structure of polyatomic molecules - Harter - Patterson - 1984
Galloping waves and their relativistic properties - ajp-1985-Harter
Asymptotic eigensolutions of fourth and sixth rank octahedral tensor operators - Harter-Patterson-JMP-1979
Nuclear spin weights and gas phase spectral structure of 12C60 and 13C60 buckminsterfullerene -Harter-Reimer-Cpl-1992 - (Alt1, Alt2 Erratum)

Theory of hyperfine and superfine levels in symmetric polyatomic molecules.
- I) Trigonal and tetrahedral molecules: Elementary spin-1/2 cases in vibronic ground states - PRA-1979-Harter-Patterson (Alt scan)
- II) Elementary cases in octahedral hexafluoride molecules - Harter-PRA-1981 (Alt scan)

Rotation-vibration scalar coupling zeta coefficients and spectroscopic band shapes of buckminsterfullerene - Weeks-Harter-CPL-1991 (Alt scan)
Fullerene symmetry reduction and rotational level fine structure/ the Buckyball isotopomer 12C 13C59 - jcp-Reimer-Harter-1997 (HiRez)
Molecular Eigensolution Symmetry Analysis and Fine Structure - IJMS-harter-mitchell-2013

Rotation–vibration spectra of icosahedral molecules.
- I) Icosahedral symmetry analysis and fine structure - harter-weeks-jcp-1989
- II) Icosahedral symmetry, vibrational eigenfrequencies, and normal modes of buckminsterfullerene - weeks-harter-jcp-1989
- III) Half-integral angular momentum - harter-reimer-jcp-1991

QTCA Unit 10 Ch 30 - 2013
Molecular Symmetry and Dynamics - Ch32-Springer Handbooks of Atomic, Molecular, and Optical Physics - Harter-2006
AMOP Ch 0 Space-Time Symmetry - 2019

RESONANCE AND REVIVALS
- I) QUANTUM ROTOR AND INFINITE-WELL DYNAMICS - ISMSLi2012 (Talk) OSU knowledge Bank
- II) Comparing Half-integer Spin and Integer Spin - Alva-ISMS-Ohio2013-R777 (Talks)
- III) Quantum Resonant Beats and Revivals in the Morse Oscillators and Rotors - (2013-Li-Diss)

Rovibrational Spectral Fine Structure Of Icosahedral Molecules - Cpl 1986 (Alt Scan)
Gas Phase Level Structure of C60 Buckyball and Derivatives Exhibiting Broken Icosahedral Symmetry - reimer-diss-1996
Resonance and Revivals in Quantum Rotors - Comparing Half-integer Spin and Integer Spin - Alva-ISMS-Ohio2013-R777 (Talk)
Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-cpl-2013
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - harter - jms - 2001
Representaions Of Multidimensional Symmetries In Networks - harter-jmp-1973

**In development - a web based A.M.O.P. oriented reference page, with thumbnail/previews, greater control over the information display, and eventually full on* Apache-SOLR *Index and search for* nuanced, whole-site content/metadata level *searching. This bad boy will be a sure force multiplier.*

Intro spin ½ coupling
Unit 8 Ch. 24 p3.

H atom hyperfine-B-level crossing
Unit 8 Ch. 24 p15.

Hyperf. theory Ch. 24 p48.

Hyperf. theory Ch. 24 p48.
Deeper theory ends p53

Intro 2p3p coupling
Unit 8 Ch. 24 p17.

Intro LS-jj coupling
Unit 8 Ch. 24 p22.

CG coupling derived (start)
Unit 8 Ch. 24 p39.
CG coupling derived (formula)
Unit 8 Ch. 24 p44.

Lande' g-factor
Unit 8 Ch. 24 p26.

Irrep Tensor building
Unit 8 Ch. 25 p5.

Irrep Tensor Tables
Unit 8 Ch. 25 p12.

Wigner-Eckart tensor Theorem.
Unit 8 Ch. 25 p17.

Tensors Applied to d,f-levels.
Unit 8 Ch. 25 p21.

Tensors Applied to high J levels.
Unit 8 Ch. 25 p63.

Intro 3-particle coupling.
Unit 8 Ch. 25 p28.

Intro 3,4-particle Young Tableaus
GrpThLect29 p42.

Young Tableau Magic Formulae
GrpThLect29 p46-48.

(Int.J.Mol.Sci, 14, 714(2013) p.755-774 , QTCA Unit 7 Ch. 23-26)
(PSDS - Ch. 5, 7)

AMOP reference links on page 2

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Rank-1 tensor

$$\phi_1' = \mathbf{u}\phi_1 = \phi_1 D_{11} + \phi_2 D_{21}$$
$$\phi_2' = \mathbf{u}\psi_2 = \phi_1 D_{12} + \phi_2 D_{22}$$

where: $D_{jk} = (\phi_j^*, \phi_k') = (\phi_j^*, \mathbf{u}\phi_k)$

Dirac notation:

$$\left|1'\right\rangle = \mathbf{u}\left|1\right\rangle = \left|1\right\rangle D_{11} + \left|2\right\rangle D_{21}$$
$$\left|2'\right\rangle = \mathbf{u}\left|2\right\rangle = \left|1\right\rangle D_{12} + \left|2\right\rangle D_{22}$$

where: $D_{jk}(\mathbf{u}) = \left\langle j \middle| k' \right\rangle = \left\langle j \middle| \mathbf{u} \middle| k \right\rangle$

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Typical U(2) transformations (Just like spin-½ irep in basis {1=+½ ,1=-½})

Rank-1 tensor

matrix representations

$$\phi_1' = \mathbf{u}\phi_1 = \phi_1 D_{11} + \phi_2 D_{21}$$

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where: $D_{jk} = (\phi_j^*, \phi_k') = (\phi_j^*, \mathbf{u}\phi_k)$

$$\left|1\right\rangle = \phi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\left|2\right\rangle = \phi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Dirac notation:

$$\left|1'\right\rangle = \mathbf{u}\left|1\right\rangle = \left|1\right\rangle D_{11} + \left|2\right\rangle D_{21}$$

$$\left|2'\right\rangle = \mathbf{u}\left|2\right\rangle = \left|1\right\rangle D_{12} + \left|2\right\rangle D_{22}$$

where: $D_{jk}(\mathbf{u}) = \left\langle j \middle| k' \right\rangle = \left\langle j \middle| \mathbf{u} \middle| k \right\rangle$

$$D_{jk}(\mathbf{u}) = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$$

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$$\phi'_1 = \mathbf{u}\phi_1 = \phi_1 D_{11} + \phi_2 D_{21}$$

$$\phi'_2 = \mathbf{u}\psi_2 = \phi_1 D_{12} + \phi_2 D_{22}$$

where: $D_{jk} = (\phi^*_j, \phi'_k) = (\phi^*_j, \mathbf{u}\phi_k)$

$$\left|1\right\rangle = \phi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \left|2\right\rangle = \phi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Dirac notation:

$$\left|1'\right\rangle = \mathbf{u}\left|1\right\rangle = \left|1\right\rangle D_{11} + \left|2\right\rangle D_{21}$$

$$\left|2'\right\rangle = \mathbf{u}\left|2\right\rangle = \left|1\right\rangle D_{12} + \left|2\right\rangle D_{22}$$

where: $D_{jk}(\mathbf{u}) = \left\langle j \middle| k' \right\rangle = \left\langle j \middle| \mathbf{u} \middle| k \right\rangle$

$$D_{jk}(\mathbf{u}) = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$$

Rank-2 tensor (2 particles each with U(2) state space)

$$\left|1\right\rangle\left|1\right\rangle = \phi_1 \otimes \phi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \left|1\right\rangle\left|2\right\rangle = \phi_1 \otimes \phi_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \left|2\right\rangle\left|1\right\rangle = \phi_2 \otimes \phi_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \left|2\right\rangle\left|2\right\rangle = \phi_2 \otimes \phi_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

2-particle U(2) transform

$$\left|j'\right\rangle\left|k'\right\rangle = \mathbf{u}\left|j\right\rangle\mathbf{u}\left|k\right\rangle$$

$$= \sum_{j,k} \left|j\right\rangle\left|k\right\rangle D_{jj'} D_{kk'}$$

$$= \sum_{j,k} \left|j\right\rangle\left|k\right\rangle D \otimes D_{jk:j'k'}$$

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Rank-1 tensor | matrix representations

$$\phi_1' = \mathbf{u}\phi_1 = \phi_1 D_{11} + \phi_2 D_{21}$$
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where: $D_{jk} = (\phi_j^*, \phi_k') = (\phi_j^*, \mathbf{u}\phi_k)$

$$|1\rangle = \phi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |2\rangle = \phi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Dirac notation:

$$|1'\rangle = \mathbf{u}|1\rangle = |1\rangle D_{11} + |2\rangle D_{21}$$
$$|2'\rangle = \mathbf{u}|2\rangle = |1\rangle D_{12} + |2\rangle D_{22}$$

where: $D_{jk}(\mathbf{u}) = \langle j|k'\rangle = \langle j|\mathbf{u}|k\rangle$

$$D_{jk}(\mathbf{u}) = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$$

Rank-2 tensor (2 particles each with U(2) state space)

$$|1\rangle|1\rangle = \phi_1 \otimes \phi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |1\rangle|2\rangle = \phi_1 \otimes \phi_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |2\rangle|1\rangle = \phi_2 \otimes \phi_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |2\rangle|2\rangle = \phi_2 \otimes \phi_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

2-particle U(2) transform and outer-product U(2) transform matrix $D_{jj'}D_{kk'} = D\otimes D_{jk:j'k'} =$

$$|j'\rangle|k'\rangle = \mathbf{u}|j\rangle\mathbf{u}|k\rangle = \sum_{j,k} |j\rangle|k\rangle D_{jj'}D_{kk'} = \sum_{j,k} |j\rangle|k\rangle D\otimes D_{jk:j'k'}$$

$$= \begin{pmatrix} D_{11}D_{11} & D_{11}D_{12} & D_{12}D_{11} & D_{12}D_{12} \\ D_{11}D_{21} & D_{11}D_{22} & D_{12}D_{21} & D_{12}D_{22} \\ D_{21}D_{11} & D_{21}D_{12} & D_{22}D_{11} & D_{22}D_{12} \\ D_{21}D_{21} & D_{21}D_{22} & D_{22}D_{21} & D_{22}D_{22} \end{pmatrix} = \begin{pmatrix} D_{11}\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} & D_{12}\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \\ D_{21}\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} & D_{22}\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \end{pmatrix}$$

AMOP reference links on page 2

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$$\left|j'\right\rangle\left|k'\right\rangle = \mathbf{u}\left|j\right\rangle\mathbf{u}\left|k\right\rangle = \sum_{j,k}\left|j\right\rangle\left|k\right\rangle D_{jj'}D_{kk'} = \sum_{j,k}\left|j\right\rangle\left|k\right\rangle D\otimes D_{jk:j'k'}$$

$$= \begin{pmatrix} D_{11}D_{11} & D_{11}D_{12} & D_{12}D_{11} & D_{12}D_{12} \\ D_{11}D_{21} & D_{11}D_{22} & D_{12}D_{21} & D_{12}D_{22} \\ D_{21}D_{11} & D_{21}D_{12} & D_{22}D_{11} & D_{22}D_{12} \\ D_{21}D_{21} & D_{21}D_{22} & D_{22}D_{21} & D_{22}D_{22} \end{pmatrix} = \begin{pmatrix} D_{11}\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} & D_{12}\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \\ D_{21}\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} & D_{22}\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \end{pmatrix}$$

2-particle permutation operation: $\mathbf{s}(ab)\left|j\right\rangle_a\left|k\right\rangle_b = \left|k\right\rangle_a\left|j\right\rangle_b$

$\mathbf{s}(ab)\left|1\right\rangle_a\left|1\right\rangle_b = \left|1\right\rangle_a\left|1\right\rangle_b$, $\mathbf{s}(ab)\left|1\right\rangle_a\left|2\right\rangle_b = \left|2\right\rangle_a\left|1\right\rangle_b$, $\mathbf{s}(ab)\left|2\right\rangle_a\left|1\right\rangle_b = \left|1\right\rangle_a\left|2\right\rangle_b$, $\mathbf{s}(ab)\left|2\right\rangle_a\left|2\right\rangle_b = \left|2\right\rangle_a\left|2\right\rangle_b$

$S_2 = \{(a)(b), (ab)\}$ represented by matrices: $S\big((a)(b)\big) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$, $S\big((ab)\big) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$

in basis: $\left|1\right\rangle\left|1\right\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$, $\left|1\right\rangle\left|2\right\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$, $\left|2\right\rangle\left|1\right\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$, $\left|2\right\rangle\left|2\right\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

U(2) tensor product states and S_n permutation symmetry

2-particle U(2) transform and outer-product U(2) transform matrix $D_{jj'}D_{kk'} = D\otimes D_{jk:j'k'} =$

$$\left|j'\right\rangle\left|k'\right\rangle = \mathbf{u}\left|j\right\rangle\mathbf{u}\left|k\right\rangle = \sum_{j,k}\left|j\right\rangle\left|k\right\rangle D_{jj'}D_{kk'} = \sum_{j,k}\left|j\right\rangle\left|k\right\rangle D\otimes D_{jk:j'k'}$$

$$= \begin{pmatrix} D_{11}D_{11} & D_{11}D_{12} & D_{12}D_{11} & D_{12}D_{12} \\ D_{11}D_{21} & D_{11}D_{22} & D_{12}D_{21} & D_{12}D_{22} \\ D_{21}D_{11} & D_{21}D_{12} & D_{22}D_{11} & D_{22}D_{12} \\ D_{21}D_{21} & D_{21}D_{22} & D_{22}D_{21} & D_{22}D_{22} \end{pmatrix} = \begin{pmatrix} D_{11}\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22}\end{pmatrix} & D_{12}\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22}\end{pmatrix} \\ D_{21}\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22}\end{pmatrix} & D_{22}\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22}\end{pmatrix} \end{pmatrix}$$

2-particle permutation operation: $\mathbf{s}(ab)\left|j\right\rangle_a\left|k\right\rangle_b = \left|k\right\rangle_a\left|j\right\rangle_b$

$\mathbf{s}(ab)\left|1\right\rangle_a\left|1\right\rangle_b=\left|1\right\rangle_a\left|1\right\rangle_b$, $\mathbf{s}(ab)\left|1\right\rangle_a\left|2\right\rangle_b=\left|2\right\rangle_a\left|1\right\rangle_b$, $\mathbf{s}(ab)\left|2\right\rangle_a\left|1\right\rangle_b=\left|1\right\rangle_a\left|2\right\rangle_b$, $\mathbf{s}(ab)\left|2\right\rangle_a\left|2\right\rangle_b=\left|2\right\rangle_a\left|2\right\rangle_b$

$S_2=\{(a)(b), (ab)\}$ represented by matrices:

in basis: $\left|1\right\rangle\left|1\right\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$, $\left|1\right\rangle\left|2\right\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$, $\left|2\right\rangle\left|1\right\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$, $\left|2\right\rangle\left|2\right\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

$$S\big((a)(b)\big) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad S\big((ab)\big) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

2-particle permutation $\mathbf{s}(ab)$ commutes with U(2) transform matrix $D\otimes D$:

$$\mathbf{s}(ab)D\otimes D\,\phi_j\phi_k = \sum_{m,n}\mathbf{s}(ab)\phi_m\phi_n D_{jm}D_{kn} = \sum_{m,n}\phi_n\phi_m D_{jm}D_{kn} = \sum_{m,n}\phi_n\phi_m D_{kn}D_{jm} = D\otimes D\,\phi_k\phi_j = D\otimes D\,\mathbf{s}(ab)\phi_j\phi_k$$

U(2) tensor product states and S_n permutation symmetry

2-particle U(2) transform and outer-product U(2) transform matrix

$$D_{jj'}D_{kk'} = D\otimes D_{jk:j'k'} =$$

$$\left|j'\right\rangle\left|k'\right\rangle = \mathbf{u}\left|j\right\rangle\mathbf{u}\left|k\right\rangle = \sum_{j,k}\left|j\right\rangle\left|k\right\rangle D_{jj'}D_{kk'} = \sum_{j,k}\left|j\right\rangle\left|k\right\rangle D\otimes D_{jk:j'k'}$$

$$= \begin{pmatrix} D_{11}D_{11} & D_{11}D_{12} & D_{12}D_{11} & D_{12}D_{12} \\ D_{11}D_{21} & D_{11}D_{22} & D_{12}D_{21} & D_{12}D_{22} \\ D_{21}D_{11} & D_{21}D_{12} & D_{22}D_{11} & D_{22}D_{12} \\ D_{21}D_{21} & D_{21}D_{22} & D_{22}D_{21} & D_{22}D_{22} \end{pmatrix} = \begin{pmatrix} D_{11}\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22}\end{pmatrix} & D_{12}\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22}\end{pmatrix} \\ D_{21}\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22}\end{pmatrix} & D_{22}\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22}\end{pmatrix} \end{pmatrix}$$

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$$\mathbf{s}(ab)\left|j\right\rangle_a\left|k\right\rangle_b = \left|k\right\rangle_a\left|j\right\rangle_b$$

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$S_2=\{(a)(b), (ab)\}$ represented by matrices:

$$S\big((a)(b)\big) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad S\big((ab)\big) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

in basis: $\left|1\right\rangle\left|1\right\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$, $\left|1\right\rangle\left|2\right\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$, $\left|2\right\rangle\left|1\right\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$, $\left|2\right\rangle\left|2\right\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

2-particle permutation s(ab) commutes with U(2) transform matrix $D\otimes D$:

$$\mathbf{s}(ab)D\otimes D\phi_j\phi_k = \sum_{m,n}\mathbf{s}(ab)\phi_m\phi_n D_{jm}D_{kn} = \sum_{m,n}\phi_n\phi_m D_{jm}D_{kn} = \sum_{m,n}\phi_n\phi_m D_{kn}D_{jm} = D\otimes D\phi_k\phi_j = D\otimes D\,\mathbf{s}(ab)\phi_j\phi_k$$

$$\mathbf{s}(ab)D\otimes D = D\otimes D\,\mathbf{s}(ab)$$

U(2) tensor product states and S_n permutation symmetry

2-particle U(2) transform and outer-product U(2) transform matrix $D_{jj'}D_{kk'} = D\otimes D_{jk:j'k'} =$

$$\left|j'\right\rangle\left|k'\right\rangle = \mathbf{u}\left|j\right\rangle\mathbf{u}\left|k\right\rangle = \sum_{j,k}\left|j\right\rangle\left|k\right\rangle D_{jj'}D_{kk'} = \sum_{j,k}\left|j\right\rangle\left|k\right\rangle D\otimes D_{jk:j'k'}$$

$$= \begin{pmatrix} D_{11}D_{11} & D_{11}D_{12} & D_{12}D_{11} & D_{12}D_{12} \\ D_{11}D_{21} & D_{11}D_{22} & D_{12}D_{21} & D_{12}D_{22} \\ D_{21}D_{11} & D_{21}D_{12} & D_{22}D_{11} & D_{22}D_{12} \\ D_{21}D_{21} & D_{21}D_{22} & D_{22}D_{21} & D_{22}D_{22} \end{pmatrix} = \begin{pmatrix} D_{11}\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} & D_{12}\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \\ D_{21}\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} & D_{22}\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \end{pmatrix}$$

2-particle permutation operation: $\mathbf{s}(ab)\left|j\right\rangle_a\left|k\right\rangle_b = \left|k\right\rangle_a\left|j\right\rangle_b$

$\mathbf{s}(ab)\left|1\right\rangle_a\left|1\right\rangle_b = \left|1\right\rangle_a\left|1\right\rangle_b$, $\mathbf{s}(ab)\left|1\right\rangle_a\left|2\right\rangle_b = \left|2\right\rangle_a\left|1\right\rangle_b$, $\mathbf{s}(ab)\left|2\right\rangle_a\left|1\right\rangle_b = \left|1\right\rangle_a\left|2\right\rangle_b$, $\mathbf{s}(ab)\left|2\right\rangle_a\left|2\right\rangle_b = \left|2\right\rangle_a\left|2\right\rangle_b$

$S_2 = \{(a)(b), (ab)\}$ represented by matrices:

$$S\big((a)(b)\big) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad S\big((ab)\big) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

in basis: $\left|1\right\rangle\left|1\right\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\left|1\right\rangle\left|2\right\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $\left|2\right\rangle\left|1\right\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $\left|2\right\rangle\left|2\right\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

2-particle permutation $\mathbf{s}(ab)$ commutes with U(2) transform matrix $D\otimes D$:

$$\mathbf{s}(ab)D\otimes D\,\phi_j\phi_k = \sum_{m,n}\mathbf{s}(ab)\phi_m\phi_n D_{jm}D_{kn} = \sum_{m,n}\phi_n\phi_m D_{jm}D_{kn} = \sum_{m,n}\phi_n\phi_m D_{kn}D_{jm} = D\otimes D\,\phi_k\phi_j = D\otimes D\,\mathbf{s}(ab)\phi_j\phi_k$$

So $S_2 = \{\mathbf{s}(ab)\}$ is *symmetry* of U(2)… $\boxed{\mathbf{s}(ab)D\otimes D = D\otimes D\,\mathbf{s}(ab)}$ …and vice-versa!

AMOP reference links on page 2

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Interwining ($S_1 \subset S_2 \subset S_3 \subset S_4 \subset S_5$ …)*(U(1)⊂U(2)⊂U(3)⊂U(4)⊂U(5) …) algebras and tensor operator applications to spinor-rotor or orbital correlations

U(2) tensor product states and S_n permutation symmetry

Rank-1 tensor (or spinor)

Rank-2 tensor (2 particles each with U(2) state space)

2-particle U(2) transform and permutation operation

→ S_2 symmetry of U(2): Trust but verify

Applying S_2 projection to build DTran

Applying DTran for S_2

Applying DTran for U(2)

S_3 permutations related to $C_{3v} \sim D_3$ geometry

S_3 permutation matrices

Hooklength formula for S_n reps

S_3 symmetry of U(2): Applying S_3 projection (Note Pauli-exclusion principle basis)

Building S_3 DTran T from projectors

Effect of S_3 DTran T: Introducing intertwining S_3 - U(2) irep matrices

Multi-spin $(1/2)^N$ product state (Comparison to previous cases)

S_2 *symmetry* of U(2): *Trust but verify*

It might help to matrix-verify the S_2 symmetry of 2-particle U(2) transformations

$$S\big((ab)\big)\cdot D\otimes D \quad ?=? \quad D\otimes D\cdot S\big((ab)\big)$$

$$\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}\begin{pmatrix} D_{11}D_{11} & D_{11}D_{12} & D_{12}D_{11} & D_{12}D_{12} \\ D_{11}D_{21} & D_{11}D_{22} & D_{12}D_{21} & D_{12}D_{22} \\ D_{21}D_{11} & D_{21}D_{12} & D_{22}D_{11} & D_{22}D_{12} \\ D_{21}D_{21} & D_{21}D_{22} & D_{22}D_{21} & D_{22}D_{22} \end{pmatrix} \overset{?}{\underset{?}{=}} \begin{pmatrix} D_{11}D_{11} & D_{11}D_{12} & D_{12}D_{11} & D_{12}D_{12} \\ D_{11}D_{21} & D_{11}D_{22} & D_{12}D_{21} & D_{12}D_{22} \\ D_{21}D_{11} & D_{21}D_{12} & D_{22}D_{11} & D_{22}D_{12} \\ D_{21}D_{21} & D_{21}D_{22} & D_{22}D_{21} & D_{22}D_{22} \end{pmatrix}\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

S_2 *symmetry* of U(2): *Trust but verify*

It might help to matrix-verify the S_2 symmetry of 2-particle U(2) transformations

$$S\big((ab)\big)\cdot D\otimes D \quad ?=? \quad D\otimes D\cdot S\big((ab)\big)$$

$$\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}\begin{pmatrix} D_{11}D_{11} & D_{11}D_{12} & D_{12}D_{11} & D_{12}D_{12} \\ D_{11}D_{21} & D_{11}D_{22} & D_{12}D_{21} & D_{12}D_{22} \\ D_{21}D_{11} & D_{21}D_{12} & D_{22}D_{11} & D_{22}D_{12} \\ D_{21}D_{21} & D_{21}D_{22} & D_{22}D_{21} & D_{22}D_{22} \end{pmatrix} \overset{?}{\underset{?}{=}} \begin{pmatrix} D_{11}D_{11} & D_{11}D_{12} & D_{12}D_{11} & D_{12}D_{12} \\ D_{11}D_{21} & D_{11}D_{22} & D_{12}D_{21} & D_{12}D_{22} \\ D_{21}D_{11} & D_{21}D_{12} & D_{22}D_{11} & D_{22}D_{12} \\ D_{21}D_{21} & D_{21}D_{22} & D_{22}D_{21} & D_{22}D_{22} \end{pmatrix}\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

(mid-rows switched)

(mid-columns switched)

$$\begin{pmatrix} D_{11}D_{11} & D_{11}D_{12} & D_{12}D_{11} & D_{12}D_{12} \\ D_{21}D_{11} & D_{21}D_{12} & D_{22}D_{11} & D_{22}D_{12} \\ D_{11}D_{21} & D_{11}D_{22} & D_{12}D_{21} & D_{12}D_{22} \\ D_{21}D_{21} & D_{21}D_{22} & D_{22}D_{21} & D_{22}D_{22} \end{pmatrix} = \begin{pmatrix} D_{11}D_{11} & D_{12}D_{11} & D_{11}D_{12} & D_{12}D_{12} \\ D_{11}D_{21} & D_{12}D_{21} & D_{11}D_{22} & D_{12}D_{22} \\ D_{21}D_{11} & D_{22}D_{11} & D_{21}D_{12} & D_{22}D_{12} \\ D_{21}D_{21} & D_{22}D_{21} & D_{21}D_{22} & D_{22}D_{22} \end{pmatrix}$$

S_2 *symmetry* of U(2): *Trust but verify*

It might help to matrix-verify the S_2 symmetry of 2-particle U(2) transformations

$$S\big((ab)\big)\cdot D\otimes D \qquad ?=? \qquad D\otimes D\cdot S\big((ab)\big)$$

$$\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}\begin{pmatrix} D_{11}D_{11} & D_{11}D_{12} & D_{12}D_{11} & D_{12}D_{12} \\ D_{11}D_{21} & D_{11}D_{22} & D_{12}D_{21} & D_{12}D_{22} \\ D_{21}D_{11} & D_{21}D_{12} & D_{22}D_{11} & D_{22}D_{12} \\ D_{21}D_{21} & D_{21}D_{22} & D_{22}D_{21} & D_{22}D_{22} \end{pmatrix} \overset{?}{\underset{?}{=}} \begin{pmatrix} D_{11}D_{11} & D_{11}D_{12} & D_{12}D_{11} & D_{12}D_{12} \\ D_{11}D_{21} & D_{11}D_{22} & D_{12}D_{21} & D_{12}D_{22} \\ D_{21}D_{11} & D_{21}D_{12} & D_{22}D_{11} & D_{22}D_{12} \\ D_{21}D_{21} & D_{21}D_{22} & D_{22}D_{21} & D_{22}D_{22} \end{pmatrix}\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

(mid-rows switched)

(mid-columns switched)

$$\begin{pmatrix} D_{11}D_{11} & D_{11}D_{12} & D_{12}D_{11} & D_{12}D_{12} \\ D_{21}D_{11} & D_{21}D_{12} & D_{22}D_{11} & D_{22}D_{12} \\ D_{11}D_{21} & D_{11}D_{22} & D_{12}D_{21} & D_{12}D_{22} \\ D_{21}D_{21} & D_{21}D_{22} & D_{22}D_{21} & D_{22}D_{22} \end{pmatrix} = \begin{pmatrix} D_{11}D_{11} & D_{12}D_{11} & D_{11}D_{12} & D_{12}D_{12} \\ D_{11}D_{21} & D_{12}D_{21} & D_{11}D_{22} & D_{12}D_{22} \\ D_{21}D_{11} & D_{22}D_{11} & D_{21}D_{12} & D_{22}D_{12} \\ D_{21}D_{21} & D_{22}D_{21} & D_{21}D_{22} & D_{22}D_{22} \end{pmatrix}$$

…but the matrices are numerically equal.

So S_2-symmetry of 2-particle U(2) tensor representation is verified.

S_2 *symmetry* of U(2): *Trust but verify*

It might help to matrix-verify the S_2 symmetry of 2-particle U(2) transformations

$S((ab))\cdot D\otimes D \quad ?\!=\!? \quad D\otimes D\cdot S((ab))$

$$\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}\begin{pmatrix} D_{11}D_{11} & D_{11}D_{12} & D_{12}D_{11} & D_{12}D_{12} \\ D_{11}D_{21} & D_{11}D_{22} & D_{12}D_{21} & D_{12}D_{22} \\ D_{21}D_{11} & D_{21}D_{12} & D_{22}D_{11} & D_{22}D_{12} \\ D_{21}D_{21} & D_{21}D_{22} & D_{22}D_{21} & D_{22}D_{22} \end{pmatrix} \overset{?}{=} \begin{pmatrix} D_{11}D_{11} & D_{11}D_{12} & D_{12}D_{11} & D_{12}D_{12} \\ D_{11}D_{21} & D_{11}D_{22} & D_{12}D_{21} & D_{12}D_{22} \\ D_{21}D_{11} & D_{21}D_{12} & D_{22}D_{11} & D_{22}D_{12} \\ D_{21}D_{21} & D_{21}D_{22} & D_{22}D_{21} & D_{22}D_{22} \end{pmatrix}\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

(mid-rows switched) (mid-columns switched)

$$\begin{pmatrix} D_{11}D_{11} & D_{11}D_{12} & D_{12}D_{11} & D_{12}D_{12} \\ D_{21}D_{11} & D_{21}D_{12} & D_{22}D_{11} & D_{22}D_{12} \\ D_{11}D_{21} & D_{11}D_{22} & D_{12}D_{21} & D_{12}D_{22} \\ D_{21}D_{21} & D_{21}D_{22} & D_{22}D_{21} & D_{22}D_{22} \end{pmatrix} = \begin{pmatrix} D_{11}D_{11} & D_{12}D_{11} & D_{11}D_{12} & D_{12}D_{12} \\ D_{11}D_{21} & D_{12}D_{21} & D_{11}D_{22} & D_{12}D_{22} \\ D_{21}D_{11} & D_{22}D_{11} & D_{21}D_{12} & D_{22}D_{12} \\ D_{21}D_{21} & D_{22}D_{21} & D_{21}D_{22} & D_{22}D_{22} \end{pmatrix}$$

…but the matrices are numerically equal.

So S_2-symmetry of 2-particle U(2) tensor representation is verified.

So also is S_2-symmetry of any 2-particle U(m) tensor.

Showing S_3-symmetry of any 3-particle U(m) tensor is treated later.

$$S\big((ab)\big)\cdot D\otimes D\cdot S\big((ab)\big)$$

$$\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}\begin{pmatrix} D_{11}D_{11} & D_{11}D_{12} & D_{12}D_{11} & D_{12}D_{12} \\ D_{11}D_{21} & D_{11}D_{22} & D_{12}D_{21} & D_{12}D_{22} \\ D_{21}D_{11} & D_{21}D_{12} & D_{22}D_{11} & D_{22}D_{12} \\ D_{21}D_{21} & D_{21}D_{22} & D_{22}D_{21} & D_{22}D_{22} \end{pmatrix}\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

If $S(ab)$ commuted with $D\otimes D$ you might assume it passes thru to give $S(ab)\ S(ab)=1$ leaving $D\otimes D$ unchanged.
That is true numerically, but all components have flipped order.

$$=\begin{pmatrix} D_{11}D_{11} & D_{11}D_{12} & D_{12}D_{11} & D_{12}D_{12} \\ D_{21}D_{11} & D_{21}D_{12} & D_{22}D_{11} & D_{22}D_{12} \\ D_{11}D_{21} & D_{11}D_{22} & D_{12}D_{21} & D_{12}D_{22} \\ D_{21}D_{21} & D_{21}D_{22} & D_{22}D_{21} & D_{22}D_{22} \end{pmatrix}\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}=\begin{pmatrix} D_{11}D_{11} & D_{12}D_{11} & D_{11}D_{12} & D_{12}D_{12} \\ D_{21}D_{11} & D_{22}D_{11} & D_{21}D_{12} & D_{22}D_{12} \\ D_{11}D_{21} & D_{12}D_{21} & D_{11}D_{22} & D_{12}D_{22} \\ D_{21}D_{21} & D_{22}D_{21} & D_{21}D_{22} & D_{22}D_{22} \end{pmatrix}$$

Each $D_{ab}D_{cd}$ has become $D_{cd}D_{ab}$

$$S\big((ab)\big)\cdot D\otimes D\cdot S\big((ab)\big)$$

$$\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}\begin{pmatrix} D_{11}D_{11} & D_{11}D_{12} & D_{12}D_{11} & D_{12}D_{12} \\ D_{11}D_{21} & D_{11}D_{22} & D_{12}D_{21} & D_{12}D_{22} \\ D_{21}D_{11} & D_{21}D_{12} & D_{22}D_{11} & D_{22}D_{12} \\ D_{21}D_{21} & D_{21}D_{22} & D_{22}D_{21} & D_{22}D_{22} \end{pmatrix}\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}\begin{pmatrix} D_{11}D_{11} & D_{12}D_{11} & D_{11}D_{12} & D_{12}D_{12} \\ D_{11}D_{21} & D_{12}D_{21} & D_{11}D_{22} & D_{12}D_{22} \\ D_{21}D_{11} & D_{22}D_{11} & D_{21}D_{12} & D_{22}D_{12} \\ D_{21}D_{21} & D_{22}D_{21} & D_{21}D_{22} & D_{22}D_{22} \end{pmatrix}=\begin{pmatrix} D_{11}D_{11} & D_{12}D_{11} & D_{11}D_{12} & D_{12}D_{12} \\ D_{21}D_{11} & D_{22}D_{11} & D_{21}D_{12} & D_{22}D_{12} \\ D_{11}D_{21} & D_{12}D_{21} & D_{11}D_{22} & D_{12}D_{22} \\ D_{21}D_{21} & D_{22}D_{21} & D_{21}D_{22} & D_{22}D_{22} \end{pmatrix}$$

AMOP reference links on page 2

4.02.18 class 20: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

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Interwining $(S_1 \subset S_2 \subset S_3 \subset S_4 \subset S_5 \ldots)*(U(1) \subset U(2) \subset U(3) \subset U(4) \subset U(5) \ldots)$ algebras and tensor operator applications to spinor-rotor or orbital correlations

U(2) tensor product states and S_n permutation symmetry
Rank-1 tensor (or spinor)

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2-particle U(2) transform and permutation operation
S_2 symmetry of U(2): Trust but verify

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Applying DTran for U(2)

S_3 permutations related to $C_{3v} \sim D_3$ geometry
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Multi-spin $(1/2)^N$ product state (Comparison to previous cases)

S_2 *symmetry* of U(2): *Applying S_2 projection*

S_2 matrix eigen-solution found by projectors: Minimal eq. $\mathbf{(ab)}^2\mathbf{-1}=0=(\mathbf{(ab)}+\mathbf{1})(\mathbf{(ab)}+\mathbf{1})$ yields:

Symmetric ($\square\square$): $\mathbf{P}^{\square\square} = \frac{1}{2}\left[\mathbf{1} + \mathbf{(ab)}\right]$

Anti-Symmetric ($\begin{smallmatrix}\square\\\square\end{smallmatrix}$): $\mathbf{P}^{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = \frac{1}{2}\left[\mathbf{1} - \mathbf{(ab)}\right]$

S_2 *symmetry* of U(2): *Applying S_2 projection*

S_2 matrix eigen-solution found by projectors: Minimal eq. $\mathbf{(ab)}^2\mathbf{-1}=0=(\mathbf{(ab)}+\mathbf{1})(\mathbf{(ab)}+\mathbf{1})$ yields:

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Anti-Symmetric ($\begin{smallmatrix}\square\\\square\end{smallmatrix}$): $\mathbf{P}^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}=\frac{1}{2}\left[\mathbf{1}-\mathbf{(ab)}\right]$

Matrix representations of projectors:

$$S(\mathbf{P}^{\square\square})=\frac{1}{2}\left[S(\mathbf{1})+S(\mathbf{(ab)})\right]=\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{2} & \frac{1}{2} & \cdot \\ \cdot & \frac{1}{2} & \frac{1}{2} & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

$$S(\mathbf{P}^{\begin{smallmatrix}\square\\\square\end{smallmatrix}})=\frac{1}{2}\left[S(\mathbf{1})-S(\mathbf{(ab)})\right]=\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{2} & \frac{-1}{2} & \cdot \\ \cdot & \frac{-1}{2} & \frac{1}{2} & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

S_2 *symmetry* of U(2): *Applying S_2 projection*

S_2 matrix eigen-solution found by projectors: Minimal eq. **(ab)**^{2}-**1**=0=(**(ab)**+**1**)(**(ab)**+**1**) yields:

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Anti-Symmetric ($\begin{smallmatrix}\square\\\square\end{smallmatrix}$): $\mathbf{P}^{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = \frac{1}{2}\left[\mathbf{1} - \mathbf{(ab)}\right]$

Matrix representation of *Diagonalizing Transform* (DTran *T*) is made by excerpting **P**-columns

$$S(\mathbf{P}^{\square\square}) = \tfrac{1}{2}\left[S(\mathbf{1}) + S(\mathbf{ab})\right] = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{2} & \frac{1}{2} & \cdot \\ \cdot & \frac{1}{2} & \frac{1}{2} & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

$$S(\mathbf{P}^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}) = \tfrac{1}{2}\left[S(\mathbf{1}) - S(\mathbf{ab})\right] = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{2} & \frac{-1}{2} & \cdot \\ \cdot & \frac{-1}{2} & \frac{1}{2} & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{\sqrt{2}} & \cdot & \frac{1}{\sqrt{2}} \\ \cdot & \frac{1}{\sqrt{2}} & \cdot & \frac{-1}{\sqrt{2}} \\ \cdot & \cdot & 1 & \cdot \end{pmatrix} = T$$

AMOP reference links on page 2

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Applying S_2 projection to build DTran
➡ Applying DTran for S_2
Applying DTran for U(2)

S_3 permutations related to $C_{3v} \sim D_3$ geometry
S_3 permutation matrices
Hooklength formula for S_n reps
S_3 symmetry of U(2): Applying S_3 projection (Note Pauli-exclusion principle basis)
Building S_3 DTran T from projectors
Effect of S_3 DTran T: Introducing intertwining S_3 - U(2) irep matrices
Multi-spin $(1/2)^N$ product state (Comparison to previous cases)

S_2 *symmetry* of U(2): *Applying S_2 projection*

S_2 matrix eigen-solution found by projectors: Minimal eq. $(\mathbf{ab})^2-\mathbf{1}=0=((\mathbf{ab})+\mathbf{1})((\mathbf{ab})+\mathbf{1})$ yields:

Symmetric ($\square\square$): $\mathbf{P}^{\square\square}=\frac{1}{2}\left[\mathbf{1}+(\mathbf{ab})\right]$ Anti-Symmetric ($\begin{smallmatrix}\square\\\square\end{smallmatrix}$): $\mathbf{P}^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}=\frac{1}{2}\left[\mathbf{1}-(\mathbf{ab})\right]$

Matrix representation of *Diagonalizing Transform* (DTran T) is made by excerpting **P**-columns

$$S(\mathbf{P}^{\square\square})=\frac{1}{2}\left[S(\mathbf{1})+S(\mathbf{ab})\right]=\begin{pmatrix}1&\cdot&\cdot&\cdot\\\cdot&\frac{1}{2}&\frac{1}{2}&\cdot\\\cdot&\frac{1}{2}&\frac{1}{2}&\cdot\\\cdot&\cdot&\cdot&1\end{pmatrix}\qquad S(\mathbf{P}^{\begin{smallmatrix}\square\\\square\end{smallmatrix}})=\frac{1}{2}\left[S(\mathbf{1})-S(\mathbf{ab})\right]=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot\\\cdot&\frac{1}{2}&\frac{-1}{2}&\cdot\\\cdot&\frac{-1}{2}&\frac{1}{2}&\cdot\\\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

$$\overset{T^\dagger}{\begin{pmatrix}1&\cdot&\cdot&\cdot\\\cdot&\frac{1}{\sqrt{2}}&\cdot\frac{1}{\sqrt{2}}&\\\cdot&&\cdot&1\\\cdot&\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}&\cdot\end{pmatrix}}\overset{S(ab)}{\begin{pmatrix}1&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot\\\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&1\end{pmatrix}}\overset{T}{\begin{pmatrix}1&\cdot&\cdot&\cdot\\\cdot&\frac{1}{\sqrt{2}}&\cdot&\frac{1}{\sqrt{2}}\\\cdot&\frac{1}{\sqrt{2}}&\cdot&\frac{-1}{\sqrt{2}}\\\cdot&\cdot&1&\cdot\end{pmatrix}}=T^\dagger S(ab)T$$

Next apply DTran T and its transpose $T^\dagger$ to the $S(ab)$ matrix to find $T^\dagger S(ab)T$.

S_2 *symmetry* of U(2): *Applying S_2 projection*

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Anti-Symmetric ($\begin{smallmatrix}\square\\\square\end{smallmatrix}$): $\mathbf{P}^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}=\frac{1}{2}\left[\mathbf{1}-(\mathbf{ab})\right]$

Matrix representation of *Diagonalizing Transform* (DTran *T*) is made by excerpting **P**-columns

$$S(\mathbf{P}^{\square\square})=\tfrac{1}{2}\left[S(\mathbf{1})+S(\mathbf{ab})\right]=\begin{pmatrix}1&\cdot&\cdot&\cdot\\ \cdot&\frac{1}{2}&\frac{1}{2}&\cdot\\ \cdot&\frac{1}{2}&\frac{1}{2}&\cdot\\ \cdot&\cdot&\cdot&1\end{pmatrix}$$

$$S(\mathbf{P}^{\begin{smallmatrix}\square\\\square\end{smallmatrix}})=\tfrac{1}{2}\left[S(\mathbf{1})-S(\mathbf{ab})\right]=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot\\ \cdot&\frac{1}{2}&\frac{-1}{2}&\cdot\\ \cdot&\frac{-1}{2}&\frac{1}{2}&\cdot\\ \cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

$$\overset{T^\dagger}{\begin{pmatrix}1&\cdot&\cdot&\cdot\\ \cdot&\frac{1}{\sqrt2}&\frac{1}{\sqrt2}&\\ \cdot&\cdot&\cdot&1\\ \cdot&\frac{1}{\sqrt2}&\frac{-1}{\sqrt2}&\cdot\end{pmatrix}}\overset{S(ab)}{\begin{pmatrix}1&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot\\ \cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&1\end{pmatrix}}\overset{T}{\begin{pmatrix}1&\cdot&\cdot&\cdot\\ \cdot&\frac{1}{\sqrt2}&\cdot&\frac{1}{\sqrt2}\\ \cdot&\frac{1}{\sqrt2}&\cdot&\frac{-1}{\sqrt2}\\ \cdot&\cdot&1&\cdot\end{pmatrix}}=T^\dagger S(ab)T$$

Next apply DTran *T* and its transpose $T^\dagger$ to the *S*(*ab*) matrix to find $T^\dagger S(ab)T$.

$$\begin{pmatrix}1&\cdot&\cdot&\cdot\\ \cdot&\frac{1}{\sqrt2}&\frac{1}{\sqrt2}&\cdot\\ \cdot&\cdot&\cdot&1\\ \cdot&\frac{-1}{\sqrt2}&\frac{-1}{\sqrt2}&\cdot\end{pmatrix}\begin{pmatrix}1&\cdot&\cdot&\cdot\\ \cdot&\frac{1}{\sqrt2}&\cdot&\frac{1}{\sqrt2}\\ \cdot&\frac{1}{\sqrt2}&\cdot&\frac{-1}{\sqrt2}\\ \cdot&\cdot&1&\cdot\end{pmatrix}=T^\dagger S(ab)T$$

S_2 *symmetry* of U(2): *Applying S_2 projection*

S_2 matrix eigen-solution found by projectors: Minimal eq. $\mathbf{(ab)}^2\text{-}\mathbf{1}=0=((\mathbf{ab})+\mathbf{1})((\mathbf{ab})+\mathbf{1})$ yields:

Symmetric ($\square\square$): $\mathbf{P}^{\square\square}=\frac{1}{2}\left[\mathbf{1}+(\mathbf{ab})\right]$ Anti-Symmetric ($\begin{smallmatrix}\square\\\square\end{smallmatrix}$): $\mathbf{P}^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}=\frac{1}{2}\left[\mathbf{1}-(\mathbf{ab})\right]$

Matrix representation of *Diagonalizing Transform* (DTran T) is made by excerpting **P**-columns

$$S(\mathbf{P}^{\square\square})=\tfrac{1}{2}\left[S(\mathbf{1})+S(\mathbf{ab})\right]=\begin{pmatrix}1&\cdot&\cdot&\cdot\\\cdot&\frac{1}{2}&\frac{1}{2}&\cdot\\\cdot&\frac{1}{2}&\frac{1}{2}&\cdot\\\cdot&\cdot&\cdot&1\end{pmatrix}\qquad S(\mathbf{P}^{\begin{smallmatrix}\square\\\square\end{smallmatrix}})=\tfrac{1}{2}\left[S(\mathbf{1})-S(\mathbf{ab})\right]=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot\\\cdot&\frac{1}{2}&\frac{-1}{2}&\cdot\\\cdot&\frac{-1}{2}&\frac{1}{2}&\cdot\\\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

$$\overset{T^\dagger}{\begin{pmatrix}1&\cdot&\cdot&\cdot\\\cdot&\frac{1}{\sqrt2}&\frac{1}{\sqrt2}&\\\cdot&\cdot&\cdot&1\\\cdot&\frac{1}{\sqrt2}&\frac{-1}{\sqrt2}&\cdot\end{pmatrix}}\overset{S(ab)}{\begin{pmatrix}1&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot\\\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&1\end{pmatrix}}\overset{T}{\begin{pmatrix}1&\cdot&\cdot&\cdot\\\cdot&\frac{1}{\sqrt2}&\cdot&\frac{1}{\sqrt2}\\\cdot&\frac{1}{\sqrt2}&\cdot&\frac{-1}{\sqrt2}\\\cdot&\cdot&1&\cdot\end{pmatrix}}=T^\dagger S(ab)T$$

Next apply DTran T and its transpose $T^\dagger$ to the $S(ab)$ matrix to find $T^\dagger S(ab)T$.

$$T^\dagger S(ab)T=\begin{pmatrix}1&\cdot&\cdot&\cdot\\\cdot&\frac{1}{\sqrt2}&\frac{1}{\sqrt2}&\cdot\\\cdot&\cdot&\cdot&1\\\cdot&\frac{-1}{\sqrt2}&\frac{1}{\sqrt2}&\cdot\end{pmatrix}\begin{pmatrix}1&\cdot&\cdot&\cdot\\\cdot&\frac{1}{\sqrt2}&\cdot&\frac{1}{\sqrt2}\\\cdot&\frac{1}{\sqrt2}&\cdot&\frac{-1}{\sqrt2}\\\cdot&\cdot&1&\cdot\end{pmatrix}=\begin{pmatrix}1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&-1\end{pmatrix}=\begin{pmatrix}D^{\square\square}&\cdot&\cdot&\cdot\\\cdot&D^{\square\square}&\cdot&\cdot\\\cdot&\cdot&D^{\square\square}&\cdot\\\cdot&\cdot&\cdot&D^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\end{pmatrix}$$

Three (3) symmetric ireps. $D^{\square\square}$ and one (1) anti-sym $D^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$

AMOP reference links on page 2

4.02.18 class 20: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

Interwining ($S_1 \subset S_2 \subset S_3 \subset S_4 \subset S_5$ …)*(U(1)⊂U(2)⊂U(3)⊂U(4)⊂U(5) …) algebras and tensor operator applications to spinor-rotor or orbital correlations

U(2) tensor product states and S_n permutation symmetry
Rank-1 tensor (or spinor)

Rank-2 tensor (2 particles each with U(2) state space)
2-particle U(2) transform and permutation operation
S_2 symmetry of U(2): Trust but verify

Applying S_2 projection to build DTran
Applying DTran for S_2
➡ Applying DTran for U(2)

S_3 permutations related to C_{3v}~D_3 geometry
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Multi-spin $(1/2)^N$ product state (Comparison to previous cases)

Finally, apply DTran$T$ to find $T^\dagger D\otimes D T$.

$$
\overset{T^\dagger}{\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \cdot \end{pmatrix}}
\overset{D\otimes D}{\begin{pmatrix} D_{11\,11} & D_{11\,12} & D_{12\,11} & D_{12\,12} \\ D_{11\,21} & D_{11\,22} & D_{12\,21} & D_{12\,22} \\ D_{11\,21} & D_{12\,21} & D_{11\,22} & D_{12\,22} \\ D_{21\,21} & D_{21\,22} & D_{21\,22} & D_{22\,22} \end{pmatrix}}
\overset{T}{\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{\sqrt{2}} & \cdot & \frac{1}{\sqrt{2}} \\ \cdot & \frac{1}{\sqrt{2}} & \cdot & \frac{-1}{\sqrt{2}} \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}}
= T^\dagger D\otimes DT
$$

Next apply DTran T and its transpose $T^\dagger$ to the $S(ab)$ matrix to find $T^\dagger S(ab)T$.

$$
\overset{T^\dagger}{\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \cdot \end{pmatrix}}
\overset{S(ab)}{\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}}
\overset{T}{\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{\sqrt{2}} & \cdot & \frac{1}{\sqrt{2}} \\ \cdot & \frac{1}{\sqrt{2}} & \cdot & \frac{-1}{\sqrt{2}} \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}}
= T^\dagger S(ab)T
$$

$$
T^\dagger S(ab)T = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdot \end{pmatrix}
\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{\sqrt{2}} & \cdot & \frac{1}{\sqrt{2}} \\ \cdot & \frac{1}{\sqrt{2}} & \cdot & \frac{-1}{\sqrt{2}} \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}
= \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & -1 \end{pmatrix}
= \begin{pmatrix} D^{\square\square} & \cdot & \cdot & \cdot \\ \cdot & D^{\square\square} & \cdot & \cdot \\ \cdot & \cdot & D^{\square\square} & \cdot \\ \cdot & \cdot & \cdot & D^{\genfrac{}{}{0pt}{}{\square}{\square}} \end{pmatrix}
$$

Three (3) symmetric ireps. $D^{\square\square}$ and one (1) anti-sym $D^{\genfrac{}{}{0pt}{}{\square}{\square}}$

Finally, apply DTran$T$ to find $T^\dagger D\otimes DT$.

$$\overset{T^\dagger}{\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \cdot \end{pmatrix}} \overset{D\otimes D}{\begin{pmatrix} D_{11\,11} & D_{11\,12} & D_{12\,11} & D_{12\,12} \\ D_{11\,21} & D_{11\,22} & D_{12\,21} & D_{12\,22} \\ D_{11\,21} & D_{12\,21} & D_{11\,22} & D_{12\,22} \\ D_{21\,21} & D_{21\,22} & D_{21\,22} & D_{22\,22} \end{pmatrix}} \overset{T}{\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{\sqrt{2}} & \cdot & \frac{1}{\sqrt{2}} \\ \cdot & \frac{1}{\sqrt{2}} & \cdot & \frac{-1}{\sqrt{2}} \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}} = T^\dagger D\otimes DT$$

$$\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \cdot \end{pmatrix} \begin{pmatrix} D_{11\,11} & \frac{\sqrt{2}}{1} D_{11\,12} & D_{12\,12} & 0 \\ D_{11\,21} & \frac{1}{\sqrt{2}}(D_{11\,22}+D_{12\,21}) & D_{12\,22} & \frac{1}{\sqrt{2}}(D_{11\,22}-D_{12\,21}) \\ D_{11\,21} & \frac{1}{\sqrt{2}}(D_{11\,22}+D_{12\,21}) & D_{12\,22} & \frac{1}{\sqrt{2}}(D_{12\,21}-D_{11\,22}) \\ D_{21\,21} & \frac{\sqrt{2}}{1} D_{21\,22} & D_{22\,22} & 0 \end{pmatrix}$$

Next apply DTran T and its transpose $T^\dagger$ to the $S(ab)$ matrix to find $T^\dagger S(ab)T$.

$$\overset{T^\dagger}{\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \cdot \end{pmatrix}} \overset{S(ab)}{\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}} \overset{T}{\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{\sqrt{2}} & \cdot & \frac{1}{\sqrt{2}} \\ \cdot & \frac{1}{\sqrt{2}} & \cdot & \frac{-1}{\sqrt{2}} \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}} = T^\dagger S(ab)T$$

$$T^\dagger S(ab)T = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdot \end{pmatrix} \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{\sqrt{2}} & \cdot & \frac{1}{\sqrt{2}} \\ \cdot & \frac{1}{\sqrt{2}} & \cdot & \frac{-1}{\sqrt{2}} \\ \cdot & \cdot & 1 & \cdot \end{pmatrix} = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & -1 \end{pmatrix} = \begin{pmatrix} D^{\square\square} & \cdot & \cdot & \cdot \\ \cdot & D^{\square\square} & \cdot & \cdot \\ \cdot & \cdot & D^{\square\square} & \cdot \\ \cdot & \cdot & \cdot & D^{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \end{pmatrix}$$

Three (3) symmetric ireps. $D^{\square\square}$ and one (1) anti-sym $D^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$

Finally, apply DTran$T$ to find $T^\dagger D\otimes DT$.

$$\overset{T^\dagger}{\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{\sqrt2} & \frac{1}{\sqrt2} & \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \frac{1}{\sqrt2} & \frac{-1}{\sqrt2} & \cdot \end{pmatrix}} \overset{D\otimes D}{\begin{pmatrix} D_{11\,11} & D_{11\,12} & D_{12\,11} & D_{12\,12} \\ D_{11\,21} & D_{11\,22} & D_{12\,21} & D_{12\,22} \\ D_{11\,21} & D_{12\,21} & D_{11\,22} & D_{12\,22} \\ D_{21\,21} & D_{21\,22} & D_{21\,22} & D_{22\,22} \end{pmatrix}} \overset{T}{\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{\sqrt2} & \cdot & \frac{1}{\sqrt2} \\ \cdot & \frac{1}{\sqrt2} & \cdot & \frac{-1}{\sqrt2} \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}} = T^\dagger D\otimes DT$$

$$\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{\sqrt2} & \frac{1}{\sqrt2} & \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \frac{1}{\sqrt2} & \frac{-1}{\sqrt2} & \cdot \end{pmatrix} \begin{pmatrix} D_{11\,11} & \frac{\sqrt2}{1}D_{11\,12} & D_{12\,12} & 0 \\ D_{11\,21} & \frac{1}{\sqrt2}(D_{11\,22}+D_{12\,21}) & D_{12\,22} & \frac{1}{\sqrt2}(D_{11\,22}-D_{12\,21}) \\ D_{11\,21} & \frac{1}{\sqrt2}(D_{11\,22}+D_{12\,21}) & D_{12\,22} & \frac{1}{\sqrt2}(D_{12\,21}-D_{11\,22}) \\ D_{21\,21} & \frac{\sqrt2}{1}D_{21\,22} & D_{22\,22} & 0 \end{pmatrix} = \begin{pmatrix} D_{11\,11} & \frac{\sqrt2}{1}D_{11\,12} & D_{12\,12} & 0 \\ \frac{\sqrt2}{1}D_{11\,21} & D_{11\,22}+D_{12\,21} & \frac{\sqrt2}{1}D_{12\,22} & 0 \\ D_{21\,21} & \frac{\sqrt2}{1}D_{21\,22} & D_{22\,22} & 0 \\ 0 & 0 & 0 & D_{11\,22}+D_{12\,21} \end{pmatrix}$$

Next apply DTran T and its transpose $T^\dagger$ to the $S(ab)$ matrix to find $T^\dagger S(ab)T$.

$$\overset{T^\dagger}{\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{\sqrt2} & \frac{1}{\sqrt2} & \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \frac{1}{\sqrt2} & \frac{-1}{\sqrt2} & \cdot \end{pmatrix}} \overset{S(ab)}{\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}} \overset{T}{\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{\sqrt2} & \cdot & \frac{1}{\sqrt2} \\ \cdot & \frac{1}{\sqrt2} & \cdot & \frac{-1}{\sqrt2} \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}} = T^\dagger S(ab)T$$

$$T^\dagger S(ab)T = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{\sqrt2} & \frac{1}{\sqrt2} & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \frac{-1}{\sqrt2} & \frac{1}{\sqrt2} & \cdot \end{pmatrix} \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{\sqrt2} & \cdot & \frac{1}{\sqrt2} \\ \cdot & \frac{1}{\sqrt2} & \cdot & \frac{-1}{\sqrt2} \\ \cdot & \cdot & 1 & \cdot \end{pmatrix} = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & -1 \end{pmatrix} = \begin{pmatrix} D^{\square\square} & \cdot & \cdot & \cdot \\ \cdot & D^{\square\square} & \cdot & \cdot \\ \cdot & \cdot & D^{\square\square} & \cdot \\ \cdot & \cdot & \cdot & D^{\square}_{\square} \end{pmatrix}$$

Three (3) symmetric ireps. $D^{\square\square}$ and one (1) anti-sym $D^{\square}_{\square}$

Finally, apply DTran$T$ to find $T^{\dagger}D\otimes DT$.

$$\overset{T^{\dagger}}{\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \cdot \end{pmatrix}} \overset{D\otimes D}{\begin{pmatrix} D_{11\,11} & D_{11\,12} & D_{12\,11} & D_{12\,12} \\ D_{11\,21} & D_{11\,22} & D_{12\,21} & D_{12\,22} \\ D_{11\,21} & D_{12\,21} & D_{11\,22} & D_{12\,22} \\ D_{21\,21} & D_{21\,22} & D_{21\,22} & D_{22\,22} \end{pmatrix}} \overset{T}{\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{\sqrt{2}} & \cdot & \frac{1}{\sqrt{2}} \\ \cdot & \frac{1}{\sqrt{2}} & \cdot & \frac{-1}{\sqrt{2}} \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}} = T^{\dagger}D\otimes DT$$

$$T^{\dagger}D\otimes DT = \begin{pmatrix} D_{11\,11} & \frac{\sqrt{2}}{1}D_{11\,12} & D_{12\,12} & 0 \\ \frac{\sqrt{2}}{1}D_{11\,21} & D_{11\,22}+D_{12\,21} & \frac{\sqrt{2}}{1}D_{12\,22} & 0 \\ D_{21\,21} & \frac{\sqrt{2}}{1}D_{21\,22} & D_{22\,22} & 0 \\ 0 & 0 & 0 & D_{11\,22}+D_{12\,21} \end{pmatrix} = \begin{pmatrix} \boxed{D^{\square\square}\text{(of U(2)}=D^{j=1}} & 0 \\ 0\quad 0\quad 0 & \boxed{D^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}=D^{j=0}} \end{pmatrix}$$

Next apply DTran T and its transpose $T^{\dagger}$ to the $S(ab)$ matrix to find $T^{\dagger}S(ab)T$.

$$\overset{T^{\dagger}}{\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \cdot \end{pmatrix}} \overset{S(ab)}{\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}} \overset{T}{\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{\sqrt{2}} & \cdot & \frac{1}{\sqrt{2}} \\ \cdot & \frac{1}{\sqrt{2}} & \cdot & \frac{-1}{\sqrt{2}} \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}} = T^{\dagger}S(ab)T$$

$$T^{\dagger}S(ab)T = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & -1 \end{pmatrix} = \begin{pmatrix} D^{\square\square} & \cdot & \cdot & \cdot \\ \cdot & D^{\square\square} & \cdot & \cdot \\ \cdot & \cdot & D^{\square\square} & \cdot \\ \cdot & \cdot & \cdot & D^{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \end{pmatrix}$$

Three (3) symmetric ireps. $D^{\square\square}$ and one (1) anti-sym $D^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$

Finally, apply DTran$T$ to find $T^\dagger D\otimes DT$.

$$\overset{T^\dagger}{\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \cdot \end{pmatrix}} \overset{D\otimes D}{\begin{pmatrix} D_{11\,11} & D_{11\,12} & D_{12\,11} & D_{12\,12} \\ D_{11\,21} & D_{11\,22} & D_{12\,21} & D_{12\,22} \\ D_{11\,21} & D_{12\,21} & D_{11\,22} & D_{12\,22} \\ D_{21\,21} & D_{21\,22} & D_{21\,22} & D_{22\,22} \end{pmatrix}} \overset{T}{\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{\sqrt{2}} & \cdot & \frac{1}{\sqrt{2}} \\ \cdot & \frac{1}{\sqrt{2}} & \cdot & \frac{-1}{\sqrt{2}} \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}} = T^\dagger D\otimes DT$$

$$T^\dagger D\otimes DT = \begin{pmatrix} D_{11\,11} & \frac{\sqrt{2}}{1}D_{11\,12} & D_{12\,12} & 0 \\ \frac{\sqrt{2}}{1}D_{11\,21} & D_{11\,22}+D_{12\,21} & \frac{\sqrt{2}}{1}D_{12\,22} & 0 \\ D_{21\,21} & \frac{\sqrt{2}}{1}D_{21\,22} & D_{22\,22} & 0 \\ 0 & 0 & 0 & D_{11\,22}+D_{12\,21} \end{pmatrix} = \begin{pmatrix} \text{D}^{\square\square}(\text{of U(2)}=D^{j=1}) & & & 0 \\ & & & 0 \\ & & & 0 \\ 0 & 0 & 0 & \text{D}^{\square}=D^{j=0} \end{pmatrix}$$

Clearly, THIS *commutes* with THIS

Next apply DTran T and its transpose $T^\dagger$ to the $S(ab)$ matrix to find $T^\dagger S(ab)T$.

$$\overset{T^\dagger}{\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \cdot \end{pmatrix}} \overset{S(ab)}{\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}} \overset{T}{\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{\sqrt{2}} & \cdot & \frac{1}{\sqrt{2}} \\ \cdot & \frac{1}{\sqrt{2}} & \cdot & \frac{-1}{\sqrt{2}} \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}} = T^\dagger S(ab)T$$

$$T^\dagger S(ab)T = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & -1 \end{pmatrix} = \begin{pmatrix} D^{\square\square} & \cdot & \cdot & \cdot \\ \cdot & D^{\square\square} & \cdot & \cdot \\ \cdot & \cdot & D^{\square\square} & \cdot \\ \cdot & \cdot & \cdot & D^{\square}_{\square} \end{pmatrix}$$

Three (3) symmetric ireps. $D^{\square\square}$ and one (1) anti-sym $D^{\square}_{\square}$

AMOP reference links on page 2

4.02.18 class 20: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

Interwining ($S_1 \subset S_2 \subset S_3 \subset S_4 \subset S_5$ …)*(U(1)⊂U(2)⊂U(3)⊂U(4)⊂U(5) …) algebras and tensor operator applications to spinor-rotor or orbital correlations

U(2) tensor product states and S_n permutation symmetry
Rank-1 tensor (or spinor)

Rank-2 tensor (2 particles each with U(2) state space)
2-particle U(2) transform and permutation operation
S_2 symmetry of U(2): Trust but verify

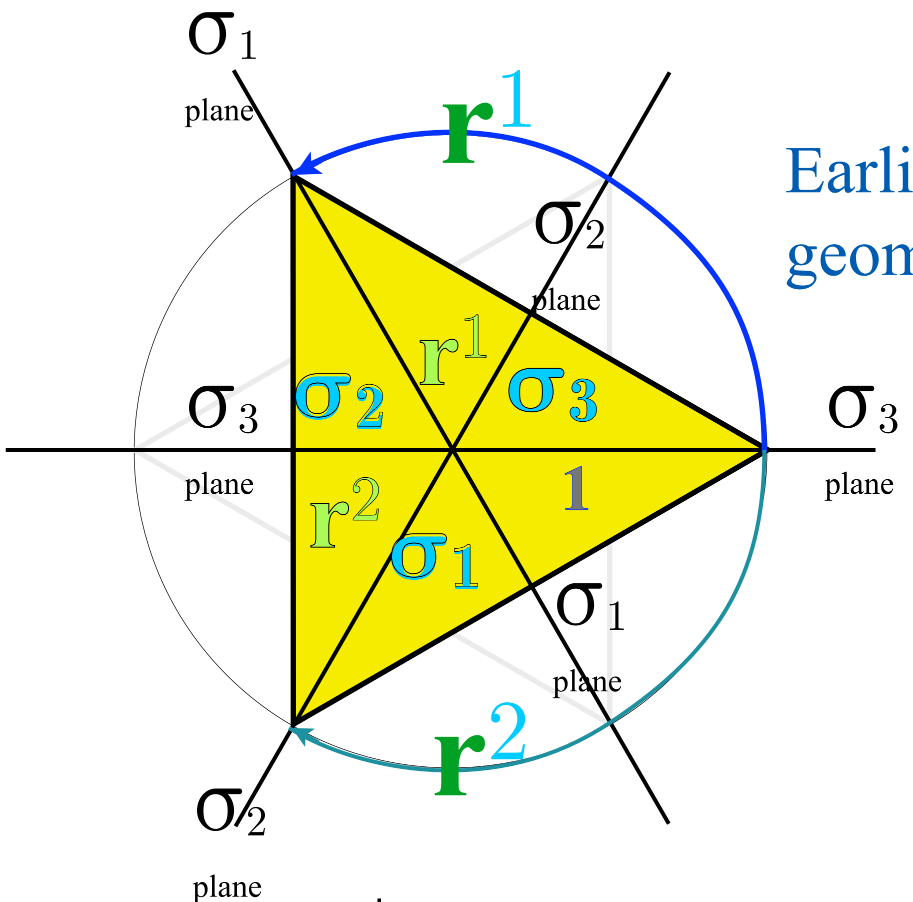
Applying S_2 projection to build DTran
Applying DTran for S_2
Applying DTran for U(2)

➡ S_3 permutations related to C_{3v}~D_3 geometry
S_3 permutation matrices
Hooklength formula for S_n reps
S_3 symmetry of U(2): Applying S_3 projection
Building S_3 DTran T from projectors
Effect of S_3 DTran T: Introducing intertwining S_3 - U(2) irep matrices
Multi-spin $(1/2)^N$ product state (Comparison to previous cases)

S_3 *permutations related to C_{3v}~D_3 geometry*

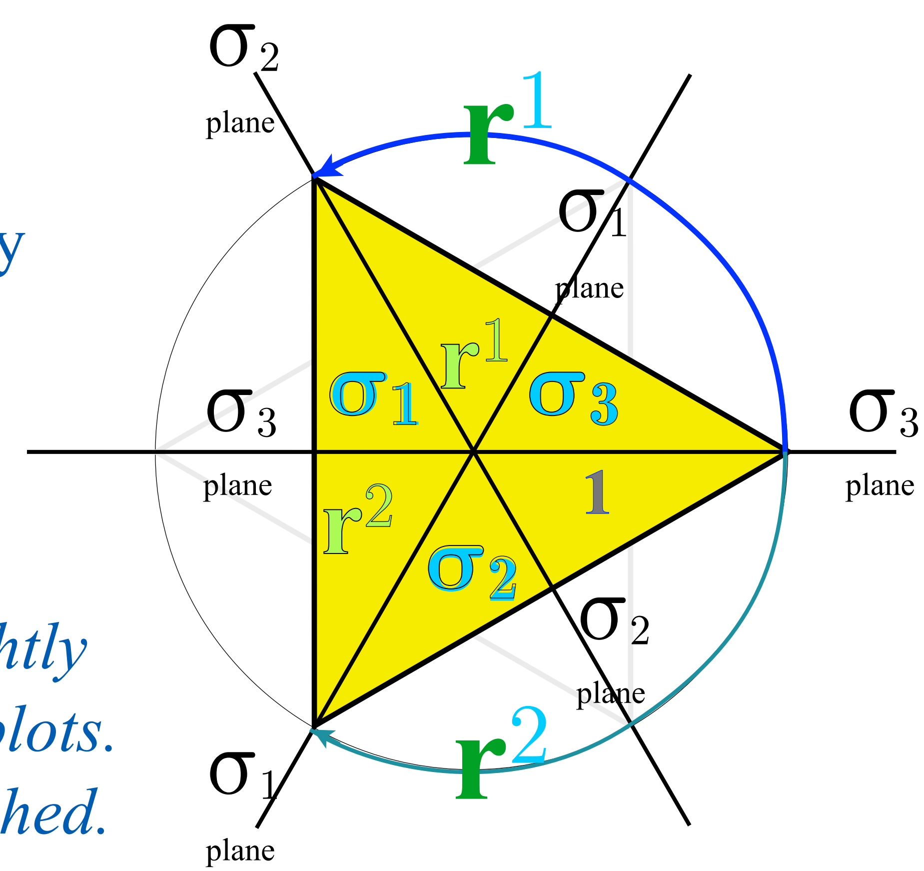
D3<C3v nomogram
AMOP Class 12 pdf p30

D3<D6 nomogram
AMOP Class 14 pdf p28



Earlier geometry

New geometry



C_{3v} geometry differs slightly from earlier Lecture 12 plots.
σ_1 and σ_2 plane are switched.

C_{3v} **gg**† *form*	**1**	r^2	r^1	σ_1	σ_2	σ_3
1	**1**	r^2	r^1	σ_1	σ_2	σ_3
r^1	r^1	**1**	r^2	σ_3	σ_1	σ_2
r^2	r^2	r^1	**1**	σ_2	σ_3	σ_1
σ_1	σ_1	σ_3	σ_2	**1**	r^1	r^2
σ_2	σ_2	σ_1	σ_3	r^2	**1**	r^1
σ_3	σ_3	σ_2	σ_1	r^1	r^2	**1**

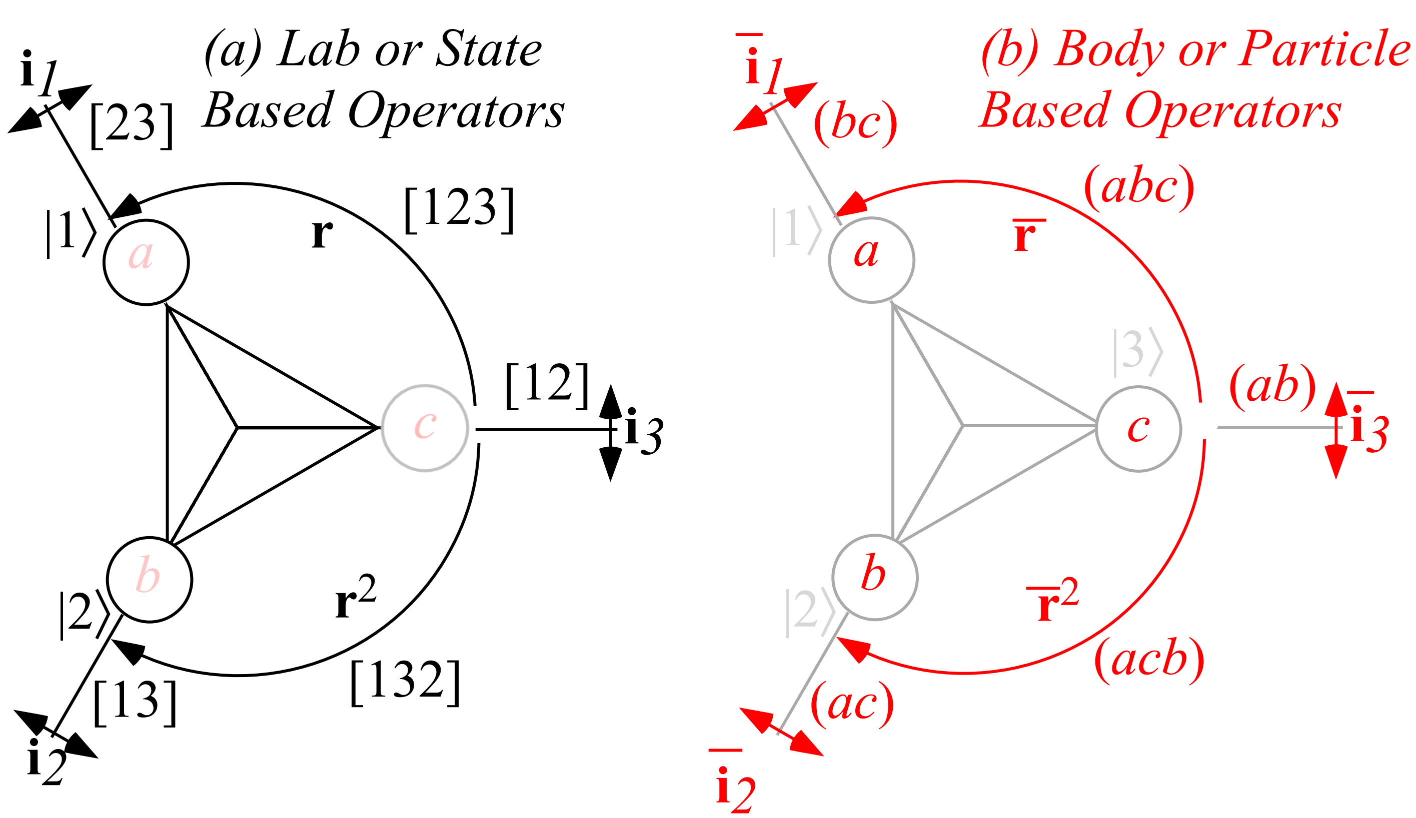
C_{3v} **gg**† *form*	**1**	r^2	r^1	σ_1	σ_2	σ_3
1	**1**	r^2	r^1	σ_1	σ_2	σ_3
r^1	r^1	**1**	r^2	σ_2	σ_3	σ_1
r^2	r^2	r^1	**1**	σ_3	σ_1	σ_2
σ_1	σ_1	σ_2	σ_3	**1**	r^2	r^1
σ_2	σ_2	σ_3	σ_1	r^1	**1**	r^2
σ_3	σ_3	σ_1	σ_2	r^2	r^1	**1**

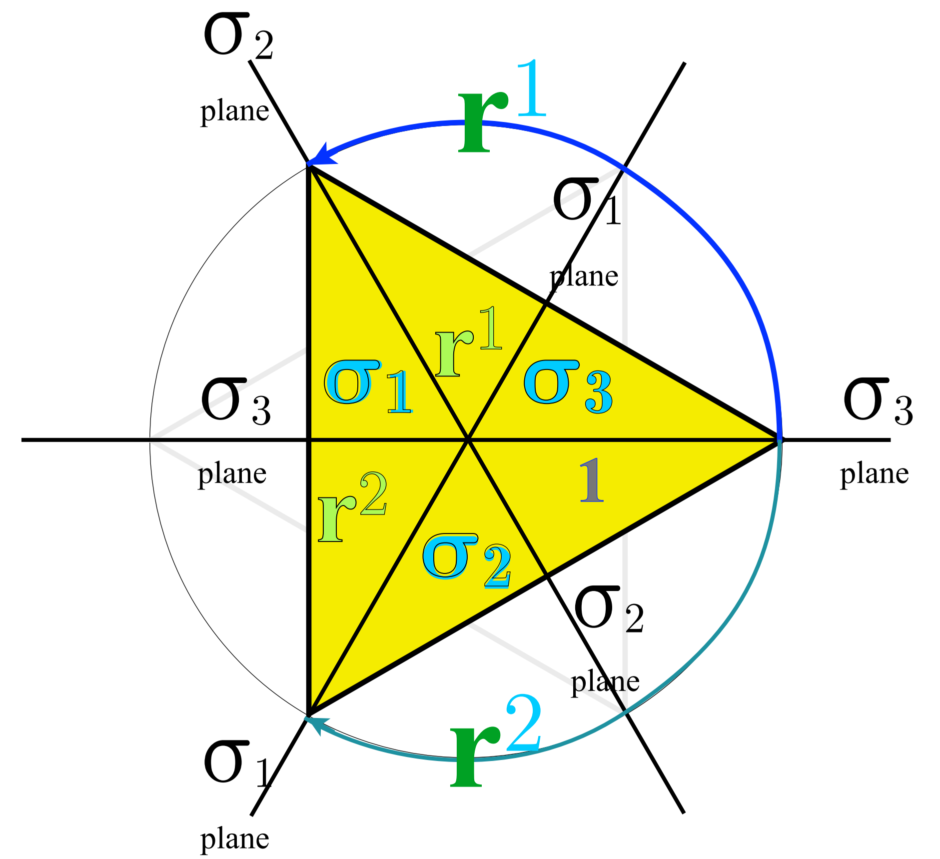
S_3 permutations related to C_{3v}~D_3 geometry

D3<C3v nomogram
AMOP Class 12 pdf p30

D3<D6 nomogram
AMOP Class 14 pdf p28

Fig. 25.3.0 QTforCA Unit 8 Ch.25 pdf p28

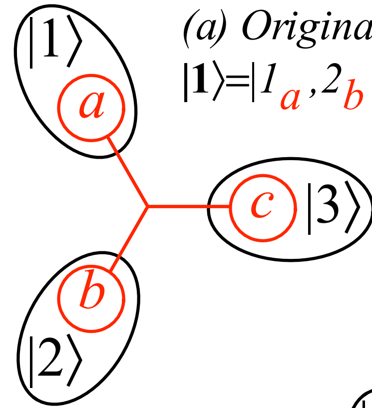




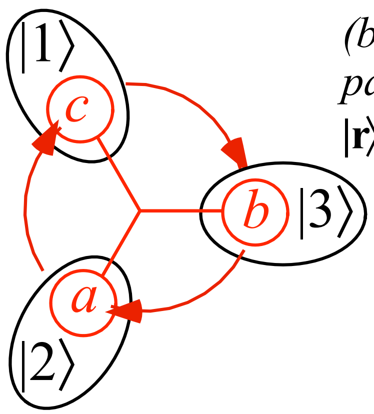
C_{3v} $\mathbf{gg}^\dagger$ form	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	σ_1	σ_2	σ_3
$(a)(b)(c)=\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	σ_1	σ_2	σ_3
$(abc)=\mathbf{r}^1$	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$	σ_2	σ_3	σ_1
$(acb)=\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$	σ_3	σ_1	σ_2
$(bc)=\sigma_1$	σ_1	σ_2	σ_3	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$
$(ac)=\sigma_2$	σ_2	σ_3	σ_1	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$
$(ab)=\sigma_3$	σ_3	σ_1	σ_2	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$

S_3 permutations related to C_{3v}~D_3 geometry

Fig. 25.3.1 QTforCA Unit 8 Ch.25 pdf p29



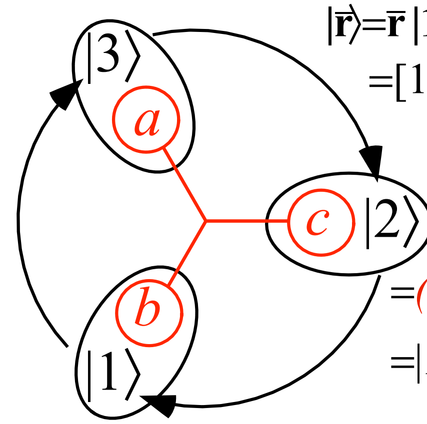
(a) Original state
$|\mathbf{1}\rangle = |1_a, 2_b, 3_c\rangle$



(b) Lab-fixed particle-3-cycle (120°rotation **r***)*

$|\mathbf{r}\rangle = \mathbf{r}\,|\mathbf{1}\rangle$
$= (abc)|1_a, 2_b, 3_c\rangle$
$= |1_c, 2_a, 3_b\rangle = |2_a, 3_b, 1_c\rangle$
$= \bar{\mathbf{r}}^2|\mathbf{1}\rangle = [132]|1_a, 2_b, 3_c\rangle$

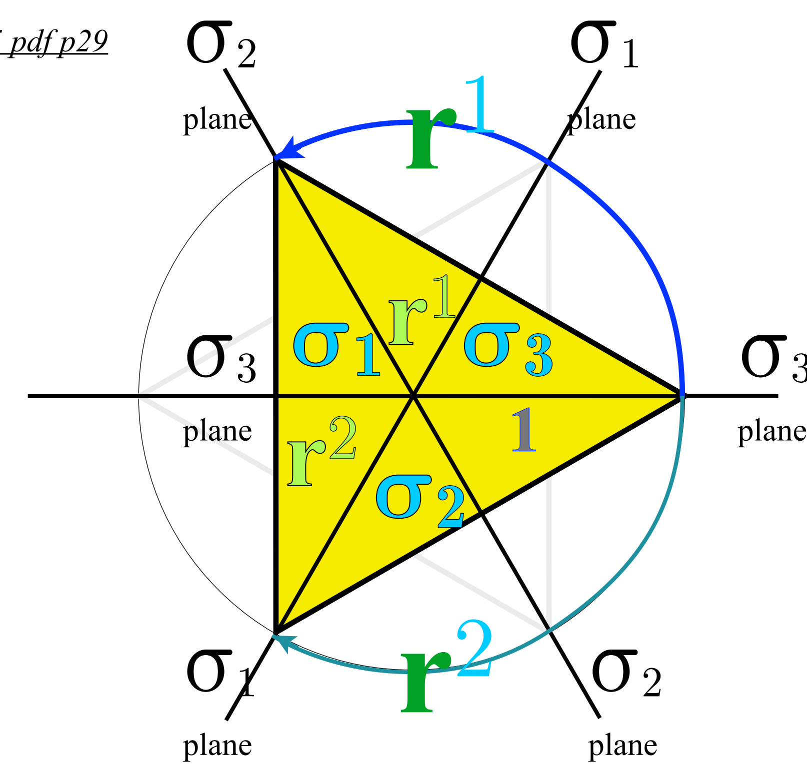
$[132]\left|1_a, 2_b, 3_c\right\rangle = \left|2_a, 3_b, 1_c\right\rangle$



(c) Particle-fixed lab-120°rotation $\bar{\mathbf{r}}$

$|\bar{\mathbf{r}}\rangle = \bar{\mathbf{r}}\,|\mathbf{1}\rangle$
$= [123]|1_a, 2_b, 3_c\rangle$
$= |3_a, 1_b, 2_c\rangle$
$= \mathbf{r}^2|\mathbf{1}\rangle$
$= (acb)|1_a, 2_b, 3_c\rangle$
$= |1_b, 2_c, 3_a\rangle$

$[123]\left|1_a, 2_b, 3_c\right\rangle = \left|3_a, 1_b, 2_c\right\rangle$



1	r^2	r	i_1	i_2	i_3
r	1	r^2	i_2	i_3	i_1
r^2	r	1	i_3	i_1	i_2
i_1	i_2	i_3	1	r^2	r
i_2	i_3	i_1	r	1	r^2
i_3	i_1	i_2	r^2	r	1

C_{3v} $\mathbf{gg}^\dagger$ form	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	σ_1	σ_2	σ_3
$(a)(b)(c) = \mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	σ_1	σ_2	σ_3
$(abc) = \mathbf{r}^1$	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$	σ_2	σ_3	σ_1
$(acb) = \mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$	σ_3	σ_1	σ_2
$(bc) = \sigma_1$	σ_1	σ_2	σ_3	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$
$(ac) = \sigma_2$	σ_2	σ_3	σ_1	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$
$(ab) = \sigma_3$	σ_3	σ_1	σ_2	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$

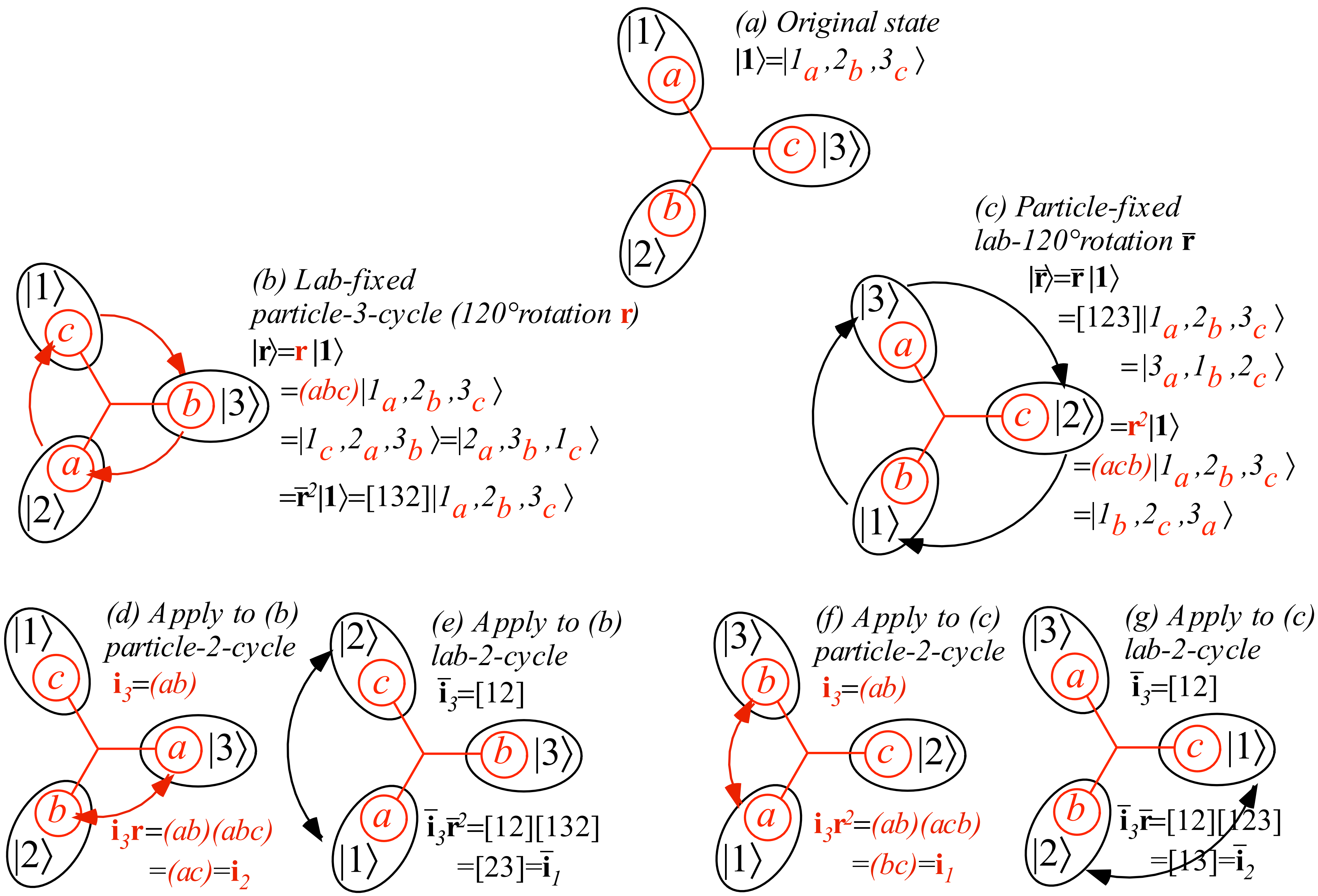
(1)	(acb)	(abc)	(bc)	(ac)	(ab)
(abc)	(1)	(acb)	(ac)	(ab)	(bc)
(acb)	(abc)	(1)	(ab)	(bc)	(ac)
(bc)	(ac)	(ab)	(1)	(acb)	(abc)
(ac)	(ab)	(bc)	(abc)	(1)	(acb)
(ab)	(bc)	(ac)	(acb)	(abc)	(1)

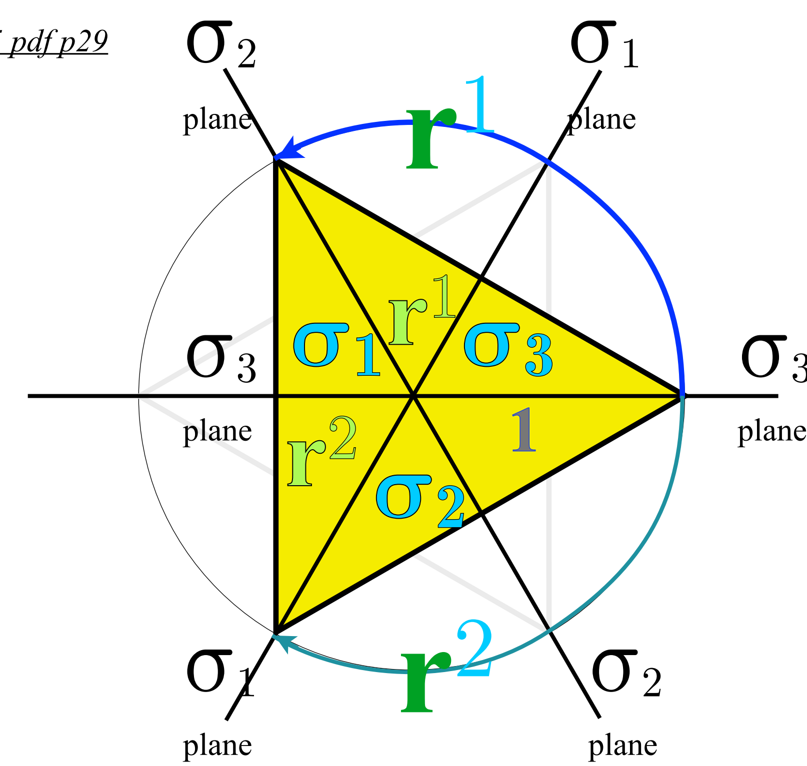
[1]	[132]	[123]	[23]	[13]	[12]
[123]	[1]	[132]	[13]	[12]	[23]
[132]	[123]	[1]	[12]	[23]	[13]
[23]	[13]	[12]	[1]	[132]	[123]
[13]	[12]	[23]	[123]	[1]	[132]
[12]	[23]	[13]	[132]	[123]	[1]

Fig. 25.3.1 Relating D_3 *and* S_3 *permutation operations*

S_3 *permutations related to C_{3v}~D_3 geometry*

Fig. 25.3.1 QTforCA Unit 8 Ch.25 pdf p29





1	r^2	r	i_1	i_2	i_3
r	1	r^2	i_2	i_3	i_1
r^2	r	1	i_3	i_1	i_2
i_1	i_2	i_3	1	r^2	r
i_2	i_3	i_1	r	1	r^2
i_3	i_1	i_2	r^2	r	1

C_{3v} $gg^\dagger$ form	**1**	$\mathbf{r}^2$	$\mathbf{r}^1$	σ_1	σ_2	σ_3
$(a)(b)(c) = \mathbf{1}$	**1**	$\mathbf{r}^2$	$\mathbf{r}^1$	σ_1	σ_2	σ_3
$(abc) = \mathbf{r}^1$	$\mathbf{r}^1$	**1**	$\mathbf{r}^2$	σ_2	σ_3	σ_1
$(acb) = \mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	**1**	σ_3	σ_1	σ_2
$(bc) = \sigma_1$	σ_1	σ_2	σ_3	**1**	$\mathbf{r}^2$	$\mathbf{r}^1$
$(ac) = \sigma_2$	σ_2	σ_3	σ_1	$\mathbf{r}^1$	**1**	$\mathbf{r}^2$
$(ab) = \sigma_3$	σ_3	σ_1	σ_2	$\mathbf{r}^2$	$\mathbf{r}^1$	**1**

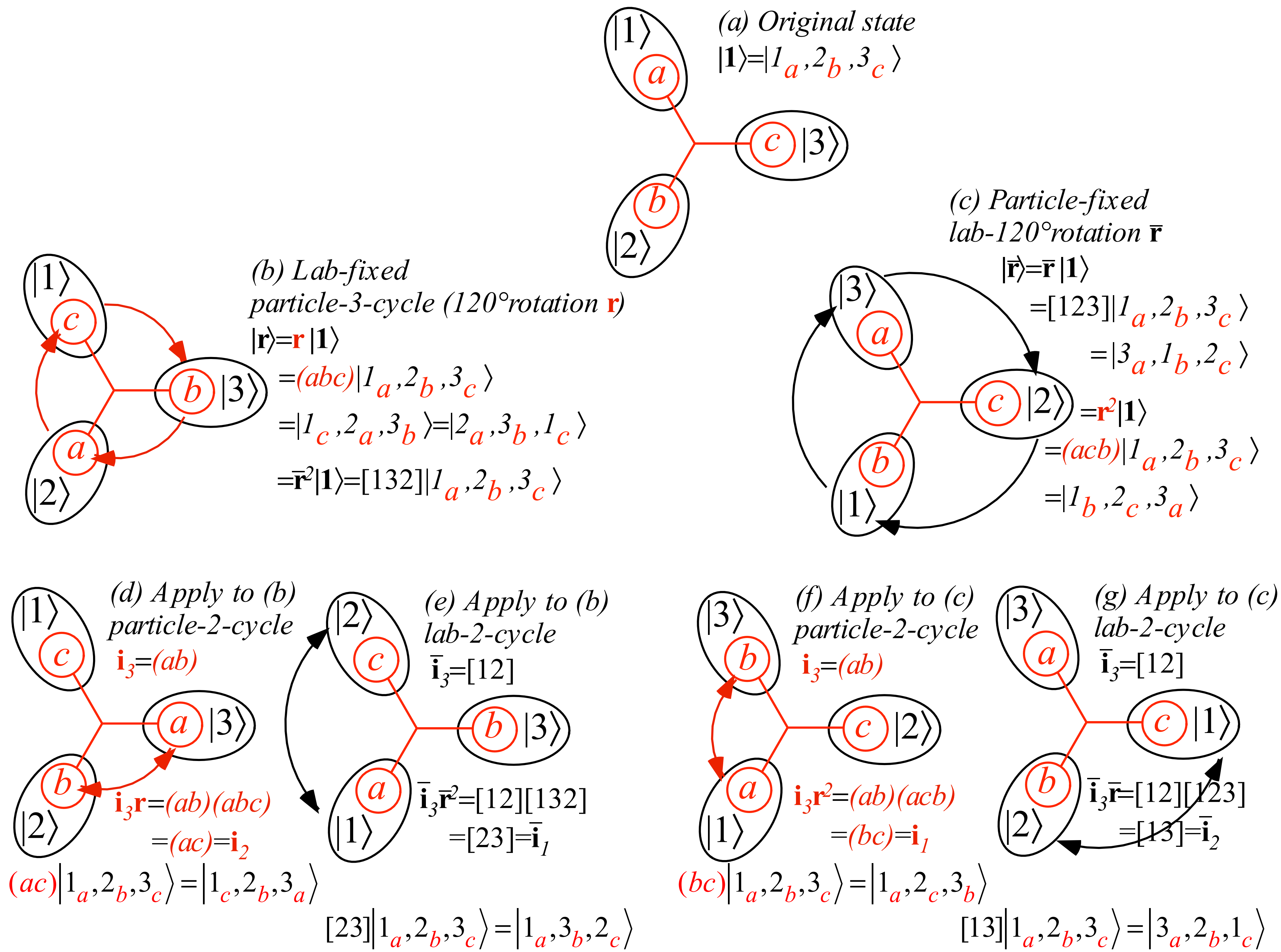
(1)	(acb)	(abc)	(bc)	(ac)	(ab)
(abc)	(1)	(acb)	(ac)	(ab)	(bc)
(acb)	(abc)	(1)	(ab)	(bc)	(ac)
(bc)	(ac)	(ab)	(1)	(acb)	(abc)
(ac)	(ab)	(bc)	(abc)	(1)	(acb)
(ab)	(bc)	(ac)	(acb)	(abc)	(1)

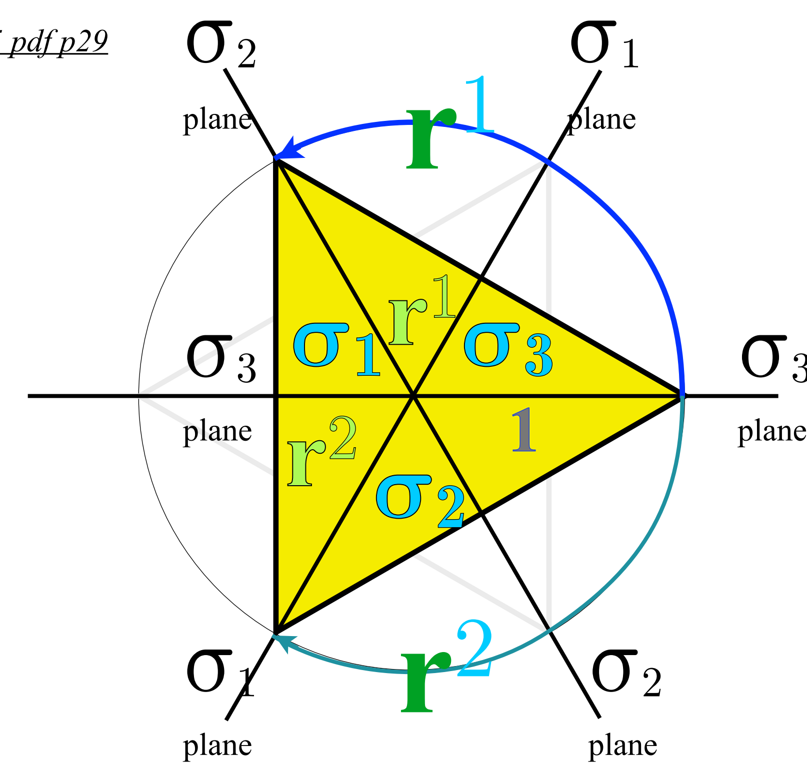
[1]	[132]	[123]	[23]	[13]	[12]
[123]	[1]	[132]	[13]	[12]	[23]
[132]	[123]	[1]	[12]	[23]	[13]
[23]	[13]	[12]	[1]	[132]	[123]
[13]	[12]	[23]	[123]	[1]	[132]
[12]	[23]	[13]	[132]	[123]	[1]

Fig. 25.3.1 Relating D_3 and S_3 permutation operations

S_3 permutations related to C_{3v}~D_3 geometry

Fig. 25.3.1 QTforCA Unit 8 Ch.25 pdf p29





1	r^2	r	i_1	i_2	i_3
r	1	r^2	i_2	i_3	i_1
r^2	r	1	i_3	i_1	i_2
i_1	i_2	i_3	1	r^2	r
i_2	i_3	i_1	r	1	r^2
i_3	i_1	i_2	r^2	r	1

C_{3v} $gg^\dagger$ form	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	σ_1	σ_2	σ_3
$(a)(b)(c)=\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	σ_1	σ_2	σ_3
$(abc)=\mathbf{r}^1$	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$	σ_2	σ_3	σ_1
$(acb)=\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$	σ_3	σ_1	σ_2
$(bc)=\sigma_1$	σ_1	σ_2	σ_3	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$
$(ac)=\sigma_2$	σ_2	σ_3	σ_1	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$
$(ab)=\sigma_3$	σ_3	σ_1	σ_2	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$

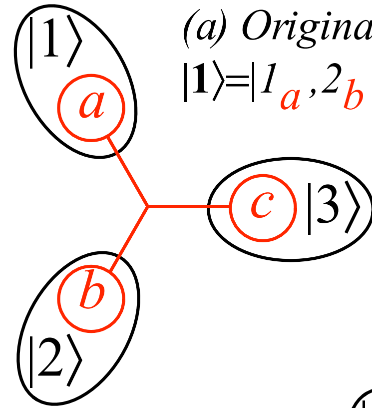
(1)	(acb)	(abc)	(bc)	(ac)	(ab)
(abc)	(1)	(acb)	(ac)	(ab)	(bc)
(acb)	(abc)	(1)	(ab)	(bc)	(ac)
(bc)	(ac)	(ab)	(1)	(acb)	(abc)
(ac)	(ab)	(bc)	(abc)	(1)	(acb)
(ab)	(bc)	(ac)	(acb)	(abc)	(1)

[1]	[132]	[123]	[23]	[13]	[12]
[123]	[1]	[132]	[13]	[12]	[23]
[132]	[123]	[1]	[12]	[23]	[13]
[23]	[13]	[12]	[1]	[132]	[123]
[13]	[12]	[23]	[123]	[1]	[132]
[12]	[23]	[13]	[132]	[123]	[1]

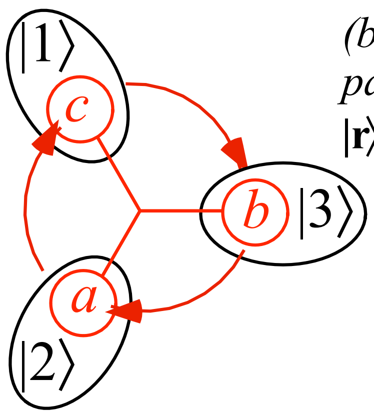
Fig. 25.3.1 Relating D_3 and S_3 permutation operations

S_3 permutations related to $C_{3v}\sim D_3$ geometry

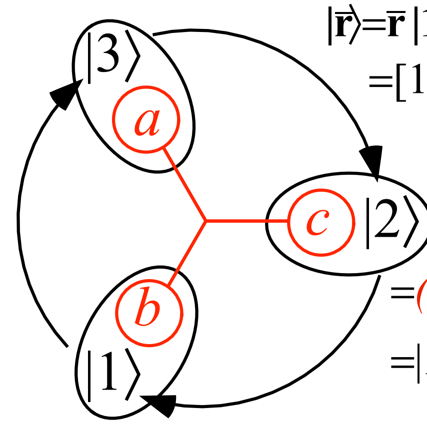
Fig. 25.3.1 QTforCA Unit 8 Ch.25 pdf p29



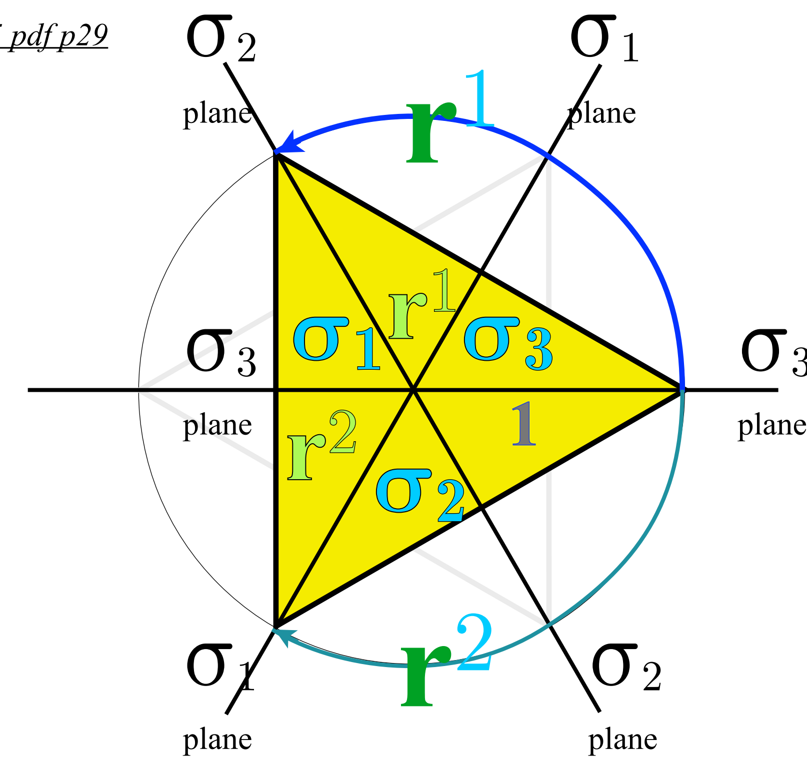
(a) Original state
$|\mathbf{1}\rangle = |1_a, 2_b, 3_c\rangle$



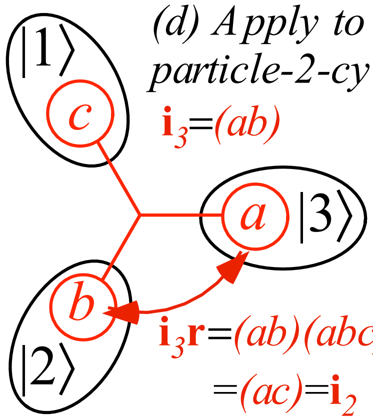
(b) Lab-fixed particle-3-cycle (120°rotation $\mathbf{r}$*)*
$|\mathbf{r}\rangle = \mathbf{r}\,|\mathbf{1}\rangle$
$= (abc)|1_a, 2_b, 3_c\rangle$
$= |1_c, 2_a, 3_b\rangle = |2_a, 3_b, 1_c\rangle$
$= \bar{\mathbf{r}}^2|\mathbf{1}\rangle = [132]|1_a, 2_b, 3_c\rangle$



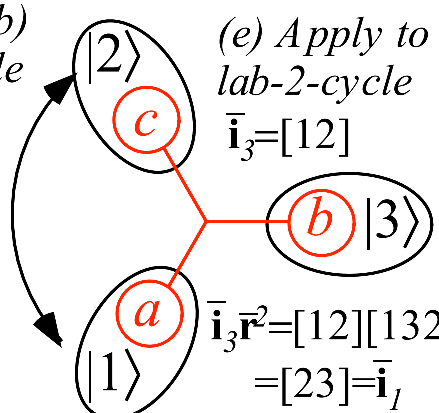
(c) Particle-fixed lab-120°rotation $\bar{\mathbf{r}}$
$|\bar{\mathbf{r}}\rangle = \bar{\mathbf{r}}\,|\mathbf{1}\rangle$
$= [123]|1_a, 2_b, 3_c\rangle$
$= |3_a, 1_b, 2_c\rangle$
$= \mathbf{r}^2|\mathbf{1}\rangle$
$= (acb)|1_a, 2_b, 3_c\rangle$
$= |1_b, 2_c, 3_a\rangle$



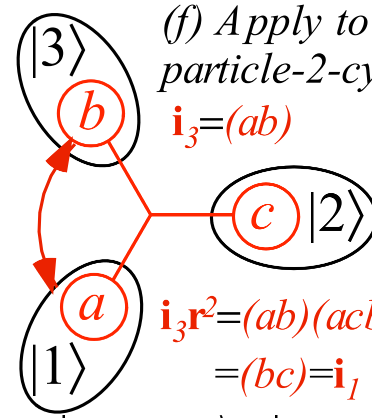
1	r^2	r	i_1	i_2	i_3
r	1	r^2	i_2	i_3	i_1
r^2	r	1	i_3	i_1	i_2
i_1	i_2	i_3	1	r^2	r
i_2	i_3	i_1	r	1	r^2
i_3	i_1	i_2	r^2	r	1



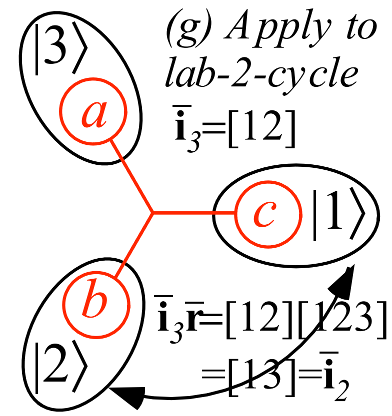
(d) Apply to (b) particle-2-cycle
$\mathbf{i}_3 = (ab)$
$\mathbf{i}_3\mathbf{r} = (ab)(abc)$
$= (ac) = \mathbf{i}_2$
$(ac)|1_a, 2_b, 3_c\rangle = |1_c, 2_b, 3_a\rangle$



(e) Apply to (b) lab-2-cycle
$\bar{\mathbf{i}}_3 = [12]$
$\bar{\mathbf{i}}_3\bar{\mathbf{r}}^2 = [12][132]$
$= [23] = \bar{\mathbf{i}}_1$
$[23]|1_a, 2_b, 3_c\rangle = |1_a, 3_b, 2_c\rangle$
$= |1_a, 2_c, 3_b\rangle$
$= |2_c, 1_a, 3_b\rangle$
$= |2_c, 3_b, 1_a\rangle$
$= |3_b, 2_c, 1_a\rangle$
$= |3_b, 1_a, 2_c\rangle$



(f) Apply to (c) particle-2-cycle
$\mathbf{i}_3 = (ab)$
$\mathbf{i}_3\mathbf{r}^2 = (ab)(acb)$
$= (bc) = \mathbf{i}_1$
$(bc)|1_a, 2_b, 3_c\rangle = |1_a, 2_c, 3_b\rangle$



(g) Apply to (c) lab-2-cycle
$\bar{\mathbf{i}}_3 = [12]$
$\bar{\mathbf{i}}_3\bar{\mathbf{r}} = [12][123]$
$= [13] = \bar{\mathbf{i}}_2$
$[13]|1_a, 2_b, 3_c\rangle = |3_a, 2_b, 1_c\rangle$

Only
relative position
counts here!

C_{3v} $\mathbf{g}\mathbf{g}^\dagger$ form	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	σ_1	σ_2	σ_3
$(a)(b)(c) = \mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	σ_1	σ_2	σ_3
$(abc) = \mathbf{r}^1$	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$	σ_2	σ_3	σ_1
$(acb) = \mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$	σ_3	σ_1	σ_2
$(bc) = \sigma_1$	σ_1	σ_2	σ_3	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$
$(ac) = \sigma_2$	σ_2	σ_3	σ_1	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$
$(ab) = \sigma_3$	σ_3	σ_1	σ_2	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$

Fig. 25.3.1 Relating D_3 *and* S_3 *permutation operations*

AMOP reference links on page 2

4.02.18 class 20: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

Interwining $(S_1 \subset S_2 \subset S_3 \subset S_4 \subset S_5 \ldots)*(U(1) \subset U(2) \subset U(3) \subset U(4) \subset U(5) \ldots)$ algebras and tensor operator applications to spinor-rotor or orbital correlations

U(2) tensor product states and S_n permutation symmetry
Rank-1 tensor (or spinor)

Rank-2 tensor (2 particles each with U(2) state space)
2-particle U(2) transform and permutation operation
S_2 symmetry of U(2): Trust but verify

Applying S_2 projection to build DTran
Applying DTran for S_2
Applying DTran for U(2)

S_3 permutations related to $C_{3v} \sim D_3$ geometry
➡ S_3 permutation matrices
Hooklength formula for S_n reps
S_3 symmetry of U(2): Applying S_3 projection
Building S_3 DTran T from projectors
Effect of S_3 DTran T: Introducing intertwining S_3 - U(2) irep matrices
Multi-spin $(1/2)^N$ product state (Comparison to previous cases)

S_3 *symmetry* of U(2): *Applying S_3 projection*

Rank-3 tensor basis |ijk⟩ (3 particles each with U(2) state space)

[1][2][3]	$\|111\rangle$	$\|112\rangle$	$\|121\rangle$	$\|122\rangle$	$\|211\rangle$	$\|212\rangle$	$\|221\rangle$	$\|222\rangle$
$\langle 111\|$	1	·	·	·	·	·	·	·
$\langle 112\|$	·	1	·	·	·	·	·	·
$\langle 121\|$	·	·	1	·	·	·	·	·
$\langle 122\|$	·	·	·	1	·	·	·	·
$\langle 211\|$	·	·	·	·	1	·	·	·
$\langle 212\|$	·	·	·	·	·	1	·	·
$\langle 221\|$	·	·	·	·	·	·	1	·
$\langle 222\|$	·	·	·	·	·	·	·	1

Representation of bicycle (ab) or [12]

[12]	$\|111\rangle$	$\|112\rangle$	$\|121\rangle$	$\|122\rangle$	$\|211\rangle$	$\|212\rangle$	$\|221\rangle$	$\|222\rangle$
$\langle 111\|$	1	·	·	·	·	·	·	·
$\langle 112\|$	·	1	·	·	·	·	·	·
$\langle 121\|$	·	·	·	·	1	·	·	·
$\langle 122\|$	·	·	·	·	·	1	·	·
$\langle 211\|$	·	·	1	·	·	·	·	·
$\langle 212\|$	·	·	·	1	·	·	·	·
$\langle 221\|$	·	·	·	·	·	·	1	·
$\langle 222\|$	·	·	·	·	·	·	·	1

S_3 *symmetry* of U(2): *Applying S_3 projection*

Rank-3 tensor basis $|ijk\rangle$ (3 particles each with U(2) state space)

[1][2][3]	$\|111\rangle$	$\|112\rangle$	$\|121\rangle$	$\|122\rangle$	$\|211\rangle$	$\|212\rangle$	$\|221\rangle$	$\|222\rangle$
$\langle 111\|$	1	·	·	·	·	·	·	·
$\langle 112\|$	·	1	·	·	·	·	·	·
$\langle 121\|$	·	·	1	·	·	·	·	·
$\langle 122\|$	·	·	·	1	·	·	·	·
$\langle 211\|$	·	·	·	·	1	·	·	·
$\langle 212\|$	·	·	·	·	·	1	·	·
$\langle 221\|$	·	·	·	·	·	·	1	·
$\langle 222\|$	·	·	·	·	·	·	·	1

Representation of bicycle (ab) or [12]

[12]	$\|111\rangle$	$\|112\rangle$	$\|121\rangle$	$\|122\rangle$	$\|211\rangle$	$\|212\rangle$	$\|221\rangle$	$\|222\rangle$
$\langle 111\|$	1	·	·	·	·	·	·	·
$\langle 112\|$	·	1	·	·	·	·	·	·
$\langle 121\|$	·	·	·	·	1	·	·	·
$\langle 122\|$	·	·	·	·	·	1	·	·
$\langle 211\|$	·	·	1	·	·	·	·	·
$\langle 212\|$	·	·	·	1	·	·	·	·
$\langle 221\|$	·	·	·	·	·	·	1	·
$\langle 222\|$	·	·	·	·	·	·	·	1

Representation of bicycle (ac) or [13]

[13]	$\|111\rangle$	$\|112\rangle$	$\|121\rangle$	$\|122\rangle$	$\|211\rangle$	$\|212\rangle$	$\|221\rangle$	$\|222\rangle$
$\langle 111\|$	1	·	·	·	·	·	·	·
$\langle 112\|$	·	·	·	·	1	·	·	·
$\langle 121\|$	·	·	1	·	·	·	·	·
$\langle 122\|$	·	·	·	·	·	·	1	·
$\langle 211\|$	·	1	·	·	·	·	·	·
$\langle 212\|$	·	·	·	·	·	1	·	·
$\langle 221\|$	·	·	·	1	·	·	·	·
$\langle 222\|$	·	·	·	·	·	·	·	1

S_3 *symmetry* of U(2): *Applying S_3 projection*

Rank-3 tensor basis |ijk⟩ (3 particles each with U(2) state space)

[1][2][3]	|111⟩	|112⟩	|121⟩	|122⟩	|211⟩	|212⟩	|221⟩	|222⟩
⟨111|	1	·	·	·	·	·	·	·
⟨112|	·	1	·	·	·	·	·	·
⟨121|	·	·	1	·	·	·	·	·
⟨122|	·	·	·	1	·	·	·	·
⟨211|	·	·	·	·	1	·	·	·
⟨212|	·	·	·	·	·	1	·	·
⟨221|	·	·	·	·	·	·	1	·
⟨222|	·	·	·	·	·	·	·	1

Representation of bicycle (ab) or [12]

[12]	|111⟩	|112⟩	|121⟩	|122⟩	|211⟩	|212⟩	|221⟩	|222⟩
⟨111|	1	·	·	·	·	·	·	·
⟨112|	·	1	·	·	·	·	·	·
⟨121|	·	·	·	·	1	·	·	·
⟨122|	·	·	·	·	·	1	·	·
⟨211|	·	·	1	·	·	·	·	·
⟨212|	·	·	·	1	·	·	·	·
⟨221|	·	·	·	·	·	·	1	·
⟨222|	·	·	·	·	·	·	·	1

Representation of bicycle (ac) or [13]

[13]	|111⟩	|112⟩	|121⟩	|122⟩	|211⟩	|212⟩	|221⟩	|222⟩
⟨111|	1	·	·	·	·	·	·	·
⟨112|	·	·	·	·	1	·	·	·
⟨121|	·	·	1	·	·	·	·	·
⟨122|	·	·	·	·	·	·	1	·
⟨211|	·	1	·	·	·	·	·	·
⟨212|	·	·	·	·	·	1	·	·
⟨221|	·	·	·	1	·	·	·	·
⟨222|	·	·	·	·	·	·	·	1

Representation of bicycle (bc) or [23]

[23]	|111⟩	|112⟩	|121⟩	|122⟩	|211⟩	|212⟩	|221⟩	|222⟩
⟨111|	1	·	·	·	·	·	·	·
⟨112|	·	·	1	·	·	·	·	·
⟨121|	·	1	·	·	·	·	·	·
⟨122|	·	·	·	1	·	·	·	·
⟨211|	·	·	·	·	1	·	·	·
⟨212|	·	·	·	·	·	·	1	·
⟨221|	·	·	·	·	·	1	·	·
⟨222|	·	·	·	·	·	·	·	1

S_3 *symmetry* of U(2): *Applying S_3 projection*

Rank-3 tensor basis |ijk⟩ (3 particles each with U(2) state space)

[1][2][3]	|111⟩	|112⟩	|121⟩	|122⟩	|211⟩	|212⟩	|221⟩	|222⟩
⟨111|	1	·	·	·	·	·	·	·
⟨112|	·	1	·	·	·	·	·	·
⟨121|	·	·	1	·	·	·	·	·
⟨122|	·	·	·	1	·	·	·	·
⟨211|	·	·	·	·	1	·	·	·
⟨212|	·	·	·	·	·	1	·	·
⟨221|	·	·	·	·	·	·	1	·
⟨222|	·	·	·	·	·	·	·	1

	111	112	121	122	211	212	221	222
111								
112								
121								
122								
211								
212								
221								
222								

Representation of tricycle (abc) or [123]

[123]	|111⟩	|112⟩	|121⟩	|122⟩	|211⟩	|212⟩	|221⟩	|222⟩
⟨111|	1	·	·	·	·	·	·	·
⟨112|	·	·	1	·	·	·	·	·
⟨121|	·	·	·	·	1	·	·	·
⟨122|	·	·	·	·	·	·	1	·
⟨211|	·	1	·	·	·	·	·	·
⟨212|	·	·	·	1	·	·	·	·
⟨221|	·	·	·	·	·	1	·	·
⟨222|	·	·	·	·	·	·	·	1

Representation of tricycle (acb) or [132]

[132] is transpose or inverse of [123]

[132]	|111⟩	|112⟩	|121⟩	|122⟩	|211⟩	|212⟩	|221⟩	|222⟩
⟨111|	1	·	·	·	·	·	·	·
⟨112|	·	·	·	·	1	·	·	·
⟨121|	·	1	·	·	·	·	·	·
⟨122|	·	·	·	·	·	1	·	·
⟨211|	·	·	1	·	·	·	·	·
⟨212|	·	·	·	·	·	·	1	·
⟨221|	·	·	·	1	·	·	·	·
⟨222|	·	·	·	·	·	·	·	1

S_3 *symmetry* of U(2): *Applying S_3 projection*

Rank-3 tensor basis $\vert ijk\rangle$ (3 particles each with U(2) state space)

[1][2][3]	$\vert 111\rangle$	$\vert 112\rangle$	$\vert 121\rangle$	$\vert 122\rangle$	$\vert 211\rangle$	$\vert 212\rangle$	$\vert 221\rangle$	$\vert 222\rangle$
$\langle 111\vert$	1	·	·	·	·	·	·	·
$\langle 112\vert$	·	1	·	·	·	·	·	·
$\langle 121\vert$	·	·	1	·	·	·	·	·
$\langle 122\vert$	·	·	·	1	·	·	·	·
$\langle 211\vert$	·	·	·	·	1	·	·	·
$\langle 212\vert$	·	·	·	·	·	1	·	·
$\langle 221\vert$	·	·	·	·	·	·	1	·
$\langle 222\vert$	·	·	·	·	·	·	·	1

	111	112	121	122	211	212	221	222
111								
112								
121								
122								
211								
212								
221								
222								

Need smaller boxes!

[12]	$\vert 111\rangle$	$\vert 112\rangle$	$\vert 121\rangle$	$\vert 122\rangle$	$\vert 211\rangle$	$\vert 212\rangle$	$\vert 221\rangle$	$\vert 222\rangle$
$\langle 111\vert$	1	·	·	·	·	·	·	·
$\langle 112\vert$	·	1	·	·	·	·	·	·
$\langle 121\vert$	·	·	·	·	1	·	·	·
$\langle 122\vert$	·	·	·	·	·	1	·	·
$\langle 211\vert$	·	·	1	·	·	·	·	·
$\langle 212\vert$	·	·	·	1	·	·	·	·
$\langle 221\vert$	·	·	·	·	·	·	1	·
$\langle 222\vert$	·	·	·	·	·	·	·	1

[13]	$\vert 111\rangle$	$\vert 112\rangle$	$\vert 121\rangle$	$\vert 122\rangle$	$\vert 211\rangle$	$\vert 212\rangle$	$\vert 221\rangle$	$\vert 222\rangle$
$\langle 111\vert$	1	·	·	·	·	·	·	·
$\langle 112\vert$	·	·	·	·	1	·	·	·
$\langle 121\vert$	·	·	1	·	·	·	·	·
$\langle 122\vert$	·	·	·	·	·	·	1	·
$\langle 211\vert$	·	1	·	·	·	·	·	·
$\langle 212\vert$	·	·	·	·	·	1	·	·
$\langle 221\vert$	·	·	·	1	·	·	·	·
$\langle 222\vert$	·	·	·	·	·	·	·	1

[23]	$\vert 111\rangle$	$\vert 112\rangle$	$\vert 121\rangle$	$\vert 122\rangle$	$\vert 211\rangle$	$\vert 212\rangle$	$\vert 221\rangle$	$\vert 222\rangle$
$\langle 111\vert$	1	·	·	·	·	·	·	·
$\langle 112\vert$	·	·	1	·	·	·	·	·
$\langle 121\vert$	·	1	·	·	·	·	·	·
$\langle 122\vert$	·	·	·	1	·	·	·	·
$\langle 211\vert$	·	·	·	·	1	·	·	·
$\langle 212\vert$	·	·	·	·	·	·	1	·
$\langle 221\vert$	·	·	·	·	·	1	·	·
$\langle 222\vert$	·	·	·	·	·	·	·	1

[123]	$\vert 111\rangle$	$\vert 112\rangle$	$\vert 121\rangle$	$\vert 122\rangle$	$\vert 211\rangle$	$\vert 212\rangle$	$\vert 221\rangle$	$\vert 222\rangle$
$\langle 111\vert$	1	·	·	·	·	·	·	·
$\langle 112\vert$	·	·	1	·	·	·	·	·
$\langle 121\vert$	·	·	·	·	1	·	·	·
$\langle 122\vert$	·	·	·	·	·	·	1	·
$\langle 211\vert$	·	1	·	·	·	·	·	·
$\langle 212\vert$	·	·	·	1	·	·	·	·
$\langle 221\vert$	·	·	·	·	·	1	·	·
$\langle 222\vert$	·	·	·	·	·	·	·	1

[132]	$\vert 111\rangle$	$\vert 112\rangle$	$\vert 121\rangle$	$\vert 122\rangle$	$\vert 211\rangle$	$\vert 212\rangle$	$\vert 221\rangle$	$\vert 222\rangle$
$\langle 111\vert$	1	·	·	·	·	·	·	·
$\langle 112\vert$	·	·	·	·	1	·	·	·
$\langle 121\vert$	·	1	·	·	·	·	·	·
$\langle 122\vert$	·	·	·	·	·	1	·	·
$\langle 211\vert$	·	·	1	·	·	·	·	·
$\langle 212\vert$	·	·	·	·	·	·	1	·
$\langle 221\vert$	·	·	·	1	·	·	·	·
$\langle 222\vert$	·	·	·	·	·	·	·	1

AMOP reference links on page 2

4.02.18 class 20: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

Interwining ($S_1 \subset S_2 \subset S_3 \subset S_4 \subset S_5$ …)*(U(1)⊂U(2)⊂U(3)⊂U(4)⊂U(5) …) algebras and tensor operator applications to spinor-rotor or orbital correlations

U(2) tensor product states and S_n permutation symmetry
Rank-1 tensor (or spinor)

Rank-2 tensor (2 particles each with U(2) state space)
2-particle U(2) transform and permutation operation
S_2 symmetry of U(2): Trust but verify

Applying S_2 projection to build DTran
Applying DTran for S_2
Applying DTran for U(2)

S_3 permutations related to C_{3v}~D_3 geometry
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➡ Hooklength formula for S_n reps
S_3 symmetry of U(2): Applying S_3 projection (Note Pauli-exclusion principle basis)
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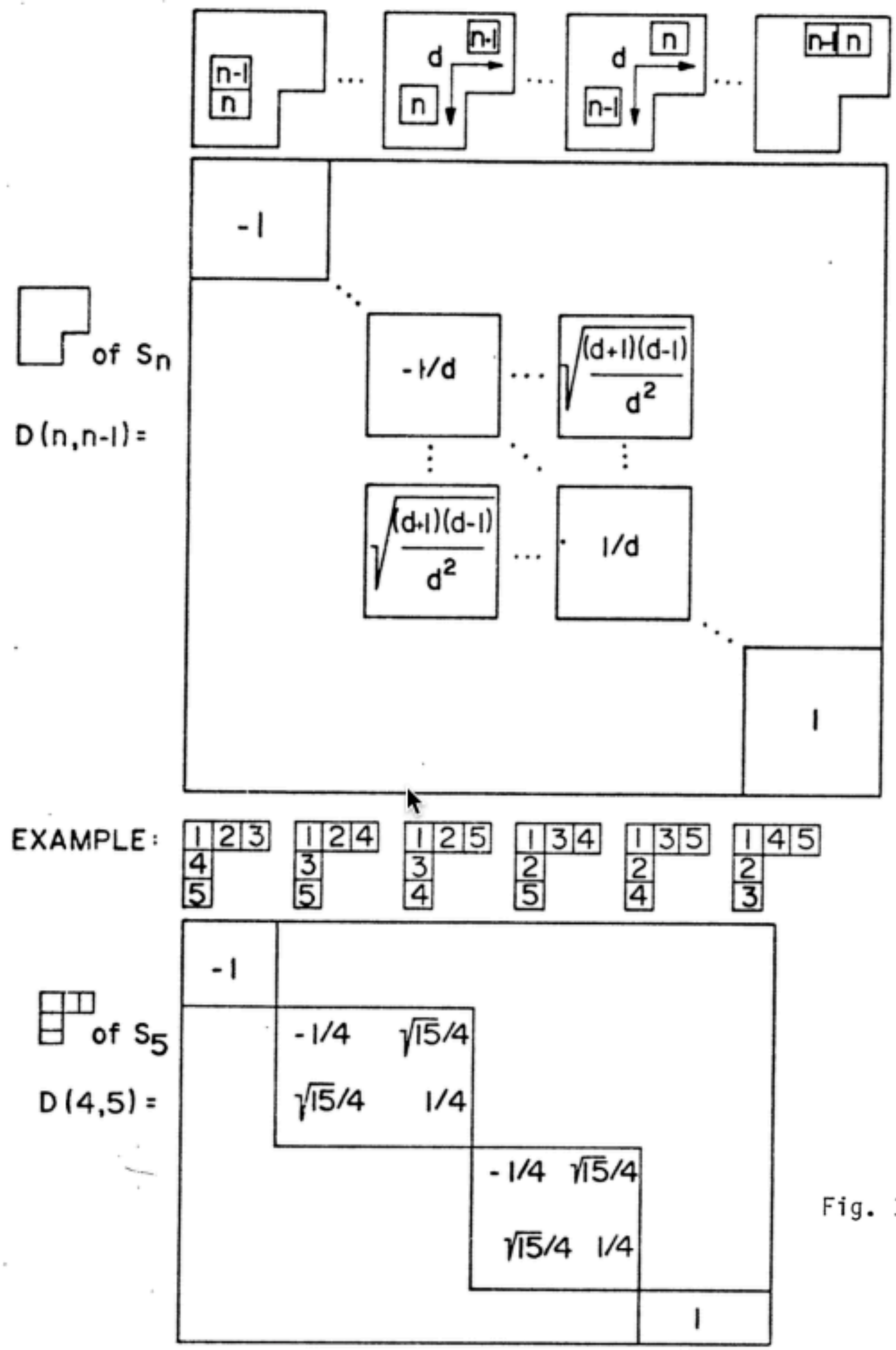
Multi-spin $(1/2)^N$ product state (Comparison to previous cases)



$$D^E_{(\sigma_2)} = D^{[2,1]}(bc) = \begin{matrix} \begin{smallmatrix} a\,b \\ c \end{smallmatrix} \\ \begin{smallmatrix} a\,c \\ b \end{smallmatrix} \end{matrix} \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$$

$$D^{[2,1]}(ab) = \begin{matrix} \begin{smallmatrix} a\,b \\ c \end{smallmatrix} \\ \begin{smallmatrix} a\,c \\ b \end{smallmatrix} \end{matrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

From unpublished Ch.10 for
Principles of Symmetry, Dynamics & Spectroscopy

Fig. 10.1.2 Yamanouchi formulas for permutation operators. Integer d is the "city block" distance between (n) and (n-1) blocks, i.e., the minimum number of streets to be crossed when traveling from one to the other. Note that when numbers (n) and (n-1) are ordered smaller above larger, the permutation is negative (anti-symmetric if d=1), and positive (symmetric if d=1) when the smaller number is left of the larger number. [The (n-1) will never be above <u>and</u> left of (n) since that arrangement would be "non-standard."]

AMOP reference links on page 2

4.02.18 class 20: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

Interwining ($S_1 \subset S_2 \subset S_3 \subset S_4 \subset S_5$ …)*(U(1)⊂U(2)⊂U(3)⊂U(4)⊂U(5) …) algebras and tensor operator applications to spinor-rotor or orbital correlations

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Multi-spin $(1/2)^N$ product state (Comparison to previous cases)

$\mathbf{g} =$	$\mathbf{1} = (1)(2)(3)$	$\mathbf{r} = (123)$	$\mathbf{r}^2 = (132)$	$\mathbf{i}_1 = (23)$	$\mathbf{i}_2 = (13)$	$\mathbf{i}_3 = (12)$
$D^{\square\square\square}(\mathbf{g}) =$						
	1	1	1	1	1	1
$D^{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}(\mathbf{g}) =$	1	1	1	-1	-1	-1
$D^{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}_{x_2 y_2}(\mathbf{g}) =$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$	$\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$	$\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$	$\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\mathbf{P}^{[\mu]}_{j,k} = \frac{\ell^{[\mu]}}{{}^{\circ}G}\left(D^{[\mu]}_{j,k}(1)\mathbf{(1)} + D^{[\mu]}_{j,k}(\mathbf{r})\mathbf{(123)} + D^{[\mu]}_{j,k}(\mathbf{r^2})\mathbf{(132)} + D^{[\mu]}_{j,k}(\mathbf{i}_1)\mathbf{(23)} + D^{[\mu]}_{j,k}(\mathbf{i}_2)\mathbf{(13)} + D^{[\mu]}_{j,k}(\mathbf{i}_3)\mathbf{(12)} \right)$$

$$\mathbf{P}^{\square\square\square} = \frac{1}{6}\left((1)\mathbf{(1)} + (1)\mathbf{(123)} + (1)\mathbf{(132)} + (1)\mathbf{(23)} + (1)\mathbf{(13)} + (1)\mathbf{(12)} \right)$$

	111	112	121	122	211	212	221	222
111	1							
112		1						
121			1					
122				1				
211					1			
212						1		
221							1	
222								1

[1][2][3]

	111	112	121	122	211	212	221	222
111	1							
112			1					
121					1			
122							1	
211		1						
212				1				
221						1		
222								1

[123]

	111	112	121	122	211	212	221	222
111	1							
112					1			
121		1						
122						1		
211			1					
212							1	
221				1				
222								1

[132]

	111	112	121	122	211	212	221	222
111	1							
112			1					
121		1						
122				1				
211					1			
212							1	
221						1		
222								1

[23]

	111	112	121	122	211	212	221	222
111	1							
112					1			
121			1					
122							1	
211		1						
212						1		
221				1				
222								1

[13]

	111	112	121	122	211	212	221	222
111	1							
112		1						
121					1			
122						1		
211			1					
212				1				
221							1	
222								1

[12]

$\mathbf{P}^{\square\square\square}$

	111	112	121	122	211	212	221	222
111	6							
112		2	2		2			
121		2	2		2			
122				2		2	2	
211		2	2		2			
212				2		2	2	
221				2		2	2	
222								6

$\frac{1}{6}$

$\mathbf{g}=$	$\mathbf{1}=(1)(2)(3)$	$\mathbf{r}=(123)$	$\mathbf{r}^2=(132)$	$\mathbf{i}_1=(23)$	$\mathbf{i}_2=(13)$	$\mathbf{i}_3=(12)$
$D^{\square\square\square}(\mathbf{g})=$	1	1	1	1	1	1
$D^{\substack{\square\\\square\\\square}}(\mathbf{g})=$	1	1	1	-1	-1	-1
$D^{\substack{\square\square\\\square}}_{x_2 y_2}(\mathbf{g})=$	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}-1/2 & -\sqrt{3}/2\\\sqrt{3}/2 & -1/2\end{pmatrix}$	$\begin{pmatrix}-1/2 & \sqrt{3}/2\\-\sqrt{3}/2 & -1/2\end{pmatrix}$	$\begin{pmatrix}-1/2 & \sqrt{3}/2\\\sqrt{3}/2 & 1/2\end{pmatrix}$	$\begin{pmatrix}-1/2 & -\sqrt{3}/2\\-\sqrt{3}/2 & 1/2\end{pmatrix}$	$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$

$$\mathbf{P}^{[\mu]}_{j,k} = \frac{\ell^{[\mu]}}{{}^{\circ}G}\left(D^{[\mu]}_{j,k}(\mathbf{1})\mathbf{(1)} + D^{[\mu]}_{j,k}(\mathbf{r})\mathbf{(123)} + D^{[\mu]}_{j,k}(\mathbf{r}^2)\mathbf{(132)} + D^{[\mu]}_{j,k}(\mathbf{i}_1)\mathbf{(23)} + D^{[\mu]}_{j,k}(\mathbf{i}_2)\mathbf{(13)} + D^{[\mu]}_{j,k}(\mathbf{i}_3)\mathbf{(12)}\right)$$

$$\mathbf{P}^{\square\square\square} = \frac{1}{6}\big((1)\mathbf{(1)} \quad + \quad (1)\mathbf{(123)} \quad + \quad (1)\mathbf{(132)} \quad + \quad (1)\mathbf{(23)} \quad + \quad (1)\mathbf{(13)} \quad + \quad (1)\mathbf{(12)}\big)$$

	111	112	121	122	211	212	221	222
111	1							
112		1						
121			1					
122				1				
211					1			
212						1		
221							1	
222								1

[1][2][3]

	111	112	121	122	211	212	221	222
111	1							
112			1					
121					1			
122							1	
211		1						
212				1				
221						1		
222								1

[123]

	111	112	121	122	211	212	221	222
111	1							
112					1			
121		1						
122						1		
211			1					
212							1	
221				1				
222								1

[132]

	111	112	121	122	211	212	221	222
111	1							
112			1					
121		1						
122				1				
211					1			
212							1	
221						1		
222								1

[23]

	111	112	121	122	211	212	221	222
111	1							
112					1			
121			1					
122							1	
211		1						
212						1		
221				1				
222								1

[13]

	111	112	121	122	211	212	221	222
111	1							
112		1						
121					1			
122						1		
211			1					
212				1				
221							1	
222								1

[12]

$\mathbf{P}^{\square\square\square}$

	111	112	121	122	211	212	221	222
111	6							
112		2	2		2			
121		2	2		2			
122				2		2	2	
211		2	2		2			
212				2		2	2	
221				2		2	2	
222								6

$\frac{1}{6}$

Difficult and tedious to sum?
Try MathType overlays (next page)

$\mathbf{g} =$	$\mathbf{1} = (1)(2)(3)$	$\mathbf{r} = (123)$	$\mathbf{r}^2 = (132)$	$\mathbf{i}_1 = (23)$	$\mathbf{i}_2 = (13)$	$\mathbf{i}_3 = (12)$
$D^{\square\square\square}(\mathbf{g}) =$						
$D^{\square\square\square\text{ (column)}}(\mathbf{g}) =$	1	1	1	1	1	1
	1	1	1	−1	−1	−1
$D^{\square\square,\square}_{x_2 y_2}(\mathbf{g}) =$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$	$\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$	$\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$	$\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

	111	112	121	122	211	212	221	222
111	1							
112		1						
121			1					
122				1				
211					1			
212						1		
221							1	
222								1

[1][2][3]

	111	112	121	122	211	212	221	222
111	1							
112		1						
121					1			
122						1		
211			1					
212				1				
221							1	
222								1

[12]

	111	112	121	122	211	212	221	222
111	1							
112					1			
121			1					
122							1	
211		1						
212						1		
221				1				
222								1

[13]

	111	112	121	122	211	212	221	222
111	1							
112			1					
121		1						
122				1				
211					1			
212							1	
221						1		
222								1

[23]

	111	112	121	122	211	212	221	222
111	1							
112			1					
121					1			
122							1	
211		1						
212				1				
221						1		
222								1

[123]

	111	112	121	122	211	212	221	222
111	1							
112					1			
121		1						
122						1		
211			1					
212							1	
221				1				
222								1

[132]

	111	112	121	122	211	212	221	222
111	111111							
112		11	11		11			
121		11	11		11			
122				11		11	11	
211		11	11		11			
212				11		11	11	
221				11		11	11	
222								111111

	6							
		2	2		2			
		2	2		2			
				2		2	2	
		2	2		2			
				2		2	2	
				2		2	2	
								6

$\mathbf{g} =$	$\mathbf{1} = (1)(2)(3)$	$\mathbf{r} = (123)$	$\mathbf{r}^2 = (132)$	$\mathbf{i}_1 = (23)$	$\mathbf{i}_2 = (13)$	$\mathbf{i}_3 = (12)$
$D^{□□□}(\mathbf{g}) =$	1	1	1	1	1	1
$D^{□□□\,(\text{column})}(\mathbf{g}) =$	1	1	1	−1	−1	−1
$D^{□□,□}_{x_2 y_2}(\mathbf{g}) =$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$	$\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$	$\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$	$\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\mathbf{P}^{[\mu]}_{j,k} = \frac{\ell^{[\mu]}}{{}^{\circ}G}\left(D^{[\mu]}_{j,k}(1)\mathbf{(1)} + D^{[\mu]}_{j,k}(\mathbf{r})\mathbf{(123)} + D^{[\mu]}_{j,k}(\mathbf{r^2})\mathbf{(132)} + D^{[\mu]}_{j,k}(\mathbf{i}_1)\mathbf{(23)} + D^{[\mu]}_{j,k}(\mathbf{i}_2)\mathbf{(13)} + D^{[\mu]}_{j,k}(\mathbf{i}_3)\mathbf{(12)} \right)$$

$$\mathbf{P}^{□□,□}_{11} = \frac{1}{6}\Big((2)\mathbf{(1)} \;+\; (-1)\mathbf{(123)} \;+\; (-1)\mathbf{(132)} \;+\; (-1)\mathbf{(23)} \;+\; (-1)\mathbf{(13)} \;+\; (+2)\mathbf{(12)} \Big)$$

	111	112	121	122	211	212	221	222
111	1							
112		1						
121			1					
122				1				
211					1			
212						1		
221							1	
222								1

[1][2][3]

	111	112	121	122	211	212	221	222
111	1							
112			1					
121					1			
122							1	
211		1						
212				1				
221						1		
222								1

[123]

	111	112	121	122	211	212	221	222
111	1							
112					1			
121		1						
122						1		
211			1					
212							1	
221				1				
222								1

[132]

	111	112	121	122	211	212	221	222
111	1							
112			1					
121		1						
122				1				
211					1			
212							1	
221						1		
222								1

[23]

	111	112	121	122	211	212	221	222
111	1							
112					1			
121			1					
122							1	
211		1						
212						1		
221				1				
222								1

[13]

	111	112	121	122	211	212	221	222
111	1							
112		1						
121					1			
122						1		
211			1					
212				1				
221							1	
222								1

[12]

$\mathbf{P}^{□□,□}_{11}$

	111	112	121	122	211	212	221	222
111								
112		1	-2		1			
121		−2	4		-2			
122				1		-2	1	
211		1	-2		1			
212				-2		4	-2	
221				1		-2	1	
222								

$\frac{1}{6}$

$\mathbf{g} =$	$\mathbf{1} = (1)(2)(3)$	$\mathbf{r} = (123)$	$\mathbf{r}^2 = (132)$	$\mathbf{i}_1 = (23)$	$\mathbf{i}_2 = (13)$	$\mathbf{i}_3 = (12)$
$D^{[3]}(\mathbf{g}) =$						
	1	1	1	1	1	1
$D^{[1^3]}(\mathbf{g}) =$	1	1	1	−1	−1	−1
$D^{[21]}_{x_2y_2}(\mathbf{g}) =$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$	$\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$	$\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$	$\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\mathbf{P}^{[\mu]}_{j,k} = \frac{\ell^{[\mu]}}{{}^{\circ}G}\Big(D^{[\mu]}_{j,k}(1)\mathbf{(1)} + D^{[\mu]}_{j,k}(\mathbf{r})\mathbf{(123)} + D^{[\mu]}_{j,k}(\mathbf{r^2})\mathbf{(132)} + D^{[\mu]}_{j,k}(\mathbf{i}_1)\mathbf{(23)} + D^{[\mu]}_{j,k}(\mathbf{i}_2)\mathbf{(13)} + D^{[\mu]}_{j,k}(\mathbf{i}_3)\mathbf{(12)}\Big)$$

$$\mathbf{P}^{[21]}_{21} = \frac{\sqrt{3}}{2}\Big((0)\mathbf{(1)} + (-1)\mathbf{(123)} + (+1)\mathbf{(132)} + (-1)\mathbf{(23)} + (0)\mathbf{(13)} + (+1)\mathbf{(12)}\Big)$$

	111	112	121	122	211	212	221	222
111	1							
112		1						
121			1					
122				1				
211					1			
212						1		
221							1	
222								1

[1][2][3]

	111	112	121	122	211	212	221	222
111	1							
112			1					
121					1			
122							1	
211		1						
212				1				
221						1		
222								1

[123]

	111	112	121	122	211	212	221	222
111	1							
112					1			
121		1						
122						1		
211			1					
212							1	
221				1				
222								1

[132]

	111	112	121	122	211	212	221	222
111	1							
112			1					
121		1						
122				1				
211					1			
212							1	
221						1		
222								1

[23]

	111	112	121	122	211	212	221	222
111	1							
112					1			
121			1					
122							1	
211		1						
212						1		
221				1				
222								1

[13]

	111	112	121	122	211	212	221	222
111	1							
112		1						
121					1			
122						1		
211			1					
212				1				
221							1	
222								1

[12]

$\mathbf{P}^{[21]}_{21}$

	111	112	121	122	211	212	221	222
111								
112		-1	2		-1			
121								
122				1		-2	1	
211		1	-2		1			
212								
221				-1		2	-1	
222								

$\frac{1}{6}$

AMOP reference links on page 2

4.02.18 class 20: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

Interwining ($S_1 \subset S_2 \subset S_3 \subset S_4 \subset S_5$ …)*(U(1)⊂U(2)⊂U(3)⊂U(4)⊂U(5) …) algebras and tensor operator applications to spinor-rotor or orbital correlations

U(2) tensor product states and S_n permutation symmetry

Rank-1 tensor (or spinor)

Rank-2 tensor (2 particles each with U(2) state space)

2-particle U(2) transform and permutation operation

S_2 symmetry of U(2): Trust but verify

Applying S_2 projection to build DTran

Applying DTran for S_2

Applying DTran for U(2)

S_3 permutations related to C_{3v}~D_3 geometry

S_3 permutation matrices

Hooklength formula for S_n reps

➡ S_3 symmetry of U(2): Applying S_3 projection (Note Pauli-exclusion principle basis)

Building S_3 DTran T from projectors

Effect of S_3 DTran T: Introducing intertwining S_3 - U(2) irep matrices

Multi-spin $(1/2)^N$ product state (Comparison to previous cases)

Note all $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}$ (totally antisymmetric) U(2) (spin-½) states $\begin{array}{|c|}\hline \uparrow \\ \hline \uparrow \\ \hline \uparrow \\ \hline\end{array}$ $\begin{array}{|c|}\hline \uparrow \\ \hline \uparrow \\ \hline \downarrow \\ \hline\end{array}$ $\begin{array}{|c|}\hline \uparrow \\ \hline \downarrow \\ \hline \downarrow \\ \hline\end{array}$ $\begin{array}{|c|}\hline \downarrow \\ \hline \downarrow \\ \hline \downarrow \\ \hline\end{array}$ are non-existent .

$$\left| \begin{array}{c} \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array} \\ \begin{array}{|c|}\hline \uparrow \\ \hline \uparrow \\ \hline \uparrow \\ \hline\end{array} \end{array} \right\rangle = \mathbf{P}^{\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}}_{\begin{array}{|c|}\hline a \\ \hline b \\ \hline c \\ \hline\end{array}\ \begin{array}{|c|}\hline \uparrow \\ \hline \uparrow \\ \hline \uparrow \\ \hline\end{array}} \left\| \uparrow,\uparrow,\uparrow \right\rangle\!\right\rangle = 0 \quad (\textit{Does not exist}), \left| \begin{array}{c} \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array} \\ \begin{array}{|c|}\hline \uparrow \\ \hline \uparrow \\ \hline \downarrow \\ \hline\end{array} \end{array} \right\rangle = \mathbf{P}^{\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}}_{\begin{array}{|c|}\hline a \\ \hline b \\ \hline c \\ \hline\end{array}\ \begin{array}{|c|}\hline \uparrow \\ \hline \uparrow \\ \hline \downarrow \\ \hline\end{array}} \left\| \uparrow,\uparrow,\downarrow \right\rangle\!\right\rangle = 0 \quad (\textit{Does not exist}), \ldots etc.$$

It takes at least 3 distinct (U(3)) states to make a 3rd rank "determinant" state $\begin{array}{|c|}\hline a \\ \hline b \\ \hline c \\ \hline\end{array}$.

This is the symmetry basis of the Pauli-exclusion principle.

AMOP reference links on page 2

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Interwining ($S_1 \subset S_2 \subset S_3 \subset S_4 \subset S_5$ …)*(U(1)⊂U(2)⊂U(3)⊂U(4)⊂U(5) …) algebras and tensor operator applications to spinor-rotor or orbital correlations

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S$_3$ *symmetry* of U(2): *Building S$_3$* DTran *T from projectors*

$\mathbf{P}^{\square\square\square} = \frac{1}{6}\times$

	111	112	121	122	211	212	221	222
111	6							
112		2	2		2			
121		2	2		2			
122				2		2	2	
211		2	2		2			
212				2		2	2	
221				2		2	2	
222								6

$\mathbf{P}^{\square\square,\square}_{11} = \frac{1}{6}\times$

	111	112	121	122	211	212	221	222
111								
112		1	-2		1			
121		−2	4		-2			
122				1		-2	1	
211		1	-2		1			
212				-2		4	-2	
221				1		-2	1	
222								

$\mathbf{P}^{\square\square,\square}_{21} = \frac{1}{6}\times$

	111	112	121	122	211	212	221	222
111								
112		-1	2		-1			
121								
122				1		-2	1	
211		1	-2		1			
212								
221				-1		2	-1	
222								

$T=$

	$\square\square\square_\alpha$	$\square\square\square_\beta$	$\square\square\square_\gamma$	$\square\square\square_\delta$	$\square\square,\square_{11}$	$\square\square,\square_{21}$	$\square\square,\square_{12}$	$\square\square,\square_{22}$
111	1							
112		$\frac{1}{\sqrt{3}}$			$\frac{1}{\sqrt{6}}$	$\frac{-1}{\sqrt{2}}$		
121		$\frac{1}{\sqrt{3}}$			$\frac{-2}{\sqrt{6}}$			
122			$\frac{1}{\sqrt{3}}$				$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{2}}$
211		$\frac{1}{\sqrt{3}}$			$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{2}}$		
212			$\frac{1}{\sqrt{3}}$				$\frac{-2}{\sqrt{6}}$	
221			$\frac{1}{\sqrt{3}}$				$\frac{1}{\sqrt{6}}$	$\frac{-1}{\sqrt{2}}$
222				1				

AMOP reference links on page 2

4.02.18 class 20: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

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Interwining ($S_1 \subset S_2 \subset S_3 \subset S_4 \subset S_5$ …)*($U(1) \subset U(2) \subset U(3) \subset U(4) \subset U(5)$ …) algebras and tensor operator applications to spinor-rotor or orbital correlations

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    - ➡ Effect of S_3 DTran T: Introducing intertwining S_3 - U(2) irep matrices
    - Multi-spin $(1/2)^N$ product state (Comparison to previous cases)

S3 *symmetry* of U(2): *Effect of* S_3 DTran T *on intertwining* S_3 *- U(2) irep matrices*

S_3 matrices:

U(2) matrices:

$$T^\dagger S(\mathbf{p}_{abc})T =$$

	$D(\mathbf{p})$							
		$D(\mathbf{p})$						
			$D(\mathbf{p})$					
				$D(\mathbf{p})$				
					$D_{11}(\mathbf{p})$	$D_{12}(\mathbf{p})$		
					$D_{21}(\mathbf{p})$	$D_{22}(\mathbf{p})$		
							$D_{11}(\mathbf{p})$	$D_{12}(\mathbf{p})$
							$D_{21}(\mathbf{p})$	$D_{22}(\mathbf{p})$

$$T^\dagger D\otimes D\otimes D(\mathbf{u})T =$$

	$D_{11}(\mathbf{u})$	$D_{12}(\mathbf{u})$	$D_{13}(\mathbf{u})$	$D_{14}(\mathbf{u})$				
	$D_{21}(\mathbf{u})$	$D_{22}(\mathbf{u})$	$D_{23}(\mathbf{u})$	$D_{24}(\mathbf{u})$				
	$D_{31}(\mathbf{u})$	$D_{31}(\mathbf{u})$	$D_{31}(\mathbf{u})$	$D_{31}(\mathbf{u})$				
	$D_{41}(\mathbf{u})$	$D_{42}(\mathbf{u})$	$D_{43}(\mathbf{u})$	$D_{44}(\mathbf{u})$				
					$D_{11}(\mathbf{u})$		$D_{12}(\mathbf{u})$	
						$D_{12}(\mathbf{u})$		$D_{12}(\mathbf{u})$
					$D_{21}(\mathbf{u})$		$D_{22}(\mathbf{u})$	
						$D_{21}(\mathbf{u})$		$D_{22}(\mathbf{u})$

S_3 *symmetry* of U(2): *Effect of S_3* DTran T *on intertwining S_3 - U(2) irep matrices*

S_3 matrices:

U(2) matrices:

$$T^{\dagger} S(\mathbf{p}_{abc}) T =$$

	$D(\mathbf{p})$							
		$D(\mathbf{p})$						
			$D(\mathbf{p})$					
				$D(\mathbf{p})$				
					$D_{11}(\mathbf{p})$	$D_{12}(\mathbf{p})$		
					$D_{21}(\mathbf{p})$	$D_{22}(\mathbf{p})$		
							$D_{11}(\mathbf{p})$	$D_{12}(\mathbf{p})$
							$D_{21}(\mathbf{p})$	$D_{22}(\mathbf{p})$

$$T^{\dagger} D \otimes D \otimes D(\mathbf{u}) T =$$

	$D_{11}(\mathbf{u})$	$D_{12}(\mathbf{u})$	$D_{13}(\mathbf{u})$	$D_{14}(\mathbf{u})$				
	$D_{21}(\mathbf{u})$	$D_{22}(\mathbf{u})$	$D_{23}(\mathbf{u})$	$D_{24}(\mathbf{u})$				
	$D_{31}(\mathbf{u})$	$D_{31}(\mathbf{u})$	$D_{31}(\mathbf{u})$	$D_{31}(\mathbf{u})$				
	$D_{41}(\mathbf{u})$	$D_{42}(\mathbf{u})$	$D_{43}(\mathbf{u})$	$D_{44}(\mathbf{u})$				
					$D_{11}(\mathbf{u})$		$D_{12}(\mathbf{u})$	
						$D_{12}(\mathbf{u})$		$D_{12}(\mathbf{u})$
					$D_{21}(\mathbf{u})$		$D_{22}(\mathbf{u})$	
						$D_{21}(\mathbf{u})$		$D_{22}(\mathbf{u})$

After flipping rows and columns (6⇔7) of T matrix

$$T_{\substack{67\\ flip}}{}^{\dagger} S(\mathbf{p}_{abc}) T_{\substack{67\\ flip}} =$$

	$D(\mathbf{p})$							
		$D(\mathbf{p})$						
			$D(\mathbf{p})$					
				$D(\mathbf{p})$				
					$D_{11}(\mathbf{p})$		$D_{12}(\mathbf{p})$	
						$D_{11}(\mathbf{p})$		$D_{12}(\mathbf{p})$
					$D_{21}(\mathbf{p})$		$D_{22}(\mathbf{p})$	
						$D_{21}(\mathbf{p})$		$D_{22}(\mathbf{p})$

$$T_{\substack{67\\ flip}}{}^{\dagger} D \otimes D \otimes D(\mathbf{u}) T_{\substack{67\\ flip}} =$$

	$D_{11}(\mathbf{u})$	$D_{12}(\mathbf{u})$	$D_{13}(\mathbf{u})$	$D_{14}(\mathbf{u})$				
	$D_{21}(\mathbf{u})$	$D_{22}(\mathbf{u})$	$D_{23}(\mathbf{u})$	$D_{24}(\mathbf{u})$				
	$D_{31}(\mathbf{u})$	$D_{31}(\mathbf{u})$	$D_{31}(\mathbf{u})$	$D_{31}(\mathbf{u})$				
	$D_{41}(\mathbf{u})$	$D_{42}(\mathbf{u})$	$D_{43}(\mathbf{u})$	$D_{44}(\mathbf{u})$				
					$D_{11}(\mathbf{u})$	$D_{12}(\mathbf{u})$		
					$D_{21}(\mathbf{u})$	$D_{22}(\mathbf{u})$		
							$D_{11}(\mathbf{u})$	$D_{12}(\mathbf{u})$
							$D_{21}(\mathbf{u})$	$D_{22}(\mathbf{u})$

S_3 *symmetry* of U(2): *Effect of S_3* DTran *T on intertwining S_3 - U(2) irep matrices*

S_3 matrices:

$T^\dagger S(\mathbf{p}_{abc})T =$

Two 2-by-2 D□□ ($\mathbf{u}$) ireps of S_3

$D(\mathbf{p})$							
	$D(\mathbf{p})$						
		$D(\mathbf{p})$					
			$D(\mathbf{p})$				
				$D_{11}(\mathbf{p})$	$D_{12}(\mathbf{p})$		
				$D_{21}(\mathbf{p})$	$D_{22}(\mathbf{p})$		
						$D_{11}(\mathbf{p})$	$D_{12}(\mathbf{p})$
						$D_{21}(\mathbf{p})$	$D_{22}(\mathbf{p})$

U(2) matrices:

$T^\dagger D\otimes D\otimes D(\mathbf{u})T =$

$D_{11}(\mathbf{u})$	$D_{12}(\mathbf{u})$	$D_{13}(\mathbf{u})$	$D_{14}(\mathbf{u})$				
$D_{21}(\mathbf{u})$	$D_{22}(\mathbf{u})$	$D_{23}(\mathbf{u})$	$D_{24}(\mathbf{u})$				
$D_{31}(\mathbf{u})$	$D_{31}(\mathbf{u})$	$D_{31}(\mathbf{u})$	$D_{31}(\mathbf{u})$				
$D_{41}(\mathbf{u})$	$D_{42}(\mathbf{u})$	$D_{43}(\mathbf{u})$	$D_{44}(\mathbf{u})$				
				$D_{11}(\mathbf{u})$		$D_{12}(\mathbf{u})$	
					$D_{12}(\mathbf{u})$		$D_{12}(\mathbf{u})$
				$D_{21}(\mathbf{u})$		$D_{22}(\mathbf{u})$	
					$D_{21}(\mathbf{u})$		$D_{22}(\mathbf{u})$

After flipping rows and columns (6⇔7) of T matrix

$T_{\substack{67\\flip}}{}^\dagger S(\mathbf{p}_{abc})T_{\substack{67\\flip}} =$

Four 1-by-1 D□□□($\mathbf{p}$) S_3 ireps

$D(\mathbf{p})$							
	$D(\mathbf{p})$						
		$D(\mathbf{p})$					
			$D(\mathbf{p})$				
				$D_{11}(\mathbf{p})$		$D_{12}(\mathbf{p})$	
					$D_{11}(\mathbf{p})$		$D_{12}(\mathbf{p})$
				$D_{21}(\mathbf{p})$		$D_{22}(\mathbf{p})$	
					$D_{21}(\mathbf{p})$		$D_{22}(\mathbf{p})$

$T_{\substack{67\\flip}}{}^\dagger D\otimes D\otimes D(\mathbf{u})T_{\substack{67\\flip}} =$

One 4-by-4 D□□□($\mathbf{u}$) $= D^{\frac{3}{2}}$ irep

$D_{11}(\mathbf{u})$	$D_{12}(\mathbf{u})$	$D_{13}(\mathbf{u})$	$D_{14}(\mathbf{u})$				
$D_{21}(\mathbf{u})$	$D_{22}(\mathbf{u})$	$D_{23}(\mathbf{u})$	$D_{24}(\mathbf{u})$				
$D_{31}(\mathbf{u})$	$D_{31}(\mathbf{u})$	$D_{31}(\mathbf{u})$	$D_{31}(\mathbf{u})$				
$D_{41}(\mathbf{u})$	$D_{42}(\mathbf{u})$	$D_{43}(\mathbf{u})$	$D_{44}(\mathbf{u})$				
				$D_{11}(\mathbf{u})$	$D_{12}(\mathbf{u})$		
				$D_{21}(\mathbf{u})$	$D_{22}(\mathbf{u})$		
						$D_{11}(\mathbf{u})$	$D_{12}(\mathbf{u})$
						$D_{21}(\mathbf{u})$	$D_{22}(\mathbf{u})$

Two 2-by-2 D□□ ($\mathbf{u}$) $= D^{\frac{1}{2}}$ ireps

AMOP reference links on page 2

4.02.18 class 20: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

Interwining ($S_1 \subset S_2 \subset S_3 \subset S_4 \subset S_5$ …)*(U(1)⊂U(2)⊂U(3)⊂U(4)⊂U(5) …) algebras and tensor operator applications to spinor-rotor or orbital correlations

U(2) tensor product states and S_n permutation symmetry

Rank-1 tensor (or spinor)

Rank-2 tensor (2 particles each with U(2) state space)

2-particle U(2) transform and permutation operation

S_2 symmetry of U(2): Trust but verify

Applying S_2 projection to build DTran

Applying DTran for S_2

Applying DTran for U(2)

S_3 permutations related to C_{3v}~D_3 geometry

S_3 permutation matrices

Hooklength formula for S_n reps

S_3 symmetry of U(2): Applying S_3 projection (Note Pauli-exclusion principle basis)

Building S_3 DTran T from projectors

Effect of S_3 DTran T: Introducing intertwining S_3 - U(2) irep matrices

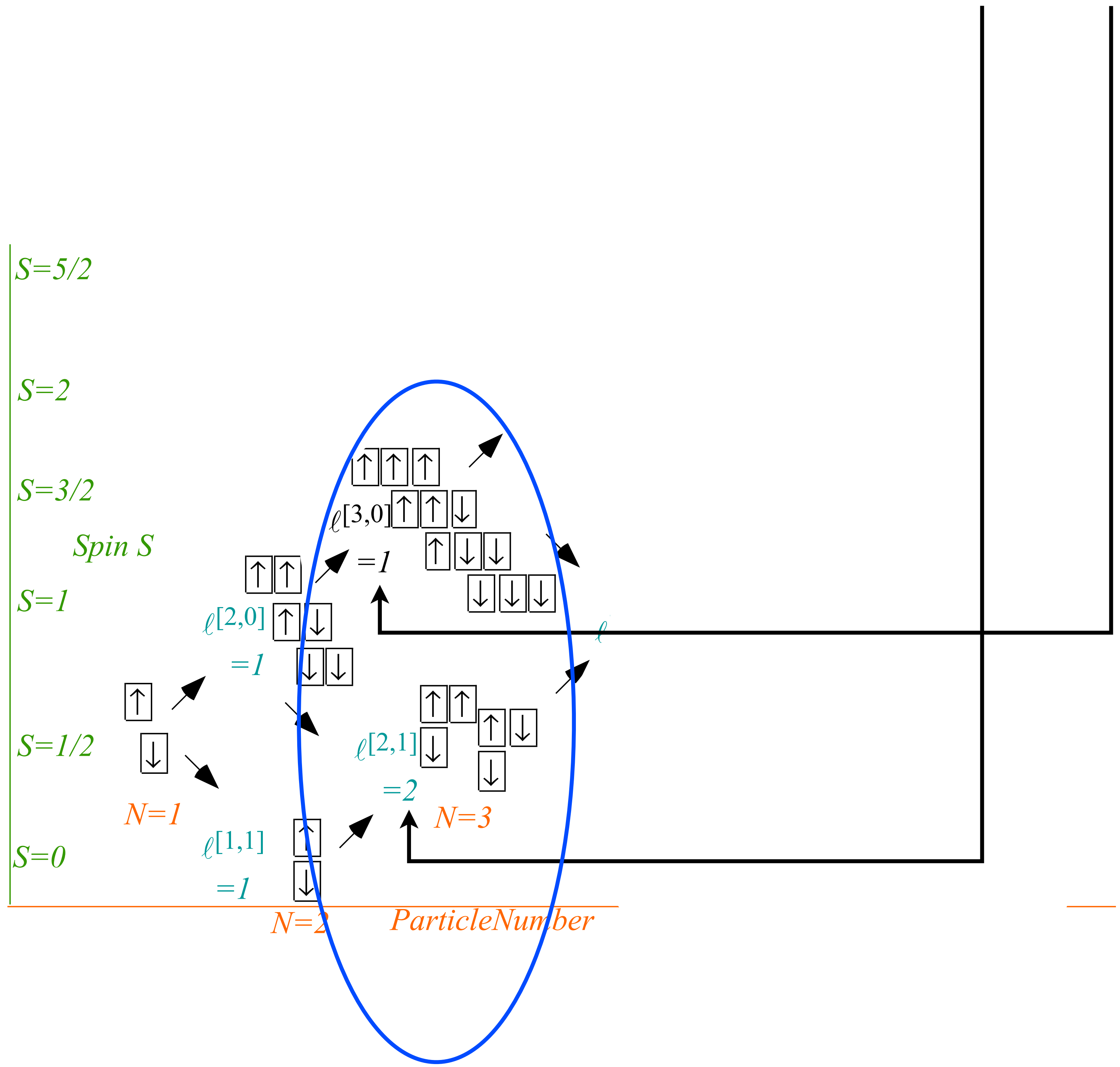
➡ Multi-spin $(1/2)^N$ product state (Comparison to previous cases)

Multi-spin (1/2)^N product states

$$(d^{\frac{1}{2}} \otimes d^{\frac{1}{2}}) = d^0 + d^1$$

$$(d^{\frac{1}{2}} \otimes d^{\frac{1}{2}}) \otimes d^{\frac{1}{2}} = (d^0 + d^1) \otimes d^{\frac{1}{2}} = d^0 \otimes d^{\frac{1}{2}} + d^1 \otimes d^{\frac{1}{2}}$$

$$= \quad d^{\frac{1}{2}} + \quad d^{\frac{1}{2}} + d^{\frac{3}{2}} = 2d^{\frac{1}{2}} + 1d^{\frac{3}{2}}$$



Multi-spin $(1/2)^N$ product states

$$2^N = \sum_S^{N/2} \ell^{[S]}\ell^{[\mu_1,\mu_2]} = \sum_S^{N/2} (2S+1)\,\ell^{\left[\frac{N+2S}{2},\frac{N-2S}{2}\right]}$$

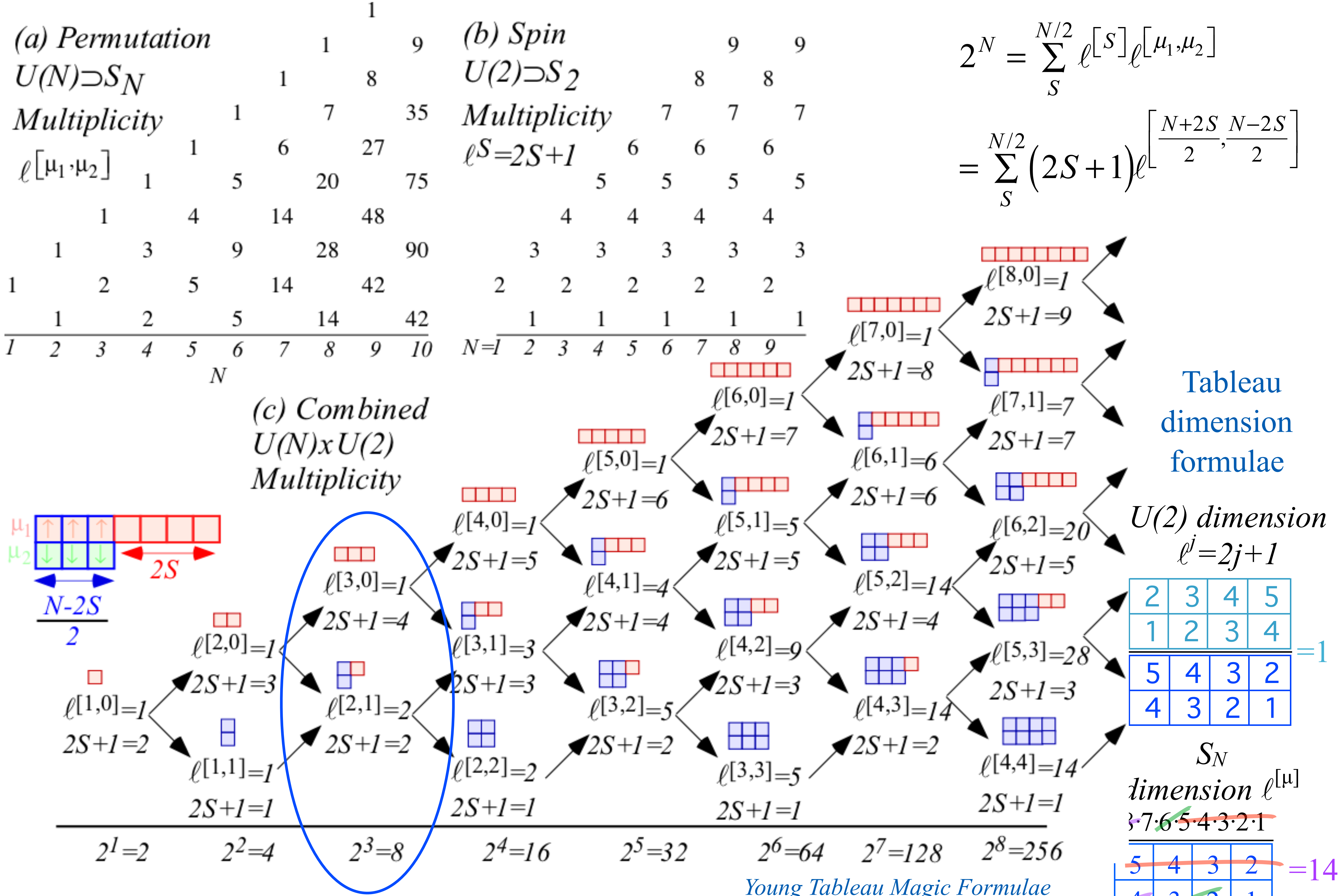


Fig. 23.3.2 *Spin-1/2 and U(2) Tableau branching diagrams*

Young Tableau Magic Formulae
GrpThLect29 p46-48.

$\mathbf{g} =$	$\mathbf{1} = (1)(2)(3)$	$\mathbf{r} = (123)$	$\mathbf{r}^2 = (132)$	$\mathbf{i}_1 = (23)$	$\mathbf{i}_2 = (13)$	$\mathbf{i}_3 = (12)$
$D^{□□□}(\mathbf{g}) =$	1	1	1	1	1	1
$D^{□}(\mathbf{g}) =$	1	1	1	-1	-1	-1
$D^{□□}_{x_2y_2}(\mathbf{g}) =$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$	$\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$	$\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$	$\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

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122				1				
211					1			
212						1		
221							1	
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	111	112	121	122	211	212	221	222
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$\langle 121\|$	·	·	·	·	·	·	·	·
$\langle 122\|$	·	·	·	·	·	·	·	·
$\langle 211\|$	·	·	·	·	·	·	·	·
$\langle 212\|$	·	·	·	·	·	·	·	·
$\langle 221\|$	·	·	·	·	·	·	·	·
$\langle 222\|$	·	·	·	·	·	·	·	·