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GLOBAL vs LOCAL symmetry of states *...and group* **H** *parameters$\{r,i_1,i_2,i_3\}$*

AMOP reference links *(Updated list given on 2nd page of each class presentation)*

Web Resources - front page | Quantum Theory for the Computer Age | 2014 AMOP

UAF Physics UTube channel | Principles of Symmetry, Dynamics, and Spectroscopy | 2017 Group Theory for QM

Classical Mechanics with a Bang! | 2018 AMOP

Modern Physics and its Classical Foundations

Frame Transformation Relations And Multipole Transitions In Symmetric Polyatomic Molecules - RMP-1978
Rotational energy surfaces and high- J eigenvalue structure of polyatomic molecules - Harter - Patterson - 1984
Galloping waves and their relativistic properties - ajp-1985-Harter
Asymptotic eigensolutions of fourth and sixth rank octahedral tensor operators - Harter-Patterson-JMP-1979
Nuclear spin weights and gas phase spectral structure of 12C60 and 13C60 buckminsterfullerene -Harter-Reimer-Cpl-1992 - (Alt1, Alt2 Erratum)

Theory of hyperfine and superfine levels in symmetric polyatomic molecules.

I) Trigonal and tetrahedral molecules: Elementary spin-1/2 cases in vibronic ground states - PRA-1979-Harter-Patterson (Alt scan)
II) Elementary cases in octahedral hexafluoride molecules - Harter-PRA-1981 (Alt scan)

Rotation-vibration scalar coupling zeta coefficients and spectroscopic band shapes of buckminsterfullerene - Weeks-Harter-CPL-1991 (Alt scan)
Fullerene symmetry reduction and rotational level fine structure/ the Buckyball isotopomer 12C 13C59 - jcp-Reimer-Harter-1997 (HiRez)
Molecular Eigensolution Symmetry Analysis and Fine Structure - IJMS-harter-mitchell-2013

Rotation–vibration spectra of icosahedral molecules.

I) Icosahedral symmetry analysis and fine structure - harter-weeks-jcp-1989
II) Icosahedral symmetry, vibrational eigenfrequencies, and normal modes of buckminsterfullerene - weeks-harter-jcp-1989
III) Half-integral angular momentum - harter-reimer-jcp-1991

QTCA Unit 10 Ch 30 - 2013
AMOP Ch 32 Molecular Symmetry and Dynamics - 2019
AMOP Ch 0 Space-Time Symmetry - 2019

RESONANCE AND REVIVALS

I) QUANTUM ROTOR AND INFINITE-WELL DYNAMICS - ISMSLi2012 (Talk) OSU knowledge Bank
II) Comparing Half-integer Spin and Integer Spin - Alva-ISMS-Ohio2013-R777 (Talks)
III) Quantum Resonant Beats and Revivals in the Morse Oscillators and Rotors - (2013-Li-Diss)

Rovibrational Spectral Fine Structure Of Icosahedral Molecules - Cpl 1986 (Alt Scan)
Gas Phase Level Structure of C60 Buckyball and Derivatives Exhibiting Broken Icosahedral Symmetry - reimer-diss-1996
Resonance and Revivals in Quantum Rotors - Comparing Half-integer Spin and Integer Spin - Alva-ISMS-Ohio2013-R777 (Talk)
Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-cpl-2013
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - harter - jms - 2001

**In development - a web based A.M.O.P. oriented reference page, with thumbnail/previews, greater control over the information display, and eventually full on* Apache-SOLR *Index and search for* nuanced, whole-site content/metadata level *searching. This bad boy will be a sure force multiplier.*

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Crystal-Point Group Zoo having 32 groups (Showing 16 Abelian Crystal Groups)

Fig. 2.11.1 PSDS

The other 16 crystal-point groups are Non-Abelian

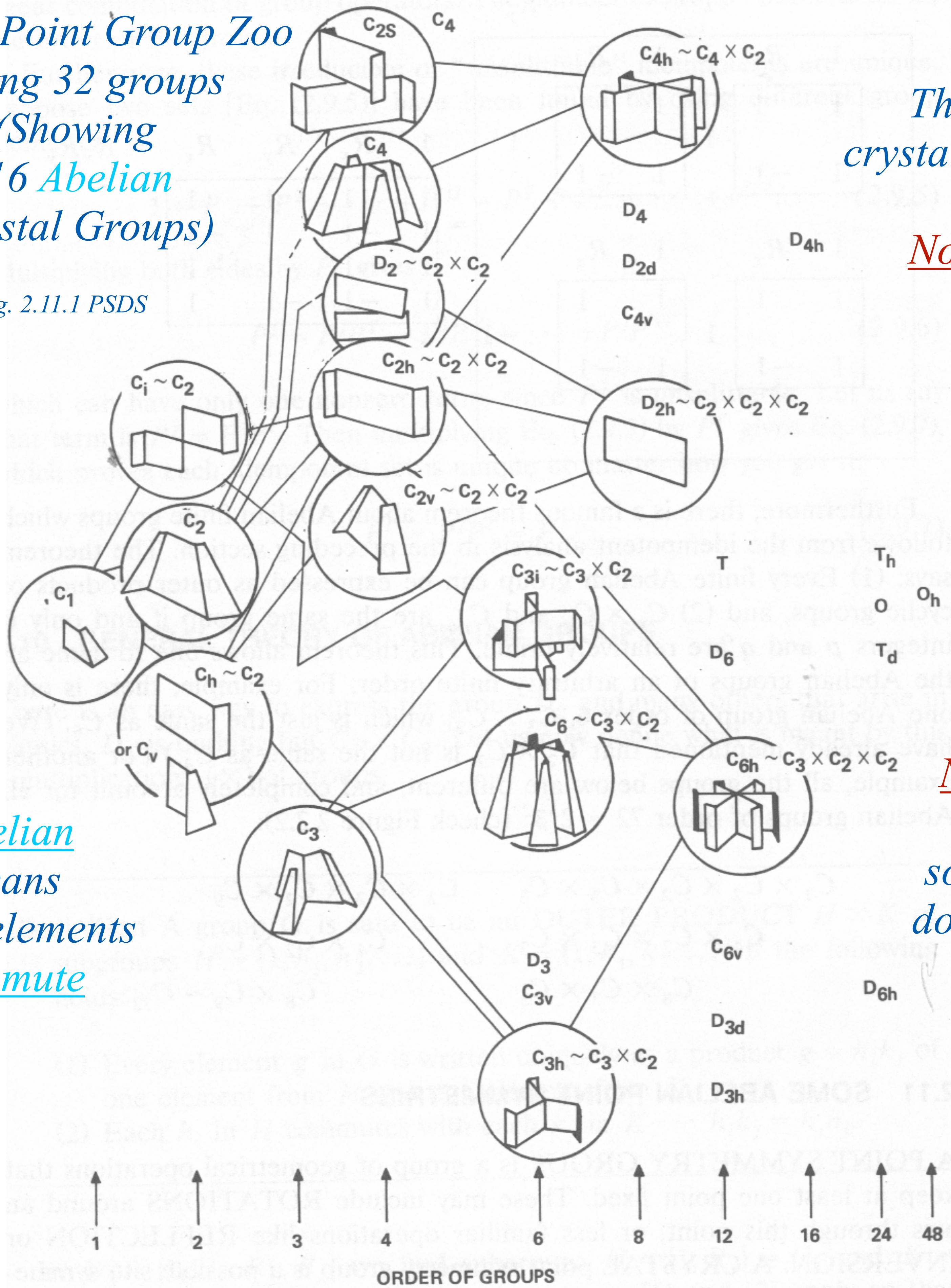

Figure 2.11.1 Abelian crystal point groups. Sixteen of the 32 crystal point groups are Abelian and are illustrated by models drawn in circles.

Abelian means all its elements commute

Non-Abelian means some elements do not commute

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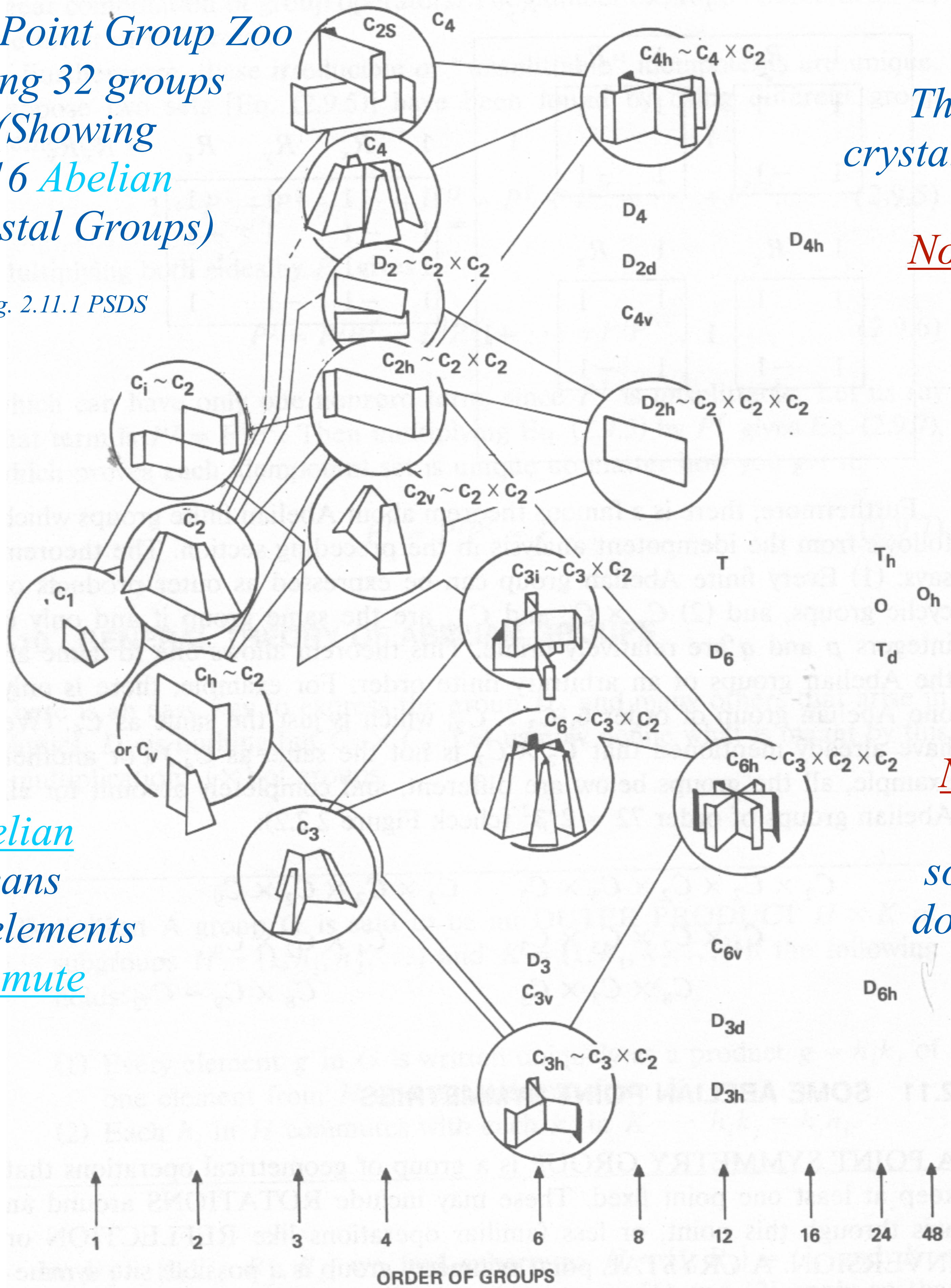
From GTQM Lecture 12.6 p. 134
Character Trace of n-fold rotation
where: $\ell^j = 2j+1$
is U(2) irrep dimension

$$\chi^j\left(\frac{2\pi}{n}\right) = \frac{\sin\frac{\pi}{n}(2j+1)}{\sin\frac{\pi}{n}} = \frac{\sin\frac{\pi\ell^j}{n}}{\sin\frac{\pi}{n}}$$



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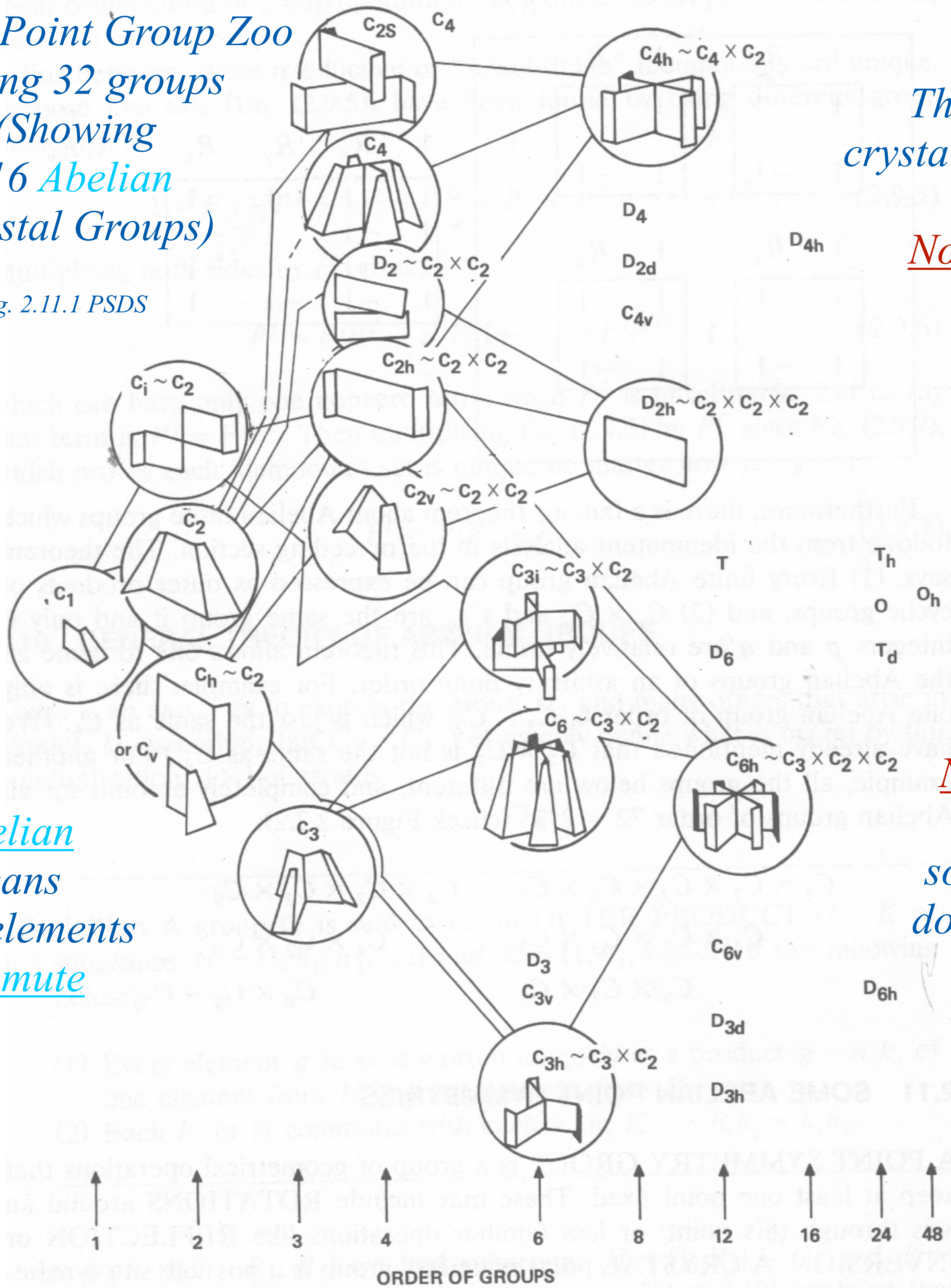

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$$\chi^j\left(\frac{2\pi}{n}\right) = \frac{\sin\frac{\pi}{n}(2j+1)}{\sin\frac{\pi}{n}} = \frac{\sin\frac{\pi\ell^j}{n}}{\sin\frac{\pi}{n}}$$

To be a crystal-point group the Character Trace of *n*-fold vector rotation
for: $\ell^1 = 2+1 = 3$
must be an integer

$$\chi^1\left(\frac{2\pi}{n}\right) = \frac{\sin\frac{\pi}{1}(2j+1)}{\sin\frac{\pi}{n}} = \frac{\sin\frac{3\pi}{n}}{\sin\frac{\pi}{n}} = \text{integer}$$

<u>Non</u>-Abelian means some elements do <u>not</u> commute

$$\frac{\sin\frac{3\pi}{2}}{\sin\frac{\pi}{2}} = -1 \ (n{=}2 \text{ ok})$$

$$\frac{\sin\frac{3\pi}{3}}{\sin\frac{\pi}{3}} = +1 \ (n{=}3 \text{ ok})$$

$$\frac{\sin\frac{3\pi}{4}}{\sin\frac{\pi}{4}} = +1 \ (n{=}4 \text{ ok})$$

$$\frac{\sin\frac{3\pi}{5}}{\sin\frac{\pi}{5}} = G^+ \ (n{=}5 \text{ NO!})$$

$$\frac{\sin\frac{3\pi}{6}}{\sin\frac{\pi}{6}} = +2 \ (n{=}6 \text{ ok})$$

<u>Abelian</u> means <u>all</u> its elements <u>commute</u>

Crystal-Point Group Zoo having 32 groups (Showing 16 Abelian Crystal Groups)

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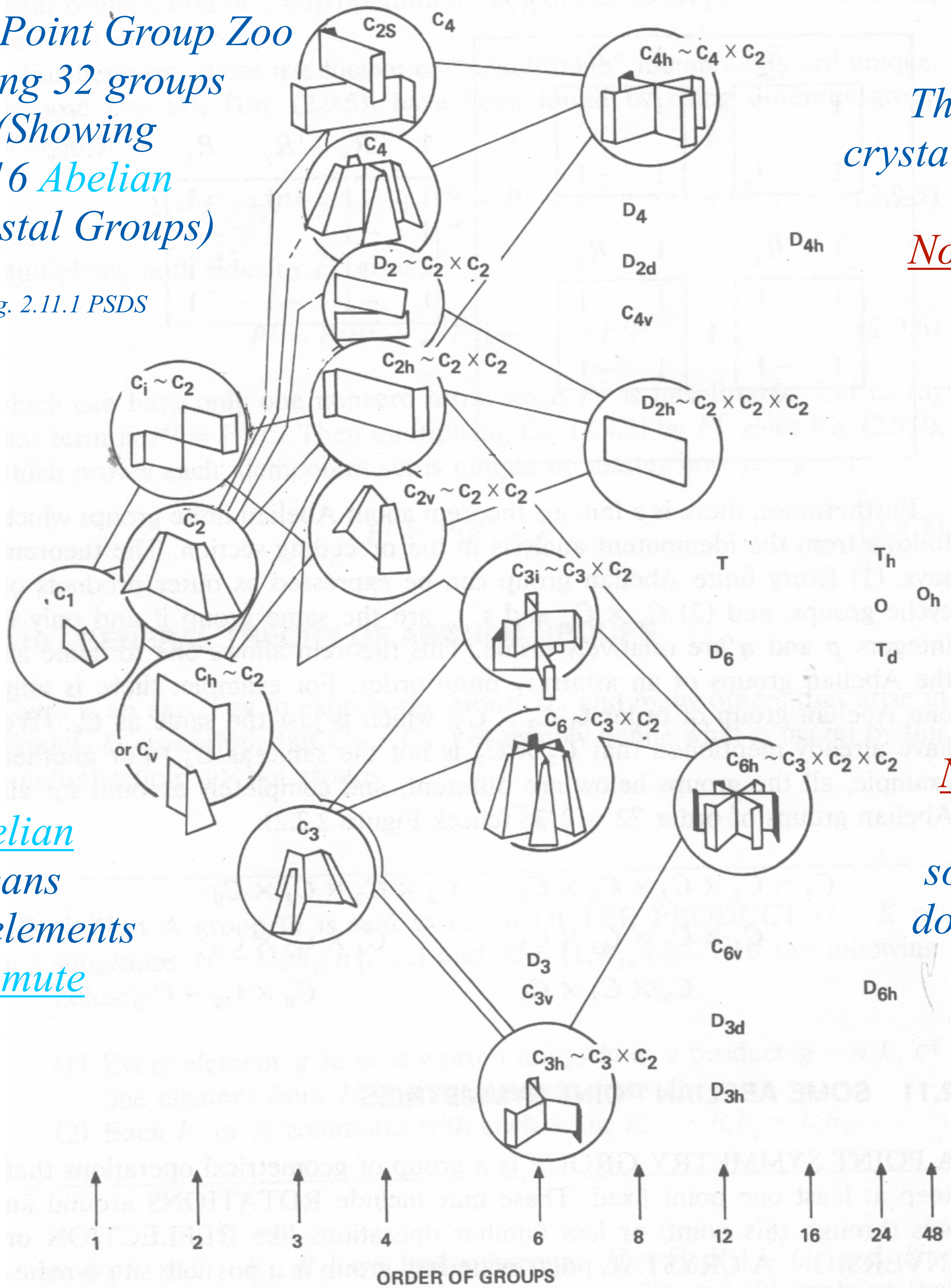

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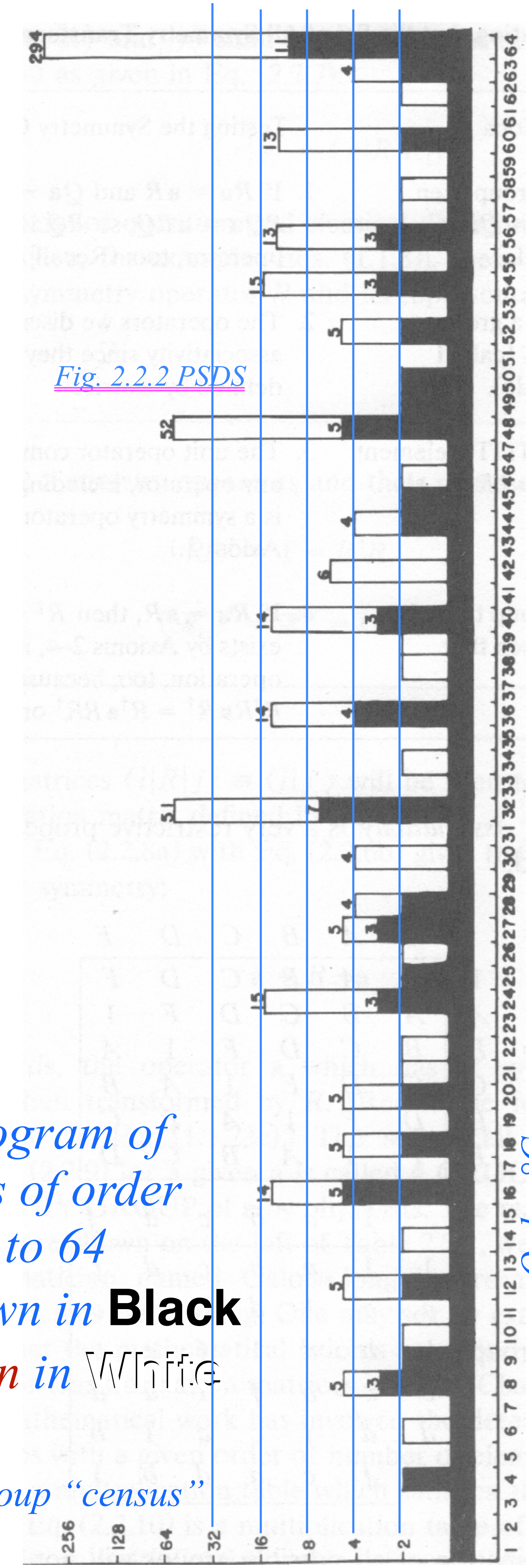


Log-histogram of all groups of order °G=1 to 64

Abelian shown in **Black**

Non-Abelian in White

Group "census"

Crystal-Point Group Zoo having 32 groups (Showing 16 Non-Abelian Crystal Groups)

Fig. 2.11.1 PSDS

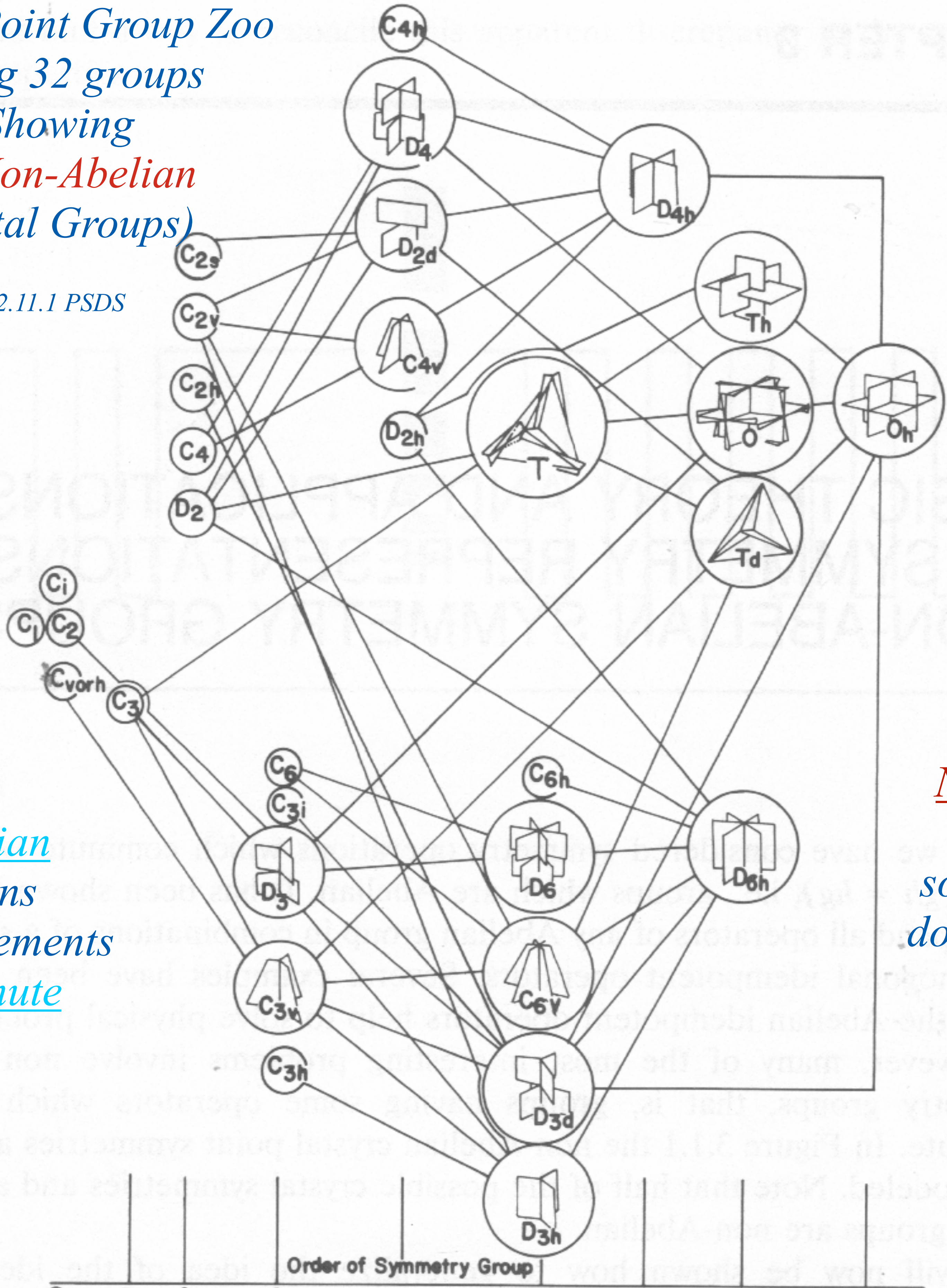


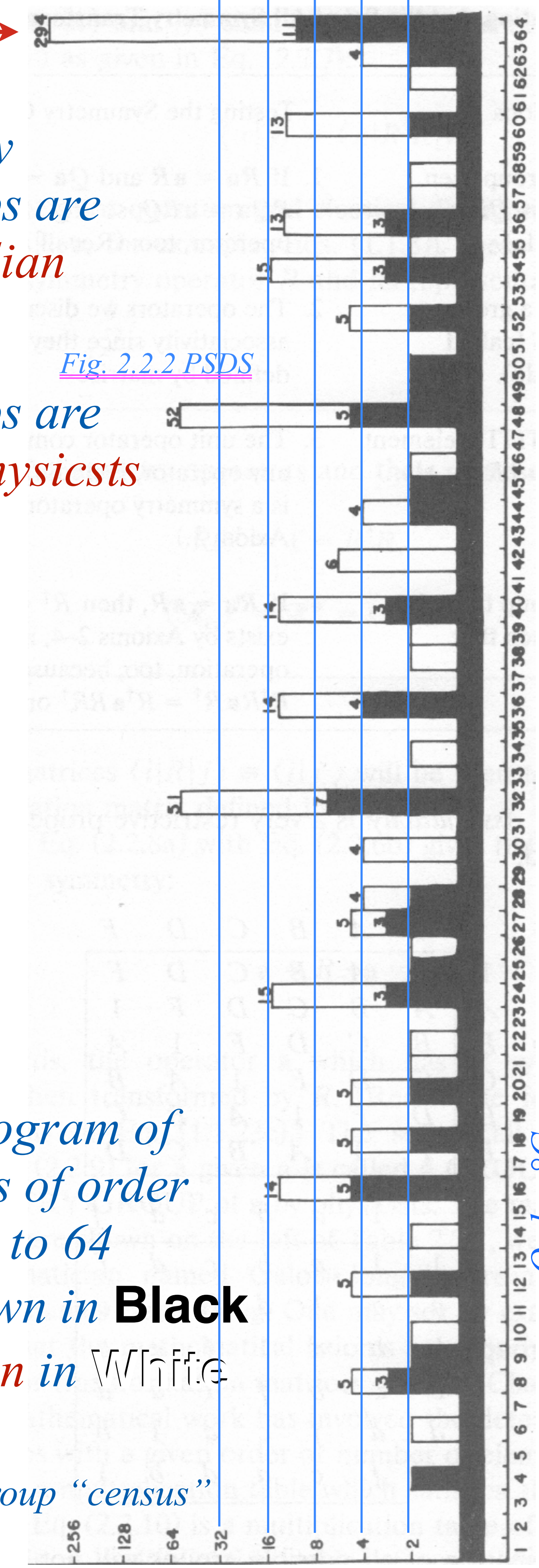
Figure 3.1.1 Crystal point symmetry groups. Models are sketched in circles for the 16 non-Abelian groups. (See also Figure 2.11.1.)

Abelian means all its elements commute

Clearly most groups are Non-Abelian

and most groups are unkown to physicsts

Fig. 2.2.2 PSDS



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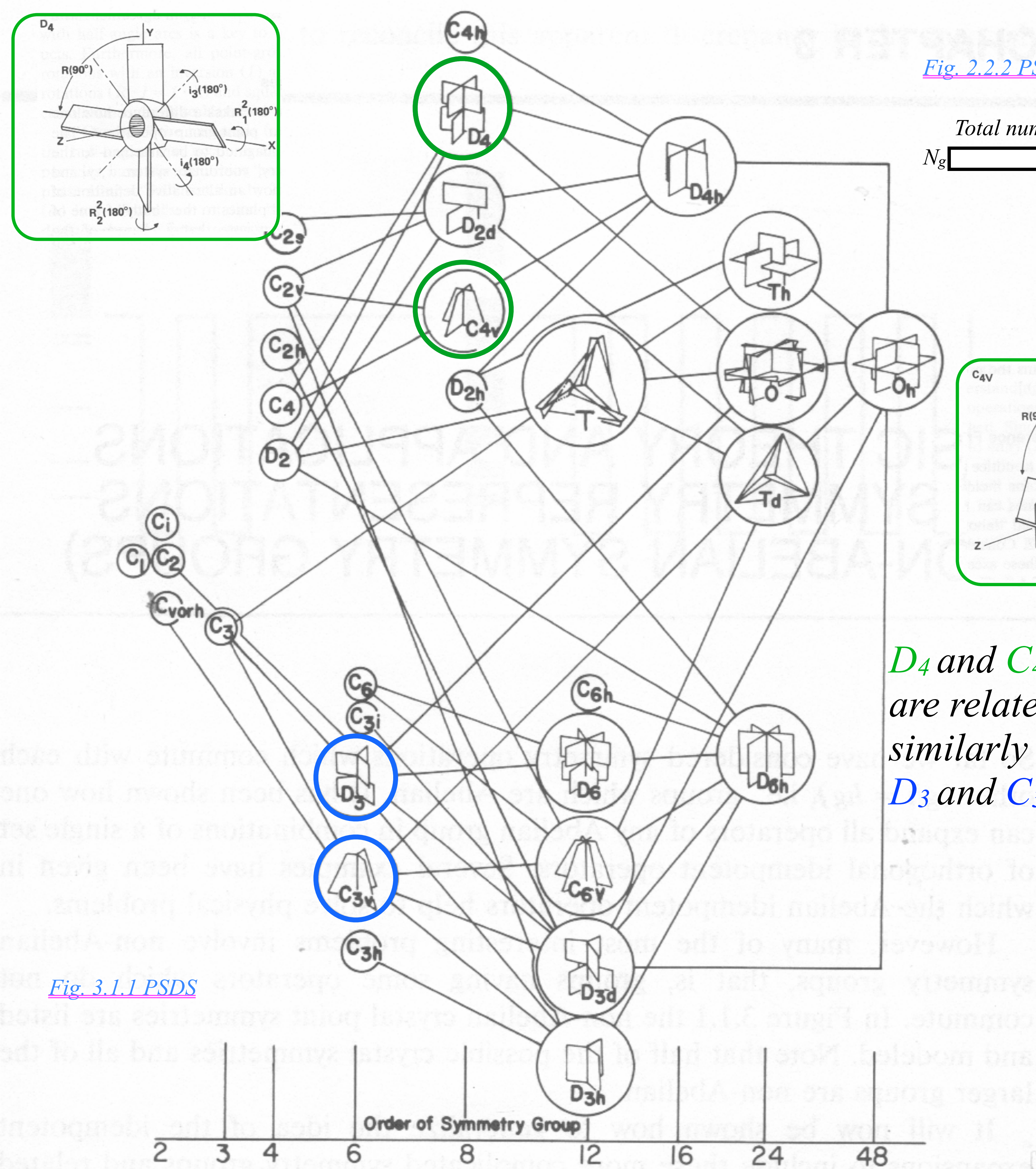
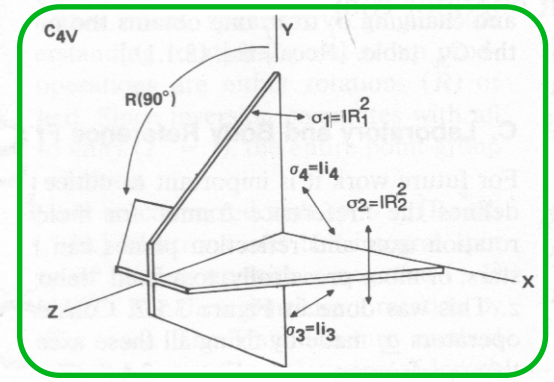
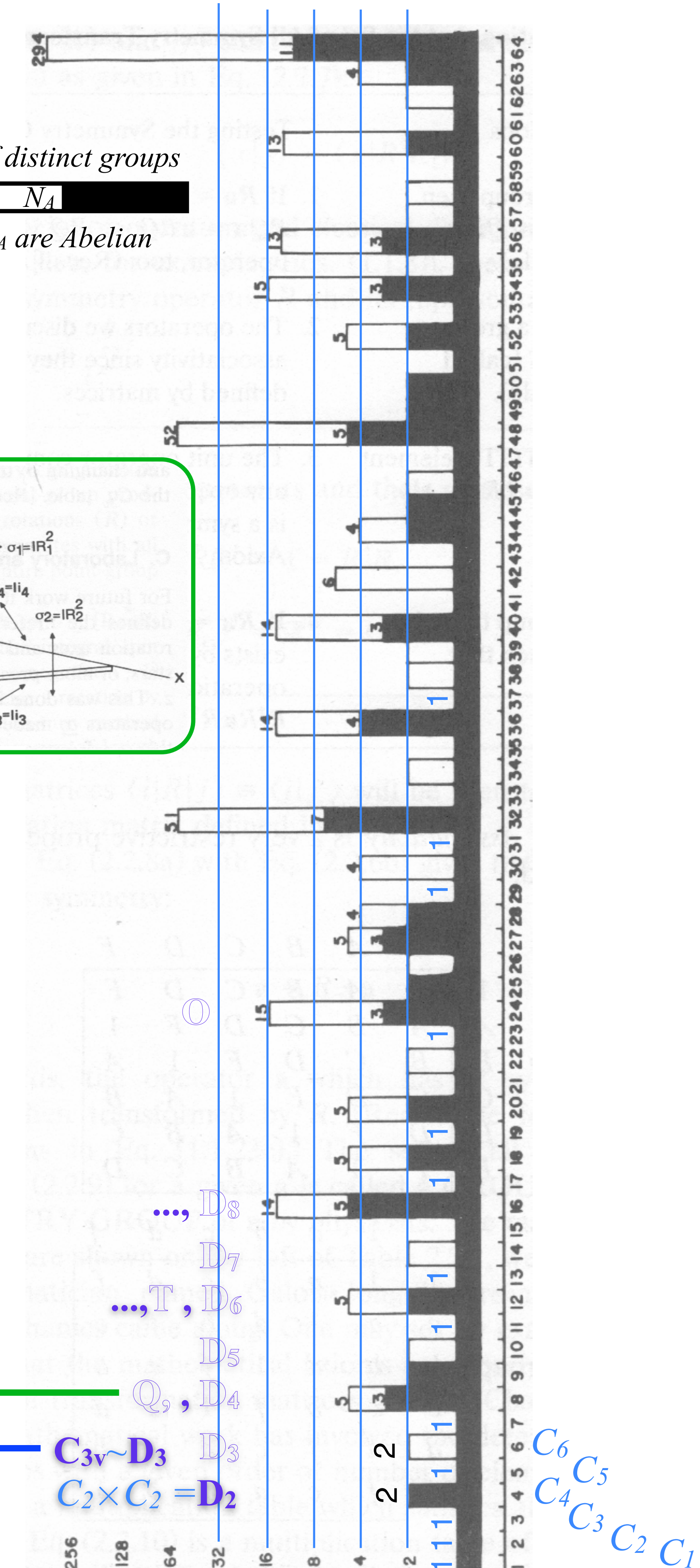


Fig. 2.2.2 PSDS

Total number N_g of distinct groups

number N_A are Abelian





D_4 and C_{4v} are related similarly to D_3 and C_{3v}

C_{3v}~D_3

$C_2 \times C_2 = D_2$

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3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

Fig. 3.1.3 PSDS

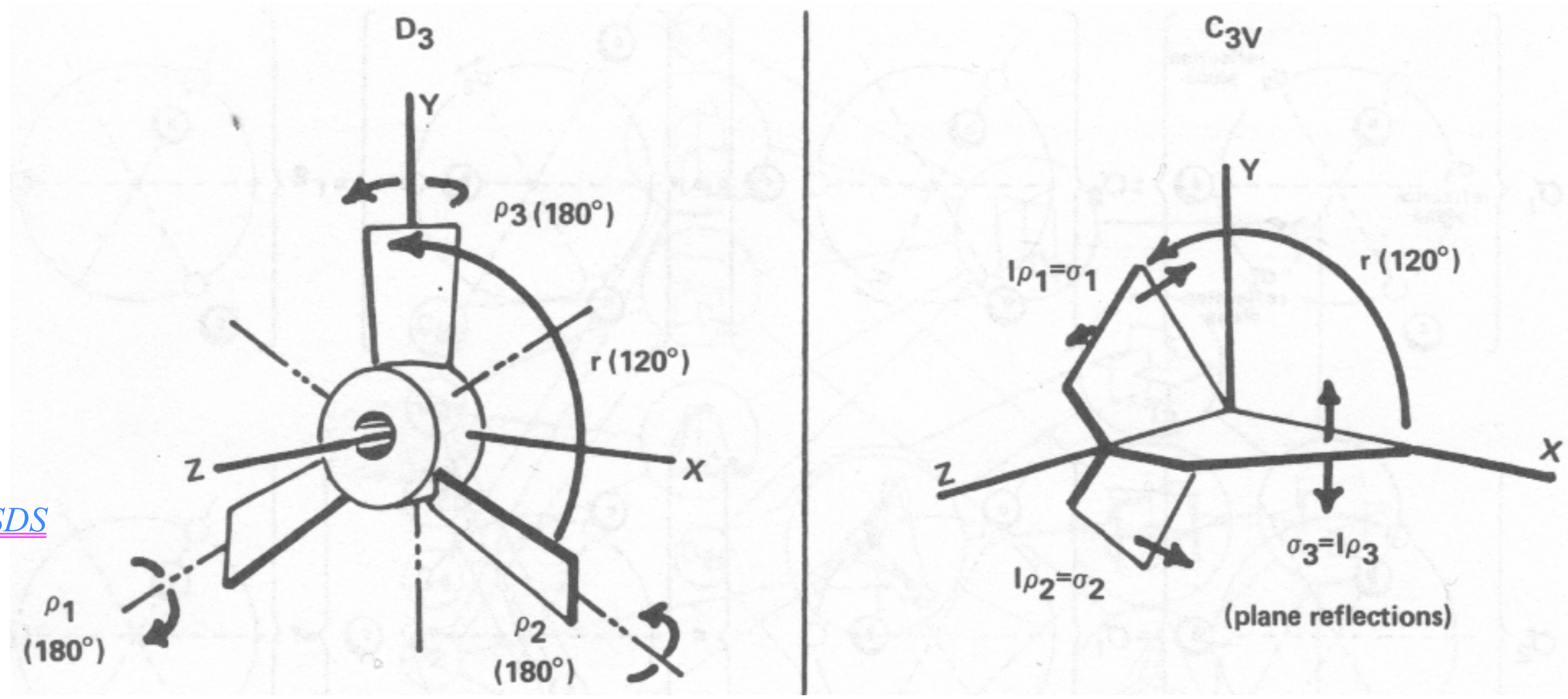
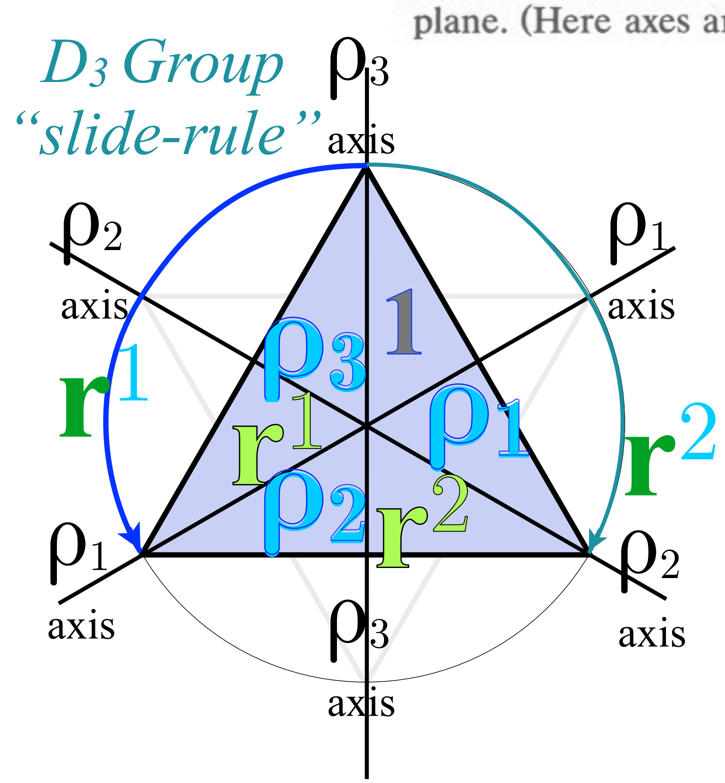
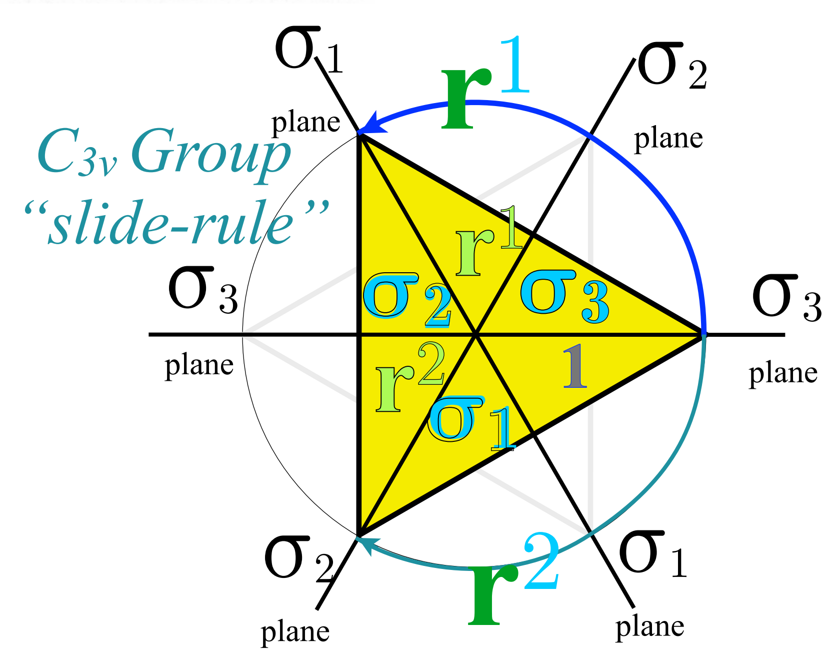


Figure 3.1.3 Pictorial comparison of D_3 and C_{3v} symmetry. A propeller having D_3 symmetry is shown next to a three-plane paddle having C_{3v} symmetry. The group operations are labeled by arrows, which indicate the effect they have. For example, ρ_3 is a 180° rotation around the y axis, while $I\rho_3 = \sigma_3$ is a reflection through the xz plane. (Here axes are fixed and the objects rotate.)

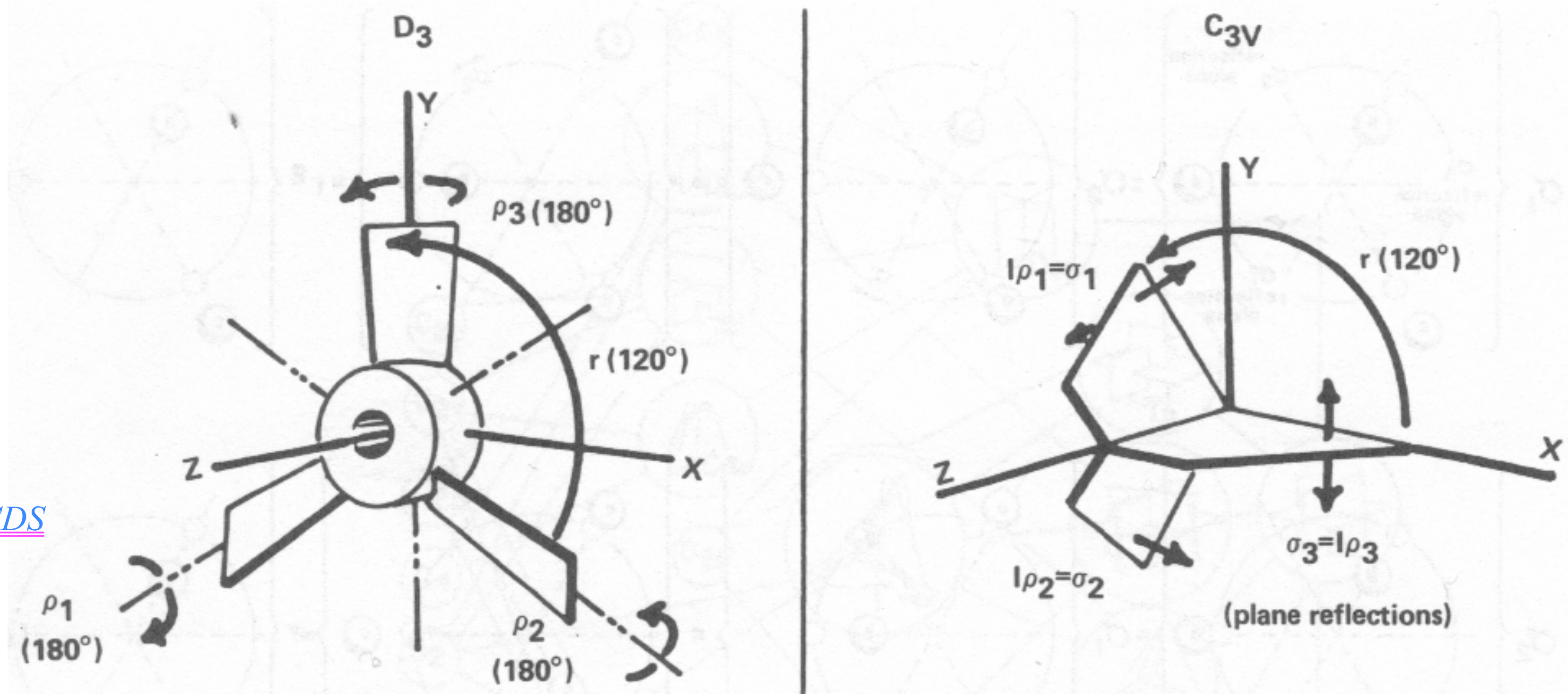




**isomorphic means mathematically the same* abstract *group even if physically different action.*

Showing that D_3 and C_{3v} are isomorphic ($D_3 \sim C_{3v}$ share product table)*

3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}



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$$180°\, D_3\text{-Y-axis-rotation: } \rho_3 = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & +1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix} \quad \textit{maps to}: \text{ XZ-mirror-plane reflection: } \sigma_3 = \begin{pmatrix} +1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & +1 \end{pmatrix}$$

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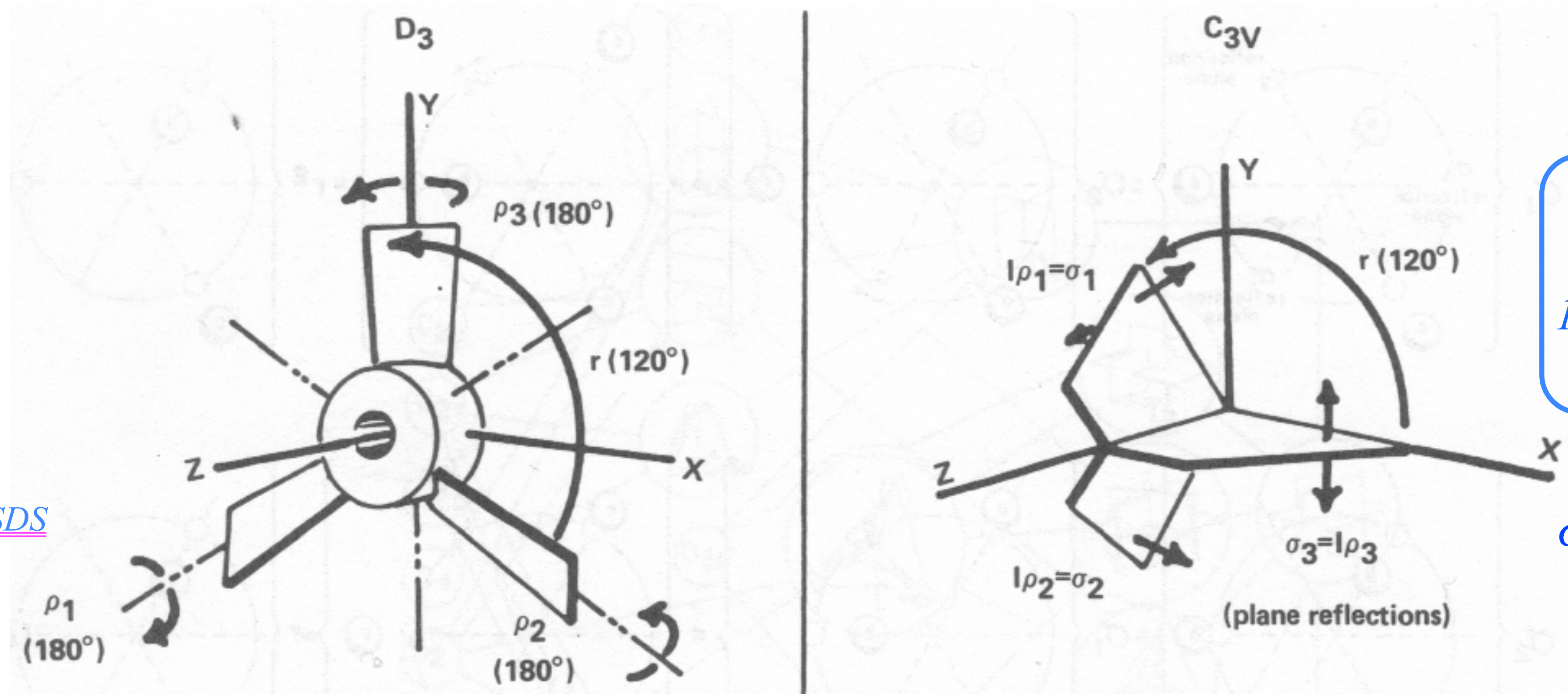
3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}



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Mirror-plane-reflection **σ** *equals*
*180° ⊥-axial-rotation-*inversion
$\sigma = \mathbf{R}\cdot\mathbf{I} = \mathbf{I}\cdot\mathbf{R}$

$$\sigma_3 = \begin{pmatrix} +1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & +1 \end{pmatrix} = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & +1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}\begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix} = \rho_3\cdot\mathbf{I} = \mathbf{I}\cdot\rho_3$$

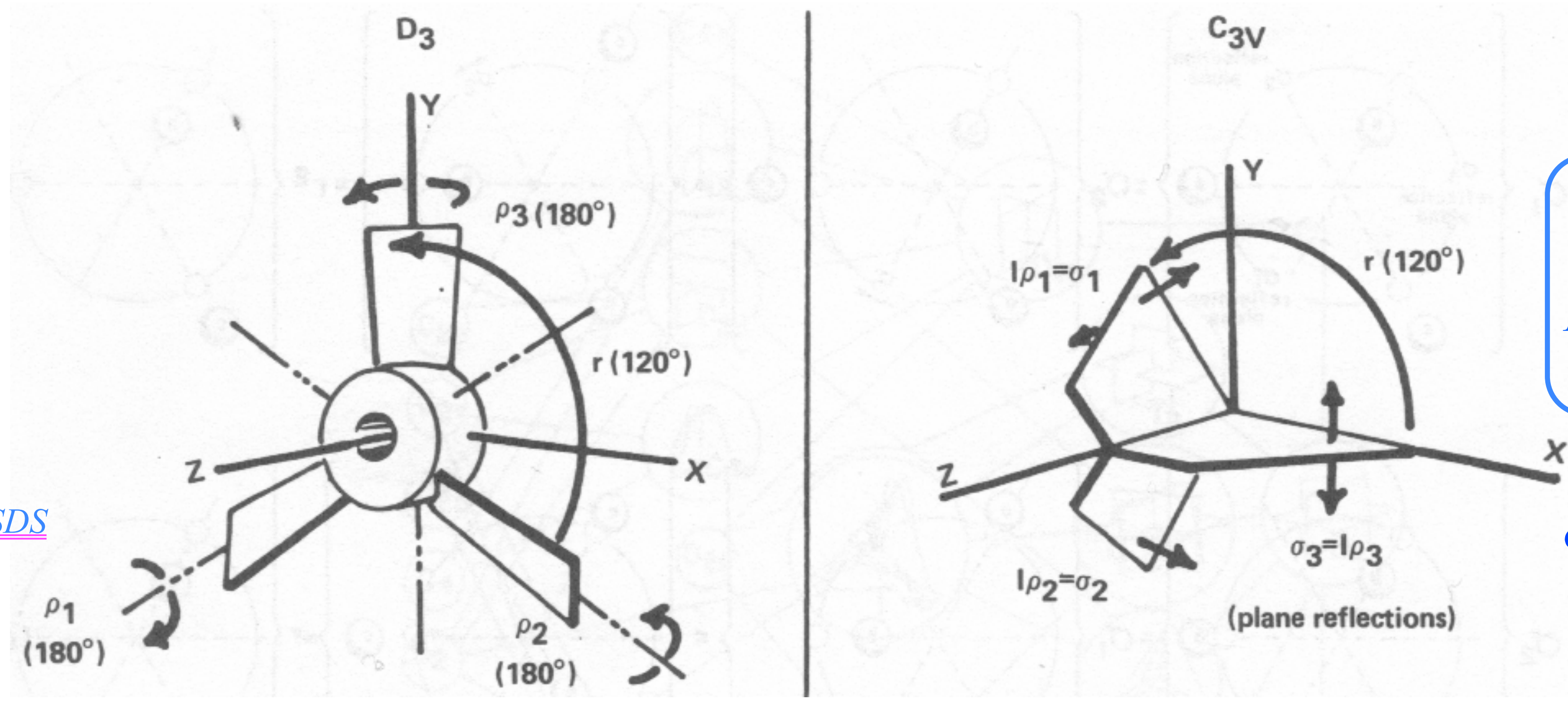
180° D_3-Y-axis-rotation: $\rho_3 = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & +1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$ *maps to*: XZ-mirror-plane reflection: $\sigma_3 = \begin{pmatrix} +1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & +1 \end{pmatrix}$

**isomorphic means mathematically the same* abstract *group even if physically different action.*

Showing that D_3 and C_{3v} are ***isomorphic**** *($D_3 \sim C_{3v}$ share product table)*

3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

Fig. 3.1.3 PSDS



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$\boldsymbol{\sigma} = \mathbf{R}\cdot\mathbf{I} = \mathbf{I}\cdot\mathbf{R}$

$$\sigma_3 = \begin{pmatrix} +1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & +1 \end{pmatrix} = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & +1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}\begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix} = \rho_3\cdot\mathbf{I} = \mathbf{I}\cdot\rho_3$$

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Inversion
$\mathbf{I} = -\mathbf{1}$
commutes with all **R**

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3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

Fig. 3.1.3 PSDS

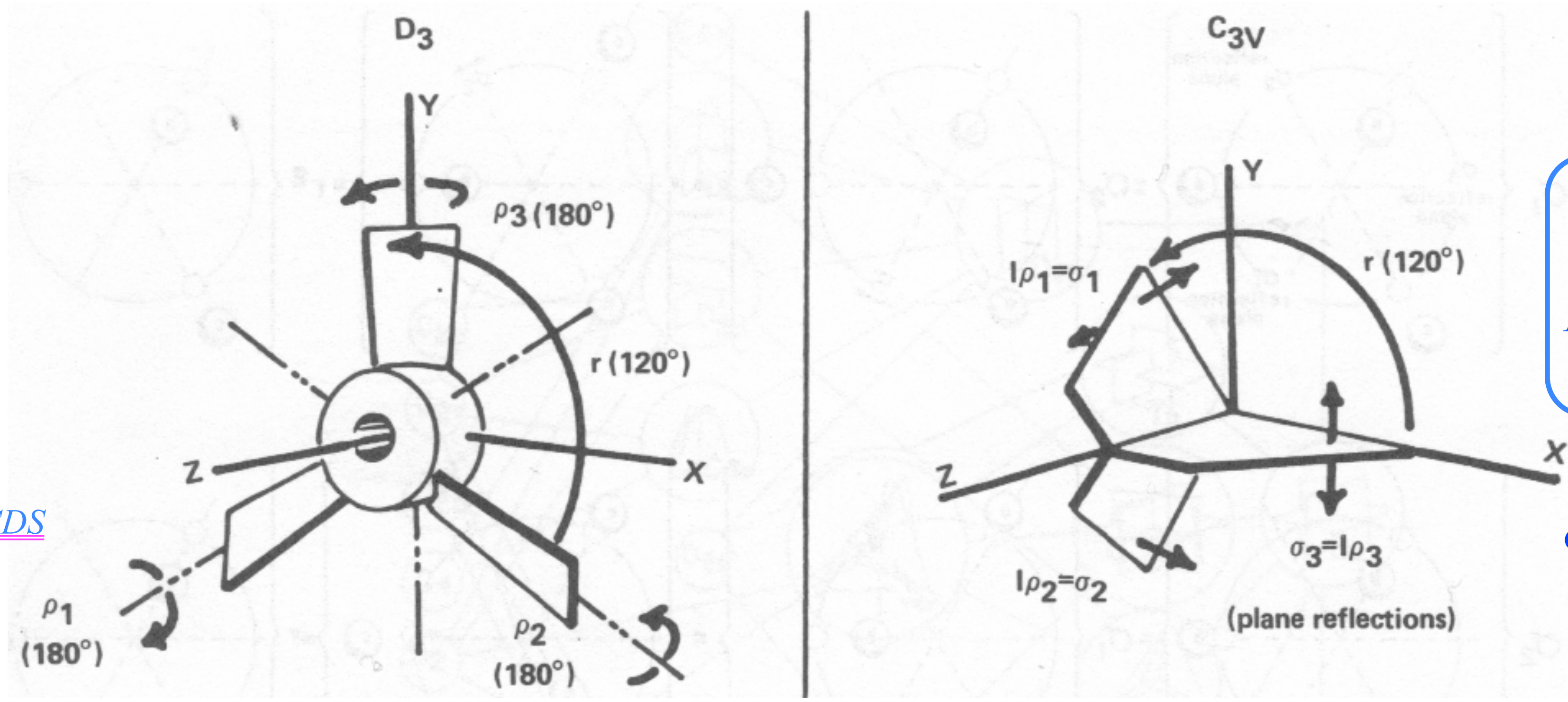

Figure 3.1.3 Pictorial comparison of D_3 and C_{3v} symmetry. A propeller having D_3 symmetry is shown next to a three-plane paddle having C_{3v} symmetry. The group operations are labeled by arrows, which indicate the effect they have. For example, ρ_3 is a 180° rotation around the y axis, while $I\rho_3 = \sigma_3$ is a reflection through the xz plane. (Here axes are fixed and the objects rotate.)

Mirror-plane-reflection $\boldsymbol{\sigma}$
equals
*180° ⊥-axial-rotation-*inversion
$\boldsymbol{\sigma} = \mathbf{R}\cdot\mathbf{I} = \mathbf{I}\cdot\mathbf{R}$

$$\sigma_3 = \begin{pmatrix} +1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & +1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & +1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}\begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

$$= \qquad \rho_3\cdot\mathbf{I} \qquad = \qquad \mathbf{I}\cdot\rho_3$$

180° D_3-Y-axis-rotation: $\rho_3 = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & +1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$ *maps to*: XZ-mirror-plane reflection: $\sigma_3 = \begin{pmatrix} +1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & +1 \end{pmatrix}$

180° D_3-ρ_2-axis-rotation: ρ_2 *maps to*: ⊥ρ_2-mirror-plane reflection: $\sigma_2 = \rho_2\cdot\mathrm{I} = \mathrm{I}\cdot\rho_2$

Inversion
$\mathbf{I} = -\mathbf{1}$
commutes with all **R**

**isomorphic means mathematically the same* abstract *group even if physically different action.*

Showing that D_3 and C_{3v} are isomorphic ($D_3 \sim C_{3v}$ share product table)*

3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

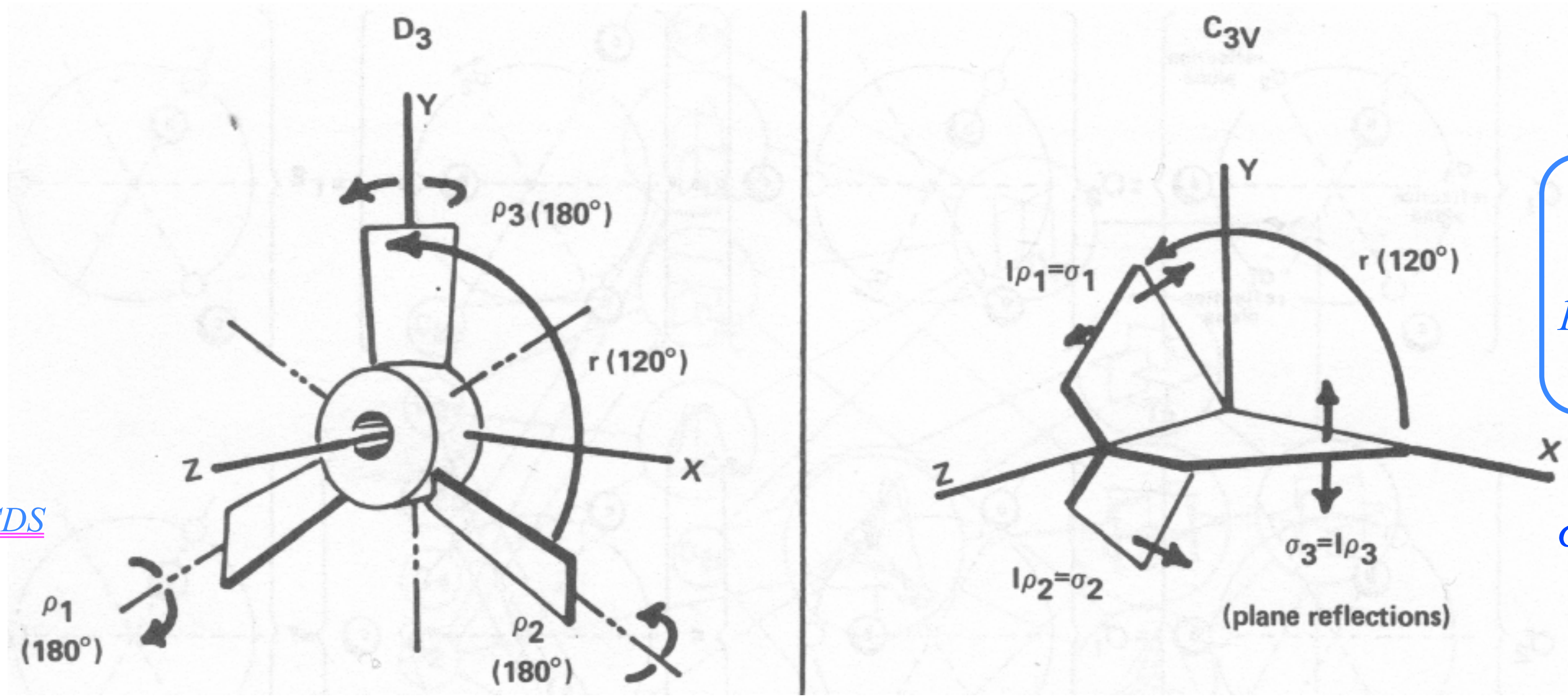

Fig. 3.1.3 PSDS

Figure 3.1.3 Pictorial comparison of D_3 and C_{3v} symmetry. A propeller having D_3 symmetry is shown next to a three-plane paddle having C_{3v} symmetry. The group operations are labeled by arrows, which indicate the effect they have. For example, ρ_3 is a 180° rotation around the y axis, while $I\rho_3 = \sigma_3$ is a reflection through the xz plane. (Here axes are fixed and the objects rotate.)

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$$180°\, D_3\text{-Y-axis-rotation: } \rho_3 = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & +1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix} \quad \textit{maps to}: \text{ XZ-mirror-plane reflection: } \sigma_3 = \begin{pmatrix} +1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & +1 \end{pmatrix}$$

Inversion
$\mathbf{I} = -\mathbf{1}$
commutes with all $\mathbf{R}$

$$180°\, D_3\text{-}\rho_2\text{-axis-rotation: } \rho_2 \quad \textit{maps to}: \perp\rho_2\text{-mirror-plane reflection: } \sigma_2 = \rho_2\cdot\mathrm{I} = \mathrm{I}\cdot\rho_2$$

$$180°\, D_3\text{-}\rho_1\text{-axis-rotation: } \rho_1 \quad \textit{maps to}: \perp\rho_1\text{-mirror-plane reflection: } \sigma_1 = \rho_1\cdot\mathrm{I} = \mathrm{I}\cdot\rho_1$$

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Showing that D_3 and C_{3v} are isomorphic ($D_3 \sim C_{3v}$ share product table)*

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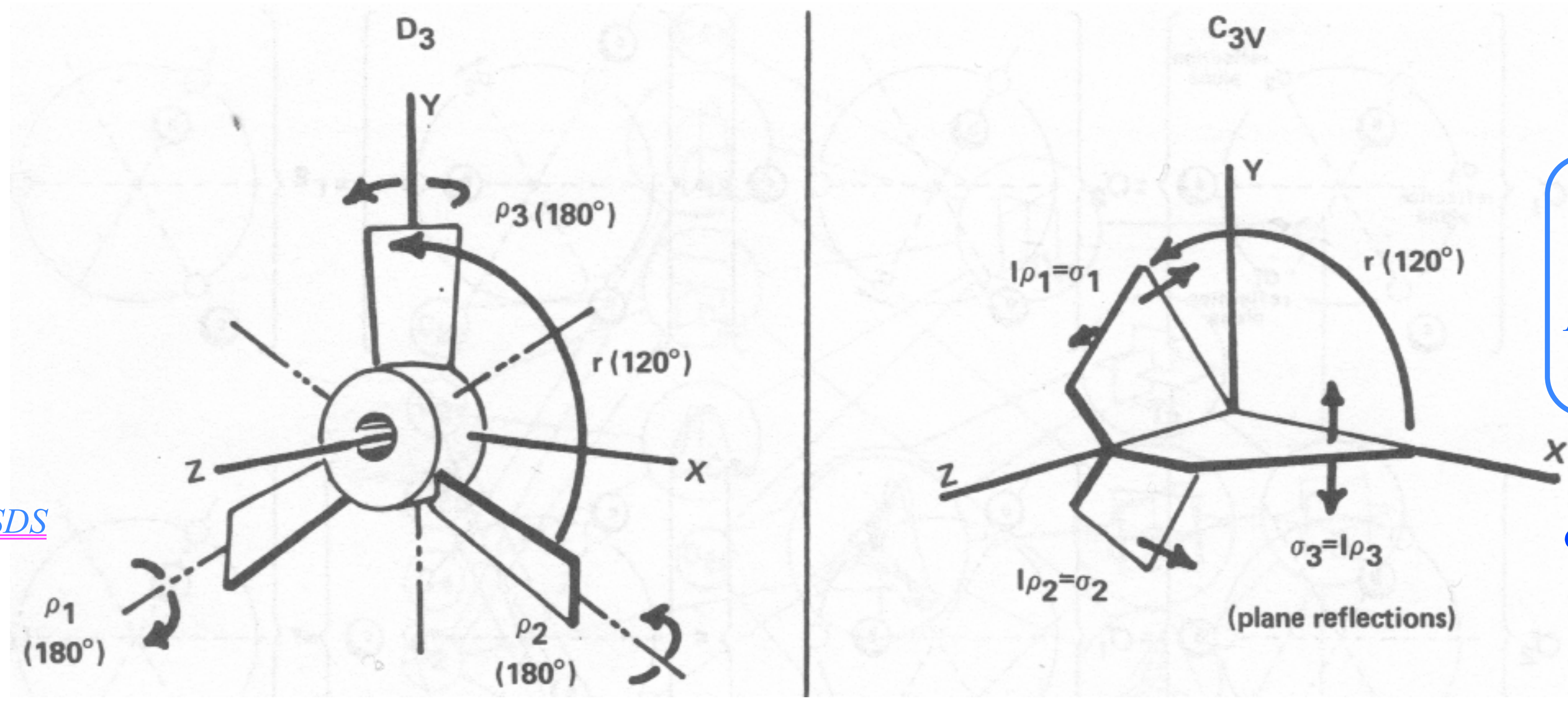

Fig. 3.1.3 PSDS

Figure 3.1.3 Pictorial comparison of D_3 and C_{3v} symmetry. A propeller having D_3 symmetry is shown next to a three-plane paddle having C_{3v} symmetry. The group operations are labeled by arrows, which indicate the effect they have. For example, ρ_3 is a 180° rotation around the y axis, while $I\rho_3 = \sigma_3$ is a reflection through the xz plane. (Here axes are fixed and the objects rotate.)

Mirror-plane-reflection **σ** *equals* *180° ⊥-axial-rotation-*inversion

$\boldsymbol{\sigma} = \mathbf{R}\cdot\mathbf{I} = \mathbf{I}\cdot\mathbf{R}$

$$\sigma_3 = \begin{pmatrix} +1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & +1 \end{pmatrix} = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & +1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}\begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix} = \rho_3\cdot\mathbf{I} = \mathbf{I}\cdot\rho_3$$

180° D_3-Y-axis-rotation: $\rho_3 = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & +1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$ *maps to*: XZ-mirror-plane reflection: $\sigma_3 = \begin{pmatrix} +1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & +1 \end{pmatrix}$

Inversion **I** = **−1** *commutes with all* **R**

180° D_3-ρ_2-axis-rotation: ρ_2 *maps to*: ⊥ρ_2-mirror-plane reflection: $\sigma_2 = \rho_2\cdot\mathrm{I} = \mathrm{I}\cdot\rho_2$

180° D_3-ρ_1-axis-rotation: ρ_1 *maps to*: ⊥ρ_1-mirror-plane reflection: $\sigma_1 = \rho_1\cdot\mathrm{I} = \mathrm{I}\cdot\rho_1$

D_3-product: $\rho_1\rho_2$ *maps to*: C_{3v}-product: $\sigma_1\sigma_2 = \rho_1\mathrm{I}\,\mathrm{I}\rho_2 = \rho_1\rho_2$

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Showing that D_3 and C_{3v} are isomorphic ($D_3 \sim C_{3v}$ share product table)*

3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

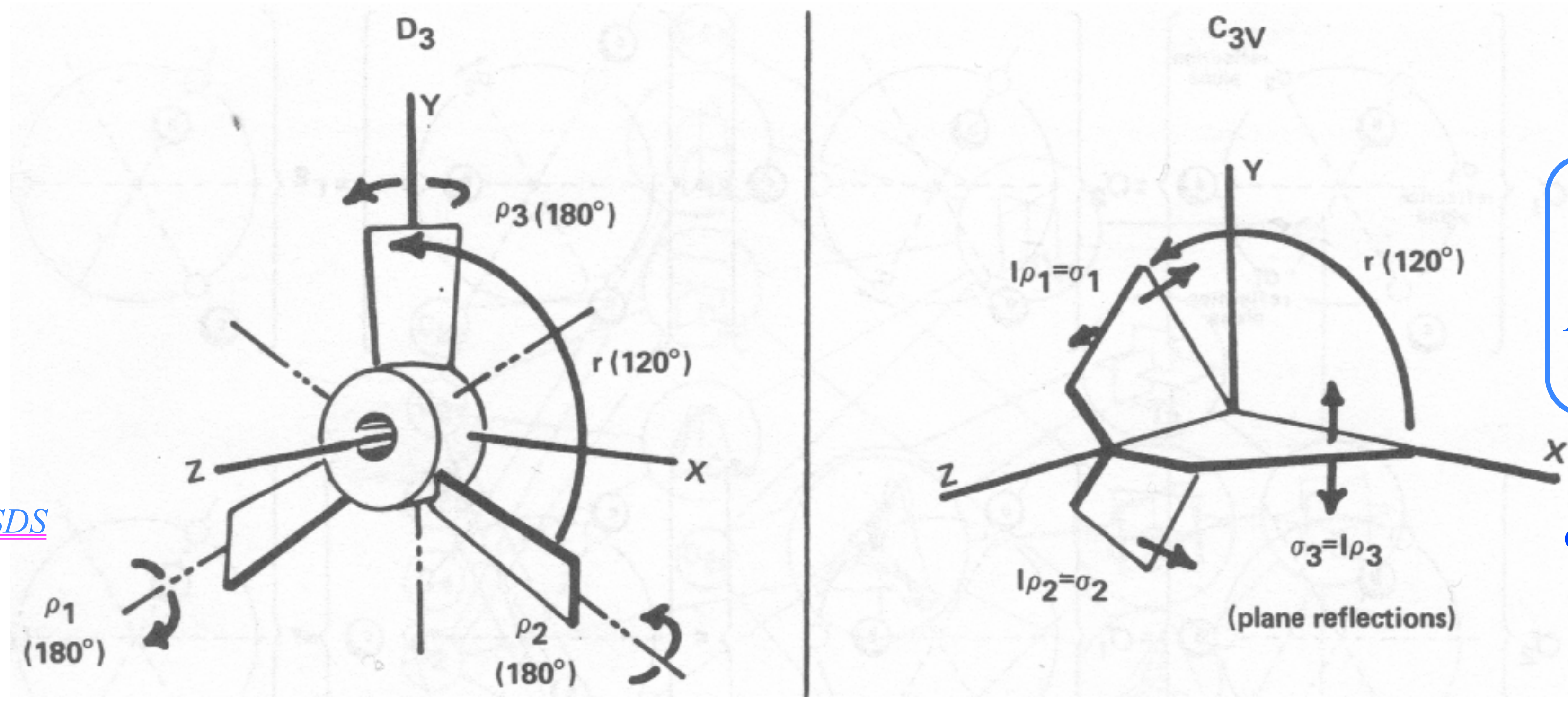

Fig. 3.1.3 PSDS

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180° D_3-Y-axis-rotation: $\rho_3 = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & +1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$ *maps to*: XZ-mirror-plane reflection: $\sigma_3 = \begin{pmatrix} +1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & +1 \end{pmatrix}$

180° D_3-ρ_2-axis-rotation: ρ_2 *maps to*: ⊥ρ_2-mirror-plane reflection: $\sigma_2 = \rho_2\cdot\mathrm{I} = \mathrm{I}\cdot\rho_2$

180° D_3-ρ_1-axis-rotation: ρ_1 *maps to*: ⊥ρ_1-mirror-plane reflection: $\sigma_1 = \rho_1\cdot\mathrm{I} = \mathrm{I}\cdot\rho_1$

D_3-product: $\rho_1\rho_2$ *maps to*: C_{3v}-product: $\sigma_1\sigma_2 = \rho_1\mathrm{I}\,\mathrm{I}\rho_2 = \rho_1\rho_2$

D_3-product: $\rho_1\mathbf{r}^p$ *maps to*: C_{3v}-product: $\sigma_1\mathbf{r}^p = \rho_1\mathrm{I}\,\mathbf{r}^p = \rho_1\mathbf{r}^p\mathrm{I} = \mathrm{I}\rho_1\mathbf{r}^p$

**isomorphic means mathematically the same* abstract *group even if physically different action.*

Showing that D_3 and C_{3v} are isomorphic ($D_3 \sim C_{3v}$ share product table)*

AMOP reference links on following page

2.21.18 class 12.0: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

Discrete symmetry subgroups of O(3) and application to tunneling and vibrational dynamics: D_3 and C_{3v} group products, classes, and irrep projection operators

32 crystal point symmetries: 16 Abelian (commutative) and 16 non-Abelian groups

Smallest non-Abelian symmetry: 3-C_2-axis D_3 vs. 3-C_v-plane C_{3v} isomorphic to permutation-S_3

Relating C_2-180°rotations $\mathbf{R}_z$, C_v-plane reflections $\boldsymbol{\sigma}_z$, and inversion $\mathbf{I}$ operators

Deriving $D_3 \sim C_{3v}$ products by group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$

Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes

Non-commutative symmetry expansion and Global-Local solution

Global vs Local symmetry and Mock-Mach principle

Global vs Local matrix duality for D_3

Global vs Local symmetry expansion of D_3 Hamiltonian

*Group theory and algebra of **D_3 Center** (Class algebra)*

Self-symmetry (Normalizer). *Lagrange Coset Theorem for classes*

1st-Stage spectral decomposition of "Group-table" Hamiltonian of D_3 symmetry

All-commuting operators $\boldsymbol{\kappa}_k$ *All-commuting projectors $\mathbf{P}^{(\alpha)}$*

D_3-invariant irep characters $\chi_k^{(\alpha)}$ *Invariant numbers: Centrum, Rank, and Order*

2nd-Stage spectral decompositions of global/local D_3

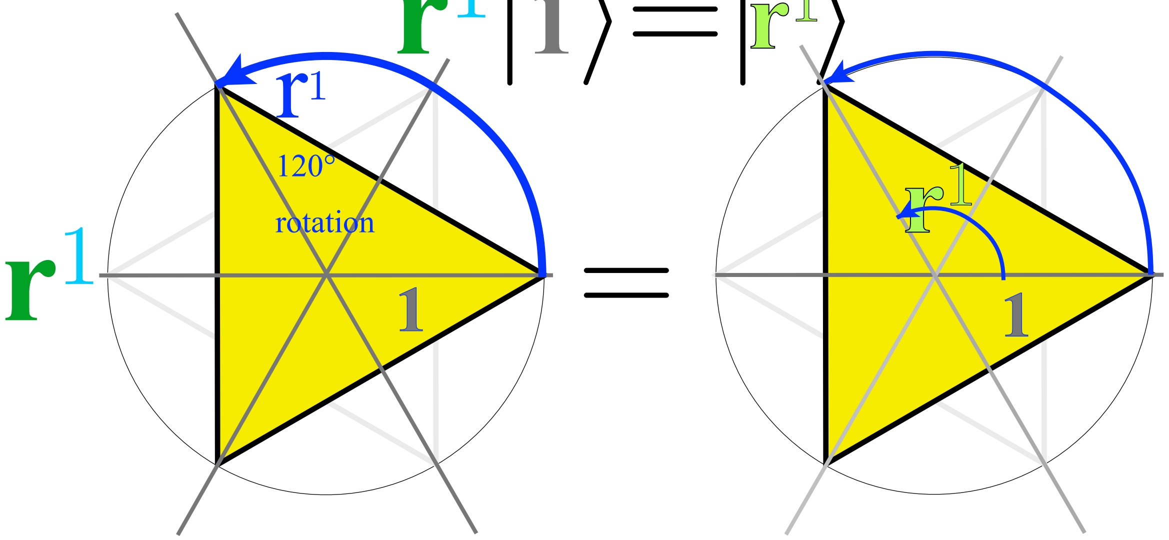
Subgroup chains $D_3 \supset C_2$ and $D_3 \supset C_3$ split class projectors *...and classes*

3rd-Stage spectral decomposition of ALL of D_3 *...and of Hamiltonian $\mathbf{H}$*

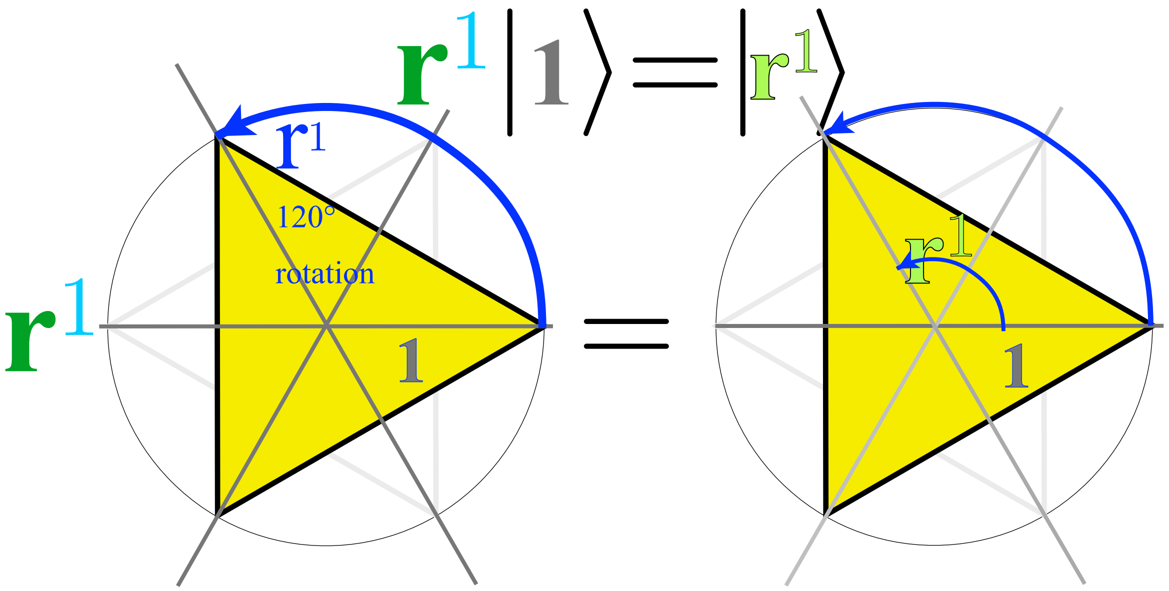
GLOBAL vs LOCAL symmetry of states *...and group $\mathbf{H}$ parameters $\{r, i_1, i_2, i_3\}$*

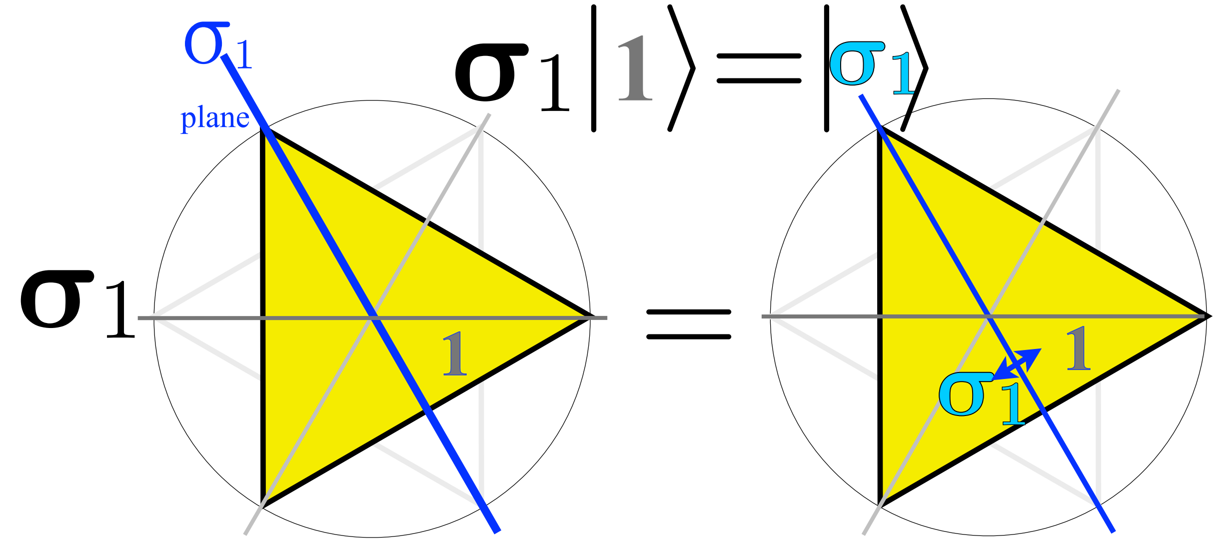
Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$

$$\mathbf{r}^1|\mathbf{1}\rangle = |\mathbf{r}^1\rangle$$

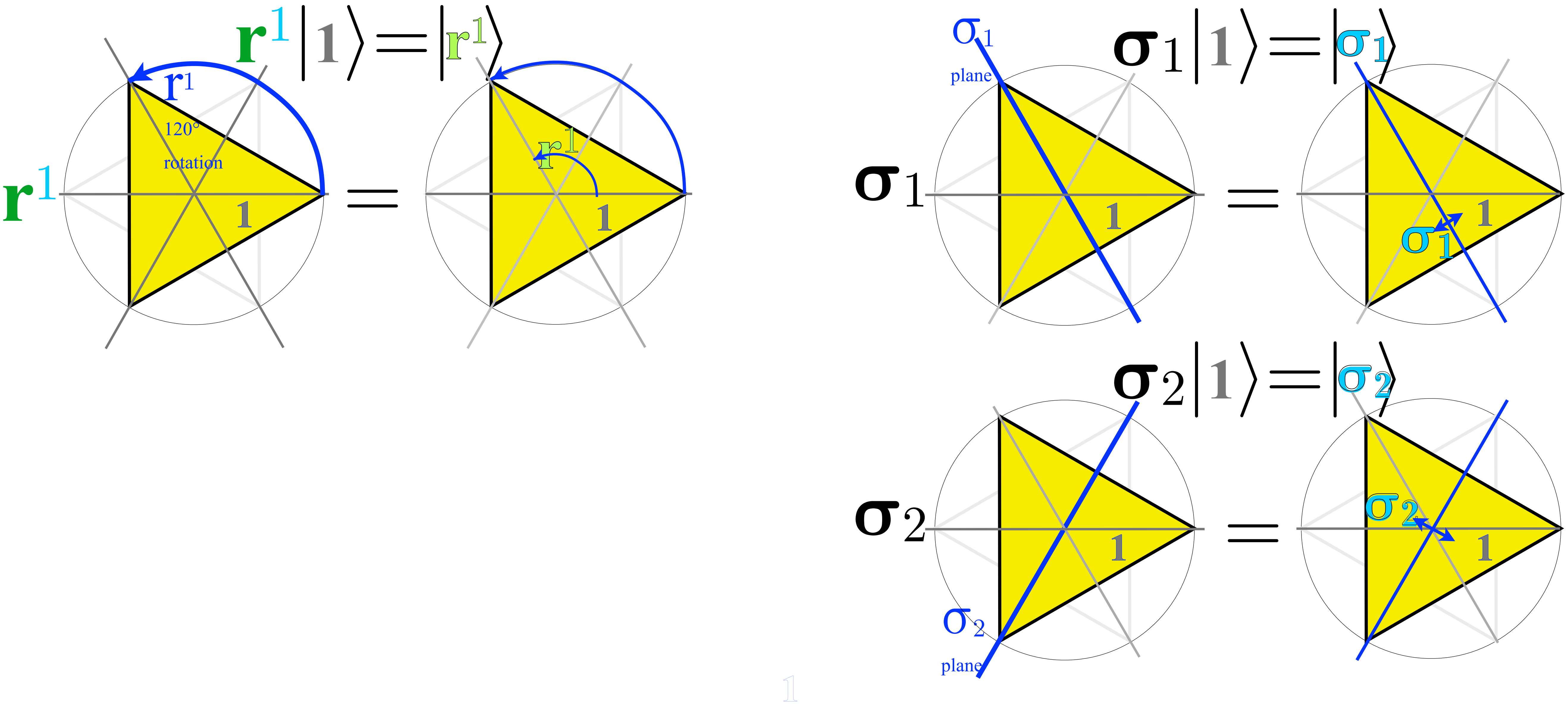


Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$





Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$



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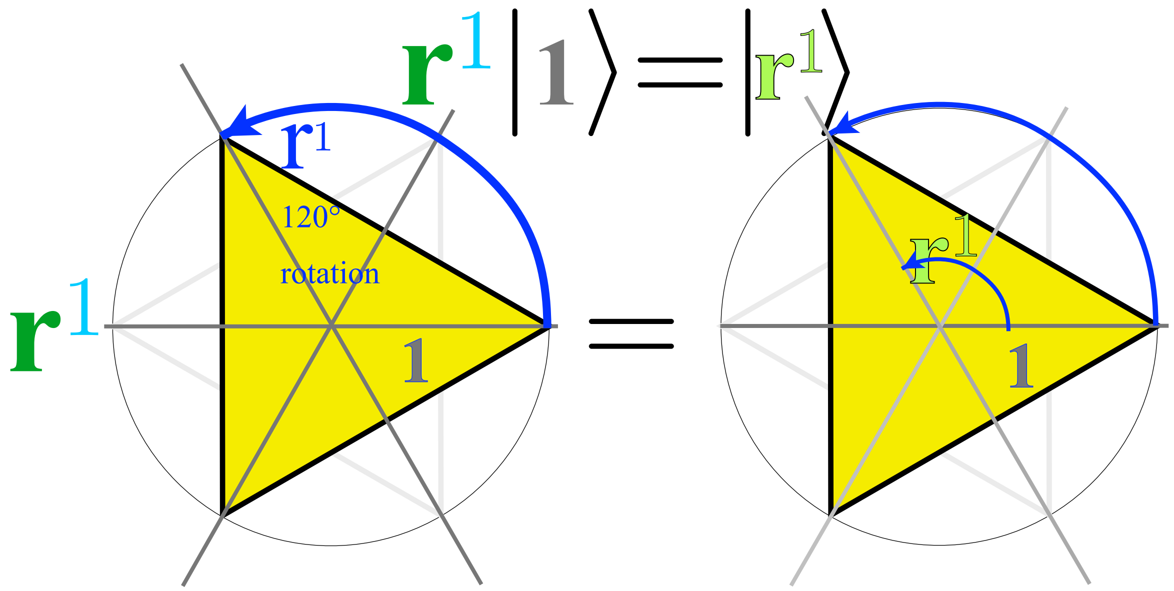
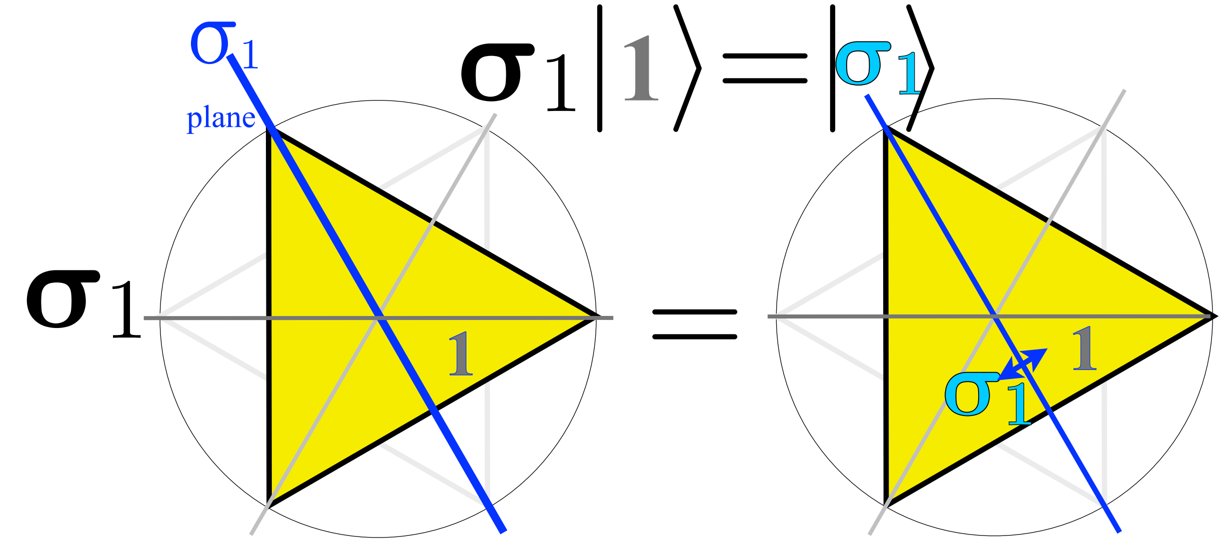
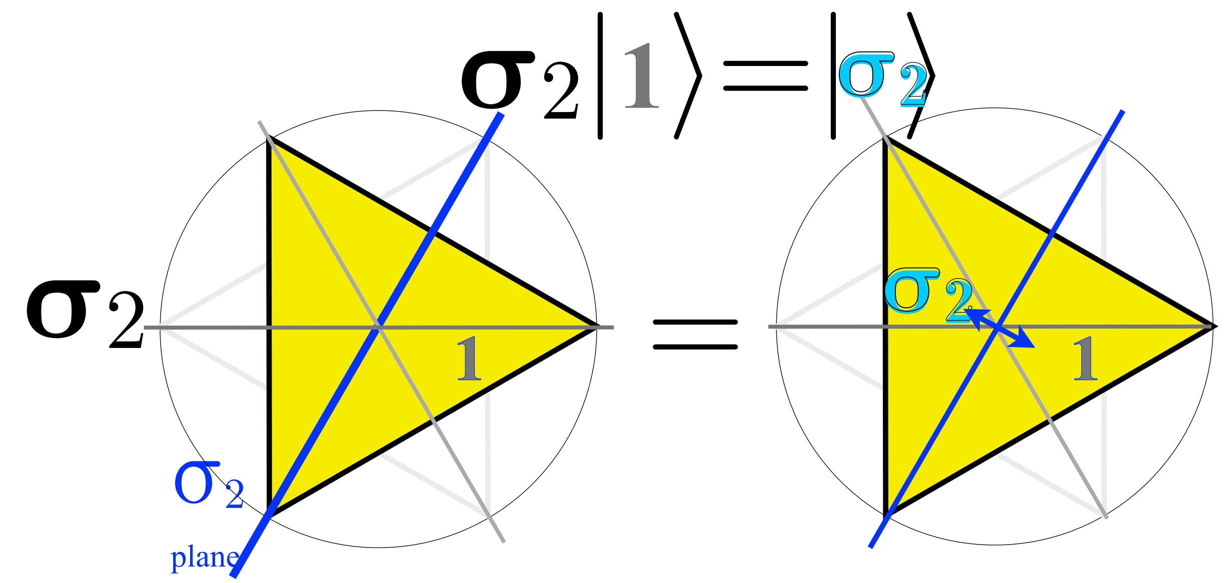
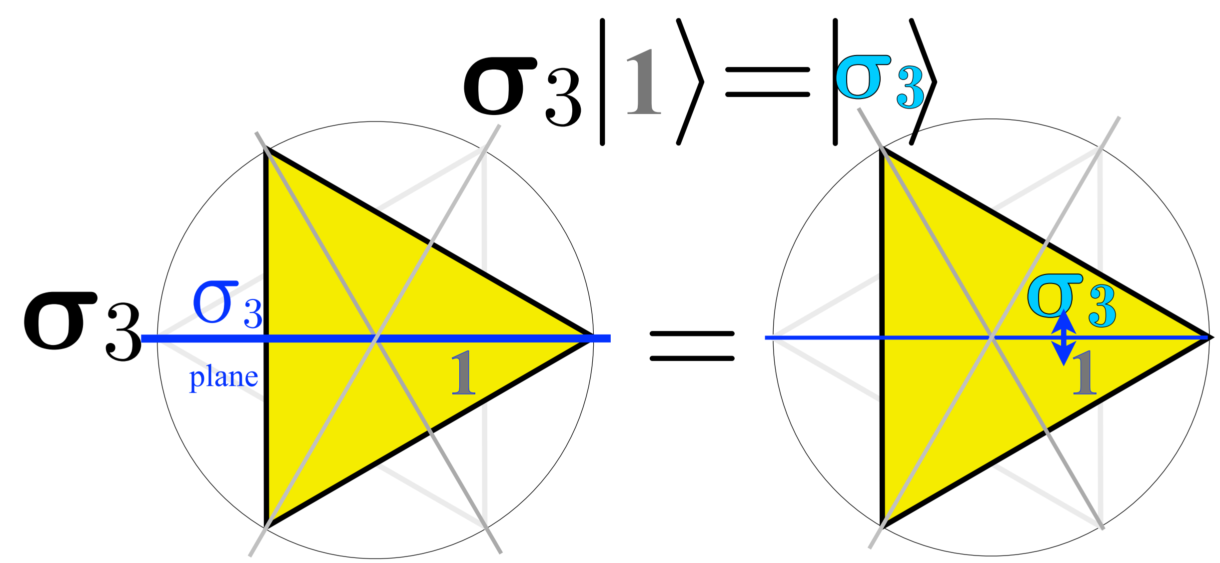
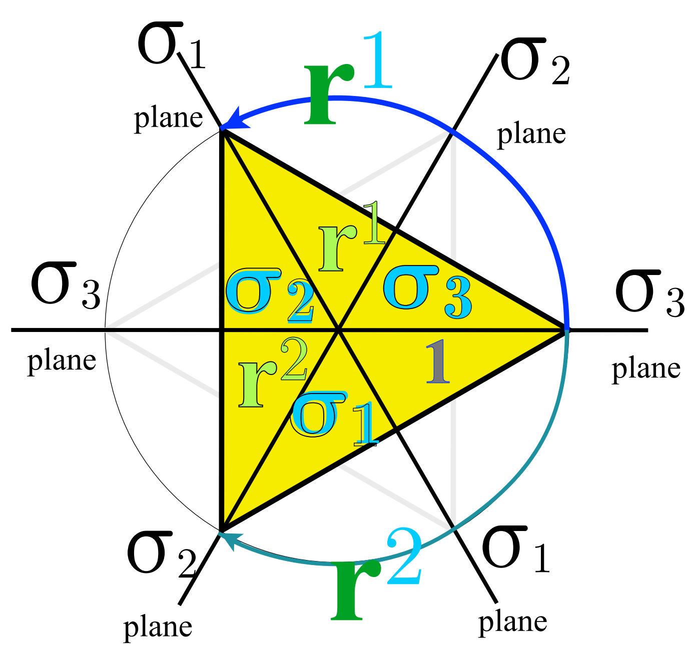




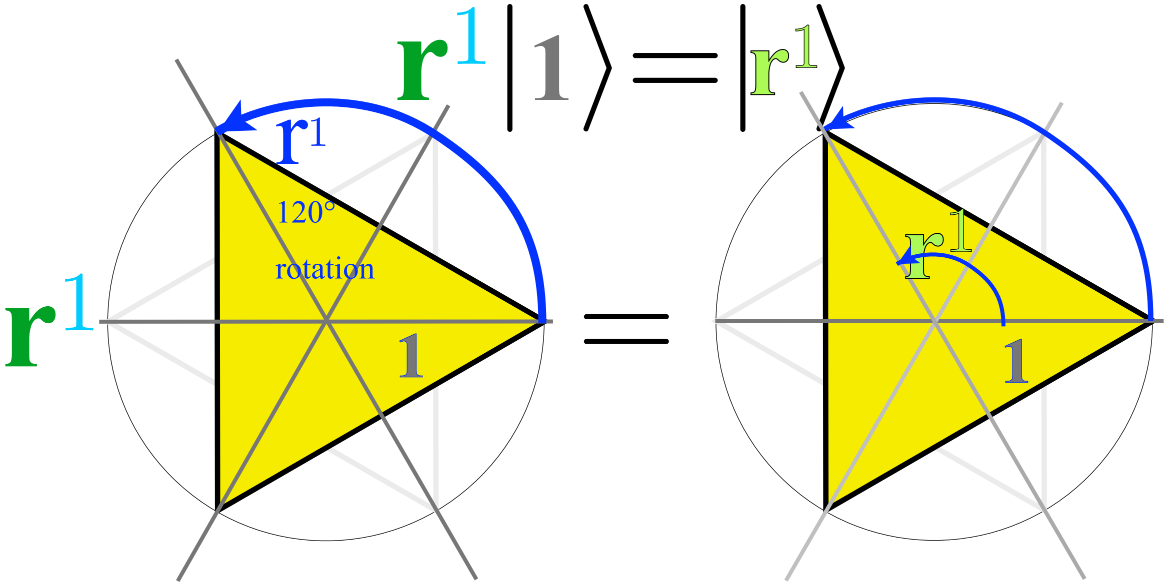


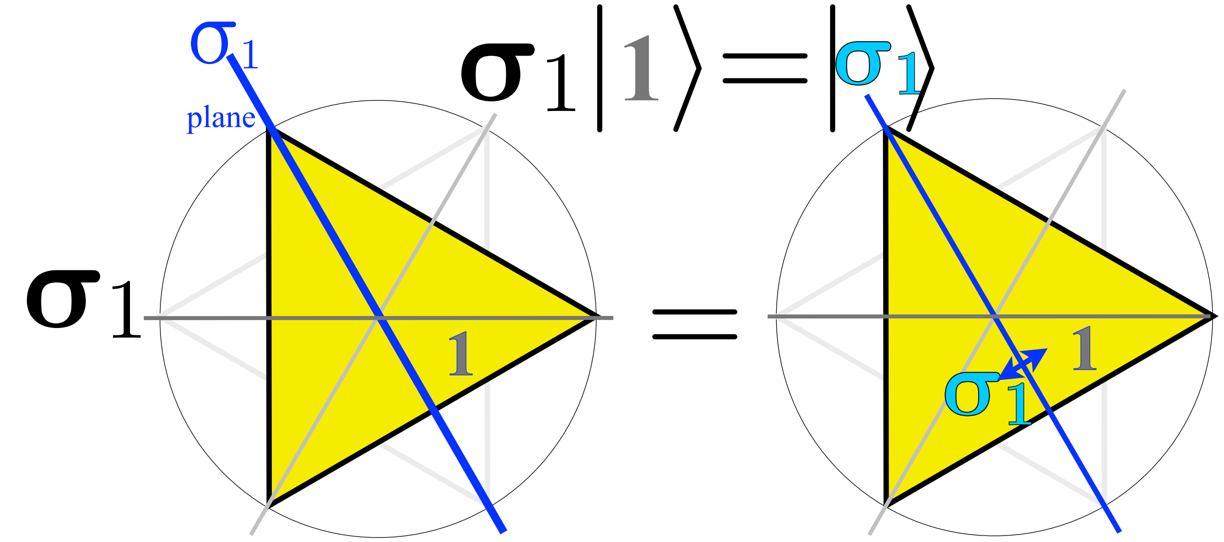


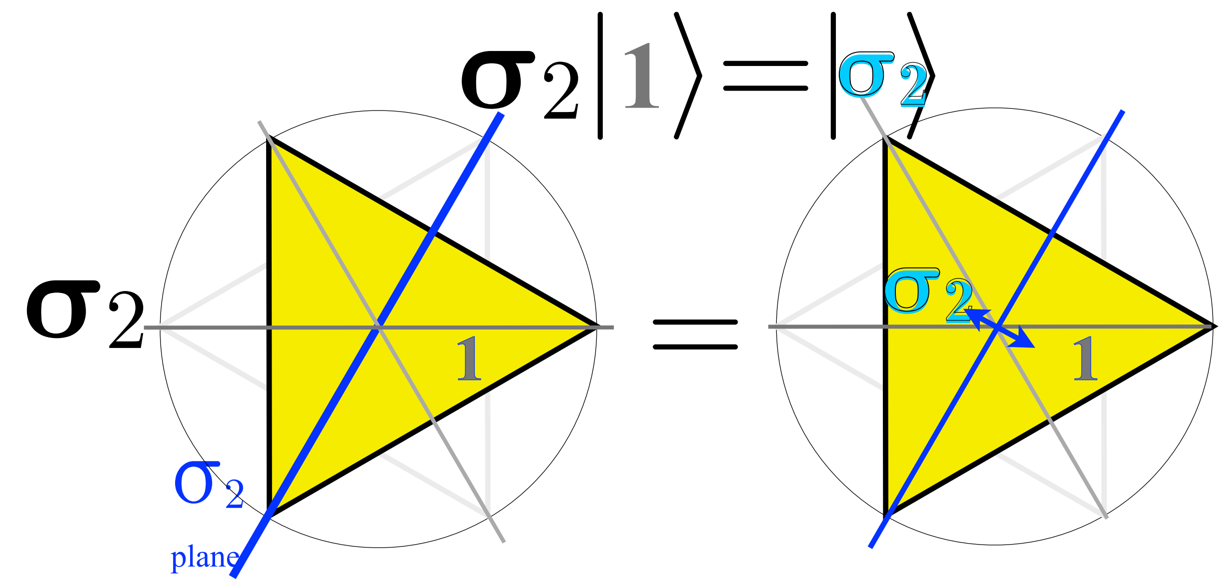
Building
C_{3v} Group
"slide-rule"



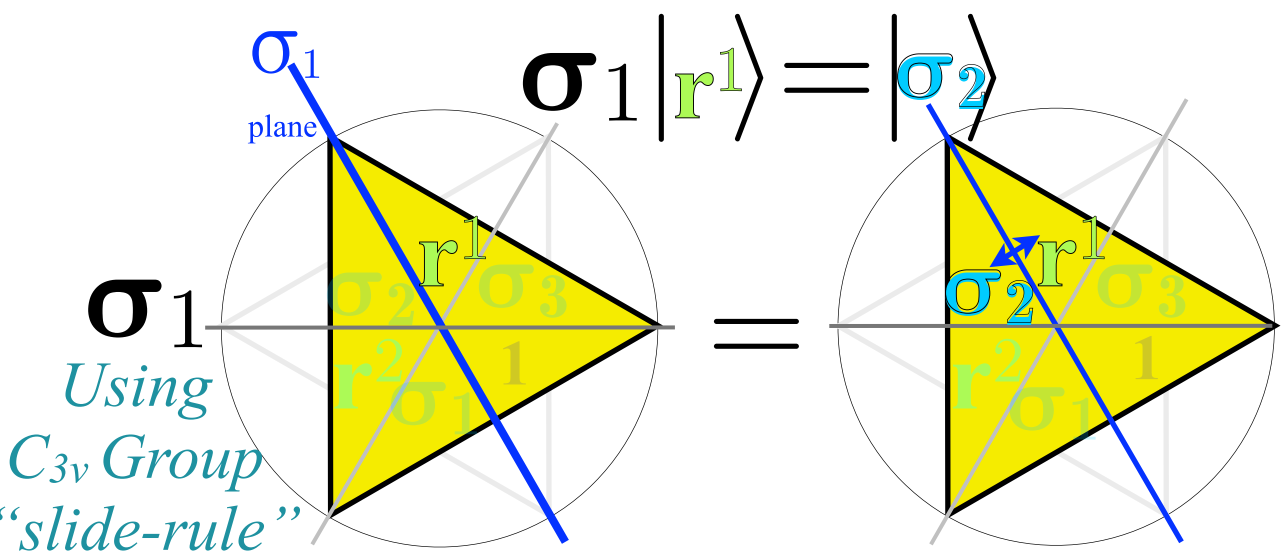
Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$

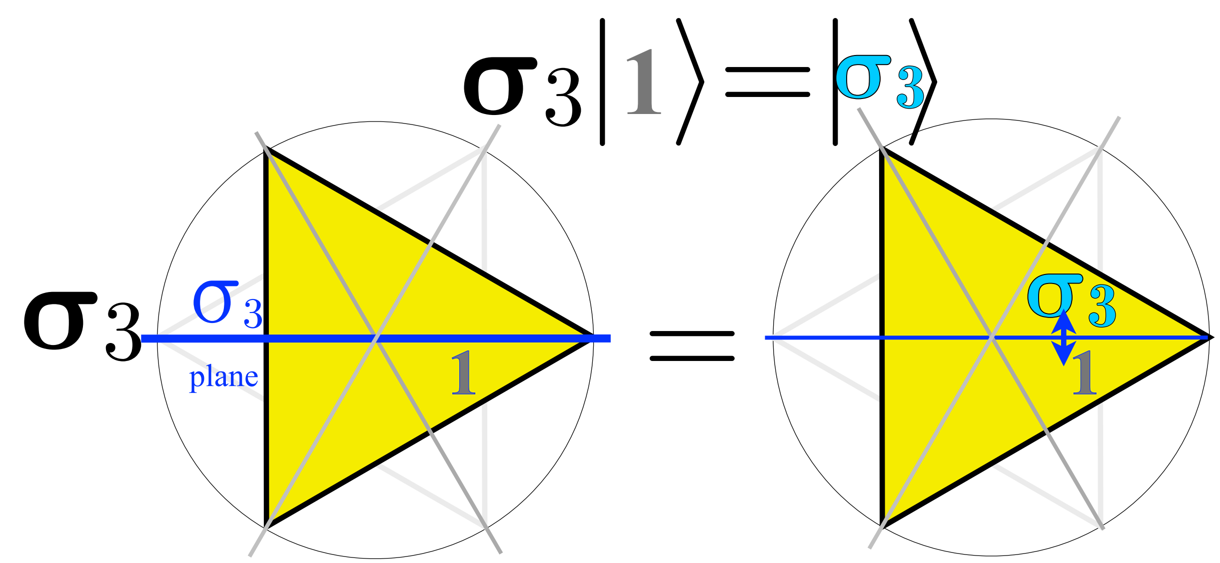




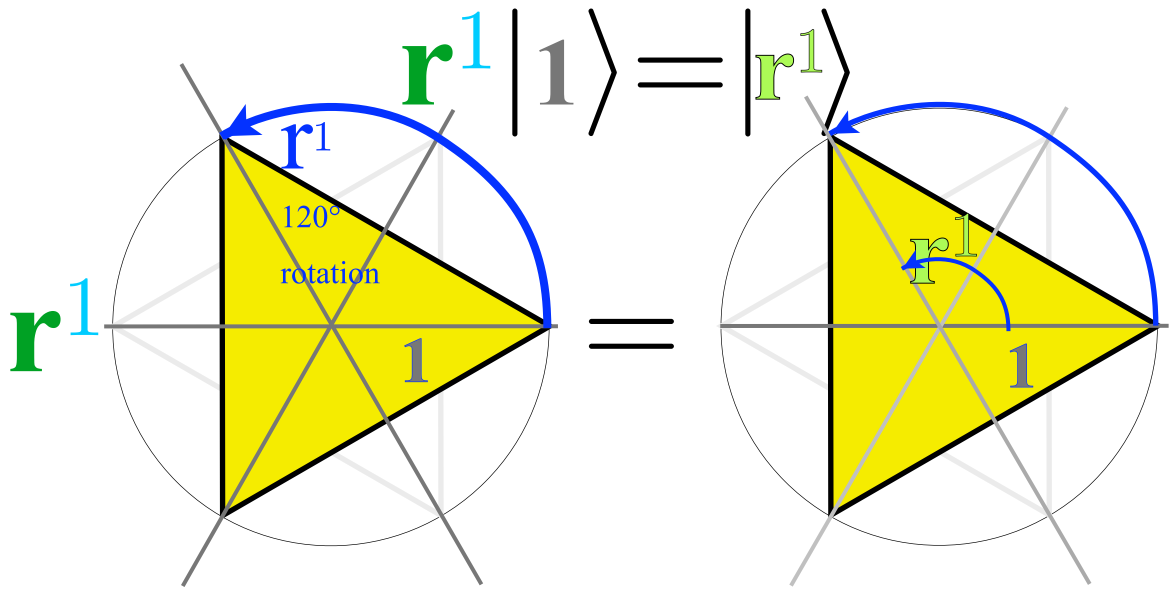


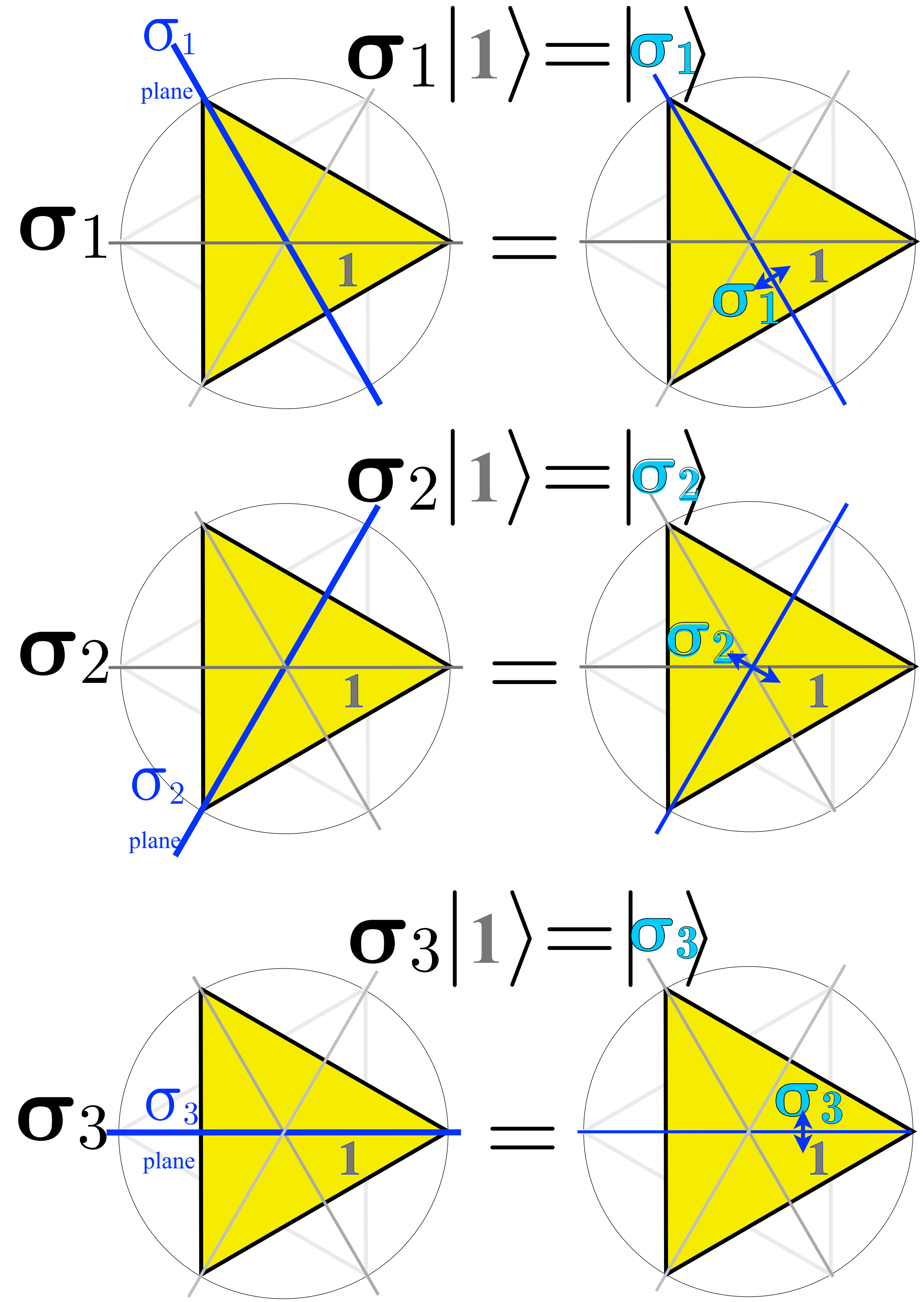
Example: Find C_{3v} product $\boldsymbol{\sigma}_1\mathbf{r}^1|1\rangle = \boldsymbol{\sigma}_1|\mathbf{r}^1\rangle$



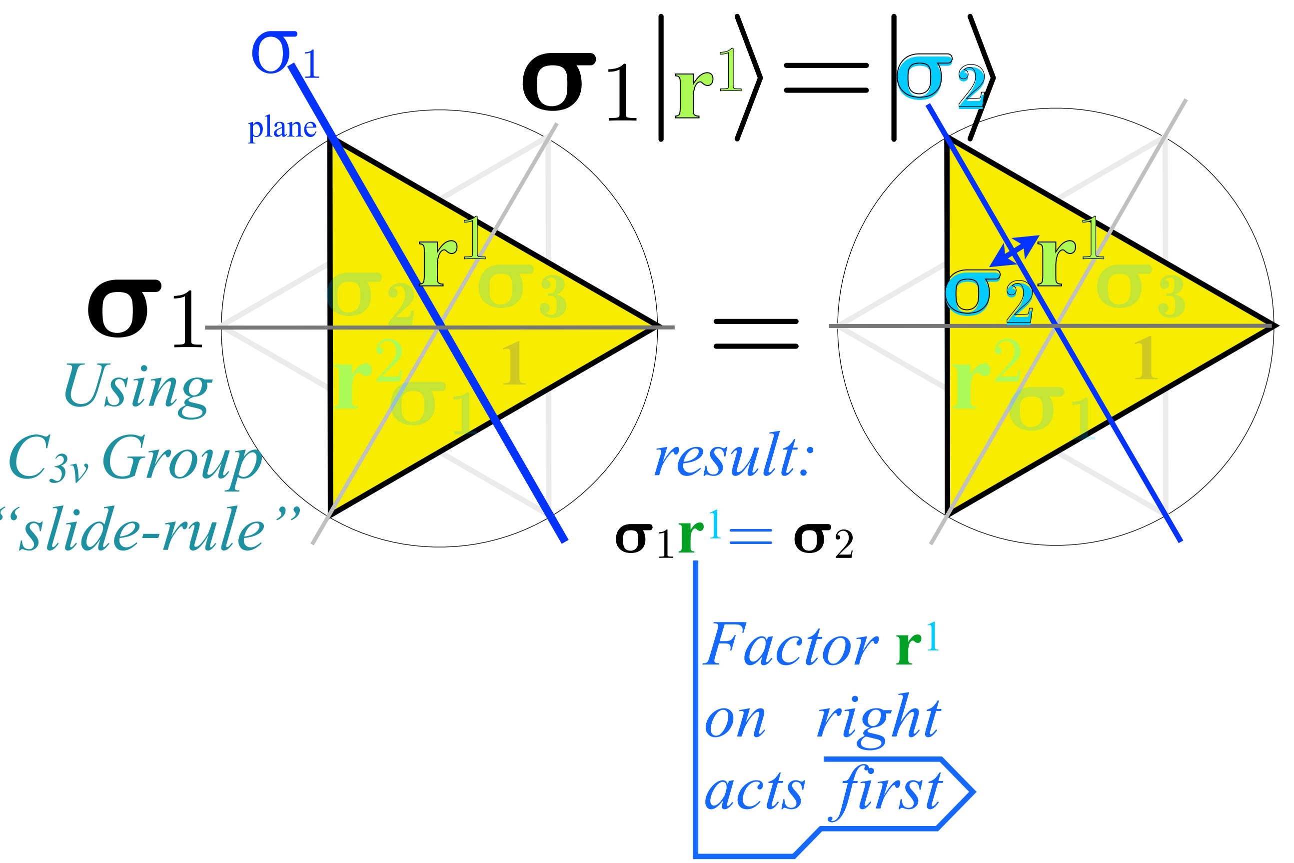


Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$





Example: Find C_{3v} product $\boldsymbol{\sigma}_1\mathbf{r}^1|1\rangle = \boldsymbol{\sigma}_1|\mathbf{r}^1\rangle$



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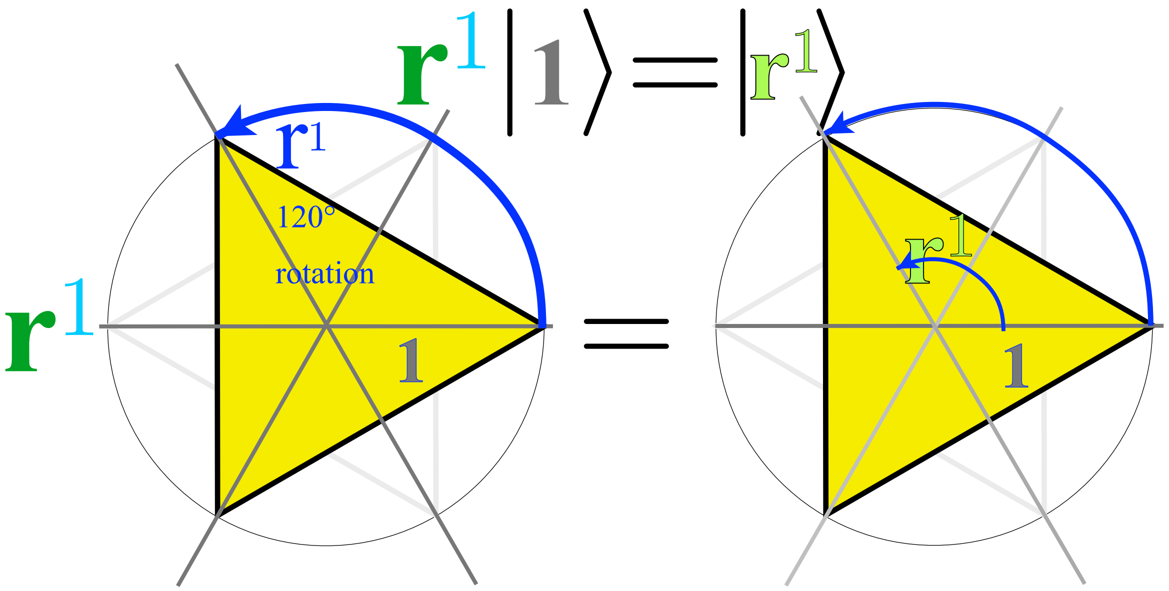
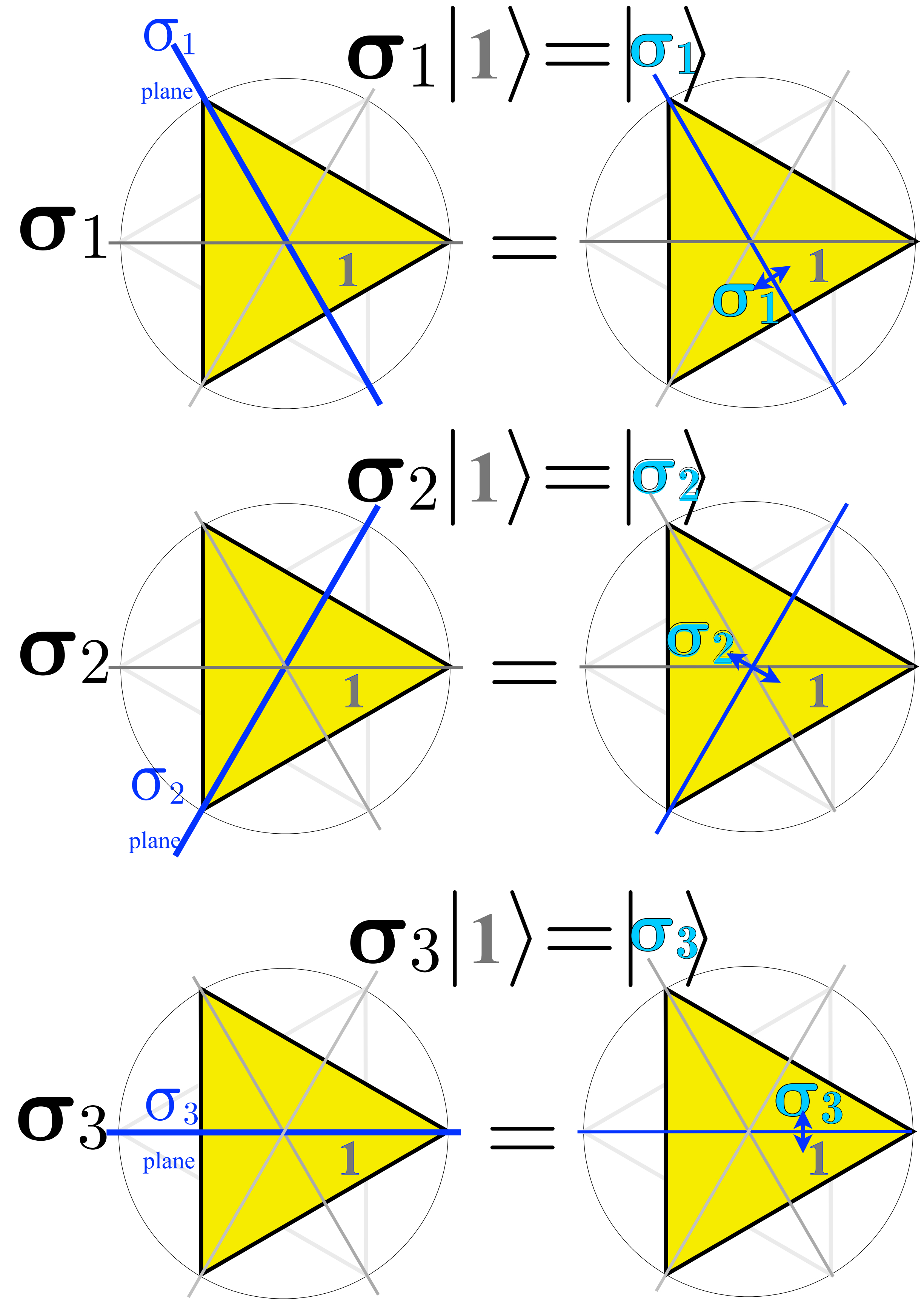
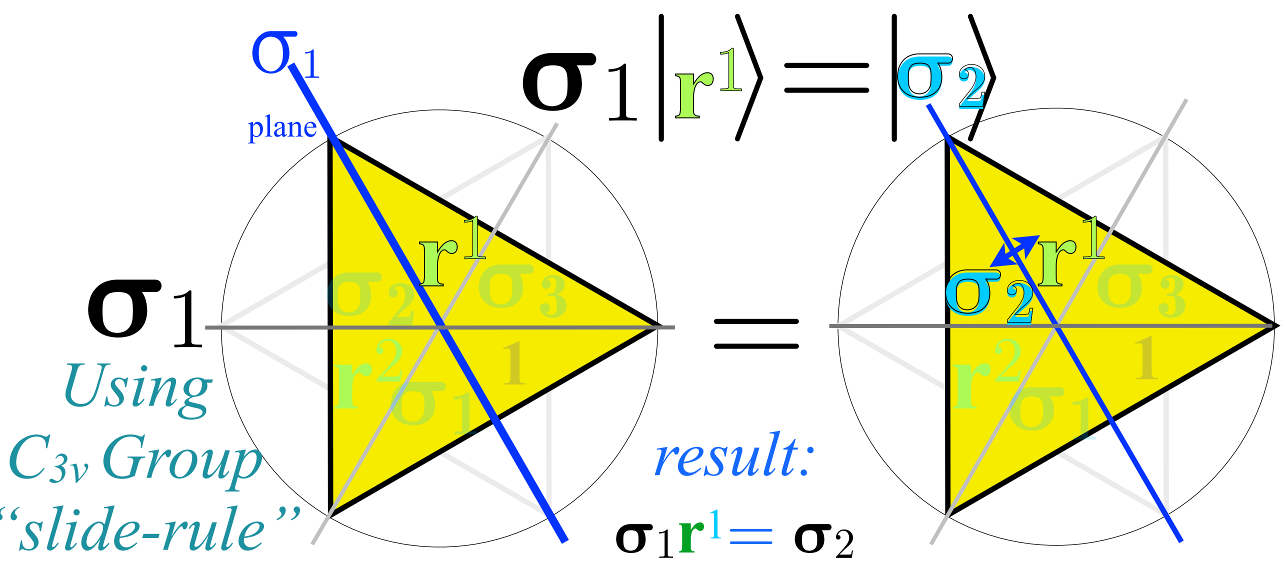
Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$





Example: Find C_{3v} product $\sigma_1\mathbf{r}^1|1\rangle = \sigma_1|\mathbf{r}^1\rangle$

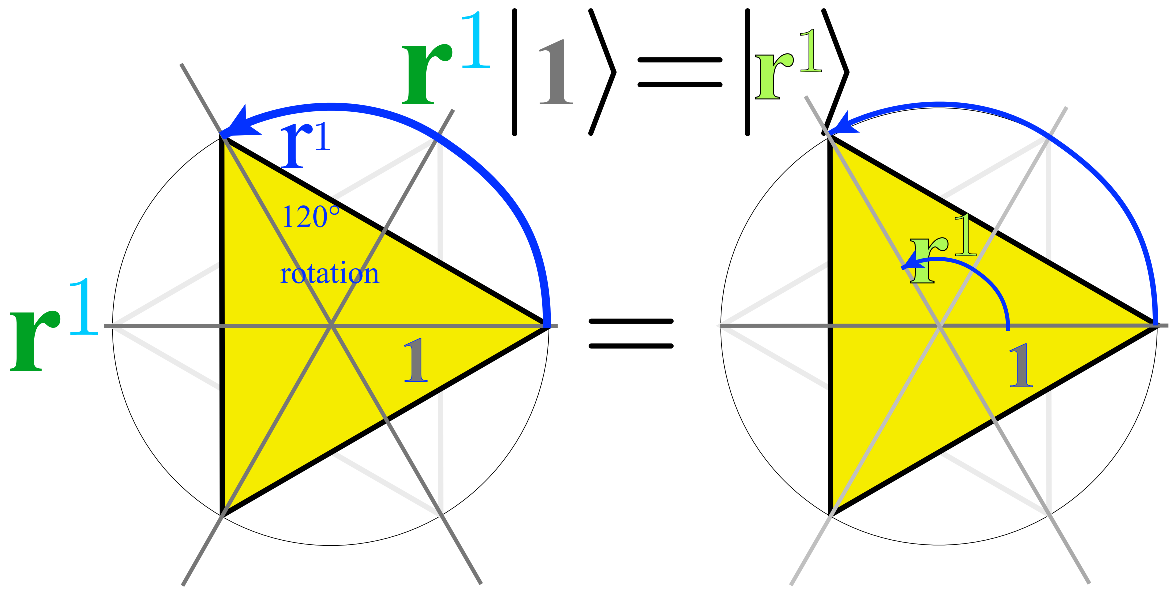


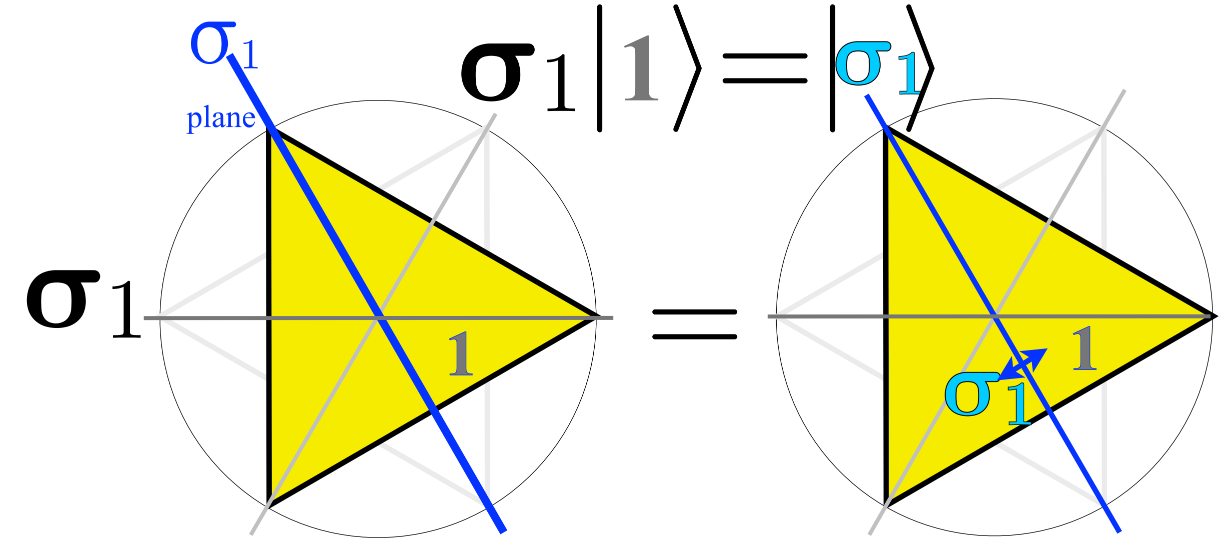
Other σ_1 *results from graph:*

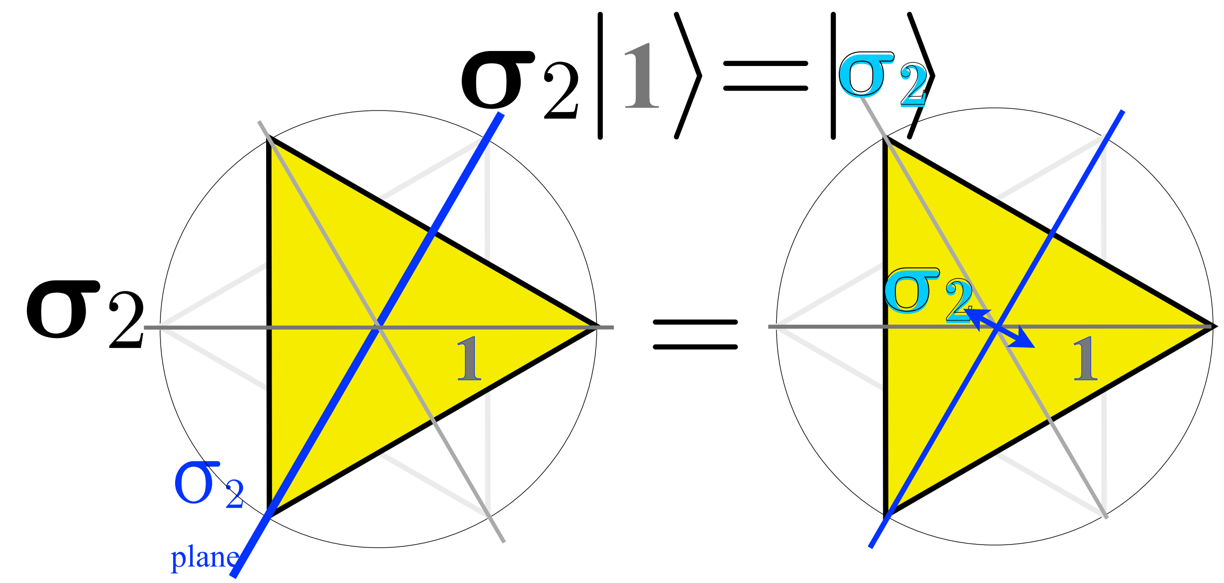
$\sigma_1\{\mathbf{1}, \mathbf{r}^1, \mathbf{r}^2, \sigma_1, \sigma_2, \sigma_3\}$

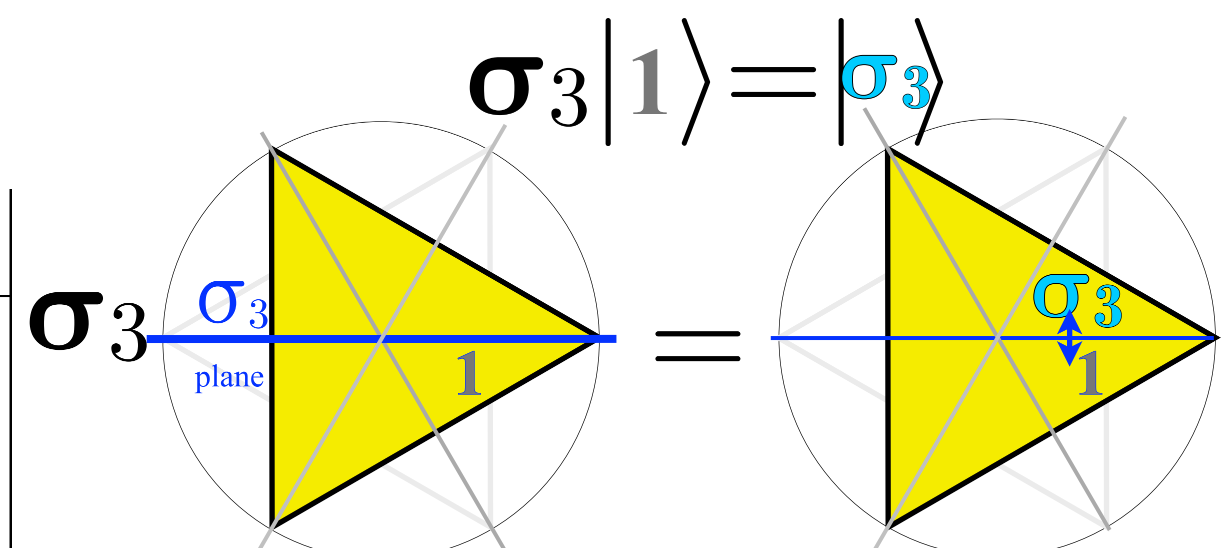
$= \{\sigma_1, \sigma_2, \sigma_3, \mathbf{1}, \mathbf{r}^1, \mathbf{r}^2\}$

Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$

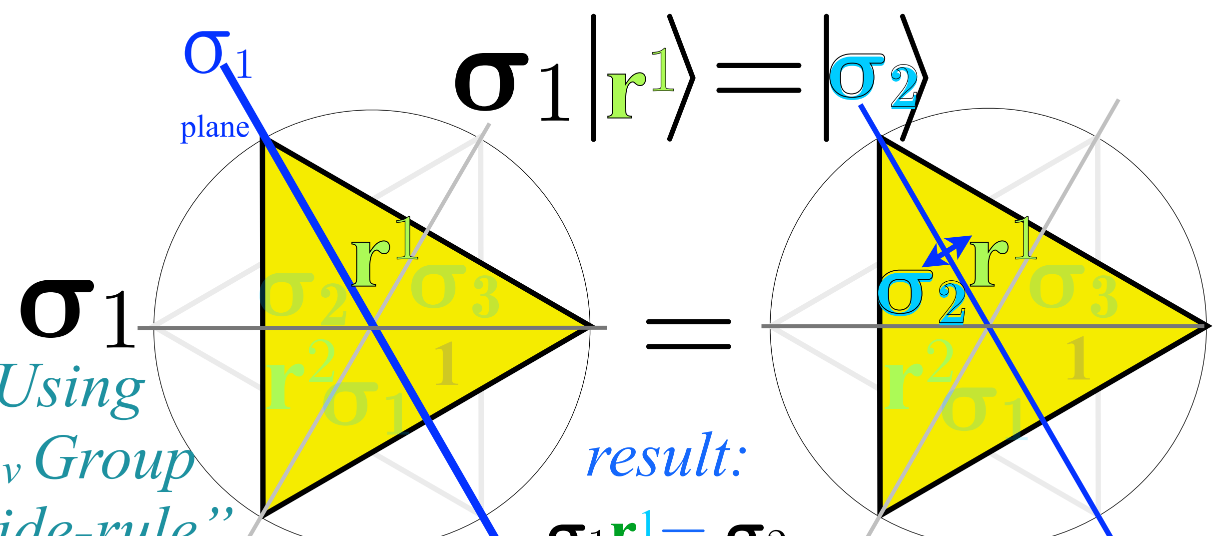








Example: Find C_{3v} product $\sigma_1\mathbf{r}^1|1\rangle = \sigma_1|\mathbf{r}^1\rangle$



Using C_{3v} Group "slide-rule"

result: $\sigma_1\mathbf{r}^1 = \sigma_2$

Other σ_1 results from graph:

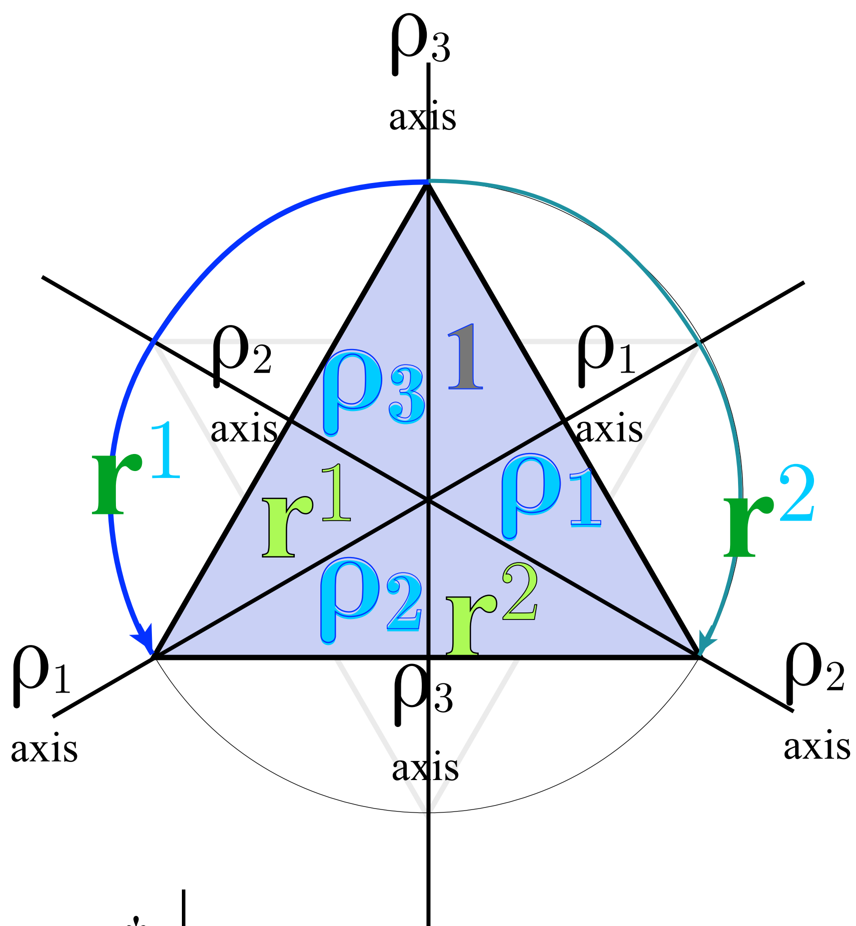
$\sigma_1\{1, \mathbf{r}^1, \mathbf{r}^2, \sigma_1, \sigma_2, \sigma_3\}$

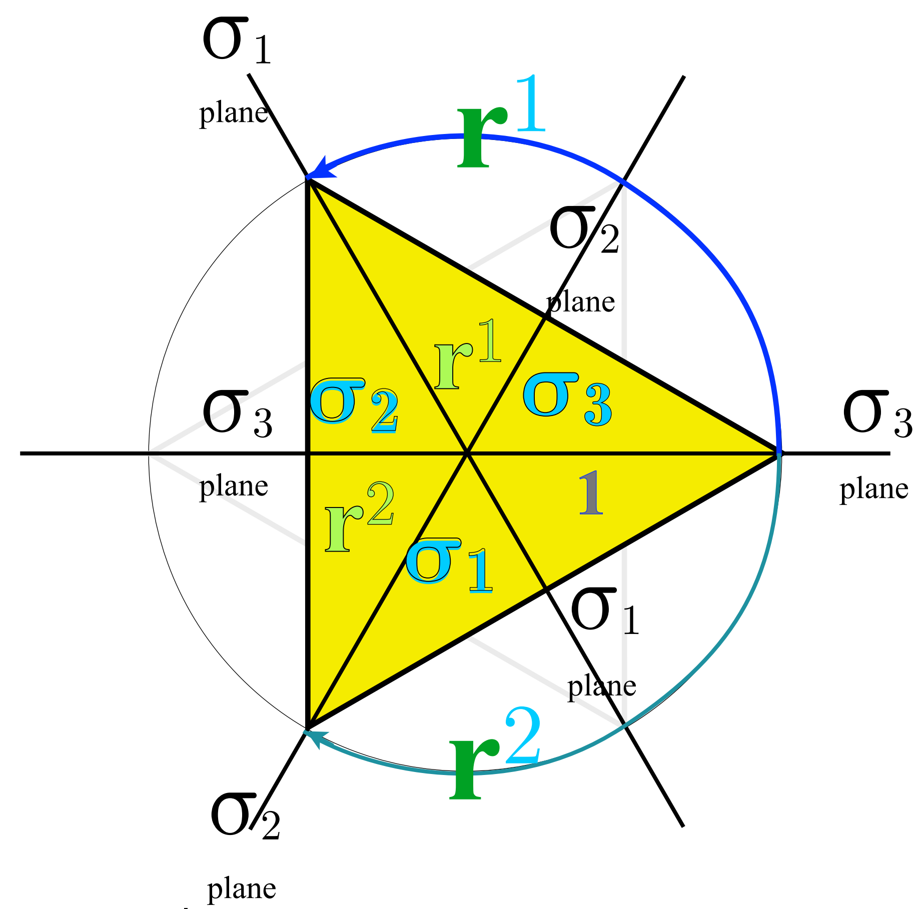
$= \{\sigma_1, \sigma_2, \sigma_3, 1, \mathbf{r}^1, \mathbf{r}^2\}$

....whole C_{3v} group table:

C_{3v} $gg^\dagger$ *form*	1	r^2	r^1	σ_1	σ_2	σ_3
1	1	r^2	r^1	σ_1	σ_2	σ_3
r^1	r^1	1	r^2	σ_3	σ_1	σ_2
r^2	r^2	r^1	1	σ_2	σ_3	σ_1
σ_1	σ_1	σ_3	σ_2	1	r^1	r^2
σ_2	σ_2	σ_1	σ_3	r^2	1	r^1
σ_3	σ_3	σ_2	σ_1	r^1	r^2	1

Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$





D_3 $\mathbf{gg}^\dagger$ *form*	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{r}^1$	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$	ρ_3	ρ_1	ρ_2
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$	ρ_2	ρ_3	ρ_1
ρ_1	ρ_1	ρ_3	ρ_2	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
ρ_2	ρ_2	ρ_1	ρ_3	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
ρ_3	ρ_3	ρ_2	ρ_1	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

D_3 and C_{3v} clearly are isomorphic $D_3 \sim C_{3v}$ share group table

↔

...except for notation
$\rho_k \leftrightarrow \sigma_k$

C_{3v} $\mathbf{gg}^\dagger$ *form*	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	σ_1	σ_2	σ_3
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	σ_1	σ_2	σ_3
$\mathbf{r}^1$	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$	σ_3	σ_1	σ_2
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$	σ_2	σ_3	σ_1
σ_1	σ_1	σ_3	σ_2	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
σ_2	σ_2	σ_1	σ_3	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
σ_3	σ_3	σ_2	σ_1	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

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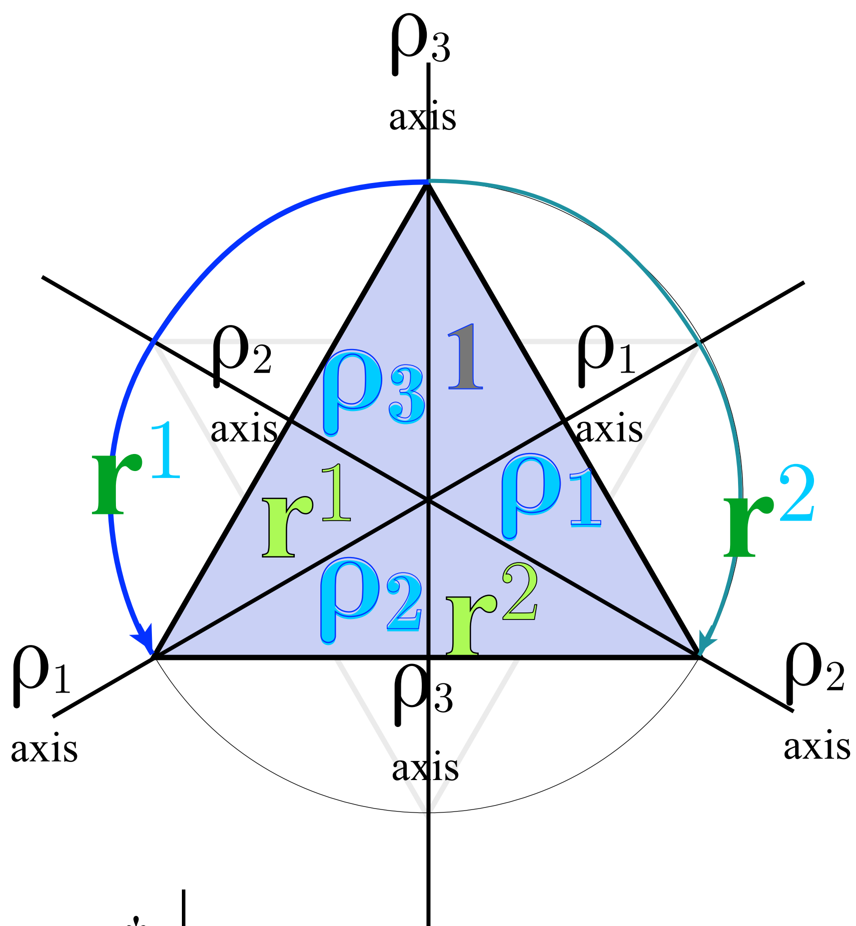
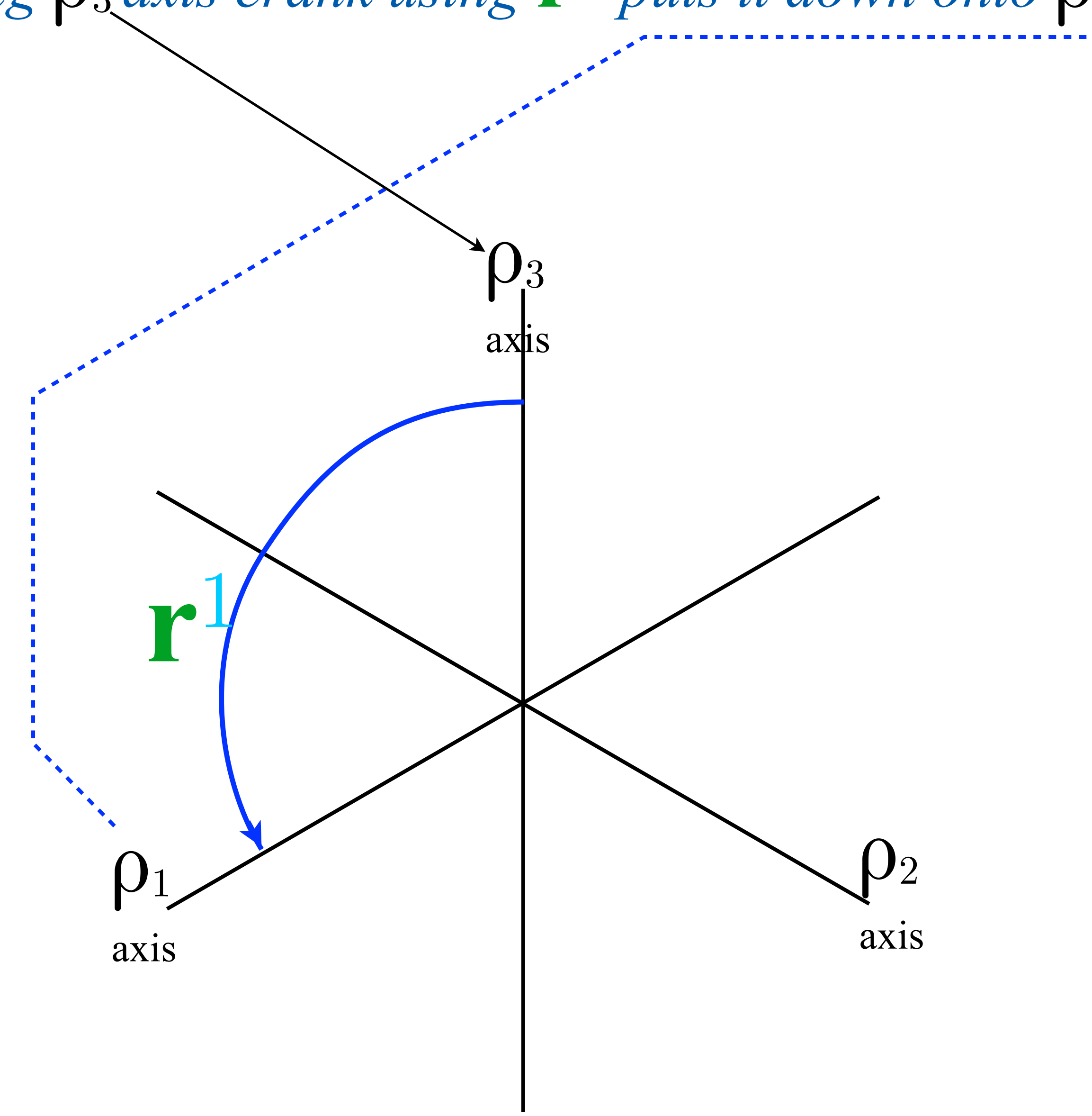


Transforming D_3 operators using D_3 operators

Example: Rotating ρ_3 axis crank using $\mathbf{r}^1$ puts it down onto ρ_1



D_3 **gg**† *form*	**1**	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
1	**1**	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{r}^1$	$\mathbf{r}^1$	**1**	$\mathbf{r}^2$	ρ_3	ρ_1	ρ_2
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	**1**	ρ_2	ρ_3	ρ_1
ρ_1	ρ_1	ρ_3	ρ_2	**1**	$\mathbf{r}^1$	$\mathbf{r}^2$
ρ_2	ρ_2	ρ_1	ρ_3	$\mathbf{r}^2$	**1**	$\mathbf{r}^1$
ρ_3	ρ_3	ρ_2	ρ_1	$\mathbf{r}^1$	$\mathbf{r}^2$	**1**

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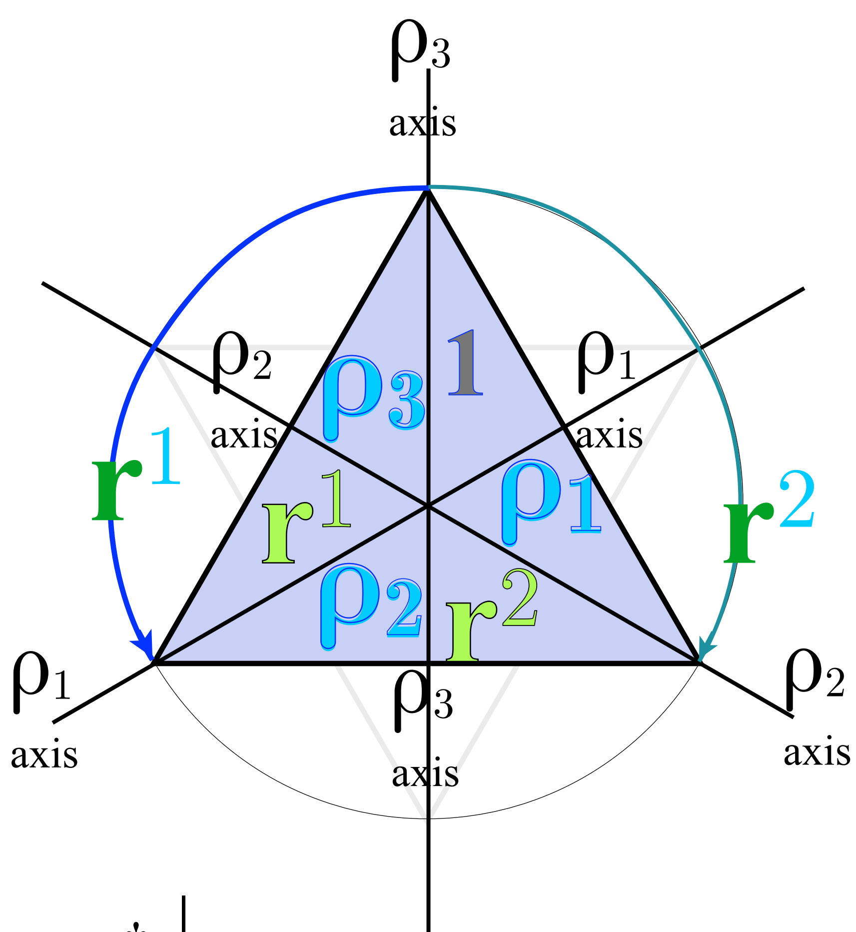

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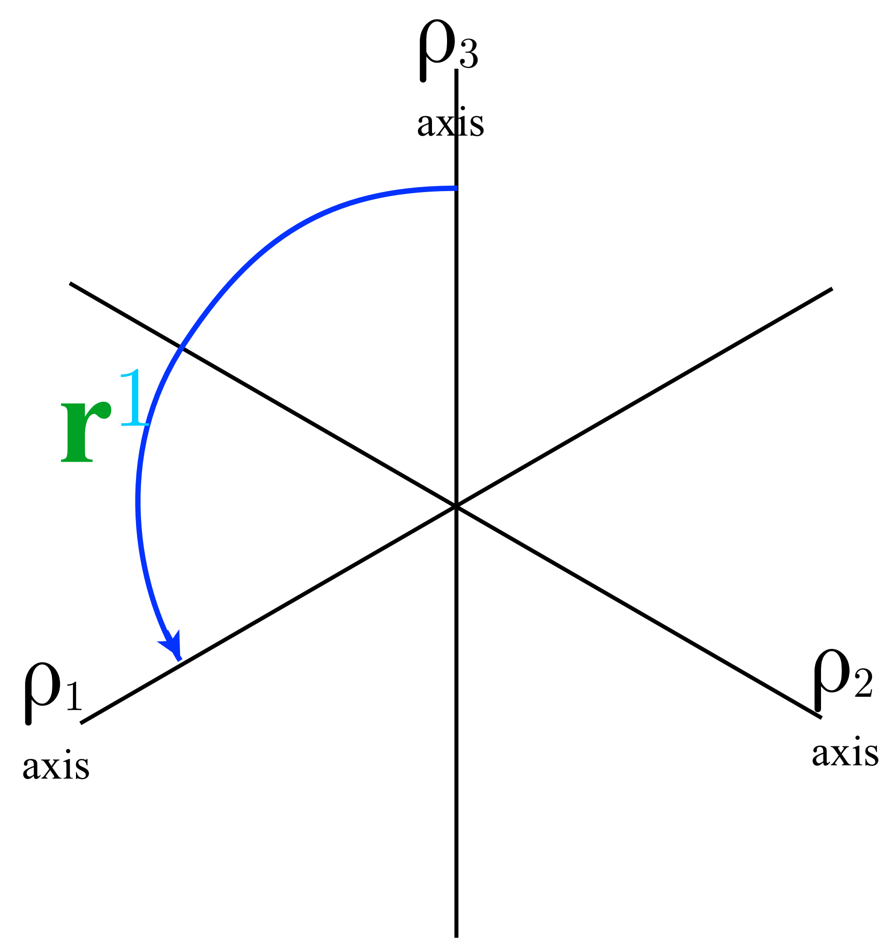
Seems to imply: $\mathbf{r}^1\rho_3(\mathbf{r}^1)^{-1} = \mathbf{r}^1\rho_3\mathbf{r}^2 = \rho_1$



D_3 $gg^\dagger$ *form*	**1**	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
1	**1**	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{r}^1$	$\mathbf{r}^1$	**1**	$\mathbf{r}^2$	ρ_3	ρ_1	ρ_2
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	**1**	ρ_2	ρ_3	ρ_1
ρ_1	ρ_1	ρ_3	ρ_2	**1**	$\mathbf{r}^1$	$\mathbf{r}^2$
ρ_2	ρ_2	ρ_1	ρ_3	$\mathbf{r}^2$	**1**	$\mathbf{r}^1$
ρ_3	ρ_3	ρ_2	ρ_1	$\mathbf{r}^1$	$\mathbf{r}^2$	**1**

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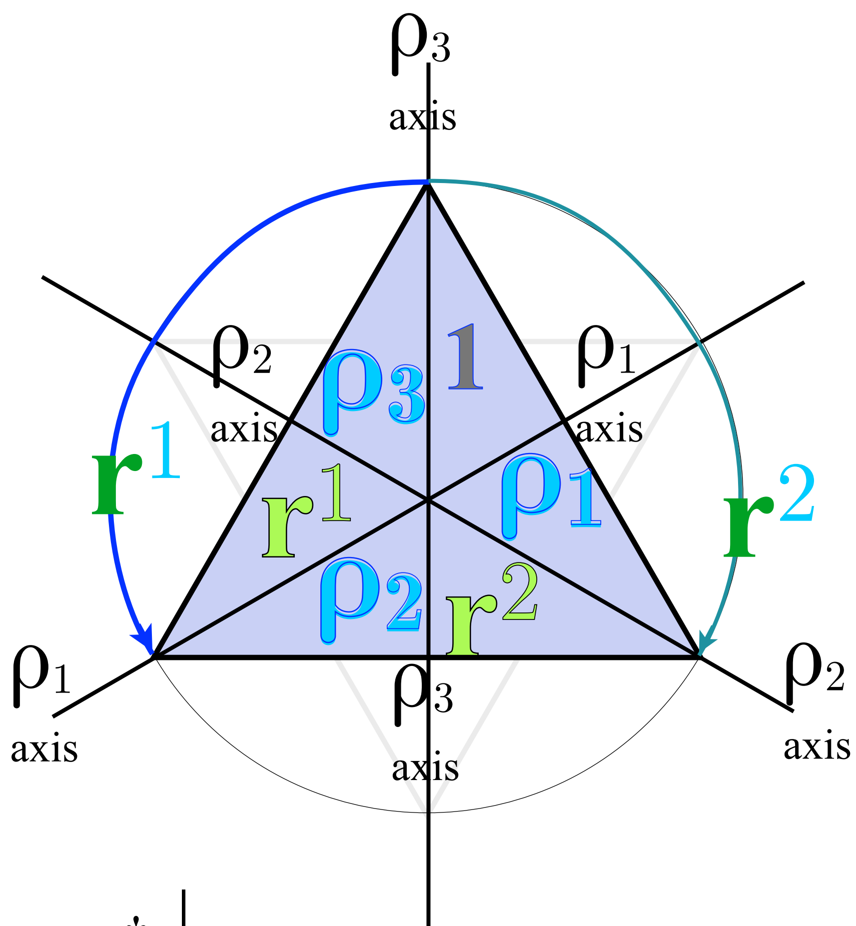

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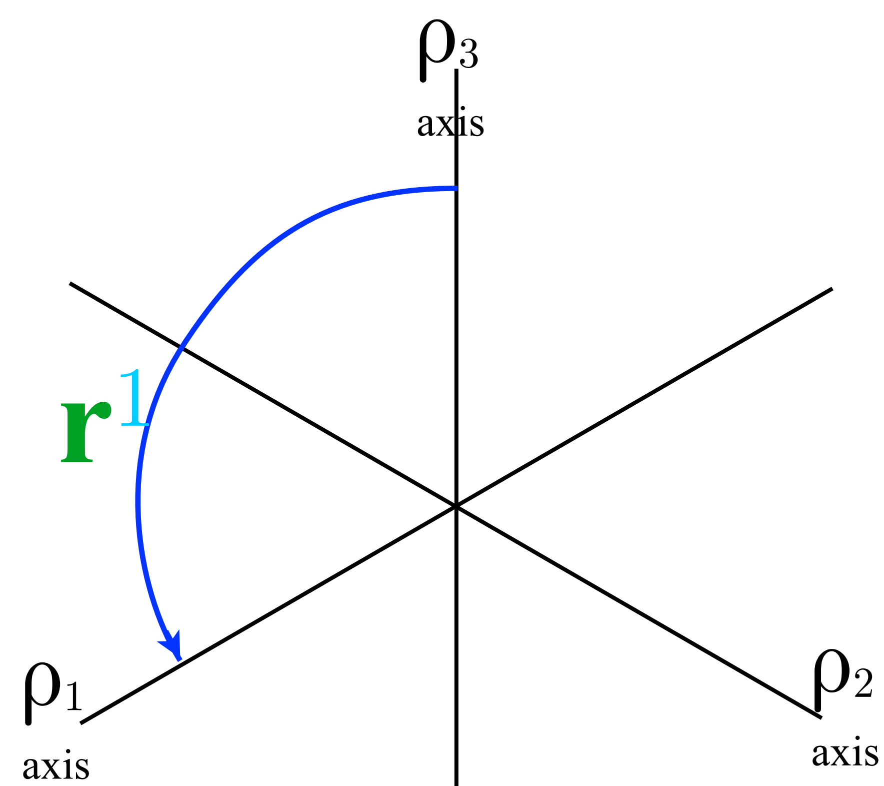


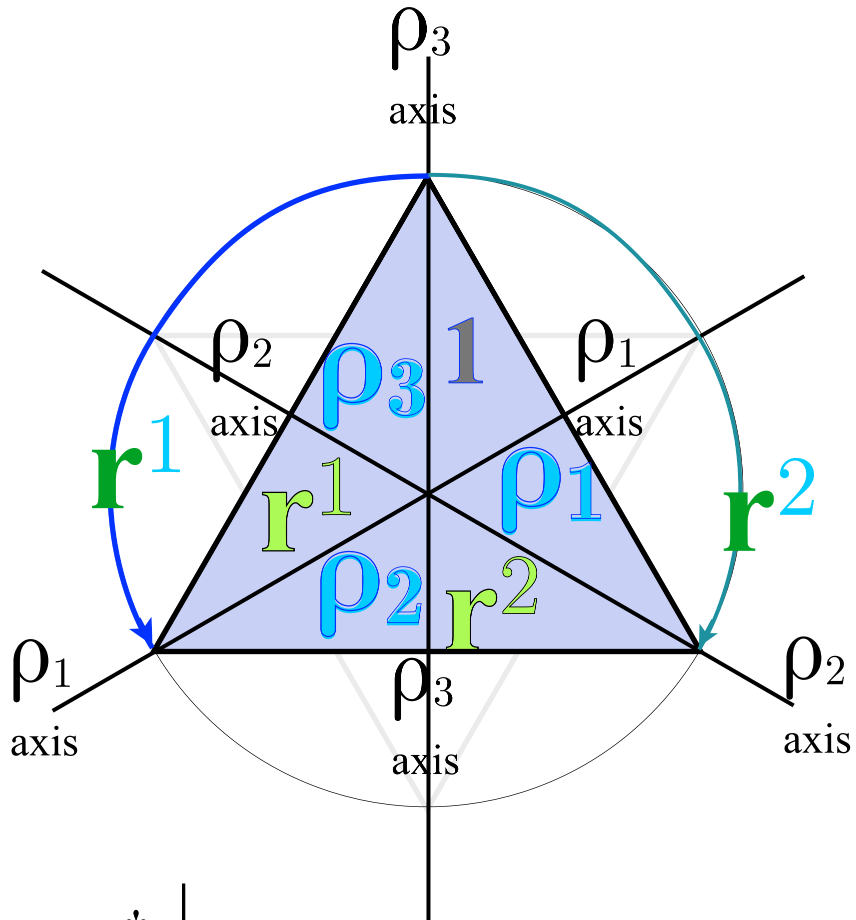
D_3 $\mathbf{gg}^\dagger$ *form*	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{r}^1$	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$	ρ_3	ρ_1	ρ_2
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$	ρ_2	ρ_3	ρ_1
ρ_1	ρ_1	ρ_3	ρ_2	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
ρ_2	ρ_2	ρ_1	ρ_3	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
ρ_3	ρ_3	ρ_2	ρ_1	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

Need to check that with table:

$\mathbf{r}^1\rho_3\mathbf{r}^2 = \rho_2\mathbf{r}^2 = \rho_1$

Checks out!

Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes



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1	**1**	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{r}^1$	$\mathbf{r}^1$	**1**	$\mathbf{r}^2$	ρ_3	ρ_1	ρ_2
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	**1**	ρ_2	ρ_3	ρ_1
ρ_1	ρ_1	ρ_3	ρ_2	**1**	$\mathbf{r}^1$	$\mathbf{r}^2$
ρ_2	ρ_2	ρ_1	ρ_3	$\mathbf{r}^2$	**1**	$\mathbf{r}^1$
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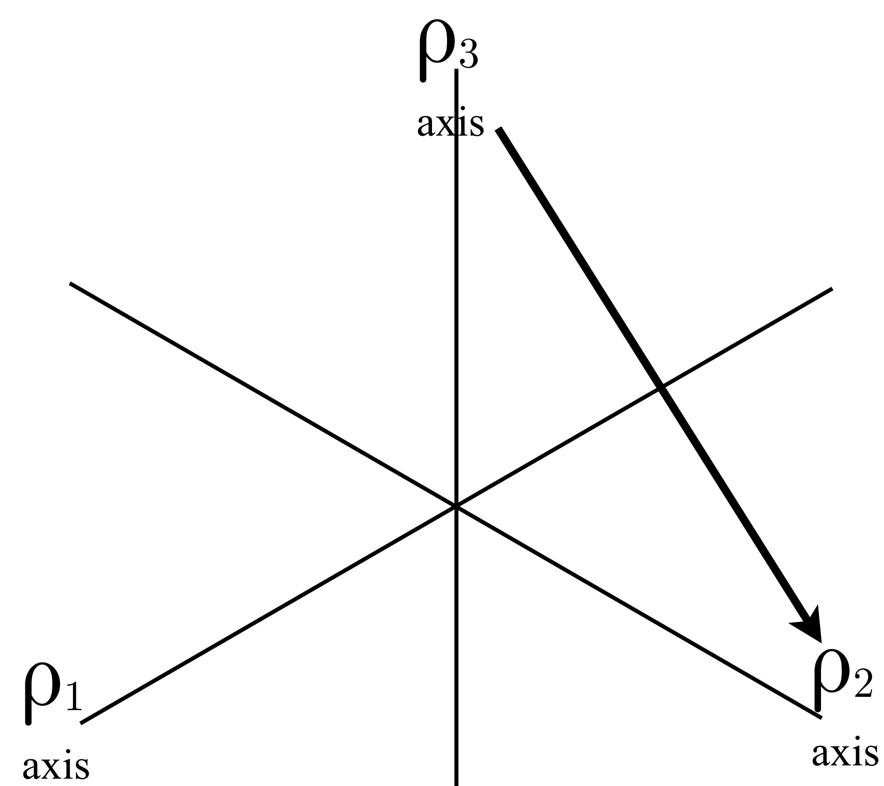

Need to check that with table:

$\rho_1\rho_3\rho_1 = \mathbf{r}^2\rho_1 = \rho_2$ *Checks out!*

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Abelian (*Commutative*) $C_2, C_3, \ldots, C_6 \ldots$

H diagonalized by r^p *symmetry operators that COMMUTE with H* $(r^p H = H r^p)$,

and with each other $(r^p r^q = r^{p+q} = r^q r^p)$.

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<u>Abelian</u> (Commutative) C_2, C_3, ...,C_6 ...

H diagonalized by r^p symmetry operators that COMMUTE with H ($r^p H = H r^p$),

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What we need to learn now:

<u>Non-Abelian</u> (do <u>not</u> commute) D_3, O_h,...

While all H symmetry operations COMMUTE with H ($\mathbf{U} H = H \mathbf{U}$)

most do <u>not</u> with each other ($\mathbf{U}\mathbf{V} \neq \mathbf{V}\mathbf{U}$).

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While all H symmetry operations COMMUTE with H $(\mathbf{U} H = H \mathbf{U})$

most do not with each other $(\mathbf{U}\mathbf{V} \neq \mathbf{V}\mathbf{U})$.

Q: *So how do we write* H *in terms of non-commutative* $\mathbf{U}$ *?*

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"Give me a place to stand...
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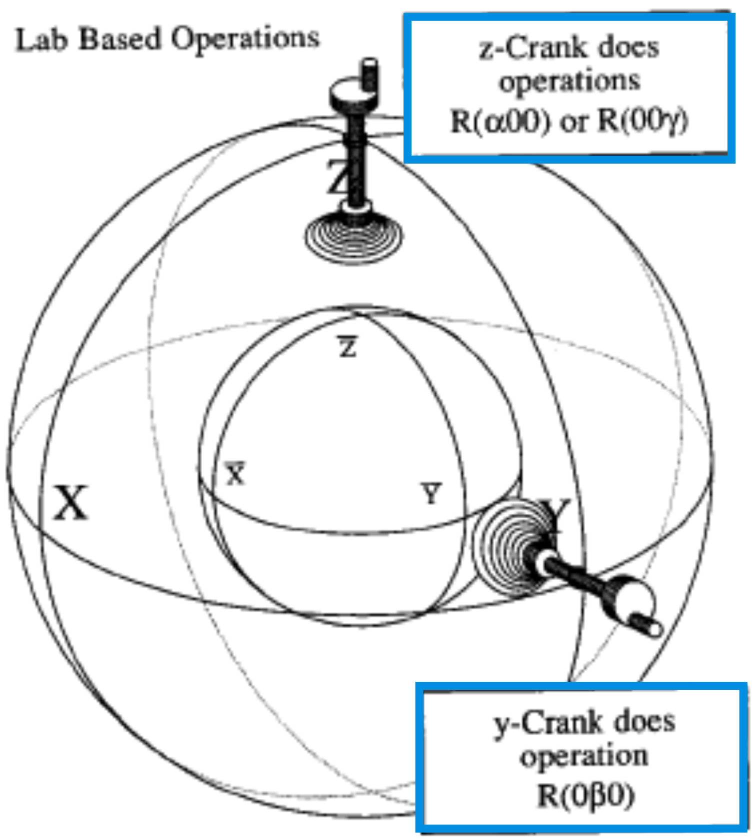
Archimedes 287-212 B.C.E

Ideas of duality/relativity go *way* back (...VanVleck, Casimir..., Mach, Newton, Archimedes...)

Lab-fixed (Extrinsic-Global)**R**



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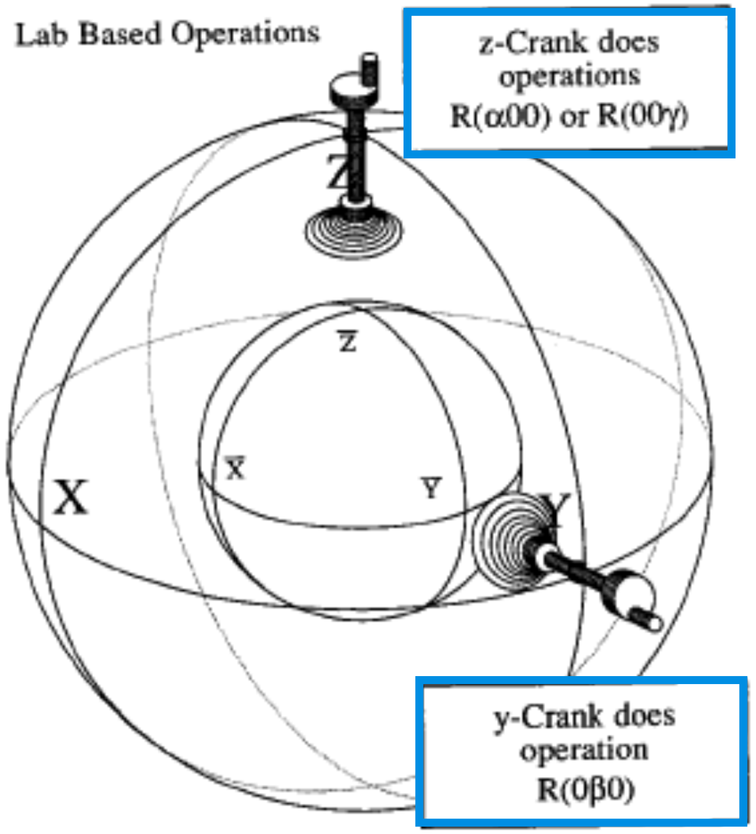

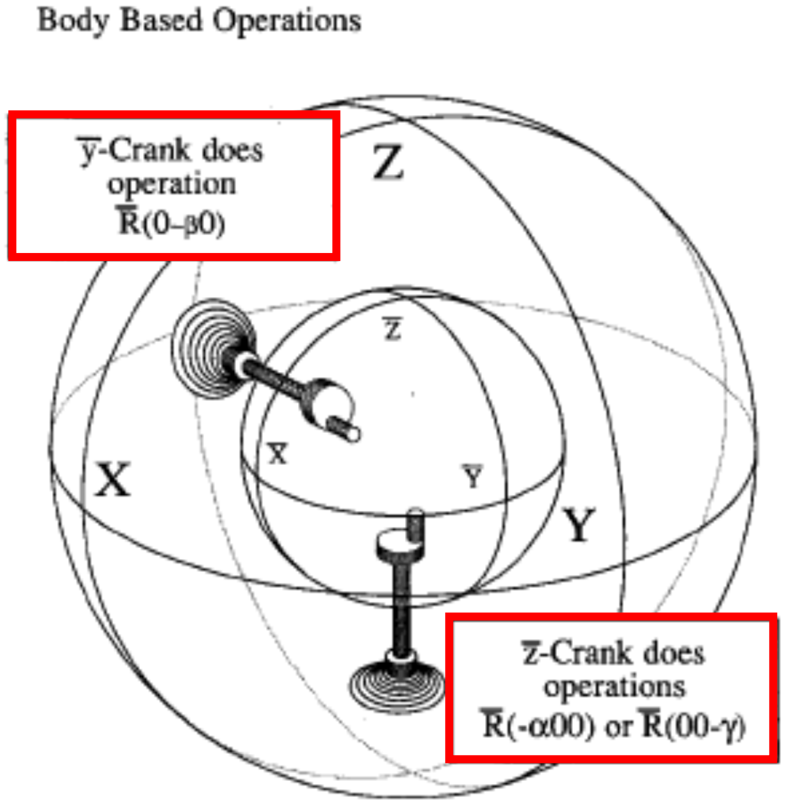

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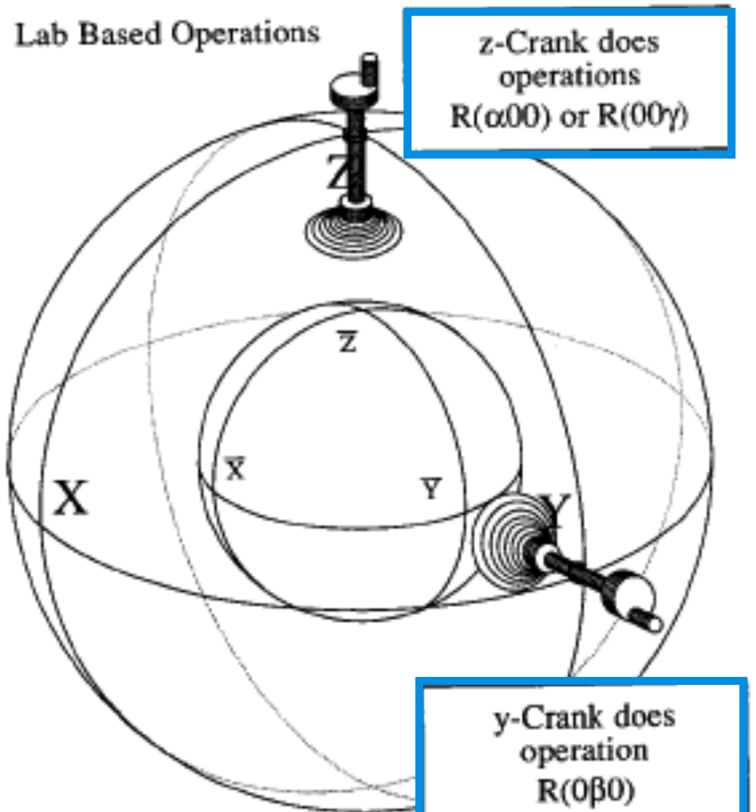
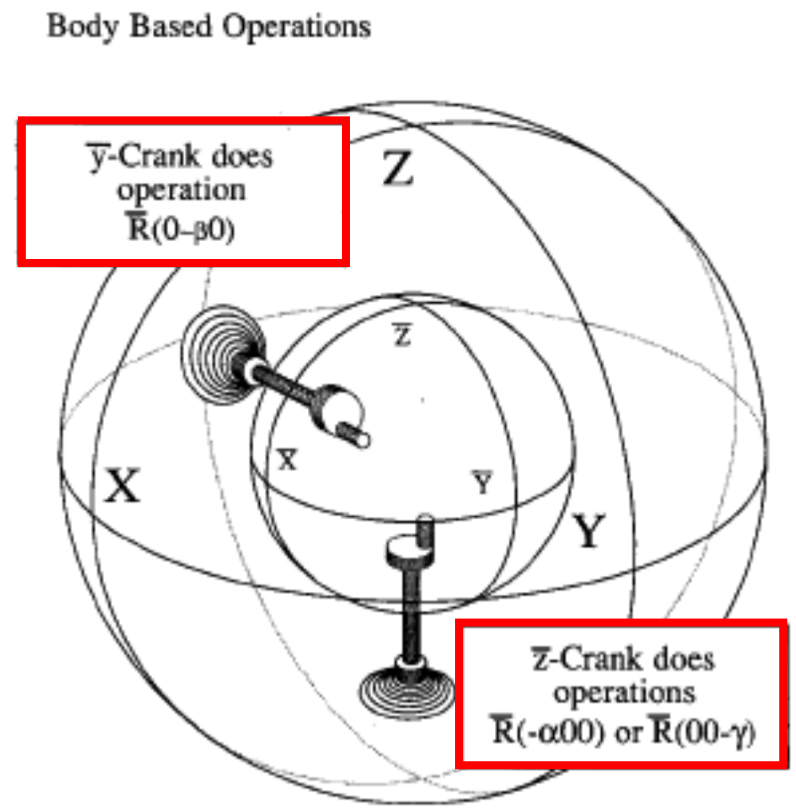


R *commutes*
with *all* $\bar{\mathbf{R}}$
(because they're independent or "unentangled")



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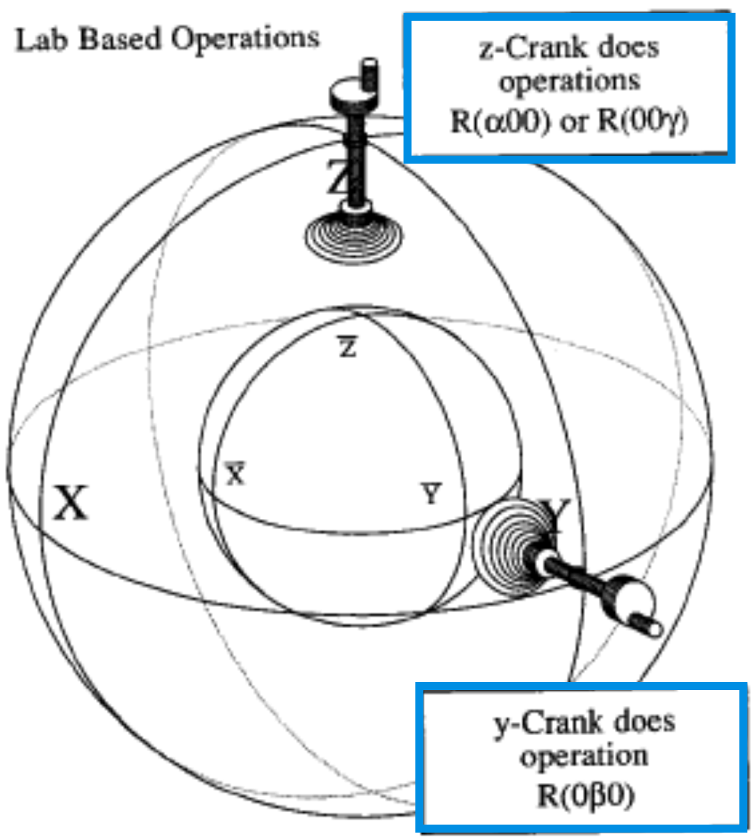

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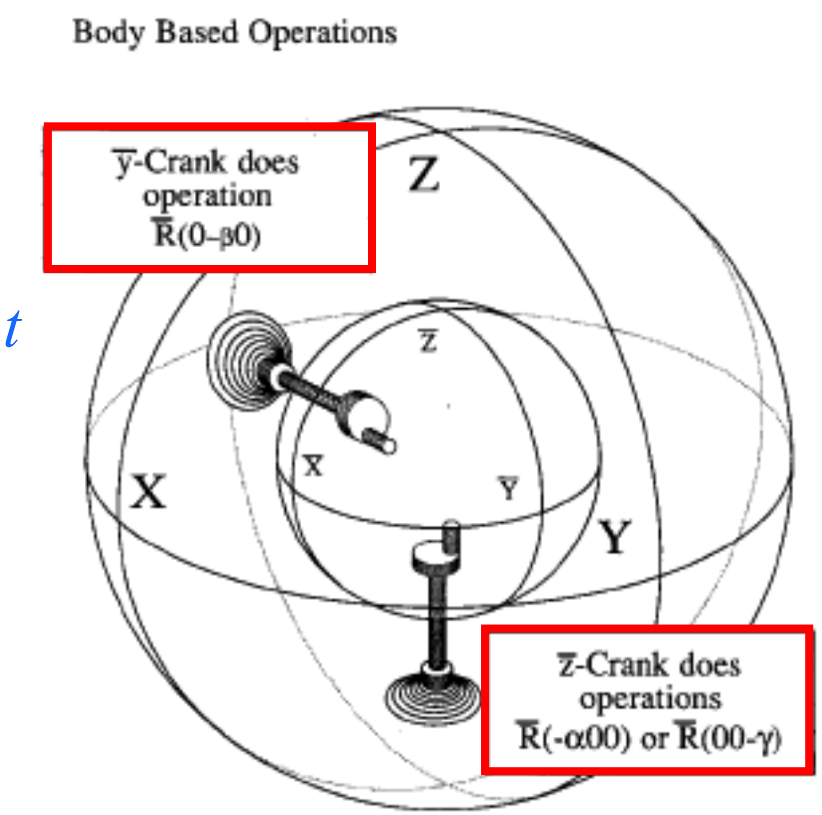
Mock-Mach relativity principle

$$\mathbf{R}|1\rangle = \bar{\mathbf{R}}^{-1}|1\rangle$$

...for *one* state |1⟩ *only!*



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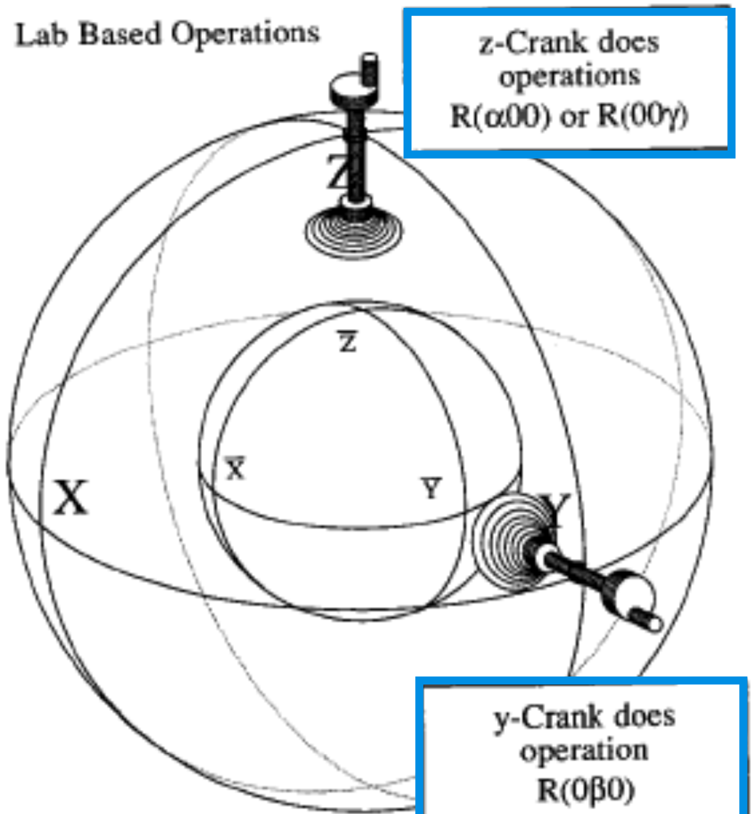
Lab Based Operations



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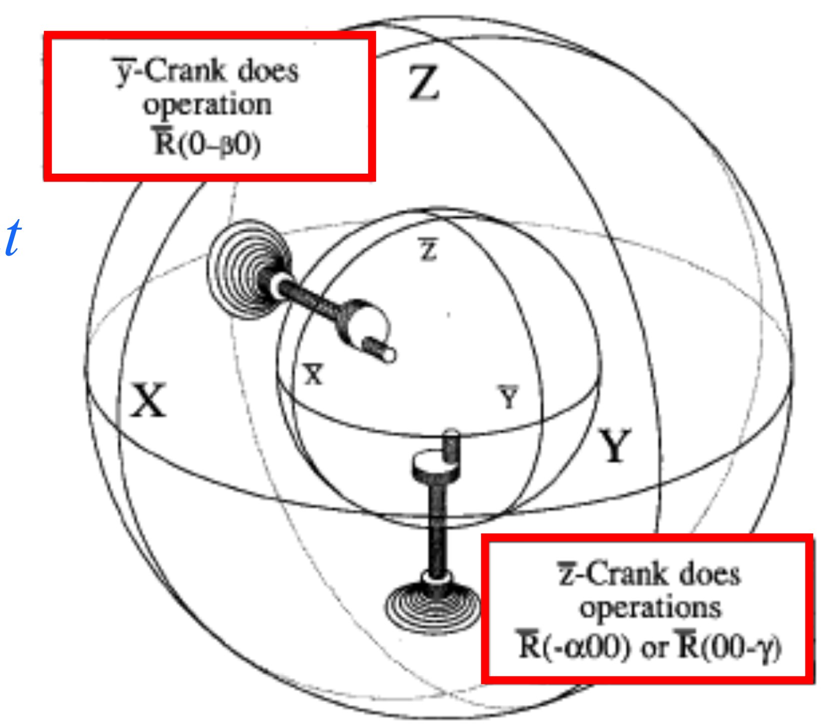
Body Based Operations



...But *how* do you actually *make* the **R** and $\bar{\mathbf{R}}$ operations?

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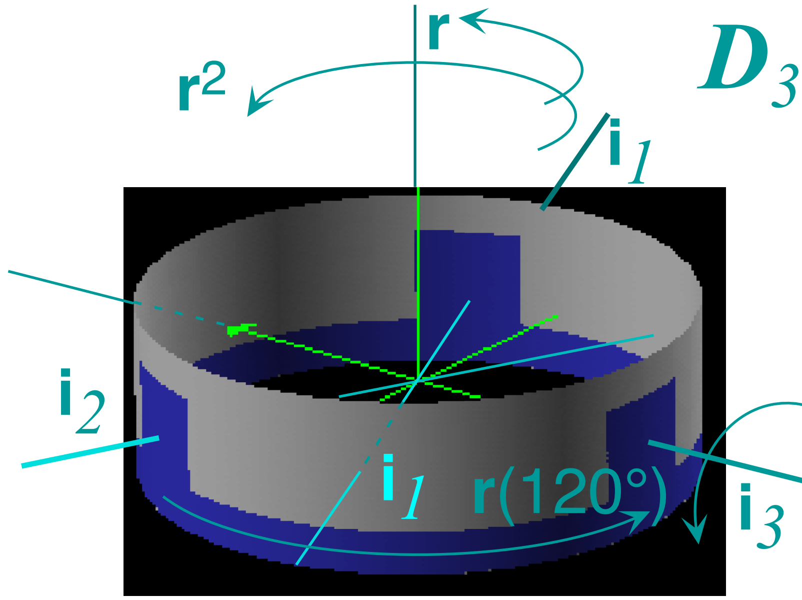
Example of GLOBAL vs LOCAL symmetry algebra for $D_3 \sim C_{3v}$

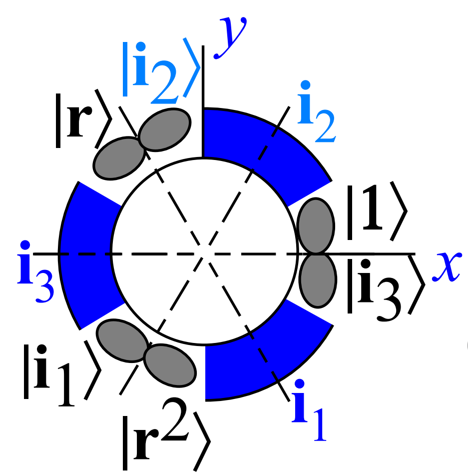


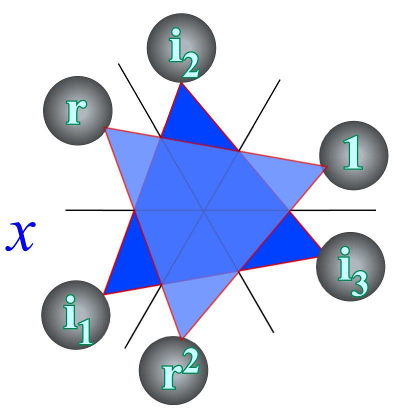
D_3

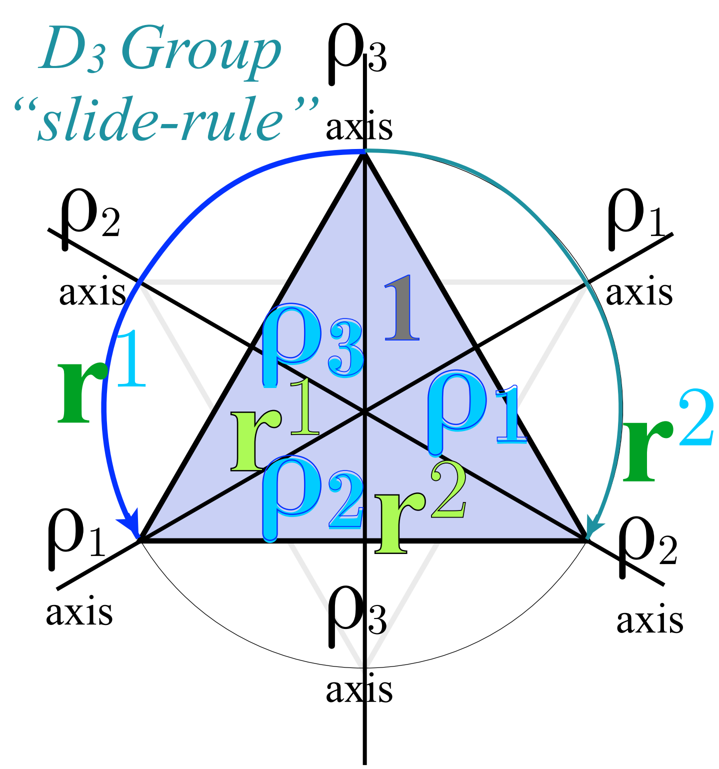
1	r^2	r	i_1	i_2	i_3
r	1	r^2	i_3	i_1	i_2
r^2	r	1	i_2	i_3	i_1
i_1	i_3	i_2	1	r	r^2
i_2	i_1	i_3	r^2	1	r
i_3	i_2	i_1	r	r^2	1

Here group operator notation 1, **r**, $\mathbf{r}^2$, $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ matches 1, $\mathbf{r}^1$, $\mathbf{r}^2$, ρ_1, ρ_2, ρ_3 used previously.

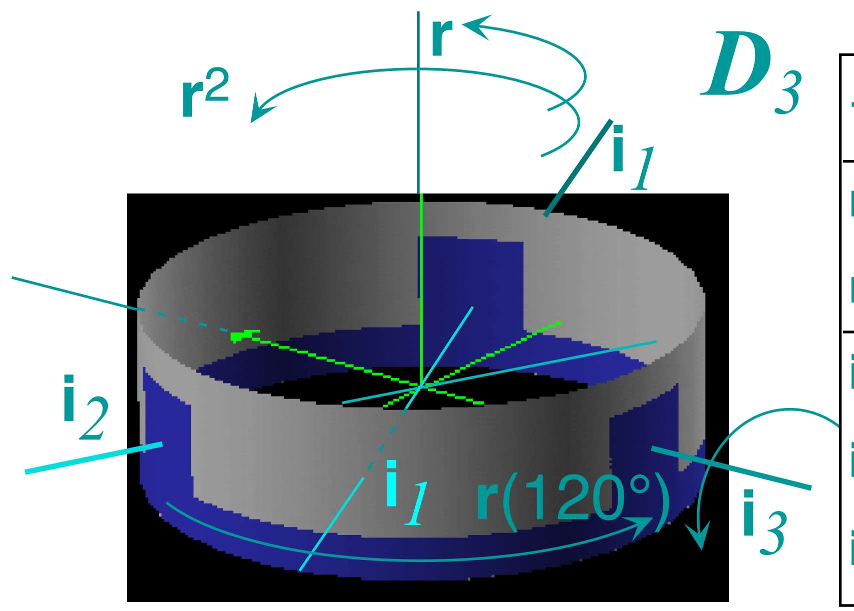
D_3-defined local-wave bases







Example of GLOBAL vs LOCAL symmetry algebra for $D_3 \sim C_{3v}$



D_3

1	r^2	r	i_1	i_2	i_3
r	1	r^2	i_3	i_1	i_2
r^2	r	1	i_2	i_3	i_1
i_1	i_3	i_2	1	r	r^2
i_2	i_1	i_3	r^2	1	r
i_3	i_2	i_1	r	r^2	1

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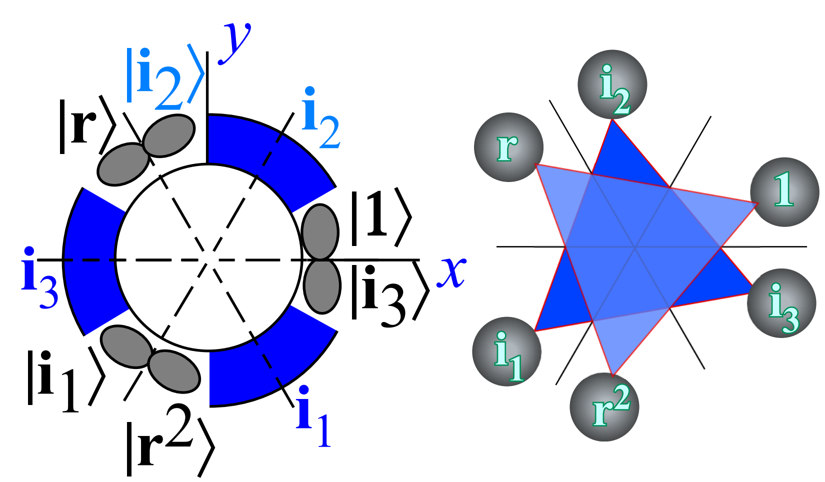
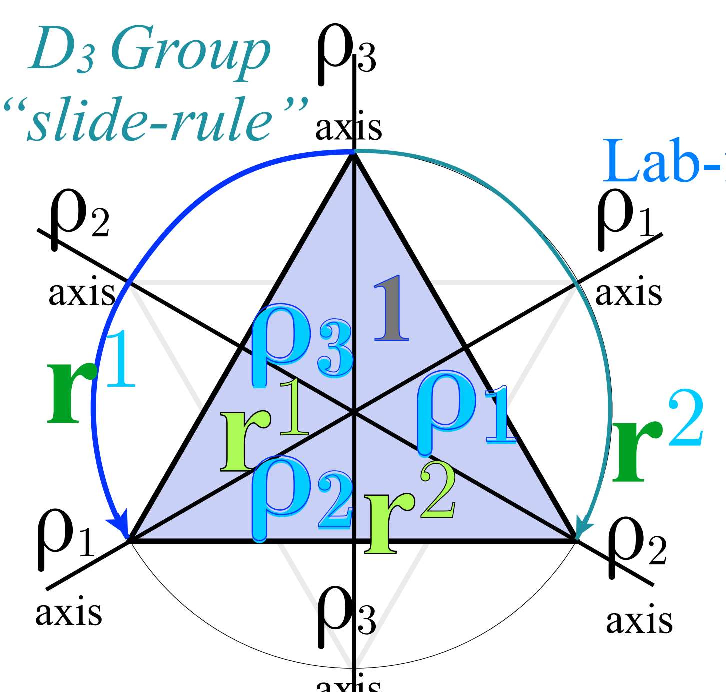
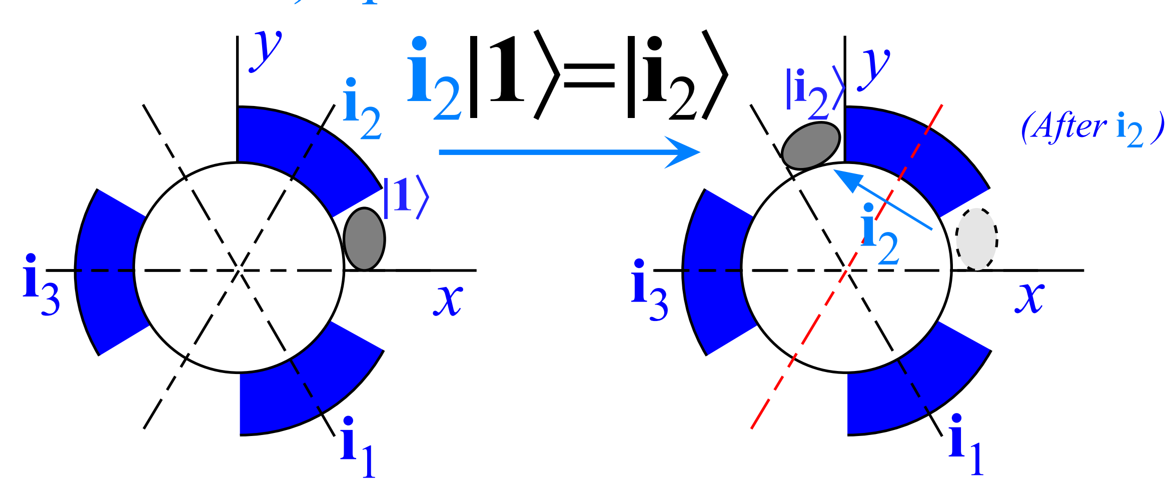


D_3 Group "slide-rule"



Lab-fixed (Extrinsic-Global) operations and rotation axes



Global vs Local symmetry matrix duality for D_3

Example of RELATIVITY-DUALITY for $D_3 \sim C_{3v}$

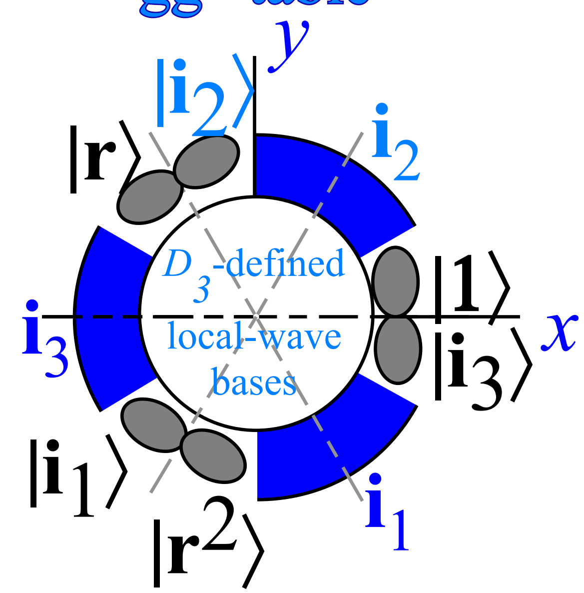
To represent *external* {..**T**,**U**,**V**,... }switch $\mathbf{g} \leftrightarrow \mathbf{g}^\dagger$ on top of group table

$$R^G(\mathbf{1}) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad R^G(\mathbf{r}) = \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix}, \quad R^G(\mathbf{r}^2) = \begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix},$$

$$R^G(\mathbf{i}_1) = \begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix}, \quad R^G(\mathbf{i}_2) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad R^G(\mathbf{i}_3) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

1	$\mathbf{r}^2$	**r**	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
r	**1**	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	**r**	**1**	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	**1**	**r**	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	**1**	**r**
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	**r**	$\mathbf{r}^2$	**1**

***D_3* global $\mathbf{gg}^\dagger$-table**



Global vs Local symmetry matrix duality for D_3

Example of RELATIVITY-DUALITY for $D_3 \sim C_{3v}$

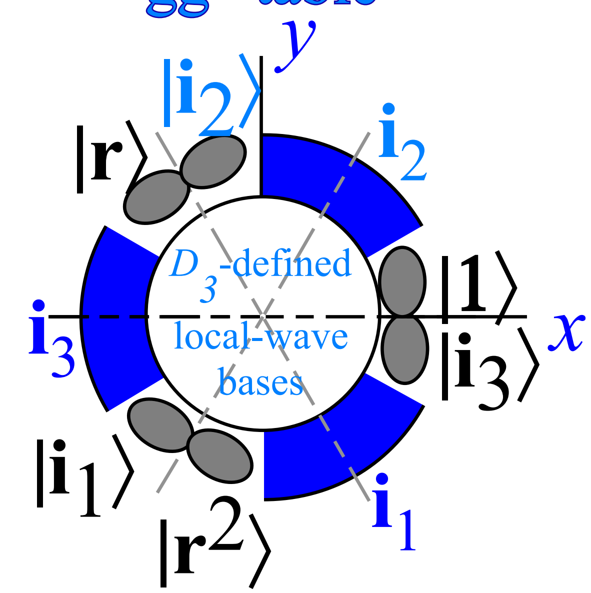
To represent *external* {..**T**,**U**,**V**,... } switch **g** ⇄ **g**† on top of group table

$$R^G(\mathbf{1})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\ R^G(\mathbf{r})=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\ R^G(\mathbf{r}^2)=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\mathbf{i}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix},\ R^G(\mathbf{i}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\ R^G(\mathbf{i}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

1	r^2	r	i_1	i_2	i_3
r	1	r^2	i_3	i_1	i_2
r^2	r	1	i_2	i_3	i_1
i_1	i_3	i_2	1	r	r^2
i_2	i_1	i_3	r^2	1	r
i_3	i_2	i_1	r	r^2	1

D_3 global $gg^\dagger$-table



D_3 local $g^\dagger g$-table

To represent *internal* {..$\bar{\mathbf{T}}$,$\bar{\mathbf{U}}$,$\bar{\mathbf{V}}$,... } switch **g** ⇄ **g**† on side of group table

$$R^G(\bar{\mathbf{1}})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\ R^G(\bar{\mathbf{r}})=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\ R^G(\bar{\mathbf{r}}^2)=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\bar{\mathbf{i}}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\ R^G(\bar{\mathbf{i}}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix},\ R^G(\bar{\mathbf{i}}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

1	r	r^2	i_1	i_2	i_3
r^2	1	r	i_2	i_3	i_1
r	r^2	1	i_3	i_1	i_2
i_1	i_2	i_3	1	r	r^2
i_2	i_3	i_1	r^2	1	r
i_3	i_1	i_2	r	r^2	1

Global vs Local symmetry matrix duality for D_3

Example of RELATIVITY-DUALITY for $D_3 \sim C_{3v}$

To represent *external* $\{..\mathbf{T},\mathbf{U},\mathbf{V},...\}$ switch $\mathbf{g} \leftrightarrow \mathbf{g}^\dagger$ on top of group table

1	$\mathbf{r}^2$	**r**	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
r	**1**	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	**r**	**1**	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	**1**	**r**	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	**1**	**r**
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	**r**	$\mathbf{r}^2$	**1**

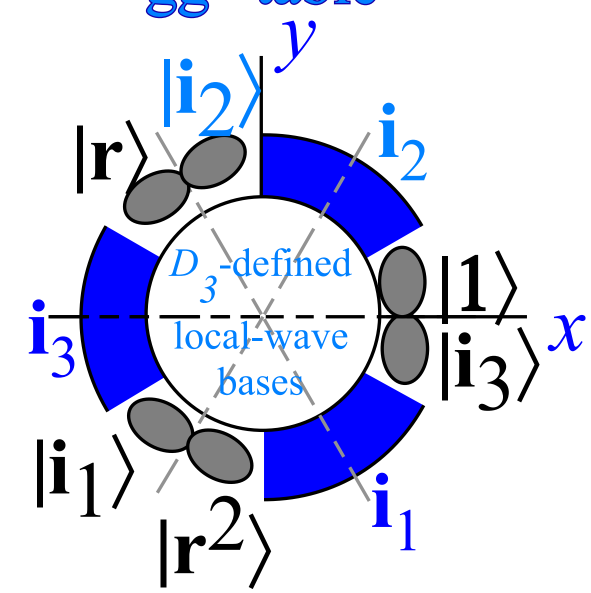
D_3 global $gg^\dagger$-table

$$R^G(\mathbf{1})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\quad R^G(\mathbf{r})=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\quad R^G(\mathbf{r}^2)=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\mathbf{i}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix},\quad R^G(\mathbf{i}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\quad R^G(\mathbf{i}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

RESULT:
Any $R(\mathbf{T})$
commute (Even if **T** and **U** do not...)
with any $R(\bar{\mathbf{U}})$...

...and **T U**=**V** if & only if $\bar{\mathbf{T}}\,\bar{\mathbf{U}}=\bar{\mathbf{V}}$.



D_3 local $g^\dagger g$-table

To represent *internal* $\{..\bar{\mathbf{T}},\bar{\mathbf{U}},\bar{\mathbf{V}},...\}$ switch $\mathbf{g} \leftrightarrow \mathbf{g}^\dagger$ on side of group table

1	**r**	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	**1**	**r**	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
r	$\mathbf{r}^2$	**1**	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	**1**	**r**	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	**1**	**r**
$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$	**r**	$\mathbf{r}^2$	**1**

$$R^G(\bar{\mathbf{1}})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\quad R^G(\bar{\mathbf{r}})=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\quad R^G(\bar{\mathbf{r}}^2)=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\bar{\mathbf{i}}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\quad R^G(\bar{\mathbf{i}}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix},\quad R^G(\bar{\mathbf{i}}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

AMOP reference links on following page

2.21.18 class 12.0: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

Discrete symmetry subgroups of O(3) and application to tunneling and vibrational dynamics: D_3 and C_{3v} group products, classes, and irrep projection operators

32 crystal point symmetries: 16 Abelian (commutative) and 16 non-Abelian groups

Smallest non-Abelian symmetry: 3-C_2-axis D_3 vs. 3-C_v-plane C_{3v} isomorphic to permutation-S_3

Relating C_2-180°rotations $\mathbf{R}_z$, C_v-plane reflections $\boldsymbol{\sigma}_z$, and inversion $\mathbf{I}$ operators

Deriving $D_3 \sim C_{3v}$ products by group definition $|g\rangle=\mathbf{g}|1\rangle$ of position ket $|g\rangle$

Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes

Non-commutative symmetry expansion and Global-Local solution

Global vs Local symmetry and Mock-Mach principle

Global vs Local matrix duality for D_3

Global vs Local symmetry expansion of D_3 Hamiltonian

*Group theory and algebra of $\mathbf{D_3}$ **Center** (Class algebra)*

Self-symmetry (Normalizer). Lagrange Coset Theorem for classes

1st-Stage spectral decomposition of "Group-table" Hamiltonian of D_3 symmetry

All-commuting operators $\boldsymbol{\kappa}_k$ All-commuting projectors $\mathbf{P}^{(\alpha)}$

D_3-invariant irep characters $\chi_k^{(\alpha)}$ Invariant numbers: Centrum, Rank, and Order

2nd-Stage spectral decompositions of global/local D_3

Subgroup chains $D_3 \supset C_2$ and $D_3 \supset C_3$ split class projectors ...and classes

3rd-Stage spectral decomposition of ALL of D_3 ...and of Hamiltonian $\mathbf{H}$

GLOBAL vs LOCAL symmetry of states ...and group $\mathbf{H}$ parameters $\{r,i_1,i_2,i_3\}$

Global vs Local symmetry expansion of D_3 Hamiltonian

Example of RELATIVITY-DUALITY for $D_3 \sim C_{3v}$

To represent *external* $\{..\mathbf{T},\mathbf{U},\mathbf{V},...\}$ switch $\mathbf{g} \leftrightarrow \mathbf{g}^\dagger$ on top of group table

$$R^G(\mathbf{1}) = \begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1 \end{pmatrix}, R^G(\mathbf{r}) = \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot \end{pmatrix}, R^G(\mathbf{r}^2) = \begin{pmatrix} \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot \end{pmatrix},$$

$$R^G(\mathbf{i}_1) = \begin{pmatrix} \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot \end{pmatrix}, R^G(\mathbf{i}_2) = \begin{pmatrix} \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot \end{pmatrix}, R^G(\mathbf{i}_3) = \begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}$$

1	r^2	r	i_1	i_2	i_3
r	1	r^2	i_3	i_1	i_2
r^2	r	1	i_2	i_3	i_1
i_1	i_3	i_2	1	r	r^2
i_2	i_1	i_3	r^2	1	r
i_3	i_2	i_1	r	r^2	1

D_3 global $gg^\dagger$-table

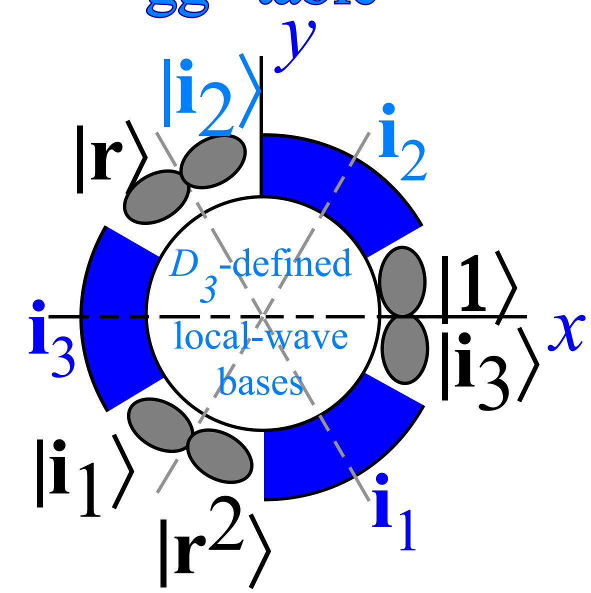
RESULT:
Any $R(\mathbf{T})$ *commute* (Even if **T** and **U** do not...) with any $R(\bar{\mathbf{U}})$...

...and **T U**=**V** if & only if $\bar{\mathbf{T}}\,\bar{\mathbf{U}}=\bar{\mathbf{V}}$.

So an $\mathbb{H}$*-matrix having* ***Global*** *symmetry* D_3

$$\mathbb{H} = H\bar{\mathbf{1}}^0 + r_1\bar{\mathbf{r}}^1 + r_2\bar{\mathbf{r}}^2 + i_1\bar{\mathbf{i}}_1 + i_2\bar{\mathbf{i}}_2 + i_3\bar{\mathbf{i}}_3$$

is made from ***Local*** *symmetry matrices*



D_3 local $g^\dagger g$-table

To represent *internal* $\{..\bar{\mathbf{T}},\bar{\mathbf{U}},\bar{\mathbf{V}},...\}$ switch $\mathbf{g} \leftrightarrow \mathbf{g}^\dagger$ on side of group table

$$R^G(\bar{\mathbf{1}}) = \begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1 \end{pmatrix}, R^G(\bar{\mathbf{r}}) = \begin{pmatrix} \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot \end{pmatrix}, R^G(\bar{\mathbf{r}}^2) = \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot \end{pmatrix},$$

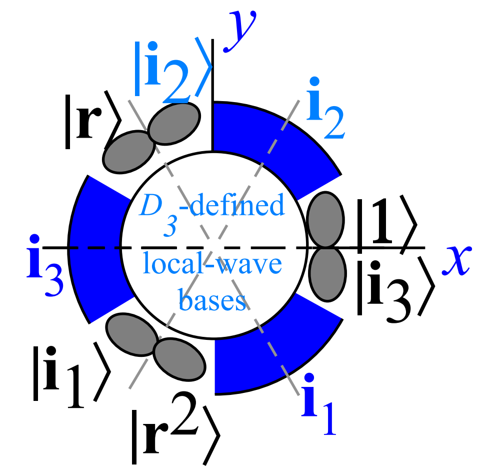
$$R^G(\bar{\mathbf{i}}_1) = \begin{pmatrix} \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot \end{pmatrix}, R^G(\bar{\mathbf{i}}_2) = \begin{pmatrix} \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot \end{pmatrix}, R^G(\bar{\mathbf{i}}_3) = \begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}$$

1	r	r^2	i_1	i_2	i_3
r^2	1	r	i_2	i_3	i_1
r	r^2	1	i_3	i_1	i_2
i_1	i_2	i_3	1	r	r^2
i_2	i_3	i_1	r^2	1	r
i_3	i_1	i_2	r	r^2	1

Global vs Local symmetry expansion of D_3 Hamiltonian

Example of RELATIVITY-DUALITY for $D_3 \sim C_{3v}$

To represent *external* $\{..\mathbf{T},\mathbf{U},\mathbf{V},...\}$ switch $\mathbf{g} \leftrightarrows \mathbf{g}^\dagger$ on top of group table

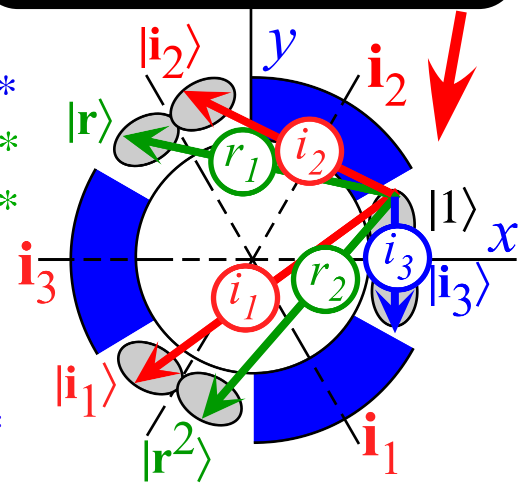


$$R^G(\mathbf{1})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\quad R^G(\mathbf{r})=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\quad R^G(\mathbf{r}^2)=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\mathbf{i}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix},\quad R^G(\mathbf{i}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\quad R^G(\mathbf{i}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

Local $\mathbb{H}$ matrix parametrized by $\bar{\mathbf{g}}$'s

$H = \langle 1|\mathbb{H}|1\rangle = H^*$
$r_1 = \langle \mathbf{r}|\mathbb{H}|1\rangle = r_2^*$
$r_2 = \langle \mathbf{r}^2|\mathbb{H}|1\rangle = r_1^*$
$i_1 = \langle \mathbf{i}_1|\mathbb{H}|1\rangle = i_1^*$
$i_2 = \langle \mathbf{i}_2|\mathbb{H}|1\rangle = i_2^*$
$i_3 = \langle \mathbf{i}_3|\mathbb{H}|1\rangle = i_3^*$



RESULT:
Any $R(\mathbf{T})$ *commute* (Even if $\mathbf{T}$ and $\mathbf{U}$ do not...)
with any $R(\bar{\mathbf{U}})$...

...and $\mathbf{T}\,\mathbf{U}=\mathbf{V}$ if & only if $\bar{\mathbf{T}}\,\bar{\mathbf{U}}=\bar{\mathbf{V}}$.

So an $\mathbb{H}$-matrix
*having **Global** symmetry D_3*

$$\mathbb{H} = H\bar{\mathbf{1}}^0 + r_1\bar{\mathbf{r}}^1 + r_2\bar{\mathbf{r}}^2 + i_1\bar{\mathbf{i}}_1 + i_2\bar{\mathbf{i}}_2 + i_3\bar{\mathbf{i}}_3$$

is made from
***Local** symmetry matrices*

To represent *internal* $\{..\bar{\mathbf{T}},\bar{\mathbf{U}},\bar{\mathbf{V}},...\}$ switch $\mathbf{g} \leftrightarrows \mathbf{g}^\dagger$ on side of group table

$$R^G(\bar{\mathbf{1}})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\quad R^G(\bar{\mathbf{r}})=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\quad R^G(\bar{\mathbf{r}}^2)=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\bar{\mathbf{i}}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\quad R^G(\bar{\mathbf{i}}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix},\quad R^G(\bar{\mathbf{i}}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

local D_3 defined Hamiltonian matrix

$\mathbb{H} =$

	$\lvert 1\rangle$	$\lvert \mathbf{r}\rangle$	$\lvert \mathbf{r}^2\rangle$	$\lvert \mathbf{i}_1\rangle$	$\lvert \mathbf{i}_2\rangle$	$\lvert \mathbf{i}_3\rangle$
$\langle 1\rvert$	H	r_1	r_2	i_1	i_2	i_3
$\langle \mathbf{r}\rvert$	r_2	H	r_1	i_2	i_3	i_1
$\langle \mathbf{r}^2\rvert$	r_1	r_2	H	i_3	i_1	i_2
$\langle \mathbf{i}_1\rvert$	i_1	i_2	i_3	H	r_1	r_2
$\langle \mathbf{i}_2\rvert$	i_2	i_3	i_1	r_2	H	r_1
$\langle \mathbf{i}_3\rvert$	i_3	i_1	i_2	r_1	r_2	H

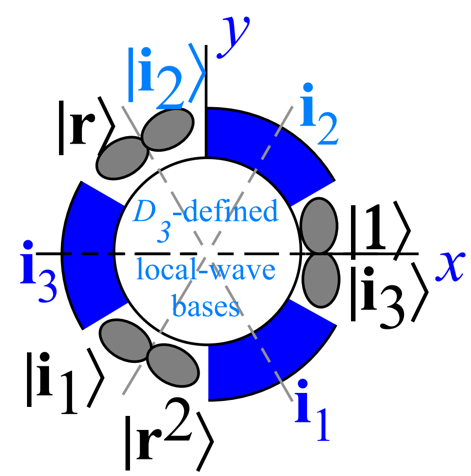
Global vs Local symmetry expansion of D_3 Hamiltonian

Example of RELATIVITY-DUALITY for $D_3 \sim C_{3v}$

To represent *external* {..**T**,**U**,**V**,... } switch $\mathbf{g} \leftrightarrows \mathbf{g}^\dagger$ on top of group table

$$R^G(\mathbf{1})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\; R^G(\mathbf{r})=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\; R^G(\mathbf{r}^2)=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\mathbf{i}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix},\; R^G(\mathbf{i}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\; R^G(\mathbf{i}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$



Local $\mathbb{H}$ *matrix parametrized by* $\bar{\mathbf{g}}$*'s*

RESULT:

Any $R(\mathbf{T})$ *commute* (Even if **T** and **U** do not...) with any $R(\bar{\mathbf{U}})$...

...and **T U**=**V** if & only if $\bar{\mathbf{T}}\,\bar{\mathbf{U}}=\bar{\mathbf{V}}$.

So an $\mathbb{H}$*-matrix having* ***Global*** *symmetry* D_3

$$\mathbb{H}=H\bar{\mathbf{1}}^0+r_1\bar{\mathbf{r}}^1+r_2\bar{\mathbf{r}}^2+i_1\bar{\mathbf{i}}_1+i_2\bar{\mathbf{i}}_2+i_3\bar{\mathbf{i}}_3$$

is made from ***Local*** *symmetry matrices*

$H=\langle 1|\mathbb{H}|1\rangle=H^*$

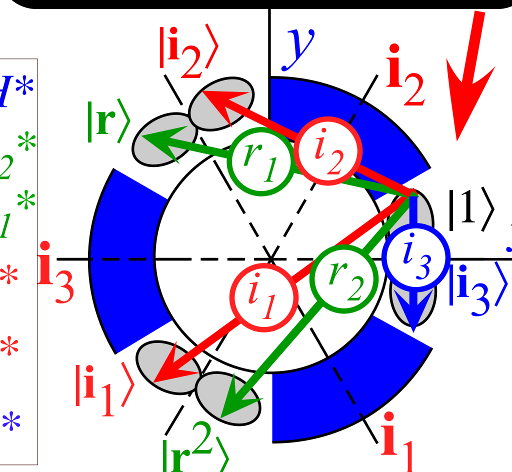
$r_1=\langle \mathbf{r}|\mathbb{H}|1\rangle=r_2^*$

$r_2=\langle \mathbf{r}^2|\mathbb{H}|1\rangle=r_1^*$

$i_1=\langle \mathbf{i}_1|\mathbb{H}|1\rangle=i_1^*$

$i_2=\langle \mathbf{i}_2|\mathbb{H}|1\rangle=i_2^*$

$i_3=\langle \mathbf{i}_3|\mathbb{H}|1\rangle=i_3^*$



All the global $\mathbf{g}$ *commute with general local* $\mathbb{H}$ *matrix.*

local D_3 defined Hamiltonian matrix

$\mathbb{H}=$	$\|\mathbf{1}\rangle$	$\|\mathbf{r}\rangle$	$\|\mathbf{r}^2\rangle$	$\|\mathbf{i}_1\rangle$	$\|\mathbf{i}_2\rangle$	$\|\mathbf{i}_3\rangle$
$\langle\mathbf{1}\|$	H	r_1	r_2	i_1	i_2	i_3
$\langle\mathbf{r}\|$	r_2	H	r_1	i_2	i_3	i_1
$\langle\mathbf{r}^2\|$	r_1	r_2	H	i_3	i_1	i_2
$\langle\mathbf{i}_1\|$	i_1	i_2	i_3	H	r_1	r_2
$\langle\mathbf{i}_2\|$	i_2	i_3	i_1	r_2	H	r_1
$\langle\mathbf{i}_3\|$	i_3	i_1	i_2	r_1	r_2	H

To represent *internal* {..$\bar{\mathbf{T}}$,$\bar{\mathbf{U}}$,$\bar{\mathbf{V}}$,... } switch $\mathbf{g} \leftrightarrows \mathbf{g}^\dagger$ on side of group table

$$R^G(\bar{\mathbf{1}})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\; R^G(\bar{\mathbf{r}})=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\; R^G(\bar{\mathbf{r}}^2)=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\bar{\mathbf{i}}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\; R^G(\bar{\mathbf{i}}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix},\; R^G(\bar{\mathbf{i}}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

Global vs Local symmetry expansion of D_3 Hamiltonian

Example of RELATIVITY-DUALITY for D

To represent *external* {..T, U, V,... }...

$R^G(\mathbf{1})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix}$, $R^G(\mathbf{r})=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix}$, $R^G(\mathbf{r}^2)=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix}$, $R^G(\mathbf{i}_1)=\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\1&\cdot&\cdot\\\cdot&1&\cdot\\\cdot&\cdot&1\end{pmatrix}$

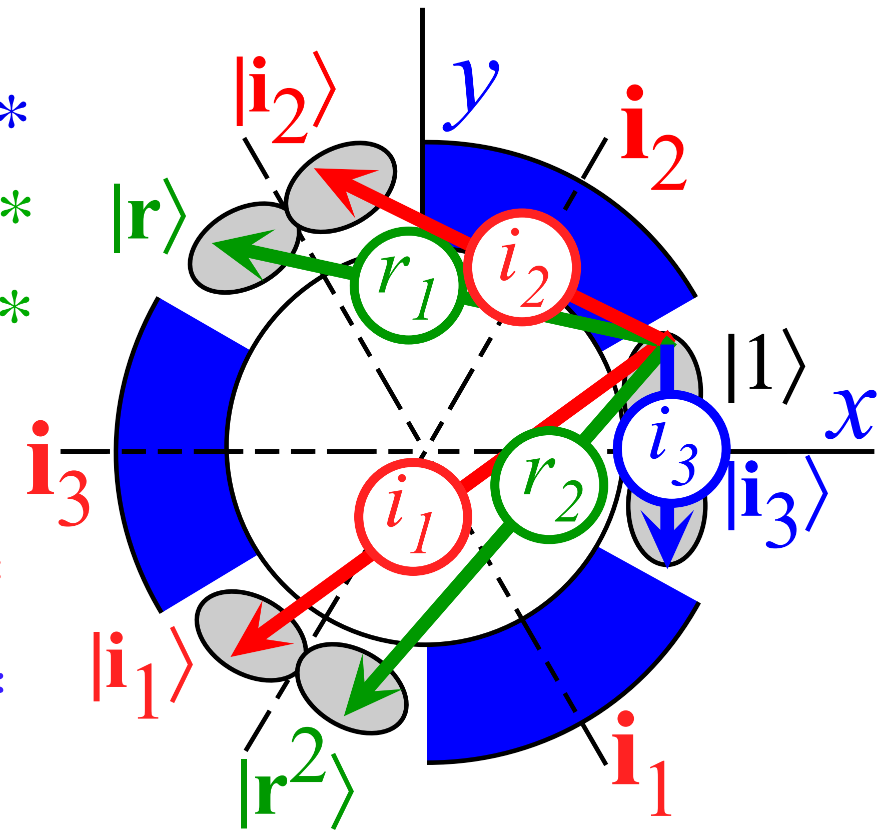
RESULT:
Any $R(\mathbf{T})$ *commute* (Even if **T** and **U** do not...)
with any $R(\bar{\mathbf{U}})$...

...and **T U**=**V** if & only if $\bar{\mathbf{T}}\,\bar{\mathbf{U}}=\bar{\mathbf{V}}$.

To represent *internal* $\{..\bar{\mathbf{T}},\bar{\mathbf{U}},\bar{\mathbf{V}},...\}$....

$R^G(\bar{\mathbf{1}})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix}$, $R^G(\bar{\mathbf{r}})=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix}$, $R^G(\bar{\mathbf{r}}^2)=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix}$, $R^G(\bar{\mathbf{i}}_1)=\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\1&\cdot&\cdot\\\cdot&\cdot&1\\\cdot&1&\cdot\end{pmatrix}$

$$H=\langle 1|\mathbb{H}|1\rangle=H^*$$
$$r_1=\langle \mathbf{r}|\mathbb{H}|1\rangle=r_2^*$$
$$r_2=\langle \mathbf{r}^2|\mathbb{H}|1\rangle=r_1^*$$
$$i_1=\langle \mathbf{i}_1|\mathbb{H}|1\rangle=i_1^*$$
$$i_2=\langle \mathbf{i}_2|\mathbb{H}|1\rangle=i_2^*$$
$$i_3=\langle \mathbf{i}_3|\mathbb{H}|1\rangle=i_3^*$$



local-D_3-defined Hamiltonian matrix

So an $\mathbb{H}$*-matrix having* ***Global*** *symmetry* D_3

$$\mathbb{H}=H\bar{\mathbf{1}}^0+r_1\bar{\mathbf{r}}^1+r_2\bar{\mathbf{r}}^2+i_1\bar{\mathbf{i}}_1+i_2\bar{\mathbf{i}}_2+i_3\bar{\mathbf{i}}_3$$

is made from ***Local*** *symmetry matrices*

$\mathbb{H}=$

	$\vert\mathbf{1})$	$\vert\mathbf{r})$	$\vert\mathbf{r}^2)$	$\vert\mathbf{i}_1)$	$\vert\mathbf{i}_2)$	$\vert\mathbf{i}_3)$
$(\mathbf{1}\vert$	H	r_1	r_2	i_1	i_2	i_3
$(\mathbf{r}\vert$	r_2	H	r_1	i_2	i_3	i_1
$(\mathbf{r}^2\vert$	r_1	r_2	H	i_3	i_1	i_2
$(\mathbf{i}_1\vert$	i_1	i_2	i_3	H	r_1	r_2
$(\mathbf{i}_2\vert$	i_2	i_3	i_1	r_2	H	r_1
$(\mathbf{i}_3\vert$	i_3	i_1	i_2	r_1	r_2	H

AMOP reference links on following page

2.21.18 class 12.0: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

Discrete symmetry subgroups of O(3) and application to tunneling and vibrational dynamics: D_3 and C_{3v} group products, classes, and irrep projection operators

32 crystal point symmetries: 16 Abelian (commutative) and 16 non-Abelian groups

Smallest non-Abelian symmetry: 3-C_2-axis D_3 vs. 3-C_v-plane C_{3v} isomorphic to permutation-S_3

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*Group theory and algebra of **D_3 Center** (Class algebra)*

Self-symmetry (Normalizer). *Lagrange Coset Theorem for classes*

1st-Stage spectral decomposition of "Group-table" Hamiltonian of D_3 symmetry

All-commuting operators $\boldsymbol{\kappa}_k$ *All-commuting projectors $\mathbf{P}^{(\alpha)}$*

D_3-invariant irep characters $\chi_k^{(\alpha)}$ *Invariant numbers: Centrum, Rank, and Order*

2nd-Stage spectral decompositions of global/local D_3

Subgroup chains $D_3 \supset C_2$ and $D_3 \supset C_3$ split class projectors *...and classes*

3rd-Stage spectral decomposition of ALL of D_3 *...and of Hamiltonian $\mathbf{H}$*

GLOBAL vs LOCAL symmetry of states *...and group $\mathbf{H}$ parameters $\{r,i_1,i_2,i_3\}$*

Class algebra $\{\kappa_1, \kappa_2, \kappa_3,\}$ *of* D_3 *Center*

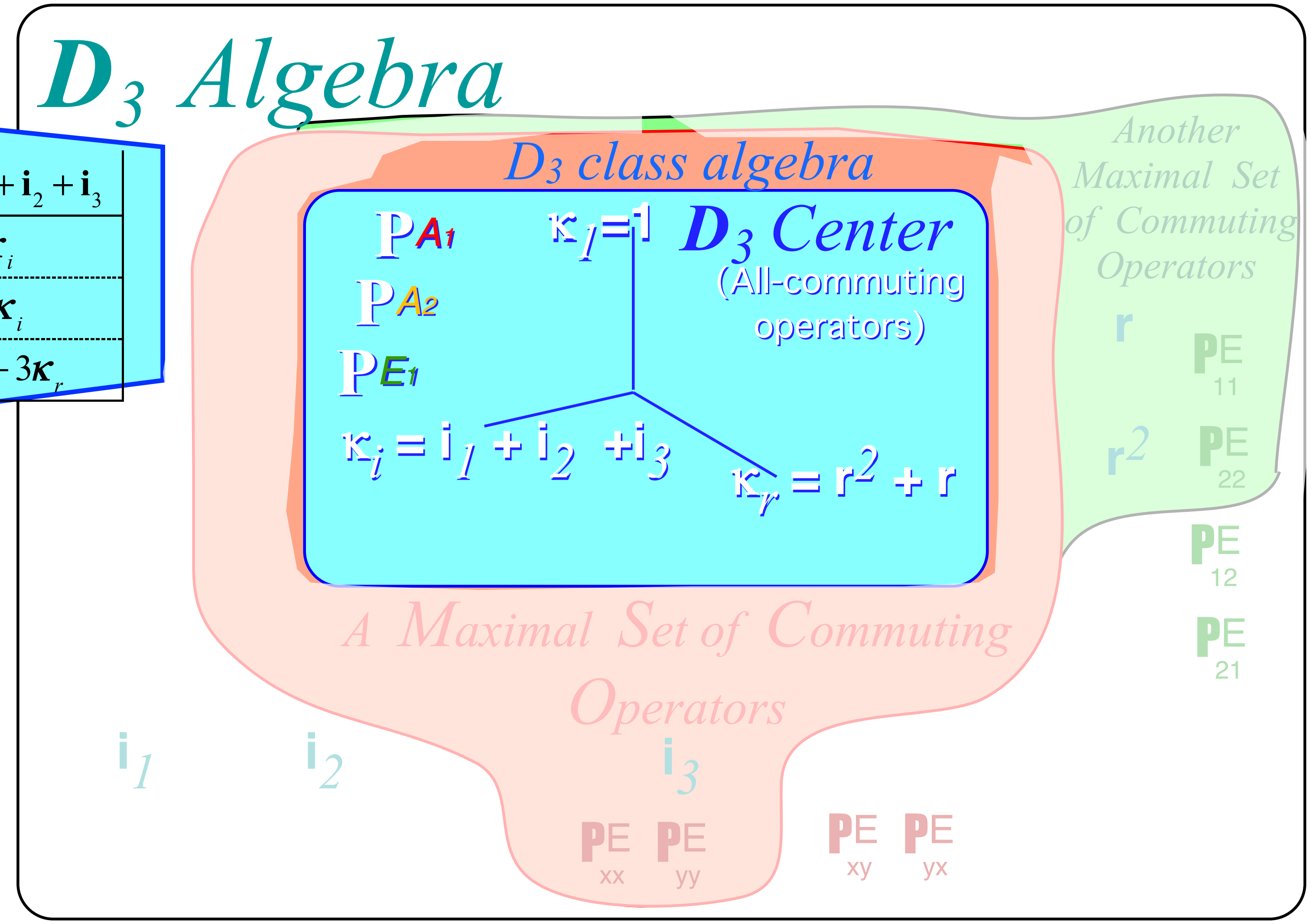
1	$\mathbf{r}^2$	**r**	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
r	**1**	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	**r**	**1**	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	**1**	**r**	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	**1**	**r**
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	**r**	$\mathbf{r}^2$	**1**

$\rightarrow$

	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r} + \mathbf{r}^2$	$\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1 + \kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1 + 3\kappa_r$

Class-sum κ_k *commutes with all* $\mathbf{g}_t$

Class-sum κ_k invariance: $\mathbf{g}_t \kappa_k = \kappa_k \mathbf{g}_t$



Review: *Spectral resolution of* ***D_3 Center*** *(Class algebra)*

1	$\mathbf{r}^2$	**r**	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
r	**1**	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	**r**	**1**	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	**1**	**r**	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	**1**	**r**
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	**r**	$\mathbf{r}^2$	**1**

$\rightarrow$

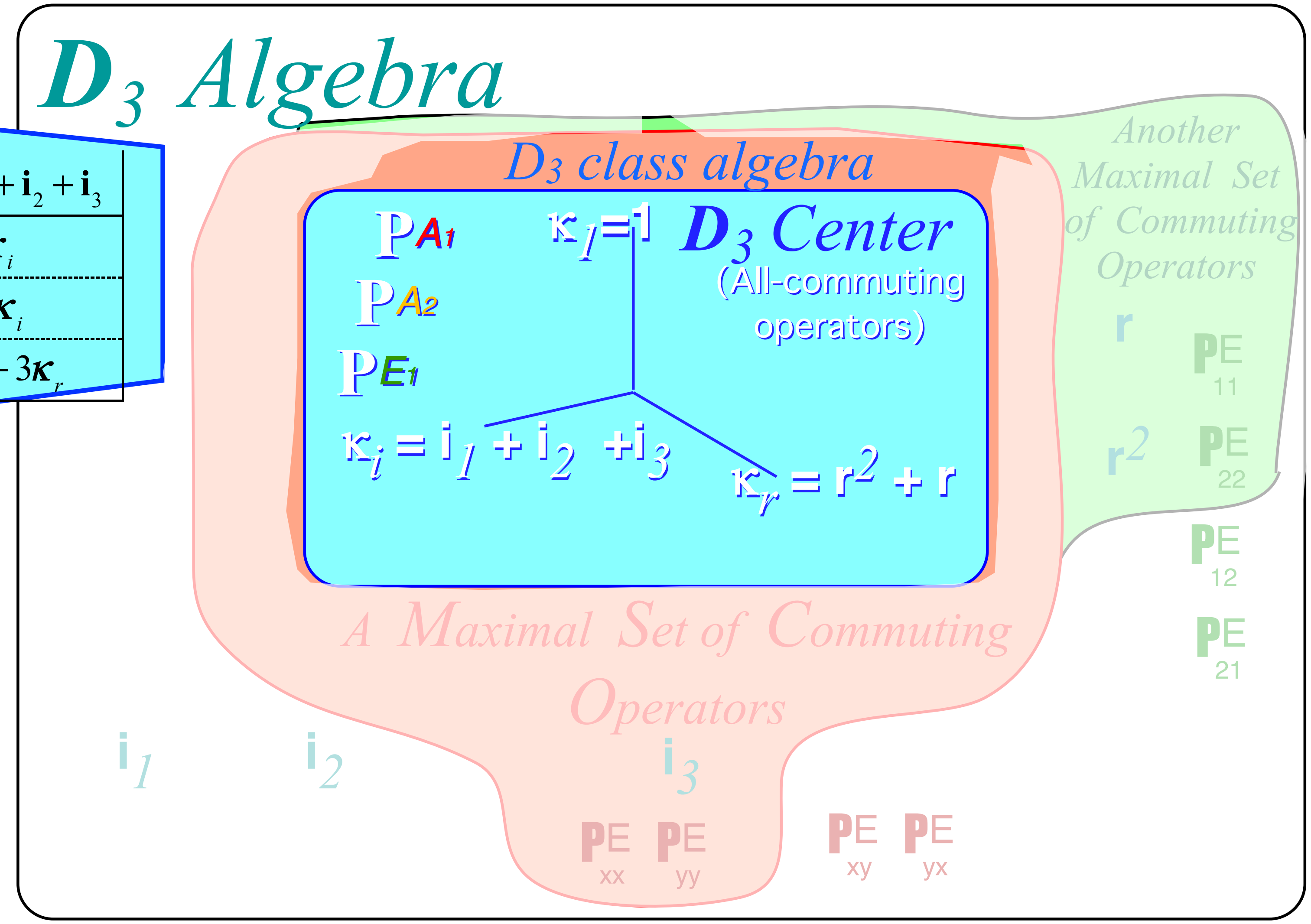
	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r} + \mathbf{r}^2$	$\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1 + \kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1 + 3\kappa_r$

Class-sum $\boldsymbol{\kappa}_k$ *commutes with all* $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t \boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k \mathbf{g}_t$

$^\circ G$ = order of group: $({}^\circ D_3 = 6)$

$^\circ \kappa_k$ = order of class κ_k: $({}^\circ \kappa_1 = 1,\ {}^\circ \kappa_r = 2,\ {}^\circ \kappa_i = 3)$



Spectral resolution of D_3 Center (Class algebra)

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

	$\kappa_1=\mathbf{1}$	$\kappa_r=\mathbf{r}+\mathbf{r}^2$	$\kappa_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1+\kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1+3\kappa_r$

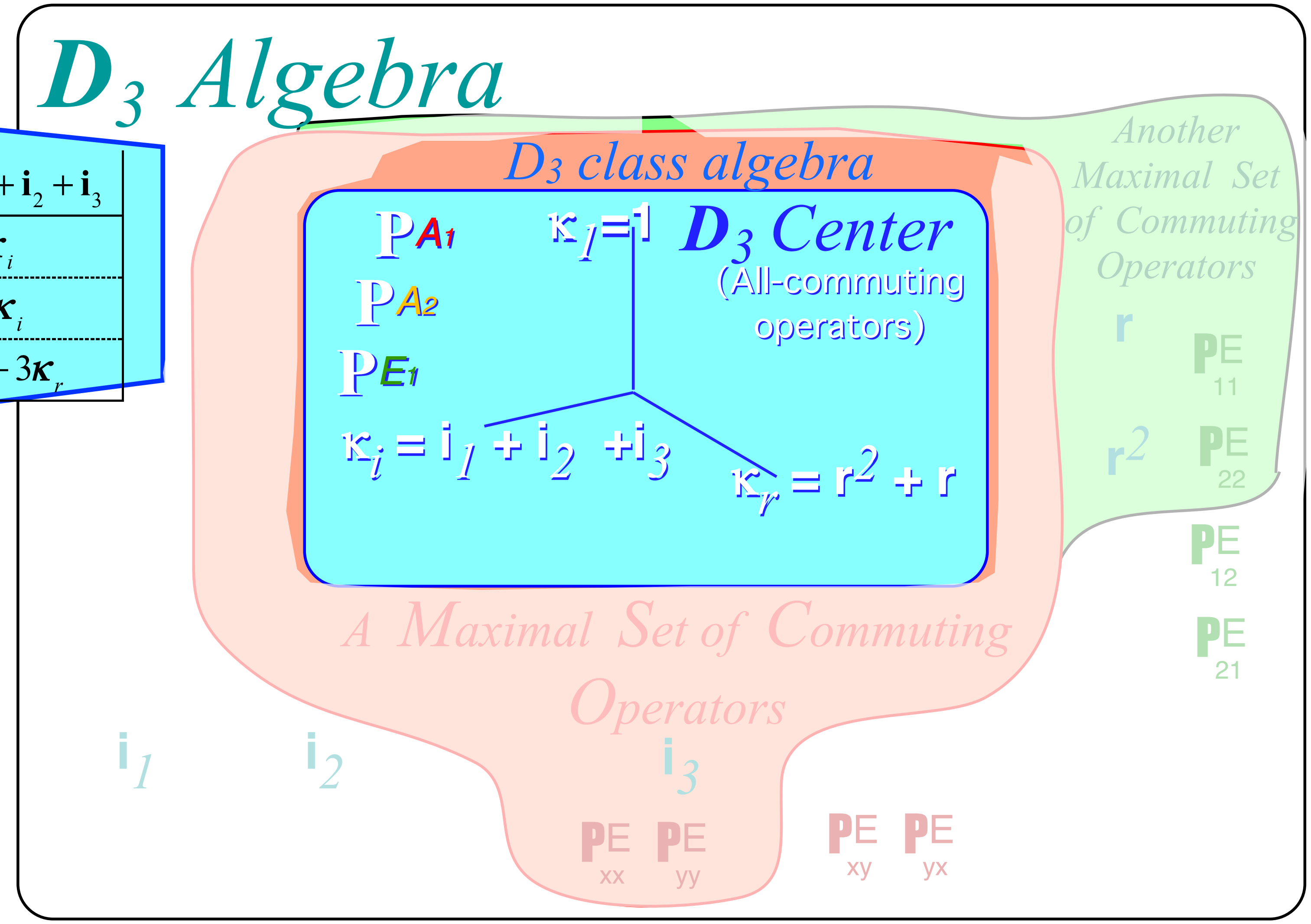
Class-sum $\boldsymbol{\kappa}_k$ *commutes with all* $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t\boldsymbol{\kappa}_k\mathbf{g}_t^{-1}=\boldsymbol{\kappa}_k$ or: $\mathbf{g}_t\boldsymbol{\kappa}_k=\boldsymbol{\kappa}_k\mathbf{g}_t$

$^\circ G$ = order of group: $(^\circ D_3=6)$

$^\circ\kappa_k$ = order of class κ_k: $(^\circ\kappa_1=1,\ ^\circ\kappa_r=2,\ ^\circ\kappa_i=3)$

$$\mathbf{g}_t\boldsymbol{\kappa}_k\mathbf{g}_t^{-1}=\boldsymbol{\kappa}_k \quad \text{where: } \boldsymbol{\kappa}_\mathbf{k}=\sum_{j=1}^{j={}^\circ\kappa_k}\mathbf{g}_j$$



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D_3-invariant irep characters $\chi_k^{(\alpha)}$ — *Invariant numbers: Centrum, Rank, and Order*

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Subgroup chains $D_3 \supset C_2$ and $D_3 \supset C_3$ split class projectors — *...and classes*

3rd-Stage spectral decomposition of ALL of D_3 — *...and of Hamiltonian $\mathbf{H}$*

GLOBAL vs LOCAL symmetry of states — *...and group $\mathbf{H}$ parameters $\{r, i_1, i_2, i_3\}$*

Review: *Spectral resolution of **D_3 Center** (Class algebra)*

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r}+\mathbf{r}^2$	$\kappa_i = \mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1+\kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1+3\kappa_r$

Class-sum $\boldsymbol{\kappa}_k$ commutes with all $\mathbf{g}_t$

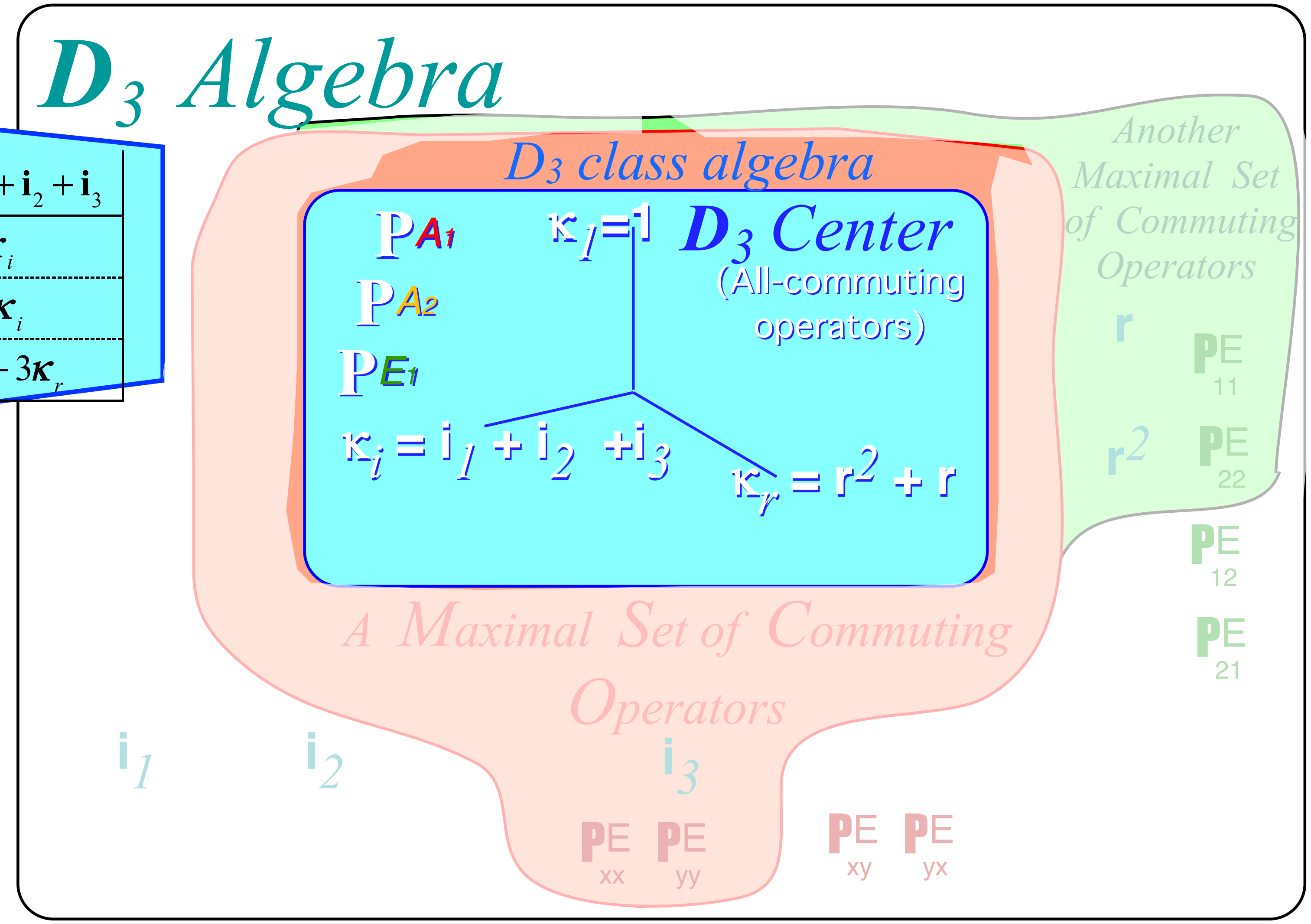
Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t\boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k\mathbf{g}_t$

$^\circ G$ = order of group: $(^\circ D_3 = 6)$

$^\circ\kappa_k$ = order of class κ_k: $(^\circ\kappa_1 = 1,\ ^\circ\kappa_r = 2,\ ^\circ\kappa_i = 3)$

$$\mathbf{g}_t\boldsymbol{\kappa}_k\mathbf{g}_t^{-1} = \boldsymbol{\kappa}_k \quad \text{where: } \boldsymbol{\kappa}_\mathbf{k} = \sum_{j=1}^{j=^\circ\kappa_k}\mathbf{g}_j = \frac{1}{^\circ s_k}\sum_{t=1}^{t=^\circ G}\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}$$

$^\circ s_k$=order of $\mathbf{g}_k$-self-symmetry: $(^\circ s_1 = 6,\ ^\circ s_r = 3,\ ^\circ s_i = 2)$



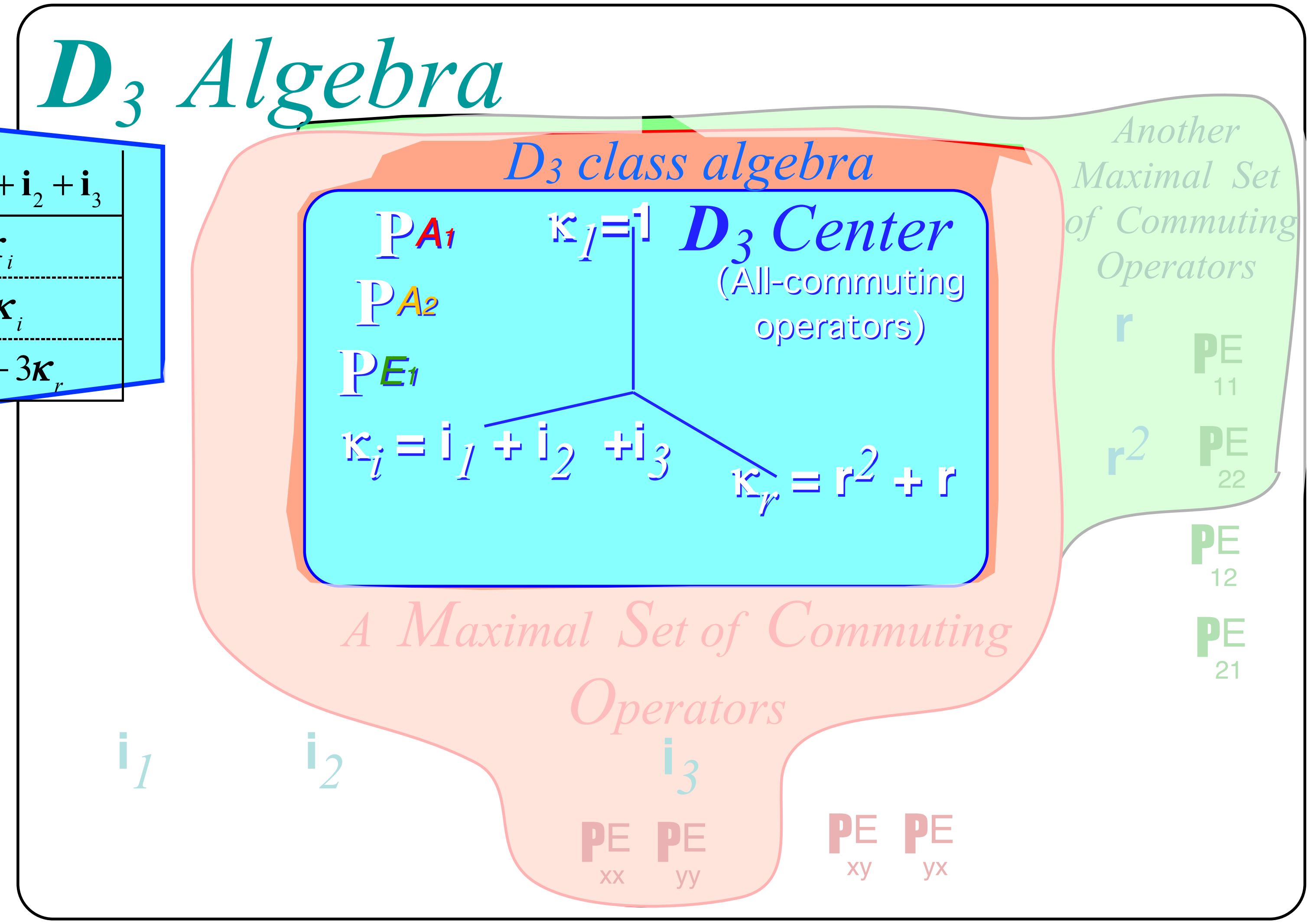
*Group theory of **D_3 Center** (Class algebra)*

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r} + \mathbf{r}^2$	$\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1 + \kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1 + 3\kappa_r$

Class-sum $\boldsymbol{\kappa}_k$ commutes with all $\mathbf{g}_t$



Class-sum $\boldsymbol{\kappa}_k$ invariance: $\quad \mathbf{g}_t \boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k \mathbf{g}_t$

$^\circ G$ = order of group: $\quad (^\circ D_3 = 6)$

$^\circ \kappa_k$ = order of class κ_k: $\quad (^\circ \kappa_1 = 1,\ ^\circ \kappa_r = 2,\ ^\circ \kappa_i = 3)$

$$\mathbf{g}_t \boldsymbol{\kappa}_k \mathbf{g}_t^{-1} = \boldsymbol{\kappa}_k \quad \text{where: } \boldsymbol{\kappa}_\mathbf{k} = \sum_{j=1}^{j={}^\circ\kappa_k} \mathbf{g}_j = \frac{1}{^\circ s_k} \sum_{t=1}^{t={}^\circ G} \mathbf{g}_t \mathbf{g}_k \mathbf{g}_t^{-1}$$

$^\circ s_k$=order of $\mathbf{g}_k$-self-symmetry:$(^\circ s_1 = 6,\ ^\circ s_r = 3,\ ^\circ s_i = 2)$

$^\circ s_k = {}^\circ G / {}^\circ \kappa_k \qquad$ *$^\circ s_k$ is an integer count of D_3 operators $\mathbf{g}_s$ that commute with $\mathbf{g}_k$.*

AMOP reference links on following page

2.21.18 class 12.0: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

Discrete symmetry subgroups of O(3) and application to tunneling and vibrational dynamics: D_3 and C_{3v} group products, classes, and irrep projection operators

32 crystal point symmetries: 16 Abelian (commutative) and 16 non-Abelian groups

Smallest non-Abelian symmetry: 3-C_2-axis D_3 vs. 3-C_v-plane C_{3v} isomorphic to permutation-S_3

Relating C_2-180°rotations $\mathbf{R}_z$, C_v-plane reflections $\boldsymbol{\sigma}_z$, and inversion $\mathbf{I}$ operators

Deriving $D_3 \sim C_{3v}$ products by group definition $|g\rangle=\mathbf{g}|1\rangle$ of position ket $|g\rangle$

Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes

Non-commutative symmetry expansion and Global-Local solution

Global vs Local symmetry and Mock-Mach principle

Global vs Local matrix duality for D_3

Global vs Local symmetry expansion of D_3 Hamiltonian

*Group theory and algebra of **D_3 Center** (Class algebra)*

Self-symmetry (Normalizer). *Lagrange Coset Theorem for classes*

1st-Stage spectral decomposition of "Group-table" Hamiltonian of D_3 symmetry

All-commuting operators $\boldsymbol{\kappa}_k$ *All-commuting projectors $\mathbf{P}^{(\alpha)}$*

D_3-invariant irep characters $\chi_k^{(\alpha)}$ *Invariant numbers: Centrum, Rank, and Order*

2nd-Stage spectral decompositions of global/local D_3

Subgroup chains $D_3 \supset C_2$ and $D_3 \supset C_3$ split class projectors *...and classes*

3rd-Stage spectral decomposition of ALL of D_3 *...and of Hamiltonian $\mathbf{H}$*

GLOBAL vs LOCAL symmetry of states *...and group $\mathbf{H}$ parameters $\{r,i_1,i_2,i_3\}$*

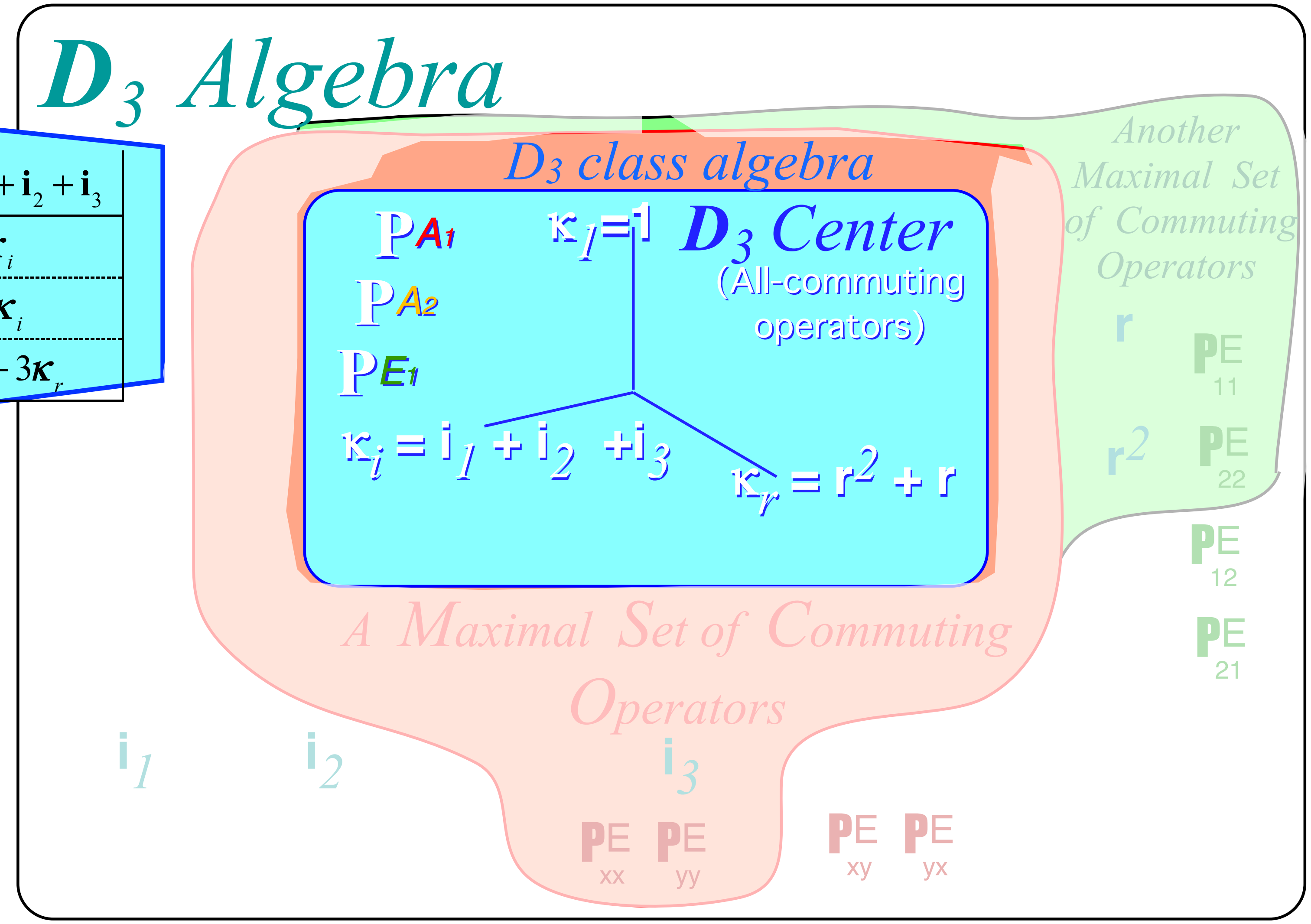
Review: Spectral resolution of D_3 Center (Class algebra)

1	$\mathbf{r}^2$	**r**	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
r	**1**	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	**r**	**1**	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	**1**	**r**	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	**1**	**r**
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	**r**	$\mathbf{r}^2$	**1**

$\rightarrow$

	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r} + \mathbf{r}^2$	$\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1 + \kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1 + 3\kappa_r$

Class-sum $\boldsymbol{\kappa}_k$ commutes with all $\mathbf{g}_t$



Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t \boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k \mathbf{g}_t$

$°G$ = order of group: $(°D_3 = 6)$

$°\kappa_k$ = order of class κ_k: $(°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$

$$\mathbf{g}_t \boldsymbol{\kappa}_k \mathbf{g}_t^{-1} = \boldsymbol{\kappa}_k \quad \text{where: } \boldsymbol{\kappa}_\mathbf{k} = \sum_{j=1}^{j=°\kappa_k} \mathbf{g}_j = \frac{1}{°s_k} \sum_{t=1}^{t=°G} \mathbf{g}_t \mathbf{g}_k \mathbf{g}_t^{-1}$$

$°s_k$=order of $\mathbf{g}_k$-self-symmetry: $(°s_1 = 6,\ °s_r = 3,\ °s_i = 2)$

$°s_k = °G / °\kappa_k$

$°s_k$ is an integer count of D_3 operators $\mathbf{g}_s$ that commute with $\mathbf{g}_k$.

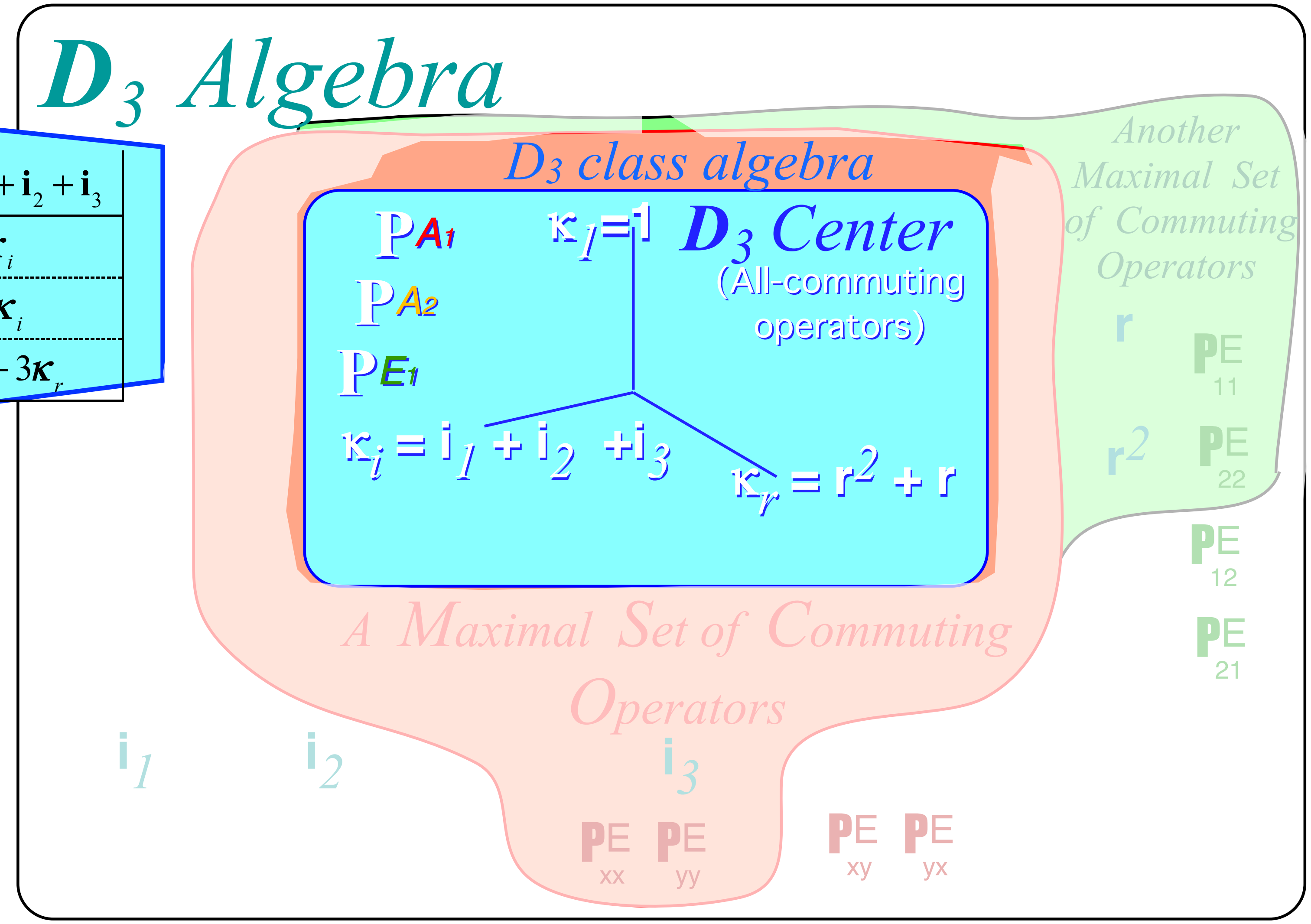
...now a few pages to prove
and apply this key integer ratio
related to Laggrange's theorems.

Review: *Spectral resolution of **D_3 Center** (Class algebra)*

1	r^2	r	i_1	i_2	i_3
r	1	r^2	i_3	i_1	i_2
r^2	r	1	i_2	i_3	i_1
i_1	i_3	i_2	1	r	r^2
i_2	i_1	i_3	r^2	1	r
i_3	i_2	i_1	r	r^2	1

$\rightarrow$

	$\kappa_1 = 1$	$\kappa_r = r + r^2$	$\kappa_i = i_1 + i_2 + i_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1 + \kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1 + 3\kappa_r$



Class-sum $\boldsymbol{\kappa}_k$ commutes with all $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t \boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k \mathbf{g}_t$

$°G$ = order of group: $(°D_3 = 6)$

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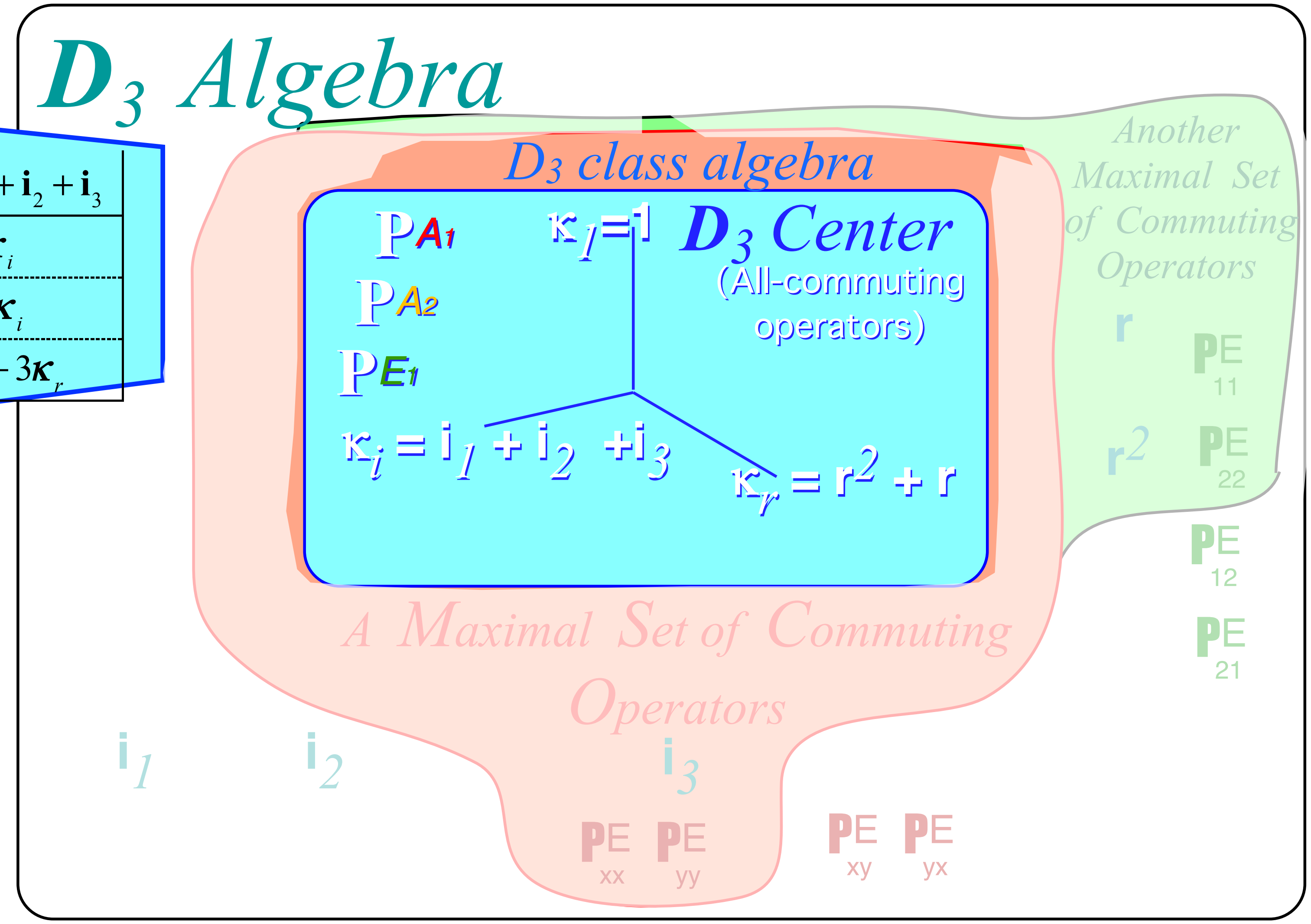
*Review: Spectral resolution of **D_3 Center** (Class algebra)*

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r} + \mathbf{r}^2$	$\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1 + \kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1 + 3\kappa_r$

Class-sum $\boldsymbol{\kappa}_k$ commutes with all $\mathbf{g}_t$



Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t \boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k \mathbf{g}_t$

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$$\mathbf{g}_t \boldsymbol{\kappa}_k \mathbf{g}_t^{-1} = \boldsymbol{\kappa}_k \quad \text{where: } \boldsymbol{\kappa}_\mathbf{k} = \sum_{j=1}^{j={}^\circ\kappa_k} \mathbf{g}_j = \frac{1}{^\circ s_k} \sum_{t=1}^{t={}^\circ G} \mathbf{g}_t \mathbf{g}_k \mathbf{g}_t^{-1}$$

$^\circ s_k$=order of $\mathbf{g}_k$-self-symmetry: $(^\circ s_1 = 6,\ ^\circ s_r = 3,\ ^\circ s_i = 2)$

$^\circ s_k = {}^\circ G / {}^\circ\kappa_k$ — *$^\circ s_k$ is an integer count of D_3 operators $\mathbf{g}_s$ that commute with $\mathbf{g}_k$.*

These operators $\mathbf{g}_s$ form the $\mathbf{g}_k$-self-symmetry group s_k. Each $\mathbf{g}_s$ transforms $\mathbf{g}_k$ into itself: $\mathbf{g}_s \mathbf{g}_k \mathbf{g}_s^{-1} = \mathbf{g}_k$

If an operator $\mathbf{g}_t$ transforms $\mathbf{g}_k$ into a different element $\mathbf{g}'_k$ of its class: $\mathbf{g}_t \mathbf{g}_k \mathbf{g}_t^{-1} = \mathbf{g}'_k$, then so does $\mathbf{g}_t \mathbf{g}_s$.
that is: $\mathbf{g}_t \mathbf{g}_s \mathbf{g}_k (\mathbf{g}_t \mathbf{g}_s)^{-1} = \mathbf{g}_t \mathbf{g}_s \mathbf{g}_k \mathbf{g}_s^{-1} \mathbf{g}_t^{-1} = \mathbf{g}_t \mathbf{g}_k \mathbf{g}_t^{-1} = \mathbf{g}'_k$,

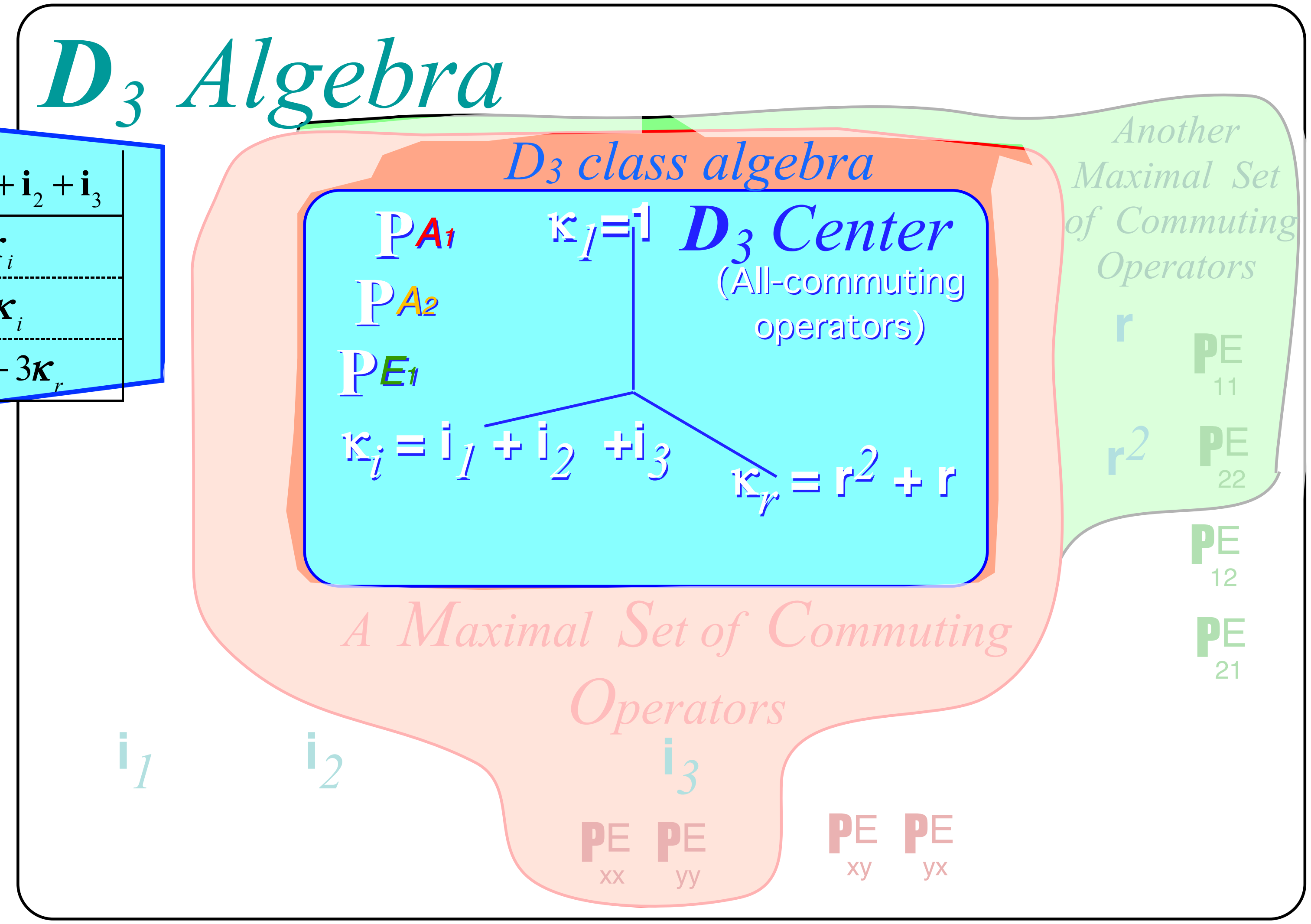
⋮
⋮

Review: *Spectral resolution of* ***D₃ Center*** *(Class algebra)*

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

→

	$\kappa_1=\mathbf{1}$	$\kappa_r=\mathbf{r}+\mathbf{r}^2$	$\kappa_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1+\kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1+3\kappa_r$



Class-sum $\boldsymbol{\kappa}_k$ *commutes with all* $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t\boldsymbol{\kappa}_k=\boldsymbol{\kappa}_k\mathbf{g}_t$

$°G$ = order of group: $(°D_3=6)$

$°\kappa_k$ = order of class κ_k: $(°\kappa_1=1,\ °\kappa_r=2,\ °\kappa_i=3)$

$$\mathbf{g}_t\boldsymbol{\kappa}_k\mathbf{g}_t^{-1}=\boldsymbol{\kappa}_k \quad \text{where: } \boldsymbol{\kappa}_\mathbf{k}=\sum_{j=1}^{j=°\kappa_k}\mathbf{g}_j=\frac{1}{°s_k}\sum_{t=1}^{t=°G}\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}$$

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$°s_k=°G/°\kappa_k$ — *$°s_k$ is an integer count of D_3 operators $\mathbf{g}_s$ that commute with $\mathbf{g}_k$.*

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that is: $\mathbf{g}_t\mathbf{g}_s\mathbf{g}_k(\mathbf{g}_t\mathbf{g}_s)^{-1}=\mathbf{g}_t\mathbf{g}_s\mathbf{g}_k\mathbf{g}_s^{-1}\mathbf{g}_t^{-1}=\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}=\mathbf{g}'_k$,

Subgroup $s_k=\{\mathbf{g}_0=\mathbf{1},\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\ldots\}$ has $\ell=(°\kappa_k-1)$ ***Left Cosets*** (one coset for each member of class κ_k).

$\mathbf{g}_1 s_k=\mathbf{g}_1\{\mathbf{g}_0=1,\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\ldots\}$,

⋮

⋮

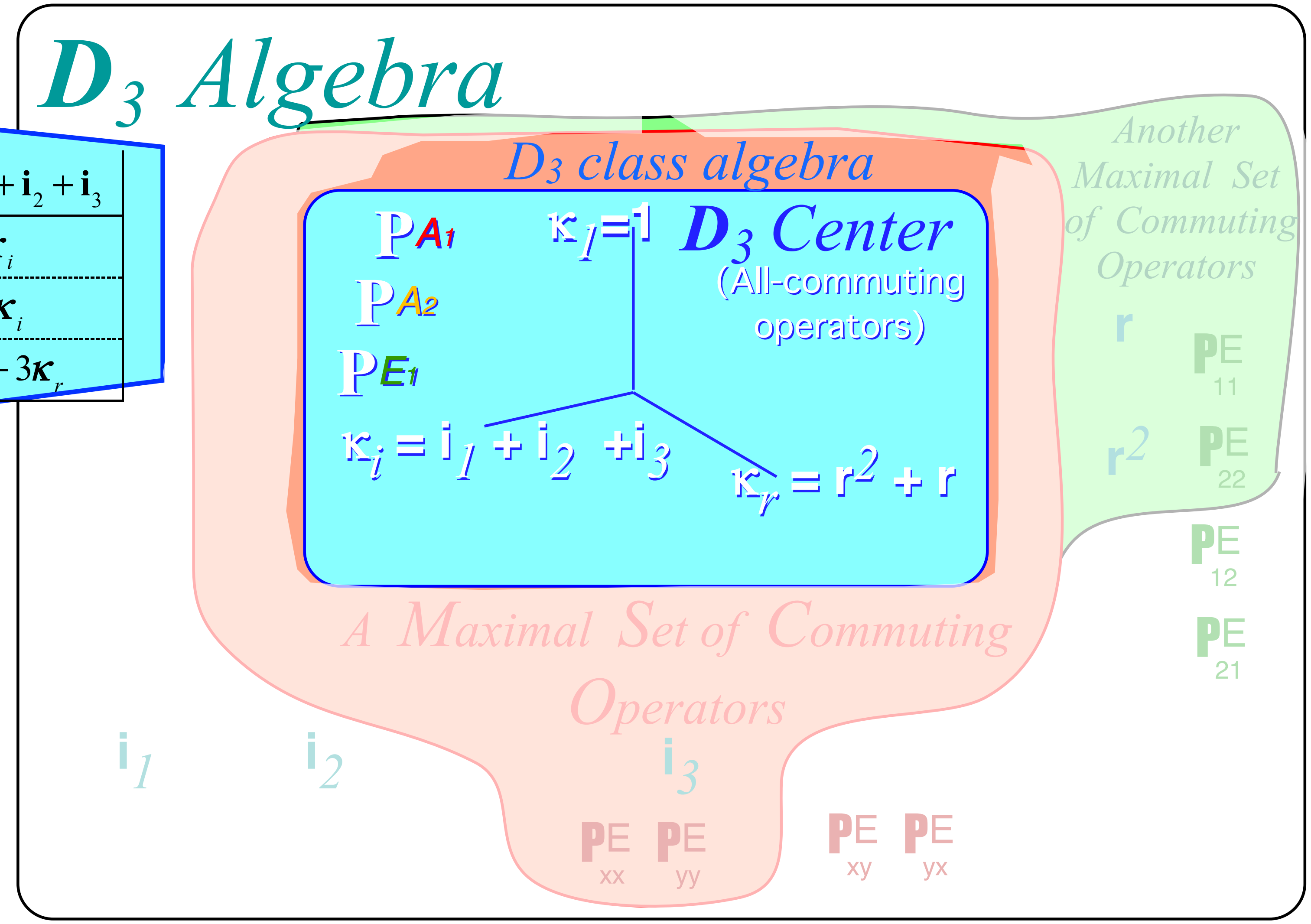
*Review: Spectral resolution of **D_3 Center** (Class algebra)*

1	$\mathbf{r}^2$	**r**	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
r	**1**	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	**r**	**1**	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	**1**	**r**	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	**1**	**r**
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	**r**	$\mathbf{r}^2$	**1**

→

	$\kappa_1=1$	$\kappa_r=\mathbf{r}+\mathbf{r}^2$	$\kappa_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1+\kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1+3\kappa_r$

Class-sum $\boldsymbol{\kappa}_k$ commutes with all $\mathbf{g}_t$



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$$\mathbf{g}_t\boldsymbol{\kappa}_k\mathbf{g}_t^{-1}=\boldsymbol{\kappa}_k \quad \text{where: } \boldsymbol{\kappa}_\mathbf{k}=\sum_{j=1}^{j=°\kappa_k}\mathbf{g}_j=\frac{1}{°s_k}\sum_{t=1}^{t=°G}\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}$$

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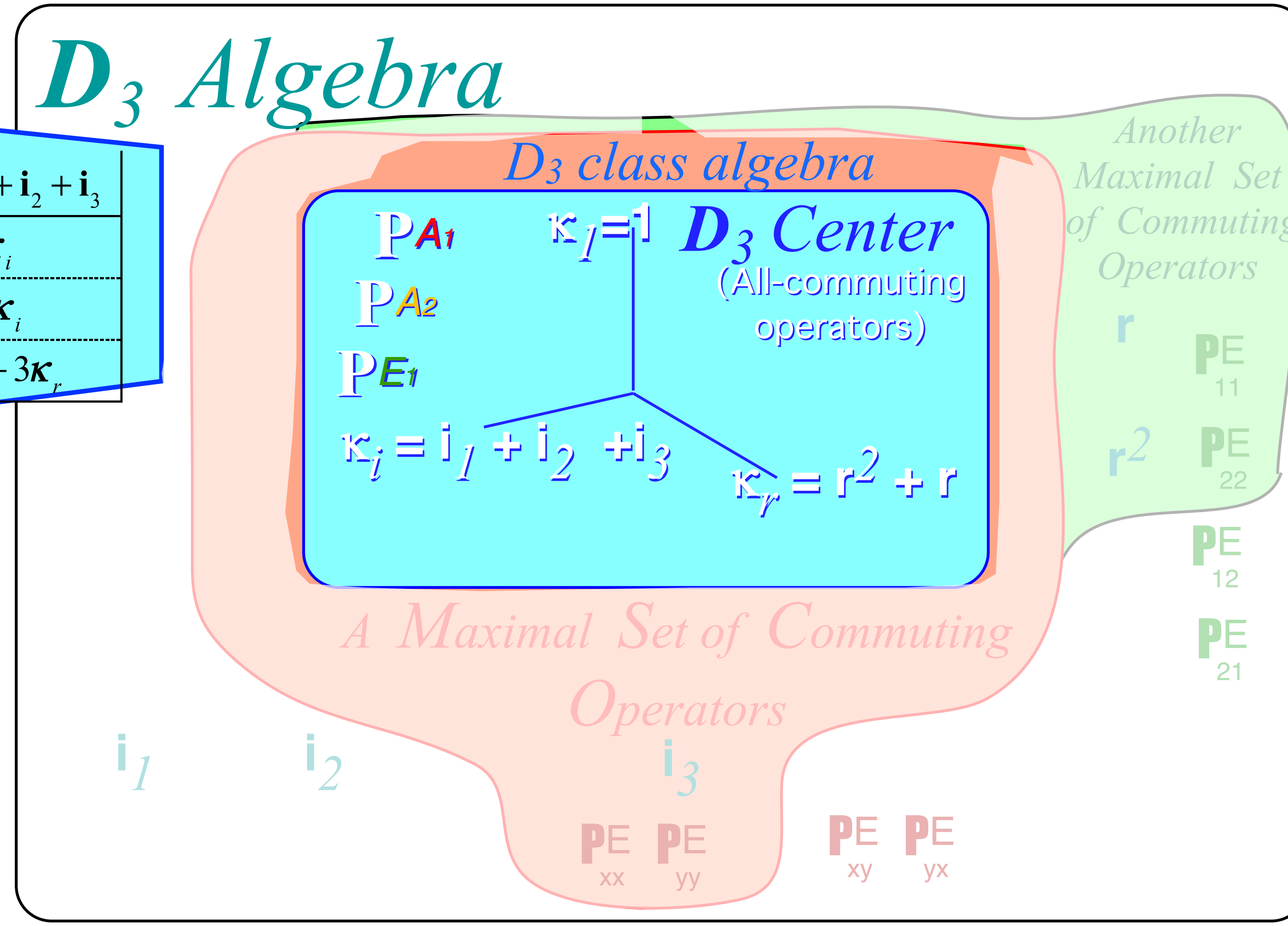
$$\mathbf{g}_1 s_k=\mathbf{g}_1\{\mathbf{g}_0=1,\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\ldots\},$$
$$\mathbf{g}_2 s_k=\mathbf{g}_2\{\mathbf{g}_0=1,\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\ldots\},\ldots$$
$$\vdots$$
$$\mathbf{g}_\ell s_k=\mathbf{g}_\ell\{\mathbf{g}_0=1,\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\ldots\}$$

Review: *Spectral resolution of* ***D_3 Center*** *(Class algebra)*

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

→

	$\kappa_1=\mathbf{1}$	$\kappa_r=\mathbf{r}+\mathbf{r}^2$	$\kappa_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1+\kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1+3\kappa_r$



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$^\circ G$ = order of group: $(^\circ D_3=6)$

$^\circ\kappa_k$ = order of class κ_k: $(^\circ\kappa_1=1,\ ^\circ\kappa_r=2,\ ^\circ\kappa_i=3)$

$$\mathbf{g}_t\boldsymbol{\kappa}_k\mathbf{g}_t^{-1}=\boldsymbol{\kappa}_k \quad \text{where: } \boldsymbol{\kappa}_\mathbf{k}=\sum_{j=1}^{j=^\circ\kappa_k}\mathbf{g}_j=\frac{1}{^\circ s_k}\sum_{t=1}^{t=^\circ G}\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}$$

$^\circ s_k$=order of $\mathbf{g}_k$-self-symmetry: $(^\circ s_1=6,\ ^\circ s_r=3,\ ^\circ s_i=2)$

$^\circ s_k = {^\circ G}/{^\circ\kappa_k}$

$^\circ s_k$ is an integer count of D_3 operators $\mathbf{g}_s$ that commute with $\mathbf{g}_k$.

These operators $\mathbf{g}_s$ form the $\mathbf{g}_k$-self-symmetry group s_k. Each $\mathbf{g}_s$ transforms $\mathbf{g}_k$ into itself: $\mathbf{g}_s\mathbf{g}_k\mathbf{g}_s^{-1}=\mathbf{g}_k$

If an operator $\mathbf{g}_t$ transforms $\mathbf{g}_k$ into a different element $\mathbf{g}'_k$ of its class: $\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}=\mathbf{g}'_k$, then so does $\mathbf{g}_t\mathbf{g}_s$.

that is: $\mathbf{g}_t\mathbf{g}_s\mathbf{g}_k(\mathbf{g}_t\mathbf{g}_s)^{-1}=\mathbf{g}_t\mathbf{g}_s\mathbf{g}_k\mathbf{g}_s^{-1}\mathbf{g}_t^{-1}=\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}=\mathbf{g}'_k$,

Subgroup $s_k=\{\mathbf{g}_0=\mathbf{1},\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\dots\}$ has $\ell=(^\circ\kappa_k-1)$ ***Left Cosets*** (one coset for each member of class κ_k).

$\mathbf{g}_1 s_k=\mathbf{g}_1\{\mathbf{g}_0=1,\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\dots\}$,

$\mathbf{g}_2 s_k=\mathbf{g}_2\{\mathbf{g}_0=1,\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\dots\}$,....

They will divide the group of order $^\circ D_3={^\circ\kappa_k}\cdot{^\circ s_k}$ evenly into $^\circ\kappa_k$ subsets each of order $^\circ s_k$.

Review: *Spectral resolution of* ***D₃ Center*** *(Class algebra)*

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

	$\kappa_1=\mathbf{1}$	$\kappa_r=\mathbf{r}+\mathbf{r}^2$	$\kappa_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1+\kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1+3\kappa_r$

Class-sum κ_k *commutes with all* $\mathbf{g}_t$

Class-sum κ_k invariance: $\mathbf{g}_t\kappa_k = \kappa_k\mathbf{g}_t$

$^\circ G$ = order of group: $(^\circ D_3 = 6)$

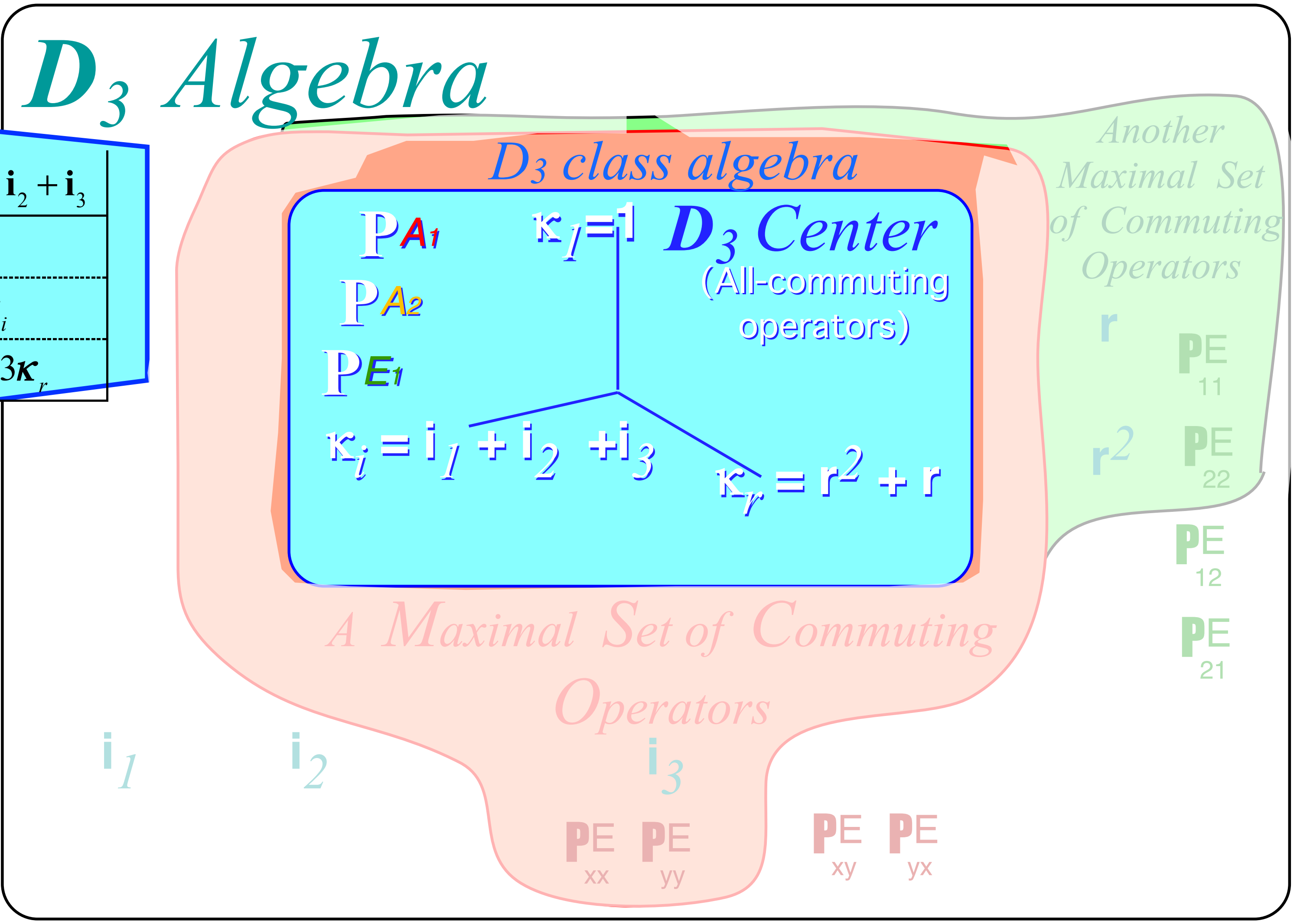
$^\circ\kappa_k$ = order of class κ_k: $(^\circ\kappa_1 = 1,\ ^\circ\kappa_r = 2,\ ^\circ\kappa_i = 3)$

$$\mathbf{g}_t\kappa_k\mathbf{g}_t^{-1} = \kappa_k \quad \text{where: } \kappa_k = \sum_{j=1}^{j=^\circ\kappa_k}\mathbf{g}_j = \frac{1}{^\circ s_k}\sum_{t=1}^{t=^\circ G}\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}$$

$^\circ s_k$=order of $\mathbf{g}_k$-self-symmetry: $(^\circ s_1 = 6,\ ^\circ s_r = 3,\ ^\circ s_i = 2)$

$^\circ s_k = {}^\circ G / {}^\circ\kappa_k$

$^\circ s_k$ is an integer count of D_3 operators $\mathbf{g}_s$ that commute with $\mathbf{g}_k$.



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$\mathbf{g}_2 s_k = \mathbf{g}_2\{\mathbf{g}_0=1,\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\dots\}$,....

These results are known as ***Lagrange's Coset Theorem(s)***

They will divide the group of order $^\circ D_3 = {}^\circ\kappa_k \cdot {}^\circ s_k$ evenly into $^\circ\kappa_k$ subsets each of order $^\circ s_k$.

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➡ *1st-Stage spectral decomposition of "Group-table" Hamiltonian of D_3 symmetry*

All-commuting operators $\boldsymbol{\kappa}_k$ ⬅ *All-commuting projectors $\mathbf{P}^{(\alpha)}$*

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Spectral analysis of non-commutative "Group-table Hamiltonian"

1st Step: Spectral resolution of D_3-Center (Class algebra of D_3)

$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

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$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

Each class-sum $\underline{\kappa}_k$ commutes with all of D_3.

$\kappa_1 = \mathbf{1}$	$\kappa_2 = \mathbf{r}^1 + \mathbf{r}^2$	$\kappa_3 = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_2	$2\kappa_1 + \kappa_2$	$2\kappa_3$
κ_3	$2\kappa_3$	$3\kappa_1 + 3\kappa_2$

$\boldsymbol{\kappa}_g$'s are *mutually commuting* with respect to themselves and *all-commuting* with respect to the whole group.

$$\mathbf{r}\,\boldsymbol{\kappa}_i\,\mathbf{r}^{-1} = \mathbf{i}_2 + \mathbf{i}_3 + \mathbf{i}_1 = \boldsymbol{\kappa}_i \quad \text{or:} \quad \mathbf{r}\,\boldsymbol{\kappa}_i = \boldsymbol{\kappa}_i\,\mathbf{r}$$

$$\sum_{\mathbf{h}=1}^{\circ G} \mathbf{hgh}^{-1} = \upsilon_g \boldsymbol{\kappa}_g\ , \qquad \text{where: } \upsilon_g = \frac{{}^\circ G}{{}^\circ \kappa_g} = \textit{integer}$$

${}^\circ\kappa g$ is order of class $\boldsymbol{\kappa} g$ and must evenly divide group order ${}^\circ G$.

Spectral analysis of non-commutative "Group-table Hamiltonian"

1st Step: Spectral resolution of D_3-Center (Class algebra of D_3)

$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

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κ_2	$2\kappa_1 + \kappa_2$	$2\kappa_3$
κ_3	$2\kappa_3$	$3\kappa_1 + 3\kappa_2$

$$\boldsymbol{\kappa}^2{}_3 = 3\cdot\boldsymbol{\kappa}_2 + 3\cdot\mathbf{1}$$

Note also:

$$\boldsymbol{\kappa}^2{}_2 - \boldsymbol{\kappa}_2 - 2\cdot\mathbf{1} = 0$$

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Spectral analysis of non-commutative "Group-table Hamiltonian"

1st Step: Spectral resolution of D_3-Center (Class algebra of D_3)

$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

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Each class-sum $\underline{\kappa}_k$ commutes with all of D_3.

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Class products give spectral polynomial and all-commuting projectors $\mathbf{P}^{(\alpha)}$

$$0 = \kappa_3^3 - 9\kappa_3 = (\kappa_3 - 3\cdot\mathbf{1})(\kappa_3 + 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})$$

$\leftarrow \boldsymbol{\kappa}^2_3 = 3\cdot\boldsymbol{\kappa}_2 + 3\cdot\mathbf{1}$

Note also:

$$\boldsymbol{\kappa}^2_2 - \boldsymbol{\kappa}_2 - 2\cdot\mathbf{1} = 0$$

$$0 = (\boldsymbol{\kappa}_2 - 2\cdot\mathbf{1})(\boldsymbol{\kappa}_2 + \mathbf{1})$$

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$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

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Class products give spectral polynomial and all-commuting projectors $\mathbf{P}^{(\alpha)} = \mathbf{P}^{A_1}$, $\mathbf{P}^{A_2}$, and $\mathbf{P}^{E}$

Note also:

$\kappa_2^2 - \kappa_2 - 2 \cdot \mathbf{1} = 0$

$$0 = \kappa_3^3 - 9\kappa_3 = (\kappa_3 - 3 \cdot \mathbf{1})(\kappa_3 + 3 \cdot \mathbf{1})(\kappa_3 - 0 \cdot \mathbf{1})$$

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Spectral analysis of non-commutative "Group-table Hamiltonian"

1st Step: Spectral resolution of D_3-Center (Class algebra of D_3)

$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
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$$0 = \kappa_3^3 - 9\kappa_3 = (\kappa_3 - 3\cdot\mathbf{1})(\kappa_3 + 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})$$

$$0 = (\kappa_3 - 3\cdot\mathbf{1})\mathbf{P}^{A_1}$$

$$\kappa_3\mathbf{P}^{A_1} = +3\cdot\mathbf{P}^{A_1}$$

$$\mathbf{P}^{A_1} = \frac{(\kappa_3 + 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})}{(+3+3)(+3-0)}$$

Spectral analysis of non-commutative "Group-table Hamiltonian"

1st Step: Spectral resolution of D_3-Center (Class algebra of D_3)

$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
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$$0 = \kappa_3^3 - 9\kappa_3 = (\kappa_3 - 3\cdot\mathbf{1})(\kappa_3 + 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})$$

$$0 = (\kappa_3 - 3\cdot\mathbf{1})\mathbf{P}^{A_1}$$

$$\kappa_3\mathbf{P}^{A_1} = +3\cdot\mathbf{P}^{A_1}$$

$$0 = (\kappa_3 + 3\cdot\mathbf{1})\mathbf{P}^{A_2}$$

$$\kappa_3\mathbf{P}^{A_2} = -3\cdot\mathbf{P}^{A_2}$$

$$\mathbf{P}^{A_1} = \frac{(\kappa_3 + 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})}{(+3+3)(+3-0)}$$

$$\mathbf{P}^{A_2} = \frac{(\kappa_3 - 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})}{(-3-3)(-3-0)}$$

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1st-Stage spectral decomposition of "Group-table" Hamiltonian of D_3 symmetry

All-commuting operators $\boldsymbol{\kappa}_k$

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D_3-invariant irep characters $\chi_k^{(\alpha)}$

Invariant numbers: Centrum, Rank, and Order

2nd-Stage spectral decompositions of global/local D_3

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3rd-Stage spectral decomposition of ALL of D_3 ...and of Hamiltonian $\mathbf{H}$

GLOBAL vs LOCAL symmetry of states ...and group $\mathbf{H}$ parameters $\{r, i_1, i_2, i_3\}$

Spectral analysis of non-commutative "Group-table Hamiltonian"

1st Step: Spectral resolution of D_3-Center (Class algebra of D_3)

$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

Each class-sum $\underline{\kappa}_k$ commutes with all of D_3.

$\kappa_1 = \mathbf{1}$	$\kappa_2 = \mathbf{r}^1 + \mathbf{r}^2$	$\kappa_3 = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_2	$2\kappa_1 + \kappa_2$	$2\kappa_3$
κ_3	$2\kappa_3$	$3\kappa_1 + 3\kappa_2$

Class products give spectral polynomial and all-commuting projectors $\mathbf{P}^{(\alpha)} = \mathbf{P}^{A_1}$, $\mathbf{P}^{A_2}$, and $\mathbf{P}^{E}$

Note also:

$\kappa_2^2 - \kappa_2 - 2\cdot\mathbf{1} = 0$

$0 = (\kappa_2 - 2\cdot\mathbf{1})(\kappa_2 + \mathbf{1})$

$$0 = \kappa_3^3 - 9\kappa_3 = (\kappa_3 - 3\cdot\mathbf{1})(\kappa_3 + 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})$$

$0 = (\kappa_3 - 3\cdot\mathbf{1})\mathbf{P}^{A_1}$

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$0 = (\kappa_3 - 0\cdot\mathbf{1})\mathbf{P}^{E}$

$\kappa_3 \mathbf{P}^{E} = +0\cdot\mathbf{P}^{E}$

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2.21.18 class 12.0: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

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$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

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Note also:

$\kappa_2^2 - \kappa_2 - 2\cdot\mathbf{1} = 0$

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$$0 = (\kappa_3 - 3\cdot\mathbf{1})\mathbf{P}^{A_1} \qquad 0 = (\kappa_3 + 3\cdot\mathbf{1})\mathbf{P}^{A_2} \qquad 0 = (\kappa_3 - 0\cdot\mathbf{1})\mathbf{P}^{E}$$

$$\kappa_3\mathbf{P}^{A_1} = +3\cdot\mathbf{P}^{A_1} \qquad \kappa_3\mathbf{P}^{A_2} = -3\cdot\mathbf{P}^{A_2} \qquad \kappa_3\mathbf{P}^{E} = +0\cdot\mathbf{P}^{E}$$

Class resolution into sum of eigenvalue · **Projector**

$$\kappa_1 = \kappa_\mathbf{1} = 1\cdot\mathbf{P}^{A_1} + 1\cdot\mathbf{P}^{A_2} + 1\cdot\mathbf{P}^{E}$$

$$\kappa_2 = \kappa_\mathbf{r} = 2\cdot\mathbf{P}^{A_1} + 2\cdot\mathbf{P}^{A_2} - 1\cdot\mathbf{P}^{E} \longleftarrow \kappa_r^2 = \kappa_r + 2\cdot\mathbf{1} \Rightarrow (\kappa_r - 2\cdot\mathbf{1})(\kappa_r + \mathbf{1}) = \mathbf{0}$$

So: κ_r has an eigenvalue 2 and -1

$$\kappa_3 = \kappa_\mathbf{i} = 3\cdot\mathbf{P}^{A_1} - 3\cdot\mathbf{P}^{A_2} + 0\cdot\mathbf{P}^{E}$$

$$\mathbf{P}^{A_1} = \frac{(\kappa_3 + 3\mathbf{1})(\kappa_3 - 0\mathbf{1})}{(+3+3)(+3-0)}$$

$$\mathbf{P}^{A_2} = \frac{(\kappa_3 - 3\mathbf{1})(\kappa_3 - 0\mathbf{1})}{(-3-3)(-3-0)}$$

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Spectral analysis of non-commutative "Group-table Hamiltonian"

1st Step: Spectral resolution of D_3-Center (Class algebra of D_3)

$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

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Each class-sum $\underline{\kappa}_k$ commutes with all of D_3.

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Class products give spectral polynomial and all-commuting projectors $\mathbf{P}^{(\alpha)} = \mathbf{P}^{A_1}$, $\mathbf{P}^{A_2}$, and $\mathbf{P}^{E}$

Note also:

$\kappa_2^2 - \kappa_2 - 2\cdot\mathbf{1} = 0$

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$\kappa_3\mathbf{P}^{A_1} = +3\cdot\mathbf{P}^{A_1}$ $\qquad$ $\kappa_3\mathbf{P}^{A_2} = -3\cdot\mathbf{P}^{A_2}$ $\qquad$ $\kappa_3\mathbf{P}^{E} = +0\cdot\mathbf{P}^{E}$

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So: κ_r has an eigenvalue 2 and -1

$$\mathbf{P}^{A_1} = \frac{(\kappa_3 + 3\mathbf{1})(\kappa_3 - 0\mathbf{1})}{(+3+3)(+3-0)}$$

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Inverse resolution gives D_3 Character Table

$$\mathbf{P}^{A_1} = (\kappa_1 + \kappa_2 + \kappa_3)/6 = (\mathbf{1} + \mathbf{r} + \mathbf{r}^2 + \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3)/6$$

$$\mathbf{P}^{A_1} = \tfrac{1}{18}\left(\kappa_3^2 + 3\kappa_3\right) = \tfrac{1}{18}\left(3\kappa_1 + 3\kappa_2 + 3\kappa_3\right)$$

Spectral analysis of non-commutative "Group-table Hamiltonian"

1st Step: Spectral resolution of D_3-Center (Class algebra of D_3)

$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
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So: κ_r has an eigenvalue 2 and -1

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Spectral analysis of non-commutative "Group-table Hamiltonian"

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$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
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$$\mathbf{P}^{E} = (2\kappa_1 - \kappa_2 + 0)/3 = (2\mathbf{1} - \mathbf{r} - \mathbf{r}^2)/3$$

$$\mathbf{P}^{A_1} = \tfrac{1}{18}\left(\kappa_3^2 + 3\kappa_3\right) = \tfrac{1}{18}\left(3\kappa_1 + 3\kappa_2 + 3\kappa_3\right)$$

$$\mathbf{P}^{A_2} = \tfrac{1}{18}\left(\kappa_3^2 - 3\kappa_3\right) = \tfrac{1}{18}\left(3\kappa_1 + 3\kappa_2 - 3\kappa_3\right)$$

$$\mathbf{P}^{E} = \tfrac{-1}{9}\left(\kappa_3^2 - 9\cdot\mathbf{1}\right) = \tfrac{-1}{9}\left(3\kappa_1 + 3\kappa_2 - 9\kappa_1\right)$$

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Global vs Local symmetry expansion of D_3 Hamiltonian

*Group theory and algebra of **D_3 Center** (Class algebra)*

Self-symmetry (Normalizer). *Lagrange Coset Theorem for classes*

→ *1st-Stage spectral decomposition of "Group-table" Hamiltonian of D_3 symmetry*

All-commuting operators $\boldsymbol{\kappa}_k$ *All-commuting projectors $\mathbf{P}^{(\alpha)}$*

D_3-invariant irep characters $\chi_k^{(\alpha)}$ ← *Invariant numbers: Centrum, Rank, and Order*

2nd-Stage spectral decompositions of global/local D_3

Subgroup chains $D_3 \supset C_2$ and $D_3 \supset C_3$ split class projectors *...and classes*

3rd-Stage spectral decomposition of ALL of D_3 *...and of Hamiltonian $\mathbf{H}$*

GLOBAL vs LOCAL symmetry of states *...and group $\mathbf{H}$ parameters$\{r, i_1, i_2, i_3\}$*

Spectral analysis of non-commutative "Group-table Hamiltonian"

1st Step: Spectral resolution of D_3-Center (Class algebra of D_3)

1	r^1	r^2	i_1	i_2	i_3
r^2	1	r^1	i_2	i_3	i_1
r^1	r^2	1	i_3	i_1	i_2
i_1	i_2	i_3	1	r^1	r^2
i_2	i_3	i_1	r^2	1	r^1
i_3	i_1	i_2	r^1	r^2	1

$\rightarrow$

Each class-sum $\underline{\kappa}_k$ commutes with all of D_3.

$\kappa_1 = \mathbf{1}$	$\kappa_2 = \mathbf{r}^1 + \mathbf{r}^2$	$\kappa_3 = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_2	$2\kappa_1 + \kappa_2$	$2\kappa_3$
κ_3	$2\kappa_3$	$3\kappa_1 + 3\kappa_2$

Class products give spectral polynomial and all-commuting projectors $\mathbf{P}^{(\alpha)} = \mathbf{P}^{A_1}$, $\mathbf{P}^{A_2}$, and $\mathbf{P}^{E}$

$$0 = \kappa_3^3 - 9\kappa_3 = (\kappa_3 - 3\cdot\mathbf{1})(\kappa_3 + 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})$$

$$0 = (\kappa_3 - 3\cdot\mathbf{1})\mathbf{P}^{A_1} \qquad 0 = (\kappa_3 + 3\cdot\mathbf{1})\mathbf{P}^{A_2} \qquad 0 = (\kappa_3 - 0\cdot\mathbf{1})\mathbf{P}^{E}$$

$$\kappa_3\mathbf{P}^{A_1} = +3\cdot\mathbf{P}^{A_1} \qquad \kappa_3\mathbf{P}^{A_2} = -3\cdot\mathbf{P}^{A_2} \qquad \kappa_3\mathbf{P}^{E} = +0\cdot\mathbf{P}^{E}$$

$$\mathbf{P}^{A_1} = \frac{(\kappa_3 + 3\mathbf{1})(\kappa_3 - 0\mathbf{1})}{(+3+3)(+3-0)}$$

$$\mathbf{P}^{A_2} = \frac{(\kappa_3 - 3\mathbf{1})(\kappa_3 - 0\mathbf{1})}{(-3-3)(-3-0)}$$

$$\mathbf{P}^{E} = \frac{(\kappa_3 - 3\mathbf{1})(\kappa_3 + 3\mathbf{1})}{(+0-3)(+0+3)}$$

Class resolution into sum of eigenvalue · Projector

$$\kappa_1 = \kappa_\mathbf{1} = 1\cdot\mathbf{P}^{A_1} + 1\cdot\mathbf{P}^{A_2} + 1\cdot\mathbf{P}^{E}$$

$$\kappa_2 = \kappa_\mathbf{r} = 2\cdot\mathbf{P}^{A_1} + 2\cdot\mathbf{P}^{A_2} - 1\cdot\mathbf{P}^{E}$$

$$\kappa_3 = \kappa_\mathbf{i} = 3\cdot\mathbf{P}^{A_1} - 3\cdot\mathbf{P}^{A_2} + 0\cdot\mathbf{P}^{E}$$

Inverse resolution gives D_3 Character Table

$$\mathbf{P}^{A_1} = (\kappa_1 + \kappa_2 + \kappa_3)/6 = (\mathbf{1} + \mathbf{r} + \mathbf{r}^2 + \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2} = (\kappa_1 + \kappa_2 - \kappa_3)/6 = (\mathbf{1} + \mathbf{r} + \mathbf{r}^2 - \mathbf{i}_1 - \mathbf{i}_2 - \mathbf{i}_3)/6$$

$$\mathbf{P}^{E} = (2\kappa_1 - \kappa_2 + 0)/3 = (2\mathbf{1} - \mathbf{r} - \mathbf{r}^2)/3$$

χ_k^α	χ_1^α	χ_2^α	χ_3^α
$\alpha = A_1$	1	1	1
$\alpha = A_2$	1	1	−1
$\alpha = E$	2	−1	0

Spectral analysis of non-commutative "Group-table Hamiltonian"

1st Step: Spectral resolution of D_3-Center (Class algebra of D_3)

$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

→

Each class-sum $\underline{\kappa}_k$ commutes with all of D_3.

$\kappa_1 = \mathbf{1}$	$\kappa_2 = \mathbf{r}^1 + \mathbf{r}^2$	$\kappa_3 = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_2	$2\kappa_1 + \kappa_2$	$2\kappa_3$
κ_3	$2\kappa_3$	$3\kappa_1 + 3\kappa_2$

Class products give spectral polynomial and all-commuting projectors $\mathbf{P}^{(\alpha)} = \mathbf{P}^{A_1}$, $\mathbf{P}^{A_2}$, and $\mathbf{P}^{E}$

$$0 = \kappa_3^3 - 9\kappa_3 = (\kappa_3 - 3\cdot\mathbf{1})(\kappa_3 + 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})$$

$$0 = (\kappa_3 - 3\cdot\mathbf{1})\mathbf{P}^{A_1} \qquad 0 = (\kappa_3 + 3\cdot\mathbf{1})\mathbf{P}^{A_2} \qquad 0 = (\kappa_3 - 0\cdot\mathbf{1})\mathbf{P}^{E}$$

$$\kappa_3\mathbf{P}^{A_1} = +3\cdot\mathbf{P}^{A_1} \qquad \kappa_3\mathbf{P}^{A_2} = -3\cdot\mathbf{P}^{A_2} \qquad \kappa_3\mathbf{P}^{E} = +0\cdot\mathbf{P}^{E}$$

$$\mathbf{P}^{A_1} = \frac{(\kappa_3 + 3\mathbf{1})(\kappa_3 - 0\mathbf{1})}{(+3+3)(+3-0)}$$

$$\mathbf{P}^{A_2} = \frac{(\kappa_3 - 3\mathbf{1})(\kappa_3 - 0\mathbf{1})}{(-3-3)(-3-0)}$$

$$\mathbf{P}^{E} = \frac{(\kappa_3 - 3\mathbf{1})(\kappa_3 + 3\mathbf{1})}{(+0-3)(+0+3)}$$

Class resolution into sum of eigenvalue · Projector

$$\kappa_1 = 1\cdot\mathbf{P}^{A_1} + 1\cdot\mathbf{P}^{A_2} + 1\cdot\mathbf{P}^{E}$$

$$\kappa_r = 2\cdot\mathbf{P}^{A_1} + 2\cdot\mathbf{P}^{A_2} - 1\cdot\mathbf{P}^{E}$$

$$\kappa_i = 3\cdot\mathbf{P}^{A_1} - 3\cdot\mathbf{P}^{A_2} + 0\cdot\mathbf{P}^{E}$$

Inverse resolution gives D_3 Character Table

$$\mathbf{P}^{A_1} = (\kappa_1 + \kappa_2 + \kappa_3)/6 = (\mathbf{1} + \mathbf{r} + \mathbf{r}^2 + \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2} = (\kappa_1 + \kappa_2 - \kappa_3)/6 = (\mathbf{1} + \mathbf{r} + \mathbf{r}^2 - \mathbf{i}_1 - \mathbf{i}_2 - \mathbf{i}_3)/6$$

$$\mathbf{P}^{E} = (2\kappa_1 - \kappa_2 + 0)/3 = (2\mathbf{1} - \mathbf{r} - \mathbf{r}^2)/3$$

*Irreducible **characters** are traces* $\chi_\kappa^{(\alpha)} = Tr\, D^{(\alpha)}(\mathbf{r}_\kappa)$ *of irreducible representations* $D^{(\alpha)}(\mathbf{r}_\kappa)$

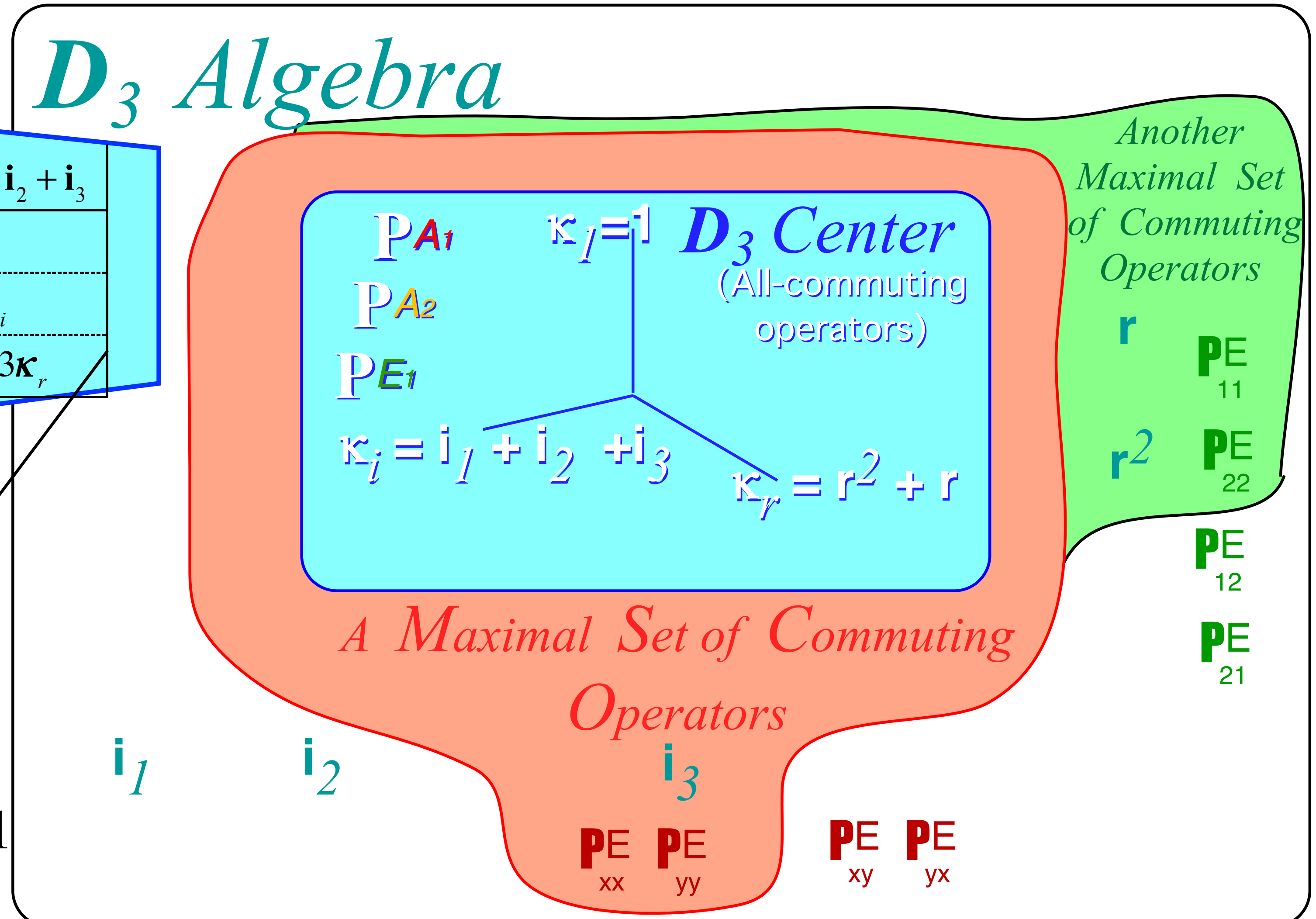
χ_k^α	χ_1^α	χ_2^α	χ_3^α
$\alpha = A_1$	1	1	1
$\alpha = A_2$	1	1	−1
$\alpha = E$	2	−1	0

Review: 1st-Stage Spectral resolution of D_3 Center (All-commuting class projectors)

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

→

	$\kappa_1=\mathbf{1}$	$\kappa_r=\mathbf{r}+\mathbf{r}^2$	$\kappa_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1+\kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1+3\kappa_r$



Class-sum $\boldsymbol{\kappa}_k$ *commutes with all* $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t\boldsymbol{\kappa}_k=\boldsymbol{\kappa}_k\mathbf{g}_t$

${}^{\circ}G$ = order of group: $({}^{\circ}D_3=6)$

${}^{\circ}\kappa_k$ = order of class κ_k: $({}^{\circ}\kappa_1=1,\ {}^{\circ}\kappa_r=2,\ {}^{\circ}\kappa_i=3)$

Class minimal equation

$$\boldsymbol{\kappa}_i^2=3\cdot\boldsymbol{\kappa}_r+3\cdot\mathbf{1}$$

$$\boldsymbol{\kappa}_i^3=3\cdot\boldsymbol{\kappa}_r\boldsymbol{\kappa}_i+3\cdot\boldsymbol{\kappa}_i=9\cdot\boldsymbol{\kappa}_i$$

$$0=\boldsymbol{\kappa}_i^3-9\cdot\boldsymbol{\kappa}_i=(\boldsymbol{\kappa}_i-3\cdot\mathbf{1})(\boldsymbol{\kappa}_i+3\cdot\mathbf{1})(\boldsymbol{\kappa}_i-0\cdot\mathbf{1})$$

Class ortho-complete $\mathbf{P}\boldsymbol{\kappa}$ *relations*

$$\boldsymbol{\kappa}_1=1\cdot\mathbf{P}^{A_1}+1\cdot\mathbf{P}^{A_2}+1\cdot\mathbf{P}^{E}=\mathbf{1}\quad\textit{(Completeness)}$$

$$\boldsymbol{\kappa}_r=2\cdot\mathbf{P}^{A_1}+2\cdot\mathbf{P}^{A_2}-1\cdot\mathbf{P}^{E}$$

$$\boldsymbol{\kappa}_i=3\cdot\mathbf{P}^{A_1}-3\cdot\mathbf{P}^{A_2}+0\cdot\mathbf{P}^{E}$$

$\chi_k^{(\alpha)}$=Trace $D^{(\alpha)}(\mathbf{g}_k)$

$\ell^{(\alpha)}$=Trace $D^{(\alpha)}(\mathbf{1})$

$$\boldsymbol{\kappa}_k=\sum_{(\alpha)}\frac{{}^{\circ}\kappa_k\chi_k^{(\alpha)}}{\ell^{(\alpha)}}\mathbf{P}^{(\alpha)}$$

irrep characters: $\chi_k^{(\alpha)}$

and dimensions: $\ell^{(\alpha)}$

$$\mathbf{P}^{(\alpha)}=\frac{\ell^{(\alpha)}}{{}^{\circ}G}\sum_k\chi_k^{(\alpha)*}\boldsymbol{\kappa}_k=\frac{\ell^{(\alpha)}}{{}^{\circ}G}\sum_{g=1}^{{}^{\circ}G}\chi_g^{(\alpha)*}\mathbf{g}$$

χ_k^α	χ_1^α	χ_r^α	χ_i^α
$\alpha=A_1$	1	1	1
$\alpha=A_2$	1	1	−1
$\alpha=E$	2	−1	0

Find $\chi_1^{(\alpha)*}=\ell^{(\alpha)}$ *using* κ_1 *coefficient*

$$\mathbf{P}^{(\alpha)}=\frac{(\ell^{(\alpha)})^2}{{}^{\circ}G}\kappa_1+\ldots$$

$$\mathbf{P}^{A_1}=(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_r+\boldsymbol{\kappa}_i)/6=\tfrac{1}{6}(\mathbf{1}+\mathbf{r}+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)$$

$$\mathbf{P}^{A_2}=(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_r-\boldsymbol{\kappa}_i)/6=\tfrac{1}{6}(\mathbf{1}+\mathbf{r}+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)$$

$$\mathbf{P}^{E}=(2\boldsymbol{\kappa}_1-\boldsymbol{\kappa}_r+0)/3=(\tfrac{2}{3}\mathbf{1}-\tfrac{1}{3}\mathbf{r}-\tfrac{1}{3}\mathbf{r}^2)$$

AMOP reference links on following page

2.21.18 class 12.0: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

Discrete symmetry subgroups of O(3) and application to tunneling and vibrational dynamics: D_3 and C_{3v} group products, classes, and irrep projection operators

32 crystal point symmetries: 16 Abelian (commutative) and 16 non-Abelian groups

Smallest non-Abelian symmetry: 3-C_2-axis D_3 vs. 3-C_v-plane C_{3v} isomorphic to permutation-S_3

Relating C_2-180°rotations $\mathbf{R}_z$, C_v-plane reflections $\boldsymbol{\sigma}_z$, and inversion $\mathbf{I}$ operators

Deriving $D_3 \sim C_{3v}$ products by group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$

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Non-commutative symmetry expansion and Global-Local solution

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1st-Stage spectral decomposition of "Group-table" Hamiltonian of D_3 symmetry

All-commuting operators $\boldsymbol{\kappa}_k$ → All-commuting projectors $\mathbf{P}^{(\alpha)}$

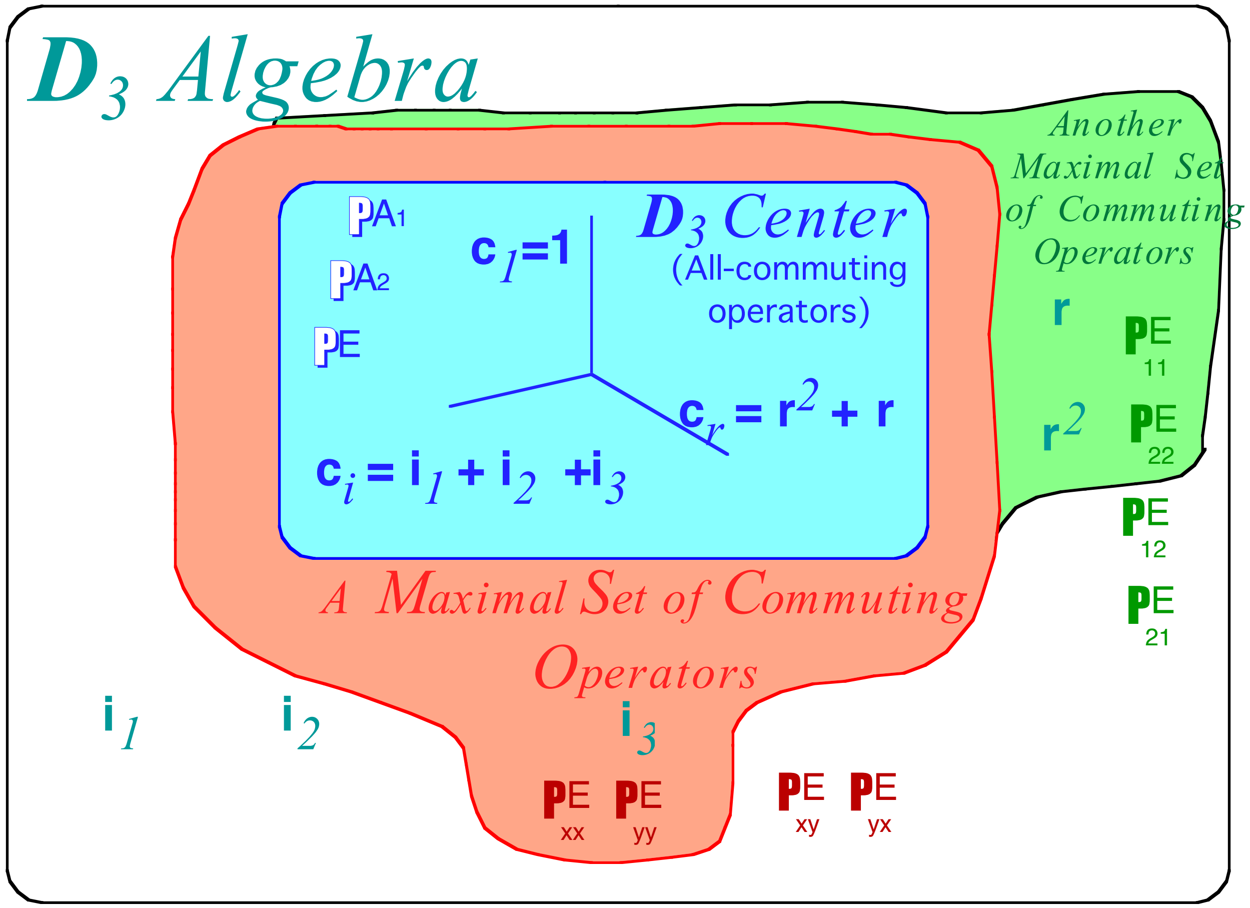
D_3-invariant irep characters $\chi_k^{(\alpha)}$ → Invariant numbers: Centrum, Rank, and Order ←

2nd-Stage spectral decompositions of global/local D_3

Subgroup chains $D_3 \supset C_2$ and $D_3 \supset C_3$ split class projectors ...and classes

3rd-Stage spectral decomposition of ALL of D_3 ...and of Hamiltonian $\mathbf{H}$

GLOBAL vs LOCAL symmetry of states ...and group $\mathbf{H}$ parameters $\{r, i_1, i_2, i_3\}$



(Fig. 15.2.1 QTCA)

D_3 $\kappa=$	$\mathbf{1}$	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
$\mathbf{P}^{A_1}=$	1	1	1	/6
$\mathbf{P}^{A_2}=$	1	1	-1	/6
$\mathbf{P}^{E}=$	2	-1	0	/3

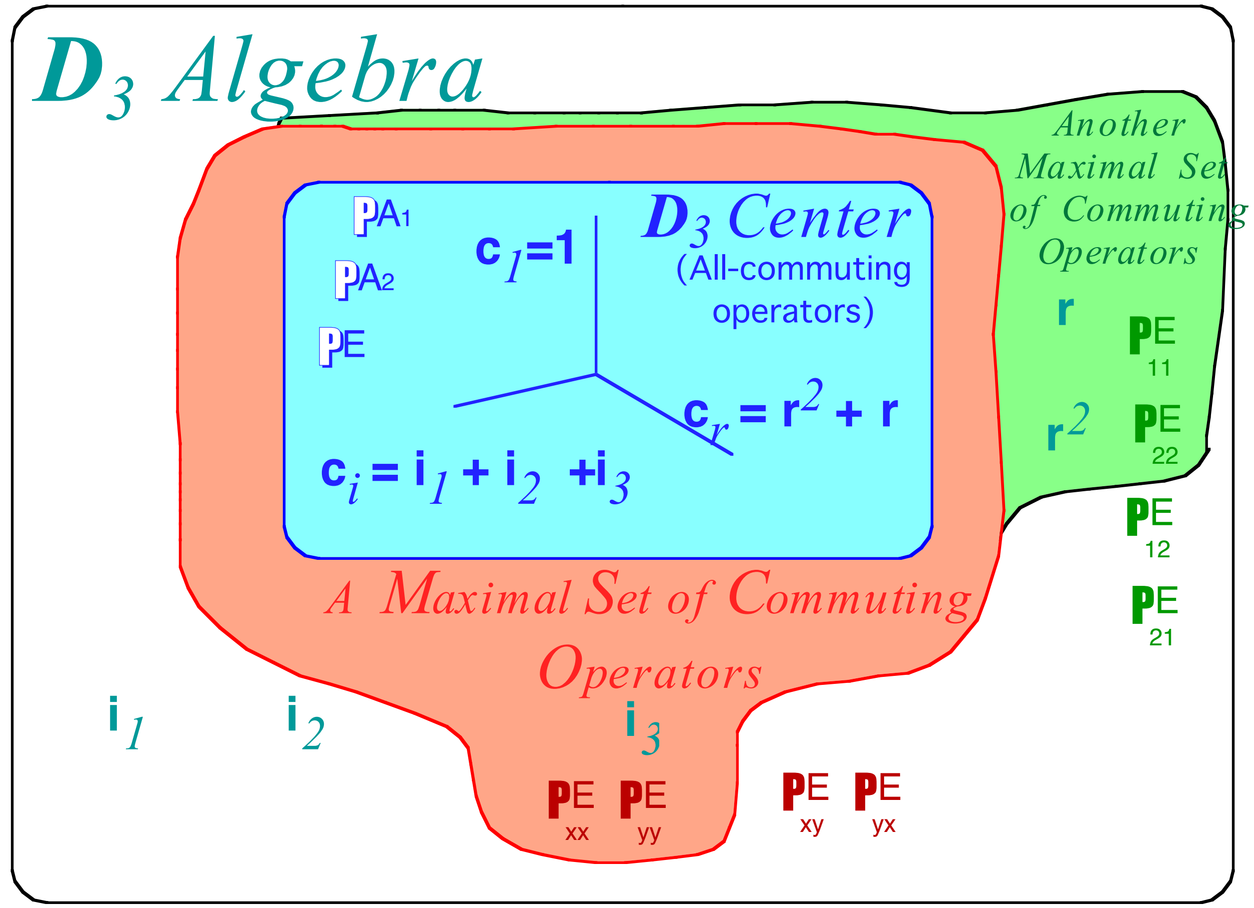
Important invariant numbers or "characters"

$\ell^\alpha =$ Irreducible representation (irrep) *dimension* or level *degeneracy*
For symmetry group or algebra G

Centrum: $\kappa(G)=\Sigma_{irrep(\alpha)} (\ell^\alpha)^0$ =Number of classes, invariants, irrep types, *all-commuting* ops

Rank: $\rho(G)=\Sigma_{irrep(\alpha)} (\ell^\alpha)^1$ =Number of irrep idempotents $\mathbf{P}^{(\alpha)}_{n,n}$, *mutually-commuting* ops

Order: $^o(G)=\Sigma_{irrep(\alpha)} (\ell^\alpha)^2$ =*Total* number of irrep projectors $\mathbf{P}^{(\alpha)}_{m,n}$ or symmetry ops



D_3	$\kappa =$ **1**	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
$\mathbf{P}^{A_1}=$	1	1	1	/6
$\mathbf{P}^{A_2}=$	1	1	-1	/6
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Order: $^{o}(G)=\Sigma_{irrep(\alpha)} (\ell^{\alpha})^2$ =*Total* number of irrep projectors $\mathbf{P}^{(\alpha)}_{m,n}$ or symmetry ops

$$\kappa(D_3)=(1)^0+(1)^0+(2)^0=3$$
$$\rho(D_3)=(1)^1+(1)^1+(2)^1=4$$
$$^{o}(D_3)=(1)^2+(1)^2+(2)^2=6$$

2.21.18 class 12.0: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

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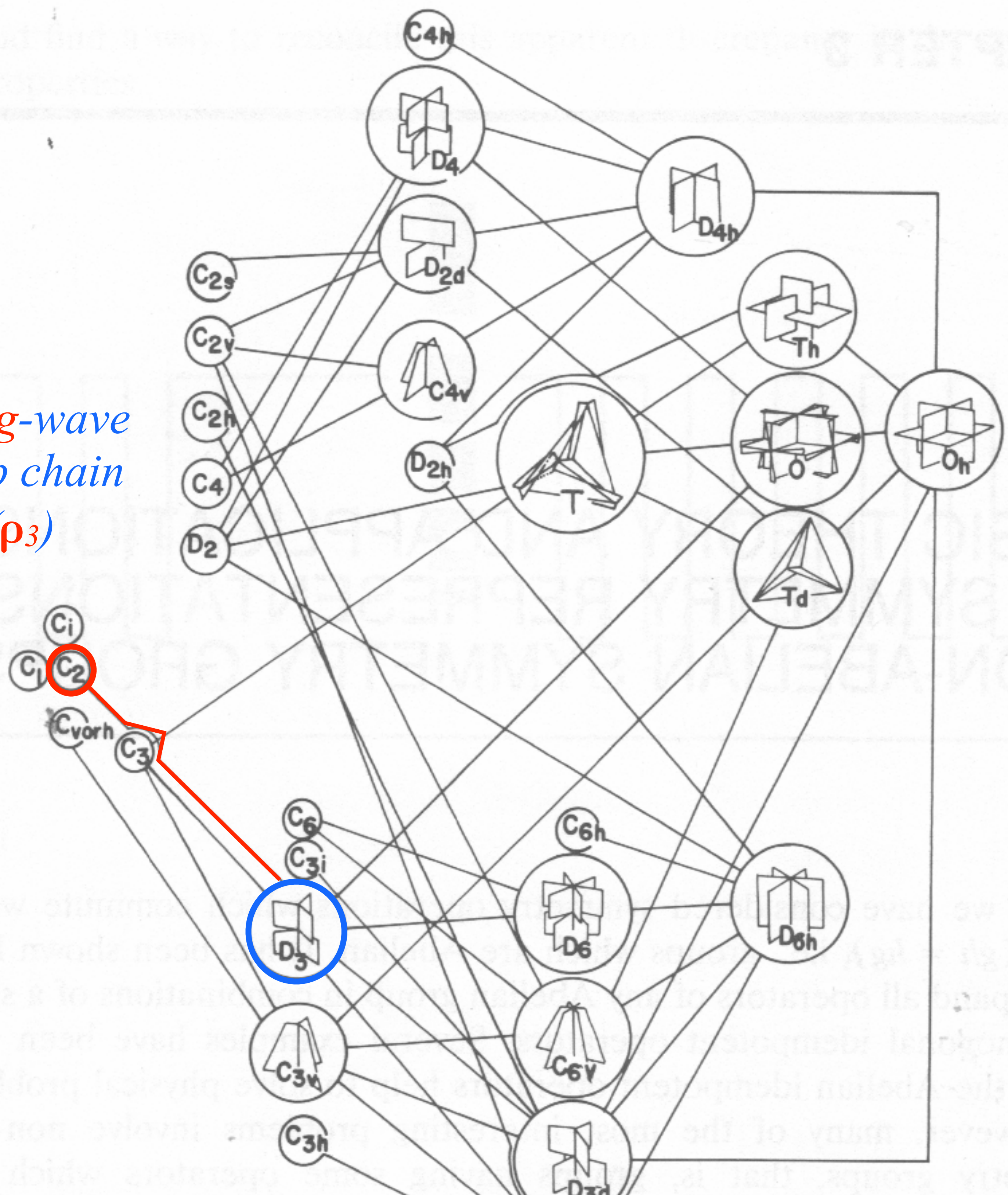
Spectral reduction of non-commutative "Group-table Hamiltonian"

D_3 Example

2nd Step: Spectral resolution of Class Projector(s) of D_3

*Correlate D_3 characters with its subgoup(s) C_2(**i**)*

Standing-wave Subroup chain $D_3 \supset C_2(\rho_3)$



Spectral reduction of non-commutative "Group-table Hamiltonian"

D_3 Example

2nd Step: Spectral resolution of Class Projector(s) of D_3

Correlate D_3 characters with its subgoup(s) $C_2(\mathbf{i})$

D_3 $\kappa =$	$\mathbf{1}$	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
$\mathbf{P}^{A_1}=$	1	1	1	/6
$\mathbf{P}^{A_2}=$	1	1	-1	/6
$\mathbf{P}^{E}=$	2	-1	0	/3

C_2 $\kappa =$	$\mathbf{1}$	$\mathbf{i}_3$	
$p^{0_2}=$	1	1	/2
$p^{1_2}=$	1	-1	/2

Spectral reduction of non-commutative "Group-table Hamiltonian"

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C_2 $\kappa =$	$\mathbf{1}$	$\mathbf{i}_3$	
$p^{0_2}=$	1	1	/2
$p^{1_2}=$	1	-1	/2

$D_3\supset C_2$ Correlation table shows which products of class projector $\mathbf{P}^{(\alpha)}$ with C_2-unit $1 = p^{0_2} + p^{1_2}$ will make IRREDUCIBLE $\mathbf{P}^{(\alpha)}_{n,n}$

$D_3\supset C_2$	0_2	1_2
$n^{A_1}=$	1	·
$n^{A_2}=$	·	1
$n^{E}=$	1	1

Spectral reduction of non-commutative "Group-table Hamiltonian"

D_3 Example

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Correlate D_3 characters with its subgoup(s) $C_2(\mathbf{i})$

D_3 $\kappa=$	$\mathbf{1}$	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
$\mathbf{P}^{A_1}=$	1	1	1	/6
$\mathbf{P}^{A_2}=$	1	1	-1	/6
$\mathbf{P}^{E}=$	2	-1	0	/3

C_2 $\kappa=$	$\mathbf{1}$	$\mathbf{i}_3$	
$p^{0_2}=$	1	1	/2
$p^{1_2}=$	1	-1	/2

$D_3 \supset C_2$ Correlation table shows which products of class projector $\mathbf{P}^{(\alpha)}$ with C_2-unit $1 = p^{0_2} + p^{1_2}$ will make IRREDUCIBLE $\mathbf{P}^{(\alpha)}_{n,n}$

$D_3 \supset C_2$	0_2	1_2
$n^{A_1}=$	1	·
$n^{A_2}=$	·	1
$n^{E}=$	1	1

Rank $\rho(D_3)$=4 implies there will be exactly 4 "C_2-friendly" irep projectors

$$\mathbf{P}^{(\alpha)}1 = \mathbf{P}^{(\alpha)}(p^{0_2} + p^{1_2})$$
$$= \mathbf{P}^{(\alpha)}_{0_2 0_2} + \mathbf{P}^{(\alpha)}_{1_2 1_2}$$

Rank :

$$\rho(G) = \sum_{irrep(\alpha)} \ell^{(\alpha)} = \text{Maximum number of } \textit{mutually} \text{ commuting operators}$$

Spectral reduction of non-commutative "Group-table Hamiltonian"

D_3 Example

2nd Step: Spectral resolution of Class Projector(s) of D_3

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D_3 $\kappa=$	$\mathbf{1}$	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
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$\mathbf{P}^{E}=$	2	-1	0	/3

C_2 $\kappa=$	$\mathbf{1}$	$\mathbf{i}_3$	
$p^{0_2}=$	1	1	/2
$p^{1_2}=$	1	-1	/2

*$D_3 \supset C_2$ Correlation table shows which products of class projector $\mathbf{P}^{(\alpha)}$ with C_2-unit $1 = p^{0_2} + p^{1_2}$ will make **IRREDUCIBLE** $\mathbf{P}^{(\alpha)}_{n,n}$*

$D_3 \supset C_2$	0_2	1_2
$n^{A_1}=$	1	·
$n^{A_2}=$	·	1
$n^{E}=$	1	1

Rank $\rho(D_3)$=4 implies there will be exactly 4 "C_2-friendly" irep projectors

$$\mathbf{P}^{(\alpha)}1 = \mathbf{P}^{(\alpha)}(p^{0_2} + p^{1_2})$$
$$= \mathbf{P}^{(\alpha)}_{0_2 0_2} + \mathbf{P}^{(\alpha)}_{1_2 1_2}$$

$$1 = p^{0_2} + p^{1_2}$$

$\mathbf{P}^{A_1}=$	$\mathbf{P}^{A_1}_{0_2 0_2}$	·
$\mathbf{P}^{A_2}=$	·	$\mathbf{P}^{A_2}_{1_2 1_2}$
$\mathbf{P}^{E}=$	$\mathbf{P}^{E}_{0_2 0_2}$	$\mathbf{P}^{E}_{1_2 1_2}$

Rank :

$$\rho(G) = \sum_{irrep(\alpha)} \ell^{(\alpha)} = \text{Maximum number of } \textit{mutually} \text{ commuting operators}$$

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D_3 $\kappa=$	$\mathbf{1}$	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
$\mathbf{P}^{A_1}=$	1	1	1	/6
$\mathbf{P}^{A_2}=$	1	1	-1	/6
$\mathbf{P}^{E}=$	2	-1	0	/3

C_2 $\kappa=$	$\mathbf{1}$	$\mathbf{i}_3$	
$p^{0_2}=$	1	1	/2
$p^{1_2}=$	1	-1	/2

*$D_3 \supset C_2$ Correlation table shows which products of class projector $\mathbf{P}^{(\alpha)}$ with C_2-unit $1 = p^{0_2} + p^{1_2}$ will make **IRREDUCIBLE** $\mathbf{P}^{(\alpha)}_{n,n}$*

$D_3 \supset C_2$	0_2	1_2
$n^{A_1}=$	1	·
$n^{A_2}=$	·	1
$n^{E}=$	1	1

Rank $\rho(D_3)$=4 implies there will be exactly 4 "C_2-friendly" irep projectors

$$\mathbf{P}^{(\alpha)}1 = \mathbf{P}^{(\alpha)}(p^{0_2} + p^{1_2})$$
$$= \mathbf{P}^{(\alpha)}_{0_2 0_2} + \mathbf{P}^{(\alpha)}_{1_2 1_2}$$

$1 = p^{0_2} + p^{1_2}$

$\mathbf{P}^{A_1}=$	$\mathbf{P}^{A_1}_{0_2 0_2}$	·
$\mathbf{P}^{A_2}=$	·	$\mathbf{P}^{A_2}_{1_2 1_2}$
$\mathbf{P}^{E}=$	$\mathbf{P}^{E}_{0_2 0_2}$	$\mathbf{P}^{E}_{1_2 1_2}$

$$\mathbf{P}^{A_1} = \mathbf{P}^{A_1} p^{0_2} = \mathbf{P}^{A_1}(1+\mathbf{i}_3)/2 = (\mathbf{1} + \mathbf{r}^1 + \mathbf{r}^2 + \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2} = \mathbf{P}^{A_2} p^{1_2} = \mathbf{P}^{A_2}(1-\mathbf{i}_3)/2 = (\mathbf{1} + \mathbf{r}^1 + \mathbf{r}^2 - \mathbf{i}_1 - \mathbf{i}_2 - \mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{0_2 0_2} = \mathbf{P}^{E} p^{0_2} = \mathbf{P}^{E}(1+\mathbf{i}_3)/2 = (2\mathbf{1} - \mathbf{r}^1 - \mathbf{r}^2 - \mathbf{i}_1 - \mathbf{i}_2 + 2\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{1_2 1_2} = \mathbf{P}^{E} p^{1_2} = \mathbf{P}^{E}(1-\mathbf{i}_3)/2 = (2\mathbf{1} - \mathbf{r}^1 - \mathbf{r}^2 + \mathbf{i}_1 + \mathbf{i}_2 - 2\mathbf{i}_3)/6$$

Rank :

$$\rho(G) = \sum_{irrep(\alpha)} \ell^{(\alpha)} = \text{Maximum number of } \textit{mutually} \text{ commuting operators}$$

AMOP reference links on following page

2.21.18 class 12.0: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

Discrete symmetry subgroups of O(3) and application to tunneling and vibrational dynamics: D_3 and C_{3v} group products, classes, and irrep projection operators

32 crystal point symmetries: 16 Abelian (commutative) and 16 non-Abelian groups
Smallest non-Abelian symmetry: 3-C_2-axis D_3 vs. 3-C_v-plane C_{3v} isomorphic to permutation-S_3
Relating C_2-180°rotations $\mathbf{R}_z$, C_v-plane reflections $\boldsymbol{\sigma}_z$, and inversion $\mathbf{I}$ operators
Deriving $D_3 \sim C_{3v}$ products by group definition $|g\rangle=\mathbf{g}|1\rangle$ of position ket $|g\rangle$
Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes
Non-commutative symmetry expansion and Global-Local solution
Global vs Local symmetry and Mock-Mach principle
Global vs Local matrix duality for D_3
Global vs Local symmetry expansion of D_3 Hamiltonian
Group theory and algebra of ***D_3 Center*** *(Class algebra)*
Self-symmetry (Normalizer). *Lagrange Coset Theorem for classes*
1st-Stage spectral decomposition of "Group-table" Hamiltonian of D_3 symmetry
All-commuting operators $\boldsymbol{\kappa}_k$ *All-commuting projectors $\mathbf{P}^{(\alpha)}$*
D_3-invariant irep characters $\chi_k^{(\alpha)}$ *Invariant numbers: Centrum, Rank, and Order*
2nd-Stage spectral decompositions of global/local D_3
Subgroup chains $D_3 \supset C_2$ and $D_3 \supset C_3$ split class projectors *...and classes*
3rd-Stage spectral decomposition of ALL of D_3 *...and of Hamiltonian $\mathbf{H}$*
GLOBAL vs LOCAL symmetry of states *...and group $\mathbf{H}$ parameters $\{r,i_1,i_2,i_3\}$*

2nd-Stage

Spectral reduction of non-commutative "Group-table Hamiltonian"

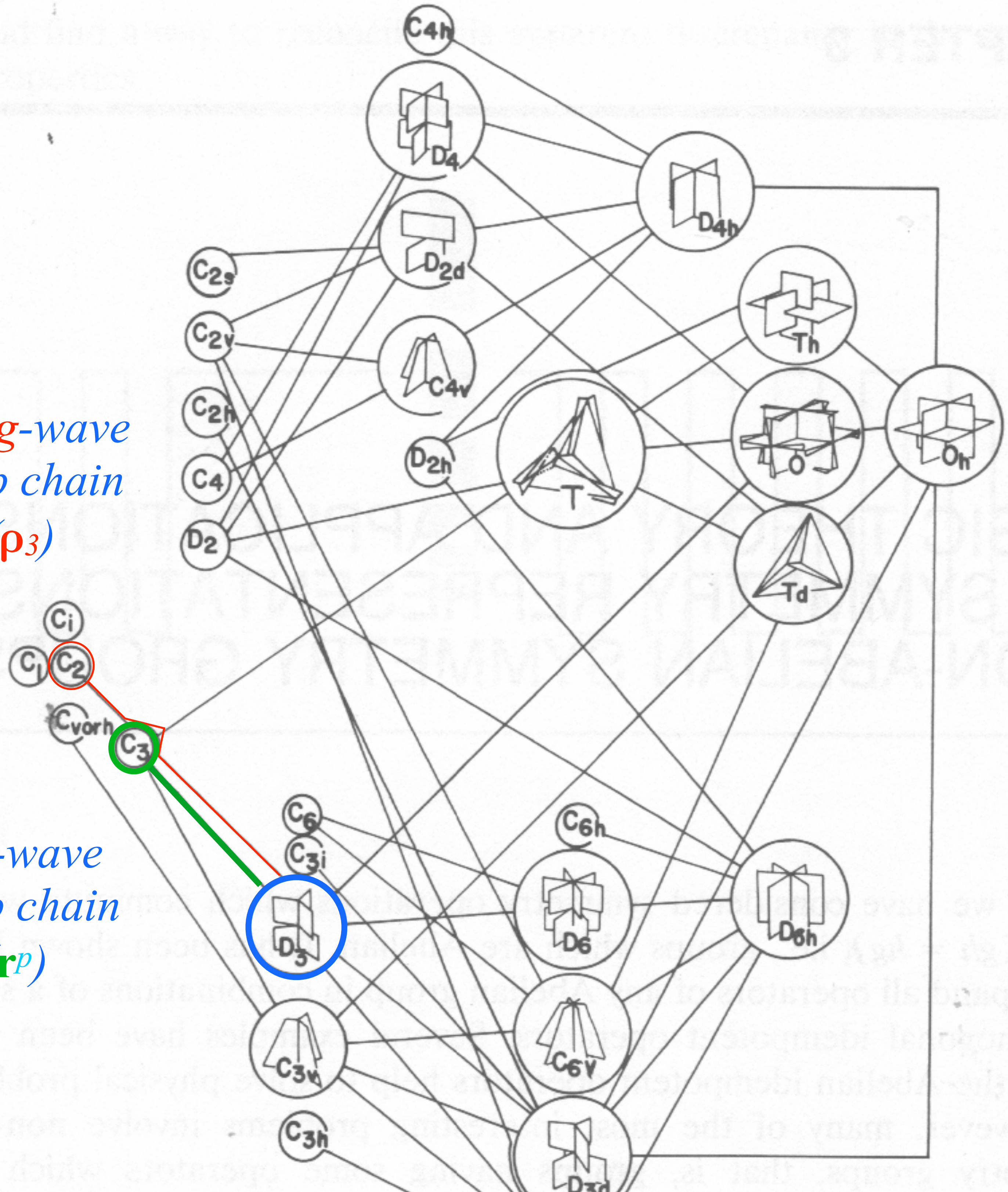
D_3 Example

2nd Step: Spectral resolution of Class Projector(s) of D_3

Correlate D_3 characters with its subgoup(s) C_2(i) or ELSE C_3(r) (C_2 and C_3 don't commute)

Standing-wave Subroup chain $D_3 \supset C_2(\rho_3)$

Moving-wave Subroup chain $D_3 \supset C_3(\mathbf{r}^p)$



2nd-Stage

Spectral reduction of non-commutative "Group-table Hamiltonian"

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2nd Step: Spectral resolution of Class Projector(s) of D_3

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$\mathbf{P}^{A_1}=$	1	1	1	/6
$\mathbf{P}^{A_2}=$	1	1	-1	/6
$\mathbf{P}^{E}=$	2	-1	0	/3

C_2 $\kappa=$	$\mathbf{1}$	$\mathbf{i}_3$	
$p^{0_2}=$	1	1	/2
$p^{1_2}=$	1	-1	/2

Let: $\varepsilon=e^{-2\pi i/3}$

C_3 $\kappa=$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	
$p^{0_3}=$	1	1	1	/3
$p^{1_3}=$	1	ε	ε^*	/3
$p^{2_3}=$	2	ε^*	ε	/3

$D_3\supset C_2$ Correlation table shows which products of class projector $\mathbf{P}^{(\alpha)}$ with C_2-unit $1=p^{0_2}+p^{1_2}$ will make ***IRREDUCIBLE*** $\mathbf{P}^{(\alpha)}_{n,n}$

$D_3\supset C_2$	0_2	1_2
$n^{A_1}=$	1	·
$n^{A_2}=$	·	1
$n^{E}=$	1	1

Rank $\rho(D_3)=4$ implies there will be exactly 4 "C_2-friendly" irep projectors

$$\mathbf{P}^{(\alpha)}1=\mathbf{P}^{(\alpha)}(p^{0_2}+p^{1_2})$$
$$=\mathbf{P}^{(\alpha)}_{0_20_2}+\mathbf{P}^{(\alpha)}_{1_21_2}$$

	$1=p^{0_2}$	$+\,p^{1_2}$
$\mathbf{P}^{A_1}=$	$\mathbf{P}^{A_1}_{0_20_2}$	·
$\mathbf{P}^{A_2}=$	·	$\mathbf{P}^{A_2}_{1_21_2}$
$\mathbf{P}^{E}=$	$\mathbf{P}^{E}_{0_20_2}$	$\mathbf{P}^{E}_{1_21_2}$

$$\mathbf{P}^{A_1}=\mathbf{P}^{A_1}p^{0_2}=\mathbf{P}^{A_1}(1+\mathbf{i}_3)/2=(\ 1+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}=\mathbf{P}^{A_2}p^{1_2}=\mathbf{P}^{A_2}(1-\mathbf{i}_3)/2=(\ 1+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{0_20_2}=\mathbf{P}^{E}p^{0_2}=\mathbf{P}^{E}(1+\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{1_21_2}=\mathbf{P}^{E}p^{1_2}=\mathbf{P}^{E}(1-\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3)/6$$

2nd-Stage

Spectral reduction of non-commutative "Group-table Hamiltonian"

D_3 Example

2nd Step: Spectral resolution of Class Projector(s) of D_3

Correlate D_3 characters with its subgoup(s) $C_2(\mathbf{i})$ or ELSE $C_3(\mathbf{r})$ (C_2 and C_3 don't commute)

D_3 κ =	$\mathbf{1}$	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
$\mathbf{P}^{A_1}=$	1	1	1	/6
$\mathbf{P}^{A_2}=$	1	1	-1	/6
$\mathbf{P}^{E}=$	2	-1	0	/3

C_2 κ =	$\mathbf{1}$	$\mathbf{i}_3$	
$p^{0_2}=$	1	1	/2
$p^{1_2}=$	1	-1	/2

Let: $\varepsilon = e^{-2\pi i/3}$

C_3 κ =	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	
$p^{0_3}=$	1	1	1	/3
$p^{1_3}=$	1	ε	ε*	/3
$p^{2_3}=$	1	ε*	ε	/3

$D_3 \supset C_2$ Correlation table shows which products of class projector $\mathbf{P}^{(\alpha)}$ with C_2-unit $1 = p^{0_2} + p^{1_2}$ will make IRREDUCIBLE $\mathbf{P}^{(\alpha)}_{n,n}$

$D_3 \supset C_2$	0_2	1_2
$n^{A_1}=$	1	·
$n^{A_2}=$	·	1
$n^{E}=$	1	1

Same for Correlation table:

$D_3 \supset C_3$	0_3	1_3	2_3
$n^{A_1}=$	1	·	·
$n^{A_2}=$	1	·	·
$n^{E}=$	·	1	1

Rank $\rho(D_3)=4$ implies there will be exactly 4 "C_2-friendly" irep projectors

$$\mathbf{P}^{(\alpha)} 1 = \mathbf{P}^{(\alpha)}(p^{0_2} + p^{1_2}) = \mathbf{P}^{(\alpha)}_{0_2 0_2} + \mathbf{P}^{(\alpha)}_{1_2 1_2}$$

$1 = p^{0_2} + p^{1_2}$

	0_2	1_2
$\mathbf{P}^{A_1}=$	$\mathbf{P}^{A_1}_{0_2 0_2}$	·
$\mathbf{P}^{A_2}=$	·	$\mathbf{P}^{A_2}_{1_2 1_2}$
$\mathbf{P}^{E}=$	$\mathbf{P}^{E}_{0_2 0_2}$	$\mathbf{P}^{E}_{1_2 1_2}$

$$\mathbf{P}^{A_1} = \mathbf{P}^{A_1} p^{0_2} = \mathbf{P}^{A_1}(1+\mathbf{i}_3)/2 = (\mathbf{1} + \mathbf{r}^1 + \mathbf{r}^2 + \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2} = \mathbf{P}^{A_2} p^{1_2} = \mathbf{P}^{A_2}(1-\mathbf{i}_3)/2 = (\mathbf{1} + \mathbf{r}^1 + \mathbf{r}^2 - \mathbf{i}_1 - \mathbf{i}_2 - \mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{0_2 0_2} = \mathbf{P}^{E} p^{0_2} = \mathbf{P}^{E}(1+\mathbf{i}_3)/2 = (2\mathbf{1} - \mathbf{r}^1 - \mathbf{r}^2 - \mathbf{i}_1 - \mathbf{i}_2 + 2\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{1_2 1_2} = \mathbf{P}^{E} p^{1_2} = \mathbf{P}^{E}(1-\mathbf{i}_3)/2 = (2\mathbf{1} - \mathbf{r}^1 - \mathbf{r}^2 + \mathbf{i}_1 + \mathbf{i}_2 - 2\mathbf{i}_3)/6$$

2nd-Stage

Spectral reduction of non-commutative "Group-table Hamiltonian"

D_3 Example

2nd Step: Spectral resolution of Class Projector(s) of D_3

Correlate D_3 characters with its subgoup(s) $C_2(\mathbf{i})$ or ELSE $C_3(\mathbf{r})$ (C_2 and C_3 don't commute)

D_3 $\kappa=$	$\mathbf{1}$	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
$\mathbf{P}^{A_1}=$	1	1	1	/6
$\mathbf{P}^{A_2}=$	1	1	-1	/6
$\mathbf{P}^{E}=$	2	-1	0	/3

C_2 $\kappa=$	$\mathbf{1}$	$\mathbf{i}_3$	
$p^{0_2}=$	1	1	/2
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Let: $\varepsilon=e^{-2\pi i/3}$

C_3 $\kappa=$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	
$p^{0_3}=$	1	1	1	/3
$p^{1_3}=$	1	ε	ε^*	/3
$p^{2_3}=$	1	ε^*	ε	/3

$D_3\supset C_2$ Correlation table shows which products of class projector $\mathbf{P}^{(\alpha)}$ with C_2-unit $1=p^{0_2}+p^{1_2}$ will make IRREDUCIBLE $\mathbf{P}^{(\alpha)}_{n,n}$

$D_3\supset C_2$	0_2	1_2
$n^{A_1}=$	1	·
$n^{A_2}=$	·	1
$n^{E}=$	1	1

Same for *Correlation table:*

$D_3\supset C_3$	0_3	1_3	2_3
$n^{A_1}=$	1	·	·
$n^{A_2}=$	1	·	·
$n^{E}=$	·	1	1

Rank $\rho(D_3)=4$ implies there will be exactly 4 "C_2-friendly" irep projectors

$$\mathbf{P}^{(\alpha)}1=\mathbf{P}^{(\alpha)}(p^{0_2}+p^{1_2})$$
$$=\mathbf{P}^{(\alpha)}_{0_20_2}+\mathbf{P}^{(\alpha)}_{1_21_2}$$

$1=p^{0_2}+p^{1_2}$

$\mathbf{P}^{A_1}=$	$\mathbf{P}^{A_1}_{0_20_2}$	·
$\mathbf{P}^{A_2}=$	·	$\mathbf{P}^{A_2}_{1_21_2}$
$\mathbf{P}^{E}=$	$\mathbf{P}^{E}_{0_20_2}$	$\mathbf{P}^{E}_{1_21_2}$

Rank $\rho(D_3)=4$ implies there will be exactly 4 "C_3-friendly" irreducible projectors

$$\mathbf{P}^{(\alpha)}1=\mathbf{P}^{(\alpha)}(p^{0_3}+p^{1_3}+p^{2_3})$$
$$=\mathbf{P}^{(\alpha)}_{0_20_2}+\mathbf{P}^{(\alpha)}_{1_31_3}+\mathbf{P}^{(\alpha)}_{2_32_3}$$

$$\mathbf{P}^{A_1}=\mathbf{P}^{A_1}p^{0_2}=\mathbf{P}^{A_1}(1+\mathbf{i}_3)/2=(1+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$
$$\mathbf{P}^{A_2}=\mathbf{P}^{A_2}p^{1_2}=\mathbf{P}^{A_2}(1-\mathbf{i}_3)/2=(1+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$
$$\mathbf{P}^{E}_{0_20_2}=\mathbf{P}^{E}p^{0_2}=\mathbf{P}^{E}(1+\mathbf{i}_3)/2=(21-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3)/6$$
$$\mathbf{P}^{E}_{1_21_2}=\mathbf{P}^{E}p^{1_2}=\mathbf{P}^{E}(1-\mathbf{i}_3)/2=(21-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3)/6$$

2nd-Stage

Spectral reduction of non-commutative "Group-table Hamiltonian"

D_3 Example

2nd Step: Spectral resolution of Class Projector(s) of D_3

Correlate D_3 characters with its subgoup(s) $C_2(\mathbf{i})$ or ELSE $C_3(\mathbf{r})$ (C_2 and C_3 don't commute)

D_3	$\kappa=\mathbf{1}$	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
$\mathbf{P}^{A_1}=$	1	1	1	/6
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$\mathbf{P}^{E}=$	2	-1	0	/3

C_2	$\kappa=\mathbf{1}$	$\mathbf{i}_3$	
$p^{0_2}=$	1	1	/2
$p^{1_2}=$	1	-1	/2

Let: $\varepsilon=e^{-2\pi i/3}$

C_3	$\kappa=\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	
$p^{0_3}=$	1	1	1	/3
$p^{1_3}=$	1	ε	ε^*	/3
$p^{2_3}=$	1	ε^*	ε	/3

*$D_3 \supset C_2$ Correlation table shows which products of class projector $\mathbf{P}^{(\alpha)}$ with C_2-unit $1 = p^{0_2} + p^{1_2}$ will make **IRREDUCIBLE** $\mathbf{P}^{(\alpha)}_{n,n}$*

$D_3 \supset C_2$	0_2	1_2
$n^{A_1}=$	1	·
$n^{A_2}=$	·	1
$n^{E}=$	1	1

Same for Correlation table:

$D_3 \supset C_3$	0_3	1_3	2_3
$n^{A_1}=$	1	·	·
$n^{A_2}=$	1	·	·
$n^{E}=$	·	1	1

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$$\mathbf{P}^{(\alpha)}1 = \mathbf{P}^{(\alpha)}(p^{0_2}+p^{1_2}) = \mathbf{P}^{(\alpha)}_{0_2 0_2} + \mathbf{P}^{(\alpha)}_{1_2 1_2}$$

$1 = p^{0_2} + p^{1_2}$

	p^{0_2}	p^{1_2}
$\mathbf{P}^{A_1}=$	$\mathbf{P}^{A_1}_{0_2 0_2}$	·
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$\mathbf{P}^{E}=$	$\mathbf{P}^{E}_{0_2 0_2}$	$\mathbf{P}^{E}_{1_2 1_2}$

$$\mathbf{P}^{A_1}=\mathbf{P}^{A_1}p^{0_2}=\mathbf{P}^{A_1}(1+\mathbf{i}_3)/2=(1+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

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$$\mathbf{P}^{E}_{0_2 0_2}=\mathbf{P}^{E}p^{0_2}=\mathbf{P}^{E}(1+\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{1_2 1_2}=\mathbf{P}^{E}p^{1_2}=\mathbf{P}^{E}(1-\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3)/6$$

Rank $\rho(D_3)=4$ implies there will be exactly 4 "C_3-friendly" irreducible projectors

$$\mathbf{P}^{(\alpha)}1 = \mathbf{P}^{(\alpha)}(p^{0_3}+p^{1_3}+p^{2_3}) = \mathbf{P}^{(\alpha)}_{0_2 0_2} + \mathbf{P}^{(\alpha)}_{1_3 1_3} + \mathbf{P}^{(\alpha)}_{2_3 2_3}$$

$1 = p^{0_3} + p^{1_3} + p^{2_3}$

	p^{0_3}	p^{1_3}	p^{2_3}
$\mathbf{P}^{A_1}=$	$\mathbf{P}^{A_1}_{0_3 0_3}$	·	·
$\mathbf{P}^{A_2}=$	$\mathbf{P}^{A_2}_{0_3 0_3}$	·	·
$\mathbf{P}^{E}=$	·	$\mathbf{P}^{E}_{1_3 1_3}$	$\mathbf{P}^{E}_{2_3 2_3}$

2nd-Stage

Spectral reduction of non-commutative "Group-table Hamiltonian"

D_3 Example

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$\mathbf{P}^{E}=$	2	-1	0	/3

C_2 $\kappa=$	$\mathbf{1}$	$\mathbf{i}_3$	
$p^{0_2}=$	1	1	/2
$p^{1_2}=$	1	-1	/2

Let:
$\varepsilon=e^{-2\pi i/3}$

C_3 $\kappa=$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	
$p^{0_3}=$	1	1	1	/3
$p^{1_3}=$	1	ε	ε^*	/3
$p^{2_3}=$	1	ε^*	ε	/3

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$D_3\supset C_2$	0_2	1_2
$n^{A_1}=$	1	·
$n^{A_2}=$	·	1
$n^{E}=$	1	1

Same for *Correlation table:*

$D_3\supset C_3$	0_3	1_3	2_3
$n^{A_1}=$	1	·	·
$n^{A_2}=$	1	·	·
$n^{E}=$	·	1	1

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$\mathbf{1}=p^{0_2}+p^{1_2}$

	p^{0_2}	p^{1_2}
$\mathbf{P}^{A_1}=$	$\mathbf{P}^{A_1}_{0_20_2}$	·
$\mathbf{P}^{A_2}=$	·	$\mathbf{P}^{A_2}_{1_21_2}$
$\mathbf{P}^{E}=$	$\mathbf{P}^{E}_{0_20_2}$	$\mathbf{P}^{E}_{1_21_2}$

Rank $\rho(D_3)=4$ implies there will be exactly 4 "C_3-friendly" irreducible projectors

$$\mathbf{P}^{(\alpha)}\mathbf{1}=\mathbf{P}^{(\alpha)}(p^{0_3}+p^{1_3}+p^{2_3})$$
$$=\mathbf{P}^{(\alpha)}_{0_20_2}+\mathbf{P}^{(\alpha)}_{1_31_3}+\mathbf{P}^{(\alpha)}_{2_32_3}$$

$\mathbf{1}=p^{0_3}+p^{1_3}+p^{2_3}$

	p^{0_3}	p^{1_3}	p^{2_3}
$\mathbf{P}^{A_1}=$	$\mathbf{P}^{A_1}_{0_30_3}$	·	·
$\mathbf{P}^{A_2}=$	$\mathbf{P}^{A_2}_{0_30_3}$	·	·
$\mathbf{P}^{E}=$	·	$\mathbf{P}^{E}_{1_31_3}$	$\mathbf{P}^{E}_{2_32_3}$

$$\mathbf{P}^{A_1}=\mathbf{P}^{A_1}p^{0_2}=\mathbf{P}^{A_1}(1+\mathbf{i}_3)/2=(\ \mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$
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$$\mathbf{P}^{E}_{0_20_2}=\mathbf{P}^{E}p^{0_2}=\mathbf{P}^{E}(1+\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3)/6$$
$$\mathbf{P}^{E}_{1_21_2}=\mathbf{P}^{E}p^{1_2}=\mathbf{P}^{E}(1-\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_1}_{0_30_3}=\mathbf{P}^{A_1}p^{0_3}=\mathbf{P}^{A_1}(1+\mathbf{r}^1+\mathbf{r}^2)/3=(\ \mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$
$$\mathbf{P}^{A_2}_{0_30_3}=\mathbf{P}^{A_2}p^{0_3}=\mathbf{P}^{A_2}(1+\mathbf{r}^1+\mathbf{r}^2)/3=(\ \mathbf{1}+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$
$$\mathbf{P}^{E}_{1_31_3}=\mathbf{P}^{E}p^{1_3}=\mathbf{P}^{E}(1+\varepsilon\mathbf{r}^1+\varepsilon^*\mathbf{r}^2)/3=(\mathbf{1}+\varepsilon\mathbf{r}^1+\varepsilon^*\mathbf{r}^2)/3$$
$$\mathbf{P}^{E}_{2_32_3}=\mathbf{P}^{E}p^{2_3}=\mathbf{P}^{E}(1+\varepsilon^*\mathbf{r}^1+\varepsilon\mathbf{r}^2)/3=(\mathbf{1}+\varepsilon^*\mathbf{r}^1+\varepsilon\mathbf{r}^2)/3$$

AMOP reference links on following page

2.21.18 class 12.0: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

Discrete symmetry subgroups of O(3) and application to tunneling and vibrational dynamics: D_3 and C_{3v} group products, classes, and irrep projection operators

32 crystal point symmetries: 16 Abelian (commutative) and 16 non-Abelian groups
Smallest non-Abelian symmetry: 3-C_2-axis D_3 vs. 3-C_v-plane C_{3v} isomorphic to permutation-S_3
- *Relating C_2-180°rotations $\mathbf{R}_z$, C_v-plane reflections $\boldsymbol{\sigma}_z$, and inversion $\mathbf{I}$ operators*
- *Deriving $D_3 \sim C_{3v}$ products by group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$*
- *Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes*

Non-commutative symmetry expansion and Global-Local solution
- *Global vs Local symmetry and Mock-Mach principle*
- *Global vs Local matrix duality for D_3*
  - *Global vs Local symmetry expansion of D_3 Hamiltonian*

*Group theory and algebra of **D_3 Center** (Class algebra)*
- *Self-symmetry (Normalizer).* *Lagrange Coset Theorem for classes*

1st-Stage spectral decomposition of "Group-table" Hamiltonian of D_3 symmetry
- *All-commuting operators $\boldsymbol{\kappa}_k$* *All-commuting projectors $\mathbf{P}^{(\alpha)}$*
- *D_3-invariant irep characters $\chi_k^{(\alpha)}$* *Invariant numbers: Centrum, Rank, and Order*

2nd-Stage spectral decompositions of global/local D_3
- *Subgroup chains $D_3 \supset C_2$ and $D_3 \supset C_3$ split class projectors* *...and classes*

3rd-Stage spectral decomposition of ALL of D_3 *...and of Hamiltonian $\mathbf{H}$*
- *GLOBAL vs LOCAL symmetry of states* *...and group $\mathbf{H}$ parameters $\{r, i_1, i_2, i_3\}$*

2nd-Stage

2nd Step: (contd.) While some class projectors $\mathbf{P}^{(\alpha)}$ split in two, so ALSO DO some classes κ_k

D_3	$\kappa=\mathbf{1}$	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
$\mathbf{P}^{A_1}=$	1	1	1	/6
$\mathbf{P}^{A_2}=$	1	1	-1	/6
$\mathbf{P}^{E}=$	2	-1	0	/3

Rank $\rho(D_3)=4$ idempotents $\mathbf{P}^{(\alpha)}$

4 *different* idempotent $\mathbf{P}^{(\alpha)}_{n,n}$

Centrum $\kappa(D_3)=3$ idempotents $\mathbf{P}^{(\alpha)}$

$$\mathbf{P}^{A_1}_{0_2 0_2}=\mathbf{P}^{A_1}p^{0_2}=\mathbf{P}^{A_1}(\mathbf{1}+\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}_{1_2 1_2}=\mathbf{P}^{A_2}p^{1_2}=\mathbf{P}^{A_2}(\mathbf{1}-\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{0_2 0_2}=\mathbf{P}^{E}p^{0_2}=\mathbf{P}^{E}(\mathbf{1}+\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{1_2 1_2}=\mathbf{P}^{E}p^{1_2}=\mathbf{P}^{E}(\mathbf{1}-\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3)/6$$

$\mathbf{P}^E$ splits into $\mathbf{P}^E=\mathbf{P}^E_{0_2 0_2}+\mathbf{P}^E_{1_2 1_2}$,

class $\kappa_{\mathbf{i}}$ splits into $\kappa_{\mathbf{i}_{12}}$ and $\kappa_{\mathbf{i}_3}$

$$\mathbf{P}^{A_1}_{0_3 0_3}=\mathbf{P}^{A_1}p^{0_3}=\mathbf{P}^{A_1}(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2)/3=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}_{0_3 0_3}=\mathbf{P}^{A_2}p^{0_3}=\mathbf{P}^{A_2}(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2)/3=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{1_3 1_3}=\mathbf{P}^{E}p^{1_3}=\mathbf{P}^{E}(\mathbf{1}+\varepsilon\mathbf{r}^1+\varepsilon^*\mathbf{r}^2)/3=(\mathbf{1}+\varepsilon\mathbf{r}^1+\varepsilon^*\mathbf{r}^2)/3$$

$$\mathbf{P}^{E}_{2_3 2_3}=\mathbf{P}^{E}p^{2_3}=\mathbf{P}^{E}(\mathbf{1}+\varepsilon^*\mathbf{r}^1+\varepsilon\mathbf{r}^2)/3=(\mathbf{1}+\varepsilon^*\mathbf{r}^1+\varepsilon\mathbf{r}^2)/3$$

$\varepsilon=e^{-2\pi i/3}$

$\mathbf{P}^E$ splits into $\mathbf{P}^E=\mathbf{P}^E_{1_3 1_3}+\mathbf{P}^E_{2_3 2_3}$,

class $\kappa_{\mathbf{r}}$ splits into $\kappa_{\mathbf{r}^1}$ and $\kappa_{\mathbf{r}^2}$

Compare ahead to Lect.17 p. 12

2nd-Stage

2nd Step: (contd.) While some class projectors $\mathbf{P}^{(\alpha)}$ split in two, so ALSO DO some classes κ_k

D_3 κ =	$\mathbf{1}$	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
$\mathbf{P}^{A_1}=$	1	1	1	/6
$\mathbf{P}^{A_2}=$	1	1	-1	/6
$\mathbf{P}^{E}=$	2	-1	0	/3

Rank $\rho(D_3)$=4 idempotents $\mathbf{P}^{(\alpha)}$

4 different idempotent $\mathbf{P}^{(\alpha)}_{n,n}$

Centrum $\kappa(D_3)$=3 idempotents $\mathbf{P}^{(\alpha)}$

$$\mathbf{P}^{A_1}_{0_2 0_2}=\mathbf{P}^{A_1}p^{0_2}=\mathbf{P}^{A_1}(1+\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}_{1_2 1_2}=\mathbf{P}^{A_2}p^{1_2}=\mathbf{P}^{A_2}(1-\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{0_2 0_2}=\mathbf{P}^{E}p^{0_2}=\mathbf{P}^{E}(1+\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{1_2 1_2}=\mathbf{P}^{E}p^{1_2}=\mathbf{P}^{E}(1-\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3)/6$$

$\mathbf{P}^E$ splits into $\mathbf{P}^E=\mathbf{P}^E_{0_2 0_2}+\mathbf{P}^E_{1_2 1_2}$,

class $\kappa_\mathbf{i}$ splits into $\kappa_{\mathbf{i}_{12}}$ and $\kappa_{\mathbf{i}_3}$

$$\mathbf{P}^{A_1}_{0_3 0_3}=\mathbf{P}^{A_1}p^{0_3}=\mathbf{P}^{A_1}(1+\mathbf{r}^1+\mathbf{r}^2)/3=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}_{0_3 0_3}=\mathbf{P}^{A_2}p^{0_3}=\mathbf{P}^{A_2}(1+\mathbf{r}^1+\mathbf{r}^2)/3=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{1_3 1_3}=\mathbf{P}^{E}p^{1_3}=\mathbf{P}^{E}(1+\varepsilon\mathbf{r}^1+\varepsilon^*\mathbf{r}^2)/3=(\mathbf{1}+\varepsilon\mathbf{r}^1+\varepsilon^*\mathbf{r}^2)/3$$

$$\mathbf{P}^{E}_{2_3 2_3}=\mathbf{P}^{E}p^{2_3}=\mathbf{P}^{E}(1+\varepsilon^*\mathbf{r}^1+\varepsilon\mathbf{r}^2)/3=(\mathbf{1}+\varepsilon^*\mathbf{r}^1+\varepsilon\mathbf{r}^2)/3$$

$\varepsilon=e^{-2\pi i/3}$

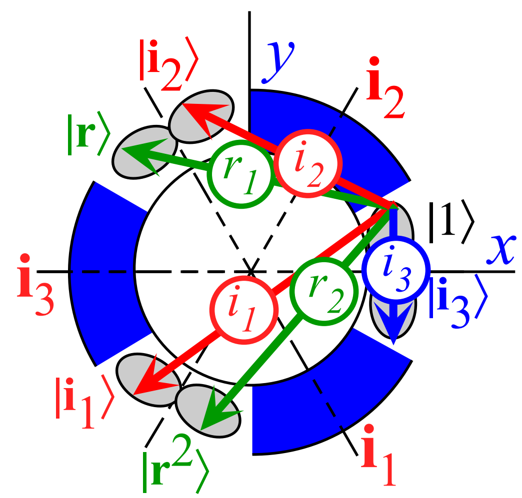
$\mathbf{P}^E$ splits into $\mathbf{P}^E=\mathbf{P}^E_{1_3 1_3}+\mathbf{P}^E_{2_3 2_3}$,

class $\kappa_\mathbf{r}$ splits into $\kappa_{\mathbf{r}^1}$ and $\kappa_{\mathbf{r}^2}$

$r=r_2$ must equal r_1 $\quad$ $i=i_2$ must equal i_1

For Local $D_3 \supset C_2(\mathbf{i}_3)$ symmetry

i_3 is free parameter



Rank $\rho(D_3)$= 4 parameters in either case

$i=i_1=i_2=i_3$

For Local $D_3 \supset C_3(\mathbf{r}^p)$ symmetry

r_1 and r_2 are free

Centrum $\kappa(D_3)=3$ idempotents $\mathbf{P}^{(\alpha)}$

D_3	$\kappa=\mathbf{1}$	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
$\mathbf{P}^{A_1}=$	1	1	1	/6
$\mathbf{P}^{A_2}=$	1	1	-1	/6
$\mathbf{P}^{E}=$	2	-1	0	/3

Rank $\rho(D_3)=4$ idempotents $\mathbf{P}^{(\alpha)}_{n,n}$

$$\mathbf{P}^{A_1}_{x,x}=\mathbf{P}^{A_1}_{0_2 0_2}=\mathbf{P}^{A_1}p^{0_2}=\mathbf{P}^{A_1}(\mathbf{1}+\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}_{y,y}=\mathbf{P}^{A_2}_{1_2 1_2}=\mathbf{P}^{A_2}p^{1_2}=\mathbf{P}^{A_2}(\mathbf{1}-\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{x,x}=\mathbf{P}^{E}_{0_2 0_2}=\mathbf{P}^{E}p^{0_2}=\mathbf{P}^{E}(\mathbf{1}+\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{y,y}=\mathbf{P}^{E}_{1_2 1_2}=\mathbf{P}^{E}p^{1_2}=\mathbf{P}^{E}(\mathbf{1}-\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3)/6$$

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*Group theory and algebra of **D_3 Center** (Class algebra)*

Self-symmetry (Normalizer). *Lagrange Coset Theorem for classes*

1st-Stage spectral decomposition of "Group-table" Hamiltonian of D_3 symmetry

All-commuting operators $\boldsymbol{\kappa}_k$ *All-commuting projectors $\mathbf{P}^{(\alpha)}$*

D_3-invariant irep characters $\chi_k^{(\alpha)}$ *Invariant numbers: Centrum, Rank, and Order*

2nd-Stage spectral decompositions of global/local D_3

Subgroup chains $D_3 \supset C_2$ and $D_3 \supset C_3$ split class projectors *...and classes*

→ *3rd-Stage spectral decomposition of ALL of D_3* ← *...and of Hamiltonian **H***

GLOBAL vs LOCAL symmetry of states *...and group **H** parameters $\{r, i_1, i_2, i_3\}$*

Centrum $\kappa(D_3)=3$
idempotents
$\mathbf{P}^{(\alpha)}$

D_3 $\kappa=$	$\mathbf{1}$	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
$\mathbf{P}^{A_1}=$	1	1	1	/6
$\mathbf{P}^{A_2}=$	1	1	-1	/6
$\mathbf{P}^{E}=$	2	-1	0	/3

Rank $\rho(D_3)=4$
idempotents
$\mathbf{P}^{(\alpha)}_{n,n}$

$$\mathbf{P}^{A_1}_{x,x}=\mathbf{P}^{A_1}_{0_2 0_2}=\mathbf{P}^{A_1}p^{0_2}=\mathbf{P}^{A_1}(\mathbf{1}+\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}_{y,y}=\mathbf{P}^{A_2}_{1_2 1_2}=\mathbf{P}^{A_2}p^{1_2}=\mathbf{P}^{A_2}(\mathbf{1}-\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{x,x}=\mathbf{P}^{E}_{0_2 0_2}=\mathbf{P}^{E}p^{0_2}=\mathbf{P}^{E}(\mathbf{1}+\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{y,y}=\mathbf{P}^{E}_{1_2 1_2}=\mathbf{P}^{E}p^{1_2}=\mathbf{P}^{E}(\mathbf{1}-\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3)/6$$

3rd and Final Step:
Spectral resolution of ALL 6 of D3 :

The old '**g**-equals-**1**-times-**g**-times-**1**' Trick

$$\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}=(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})\cdot\mathbf{g}\cdot(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})$$

Compare '*ahead*' to Lect.17 p.14

Centrum $\kappa(D_3)=3$
idempotents
$\mathbf{P}^{(\alpha)}$

D_3 $\kappa =$	$\mathbf{1}$	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
$\mathbf{P}^{A_1}=$	1	1	1	/6
$\mathbf{P}^{A_2}=$	1	1	-1	/6
$\mathbf{P}^{E}=$	2	-1	0	/3

Rank $\rho(D_3)=4$
idempotents
$\mathbf{P}^{(\alpha)}_{n,n}$

$$\mathbf{P}^{A_1}_{x,x}= \mathbf{P}^{A_1}_{0_2 0_2}=\mathbf{P}^{A_1}\mathbf{p}^{0_2}=\mathbf{P}^{A_1}(\mathbf{1}+\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}_{y,y}= \mathbf{P}^{A_2}_{1_2 1_2}=\mathbf{P}^{A_2}\mathbf{p}^{1_2}=\mathbf{P}^{A_2}(\mathbf{1}-\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{x,x}= \mathbf{P}^{E}_{0_2 0_2}=\mathbf{P}^{E}\mathbf{p}^{0_2}=\mathbf{P}^{E}(\mathbf{1}+\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{y,y}= \mathbf{P}^{E}_{1_2 1_2}=\mathbf{P}^{E}\mathbf{p}^{1_2}=\mathbf{P}^{E}(\mathbf{1}-\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3)/6$$

3rd and Final Step:
Spectral resolution of ALL 6 of D3 :

The old '**g**-equals-**1**-times-**g**-times-**1**' Trick

$$\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}=(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})\cdot\mathbf{g}\cdot(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})$$

$$\begin{aligned}\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}=\mathbf{P}^{A_1}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{A_1}_{x,x} &+ 0 + 0 + 0\\
+\;0\; &+ \mathbf{P}^{A_2}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{A_2}_{y,y} + 0 + 0\\
+\;0\; &+ 0 + \mathbf{P}^{E}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{x,x} + \mathbf{P}^{E}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{y,y}\\
+\;0\; &+ 0 + \mathbf{P}^{E}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{x,x} + \mathbf{P}^{E}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{y,y}\end{aligned}$$

$\mathbf{P}^{\mu}_{mn}$ **g**-*expansion in Lect.17 p. 35-51*

Compare '*ahead*' to Lect.17 p.14

Centrum $\kappa(D_3)=3$ idempotents $\mathbf{P}^{(\alpha)}$

D_3	$\kappa=\mathbf{1}$	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
$\mathbf{P}^{A_1}=$	1	1	1	/6
$\mathbf{P}^{A_2}=$	1	1	-1	/6
$\mathbf{P}^{E}=$	2	-1	0	/3

Rank $\rho(D_3)=4$ idempotents $\mathbf{P}^{(\alpha)}_{n,n}$

$$\mathbf{P}^{A_1}_{x,x}=\mathbf{P}^{A_1}_{0_2 0_2}=\mathbf{P}^{A_1}p^{0_2}=\mathbf{P}^{A_1}(1+\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}_{y,y}=\mathbf{P}^{A_2}_{1_2 1_2}=\mathbf{P}^{A_2}p^{1_2}=\mathbf{P}^{A_2}(1-\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{x,x}=\mathbf{P}^{E}_{0_2 0_2}=\mathbf{P}^{E}p^{0_2}=\mathbf{P}^{E}(1+\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{y,y}=\mathbf{P}^{E}_{1_2 1_2}=\mathbf{P}^{E}p^{1_2}=\mathbf{P}^{E}(1-\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3)/6$$

3rd and Final Step:

Spectral resolution of ALL 6 of D3 :

The old '$\mathbf{g}$-equals-$\mathbf{1}$-times-$\mathbf{g}$-times-$\mathbf{1}$' Trick

$$\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}=(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})\cdot\mathbf{g}\cdot(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})$$

$$\begin{aligned}\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}=\;&\mathbf{P}^{A_1}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{A_1}_{x,x}+0+0+0\\&+0+\mathbf{P}^{A_2}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{A_2}_{y,y}+0+0\\&+0+0+\mathbf{P}^{E}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{y,y}\\&+0+0+\mathbf{P}^{E}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{y,y}\end{aligned}$$

where:

$$\mathbf{P}^{A_1}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{A_1}_{x,x}=D^{A_1}(\mathbf{g})\mathbf{P}^{A_1}_{x,x}$$

$$\mathbf{P}^{A_2}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{A_2}_{y,y}=D^{A_2}(\mathbf{g})\mathbf{P}^{A_2}_{y,y}$$

$$\mathbf{P}^{E}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{x,x}=D^{E}_{x,x}(\mathbf{g})\mathbf{P}^{E}_{x,x}\qquad\mathbf{P}^{E}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{y,y}=D^{E}_{x,y}(\mathbf{g})\mathbf{P}^{E}_{x,y}$$

$$\mathbf{P}^{E}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{x,x}=D^{E}_{y,x}(\mathbf{g})\mathbf{P}^{E}_{y,x}\qquad\mathbf{P}^{E}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{y,y}=D^{E}_{y,y}(\mathbf{g})\mathbf{P}^{E}_{y,y}$$

Need to Define
6 Irreducible
Projectors $\mathbf{P}^{(\alpha)}_{m,n}$
Order $°(D_3)=6$

Centrum $\kappa(D_3)=3$ idempotents $\mathbf{P}^{(\alpha)}$

D_3	$\kappa =$ **1**	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
$\mathbf{P}^{A_1}=$	1	1	1	/6
$\mathbf{P}^{A_2}=$	1	1	-1	/6
$\mathbf{P}^{E}=$	2	-1	0	/3

Rank $\rho(D_3)=4$ idempotents $\mathbf{P}^{(\alpha)}_{n,n}$

$$\mathbf{P}^{A_1}_{x,x}= \mathbf{P}^{A_1}_{0_2 0_2}=\mathbf{P}^{A_1}p^{0_2}=\mathbf{P}^{A_1}(\mathbf{1}+\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}_{y,y}= \mathbf{P}^{A_2}_{1_2 1_2}=\mathbf{P}^{A_2}p^{1_2}=\mathbf{P}^{A_2}(\mathbf{1}-\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{x,x}= \mathbf{P}^{E}_{0_2 0_2}=\mathbf{P}^{E}p^{0_2}=\mathbf{P}^{E}(\mathbf{1}+\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{y,y}= \mathbf{P}^{E}_{1_2 1_2}=\mathbf{P}^{E}p^{1_2}=\mathbf{P}^{E}(\mathbf{1}-\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3)/6$$

3rd and Final Step:

Spectral resolution of ALL 6 of D3 :

The old ‘g-equals-1-times-g-times-1’ Trick

$$\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}=(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})\cdot\mathbf{g}\cdot(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})$$

$$\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}=D^{A_1}(\mathbf{g})\mathbf{P}^{A_1}_{x,x}+0+0+0$$
$$+0+D^{A_2}(\mathbf{g})\mathbf{P}^{A_2}_{y,y}+0+0$$
$$+0+0+D^{E}_{x,x}(\mathbf{g})\mathbf{P}^{E}_{x,x}+D^{E}_{x,y}(\mathbf{g})\mathbf{P}^{E}_{x,y}$$
$$+0+0+D^{E}_{y,x}(\mathbf{g})\mathbf{P}^{E}_{y,x}+D^{E}_{y,y}(\mathbf{g})\mathbf{P}^{E}_{y,y}$$

where:

$$\mathbf{P}^{A_1}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{A_1}_{x,x}=D^{A_1}(\mathbf{g})\mathbf{P}^{A_1}_{x,x}$$

$$\mathbf{P}^{A_2}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{A_2}_{y,y}=D^{A_2}(\mathbf{g})\mathbf{P}^{A_2}_{y,y}$$

$$\mathbf{P}^{E}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{x,x}=D^{E}_{x,x}(\mathbf{g})\mathbf{P}^{E}_{x,x} \qquad \mathbf{P}^{E}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{y,y}=D^{E}_{x,y}(\mathbf{g})\mathbf{P}^{E}_{x,y}$$

$$\mathbf{P}^{E}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{x,x}=D^{E}_{y,x}(\mathbf{g})\mathbf{P}^{E}_{y,x} \qquad \mathbf{P}^{E}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{y,y}=D^{E}_{y,y}(\mathbf{g})\mathbf{P}^{E}_{y,y}$$

Need to Define 6 Irreducible Projectors $\mathbf{P}^{(\alpha)}_{m,n}$

Order $°(D_3)=6$

Centrum $\kappa(D_3)=3$ idempotents $\mathbf{P}^{(\alpha)}$

D_3	$\kappa=\mathbf{1}$	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
$\mathbf{P}^{A_1}=$	1	1	1	/6
$\mathbf{P}^{A_2}=$	1	1	-1	/6
$\mathbf{P}^{E}=$	2	-1	0	/3

Rank $\rho(D_3)=4$ idempotents $\mathbf{P}^{(\alpha)}_{n,n}$

$$\mathbf{P}^{A_1}_{x,x}=\mathbf{P}^{A_1}_{0_2 0_2}=\mathbf{P}^{A_1}p^{0_2}=\mathbf{P}^{A_1}(\mathbf{1}+\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}_{y,y}=\mathbf{P}^{A_2}_{1_2 1_2}=\mathbf{P}^{A_2}p^{1_2}=\mathbf{P}^{A_2}(\mathbf{1}-\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{x,x}=\mathbf{P}^{E}_{0_2 0_2}=\mathbf{P}^{E}p^{0_2}=\mathbf{P}^{E}(\mathbf{1}+\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{y,y}=\mathbf{P}^{E}_{1_2 1_2}=\mathbf{P}^{E}p^{1_2}=\mathbf{P}^{E}(\mathbf{1}-\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3)/6$$

3rd and Final Step:
Spectral resolution of ALL 6 of D3 :

The old '**g**-equals-**1**-times-**g**-times-**1**' Trick

$$\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}=(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})\cdot\mathbf{g}\cdot(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})$$

$$\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}=D^{A_1}(\mathbf{g})\mathbf{P}^{A_1}_{x,x}+D^{A_2}(\mathbf{g})\mathbf{P}^{A_2}_{y,y}+D^{E}_{x,x}(\mathbf{g})\mathbf{P}^{E}_{x,x}+D^{E}_{x,y}(\mathbf{g})\mathbf{P}^{E}_{x,y}$$

$$+D^{E}_{y,x}(\mathbf{g})\mathbf{P}^{E}_{y,x}+D^{E}_{y,y}(\mathbf{g})\mathbf{P}^{E}_{y,y}$$

where:

$$\mathbf{P}^{A_1}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{A_1}_{x,x}=D^{A_1}(\mathbf{g})\mathbf{P}^{A_1}_{x,x}$$

$$\mathbf{P}^{A_2}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{A_2}_{y,y}=D^{A_2}(\mathbf{g})\mathbf{P}^{A_2}_{y,y}$$

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$$\mathbf{P}^{A_1}_{x,x}=\mathbf{P}^{A_1}_{0_2 0_2}=\mathbf{P}^{A_1}p^{0_2}=\mathbf{P}^{A_1}(\mathbf{1}+\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

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$$\mathbf{g}=D^{A_1}(\mathbf{g})\mathbf{P}^{A_1}_{x,x}+D^{A_2}(\mathbf{g})\mathbf{P}^{A_2}_{y,y}+D^{E}_{x,x}(\mathbf{g})\mathbf{P}^{E}_{x,x}+D^{E}_{y,y}(\mathbf{g})\mathbf{P}^{E}_{y,y}+D^{E}_{x,y}(\mathbf{g})\mathbf{P}^{E}_{x,y}+D^{E}_{y,x}(\mathbf{g})\mathbf{P}^{E}_{y,x}$$

Six D_3 projectors: 4 idempotents + 2 nilpotents (off-diag.)

	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	
$\mathbf{P}^{A_1}_{x,x}=$	(1	1	1	1	1	1)	/6
$\mathbf{P}^{A_2}_{y,y}=$	(1	1	1	-1	-1	-1)	/6

	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	
$\mathbf{P}^{E}_{x,x}=$	(2	-1	-1	-1	-1	+2)	/6
$\mathbf{P}^{E}_{y,x}=$	(0	1	-1	-1	+1	0)	$/\sqrt{3}/2$

	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	
$\mathbf{P}^{E}_{x,y}=$	(0	-1	1	-1	+1	0)	$/\sqrt{3}/2$
$\mathbf{P}^{E}_{y,y}=$	(2	-1	-1	+1	+1	-2)	/6

Compare ‘*ahead*’ to Lect.17 p.18-21

Centrum $\kappa(D_3)=3$
idempotents
$\mathbf{P}^{(\alpha)}$

D_3	$\kappa=\mathbf{1}$	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
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$\mathbf{P}^{E}=$	2	-1	0	/3

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$\mathbf{P}^{(\alpha)}_{n,n}$

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$$\mathbf{P}^{E}_{x,x}=\mathbf{P}^{E}_{0_2 0_2}=\mathbf{P}^{E}p^{0_2}=\mathbf{P}^{E}(\mathbf{1}+\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{y,y}=\mathbf{P}^{E}_{1_2 1_2}=\mathbf{P}^{E}p^{1_2}=\mathbf{P}^{E}(\mathbf{1}-\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3)/6$$

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Spectral resolution of ALL 6 of D3 :

$$\mathbf{g}=\Sigma_m\Sigma_e\Sigma_b D^{(m)}_{eb}(\mathbf{g})\,\mathbf{P}^{(m)}_{eb}$$

$$\mathbf{P}^{(m)}_{eb}={}_{(norm)}\Sigma_{\mathbf{g}} D^{(m)*}_{eb}(\mathbf{g})\,\mathbf{g}$$

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$$\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}=(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})\cdot\mathbf{g}\cdot(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})$$

$$\mathbf{g}=D^{A_1}(\mathbf{g})\mathbf{P}^{A_1}_{x,x}+D^{A_2}(\mathbf{g})\mathbf{P}^{A_2}_{y,y}+D^{E}_{x,x}(\mathbf{g})\mathbf{P}^{E}_{x,x}+D^{E}_{y,y}(\mathbf{g})\mathbf{P}^{E}_{y,y}+D^{E}_{x,y}(\mathbf{g})\mathbf{P}^{E}_{x,y}+D^{E}_{y,x}(\mathbf{g})\mathbf{P}^{E}_{y,x}$$

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	1	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	
$\mathbf{P}^{A_1}_{x,x}=$	(1	1	1	1	1	1)	/6
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	1	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	
$\mathbf{P}^{E}_{x,x}=$	(2	-1	-1	-1	-1	+2)	/6
$\mathbf{P}^{E}_{y,x}=$	(0	1	-1	-1	+1	0)	/√3/2

	1	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	
$\mathbf{P}^{E}_{x,y}=$	(0	-1	1	-1	+1	0)	/√3/2
$\mathbf{P}^{E}_{y,y}=$	(2	-1	-1	+1	+1	-2)	/6

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$$\mathbf{P}^{A_2}_{y,y}=\mathbf{P}^{A_2}_{1_2 1_2}=\mathbf{P}^{A_2}p^{1_2}=\mathbf{P}^{A_2}(\mathbf{1}-\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$
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$$\mathbf{P}^{E}_{y,y}=\mathbf{P}^{E}_{1_2 1_2}=\mathbf{P}^{E}p^{1_2}=\mathbf{P}^{E}(\mathbf{1}-\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3)/6$$

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$$\mathbf{P}^{(m)}_{eb}={}_{(norm)}\Sigma_{\mathbf{g}} D^{(m)*}_{eb}(\mathbf{g})\,\mathbf{g}$$

$$\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}=(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})\cdot\mathbf{g}\cdot(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})$$

$$\mathbf{g}=D^{A_1}(\mathbf{g})\mathbf{P}^{A_1}_{x,x}+D^{A_2}(\mathbf{g})\mathbf{P}^{A_2}_{y,y}+D^{E}_{x,x}(\mathbf{g})\mathbf{P}^{E}_{x,x}+D^{E}_{y,y}(\mathbf{g})\mathbf{P}^{E}_{y,y}+D^{E}_{x,y}(\mathbf{g})\mathbf{P}^{E}_{x,y}+D^{E}_{y,x}(\mathbf{g})\mathbf{P}^{E}_{y,x}$$

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	1	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	
$\mathbf{P}^{A_1}_{x,x}=$	(1	1	1	1	1	1)	/6
$\mathbf{P}^{A_2}_{y,y}=$	(1	1	1	-1	-1	-1)	/6

	1	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	
$\mathbf{P}^{E}_{x,x}=$	(2	-1	-1	-1	-1	+2)	/6
$\mathbf{P}^{E}_{y,x}=$	(0	1	-1	-1	+1	0)	/√3/2

	1	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	
$\mathbf{P}^{E}_{x,y}=$	(0	-1	1	-1	+1	0)	/√3/2
$\mathbf{P}^{E}_{y,y}=$	(2	-1	-1	+1	+1	-2)	/6

where D_3 irreducible representations are: $D^{A_1}(\mathbf{g})=+1$, $D^{A_2}(\mathbf{g})=\pm 1$,

$$D^E(1)=\begin{pmatrix}1&0\\0&1\end{pmatrix},D^E(\mathbf{r})=\begin{pmatrix}-\frac{1}{2}&-\sqrt{\frac{3}{4}}\\\sqrt{\frac{3}{4}}&-\frac{1}{2}\end{pmatrix},D^E(\mathbf{r}^2)=\begin{pmatrix}-\frac{1}{2}&\sqrt{\frac{3}{4}}\\-\sqrt{\frac{3}{4}}&-\frac{1}{2}\end{pmatrix},D^E(\mathbf{i}_1)=\begin{pmatrix}-\frac{1}{2}&-\sqrt{\frac{3}{4}}\\-\sqrt{\frac{3}{4}}&\frac{1}{2}\end{pmatrix},D^E(\mathbf{i}_2)=\begin{pmatrix}-\frac{1}{2}&\sqrt{\frac{3}{4}}\\\sqrt{\frac{3}{4}}&\frac{1}{2}\end{pmatrix},D^E(\mathbf{i}_3)=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

AMOP reference links on following page

2.21.18 class 12.0: *Symmetry Principles for Advanced Atomic-Molecular-Optical-Physics*

William G. Harter - University of Arkansas

Discrete symmetry subgroups of O(3) and application to tunneling and vibrational dynamics: D_3 and C_{3v} group products, classes, and irrep projection operators

32 crystal point symmetries: 16 Abelian (commutative) and 16 non-Abelian groups

Smallest non-Abelian symmetry: 3-C_2-axis D_3 vs. 3-C_v-plane C_{3v} isomorphic to permutation-S_3

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Global vs Local matrix duality for D_3

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1st-Stage spectral decomposition of "Group-table" Hamiltonian of D_3 symmetry

All-commuting operators $\boldsymbol{\kappa}_k$ *All-commuting projectors $\mathbf{P}^{(\alpha)}$*

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2nd-Stage spectral decompositions of global/local D_3

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3rd-Stage spectral decomposition of ALL of D_3 *...and of Hamiltonian **H***

GLOBAL vs LOCAL symmetry of states *...and group **H** parameters $\{r,i_1,i_2,i_3\}$*

$D_3 > C_2\,\mathbf{i}_3$ projector states

$$\left|{}^{(m)}_{eb}\right\rangle = \mathbf{P}^{(m)}_{eb}\left|1\right\rangle$$

Global (LAB) symmetry

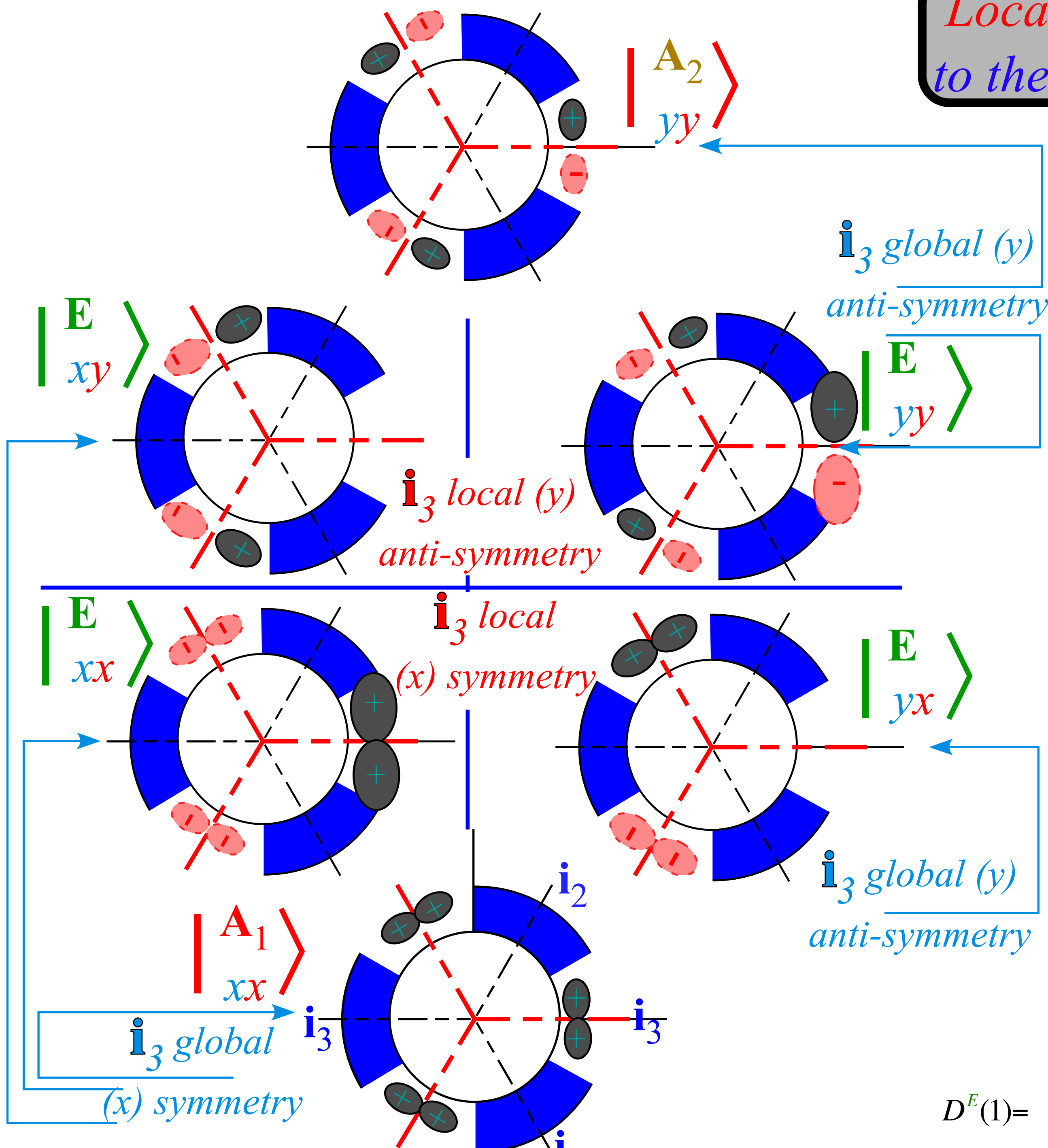
$$\mathbf{i}_3\left|{}^{(m)}_{eb}\right\rangle = \mathbf{i}_3\mathbf{P}^{(m)}_{eb}\left|1\right\rangle = (-1)^e\left|{}^{(m)}\right\rangle$$

Local (BOD) symmetry

$$\bar{\mathbf{i}}_3\left|{}^{(m)}_{eb}\right\rangle = \bar{\mathbf{i}}_3\mathbf{P}^{(m)}_{eb}\left|1\right\rangle = \mathbf{P}^{(m)}_{eb}\bar{\mathbf{i}}_3\left|1\right\rangle = \mathbf{P}^{(m)}_{eb}\mathbf{i}_3^{\dagger}\left|1\right\rangle = (-1)^b\left|{}^{(m)}\right\rangle$$

Local $\bar{\mathbf{g}}$ commute through to the "inside" to be a $\mathbf{g}^{\dagger}$

Here the "Mock-Mach" is being applied!



	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	
$\mathbf{P}^{A_2}_{y,y}=$	(1	1	1	-1	-1	-1)	/6
$\mathbf{P}^{E}_{x,y}=$	(0	-1	1	-1	+1	0)	/√3/2
$\mathbf{P}^{E}_{y,y}=$	(2	-1	-1	+1	+1	-2)	/6
$\mathbf{P}^{E}_{x,x}=$	(2	-1	-1	-1	-1	+2)	/6
$\mathbf{P}^{E}_{y,x}=$	(0	1	-1	-1	+1	0)	/√3/2
$\mathbf{P}^{A_1}_{x,x}=$	(1	1	1	1	1	1)	/6

Ket norm factors detailed in Lect.17 p.23-30

$D^{A_1}(\mathbf{g}) = +1,\; D^{A_2}(\mathbf{r}^p) = +1,\; D^{A_2}(\mathbf{i}_q) = -1$

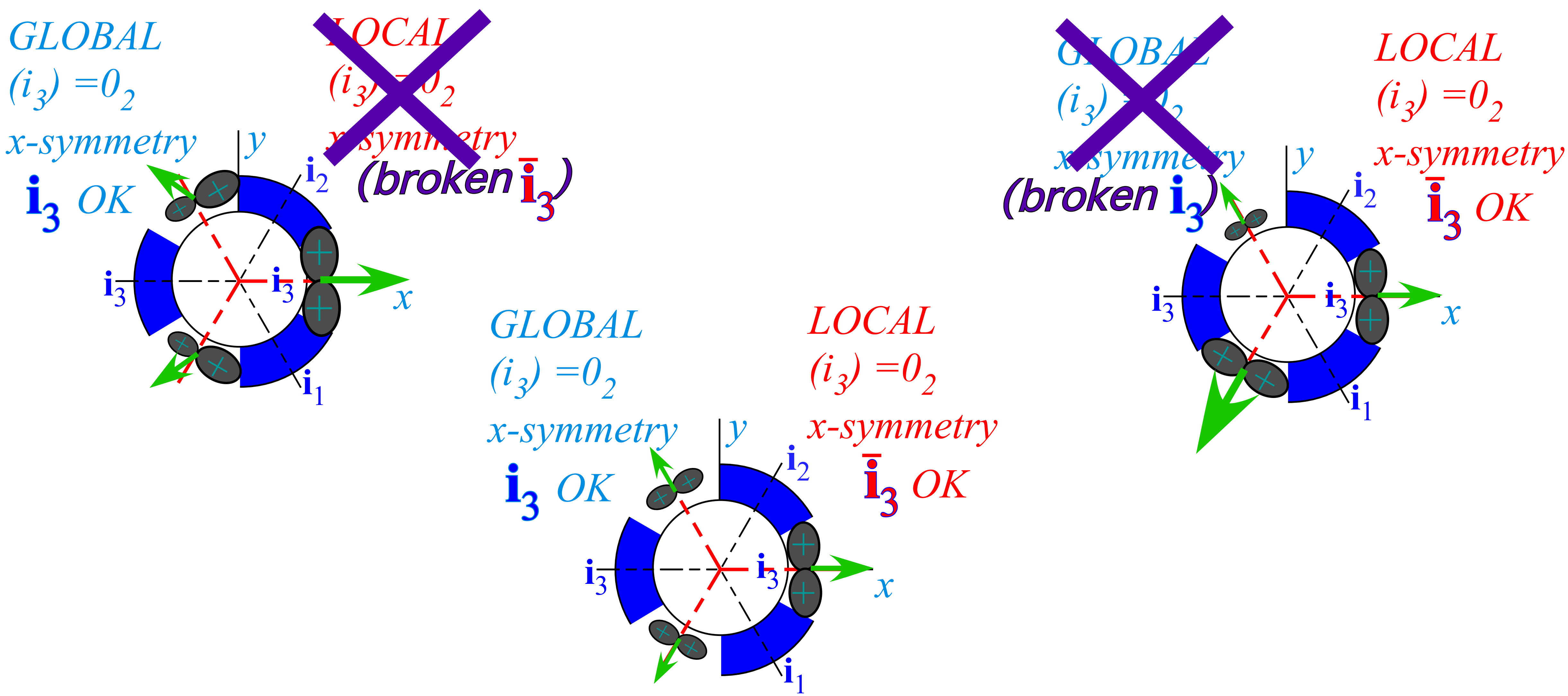
$D^E(1)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ $D^E(\mathbf{r})=\begin{pmatrix}-\frac{1}{2} & -\sqrt{\frac{3}{4}}\\ \sqrt{\frac{3}{4}} & -\frac{1}{2}\end{pmatrix}$ $D^E(\mathbf{r}^2)=\begin{pmatrix}-\frac{1}{2} & \sqrt{\frac{3}{4}}\\ -\sqrt{\frac{3}{4}} & -\frac{1}{2}\end{pmatrix}$ $D^E(\mathbf{i}_1)=\begin{pmatrix}-\frac{1}{2} & -\sqrt{\frac{3}{4}}\\ -\sqrt{\frac{3}{4}} & \frac{1}{2}\end{pmatrix}$ $D^E(\mathbf{i}_2)=\begin{pmatrix}-\frac{1}{2} & \sqrt{\frac{3}{4}}\\ \sqrt{\frac{3}{4}} & \frac{1}{2}\end{pmatrix}$ $D^E(\mathbf{i}_3)=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

$$\left|{}^{(m)}_{eb}\right\rangle = \mathbf{P}^{(m)}_{eb}\left|\mathbf{1}\right\rangle$$

external LAB — *internal BOD*

symmety label-e — *symmety label-b*

GLOBAL — *LOCAL*



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GLOBAL vs LOCAL symmetry of states *...and group **H** parameters $\{r, i_1, i_2, i_3\}$*

$$\mathbf{P}^{(\mu)}_{mn} = \frac{l^{(\mu)}}{{}^\circ G} \Sigma_{\mathbf{g}} D^{(\mu)*}_{mn}(\mathbf{g})\, \mathbf{g}$$

Spectral Efficiency: Same $D(a)_{mn}$ projectors give a lot!

	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	
$\mathbf{P}^{A_1}_{x,x}=$	(1	1	1	1	1	1)	/6
$\mathbf{P}^{A_2}_{y,y}=$	(1	1	1	-1	-1	-1)	/6

	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	
$\mathbf{P}^{E}_{x,x}=$	(2	-1	-1	-1	-1	+2)	/6
$\mathbf{P}^{E}_{y,x}=$	(0	1	-1	-1	+1	0)	/√3/2

	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	
$\mathbf{P}^{E}_{x,y}=$	(0	-1	1	-1	+1	0)	/√3/2
$\mathbf{P}^{E}_{y,y}=$	(2	-1	-1	+1	+1	-2)	/6

•Eigenstates (shown before)

•Complete Hamiltonian

$$\begin{pmatrix} H+r_1+r_2+i_1+i_2+i_3 & & & \\ & H+r_1+r_2-i_1-i_2-i_3 & & \\ & & H-\tfrac{1}{2}r_1-\tfrac{1}{2}r_2-\tfrac{1}{2}i_1-\tfrac{1}{2}i_2+i_3 & \tfrac{\sqrt{3}}{2}(-r_1+r_2-i_1+i_2) \\ & & \tfrac{\sqrt{3}}{2}(+r_1-r_2-i_1+i_2) & H-\tfrac{1}{2}r_1-\tfrac{1}{2}r_2+\tfrac{1}{2}i_1+\tfrac{1}{2}i_2-i_3 \end{pmatrix}$$

A_1-block

A_2-block

•Local symmetery eigenvalue formulae (Local Symmetry => off-diagonal=0)

$r_1 = r_2 = r_1^* = r$, $\quad i_1 = i_2 = i_1^* = i$

gives:

A_1-level: $H + 2r + 2i + i_3$

A_2-level: $H + 2r - 2i - i_3$

E_x-level: $H - r - i + i_3$

E_y-level: $H - r + i - i_3$

Rigorous Global vs Local Calculus begins on p.90 of Lecture 17. Matrix forms on p. 125-129 and p. 130-146.

$\mathbf{P}^{\mu}_{mn}$ g-expansion in Lect.17 p. 35-51

$D_3 > C_2\, \mathbf{i}_3$ projector states

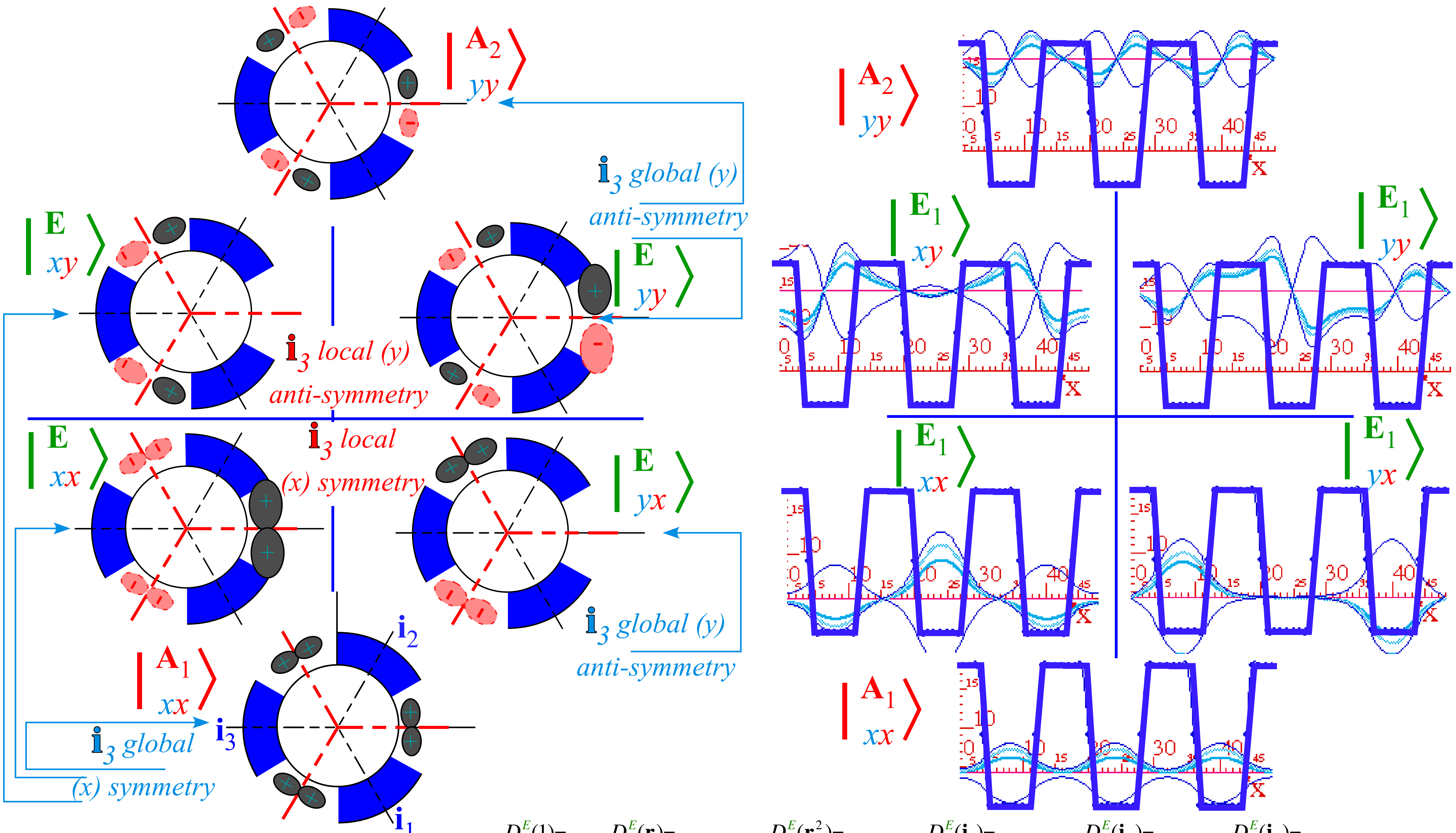
$$\left|{}^{(m)}_{eb}\right\rangle = \mathbf{P}^{(m)}_{eb}\left|1\right\rangle$$

Global (LAB) symmetry

$$\mathbf{i}_3\left|{}^{(m)}_{eb}\right\rangle = \mathbf{i}_3\mathbf{P}^{(m)}_{eb}\left|1\right\rangle = (-1)^e \left|{}^{(m)}\right\rangle$$

Local (BOD) symmetry

$$\bar{\mathbf{i}}_3\left|{}^{(m)}_{eb}\right\rangle = \bar{\mathbf{i}}_3\mathbf{P}^{(m)}_{eb}\left|1\right\rangle = \mathbf{P}^{(m)}_{eb}\bar{\mathbf{i}}_3\left|1\right\rangle = \mathbf{P}^{(m)}_{eb}\mathbf{i}_3^{\dagger}\left|1\right\rangle = (-1)^b \left|{}^{(m)}\right\rangle$$

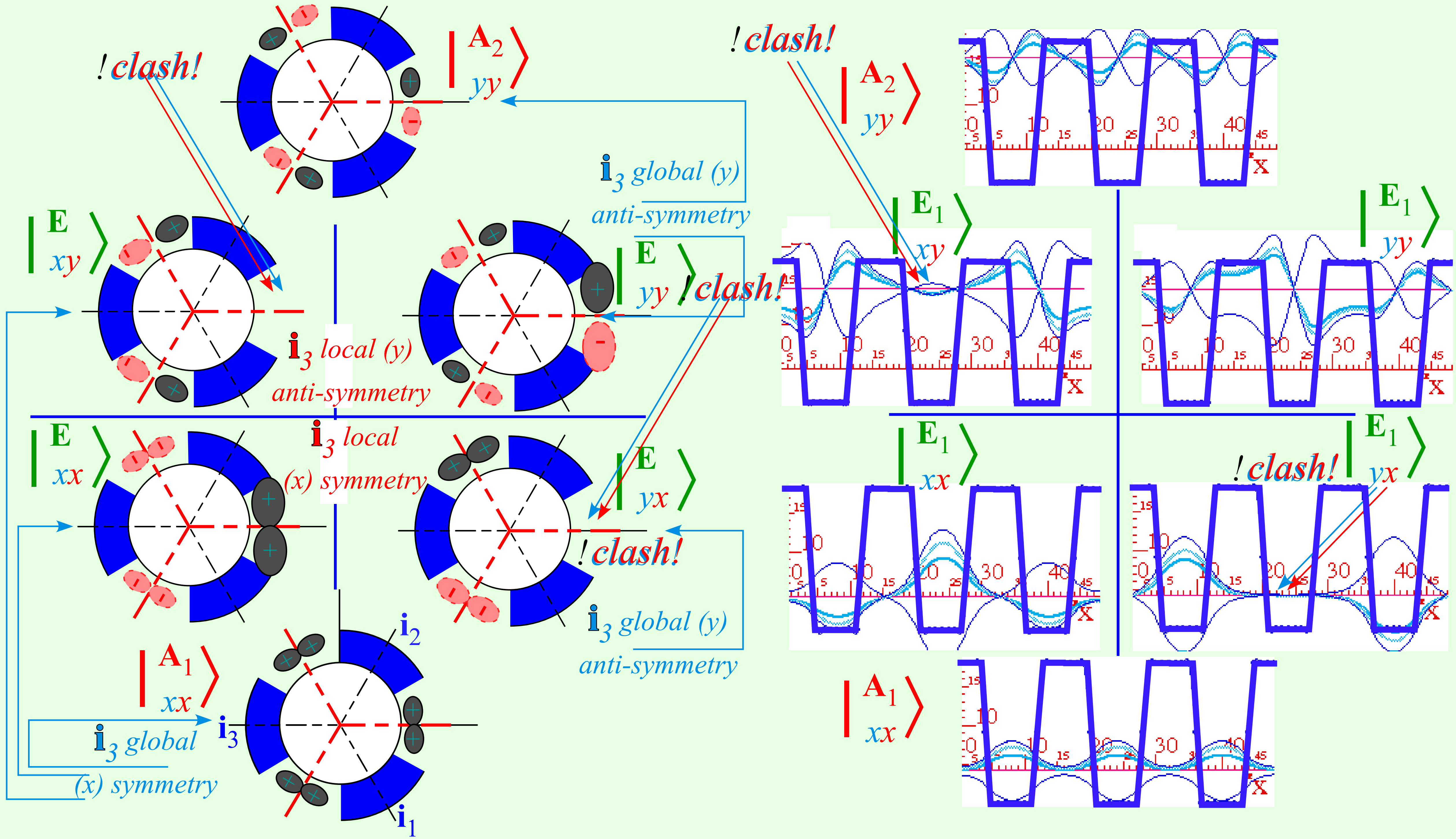


$D^{A_1}(\mathbf{g}) = +1,\; D^{A_2}(\mathbf{r}^p) = +1,\; D^{A_2}(\mathbf{i}_q) = -1$

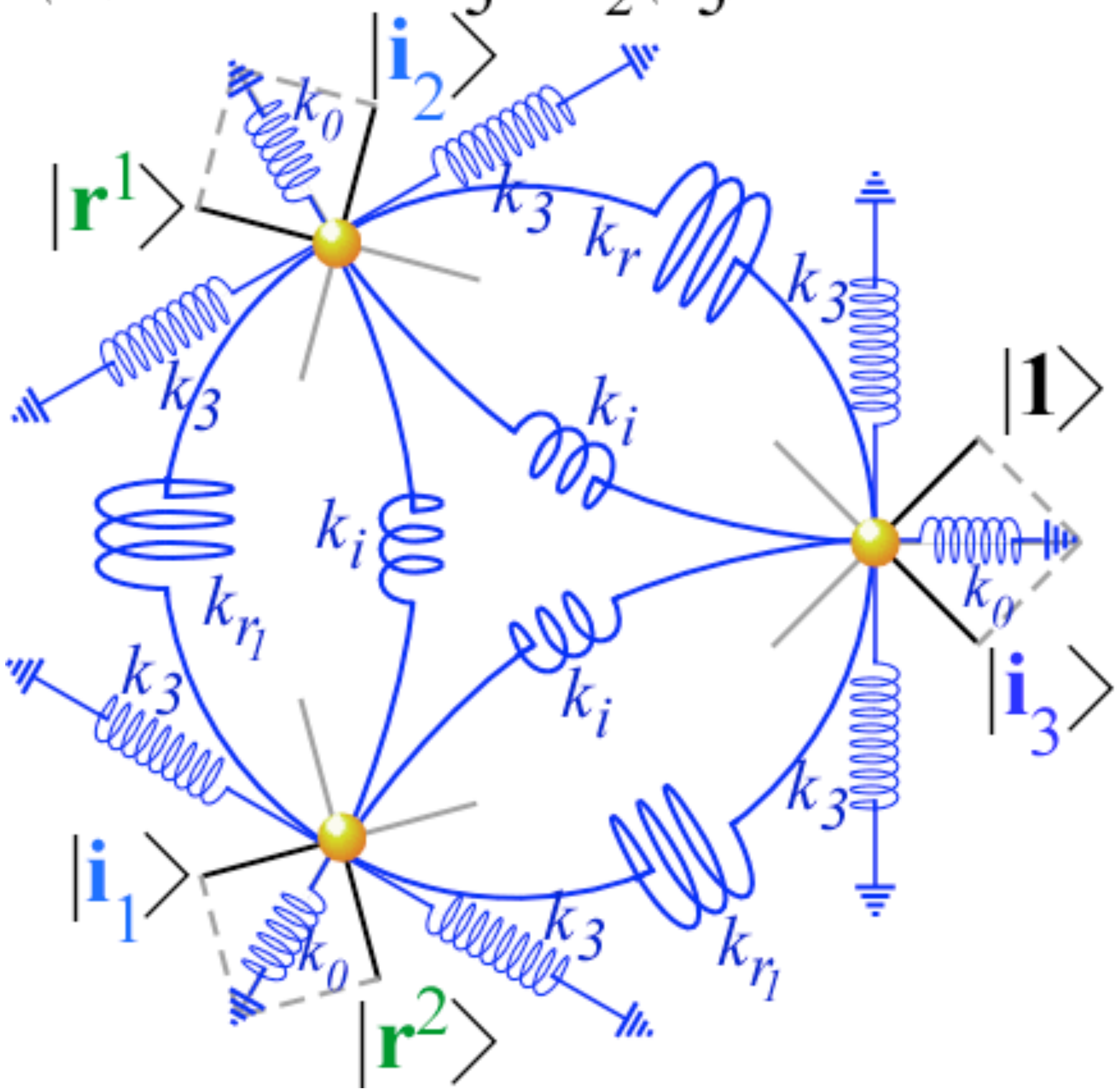
$D^E(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $D^E(\mathbf{r}) = \begin{pmatrix} -\frac{1}{2} & -\sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{4}} & -\frac{1}{2} \end{pmatrix}$, $D^E(\mathbf{r}^2) = \begin{pmatrix} -\frac{1}{2} & \sqrt{\frac{3}{4}} \\ -\sqrt{\frac{3}{4}} & -\frac{1}{2} \end{pmatrix}$, $D^E(\mathbf{i}_1) = \begin{pmatrix} -\frac{1}{2} & -\sqrt{\frac{3}{4}} \\ -\sqrt{\frac{3}{4}} & \frac{1}{2} \end{pmatrix}$, $D^E(\mathbf{i}_2) = \begin{pmatrix} -\frac{1}{2} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{4}} & \frac{1}{2} \end{pmatrix}$, $D^E(\mathbf{i}_3) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

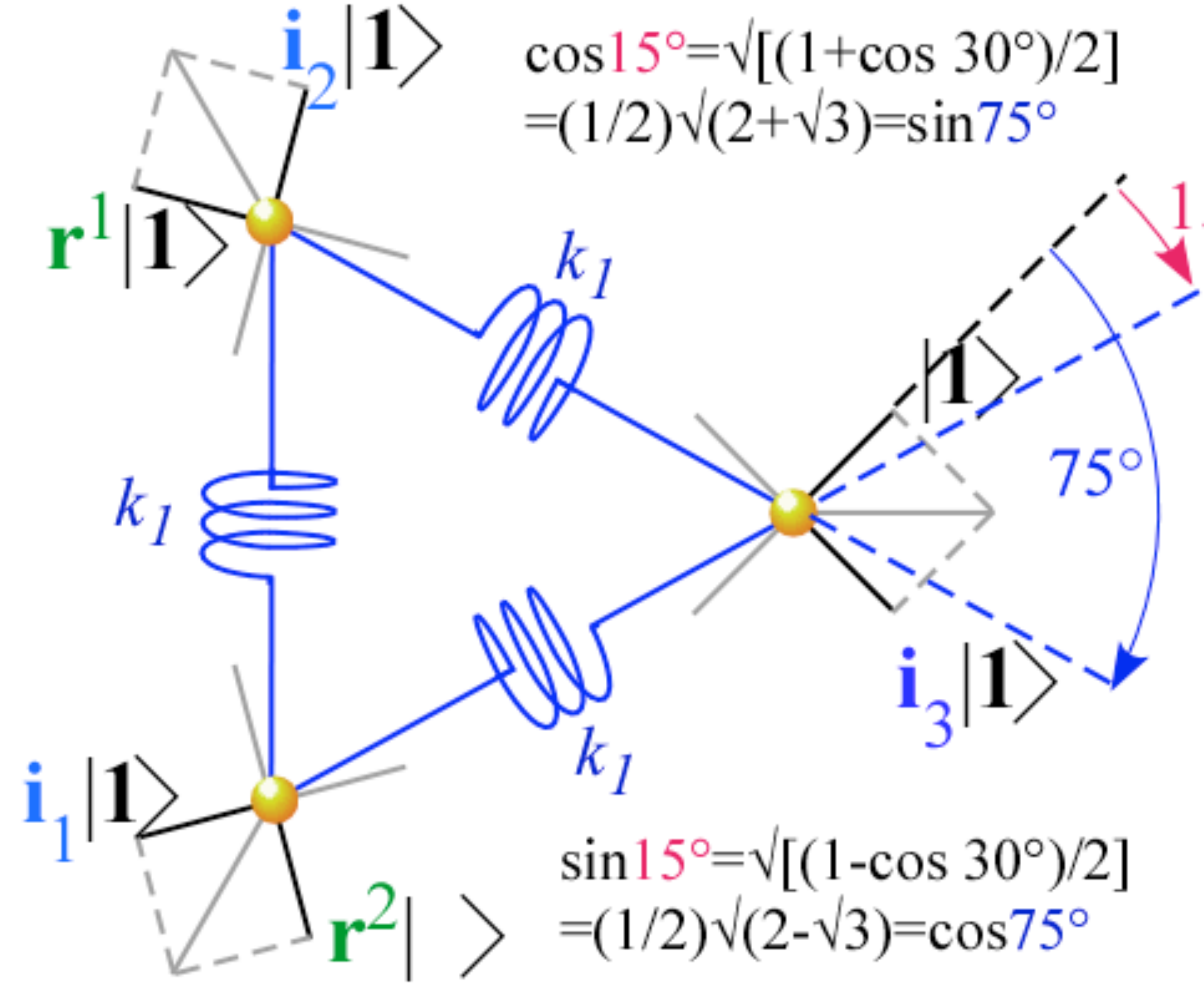
When there is no there, there...

Nobody Home
where LOCAL
and GLOBAL



(a) Local $D_3 \supset C_2(i_3)$ model   (b) Mixed local symmetry D_3 model





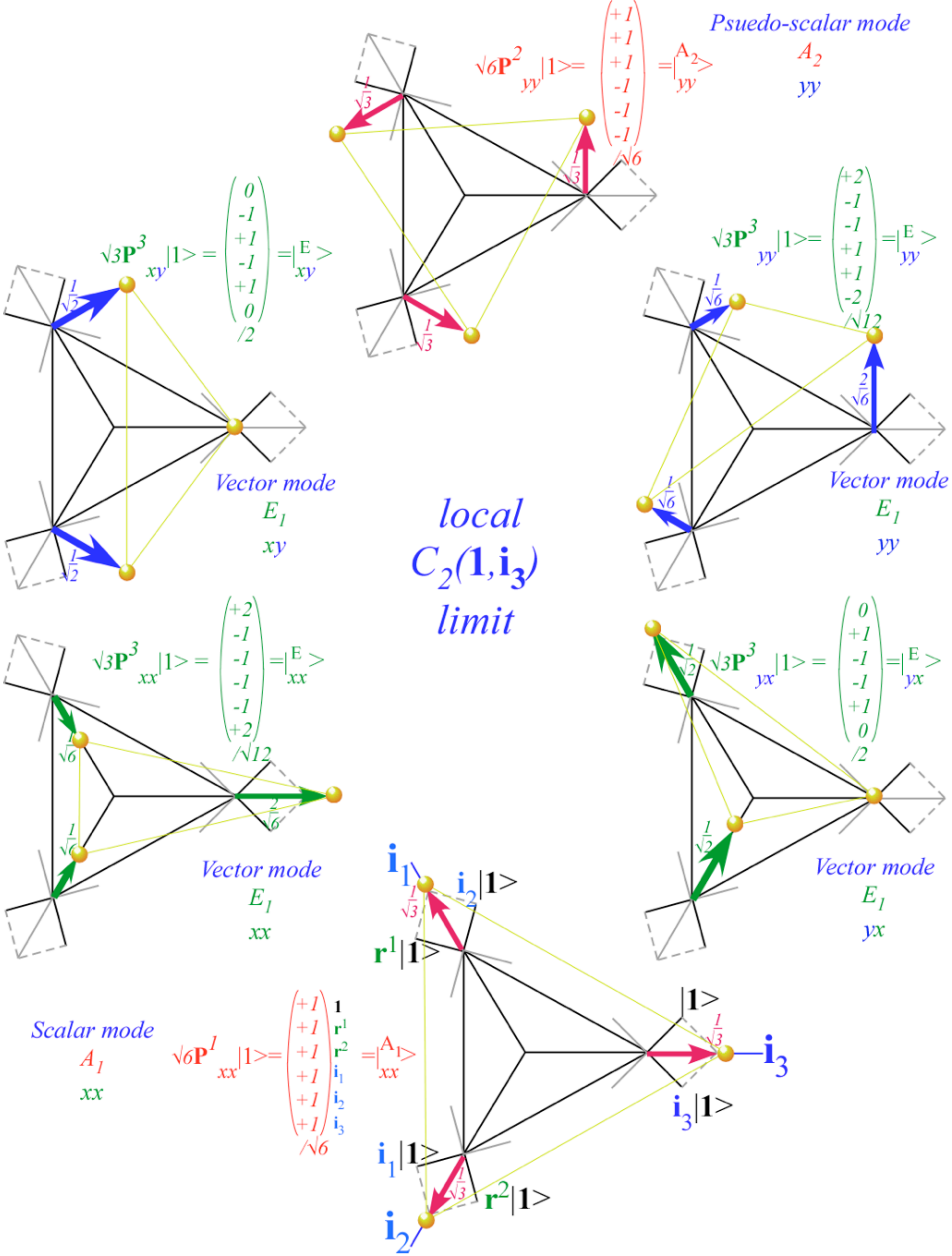
$\cos 15° = \sqrt{[(1+\cos 30°)/2]}$
$= (1/2)\sqrt{(2+\sqrt{3})} = \sin 75°$

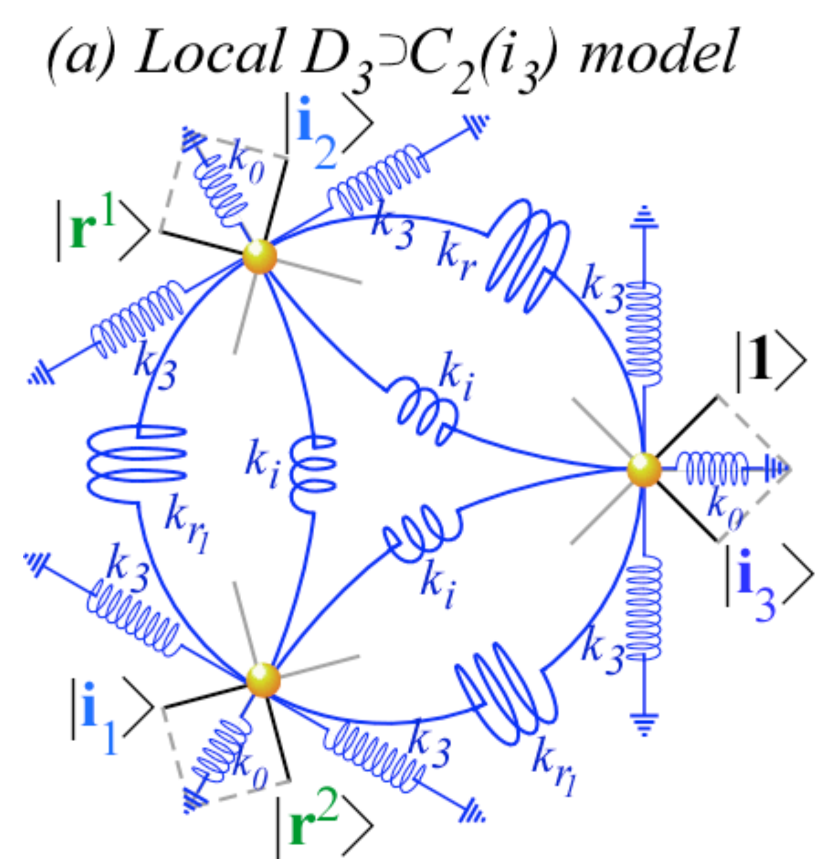
$\sin 15° = \sqrt{[(1-\cos 30°)/2]}$
$= (1/2)\sqrt{(2-\sqrt{3})} = \cos 75°$

See p. 10-41 of Lecture 18

MolVibes Web Application: https://modphys.hosted.uark.edu/markup/MolVibesWeb.html

MolVibes Web Simulation
3 Atom with C3v symmetry





See p. 10-41 of Lecture 18

MolVibes Web Simulation
3 Atom with C3v symmetry