Lecture 25.

Relativity of lightwaves and Lorentz-Minkowski coordinates IV.

(Ch. 0-3 of Unit 2  4.02.12)

5. That “old-time” relativity (Circa 600BCE- 1905CE)  (Includes Lecture 24 review)
   (“Bouncing-photons” in smoke & mirrors and Thales, again)
   The Ship and Lighthouse saga
      Light-conic-sections make invariants
   A politically incorrect analogy of rotational transformation and Lorentz transformation
      The straight scoop on “angle” and “rapidity” (They’re area!)
      Galilean velocity addition becomes rapidity addition
   Introducing the “Sin-Tan Rosetta Stone” (Thanks, Thales!)
      Introducing the stellar aberration angle $\sigma$ vs. rapidity $\rho$
   How Minkowski’s space-time graphs help visualize relativity
      Group vs. phase velocity and tangent contacts

Lecture 24 ended (about) here

Lecture 25 ended here
5. That “old-time” relativity (Circa 600BCE- 1905CE)

(“Bouncing-photons” in smoke & mirrors and Thales, again)

The Ship and Lighthouse saga

Light-conic-sections make invariants

A politically incorrect analogy of rotational transformation and Lorentz transformation

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Introducing the stellar aberration angle $\sigma$ vs. rapidity $\rho$

How Minkowski’s space-time graphs help visualize relativity

Group vs. phase velocity and tangent contacts
### Happening 0:
Ship passes Main Lighthouse.

<table>
<thead>
<tr>
<th>(Lighthouse space)</th>
<th>x = 0</th>
<th>(Lighthouse time)</th>
<th>t = 0</th>
<th>(Ship space)</th>
<th>x' = 0</th>
<th>(Ship time)</th>
<th>t' = 0</th>
</tr>
</thead>
</table>

### Happening 1:
Ship gets hit by first blink from Main Lighthouse.

<table>
<thead>
<tr>
<th>(Lighthouse space)</th>
<th>x = -1.00 c</th>
<th>(Lighthouse time)</th>
<th>t = 2.00</th>
<th>(Ship space)</th>
<th>x' = 0</th>
<th>(Ship time)</th>
<th>t' = 1.75</th>
</tr>
</thead>
</table>

### Happening 2:
Main Lighthouse blinks second time.

<table>
<thead>
<tr>
<th>(Lighthouse space)</th>
<th>x = 0</th>
<th>(Lighthouse time)</th>
<th>t = 2.00</th>
<th>(Ship space)</th>
<th>x' = c Δ</th>
<th>(Ship time)</th>
<th>t' = 2Δ = 2.30</th>
</tr>
</thead>
</table>

Fig. 2.A.3 Happening 1 (1st blink hits ship) and 2 (2nd blink at Main) both happen at t=2.
The ship and lighthouse saga

Happening 0.5: Main Lite blinks first time.

<table>
<thead>
<tr>
<th>Lighthouse</th>
<th>Ship</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = 0$</td>
<td>$x' = 0$</td>
</tr>
<tr>
<td>$t = 1.00$</td>
<td>$t' = \Delta = ???$</td>
</tr>
</tbody>
</table>

Ship v/c (rel. to lighthouse) = -0.50

Comparing Ship and Lighthouse views: Happening tables

<table>
<thead>
<tr>
<th>Happening 0: Ship passes Main Lighthouse.</th>
<th>Happening 1: Ship gets hit by first blink from Main Lighthouse.</th>
<th>Happening 2: Main Lighthouse blinks second time.</th>
</tr>
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<td>(Lighthouse space) $x = 0$</td>
<td>$x = -1.00c$</td>
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</tr>
<tr>
<td>(Lighthouse time) $t = 0$</td>
<td>$t = 2.00$</td>
<td>$t = 2.00$</td>
</tr>
<tr>
<td>(Ship space) $x' = 0$</td>
<td>$x' = 0$</td>
<td>$x' = c\Delta$</td>
</tr>
<tr>
<td>(Ship time) $t' = 0$</td>
<td>$t' = 1.75$</td>
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</tbody>
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Fig. 2.A.3 Happening 1 (1st blink hits ship) and 2 (2nd blink at Main) both happen at $t = 2$.  

Monday, April 2, 2012
The ship and lighthouse saga

Happening 0.5:
Main Lite blinks first time.

| Lighthouse: | x = 0 | Lighthouse: | t = 1.00 |
| Ship:       | x' = 0 | Ship:       | t' = Δ = ???

Lighthouse t = 1.00
Ship v/c (rel. to lighthouse) = -0.50

Ship Time t' = Δ = ???

Lighthouse t = 1.00
North Lighthouse

Comparing Ship and Lighthouse views: Happening tables

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Fig. 2.A.3 Happening 1 (1st blink hits ship) and 2 (2nd blink at Main) both happen at t = 2.
The ship and lighthouse saga

Happening 0.5:
Main Lite
blinks first time.

| Lighthouse: | x = 0 |
| Lighthouse: | t = 1.00 |

| Ship: | x' = 0 |
| Ship: | t' = Δ = ?? |

\[ c^2 \Delta^2 = c^2 + v^2 \Delta^2 \]
\[ (c^2 - v^2) \Delta^2 = c^2 \]

Ship Time \( t' = \Delta = ??? \)

Lighthouse t= 1.00
Ship v/c (rel. to lighthouse) = -0.50

Ship: \( x' = 0 \)
Ship: \( t' = \Delta = ??? \)

Comparing Ship and Lighthouse views: Happening tables

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<tr>
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</tr>
<tr>
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Fig. 2.A.3 Happening 1 (1st blink hits ship) and 2 (2nd blink at Main) both happen at t=2.
The ship and lighthouse saga

Happening 0.5:
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Ship Time \( t' = \Delta = \frac{1}{\sqrt{1 - v^2/c^2}} = \cosh \rho \)

\[
c^2 \Delta^2 = c^2 + v^2 \Delta^2
\]

\[
(c^2 - v^2) \Delta^2 = c^2
\]

\[
\Delta^2 = \frac{c^2}{(c^2 - v^2)} = \frac{1}{1 - \frac{v^2}{c^2}}
\]

Lighthouse: \( x = 0 \)

Ship: \( x' = 0 \)

Ship: \( t' = \Delta = ??? \)

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<td>x = 0, t = 2.00</td>
</tr>
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</tr>
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Fig. 2.A.3 Happening 1 (1st blink hits ship) and 2 (2nd blink at Main) both happen at \( t = 2 \).
**The ship and lighthouse saga**

Happening 0.5: Main Lite blinks first time.

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<td>Ship:</td>
<td>$t' = \Delta = 1.15$</td>
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**Ship Time**

$t' = \Delta = 1/\sqrt{1-v^2/c^2} = \cosh \rho = 1.15$

$c^2 \Delta^2 = c^2 + v^2 \Delta^2$

$(c^2 - v^2) \Delta^2 = c^2$

$\Delta^2 = \frac{c^2}{(c^2 - v^2)} = \frac{1}{1 - v^2/c^2}$

For $u/c = 1/2$

$\Delta = 1/\sqrt{1-1/4} = 2/\sqrt{3} = 1.15..$

**Comparing Ship and Lighthouse views: Happening tables**

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Fig. 2.A.3 Happening 1 (1st blink hits ship) and 2 (2nd blink at Main) both happen at $t=2$. 
Ship v/c (relative to Earth) = -0.50

Happening 0:
Ship passes Main Lighthouse.

Happening 1:
Ship gets hit by first blink from Main Lighthouse.

Happening 2:
Main Lighthouse blinks second time.

\[
\Delta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \cosh \rho = 1.15
\]

For \( u/c = 1/2 \)
\[
\Delta = \frac{1}{\sqrt{1 - 1/4}} = \frac{2}{\sqrt{3}} = 1.15\ldots
\]

Fig. 2.A.3 Happening 1 (1st blink hits ship) and 2 (2nd blink at Main) both happen at \( t = 2 \).
Ship v/c (rel. to lthse.) = -0.50

Lighthouse t = 2.00

Ship v/c = 1.0

Comparing Ship and Lighthouse views:

Happening tables

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<td>(Lighthouse space) ( x = 0 ) ( t = 0 )</td>
<td>( x = -vc/(c-v) ) ( t = c/(c-v) )</td>
<td>( x = 0 )( t = 2.00 )</td>
</tr>
<tr>
<td>(Ship space) ( x' = 0 ) ( t' = 0 )</td>
<td>( x' = 0 ) ( t' = (v+c)\Delta/c )</td>
<td>( x' = 2v\Delta ) ( t' = 2\Delta )</td>
</tr>
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Fig. 2.A.3 Happening 1 (1st blink hits ship) and 2 (2nd blink at Main) both happen at t = 2.

Lecture 24 ended here
5. That “old-time” relativity (Circa 600BCE- 1905CE)

(“Bouncing-photons” in smoke & mirrors and Thales, again)

The Ship and Lighthouse saga

Light-conic-sections make invariants

A politically incorrect analogy of rotational transformation and Lorentz transformation

The straight scoop on “angle” and “rapidity” (They’re area!)

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Introducing the stellar aberration angle $\sigma$ vs. rapidity $\rho$

How Minkowski’s space-time graphs help visualize relativity

Group vs. phase velocity and tangent contacts
Light-conic-sections make invariants

**Fig. 2.B.5** Space-Space-Time plot of world lines for Lighthouses. North Lighthouse blink waves trace light cones.
5. That “old-time” relativity (Circa 600BCE- 1905CE)

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Fig. 2.B.1 Town map according to a "tipsy" surveyor.

<table>
<thead>
<tr>
<th></th>
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</tr>
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<tbody>
<tr>
<td>(US surveyor)</td>
<td>U.S. surveyor</td>
<td>(U.S. surveyor)</td>
</tr>
<tr>
<td>x = 0</td>
<td>x = 0.5</td>
<td>x = 0</td>
</tr>
<tr>
<td>y = 0</td>
<td>y = 1.0</td>
<td>y = 1.0</td>
</tr>
<tr>
<td>(French surveyor)</td>
<td>French surveyor</td>
<td>French surveyor</td>
</tr>
<tr>
<td>x' = 0</td>
<td>x' = 0</td>
<td>x' = -0.45</td>
</tr>
<tr>
<td>y' = 0</td>
<td>y' = 1.1</td>
<td>y' = 0.89</td>
</tr>
</tbody>
</table>
A politically incorrect analogy of rotational transformation and Lorentz transformation

Fig. 2.B.1 Town map according to a "tipsy" surveyor.  
Fig. 2.B.2 Diagram and formulas for reconciliation of the two surveyor's data.

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<td></td>
</tr>
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<td>(x = 0)</td>
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<td>(x = 0)</td>
</tr>
<tr>
<td>(y = 0)</td>
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\[
\begin{align*}
\cos \theta &= \frac{1}{\sqrt{1 + \frac{b^2}{c^2}}} \\
\sin \theta &= \frac{b/c}{\sqrt{1 + \frac{b^2}{c^2}}} \\
\end{align*}
\]

\[
x' = x \cos \theta - y \sin \theta = \frac{x}{\sqrt{1 + \frac{b^2}{c^2}}} + \frac{(b/c)y}{\sqrt{1 + \frac{b^2}{c^2}}} \\
y' = x \sin \theta + y \cos \theta = \frac{(b/c)x}{\sqrt{1 + \frac{b^2}{c^2}}} + \frac{y}{\sqrt{1 + \frac{b^2}{c^2}}}
\]
A politically incorrect analogy of rotational transformation and Lorentz transformation

Fig. 2.B.1 Town map according to a "tipsy" surveyor. Fig. 2.B.2 Diagram and formulas for reconciliation of the two surveyor's data.

Reminder: Component-based derivation is clumsy!

\[
x = x' \cos \theta + y' \sin \theta
\]
\[
y = -x' \sin \theta + y' \cos \theta
\]

\[
\cos \theta = \frac{1}{\sqrt{1 + \frac{b^2}{c^2}}}
\]
\[
\sin \theta = \frac{b / c}{\sqrt{1 + \frac{b^2}{c^2}}}
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A politically incorrect analogy of rotational transformation and Lorentz transformation

Fig. 2.B.1 Town map according to a "tipsy" surveyor.

Fig. 2.B.2 Diagram and formulas for reconciliation of the two surveyor's data.

Object 0: Town Square.
Object 1: Saloon.
Object 2: Gun Shoppe.

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</tr>
<tr>
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</tr>
<tr>
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\[
\cos \theta = \frac{1}{\sqrt{1 + \frac{b^2}{c^2}}} \\
\sin \theta = \frac{b / c}{\sqrt{1 + \frac{b^2}{c^2}}}
\]

\[
x = x' \cos \theta + y' \sin \theta \\
y = -x' \sin \theta + y' \cos \theta
\]

Forget this!! It's too clumsy to generalize to 3D, 4D,...

Instead, use Dirac unit vectors |x⟩, |y⟩ and |x'⟩, |y'⟩

Reminder: Component-based derivation is clumsy!
A politically incorrect analogy of rotational transformation and Lorentz transformation

Fig. 2.B.1 Town map according to a "tipsy" surveyor.

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### Reminder: Component-based derivation is clumsy!

### Instead, use Dirac unit vectors $|x\rangle, |y\rangle$ and $|x'\rangle, |y'\rangle$

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You may apply (Jacobian) transform matrix:

\[
\begin{pmatrix}
|x|\langle x'| \langle x|\langle y' \rangle
\end{pmatrix} =
\begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}
\]

or the inverse (Kajobian) transformation:

\[
\begin{pmatrix}
|x'|\langle x| \langle x'|y\rangle
\end{pmatrix} =
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\]

to any vector $V = |V\rangle = |x\rangle\langle x|V\rangle + |y\rangle\langle y|V\rangle = |x'\rangle\langle x'|V\rangle + |y'\rangle\langle y'|V\rangle$
A politically incorrect analogy of rotational transformation and Lorentz transformation

Fig. 2.B.1 Town map according to a "tipsy" surveyor.

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Instead, use Dirac unit vectors \(|x\rangle, |y\rangle\) and \(|x'\rangle, |y'\rangle\)

You may apply (Jacobian) transform matrix:

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\begin{pmatrix}
|x| & |x'| \\
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or the inverse (Kajobian) transformation:

\[
\begin{pmatrix}
|x'| & |x| \\
|y'| & |y|
\end{pmatrix} =
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\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\]

to any vector \(|V\rangle = |x\rangle |x\rangle + |y\rangle |y\rangle\)

\[
= |x\rangle |x\rangle |V\rangle + |y\rangle |y\rangle |V\rangle
\]

Monday, April 2, 2012
**A politically incorrect analogy of rotational transformation and Lorentz transformation**

**Fig. 2.B.1 Town map according to a "tipsy" surveyor.**

\[ x = x' \cos \theta + y' \sin \theta \]
\[ y = -x' \sin \theta + y' \cos \theta \]

**Object 0:** Town Square.

- (US surveyor) \( x = 0 \)
- (2nd surveyor) \( x' = 0 \)

**Object 1:** Saloon.

- (US surveyor) \( y = 0 \)
- (2nd surveyor) \( y' = 0 \)

**Object 2:** Gun Shoppe.

- (US surveyor) \( x = 0 \)
- (2nd surveyor) \( y' = 0 \)

(Jacobian) transformation \( \{V_x, V_y\} \) from \( \{V_x', V_y'\} \):

\[
V_x = \langle x | V \rangle = \langle x | x \rangle \langle x | V \rangle + \langle x | y \rangle \langle y | V \rangle
\]
\[
V_y = \langle y | V \rangle = \langle y | x \rangle \langle x | V \rangle + \langle y | y \rangle \langle y | V \rangle
\]

You may apply (Jacobian) transform matrix:

\[
\begin{pmatrix}
\langle x | x' \rangle & \langle x | y' \rangle \\
\langle y | x' \rangle & \langle y | y' \rangle
\end{pmatrix} =
\begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}
\]

or the inverse (Kajobian) transformation:

\[
\begin{pmatrix}
\langle x' | x \rangle & \langle x' | y \rangle \\
\langle y' | x \rangle & \langle y' | y \rangle
\end{pmatrix} =
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\]

to any vector \( V = |V\rangle = |x\rangle \langle x | V \rangle + |y\rangle \langle y | V \rangle \)

\( = \langle x' | x' \rangle |x\rangle + \langle y' | y' \rangle |y\rangle \)

**Fig. 2.B.2 Diagram and formulas for reconciliation of the two surveyor's data.**

Instead, use Dirac unit vectors \( |x\rangle, |y\rangle \) and \( |x'\rangle, |y'\rangle \)

**Reminder:** Component-based derivation is clumsy!

Forget this!! It's too clumsy to generalize to 3D, 4D,...

\[ \theta = \frac{1}{\sqrt{1 + \frac{b^2}{c^2}}} \]
\[ \sin \theta = \frac{b}{c \sqrt{1 + \frac{b^2}{c^2}}} \]

\[ e_x = |x\rangle = \cos \theta |x\rangle + \sin \theta |y\rangle \]
\[ e_y = |y\rangle = -\sin \theta |x\rangle + \cos \theta |y\rangle \]
PLEASE!

Do NOT ever write this:

\[ e_x = |x'\rangle = \cos \theta |x\rangle - \sin \theta |y\rangle \]

\[ e_y = |y'\rangle = \sin \theta |x\rangle + \cos \theta |y\rangle \]

like this:

\[
\begin{pmatrix}
    e_x \\
    e_y
\end{pmatrix} =
\begin{pmatrix}
    |x'\rangle \\
    |y'\rangle
\end{pmatrix} =
\begin{pmatrix}
    \cos \theta & -\sin \theta \\
    \sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
    |x\rangle \\
    |y\rangle
\end{pmatrix}
\]
**PLEASE!**

**Do NOT ever write**

this:

\[
e_{x'} = \ket{x'} = \cos \theta \ket{x} - \sin \theta \ket{y}
\]

\[
e_{y'} = \ket{y'} = \sin \theta \ket{x} + \cos \theta \ket{y}
\]

(This is an abstract definition.)

**like this:**

\[
\begin{pmatrix}
e_{x'} \\
e_{y'}
\end{pmatrix} =
\begin{pmatrix}
\ket{x'} \\
\ket{y'}
\end{pmatrix} =
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
\ket{x} \\
\ket{y}
\end{pmatrix}
\]

This is GARBAGE!
PLEASE!

Do NOT ever write this:

\[ e_x' = \left| x' \right> = \cos \theta \left| x \right> - \sin \theta \left| y \right> \equiv R \left| x \right> \]
\[ e_y' = \left| y' \right> = \sin \theta \left| x \right> + \cos \theta \left| y \right> \equiv R \left| y \right> \]

(This is an abstract definition.)

like this:

\[ \begin{pmatrix} e_x' \\ e_y' \end{pmatrix} = \begin{pmatrix} \left| x' \right> \\ \left| y' \right> \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \left| x \right> \\ \left| y \right> \end{pmatrix} \]

(This is GARBAGE!)

Here is a matrix representation of abstract definitions: 
\[ \left| x' \right> \equiv R \left| x \right>, \left| y' \right> \equiv R \left| y \right> \]
(a) Rotation Transformation and Invariants

\[ x' = x \cos \theta - y \sin \theta = \frac{x}{\sqrt{1 + \frac{b^2}{c^2}}} + \frac{-\left(\frac{b}{c}\right)y}{\sqrt{1 + \frac{b^2}{c^2}}} \]

\[ y' = x \sin \theta + y \cos \theta = \frac{\left(\frac{b}{c}\right)x}{\sqrt{1 + \frac{b^2}{c^2}}} + \frac{y}{\sqrt{1 + \frac{b^2}{c^2}}} \]

(b) Lorentz Transformation and Invariants

\[ x' = \frac{x}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{\frac{v}{c} t}{\sqrt{1 - \frac{v^2}{c^2}}} = x \cosh \rho + y \sinh \rho \]

\[ c t' = \frac{\frac{v}{c} x}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{c t}{\sqrt{1 - \frac{v^2}{c^2}}} = x \sinh \rho + y \cosh \rho \]
5. That “old-time” relativity (Circa 600BCE- 1905CE)

(“Bouncing-photons” in smoke & mirrors and Thales, again)

The Ship and Lighthouse saga
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The straight scoop on “angle” and “rapidity” (They’re area!)
Galilean velocity addition becomes rapidity addition
Introducing the “Sin-Tan Rosetta Stone” (Thanks, Thales!)
Introducing the stellar aberration angle \( \sigma \) vs. rapidity \( \rho \)

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\[ y/x = \tanh \theta = \frac{v}{c} \]

\[ y = \sinh \rho \]
\[ x = \cosh \rho \]

Area = \left( \frac{1}{2} \right) \text{base} \cdot \text{altitude} - \text{area under curve} = \frac{1}{2} \times y - \int y \, dx

The “Area” being calculated is the total Gray Area between hyperbola pairs, X axis, and sloping u-line.

Monday, April 2, 2012
The straight scoop on “angle” and “rapidity” (They’re area!)

The “Area” being calculated is the total Gray Area between hyperbola pairs, X axis, and sloping u-line

Useful hyperbolic identities

\[
\sinh^2 \rho = \frac{\left( \frac{e^\rho - e^{-\rho}}{2} \right)^2}{2} = \frac{1}{4} \left( e^{2\rho} - e^{-2\rho} - 2 \right) = \frac{\cosh 2\rho - 1}{2}
\]

\[
\sinh \rho \cosh \rho = \left( \frac{e^\rho - e^{-\rho}}{2} \right) \left( \frac{e^\rho + e^{-\rho}}{2} \right) = \frac{1}{4} \left( e^{2\rho} - e^{-2\rho} \right) = \frac{1}{2} \sinh 2\rho
\]
The straight scoop on “angle” and “rapidity” (They’re area!)

\[ y = \tanh \theta = \frac{v}{c} \]

\[ x = \cosh \theta \]

\[ y = \sinh \rho \]

\[ x = \cosh \rho \]

The “Area” being calculated is the total Gray Area between hyperbola pairs, X axis, and sloping u-line

Useful hyperbolic identities

\[ \sinh^2 \rho = \left( \frac{e^\rho - e^{-\rho}}{2} \right)^2 = \frac{1}{4} (e^{2\rho} + e^{-2\rho} - 2) = \frac{\cosh 2 \rho - 1}{2} \]

\[ \sinh \theta \cosh \theta = \left( \frac{e^\theta - e^{-\theta}}{2} \right) \left( \frac{e^\theta + e^{-\theta}}{2} \right) = \frac{1}{4} (e^{2\theta} - e^{-2\theta}) = \frac{1}{2} \sinh 2\theta \]

\[ \int \cosh a\rho \, d\rho = \frac{1}{a} \sinh a\rho \]
The straight scoop on “angle” and “rapidity” (They’re area!)

\[
x = \cosh \theta \\
y = \sinh \theta \\
y/x = \tanh \theta = v/c
\]

\[\text{Area} = \frac{1}{2} \text{base \ altitude} - \text{area under curve} = \frac{1}{2} xy - \int y \, dx\]

\[\frac{\text{Area}}{2} = \frac{1}{2} \sinh \rho \cosh \rho - \int \sinh \rho \, d(\cosh \rho)\]

\[\int \cosh a\theta \, d\theta = \frac{1}{a} \sinh a\theta\]

\[\sinh^2 \rho = \left(\frac{e^\rho - e^{-\rho}}{2}\right)^2 = \frac{1}{4} (e^{2\rho} + e^{-2\rho} - 2) = \frac{\cosh 2\rho - 1}{2}\]

\[\sinh \rho \cosh \rho = \left(\frac{e^\rho - e^{-\rho}}{2}\right) \left(\frac{e^\rho + e^{-\rho}}{2}\right) = \frac{1}{4} (e^{2\rho} - e^{-2\rho}) = \frac{1}{2} \sinh 2\rho\]

\[\text{Amazing result: Area} = \rho \text{ is rapidity}\]
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Group vs. phase velocity and tangent contacts
**Galilean velocity addition becomes rapidity addition**

From Lect. 22 p. 27 or eq. (3.6) in Ch. 3 of Unit 2:

Evenson axiom requires geometric Doppler transform: \[ e^{\rho_{AB}} \cdot e^{\rho_{BC}} = e^{\rho_{AC}} = e^{\rho_{AB} + \rho_{BC}} \]

Easy to combine frame velocities using rapidity addition: \[ \rho_{u+v} = \rho_u + \rho_v \]

\[
\begin{align*}
\rho_{AB} &= \ln(2) = 0.69 \\
\rho_{BC} &= \ln(1/4) = -1.38 \\
\rho_{AC} &= \ln(2) = 0.69
\end{align*}
\]

\[ \rho_{AB} + \rho_{BC} = \rho_{AC} = -\rho_{CA} \]

\[ 0.69 - 1.38 = -0.69 \]
Galilean velocity addition becomes rapidity addition

From Lect. 22 p. 27 or eq. (3.6) in Ch. 3 of Unit 2:

Evenson axiom requires geometric Doppler transform: 
\[ e^{\rho_{AB}} \cdot e^{\rho_{BC}} = e^{\rho_{AC}} = e^{\rho_{AB} + \rho_{BC}} \]

Easy to combine frame velocities using rapidity addition: 
\[ \rho_{u+v} = \rho_u + \rho_v \]

\[
\frac{u'}{c} = \text{tanh}(\rho_u + \rho_v) = \frac{\tanh \rho_u + \tanh \rho_v}{1 + \tanh \rho_u \tanh \rho_v} = \frac{\frac{u}{c} + \frac{v}{c}}{1 + \frac{u}{c} \cdot \frac{v}{c}}
\]

or: 
\[
\frac{u'}{1 + \frac{u \cdot v}{c^2}}
\]

\[ \text{tanh}(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y} \]
Galilean velocity addition becomes rapidity addition

From Lect. 22 p. 27 or eq. (3.6) in Ch. 3 of Unit 2:

Evenson axiom requires geometric Doppler transform: \( e^{\rho_{AB}} \cdot e^{\rho_{BC}} = e^{\rho_{AC}} = e^{\rho_{AB} + \rho_{BC}} \)

Easy to combine frame velocities using rapidity addition:

\[
\frac{u'}{c} = \tanh(\rho_u + \rho_v) = \frac{\tanh \rho_u + \tanh \rho_v}{1 + \tanh \rho_u \tanh \rho_v} = \frac{\frac{u}{c} + \frac{v}{c}}{1 + \frac{u}{c} \cdot \frac{v}{c}}
\]

or:

\[
\frac{u'}{c} = \frac{u + v}{1 + \frac{u \cdot v}{c^2}}
\]

No longer does \((1/2 + 1/2)c\) equal \((1)c\)…

Relativistic result is:

\[
\frac{1}{\frac{1}{2} + \frac{1}{2}} c = \frac{1}{1 + \frac{1}{4}} c = \frac{1}{\frac{5}{4}} = \frac{4}{5} c
\]

\( \tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y} \)
**Galilean velocity addition becomes rapidity addition**

*From Lect. 22 p. 27 or eq. (3.6) in Ch. 3 of Unit 2:*

Evenson axiom requires geometric Doppler transform:

\[ e^{\rho_{AB}} \cdot e^{\rho_{BC}} = e^{\rho_{AC}} = e^{\rho_{AB} + \rho_{BC}} \]

Easy to combine frame velocities using rapidity addition:

\[ \rho_{u+v} = \rho_u + \rho_v \]

\[
\frac{u'}{c} = \tanh(\rho_u + \rho_v) = \frac{\tanh \rho_u + \tanh \rho_v}{1 + \tanh \rho_u \tanh \rho_v} = \frac{\frac{u}{c} + \frac{v}{c}}{1 + \frac{u}{c} \frac{v}{c}}
\]

or:

\[ u' = \frac{u + v}{1 + \frac{u \cdot v}{c^2}} \]

No longer does \((1/2+1/2)c\) equal \((1)c\)…

Relativistic result is:

\[ \frac{1/2 + 1/2}{1 + \frac{1}{2} \frac{1}{2}} c = \frac{1}{5} c = \frac{4}{5} c \]

…but, \((1/2+1)c\) **does** equal \((1)c\)…

\[ \frac{1/2 + 1}{1 + \frac{1}{2}} c = c \]
5. That “old-time” relativity (Circa 600BCE- 1905CE)

(“Bouncing-photons” in smoke & mirrors and Thales, again)

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Introducing the stellar aberration angle $\sigma$ vs. rapidity $\rho$

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Introducing the “Sin-Tan Rosetta Stone”

(a) Circular Functions

(plane geometry)

NOTE: Angle $\phi$ is now called stellar aberration angle $\sigma$

Fig. C.2-3

and

Fig. 5.4

in Unit 2
Introducing the “Sin-Tan Rosetta Stone”  NOTE: Angle $\phi$ is now called stellar aberration angle $\sigma$

Fig. C.2-3 and Fig. 5.4 in Unit 2

Hyperbolic arc area
$\rho = 1.0434 = \text{rapidity}$
$\sinh \rho = 1.2433$
$cosh \rho = 1.5955$
$tanh \rho = 0.7792$
$csch \rho = 0.8043$
$sech \rho = 0.6267$
$coth \rho = 1.2833$

Circular arc area
$\phi = 0.8934 = \text{angle}$
$\sin \phi = 0.7792$
$\cos \phi = 0.6267$
$tan \phi = 1.2433$
$csc \phi = 1.2833$
$sec \phi = 1.5955$
$cot \phi = 0.8043$

https://www.uark.edu/ua/pirelli/php/hyper_constrct.php

Monday, April 2, 2012
Introducing the “Sin-Tan Rosetta Stone” NOTE: Angle $\phi$ is now called stellar aberration angle $\sigma$
Hyperbolic Function Values

More about the “Sin-Tan Rosetta”

Arc Area = $\rho = 1.1758 \{ \text{radii} \}^2$

- $\sinh \rho = 1.4660$
- $\cosh \rho = 1.7746$
- $\tanh \rho = 0.8261$
- $\csch \rho = 0.6821$
- $\sech \rho = 0.5635$
- $\coth \rho = 1.2105$
- $\exp(\rho) = 3.2406$
- $\exp(-\rho) = 0.3086$
Circular Function Values

\[ m \angle(\sigma) = 0.9722 \text{ radians} \]

Arclength(\(\sigma\)) = 0.9722 \text{ radii}

Section Area(\(\sigma\)) = 0.9722 \text{ radii}^2

\[ \sin \sigma = 0.8261 \]

\[ \cos \sigma = 0.5635 \]

\[ \tan \sigma = 1.4660 \]

\[ \csc \sigma = 1.2105 \]

\[ \sec \sigma = 1.7746 \]

\[ \cot \sigma = 0.6821 \]
More about the "Sin-Tan Rosetta"
More about the "Sin-Tan Rosetta"

Hyperbolic Function Values

Arc Area $= \rho = 1.1758 \text{ radians}^2$

- $\sinh \rho = 1.4660$
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- $\coth \rho = 1.2105$
- $\exp(\rho) = 3.2406$
- $\exp(-\rho) = 0.3086$

Circular Function Values

- $m(\sigma) = 0.9722 \text{ radians}$
- $\text{Arc Length}(\sigma) = 0.9722 \text{ radians}$
- $\text{Sector Area}(\sigma) = 0.9722 \text{ radians}^2$

- $\sin \sigma = 0.8261$
- $\cos \sigma = 0.5635$
- $\tan \sigma = 1.4660$
- $\csc \sigma = 1.2105$
- $\sec \sigma = 1.7746$
- $\cot \sigma = 0.6821$

Note identities.
More about the “Sin-Tan Rosetta”

Hyperbolic Function Values

- Arc Area = $\rho = 1.1758 \text{ radians}^2$
- $\sinh \rho = 1.4660$
- $\cosh \rho = 1.7746$
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Circular Function Values

- $\sin \sigma = 0.8261$
- $\cos \sigma = 0.5635$
- $\tan \sigma = 1.4660$
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- $\sec \sigma = 1.7746$
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Note identities
5. That “old-time” relativity (Circa 600BCE- 1905CE)

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Group vs. phase velocity and tangent contacts
Introducing the *stellar aberration angle* $\sigma$ vs. *rapidity* $\rho$

Together, rapidity $\rho = \ln b$ and stellar aberration angle $\sigma$ are parameters of relative velocity.

The rapidity $\rho = \ln b$ is based on longitudinal wave Doppler shift $b = e^{\rho}$ defined by $u/c = \tanh(\rho)$.

At low speed: $u/c \sim \rho$.

The stellar aberration angle $\sigma$ is based on the transverse wave rotation $R = e^{i\sigma}$ defined by $u/c = \sin(\sigma)$.

At low speed: $u/c \sim \sigma$.

---

Fig. 5.6 Epstein's cosmic speedometer with aberration angle $\sigma$ and transverse Doppler shift $\cosh \upsilon$.
5. That “old-time” relativity (Circa 600BCE- 1905CE)

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How Minkowski’s space-time graphs help visualize relativity

Group vs. phase velocity and tangent contacts
How Minkowski’s space-time graphs help visualize relativity

Note that in Lighthouse frame Happening 1 is simultaneous with Happening 2 at $t=2.00$ sec.

Ref time $t = 2.00$ sec.

Happening 1: Ship 1 is hit by Blink 1
Happening 2: Lighthouse emits Blink 2

www.uark.edu/ua/pirelli/php/lighthouse_scenarios.php
How Minkowski’s space-time graphs help visualize relativity (Here: $r=\tanh(1/2)=0.549$,

Note that in Lighthouse frame Happening 1 is simultaneous with Happening 2 at $t=2.00\text{sec.}$

...but, in Ship frame Happening 1 is at $t'=1.74$ and Happening 2 is at $t'=2.30\text{sec.}$

Happening 1: Ship 1 is hit by Blink 1

Happening 2: Lighthouse emits Blink 2

(Here: $\rho=\tanh(1/2)=0.55$, and: $\sigma=\sin(1/2)=0.52 \text{ or } 30^{\circ}$)
How Minkowski’s space-time graphs help visualize relativity

Note that in Lighthouse frame Happening 1 is simultaneous with Happening 2 at $t=2.00$ sec.

...but, in Ship frame Happening 1 is at $t'=1.74$ and Happening 2 is at $t'=2.30$ sec.

That is $t'=2.30$ ship time

www.uark.edu/ua/pirelli/php/lighthouse_scenarios.php

Here: $\rho=\text{Atanh}(1/2)=0.55$, and: $\sigma=\text{Asin}(1/2)=0.52$ or 30°
5. That “old-time” relativity (Circa 600BCE- 1905CE)

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Group vs. phase velocity and tangent contacts
**Group vs. phase velocity and tangent contacts**

**Group velocity** $u$ and **phase velocity** $c^2/u$ are hyperbolic tangent slopes

(From Fig. 2.3.4)

$$\frac{\omega}{ck} = \frac{c}{u}$$

**Phase velocity**

$$\frac{d\omega}{dk} = \frac{u}{c} = \frac{ck}{\omega}$$

**Group velocity**

$$\omega = B \cosh \rho$$

$$k = B \sinh \rho$$

Rare but important case where

$$\frac{d\omega}{dk} = \frac{\Delta \omega}{\Delta k}$$

with LARGE $\Delta k$

(not infinitesimal)

Relativistic group wave speed $u = c \tanh \rho$

Newtonian speed $u \sim c \rho$

Low speed approximation

Rapidity $\rho$ approaches $u/c$

Lecture 25 ended here
Group vs. phase velocity and tangent contacts
**CW Axioms** ("All colors go c." and "r=1/b") imply hyperbolic dispersion then mechanics of matter

\[
\omega = B \cosh \rho \equiv B + \frac{1}{2} \frac{B}{c^2} u^2
\]

\[
E = \text{constant} + \frac{1}{2} Mu^2 \quad \text{(Newton's energy)}
\]

So 2-CW-light frequency \(\omega\) is like energy \(E\) while \(k\)-number is like momentum \(p\), implies Planck's \(E = s \cdot \omega\) scaling with factors: \(s = \hbar = s\) equal to DeBroglie's \(p = s \cdot k\).

\[
E = s \omega = sB \cosh \rho \equiv sB + \frac{1}{2} \frac{sB}{c^2} u^2
\]

Both relations imply: \(M = \frac{sB}{c^2}\) giving a (famous) *rest energy constant* \(sB = Mc^2\).

This then gives the famous Einstein energy \(E\) and also the Einstein momentum \(p\)

\[
E = s \omega = Mc^2 \cosh \rho \equiv Mc^2 + \frac{1}{2} Mu^2
\]

\[
p = sk = Mc \sinh \rho \equiv Mu
\]

Scale factors determined by experiment
Planck's constant
\(s = \hbar = 1.054572 \cdot 10^{-34}\) Joule·s
\(\hbar = 6.626069 \cdot 10^{-34}\) J·s = \(2\pi \hbar\)

Rest energy \((u=0)\): \(\hbar B = Mc^2\)

Rest momentum \((u=0)\): \(p = 0\)
Summary of geometry $\omega$-vs-$ck$ or $E$-vs-$cp$ relations with velocity $u$ or rapidity $\rho$

**Exact**

Relativistic group wave speed $u = c \tanh \rho$

- Circle approximates low speed wave
- Hyperbola only
- Newton's parabola only a little better than circle

- Newtonian speed $u \approx cp$

**Low speed approximation**

- Rest energy
- Newtonian energy

\[ E = \hbar \omega = Mc^2 \cosh \rho \approx Mc^2 + \frac{1}{2} Mu^2 \]

\[ E = \sqrt{Mc^2 \omega^2 - \hbar^2} \]

\[ p = \hbar k = Mc \sinh \rho \approx Mu \]

\[ \text{Relativistic DeBroglie momentum} \]

\[ p = \hbar k = \frac{Mu}{\sqrt{1 - u^2/c^2}} \]

Where: $\frac{\hbar B}{c^2} = M$ is rest mass
What’s the Matter With Light? Three definitions of optical mass

1. Rest mass $M_N = \frac{h\nu_N}{c^2}$ based on Planck’s law $E = h\nu_N = Nh\nu_f$
   
   Rest mass: $M_{\text{rest}} = \frac{E}{c^2} = \frac{h\nu_N}{c^2}$ (Is invariant)

2. Momentum mass is defined by Galileo’s old formula $p = Mu$ with newer forms for momentum $p = M_{\text{rest}} \omega \cosh \rho = M_{\text{rest}} \omega / (1 - u^2/c^2)^{1/2}$ and group velocity $u = d\omega/dk$. It is the ratio $p/u$ of momentum to velocity.
   
   Momentum mass: $M_{\text{momentum}} = \frac{p}{u} = M_{\text{rest}} \cosh \rho = M_{\text{rest}} / (1 - u^2/c^2)^{1/2}$ (Not invariant)

3. Effective mass is defined by Newton’s old formula $F = Ma$ with newer forms for $F = dp/dt = \hbar dk/dt$ and $a = du/dt$ to give $F/a = (\hbar dk/dt)(dt/du) = \hbar dk/du = \hbar / (du/dk)$. It is the ratio $F/a$ of change of momentum to the change of velocity,
   
   Effective mass: $M_{\text{effective}} = \frac{\hbar}{(du/dk)} = \hbar / (d^2 \omega / dk^2)$ = $M_{\text{rest}} \cosh^3 \rho = M_{\text{rest}} / (1 - u^2/c^2)^{3/2}$ (Not invariant)