Complex Variables, Series, and Field Coordinates I.
(Ch. 10 of Unit 1)

1. The Story of e (A Tale of Great $Interest$)
   How good are those power series?

2. What good are complex exponentials?
   Easy trig
   Easy 2D vector analysis
   Easy oscillator phase analysis
   Easy 2D vector derivatives
   Easy 2D source-free field theory
   Easy 2D vector field-potential theory
   The half-n'-half results: (Riemann-Cauchy Derivative Relations)

End of Part I. Lecture 19 Thur. 3.08.2012

1. Complex numbers provide "automatic trigonometry"
2. Complex numbers add like vectors.
3. Complex exponentials $e^{\omega t}$ track position and velocity using Phasor Clock.
4. Complex products provide 2D rotation operations.
5. Complex products provide 2D "dot"($\cdot$) and "cross"($\times$) products.
6. Complex derivative contains “divergence”($\nabla \cdot F$) and “curl”($\nabla \times F$) of 2D vector field
7. Invent source-free 2D vector fields [$\nabla \cdot F=0$ and $\nabla \times F=0$]
8. Complex potential $\phi$ contains “scalar”($F=\nabla \phi$) and “vector”($F=\nabla \times A$) potentials
9. Complex integrals $\int f(z)dz$ count 2D “circulation”($\int F \cdot dr$) and “flux”($\int F \times dr$)
10. Complex integrals define 2D monopole fields and potentials
11. Complex potentials define 2D Orthogonal Curvilinear Coordinates (OCC) of field
12. Complex derivatives give 2D dipole fields

Friday, March 9, 2012
The Story of $e$ (A Tale of Great $\$Interest$)

Simple interest at some rate $r$ based on a 1 year period.
You gave a principal $p(0)$ to the bank and some time $t$ later they would pay you $p(t) = (1 + r \cdot t)p(0)$.
$1.00 at rate $r=1$ (like Israel and Brazil that once had 100% interest.) gives $2.00 at $t=1$ year.
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**Semester compounded** interest gives $p(\frac{t}{2}) = (1 + r \cdot \frac{t}{2})p(0)$ at the half-period $\frac{t}{2}$ and then use $p(\frac{t}{2})$ during the last half to figure final payment. Now $1.00$ at rate $r=1$ earns $2.25$.

$$p^{\frac{1}{2}}(t) = (1 + r \cdot \frac{t}{2})p(\frac{t}{2}) = (1 + r \cdot \frac{t}{2}) \cdot (1 + r \cdot \frac{t}{2})p(0) = \frac{3}{2} \cdot \frac{3}{2} \cdot 1 = \frac{9}{4} = 2.25$$
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*Trimester compounded* interest gives $p\left(\frac{t}{3}\right)=(1+r\cdot \frac{t}{3})p(0)$ at the $1/3^{rd}$-period $\frac{t}{3}$ or $1^{st}$ trimester and then use that to figure the $2^{nd}$ trimester and so on. Now $1.00$ at rate $r=1$ earns $2.37$.

$$p^{\frac{1}{3}}(t) = (1 + r\cdot \frac{t}{3})p\left(\frac{2t}{3}\right) = (1 + r\cdot \frac{t}{3}) \cdot (1 + r\cdot \frac{t}{3})p\left(\frac{t}{3}\right) = (1 + r\cdot \frac{t}{3}) \cdot (1 + r\cdot \frac{t}{3}) \cdot (1 + r\cdot \frac{t}{3})p(0) = \frac{4}{3} \cdot \frac{4}{3} \cdot \frac{4}{3} \cdot 1 = \frac{64}{27} = 2.37$$
**The Story of e (A Tale of Great $Interest$)**

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$$p^{\frac{1}{2}}(t) = (1+r\frac{t}{2})p(\frac{t}{2}) = (1+r\frac{t}{2})(1+r\frac{t}{2})p(0) = \frac{3}{2} \cdot \frac{3}{2} \cdot 1 = \frac{9}{4} = 2.25$$

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$$p^{\frac{1}{3}}(t) = (1+r\frac{t}{3})p(2\frac{t}{3}) = (1+r\frac{t}{3})(1+r\frac{t}{3})p(\frac{t}{3}) = (1+r\frac{t}{3})(1+r\frac{t}{3})(1+r\frac{t}{3})p(0) = \frac{4}{3} \cdot \frac{4}{3} \cdot \frac{4}{3} \cdot 1 = \frac{64}{27} = 2.37$$

So if you compound interest more and more frequently, do you approach $\text{INFININTEREST}$?
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$$p^{1}(t) = (1 + r \cdot \frac{t}{2})p(\frac{t}{2}) = (1 + r \cdot \frac{t}{2}) \cdot (1 + r \cdot \frac{t}{2})p(0) = \frac{3}{2} \cdot \frac{3}{2} \cdot 1 = \frac{9}{4} = 2.25$$

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$$p^{\frac{1}{3}}(t) = (1 + r \cdot \frac{t}{3})p(\frac{t}{3}) = (1 + r \cdot \frac{t}{3}) \cdot (1 + r \cdot \frac{t}{3})p(\frac{t}{3}) = (1 + r \cdot \frac{t}{3}) \cdot (1 + r \cdot \frac{t}{3}) \cdot (1 + r \cdot \frac{t}{3})p(0) = \frac{4}{3} \cdot \frac{4}{3} \cdot \frac{4}{3} \cdot 1 = \frac{64}{27} = 2.37$$

So if you compound interest more and more frequently, do you approach INFININTEREST?

NOT!!
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\[
p^{\frac{1}{3}}(t) = (1+r\cdot \frac{t}{3})p\left(2\frac{t}{3}\right) = (1+r\cdot \frac{t}{3})\cdot (1+r\cdot \frac{t}{3})p\left(\frac{t}{3}\right) = (1+r\cdot \frac{t}{3})\cdot (1+r\cdot \frac{t}{3})\cdot (1+r\cdot \frac{t}{3})p(0) = \frac{4}{3} \cdot \frac{4}{3} \cdot \frac{4}{3} \cdot 1 = \frac{64}{27} = 2.37
\]

So if you compound interest more and more frequently, do you approach INFININTEREST?

\[
p^{\frac{1}{4}}(t) = (1+r\cdot \frac{t}{4})^4 p(0) = \left(\frac{5}{4}\right)^4 \cdot 1 = \frac{625}{256} = 2.44
\]

\[
p^{\frac{1}{1}}(t) = (1+r\cdot \frac{t}{1})^1 p(0) = \left(\frac{1}{4}\right)^1 \cdot 1 = \frac{25}{4} + 25\phi
\]

\[
p^{\frac{1}{2}}(t) = (1+r\cdot \frac{t}{2})^2 p(0) = \left(\frac{3}{2}\right)^2 \cdot 1 = \frac{9}{4} + 12\phi
\]

\[
p^{\frac{1}{3}}(t) = (1+r\cdot \frac{t}{3})^3 p(0) = \left(\frac{4}{3}\right)^3 \cdot 1 = \frac{64}{27} + 7\phi
\]
**The Story of e (A Tale of Great $Interest$)**

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$$p^{\frac{1}{3}}(t) = (1 + r\cdot \frac{t}{3}) \cdot (1+r\cdot \frac{t}{3})\cdot (1+r\cdot \frac{t}{3})p\left(\frac{t}{3}\right) = (1 + r\cdot \frac{t}{3})\cdot (1+r\cdot \frac{t}{3})\cdot (1+r\cdot \frac{t}{3})p(0) = \frac{4}{3} \cdot \frac{4}{3} \cdot \frac{4}{3} \cdot 1 = \frac{64}{27} = 2.37$$

**So if you compound interest more and more frequently, do you approach INFININTEREST?**

$$p^{\frac{1}{12}}(t) = (1 + r\cdot \frac{t}{12})^{12}p(0) = \left(1 + \frac{13}{12}\right)^{12} \cdot 1 = 2.613$$

**Monthly:**

$$p^{\frac{1}{52}}(t) = (1 + r\cdot \frac{t}{52})^{52}p(0) = \left(1 + \frac{53}{52}\right)^{52} \cdot 1 = 2.693$$

**Weekly:**

$$p^{\frac{1}{365}}(t) = (1 + r\cdot \frac{t}{365})^{365}p(0) = \left(1 + \frac{366}{365}\right)^{365} \cdot 1 = 2.7145$$

**Daily:**

$$p^{\frac{1}{8760}}(t) = (1 + r\cdot \frac{t}{8760})^{8760}p(0) = \left(1 + \frac{8761}{8760}\right)^{8760} \cdot 1 = 2.7181$$

**Hrly:**
Interest product formula is really inefficient: $10^6$ products for 6-figures! .. $10^9$ products for 9 ...

\[ p^{1/m}(1) = (1 + \frac{1}{m})^m \xrightarrow{m \to \infty} 2.718281828459. \]

\[ p^{1/m}(1) = 2.7169239322 \quad \text{for } m = 1,000 \]
\[ p^{1/m}(1) = 2.7181459268 \quad \text{for } m = 10,000 \]
\[ p^{1/m}(1) = 2.7182682372 \quad \text{for } m = 100,000 \]
\[ p^{1/m}(1) = 2.7182804693 \quad \text{for } m = 1,000,000 \]
\[ p^{1/m}(1) = 2.7182816925 \quad \text{for } m = 10,000,000 \]
\[ p^{1/m}(1) = 2.7182818149 \quad \text{for } m = 100,000,000 \]
\[ p^{1/m}(1) = 2.7182818271 \quad \text{for } m = 1,000,000,000 \]

Let: \( m \cdot r \cdot t = n \)

or: \( \frac{1}{m} = \frac{r \cdot t}{n} \)

\[ (1 + \frac{1}{n})^{m \cdot r \cdot t} \xrightarrow{m \to \infty} e^{r \cdot t} \]

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**Interest product formula is really inefficient:** $10^6$ products for 6-figures! .. $10^9$ products for 9 ...

\[
p^{1/m}(1) = (1 + \frac{1}{m})^m \quad \xrightarrow{m \to \infty} \quad 2.718281828459.. \]

Let: \( m \cdot r \cdot t = n \)

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\[
\left(1 + \frac{1}{m}\right)^{m \cdot r \cdot t} \xrightarrow{m \to \infty} e^{r \cdot t}
\]

\[
\left(1 + \frac{r \cdot t}{n}\right)^n \xrightarrow{n \to \infty} e^{r \cdot t}
\]

**Can improve efficiency using binomial theorem:**

\[
(x + y)^n = x^n + n \cdot x^{n-1} y + \frac{n(n-1)}{2!} x^{n-2} y^2 + \frac{n(n-1)(n-2)}{3!} x^{n-3} y^3 + ... + n \cdot xy^{n-1} + y^n
\]

\[
(1 + \frac{r \cdot t}{n})^n = 1 + n \cdot \left(\frac{r \cdot t}{n}\right) + \frac{n(n-1)}{2!} \left(\frac{r \cdot t}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!} \left(\frac{r \cdot t}{n}\right)^3 + ...
\]

Define: **Factorials (!):**

0! = 1!, 2! = 1·2, 3! = 1·2·3, ...

Friday, March 9, 2012
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\[
p^{1/m}(1) = (1 + \frac{1}{m})^m \quad \xrightarrow{m \to \infty} \quad 2.718281828459.. = e
\]

Let: \(m \cdot r \cdot t = n\)

or: \(\frac{1}{m} = \frac{r \cdot t}{n}\)

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\]

\[
(1 + \frac{r \cdot t}{n})^n = 1 + n \cdot \left(\frac{r \cdot t}{n}\right) + \frac{n(n-1)}{2!} \left(\frac{r \cdot t}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!} \left(\frac{r \cdot t}{n}\right)^3 + ...
\]

\[
e^{r \cdot t} = 1 + r \cdot t + \frac{1}{2!} (r \cdot t)^2 + \frac{1}{3!} (r \cdot t)^3 + ... = \sum_{p=0}^{\infty} \frac{(r \cdot t)^p}{p!}
\]

Define: Factorials(!):

\(0! = 1 = 1!, \quad 2! = 1 \cdot 2, \quad 3! = 1 \cdot 2 \cdot 3, ...\)

As \(n \to \infty\) let:

\(n(n - 1) \to n^2,\)

\(n(n - 1)(n - 2) \to n^3, \text{ etc.}\)
\[ \frac{1}{m}(1) = (1 + \frac{1}{m})^m \xrightarrow{m \to \infty} 2.718281828459... \]

**Can improve efficiency using binomial theorem:**

\[(x + y)^n = x^n + n \cdot x^{n-1} y + \frac{n(n-1)}{2!} x^{n-2} y^2 + \frac{n(n-1)(n-2)}{3!} x^{n-3} y^3 + \ldots + n \cdot xy^{n-1} + y^n\]

\[(1 + \frac{r \cdot t}{n})^n = 1 + n \cdot \left( \frac{r \cdot t}{n} \right) + \frac{n(n-1)}{2!} \left( \frac{r \cdot t}{n} \right)^2 + \frac{n(n-1)(n-2)}{3!} \left( \frac{r \cdot t}{n} \right)^3 + \ldots \]

Define: **Factorials(!):**

\[0! = 1!, \quad 2! = 1 \cdot 2!, \quad 3! = 1 \cdot 2 \cdot 3, \ldots\]

As \( n \to \infty \) let:

\[n(n-1) \to n^2, \quad n(n-1)(n-2) \to n^3, \text{ etc.}\]

**Precision order:**

(o=1)-e-series = 2.00000
(o=2)-e-series = 2.50000
(o=3)-e-series = 2.66667
(o=4)-e-series = 2.70833
(o=5)-e-series = 2.71667
(o=6)-e-series = 2.71805
(o=7)-e-series = 2.71825
(o=8)-e-series = 2.71828

About 12 summed quotients

for 6-figure precision (A lot better!)
Start with a general power series with constant coefficients $c_0, c_1, etc.$

\[ x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + \ldots + c_n t^n + \]

Set $t=0$ to get $c_0 = x(0)$. 

Power Series Good! Need general power series development
Power Series Good!  Need general power series development

Start with a general power series with constant coefficients \( c_0, c_1, \text{ etc.} \)

\[
x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + \ldots + c_n t^n + \]

Rate of change of position \( x(t) \) is \textit{velocity} \( v(t) \).

\[
v(t) = \frac{d}{dt} x(t) = 0 + c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + 5c_5 t^4 + \ldots + nc_n t^{n-1} +
\]

Set \( t=0 \) to get \( c_0 = x(0) \).

Set \( t=0 \) to get \( c_1 = v(0) \).
Power Series Good!  Need general power series development

Start with a general power series with constant coefficients $c_0, c_1, \text{ etc.}$  

\[ x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + \ldots + c_n t^n + \]

Rate of change of position $x(t)$ is velocity $v(t)$.  

\[ v(t) = \frac{d}{dt} x(t) = 0 + c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + 5c_5 t^4 + \ldots + nc_n t^{n-1} + \]

Change of velocity $v(t)$ is acceleration $a(t)$.  

\[ a(t) = \frac{d}{dt} v(t) = 0 + 2c_2 + 2 \cdot 3c_3 t + 3 \cdot 4c_4 t^2 + 4 \cdot 5c_5 t^3 + \ldots + n(n-1)c_n t^{n-2} + \]

Set $t=0$ to get $c_0 = x(0)$.  

Set $t=0$ to get $c_1 = v(0)$.  

Set $t=0$ to get $c_2 = \frac{1}{2} a(0)$.  

Friday, March 9, 2012
Power Series Good! Need general power series development

Start with a general power series with constant coefficients \( c_0, c_1, \text{ etc.} \). Set \( t=0 \) to get \( c_0 = x(0) \).

\[
x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + \ldots + c_n t^n +
\]

Rate of change of position \( x(t) \) is velocity \( v(t) \). Set \( t=0 \) to get \( c_1 = v(0) \).

\[
v(t) = \frac{d}{dt} x(t) = 0 + c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + 5c_5 t^4 + \ldots + nc_n t^{n-1} +
\]

Change of velocity \( v(t) \) is acceleration \( a(t) \). Set \( t=0 \) to get \( c_2 = \frac{1}{2} a(0) \).

\[
a(t) = \frac{d}{dt} v(t) = 0 + 2c_2 + 2\cdot 3c_3 t + 3\cdot 4c_4 t^2 + 4\cdot 5c_5 t^3 + \ldots + n(n-1)c_n t^{n-2} +
\]

Change of acceleration \( a(t) \) is jerk \( j(t) \). (Jerk is NASA term.) Set \( t=0 \) to get \( c_3 = \frac{1}{3!} j(0) \).

\[
j(t) = \frac{d}{dt} a(t) = 0 + 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4 t + 3\cdot 4\cdot 5c_5 t^2 + \ldots + n(n-1)(n-2)c_n t^{n-3} +
\]

Change of jerk \( j(t) \) is inauguration \( i(t) \). (Be silly like NASA!) Set \( t=0 \) to get \( c_4 = \frac{1}{4!} i(0) \).

\[
i(t) = \frac{d}{dt} j(t) = 0 + 2\cdot 3\cdot 4c_4 + 2\cdot 3\cdot 4\cdot 5c_5 t + \ldots + n(n-1)(n-2)(n-3)c_n t^{n-4} +
\]
Power Series Good! Need general power series development

Start with a general power series with constant coefficients \( c_0, c_1, \text{ etc.} \)

\[
x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + \ldots + c_n t^n +
\]

Rate of change of position \( x(t) \) is velocity \( v(t) \).

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v(t) = \frac{d}{dt} x(t) = 0 + c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + 5c_5 t^4 + \ldots + nc_n t^{n-1} +
\]

Change of velocity \( v(t) \) is acceleration \( a(t) \).

\[
a(t) = \frac{d}{dt} v(t) = 0 + 2c_2 + 2 \cdot 3 c_3 t + 3 \cdot 4 c_4 t^2 + 4 \cdot 5 c_5 t^3 + \ldots + n(n-1)c_n t^{n-2} +
\]

Change of acceleration \( a(t) \) is jerk \( j(t) \). (Jerk is NASA term.)

\[
j(t) = \frac{d}{dt} a(t) = 0 + 2 \cdot 3 c_3 + 2 \cdot 3 \cdot 4 c_4 t + 3 \cdot 4 \cdot 5 c_5 t^2 + \ldots + n(n-1)(n-2)c_n t^{n-3} +
\]

Change of jerk \( j(t) \) is inauguration \( i(t) \). (Be silly like NASA!)

\[
i(t) = \frac{d}{dt} j(t) = 0 + 2 \cdot 3 \cdot 4 c_4 + 2 \cdot 3 \cdot 4 \cdot 5 c_5 t + \ldots + n(n-1)(n-2)(n-3)c_n t^{n-4} +
\]

Gives Maclaurin (or Taylor) power series

\[
x(t) = x(0) + v(0)t + \frac{1}{2!} a(0)t^2 + \frac{1}{3!} j(0)t^3 + \frac{1}{4!} i(0)t^4 + \frac{1}{5!} r(0)t^5 + \ldots + \frac{1}{n!} x^{(n)} t^n +
\]
Power Series Good! Need general power series development

Start with a general power series with constant coefficients $c_0, c_1, \text{etc.}$

$$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + ... + c_n t^n +$$

Rate of change of position $x(t)$ is velocity $v(t)$.

$$v(t) = \frac{d}{dt} x(t) = 0 + c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + 5c_5 t^4 + ... + nc_n t^{n-1} +$$

Change of velocity $v(t)$ is acceleration $a(t)$.

$$a(t) = \frac{d}{dt} v(t) = 0 + 2c_2 + 2 \cdot 3c_3 t + 3 \cdot 4c_4 t^2 + 4 \cdot 5c_5 t^3 + ... + n(n-1)c_n t^{n-2} +$$

Change of acceleration $a(t)$ is jerk $j(t)$. (Jerk is NASA term.)

$$j(t) = \frac{d}{dt} a(t) = 0 + 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4 t + 3 \cdot 4 \cdot 5c_5 t^2 + ... + n(n-1)(n-2)c_n t^{n-3} +$$

Change of jerk $j(t)$ is inauguration $i(t)$. (Be silly like NASA!)

$$i(t) = \frac{d}{dt} j(t) = 0 + 2 \cdot 3 \cdot 4c_4 + 2 \cdot 3 \cdot 4 \cdot 5c_5 t + ... + n(n-1)(n-2)(n-3)c_n t^{n-4} +$$

Gives Maclaurin (or Taylor) power series

$$x(t) = x(0) + v(0)t + \frac{1}{2!} a(0)t^2 + \frac{1}{3!} j(0)t^3 + \frac{1}{4!} i(0)t^4 + \frac{1}{5!} r(0)t^5 + ... + \frac{1}{n!} x^{(n)}(0)t^n +$$

Good old UP I formula!
Power Series Good!  Need general power series development

Start with a general power series with constant coefficients $c_0, c_1, etc.$  Set $t=0$ to get $c_0 = x(0)$.

$$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + ... + c_n t^n +$$

Rate of change of position $x(t)$ is velocity $v(t)$.  Set $t=0$ to get $c_1 = v(0)$.

$$v(t) = \frac{d}{dt} x(t) = 0 + c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + 5c_5 t^4 + ... + nc_n t^{n-1} +$$

Change of velocity $v(t)$ is acceleration $a(t)$.  Set $t=0$ to get $c_2 = \frac{1}{2} a(0)$.

$$a(t) = \frac{d}{dt} v(t) = 0 + 2c_2 + 2 \cdot 3c_3 t + 3 \cdot 4c_4 t^2 + 4 \cdot 5c_5 t^3 + ... + n(n-1)c_n t^{n-2} +$$

Change of acceleration $a(t)$ is jerk $j(t)$.  (Jerk is NASA term.)  Set $t=0$ to get $c_3 = \frac{1}{3!} j(0)$.

$$j(t) = \frac{d}{dt} a(t) = 0 + 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4 t + 3 \cdot 4 \cdot 5c_5 t^2 + ... + n(n-1)(n-2)c_n t^{n-3} +$$

Change of jerk $j(t)$ is inauguration $i(t)$.  (Be silly like NASA!)  Set $t=0$ to get $c_4 = \frac{1}{4!} i(0)$.

$$i(t) = \frac{d}{dt} j(t) = 0 + 2 \cdot 3 \cdot 4c_4 + 2 \cdot 3 \cdot 4 \cdot 5c_5 t + ... + n(n-1)(n-2)(n-3)c_n t^{n-4} +$$

Gives Maclaurin (or Taylor) power series

$$x(t) = x(0) + v(0)t + \frac{1}{2!} a(0)t^2 + \frac{1}{3!} j(0)t^3 + \frac{1}{4!} i(0)t^4 + \frac{1}{5!} r(0)t^5 + ... + \frac{1}{n!} x^{(n)} t^n +$$

Setting all initial values to 1 = $x(0) = v(0) = a(0) = j(0) = i(0) = ....$

gives exponential:  $e^t = 1 + t + \frac{1}{2!} t^2 + \frac{1}{3!} t^3 + \frac{1}{4!} t^4 + \frac{1}{5!} t^5 + ... + \frac{1}{n!} t^n +$
But, how good are power series?

\[ x(t) = e^t \]

Gives Maclaurin (or Taylor) power series

\[ x(t) = x(0) + v(0)t + \frac{1}{2!} a(0)t^2 + \frac{1}{3!} j(0)t^3 + \frac{1}{4!} i(0)t^4 + \frac{1}{5!} r(0)t^5 + \ldots + \frac{1}{n!} x^{(n)}t^n + \]

Setting all initial values to \( I = x(0) = v(0) = a(0) = j(0) = i(0) = \ldots \)

gives exponential:

\[ e^t = 1 + t + \frac{1}{2!} t^2 + \frac{1}{3!} t^3 + \frac{1}{4!} t^4 + \frac{1}{5!} t^5 + \ldots + \frac{1}{n!} t^n + \]
How good are power series? Depends...

\[ x(t) = \cos t = 1 + 0 - \frac{t^2}{2!} + 0 + \frac{t^4}{4!} + 0 - \frac{t^6}{6!} + 0 + \frac{t^8}{8!} \ldots \]

\[ x(t) = \sin t = 0 + t + 0 - \frac{t^3}{3!} + 0 + \frac{t^5}{5!} + 0 - \frac{t^7}{7!} + 0 + \frac{t^9}{9!} \ldots \]
Suppose the fancy bankers really went bonkers and made interest rate $r$ an *imaginary number* $r = i\theta$.

Imaginary number $i = \sqrt{-1}$ powers have *repeat-after-4-pattern*: $i^0 = 1$, $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, etc...

\[ e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + ... \]  

(From exponential series)

\[ = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - ... \]  

($i = \sqrt{-1}$ implies: $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = +1$, $i^5 = i$, ...)

\[
= \left( 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - ... \right) + \left( i\theta - i\frac{\theta^3}{3!} + i\frac{\theta^5}{5!} - ... \right)
\]
Suppose the fancy bankers really went bonkers and made interest rate $r$ an *imaginary number* $r = i\theta$.

Imaginary number $i = \sqrt{-1}$ powers have *repeat-after-4-pattern*: $i^0 = 1$, $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, etc…

\[
e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \ldots
\]

(From exponential series)

\[
e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \ldots
\]

($i = \sqrt{-1}$ implies: $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = +1$, $i^5 = i$, …)

\[
e^{i\theta} = \cos \theta + i \sin \theta
\]

*Euler-DeMoivre Theorem*
Suppose the fancy bankers really went bonkers and made interest rate \( r \) an **imaginary number** \( r = i \theta \).

Imaginary number \( i = \sqrt{-1} \) powers have **repeat-after-4-pattern**: \( i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1, \text{etc...} \)

\[
e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \ldots
\]

(From exponential series)

\[
= 1 + i\theta - \frac{\theta^2}{2!} - i \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i \frac{\theta^5}{5!} - \ldots
\]

\( (i = \sqrt{-1} \text{ imples: } i^1 = i, i^2 = -1, i^3 = -i, i^4 = +1, i^5 = i, \ldots) \)

\[
= \left( 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots \right) + \left( i\theta - i \frac{\theta^3}{3!} + i \frac{\theta^5}{5!} - \ldots \right)
\]

To match series for

\[
\cos: \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots
\]

\[
\sin: \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots
\]

\[
e^{i\theta} = \cos \theta + i \sin \theta
\]

**Euler-DeMoivre Theorem**

**Imaginary axis**

\( (i \text{ axis}) \)

\[
z = r e^{i\theta} = x + iy
\]

\[
x = r \cos \theta
\]

\[
y = r \sin \theta
\]

**Real axis**

\( (1 \text{ axis}) \)

\[
re^{i\theta} = r \cos \theta + i \sin \theta
\]
2. What Good Are Complex Exponentials?
1. Complex numbers provide "automatic trigonometry"

Can't remember is \( \cos(a+b) \) or \( \sin(a+b) \)? Just factor \( e^{i(a+b)} = e^{ia}e^{ib} \)...

\[
e^{i(a+b)} = e^{ia}e^{ib}
\]

\[
\cos(a+b) + i \sin(a+b) = (\cos a + i \sin a) (\cos b + i \sin b)
\]

\[
\cos(a+b) + i \sin(a+b) = [\cos a \cos b - \sin a \sin b] + i[\sin a \cos b + \cos a \sin b]
\]
What Good Are Complex Exponentials?

1. Complex numbers provide "automatic trigonometry"

Can't remember is \( \cos(a+b) \) or \( \sin(a+b) \)? Just factor \( e^{i(a+b)} = e^{ia} e^{ib} \)...

\[
\begin{align*}
\cos(a+b) + i \sin(a+b) &= (\cos a + i \sin a) (\cos b + i \sin b) \\
\cos(a+b) + i \sin(a+b) &= [\cos a \cos b - \sin a \sin b] + i[\sin a \cos b + \cos a \sin b]
\end{align*}
\]

2. Complex numbers add like vectors.

\( z_{\text{sum}} = z + z' = (x + iy) + (x' + iy') = (x + x') + i(y + y') \)

\( z_{\text{diff}} = z - z' = (x + iy) - (x' + iy') = (x - x') + i(y - y') \)

\[
\begin{align*}
|z_{\text{SUM}}| &= \sqrt{(z + z')^*(z + z')} = \sqrt{(re^{i\phi} + r'e^{i\phi'})^*(re^{i\phi} + r'e^{i\phi'})} = \sqrt{\left(re^{-i\phi} + r'e^{-i\phi'}\right)\left(re^{i\phi} + r'e^{i\phi'}\right)} \\
&= \sqrt{r^2 + r'^2 + 2rr'(e^{i(\phi-\phi')} + e^{-i(\phi-\phi')})} = \sqrt{r^2 + r'^2 + 2rr' \cos(\phi - \phi')} \quad \text{(quick derivation of Cosine Law)}
\end{align*}
\]
What Good Are Complex Exponentials? (contd.)

3. Complex exponentials \( Ae^{-i\omega t} \) track position and velocity using Phasor Clock.

(a) Complex plane and unit vectors

\[ e^{i\theta} = x + iy \]
\[ y = \sin \theta \]
\[ x = \cos \theta \]

(b) Quantum Phasor Clock \( \psi = Ae^{-i\omega t} = A\cos \omega t - iA\sin \omega t = x + iy \)

Im \( \psi \) (The “Gonna’be”)

Re \( \psi \)

Phase angle or Argument
\[ \theta = -\omega t = \text{ATAN}[v(t)/\omega x(t)] \]

Polar Components

Magnitude or Modulus
\[ A = |\psi| = \sqrt{\psi^*\psi} \]
What Good Are Complex Exponentials? (contd.)

3. Complex exponentials $Ae^{-i\omega t}$ track position and velocity using Phasor Clock.

(a) Complex plane and unit vectors

(b) Quantum Phasor Clock $\psi = Ae^{-i\omega t} = A \cos \omega t - i \sin \omega t = x + iy$

Some Rect-vs-Polar relations worth remembering

Cartesian

\[
\begin{align*}
\psi_x &= \text{Re} \psi(t) = x(t) = A \cos \omega t \frac{\psi + \psi^*}{2} \\
\psi_y &= \text{Im} \psi(t) = \frac{v(t)}{\omega} = -A \sin \omega t \frac{\psi - \psi^*}{2i}
\end{align*}
\]

Polar

\[
\begin{align*}
\psi &= r e^{+i\theta} = r e^{-i\omega t} = r(\cos \omega t - i \sin \omega t) \\
\psi^* &= r e^{-i\theta} = r e^{+i\omega t} = r(\cos \omega t + i \sin \omega t)
\end{align*}
\]

Unit 1

Fig. 10.5
4. **Complex products provide 2D rotation operations.**

\[ e^{i\phi} \cdot z = (\cos\phi + i \sin\phi) \cdot (x + iy) = x \cos\phi - y \sin\phi + i \ (x \sin\phi + y \cos\phi) \]

\[ \mathbf{R}_{+\phi} \cdot \mathbf{r} = (x \cos \phi - y \sin \phi) \hat{e}_x + (x \sin \phi + y \cos \phi) \hat{e}_y \]

\[
\begin{pmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix} =
\begin{pmatrix}
x \cos \phi - y \sin \phi \\
x \sin \phi + y \cos \phi
\end{pmatrix} \]
4. Complex products provide 2D rotation operations.

\[ e^{i\phi} \cdot z = (\cos \phi + i \sin \phi) \cdot (x + iy) = x \cos \phi - y \sin \phi + i (x \sin \phi + y \cos \phi) \]

\[ R_{\phi} \cdot r = (x \cos \phi - y \sin \phi) \hat{e}_x + (x \sin \phi + y \cos \phi) \hat{e}_y \]

\[ \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \cos \phi - y \sin \phi \\ x \sin \phi + y \cos \phi \end{pmatrix} \]

The matrix \( e^{i\phi} \) acts on this: \( z = re^{i\theta} \)

to give this: \( e^{i\phi} e^{i\phi} z = re^{i\phi} e^{i\theta} \)
What Good Are Complex Exponentials? (contd.)

4. Complex products provide 2D rotation operations.

\[ e^{i\phi}z = (\cos \phi + i \sin \phi) \cdot (x + iy) = x \cos \phi - y \sin \phi + i (x \sin \phi + y \cos \phi) \]

\[ \mathbf{R}_{\phi} \cdot \mathbf{r} = (x \cos \phi - y \sin \phi) \hat{e}_x + (x \sin \phi + y \cos \phi) \hat{e}_y \]

\[ \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \cos \phi - y \sin \phi \\ x \sin \phi + y \cos \phi \end{pmatrix} \]

5. Complex products provide 2D “dot”(•) and “cross”(x) products.

Two complex numbers \( A = A_x + iA_y \) and \( B = B_x + iB_y \) and their “star” (*)-product \( A \ast B \).

\[ A \ast B = (A_x + iA_y) \ast (B_x + iB_y) = (A_x - iA_y)(B_x + iB_y) \]

\[ = (A_x B_x + A_y B_y) + i(A_x B_y - A_y B_x) = A \cdot B + i |A \times B| Z_{\perp(x,y)} \]

\[ \text{Real part is scalar or “dot”(•) product } A \cdot B. \]
\[ \text{Imaginary part is vector or “cross”(x) product, but just the Z-component normal to xy-plane.} \]

Rewrite \( A \ast B \) in polar form.

\[ A \ast B = (|A|e^{i\theta_A}) \ast (|B|e^{i\theta_B}) = |A|e^{-i\theta_A} |B|e^{i\theta_B} = |A||B|e^{i(\theta_B - \theta_A)} \]

\[ = |A||B|\cos(\theta_B - \theta_A) + i |A||B|\sin(\theta_B - \theta_A) = A \cdot B + i |A \times B| Z_{\perp(x,y)} \]
What Good Are Complex Exponentials? (contd.)

4. Complex products provide 2D rotation operations.

\[ e^{i\phi} \cdot z = (\cos \phi + i \sin \phi) \cdot (x + iy) = x \cos \phi - y \sin \phi + i (x \sin \phi + y \cos \phi) \]

\[ \mathbf{R}_{+\phi} \cdot \mathbf{r} = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \cos \phi - y \sin \phi \\ x \sin \phi + y \cos \phi \end{pmatrix} \]

5. Complex products provide 2D “dot”(•) and “cross”(×) products.

Two complex numbers \( \mathbf{A} = A_x + iA_y \) and \( \mathbf{B} = B_x + iB_y \) and their “star” (*)-product \( \mathbf{A} \ast \mathbf{B} \).

\[ \mathbf{A} \ast \mathbf{B} = (A_x + iA_y) \ast (B_x + iB_y) = (A_x - iA_y)(B_x + iB_y) \]

\[ = (A_x B_x + A_y B_y) + i(A_x B_y - A_y B_x) = \mathbf{A} \cdot \mathbf{B} + i |\mathbf{A} \times \mathbf{B}| \]

Real part is scalar or “dot”(•) product \( \mathbf{A} \cdot \mathbf{B} \).

Imaginary part is vector or “cross”(×) product, but just the Z-component normal to xy-plane.

Rewrite \( \mathbf{A} \ast \mathbf{B} \) in polar form.

\[ \mathbf{A} \ast \mathbf{B} = (|A| e^{i\theta_A}) \ast (|B| e^{i\theta_B}) = |A| e^{-i\theta_A} |B| e^{i\theta_B} = |A||B| e^{i(\theta_B - \theta_A)} \]

\[ = |A||B| \cos(\theta_B - \theta_A) + i |A||B| \sin(\theta_B - \theta_A) = \mathbf{A} \cdot \mathbf{B} + i |\mathbf{A} \times \mathbf{B}| \]

\[ \mathbf{A} \cdot \mathbf{B} = |A||B| \cos(\theta_B - \theta_A) \]

\[ |\mathbf{A} \times \mathbf{B}| = |A||B| \sin(\theta_B - \theta_A) \]

\[ \mathbf{A} \cdot \mathbf{B} = |A| \cos \theta_A |B| \cos \theta_B + |A| \sin \theta_A |B| \sin \theta_B \]

\[ = A_x B_x + A_y B_y \]

\[ |\mathbf{A} \times \mathbf{B}| = |A| \cos \theta_A |B| \sin \theta_B - |A| \sin \theta_A |B| \cos \theta_B \]

\[ = A_x B_y - A_y B_x \]
What Good Are Complex Exponentials? (contd.)

6. Complex derivative contains “divergence” ($\nabla \cdot \mathbf{F}$) and “curl” ($\nabla \times \mathbf{F}$) of 2D vector field

Relation of $(z,z^*)$ to $(x=\text{Re}z,y=\text{Im}z)$ defines a $z$-derivative $\frac{df}{dz}$ and “star” $z^*$-derivative. $\frac{df}{dz^*}$

\[
\begin{align*}
z &= x + iy & x &= \frac{1}{2} \left( z + z^* \right) & d\frac{f}{dz} &= \frac{\partial x}{\partial z} \frac{df}{dx} + \frac{\partial y}{\partial z} \frac{df}{dy} = \frac{1}{2} \frac{df}{dx} - i \frac{df}{dy} \\
z^* &= x - iy & y &= \frac{1}{2i} \left( z - z^* \right) & d\frac{f}{dz^*} &= \frac{\partial x}{\partial z^*} \frac{df}{dx} + \frac{\partial y}{\partial z^*} \frac{df}{dy} = \frac{1}{2} \frac{df}{dx} + i \frac{df}{dy}
\end{align*}
\]

Applying chain-rule
What Good Are Complex Exponentials? (contd.)

6. Complex derivative contains “divergence”($\nabla \cdot \mathbf{F}$) and “curl”($\nabla \times \mathbf{F}$) of 2D vector field

Relation of $(z,z^*)$ to $(x=\text{Re} z, y=\text{Im} z)$ defines a $z$-derivative $\frac{df}{dz}$ and “star” $z^*$-derivative. $\frac{df}{dz^*}$

\[
\begin{align*}
z &= x + iy & x &= \frac{1}{2} (z + z^*) \\
z^* &= x - iy & y &= \frac{1}{2i} (z - z^*) \\
\end{align*}
\]

Derivative chain-rule shows real part of $\frac{df}{dz}$ has 2D divergence $\nabla \cdot \mathbf{f}$ and imaginary part has curl $\nabla \times \mathbf{f}$.

\[
\frac{df}{dz} = \frac{d}{dz} \left( f_x + if_y \right) = \frac{1}{2} \left( \frac{\partial f_x}{\partial x} - i \frac{\partial f_y}{\partial y} \right) (f_x + if_y) = \frac{1}{2} \left( \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} \right) + i \left( \frac{\partial f_x}{\partial x} - \frac{\partial f_y}{\partial y} \right) = \frac{1}{2} \nabla \cdot \mathbf{f} + i \frac{1}{2} |\nabla \times \mathbf{f}|_{z \perp (x,y)}
\]
What Good Are Complex Exponentials? (contd.)

6. Complex derivative contains “divergence” ($\nabla \cdot \mathbf{F}$) and “curl” ($\nabla \times \mathbf{F}$) of 2D vector field

Relation of $(z, z^*)$ to $(x=\text{Re} z, y=\text{Im} z)$ defines a $z$-derivative $\frac{df}{dz}$ and “star” $z^*$-derivative. $\frac{df}{dz^*}$

\[
\begin{align*}
z &= x + iy \\
z^* &= x - iy
\end{align*}
\]

Derivative chain-rule shows real part of $\frac{df}{dz}$ has 2D divergence $\nabla \cdot \mathbf{f}$ and imaginary part has curl $\nabla \times \mathbf{f}$.

\[
\begin{align*}
\frac{df}{dz} &= \frac{\partial f_x}{\partial x} (f_x + i f_y) = \frac{1}{2} \left(\frac{\partial f_x}{\partial x} - i \frac{\partial f_y}{\partial y}\right) (f_x + i f_y) = \frac{1}{2} \left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right) + i \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) = \frac{1}{2} \nabla \cdot \mathbf{f} + i \frac{1}{2} \nabla \times \mathbf{f}
\end{align*}
\]

7. Invent source-free 2D vector fields [$\nabla \cdot \mathbf{F} = 0$ and $\nabla \times \mathbf{F} = 0$]

We can invent source-free 2D vector fields that are both zero-divergence and zero-curl. Take any function $f(z)$, conjugate it (change all $i$’s to $-i$) to give $f^*(z^*)$ for which $\frac{df^*}{dz} = 0$. 

Friday, March 9, 2012
We can invent source-free 2D vector fields that are both zero-divergence and zero-curl.

Take any function \( f(z) \), conjugate it (change all \( i \)'s to \(-i\)) to give \( f^*(z^*) \) for which \( \frac{df}{dz^*} = 0 \). Derivative chain-rule shows real part of \( \frac{df}{dz} \) has 2D divergence \( \nabla \cdot \mathbf{f} \) and imaginary part has curl \( \nabla \times \mathbf{f} \).

Relation of \((z,z^*)\) to \((x=\text{Re}z, y=\text{Im}z)\) defines a \( z \)-derivative \( \frac{df}{dz} \) and "star" \( z^* \)-derivative. 

\[
\frac{df}{dz} = \frac{\partial x}{\partial z} \frac{df}{dx} + \frac{\partial y}{\partial z} \frac{df}{dy} = \frac{1}{2} \frac{\partial f}{\partial x} - \frac{i}{2} \frac{\partial f}{\partial y} \\
\frac{df}{dz^*} = \frac{\partial x}{\partial z^*} \frac{df}{dx} + \frac{\partial y}{\partial z^*} \frac{df}{dy} = \frac{1}{2} \frac{\partial f}{\partial x} + \frac{i}{2} \frac{\partial f}{\partial y}
\]

Derivative chain-rule shows real part of \( \frac{df}{dz} \) has 2D divergence \( \nabla \cdot \mathbf{f} \) and imaginary part has curl \( \nabla \times \mathbf{f} \).

For example: if \( f(z) = a \cdot z \) then \( f^*(z^*) = a \cdot z^* = a(x-iy) \) is not function of \( z \) so it has zero \( z \)-derivative. 

\[
\mathbf{F} = (F_x, F_y) = (f^*_x, f^*_y) = (a \cdot x, -a \cdot y) \text{ has zero divergence: } \nabla \cdot \mathbf{F} = 0 \text{ and has zero curl: } |\nabla \times \mathbf{F}| = 0.
\]
What Good Are Complex Exponentials? (contd.)

7. (contd.) Invent source-free 2D vector fields \([\nabla \cdot \mathbf{F} = 0 \text{ and } \nabla \times \mathbf{F} = 0]\)

We can invent \textit{source-free 2D vector fields} that are both \textit{zero-divergence} and \textit{zero-curl}.

Take any function \(f(z)\), conjugate it (change all \(i\)'s to \(-i\)) to give \(f^*(z^*)\) for which \(\frac{df^*}{dz} = 0\).

For example, if \(f(z) = a \cdot z\) then \(f^*(z^*) = a \cdot z^* = a(x - iy)\) is not function of \(z\) so it has \textit{zero z-derivative}.

\[\mathbf{F} = (F_x, F_y) = (f^*_x, f^*_y) = (a \cdot x, -a \cdot y)\] has \textit{zero divergence}: \(\nabla \cdot \mathbf{F} = 0\) and has \textit{zero curl}: \(|\nabla \times \mathbf{F}| = 0\).

\[\mathbf{F} = (f^*_x, f^*_y) = (a \cdot x, -a \cdot y)\] is a \textit{divergence-free laminar (DFL) field}. 

precursor to
Unit 1
Fig. 10.7
8. Complex potential \( \phi \) contains “scalar”\((F=\nabla \Phi)\) and “vector”\((F=\nabla \times A)\) potentials

Any DFL field \( F \) is a gradient of a scalar potential field \( \Phi \) or a curl of a vector potential field \( A \).

\[
F = \nabla \Phi \quad \quad F = \nabla \times A
\]

A complex potential \( \phi(z) = \Phi(x,y) + iA(x,y) \) exists whose \( z \)-derivative is \( f(z) = d\phi/dz \).
Its complex conjugate \( \phi^*(z^*) = \Phi(x,y) - iA(x,y) \) has \( z^* \)-derivative \( f^*(z^*) = d\phi^*/dz^* \) giving DFL field \( F \).
8. Complex potential $\phi$ contains “scalar” ($F = \nabla \Phi$) and “vector” ($F = \nabla \times A$) potentials

Any DFL field $F$ is a gradient of a scalar potential field $\Phi$ or a curl of a vector potential field $A$.

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A complex potential $\phi(z) = \Phi(x,y) + iA(x,y)$ exists whose $z$-derivative is $f(z) = d\phi/dz$.

Its complex conjugate $\phi^*(z^*) = \Phi(x,y) - iA(x,y)$ has $z^*$-derivative $f^*(z^*) = d\phi^*/dz^*$ giving DFL field $F$.

To find $\phi = \Phi + iA$ integrate $f(z) = a \cdot z$ to get $\phi$ and isolate real ($\text{Re}\phi = \Phi$) and imaginary ($\text{Im}\phi = A$) parts.

\[
\phi = \Phi + iA = \int f \cdot dz = \int az \cdot dz = \frac{1}{2} az^2 = \frac{1}{2} a(x + iy)^2
\]

\[
= \frac{1}{2} a(x^2 - y^2) + iaxy
\]
What Good Are Complex Exponentials? (contd.)

8. Complex potential \( \phi \) contains “scalar” \((F=\nabla \Phi)\) and “vector” \((F=\nabla \times \mathbf{A})\) potentials

Any DFL field \( F \) is a gradient of a scalar potential field \( \Phi \) or a curl of a vector potential field \( \mathbf{A} \).

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A complex potential \( \phi(z) = \Phi(x,y) + i\mathbf{A}(x,y) \) exists whose \( z \)-derivative is \( f(z) = d\phi/dz \).

Its complex conjugate \( \phi^*(z^*) = \Phi(x,y) - i\mathbf{A}(x,y) \) has \( z^* \)-derivative \( f^*(z^*) = d\phi^*/dz^* \) giving DFL field \( F \).

To find \( \phi = \Phi + i\mathbf{A} \) integrate \( f(z) = a \cdot z \) to get \( \phi \) and isolate real \( (\text{Re} \phi = \Phi) \) and imaginary \( (\text{Im} \phi = \mathbf{A}) \) parts.

\[
\phi = \Phi + i \left( \frac{1}{2} a(x^2 - y^2) + iaxy \right)
\]
8. Complex potential \( \phi \) contains "scalar"(\( F=\nabla \Phi \)) and "vector"(\( F=\nabla \times \mathbf{A} \)) potentials

Any DFL field \( \mathbf{F} \) is a gradient of a scalar potential field \( \Phi \) or a curl of a vector potential field \( \mathbf{A} \).

\[
    \mathbf{F} = \nabla \Phi \\
    \mathbf{F} = \nabla \times \mathbf{A}
\]

A complex potential \( \phi(z) = \Phi(x,y) + iA(x,y) \) exists whose \( z \)-derivative is \( f(z) = d\phi/dz \).

Its complex conjugate \( \phi^*(z*) = \Phi(x,y) - iA(x,y) \) has \( z^* \)-derivative \( f^*(z*) = d\phi^*/dz^* \) giving DFL field \( \mathbf{F} \).

To find \( \phi = \Phi + iA \) integrate \( f(z) = a \cdot z \) to get \( \phi \) and isolate real (\( \text{Re} \phi = \Phi \)) and imaginary (\( \text{Im} \phi = A \)) parts.

\[
    \phi = \Phi + iA = \int f \cdot dz = \int az \cdot dz = \frac{1}{2} a z^2 = \frac{1}{2} a(x + iy)^2
\]

**BONUS!**
Get a free coordinate system!

The \((\Phi, A)\) grid is a GCC coordinate system*:

\( q^1 = \Phi = (x^2 - y^2)/2 = \text{const.} \)

\( q^2 = A = (xy) = \text{const.} \)

*Actually it’s OCC.
8. (contd.) Complex potential $\phi$ contains "scalar"($F=\nabla \Phi$) and "vector"($F=\nabla \times A$) potentials

...and either one (or half-n'-'half!) works just as well.

Derivative $\frac{d\phi^*}{dz^*}$ has 2D gradient $\nabla \Phi = \begin{pmatrix} \frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial y} \end{pmatrix}$ of scalar $\Phi$ and curl $\nabla \times A = \begin{pmatrix} \frac{\partial A}{\partial y} \\ -\frac{\partial A}{\partial x} \end{pmatrix}$ of vector $A$ (and they’re equal!)

$$\frac{d}{dz^*} \phi^* \frac{d}{dz^*} (\Phi - iA) = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (\Phi - iA) = \frac{1}{2} \left( \frac{\partial \Phi}{\partial x} + i \frac{\partial \Phi}{\partial y} \right) + \frac{1}{2} \left( \frac{\partial A}{\partial y} - i \frac{\partial A}{\partial x} \right) = \frac{1}{2} \nabla \Phi + \frac{1}{2} \nabla \times A$$
What Good Are Complex Exponentials? (contd.)

8. (contd.) Complex potential $\phi$ contains “scalar”($F=\nabla \Phi$) and “vector”($F=\nabla \times A$) potentials
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$$\frac{d}{dz^*} \phi^* = \frac{d}{dz^*} (\Phi - iA) = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (\Phi - iA) = \frac{1}{2} \left( \frac{\partial \Phi}{\partial x} + i \frac{\partial \Phi}{\partial y} \right) + \frac{1}{2} \left( \frac{\partial A}{\partial y} - i \frac{\partial A}{\partial x} \right) = \frac{1}{2} \nabla \Phi + \frac{1}{2} \nabla \times A$$

Note, mathematician definition of force field $F=+\nabla \Phi$ replaces usual physicist’s definition $F=-\nabla \Phi$
8. (contd.) Complex potential $\phi$ contains “scalar” ($F = \nabla \phi$) and “vector” ($F = \nabla \times A$) potentials …and either one (or half-n’-half!) works just as well.

Derivative $\frac{d\phi^*}{dz^*}$ has 2D gradient $\nabla \phi = \begin{pmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \end{pmatrix}$ of scalar $\Phi$ and curl $\nabla \times A = \begin{pmatrix} \frac{\partial A}{\partial y} \\ -\frac{\partial A}{\partial x} \end{pmatrix}$ of vector $A$ (and they’re equal!)

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Note, mathematician definition of force field $F = +\nabla \Phi$ replaces usual physicist’s definition $F = -\nabla \Phi$

Given $\phi$: $\text{find:}$

$$\nabla \Phi = \begin{pmatrix} \frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} a(x^2 - y^2) \\ \frac{\partial}{\partial y} a(x^2 - y^2) \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = F$$

or find:

$$\nabla \times A = \begin{pmatrix} \frac{\partial A}{\partial y} \\ -\frac{\partial A}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y} axy \\ -\frac{\partial}{\partial x} axy \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = F$$

The half-n’-half result
8. (contd.) *Complex potential* $\phi$ contains “scalar” ($F = \nabla \Phi$) and “vector” ($F = \nabla \times A$) potentials
...and either one (or half-n'-half!) works just as well.

Derivative $\frac{d\phi^*}{dz^*}$ has 2D gradient $\nabla \Phi = \begin{pmatrix} \frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial y} \end{pmatrix}$ of scalar $\Phi$ and curl $\nabla \times A = \begin{pmatrix} \frac{\partial A}{\partial y} \\ -\frac{\partial A}{\partial x} \end{pmatrix}$ of vector $A$ (and they’re equal!)

$$\frac{d}{dz^*} \phi^* = \frac{d}{dz^*} (\Phi - iA) = \frac{1}{2} \left( \frac{\partial \Phi}{\partial x} + i \frac{\partial \Phi}{\partial y} \right) (\Phi - iA) = \frac{1}{2} \left( \frac{\partial \Phi}{\partial x} + i \frac{\partial \Phi}{\partial y} \right) + \frac{1}{2} \left( \frac{\partial A}{\partial y} - i \frac{\partial A}{\partial x} \right) = \frac{1}{2} \nabla \Phi + \frac{1}{2} \nabla \times A$$

Note, mathematician definition of force field $F = +\nabla \Phi$ replaces usual physicist’s definition $F = -\nabla \Phi$

**Given $\phi$:**

$$\phi = \Phi + iA$$

$$= \frac{1}{2} a(x^2 - y^2) + iaxy$$

or find:

$$\nabla \times A = \begin{pmatrix} \frac{\partial A}{\partial y} \\ -\frac{\partial A}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} axy \\ -\frac{\partial}{\partial y} axy \end{pmatrix} = \begin{pmatrix} ax \\ ay \end{pmatrix} = F$$

**Scalar static potential lines** $\Phi = \text{const.}$ and vector flux potential lines $A = \text{const.}$ define DFL field-net.
Derivative $\frac{d\phi^*}{dz^*}$ has 2D gradient $\nabla \Phi = \left( \frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y} \right)$ of scalar $\Phi$ and curl $\nabla \times A = \left( \frac{\partial A}{\partial y}, -\frac{\partial A}{\partial x} \right)$ of vector $A$ (and they’re equal!) The half-n’-half result

$$\frac{d\phi^*}{dz^*} = \frac{d}{dz^*} (\Phi - iA) = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (\Phi - iA) = \frac{1}{2} \left( \frac{\partial \Phi}{\partial x} + i \frac{\partial \Phi}{\partial y} \right) + \frac{1}{2} \left( \frac{\partial A}{\partial y} - i \frac{\partial A}{\partial x} \right) = \frac{1}{2} \nabla \Phi + \frac{1}{2} \nabla \times A$$

Note, mathematician definition of force field $F=+\nabla \Phi$ replaces usual physicist’s definition $F=-\nabla \Phi$

Scalar static potential lines $\Phi=\text{const.}$ and vector flux potential lines $A=\text{const.}$ define DFL field-net.

The half-n’-half results are called Riemann-Cauchy Derivative Relations

$$\frac{\partial \Phi}{\partial x} = \frac{\partial A}{\partial y} \text{ is: } \frac{\partial \text{Re} f(z)}{\partial x} = \frac{\partial \text{Im} f(z)}{\partial y}$$

$$\frac{\partial \Phi}{\partial y} = -\frac{\partial A}{\partial x} \text{ is: } \frac{\partial \text{Re} f(z)}{\partial y} = -\frac{\partial \text{Im} f(z)}{\partial x}$$
9. Complex integrals $\int f(z)dz$ count 2D “circulation”($\int \mathbf{F} \cdot d\mathbf{r}$) and “flux”($\int \mathbf{F} \times d\mathbf{r}$)

Integral of $f(z)$ between point $z_1$ and point $z_2$ is potential difference $\Delta \phi = \phi(z_2) - \phi(z_1)$

$$\Delta \phi = \phi(z_2) - \phi(z_1) = \int_{z_1}^{z_2} f(z)dz = \Phi(x_2, y_2) - \Phi(x_1, y_1) + i[A(x_2, y_2) - A(x_1, y_1)]$$

In $DFL$ field $\mathbf{F}$, $\Delta \phi$ is independent of the integration path $z(t)$ connecting $z_1$ and $z_2$. 