Lecture 18
Thur. 3.01 - Mon 3.05.2012

Lagrangian and Hamiltonian dynamics:
Living with duality in GCC cells and vectors Part III.
(Ch. 12 of Unit 1)

0. Discussion of trajectory-contact-envelope problems/midterm exam

2. Examples of Hamiltonian dynamics and phase plots
   1D Pendulum and phase plot (Simulation)
   Phase control (Simulation)

3. Exploring phase space and Lagrangian mechanics more deeply
   A weird “derivation” of Lagrange’s equations
   Poincare identity and Action
   How Classicists might have “derived” quantum equations
   Huygen’s contact transformations enforce minimum action
   How to do quantum mechanics if you only know classical mechanics

Monday, March 5, 2012
Say $\alpha=90^\circ$ path rises to 1.0 then drops. When at $y=1.0$...
Q1. ...where is its focus?
Q2. ...where is the blast wave?
Q3. ...how high can $\alpha=45^\circ$ path path rise?
Say $\alpha=90^\circ$ path rises to 1.0 then drops. When at $y=1.0$
Q1. ...where is its focus?
Q2. ...where is the blast wave?
Q3. How high can $\alpha=45^\circ$ path rise?
Q4. Where on $x$-axis does $\alpha=45^\circ$ path hit?
Say $\alpha=90^\circ$ path rises to 1.0 then drops. When at $y=1.0$...

Q1. Where is its focus?
Q2. Where is the blast wave center falls as far as $50^\circ$ ball rise?
Q3. How high can $\alpha=45^\circ$ path rise? $1/2$ as high
Q4. Where on $x$-axis does $\alpha=45^\circ$ path hit? $x=2$
Q5. Where is blast wave then?
Q6. Where is $\alpha=45^\circ$ path focus?
Q7. Guess for all-path envelope and its focus? directrix?
Say $\alpha=90^\circ$ path rises to 1.0 then drops. When at $y=1.0$...
Q1. ...where is its focus?
Q2. ...where is the blast wave? center falls as far as $90^\circ$ ball rises.
Q3. How high can $\alpha=45^\circ$ path rise? $1/2$ as high.
Q4. Where on $x$-axis does $\alpha=45^\circ$ path hit? $x=2$
Q5. Where is blast wave then? centered on $45^\circ$ normal
Q6 Where is $\alpha=45^\circ$ path focus? $x=1$, $y=0$
Q7 Guess for all-path envelope and its focus? directrix?
Q7 Where is $\alpha=45^\circ$ “kite” geometry?
Q8 Where is $\alpha=0^\circ$ path focus? directrix?
Say $\alpha=90^\circ$ path rises to 1.0 then drops. When at $y=1.0$...

Q1. ...where is its focus?
Q2. ...where is the blast wave? center falls as far as 90\° ball rise.
Q3. How high can $\alpha=45^\circ$ path rise? $\frac{1}{2}$ as high.
Q4. Where on $x$-axis does $\alpha=45^\circ$ path hit? $x=2$
Q5. Where is blast wave then? centered on 45\° normal.
Q6 Where is $\alpha=45^\circ$ path focus? $x=1, y=0$
Q7 Guess for all-path envelope? and its focus? directrix?
Q7 Where is $\alpha=45^\circ$ “kite” geometry?
Q8 Where is $\alpha=0^\circ$ path focus? directrix?
Say \( \alpha=90^\circ \) path rises to \( 1.0 \) then drops. When at \( y=1.0 \)...
Q1. ....where is its focus?
Q2. ....where is the blast wave?
Q3. How high can \( \alpha=45^\circ \) path rise?
Q4. Where on x-axis does \( \alpha=45^\circ \) path hit?
Q5. Where is blast wave then?
Q6 Where is \( \alpha=45^\circ \) path focus?
Q7 Guess for all-path envelope and its focus’ directrix?
Q7 Where is \( \alpha=45^\circ \) “kite” geometry?
Q8 Where is \( \alpha=0^\circ \) path focus?
Say $\alpha=90^\circ$ path rises to 1.0 then drops. When at $y=1.0$...

Q1. Where is its focus?
Q2. Where is the blast wave?
Q3. How high can $\alpha=45^\circ$ path rise?
Q4. Where on $x$-axis does $\alpha=45^\circ$ path end?
Q5. Where is blast wave then?
Q6. Where is $\alpha=45^\circ$ path focus?
Q7. Guess for all-path envelope and its focus’ directrix?
Q8. Where is $\alpha=45^\circ$ “kite” geometry?
Exploding-starlet elliptical envelope and contacting elliptical trajectories
2. Examples of Hamiltonian dynamics and phase plots

1D Pendulum and phase plot (Simulation)
Phase space control (Simulation)
Lagrangian function $L = KE - PE = T - U$ where potential energy is $U(\theta) = -MgR \cos \theta$

$$L(\dot{\theta}, \theta) = \frac{1}{2} I \dot{\theta}^2 - U(\theta) = \frac{1}{2} I \dot{\theta}^2 + MgR \cos \theta$$
Lagrangian function \( L = KE - PE = T - U \) where potential energy is \( U(\theta) = -MgR\cos \theta \)

\[
L(\dot{\theta}, \theta) = \frac{1}{2} I \dot{\theta}^2 - U(\theta) = \frac{1}{2} I \dot{\theta}^2 + MgR\cos \theta
\]

Hamiltonian function \( H = KE + PE = T + U \) where potential energy is \( U(\theta) = -MgR\cos \theta \)

\[
H(p_\theta, \theta) = \frac{1}{2I} p_\theta^2 + U(\theta) = \frac{1}{2I} p_\theta^2 - MgR\cos \theta = E = \text{const.}
\]
Lagrangian function $L = KE - PE = T - U$ where potential energy is $U(\theta) = -MgR\cos\theta$

$$L(\dot{\theta}, \theta) = \frac{1}{2} I \dot{\theta}^2 - U(\theta) = \frac{1}{2} I \dot{\theta}^2 + MgR\cos\theta$$

Hamiltonian function $H = KE + PE = T + U$ where potential energy is $U(\theta) = -MgR\cos\theta$

$$H(p_\theta, \theta) = \frac{1}{2I} p_\theta^2 + U(\theta) = \frac{1}{2I} p_\theta^2 - MgR\cos\theta = E \implies p_\theta = \sqrt{2I(E + MgR\cos\theta)}$$
Example of plot of Hamilton for 1D-solid pendulum in its Phase Space \((\theta, p_\theta)\)

\[
H(p_\theta, \theta) = E = \frac{1}{2I} p_\theta^2 - MgR \cos \theta, \quad \text{or} \quad p_\theta = \sqrt{2I(E + MgR \cos \theta)}
\]
Example of plot of Hamilton for 1D-solid pendulum in its Phase Space \((\theta,p_\theta)\)

\[
H(p_\theta, \theta) = E = \frac{1}{2I} p_\theta^2 - MgR \cos \theta , \quad \text{or:} \quad p_\theta = \sqrt{2I\left(E + MgR \cos \theta\right)}
\]

Funny way to look at Hamilton’s equations:

\[
\begin{pmatrix}
\dot{q} \\
\dot{p}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial}{\partial q} H \\
-\frac{\partial}{\partial p} H
\end{pmatrix} = e_H \times (-\nabla H) = (\text{H-axis}) \times (\text{fall line}), \quad \text{where:} \begin{cases}
(H\text{-axis}) = e_H = e_q \times e_p \\
(\text{fall line}) = -\nabla H
\end{cases}
\]
2. Examples of Hamiltonian dynamics and phase plots

*1D Pendulum and phase plot (Simulation)*

*Phase control (Simulation)*
Simulation of atomic classical (or semi-classical) dynamics under varying phase control
3. Exploring phase space and Lagrangian mechanics more deeply

A weird “derivation” of Lagrange’s equations
Poincare identity and Action
How Classicists might have “derived” quantum equations
Huygen’s contact transformations enforce minimum action
How to do quantum mechanics if you only know classical mechanics
A strange “derivation” of Lagrange’s equations by Calculus of Variation

Variational calculus finds extreme (minimum or maximum) values to entire integrals

Minimize (or maximize): \( S(q) = \int_{t_0}^{t_1} dt \ L(q(t), \dot{q}(t), t) \).

An arbitrary but small variation function \( \delta q(t) \) is allowed at every point \( t \) in the figure along the curve except at the end points \( t_0 \) and \( t_1 \). There we demand it not vary at all.(1)

\[
\delta q(t_0) = 0 = \delta q(t_1)
\]

1st order \( L(q+\delta q) \) approximate:

\[
S(q + \delta q) = \int_{t_0}^{t_1} dt \left[ L(q, \dot{q}, t) + \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right] \text{ where: } \delta \dot{q} = \frac{d}{dt} \delta q
\]
Variational calculus finds extreme (minimum or maximum) values to entire integrals

\[ S(q) = \int_{t_0}^{t_1} dt \, L(q(t), \dot{q}(t), t) . \]

An arbitrary but small variation function \( \delta q(t) \) is allowed at every point \( t \) in the figure along the curve except at the end points \( t_0 \) and \( t_1 \). There we demand it not vary at all. (1)

**1st order \( L(q+\delta q) \) approximate:**

\[ S(q + \delta q) = \int_{t_0}^{t_1} dt \left[ L(q, \dot{q}, t) + \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right] \]

where: \( \delta \dot{q} = \frac{d}{dt} \delta q \)  

Replace \( \frac{\partial L}{\partial q} \delta q \) with \( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \delta q \right) - \frac{d}{dt} \left( \frac{\partial L}{\partial q} \right) \delta q \)
A weird “derivation” of Lagrange’s equations

Variational calculus finds extreme (minimum or maximum) values to entire integrals

\[
S(q) = \int_{t_0}^{t_1} dt \ L(q(t), \dot{q}(t), t).
\]

An arbitrary but small variation function \(\delta q(t)\) is allowed at every point \(t\) in the figure along the curve except at the end points \(t_0\) and \(t_1\). There we demand it not vary at all. (1)

1st order \(L(q+\delta q)\) approximate:

\[
\delta q(t_0) = 0 = \delta q(t_1)
\]

\[
S(q + \delta q) = \int_{t_0}^{t_1} dt \ \left[ L(q, \dot{q}, t) + \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right] \quad \text{where:} \quad \delta \dot{q} = \frac{d}{dt} \delta q
\]

\[
S(q + \delta q) = \int_{t_0}^{t_1} dt \ \left[ L(q, \dot{q}, t) + \frac{\partial L}{\partial q} \delta q - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \delta q \right] + \int_{t_0}^{t_1} dt \ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \delta q \right)
\]
An arbitrary but small variation function $\delta q(t)$ is allowed at every point $t$ in the figure along the curve except at the end points $t_0$ and $t_1$. There we demand it not vary at all. (1)

$S(q + \delta q)(t) = \int_{t_0}^{t_1} dt \ L(q, \dot{q}, t)$

1st order $L(q + \delta q)$ approximate:

$S(q + \delta q) = \int_{t_0}^{t_1} \left[ L(q, \dot{q}, t) + \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right] + \int_{t_0}^{t_1} \left[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \delta q \right) \right] dt$

$\Delta S = \frac{d}{dt} \left[ \frac{\partial L}{\partial \dot{q}} \delta q \right] |_{t_0}^{t_1}$
An arbitrary but small variation function $\delta q(t)$ is allowed at every point $t$ in the figure along the curve except at the end points $t_0$ and $t_1$. There we demand it not vary at all.

**1st order $L(q+\delta q)$ approximate:**

$$S(q+\delta q) = \int_{t_0}^{t_1} dt \left[ L(q,\dot{q},t) + \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right]$$

where: $\dot{q} = \frac{dq}{dt}$

Replace $\frac{\partial L}{\partial \dot{q}} \delta \dot{q}$ with $\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \delta q \right) - \frac{d}{dt} \left( \frac{\partial L}{\partial q} \right) \delta q$

Third term vanishes by (1). This leaves first order variation: $\delta S = S(q+\delta q) - S(q) = \int_{t_0}^{t_1} dt \left[ \frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \right] \delta q$

Extreme value (actually minimum value) of $S(q)$ occurs if and only if Lagrange equation is satisfied!

$$\delta S = 0 \Rightarrow \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \quad \text{Euler-Lagrange equation(s)}$$
A weird “derivation” of Lagrange’s equations

Variational calculus finds extreme (minimum or maximum) values to entire integrals

\[ S(q) = \int_{t_0}^{t_1} dt \ L(q(t), \dot{q}(t), t). \]

An arbitrary but small variation function \( \delta q(t) \) is allowed at every point \( t \) in the figure along the curve except at the end points \( t_0 \) and \( t_1 \). There we demand it not vary at all. (1)

1st order \( L(q+\delta q) \) approximate:

\[
\delta q(t_0) = 0 = \delta q(t_1)
\]

\[
S(q+\delta q) = \int_{t_0}^{t_1} dt \ \left[ L(q, \dot{q}, t) + \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right]
\]

where: \( \delta \dot{q} = \frac{d}{dt} \delta q \)

Replace \( \frac{\partial L}{\partial \dot{q}} \delta q \) with \( \frac{d}{dt} \left( \frac{\partial L}{\partial q} \delta q \right) - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \delta q \)

Third term vanishes by (1). This leaves first order variation: \( \delta S = S(q+\delta q) - S(q) = \int_{t_0}^{t_1} dt \ \left[ \frac{\partial L}{\partial q} \delta q - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \delta q \right] \)

Extreme value (actually minimum value) of \( S(q) \) occurs if and only if Lagrange equation is satisfied!

\[
\delta S = 0 \implies \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \quad \text{Euler-Lagrange equation(s)}
\]

But, WHY is nature so inclined to fly JUST SO as to minimize the Lagrangian \( L = T - U \)???
3. Exploring phase space and Lagrangian mechanics more deeply

A weird “derivation” of Lagrange’s equations

**Poincaré identity and Action**

How Classicists might have “derived” quantum equations

Huygen’s contact transformations enforce minimum action

How to do quantum mechanics if you only know classical mechanics
Legendre-Poincare identity and Action

Legendre transform $L(v) = p \cdot v - H(p)$ becomes *Poincare’s invariant differential* if $dt$ is cleared.

\[
L \cdot dt = p \cdot v \cdot dt - H \cdot dt = p \cdot dr - H \cdot dt
\]

\[
\left( v = \frac{dr}{dt} \text{ implies: } v \cdot dt = dr \right)
\]
Legendre-Poincare identity and Action

Legendre transform $L(v) = p \cdot v - H(p)$ becomes *Poincare’s invariant differential* if $dt$ is cleared.

$$L \cdot dt = p \cdot v \cdot dt - H \cdot dt = p \cdot dr - H \cdot dt$$

$$\left( v = \frac{dr}{dt} \text{ implies: } v \cdot dt = dr \right)$$

This is the time differential $dS$ of *action* $S = \int L \cdot dt$ whose time derivative is rate $L$ of *quantum phase*.

$$dS = L \cdot dt = p \cdot dr - H \cdot dt$$

where: $L = \frac{dS}{dt}$
Legendre-Poincare identity and Action

Legendre transform \( L(v) = p \cdot v - H(p) \) becomes Poincare’s invariant differential if \( dt \) is cleared.

\[
L \cdot dt = p \cdot v \cdot dt - H \cdot dt = p \cdot dr - H \cdot dt \quad \quad v = \frac{dr}{dt}
\]

This is the time differential \( dS \) of \textit{action} \( S = \int L \cdot dt \) whose time derivative is rate \( L \) of quantum phase.

\[
dS = L \cdot dt = p \cdot dr - H \cdot dt \quad \text{where:} \quad L = \frac{dS}{dt}
\]

Unit 2 shows \textit{DeBroglie law} \( p = \hbar k \) and \textit{Planck law} \( H = \hbar \omega \) make quantum plane wave phase \( \Phi \):

\[
\Phi = \frac{S}{\hbar} = \int L \cdot dt / \hbar
\]
Legendre-Poincare identity and Action

Legendre transform \( L(v) = p \cdot v - H(p) \) becomes Poincare’s invariant differential if \( dt \) is cleared.

\[
L \cdot dt = p \cdot v \cdot dt - H \cdot dt = p \cdot dr - H \cdot dt \quad \quad \quad \quad \quad v = \frac{dr}{dt}
\]

This is the time differential \( dS \) of action \( S = \int L \cdot dt \) whose time derivative is rate \( L \) of quantum phase.

\[
dS = L \cdot dt = p \cdot dr - H \cdot dt \quad \text{where:} \quad L = \frac{dS}{dt}
\]

Unit 2 shows DeBroglie law \( p = \hbar k \) and Planck law \( H = \hbar \omega \) make quantum plane wave phase \( \Phi \):

\[
\psi(r,t) = e^{iS/\hbar} = e^{i(p \cdot r - H \cdot t)/\hbar} = e^{i(k \cdot r - \omega \cdot t)}
\]

\[
\Phi = S/\hbar = \int L \cdot dt/\hbar
\]
Legendre-Poincare identity and Action

Legendre transform $L(v) = p \cdot v - H(p)$ becomes Poincare’s invariant differential if $dt$ is cleared.

\[
L \cdot dt = p \cdot v \cdot dt - H \cdot dt = p \cdot dr - H \cdot dt
\]

This is the time differential $dS$ of action $S = \int L \cdot dt$ whose time derivative is rate $L$ of quantum phase.

\[
dS = L \cdot dt = p \cdot dr - H \cdot dt \quad \text{where:} \quad L = \frac{dS}{dt}
\]

Unit 2 shows DeBroglie law $p = \hbar k$ and Planck law $H = \hbar \omega$ make quantum plane wave phase $\Phi$:

\[
\psi(r,t) = e^{iS/\hbar} = e^{i(p \cdot r - H \cdot t)/\hbar} = e^{i(k \cdot r - \omega \cdot t)}
\]

Q: When is the Action-differential $dS$ integrable?
A: Differential $dW = f_x(x,y)dx + f_y(x,y)dy$ is integrable to a $W(x,y)$ if: $f_x = \frac{\partial W}{\partial x}$ and: $f_y = \frac{\partial W}{\partial y}$
Legendre-Poincare identity and Action

Legendre transform $L(\mathbf{v}) = \mathbf{p} \cdot \mathbf{v} - H(\mathbf{p})$ becomes Poincare’s invariant differential if $dt$ is cleared.

$$L \cdot dt = \mathbf{p} \cdot \mathbf{v} \cdot dt - H \cdot dt = \mathbf{p} \cdot d\mathbf{r} - H \cdot dt \quad \mathbf{v} = \frac{d\mathbf{r}}{dt}$$

This is the time differential $dS$ of action $S = \int L \cdot dt$ whose time derivative is rate $L$ of quantum phase.

$$dS = L \cdot dt = \mathbf{p} \cdot d\mathbf{r} - H \cdot dt \quad \text{where:} \quad L = \frac{dS}{dt}$$

Unit 2 shows DeBroglie law $\mathbf{p} = \hbar \mathbf{k}$ and Planck law $H = \hbar \omega$ make quantum plane wave phase $\Phi$:

$$\psi(\mathbf{r}, t) = e^{iS/\hbar} = e^{i(\mathbf{p} \cdot \mathbf{r} - H \cdot t)/\hbar} = e^{i(\mathbf{k} \cdot \mathbf{r} - \omega \cdot t)}$$

Q: When is the Action-differential $dS$ integrable?

A: Differential $dW = f_x(x,y)dx + f_y(x,y)dy$ is integrable to a $W(x,y)$ if: $f_x = \frac{\partial W}{\partial x}$ and: $f_y = \frac{\partial W}{\partial y}$.

$\nabla \times f = 0$ or $\partial$-symmetry of $W$:

$$\frac{\partial f_x}{\partial y} = \frac{\partial^2 W}{\partial y \partial x} = \frac{\partial^2 W}{\partial x \partial y} = \frac{\partial f_y}{\partial x}$$
Legendre-Poincare identity and Action

Legendre transform  \( L(\mathbf{v}) = \mathbf{p} \cdot \mathbf{v} - H(\mathbf{p}) \) becomes \textit{Poincare’s invariant differential} if \( dt \) is cleared.

\[
L \cdot dt = \mathbf{p} \cdot \mathbf{v} \cdot dt - H \cdot dt = \mathbf{p} \cdot d\mathbf{r} - H \cdot dt
\]

\( \mathbf{v} = \frac{d\mathbf{r}}{dt} \)

This is the time differential \( dS \) of \textit{action} \( S = \int L \cdot dt \) whose time derivative is rate \( L \) of \textit{quantum phase}.

\[
dS = L \cdot dt = \mathbf{p} \cdot d\mathbf{r} - H \cdot dt \quad \text{where:} \quad L = \frac{dS}{dt}
\]

Unit 2 shows \textit{DeBroglie law} \( \mathbf{p} = \hbar \mathbf{k} \) and \textit{Planck law} \( H = \hbar \omega \) make \textit{quantum plane wave phase} \( \Phi \):

\[
\psi(\mathbf{r},t) = e^{iS/\hbar} = e^{i(\mathbf{p} \cdot \mathbf{r} - H \cdot t)/\hbar} = e^{i(\mathbf{k} \cdot \mathbf{r} - \omega \cdot t)}
\]

Q: When is the \textit{Action}-differential \( dS \) integrable?
A: Differential \( dW = f_x(x,y)dx + f_y(x,y)dy \) is \textit{integrable} to a \( W(x,y) \) if: \( f_x = \frac{\partial W}{\partial x} \) and: \( f_y = \frac{\partial W}{\partial y} \)

\( dS \) is integrable if:

\[
\frac{\partial S}{\partial \mathbf{r}} = \mathbf{p} \quad \text{and:} \quad \frac{\partial S}{\partial t} = -H
\]

These conditions are known as \textit{Jacobi-Hamilton equations}.

\[
\text{Similar to conditions for integrating work differential } dW = f \cdot d\mathbf{r} \text{ to get potential } W(\mathbf{r}). \text{ That condition is no curl allowed: } \nabla \times \mathbf{f} = 0 \text{ or } \partial \text{-symmetry of } W:
\]

\[
\frac{\partial f_x}{\partial y} = \frac{\partial^2 W}{\partial y \partial x} = \frac{\partial^2 W}{\partial x \partial y} = \frac{\partial f_y}{\partial x}
\]
3. Exploring phase space and Lagrangian mechanics more deeply
   A weird “derivation” of Lagrange’s equations
   Poincare identity and Action
   How Classicists might have “derived” quantum equations
   Huygen’s contact transformations enforce minimum action
   How to do quantum mechanics if you only know classical mechanics
How Jacobi-Hamilton could have “derived” Schrodinger equations

(Given “quantum wave”)

$$\psi(r,t) = e^{iS/\hbar} = e^{i(p \cdot r - H \cdot t)/\hbar} = e^{i(k \cdot r - \omega \cdot t)}$$

$dS$ is integrable if: $\frac{\partial S}{\partial r} = p$ and: $\frac{\partial S}{\partial t} = -H$

These conditions are known as Jacobi-Hamilton equations
How Jacobi-Hamilton could have “derived” Schrödinger equations

\[ \psi(r, t) = e^{iS/\hbar} = e^{i(p \cdot r - H \cdot t)/\hbar} = e^{i(k \cdot r - \omega \cdot t)} \]

\(dS\) is integrable if: \[ \frac{\partial S}{\partial r} = p \quad \text{and:} \quad \frac{\partial S}{\partial t} = -H \]

These conditions are known as Jacobi-Hamilton equations

Try 1st \( r \)-derivative of wave \( \psi \)

\[ \frac{\partial}{\partial r} \psi(r, t) = \frac{\partial}{\partial r} e^{iS/\hbar} = \frac{\partial}{\partial r} \left( \frac{iS}{\hbar} \right) e^{iS/\hbar} = \left( \frac{i}{\hbar} \right) \frac{\partial S}{\partial r} \psi(r, t) \]

\[ \frac{\partial}{\partial r} \psi(r, t) = \left( \frac{i}{\hbar} \right) p \psi(r, t) \quad \text{or:} \quad \frac{\hbar}{i} \frac{\partial}{\partial r} \psi(r, t) = p \psi(r, t) \]
How Jacobi-Hamilton could have “derived” Schrodinger equations

(Given “quantum wave”)

\[ \psi(r,t) = e^{iS/\hbar} = e^{i(p \cdot r - H \cdot t)/\hbar} = e^{i(k \cdot r - \omega \cdot t)} \]

\[ dS \text{ is integrable if: } \frac{\partial S}{\partial r} = p \quad \text{and:} \quad \frac{\partial S}{\partial t} = -H \]

These conditions are known as Jacobi-Hamilton equations

Try 1\textsuperscript{st} \textit{r}-derivative of wave $\psi$

\[ \frac{\partial}{\partial r} \psi(r,t) = \frac{\partial}{\partial r} e^{iS/\hbar} = \frac{\partial (iS/\hbar)}{\partial r} e^{iS/\hbar} = \left(\frac{i}{\hbar}\right) \frac{\partial S}{\partial r} \psi(r,t) \]

\[ \frac{\partial}{\partial r} \psi(r,t) = \left(\frac{i}{\hbar}\right) p \psi(r,t) \quad \text{or:} \quad \frac{\hbar}{i} \frac{\partial}{\partial r} \psi(r,t) = p \psi(r,t) \]

Try 1\textsuperscript{st} \textit{t}-derivative of wave $\psi$

\[ \frac{\partial}{\partial t} \psi(r,t) = \frac{\partial}{\partial t} e^{iS/\hbar} = \frac{\partial (iS/\hbar)}{\partial t} e^{iS/\hbar} = \left(\frac{i}{\hbar}\right) \frac{\partial S}{\partial t} \psi(r,t) \]

\[ \frac{\partial}{\partial t} \psi(r,t) = \left(\frac{i}{\hbar}\right) (-H) \psi(r,t) \quad \text{or:} \quad i\hbar \frac{\partial}{\partial t} \psi(r,t) = H \psi(r,t) \]
3. Exploring phase space and Lagrangian mechanics more deeply
   A weird “derivation” of Lagrange’s equations
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   How to do quantum mechanics if you only know classical mechanics
Huygen’s contact transformations enforce minimum action

Each point $\mathbf{r}_k$ on a wavefront “broadcasts” in all directions.
Only **minimum action** path interferes constructively

**Time-independent action**

(Hamilton’s *reduced action*)

$$S_H = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{p} \cdot d\mathbf{r}$$

is a purely spatial integral.

**Time-dependent action**

(Hamilton’s *principle action*)

$$S_p = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \left( \mathbf{p} \cdot d\mathbf{r} - H \cdot dt \right)$$

is a space-time integral.

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Fig. 12.12
Huygen’s contact transformations enforce minimum action

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**Time-independent action** (Hamilton’s *reduced action*) is a purely spatial integral.

$$S_H = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{p} \cdot d\mathbf{r}$$

$$S_H(\mathbf{r}_0: \mathbf{r}) = \begin{cases} 30 & \text{for } \mathbf{r}_1 \text{ to } \mathbf{r}_2 \\ 20 & \text{for } \mathbf{r}_0 \text{ to } \mathbf{r}_1 \text{ and } \mathbf{r}_2 \text{ to } \mathbf{r}_3 \end{cases}$$

**Time-dependent action** (Hamilton’s *principle action*) is space-time integral.

$$S_p = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \left( \mathbf{p} \cdot d\mathbf{r} - H \cdot dt \right)$$

$$\langle \mathbf{r}_1 | \mathbf{r}_0 \rangle = e^{i \frac{S_H(\mathbf{r}_0: \mathbf{r})}{\hbar}}$$

...because action is quantum wave phase

**Feynman’s path-sum closure relation**

$$\sum_{\mathbf{r}'} \langle \mathbf{r}_1 | \mathbf{r}' \rangle \langle \mathbf{r}' | \mathbf{r}_0 \rangle \cong \sum_{\mathbf{r}'} e^{i \frac{S_H(\mathbf{r}_0: \mathbf{r}') + S_H(\mathbf{r}': \mathbf{r}_1)}{\hbar}} = e^{i \frac{S_H(\mathbf{r}_0: \mathbf{r}_1)}{\hbar}} = \langle \mathbf{r}_1 | \mathbf{r}_0 \rangle$$
3. Exploring phase space and Lagrangian mechanics more deeply
   A weird “derivation” of Lagrange’s equations
   Poincare identity and Action
   How Classicists might have “derived” quantum equations
   Huygen’s contact transformations enforce minimum action

   How to do quantum mechanics if you only know classical mechanics
**Bohr quantization** requires quantum phase $s_n/h$ in amplitude to be an integral multiple $n$ of $2\pi$ after a closed loop integral $S_H(r_0 : r_0) = \int_{r_0}^{r_0} p \cdot dr$. The integer $n (n = 0, 1, 2, ...)$ is a *quantum number*.

$$l = \langle r_0 | r_0 \rangle = e^{i S_H(r_0 : r_0)/\hbar} = e^{i \Sigma_H / \hbar} = 1 \quad \text{for:} \quad \Sigma_H = 2\pi \hbar n = \hbar n$$

Numerically integrate Hamilton's equations and Lagrangian $L$. Color the trajectory according to the current accumulated value of action $S_H(0 : r)/\hbar$. Adjust energy to quantized pattern (if closed system*)

$$S_H(0 : r) = S_p(0, 0 : r, t) + Ht = \int_0^t L dt + Ht.$$
Bohr quantization requires quantum phase $S_H/\hbar$ in amplitude to be an integral multiple $n$ of $2\pi$ after a closed loop integral $S_H(r:0) = \int_{r_0}^{r} p \cdot dr$. The integer $n$ ($n = 0, 1, 2, \ldots$) is a quantum number.

$$I = \langle r_0 | r_0 \rangle = e^{i S_H(r_0:0)/\hbar} = e^{i \Sigma H/\hbar} = 1$$

for: $\Sigma H = 2\pi \hbar n = \hbar n$

Numerically integrate Hamilton's equations and Lagrangian $L$. Color the trajectory according to the current accumulated value of action $S_H(0: r)/\hbar$. Adjust energy to quantized pattern (if closed system*)

$$S_H(0: r) = S_p(0, 0: r, t) + Ht = \int_{0}^{t} L dt + Ht.$$
Bohr quantization requires quantum phase \( s_{\pi}/\hbar \) in amplitude to be an integral multiple \( n \) of \( 2\pi \) after a closed loop integral \( s_{\pi}(r_0: r_0) = \int_{r_0}^{r_0} p \cdot dr \). The integer \( n \) \( (n = 0, 1, 2, \ldots) \) is a quantum number.

\[
I = \langle r_0 | r_0 \rangle = e^{i S_{\pi}(r_0: r_0)/\hbar} = e^{i \Sigma_{H}/\hbar} = 1 \quad \text{for: } \Sigma_{H} = 2\pi \hbar n = \hbar n
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Numerically integrate Hamilton's equations and Lagrangian \( L \). Color the trajectory according to the current accumulated value of action \( S_{\pi}(0: r)/\hbar \). Adjust energy to quantized pattern (if closed system*)

\[
S_{\pi}(0: r) = S_p(0, 0: r, t) + Ht = \int_0^t L dt + Ht.
\]

The hue should represent the phase angle \( S_{\pi}(0: r)/\hbar \) modulo \( 2\pi \) as, for example, \( 0=\text{red}, \pi/4=\text{orange}, \pi/2=\text{yellow}, 3\pi/4=\text{green}, \pi=\text{cyan} \) (opposite of \text{red}), \( 5\pi/4=\text{indigo}, 3\pi/2=\text{blue}, 7\pi/4=\text{purple}, \) and \( 2\pi=\text{red} \) (full color circle). Interpolating action on a palette of 32 colors is enough precision for low quanta.

*open system has continuous energy
A moving wave has a *quantum phase velocity* found by setting $S = \text{const.}$ or $dS(0,0;r,t) = 0 = p \cdot dr - H dt$.

$$V_{\text{phase}} = \frac{dr}{dt} = \frac{H}{p} = \frac{\omega}{k}$$

**Unit 1**

Fig. 12.15

Quantum "phase wavefronts"

(a) $S_H = 0.3$

(b) $S_H = 0.35$

(c) $S_H = 0.4$

(d) $S_H = 0.9$

quantum *phase* velocity
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This is quite the opposite of classical particle velocity which is *quantum group velocity*.

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Fig. 12.15

quantum phase velocity

Note: This is Hamilton’s 1st Equation
A moving wave has a \textit{quantum phase velocity} found by setting \( S = \text{const.} \) or \( dS(0,0;r,t) = 0 = \mathbf{p} \cdot d\mathbf{r} - Hdt \).

\[
\nabla S_H = \mathbf{p}
\]

This is quite the opposite of classical particle velocity which is \textit{quantum group velocity}.

\[
V_{\text{phase}} = \frac{d\mathbf{r}}{dt} = \frac{H}{\mathbf{p}} = \frac{\omega}{\mathbf{k}}
\]

\[
V_{\text{group}} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\partial \omega}{\partial \mathbf{k}}
\]

Note: This is Hamilton’s 1\textsuperscript{st} Equation

\textit{Classical “blast wavefronts”}

\textit{Quantum “phase wavefronts”}

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(b) \( S_H = 0.35 \)

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