Lecture 18
Thur. 3.01 - Mon 3.05.2012

Lagrangian and Hamiltonian dynamics: Living with duality in GCC cells and vectors Part III.

(Ch. 12 of Unit 1)

0. Discussion of trajectory-contact-envelope problems/midterm exam ← Topic for 3.01.2012

2. Examples of Hamiltonian dynamics and phase plots ← Topics for 3.05.2012
- *1D Pendulum and phase plot (Simulation)*
- *Phase control (Simulation)*

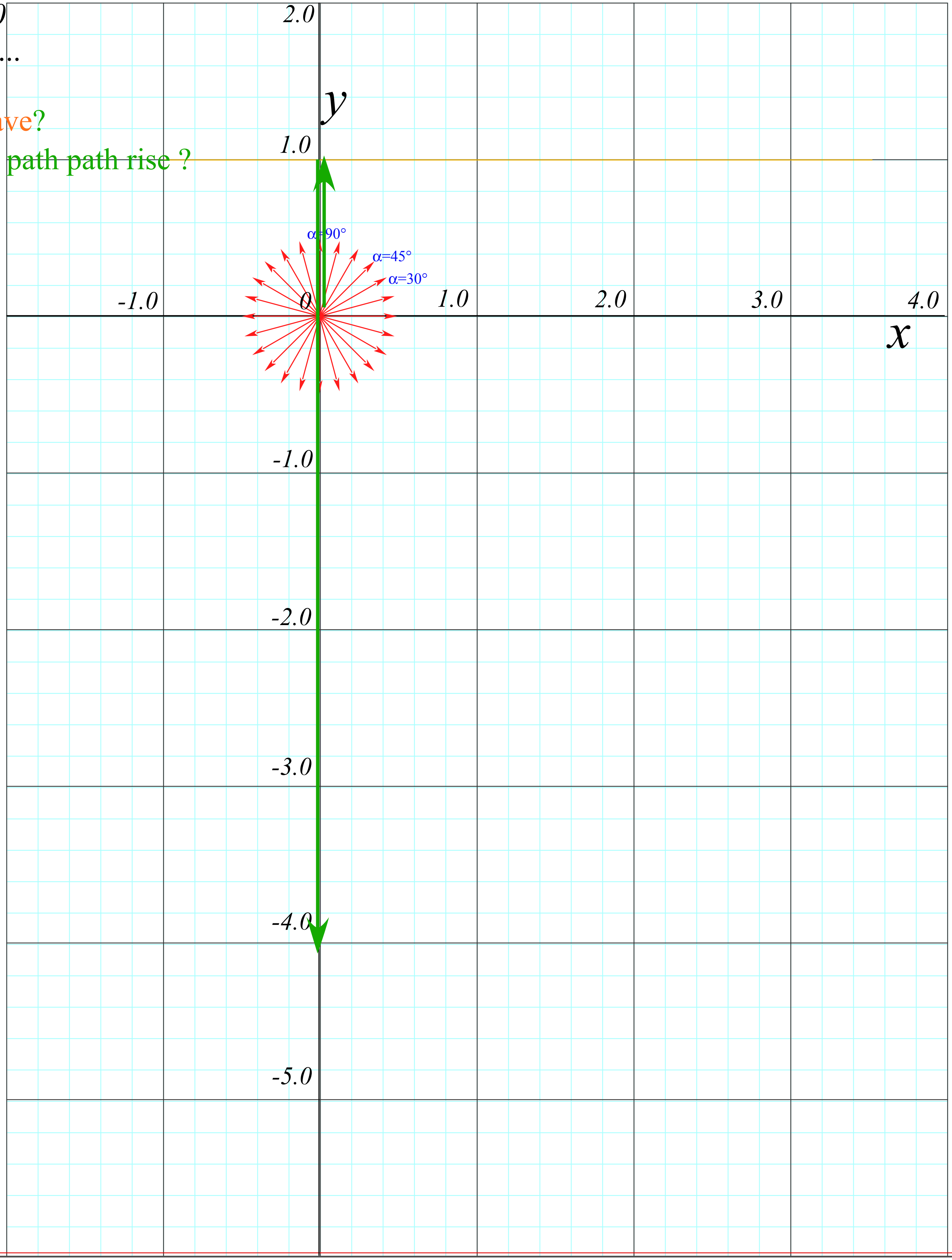
3. Exploring phase space and Lagrangian mechanics more deeply
- *A weird "derivation" of Lagrange's equations*
- *Poincare identity and Action*
- *How Classicists might have "derived" quantum equations*
- *Huygen's contact transformations enforce minimum action*
- *How to do quantum mechanics if you only know classical mechanics*

Say $\alpha=90°$ path rises to *1.0* then drops. When at *y=1.0*...

Q1. ...where is its focus?

Q2. ...where is the blast wave?

Q3. ...how high can $\alpha=45°$ path path rise ?



Say $\alpha=90°$ path rises to *1.0*
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Q1. ...where is its focus?
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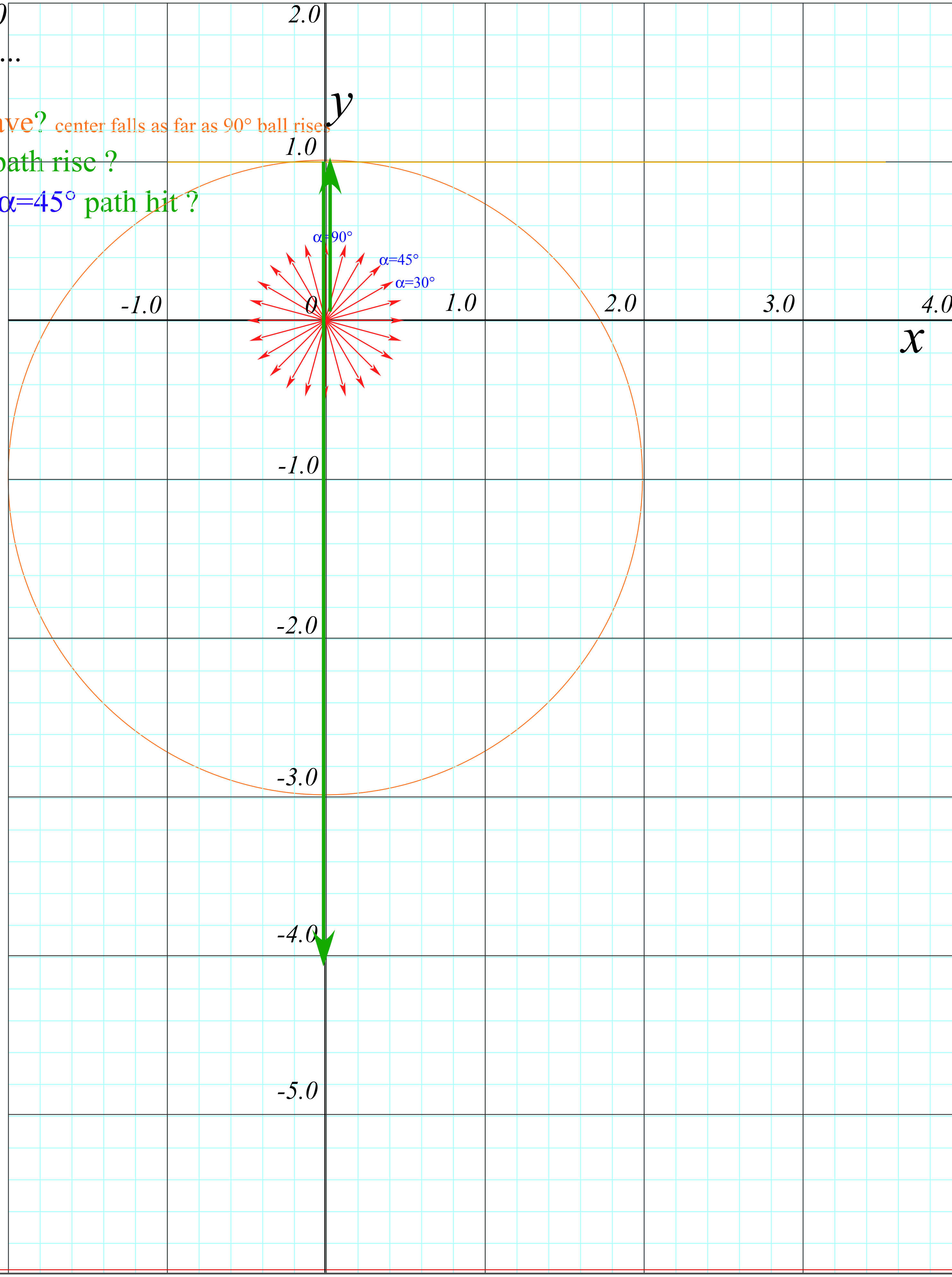
Q4. Where on *x-axis* does $\alpha=45°$ path hit ?



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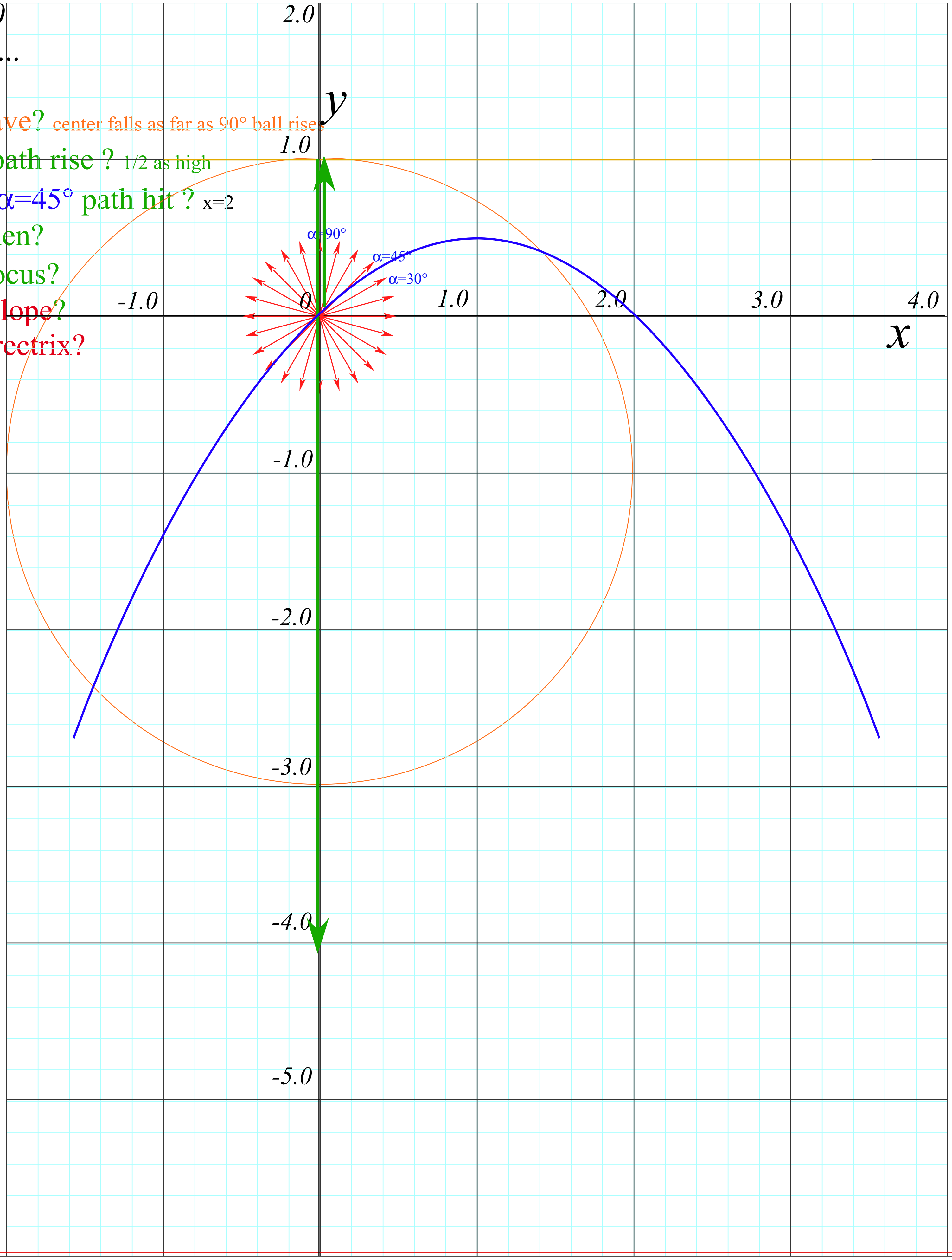
Q3. How high can α=45° path rise ? 1/2 as high
Q4. Where on *x-axis* does α=45° path hit ? x=2
Q5. Where is blast wave then?
Q6 Where is α=45° path focus?
Q7 Guess for all-path envelope?
and its focus? directrix?



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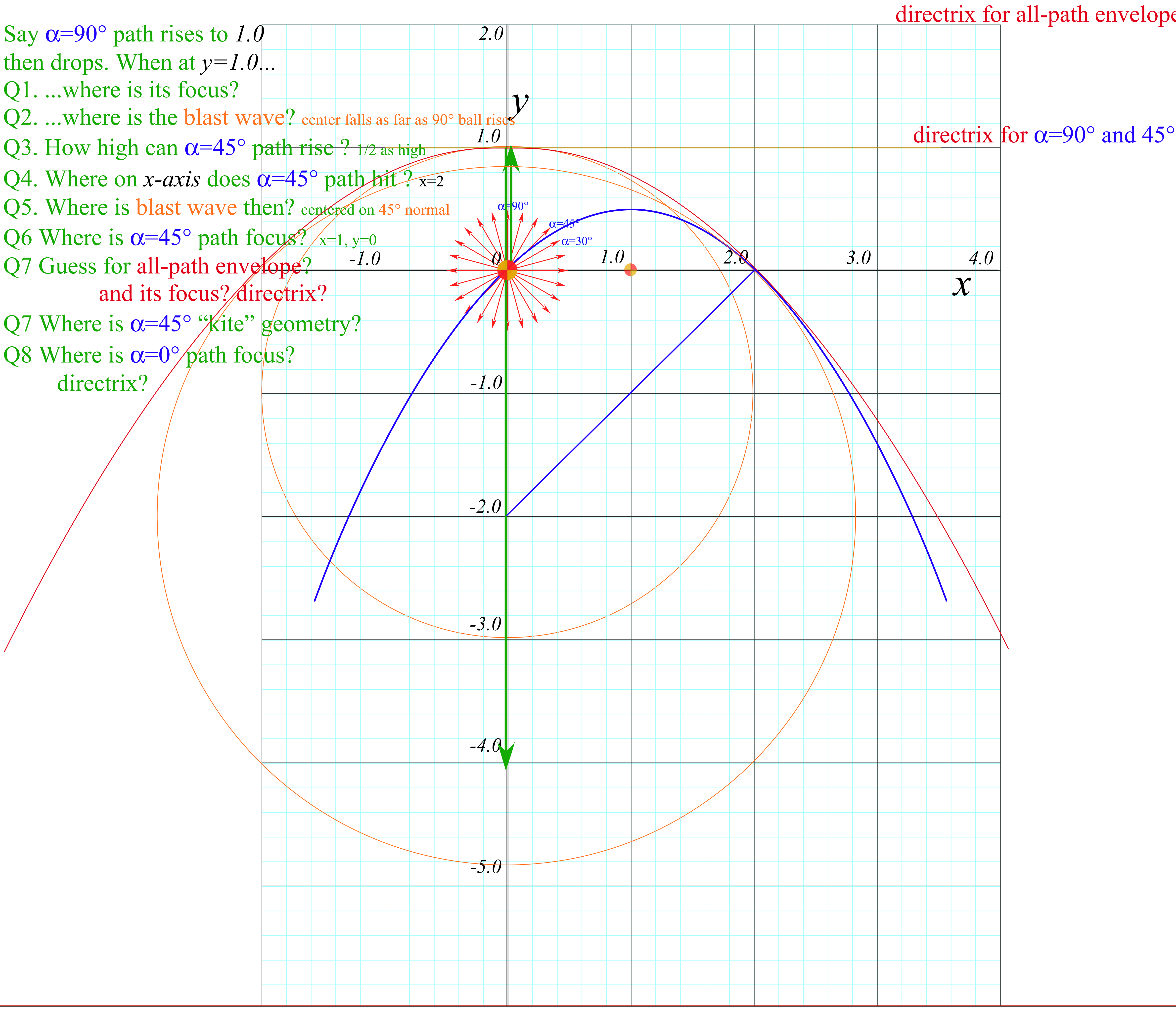
Q5. Where is blast wave then? centered on 45° normal

Q6 Where is α=45° path focus? x=1, y=0

Q7 Guess for all-path envelope? and its focus? directrix?

Q7 Where is α=45° "kite" geometry?

Q8 Where is α=0° path focus? directrix?



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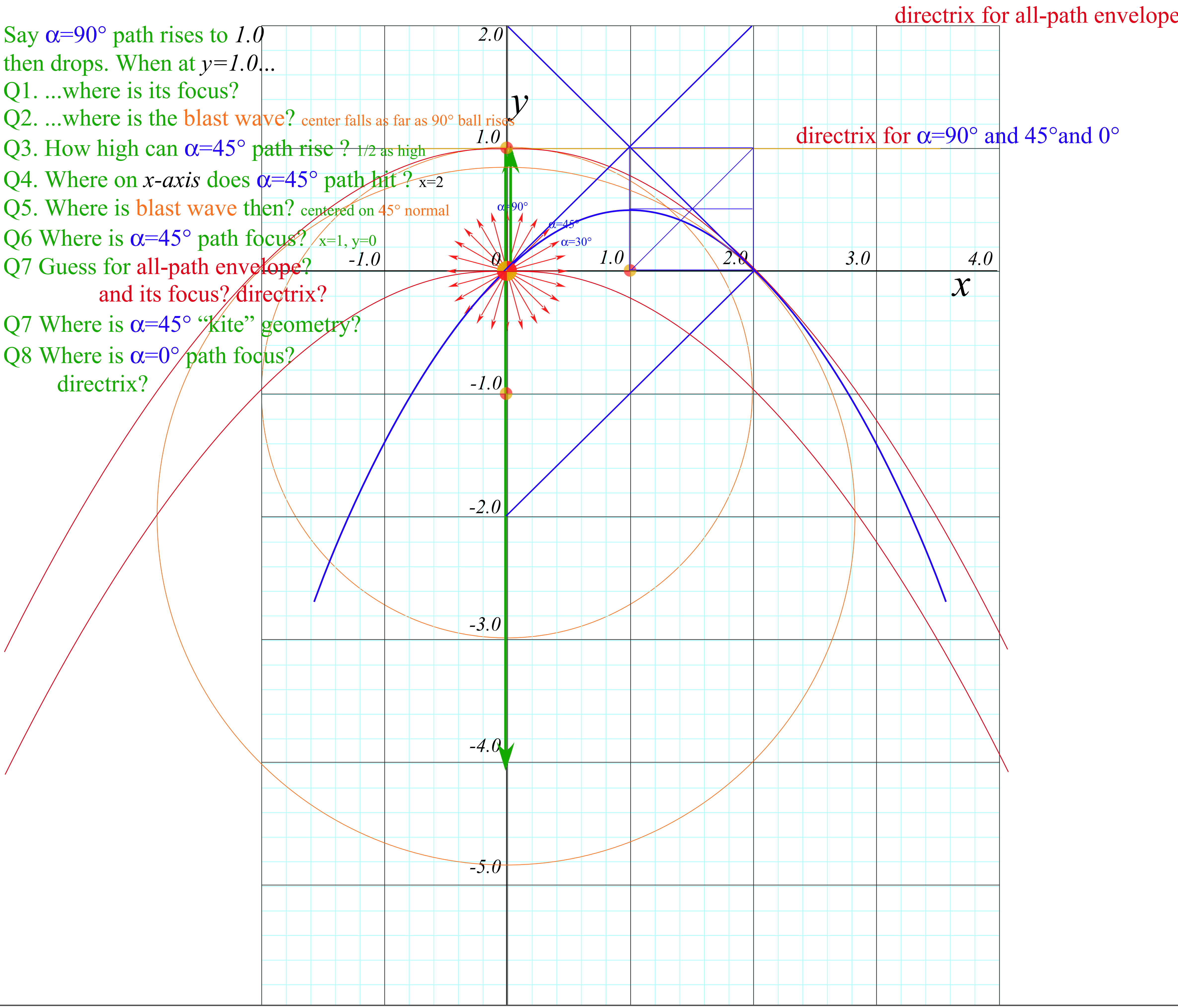

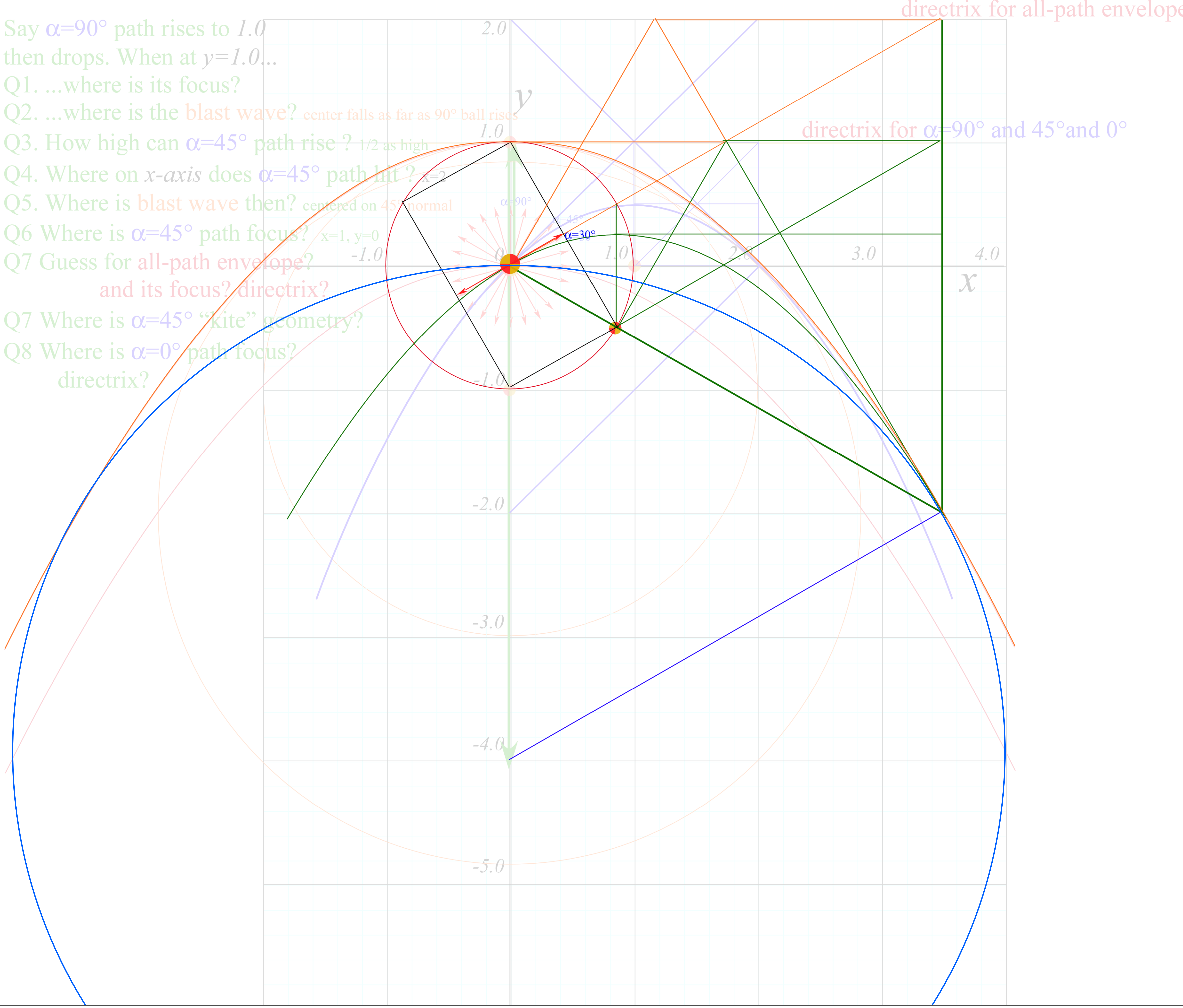
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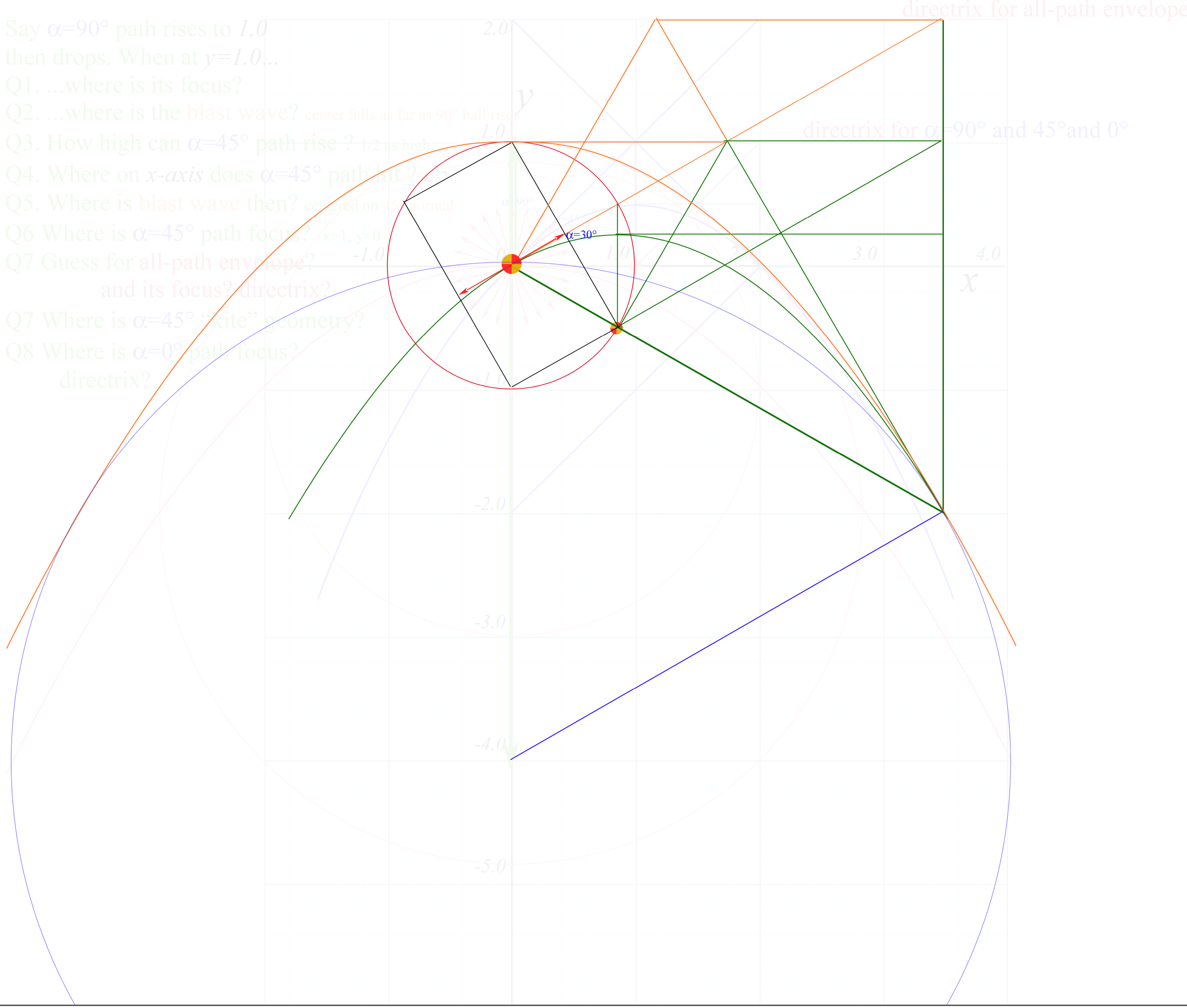
directrix?
directrix for all-path envelope
directrix for α=90° and 45° and 0°
α=90°
α=45°
α=30°
y
x
2.0
1.0
0
-1.0
-2.0
-3.0
-4.0
-5.0
1.0
2.0
3.0
4.0



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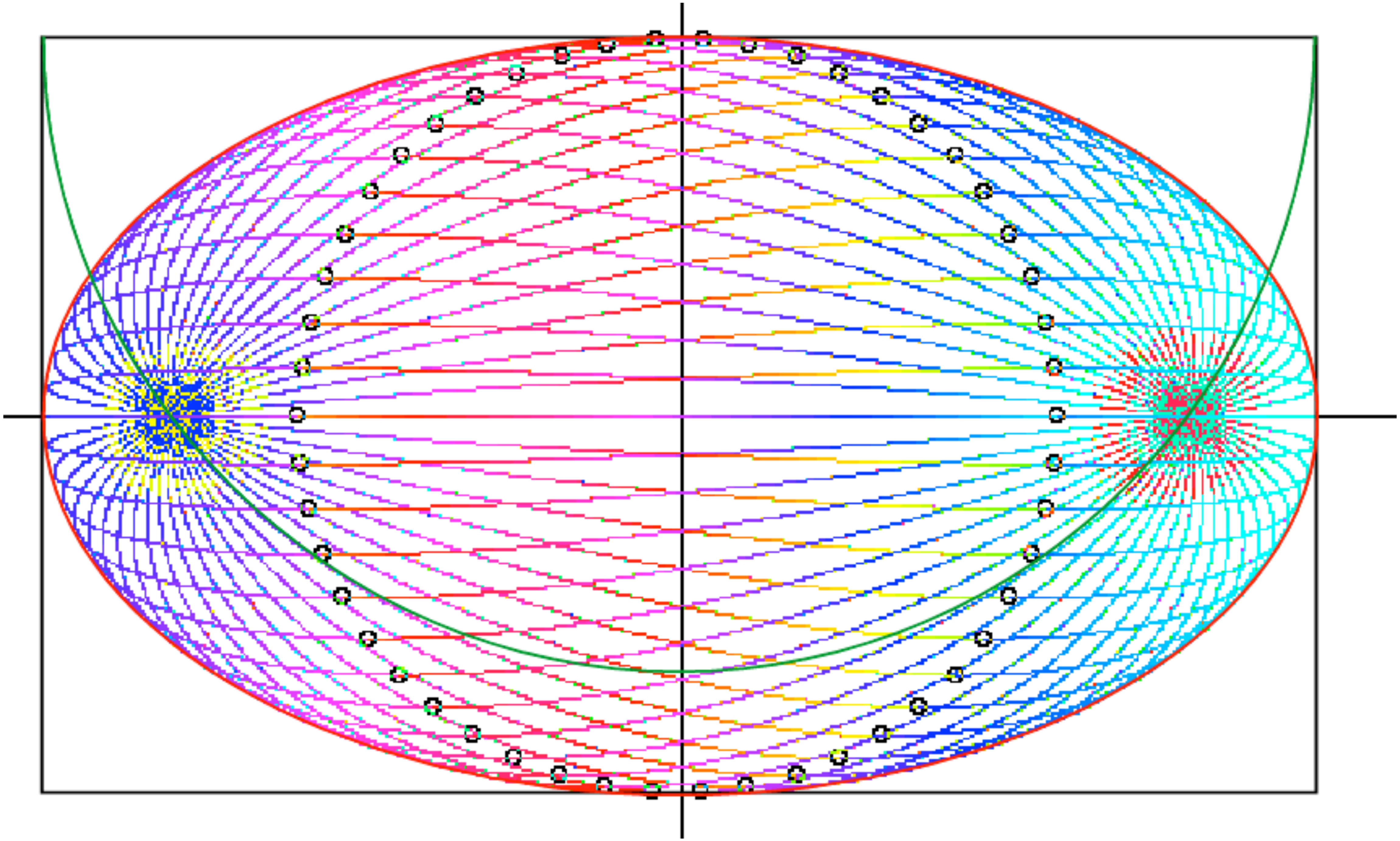
directrix for α=90° and 45°and 0°
α=30°
y
x
2.0
1.0
-1.0
0
1.0
3.0
4.0
-2.0
-3.0
-4.0
-5.0

Exploding-starlet elliptical envelope and contacting elliptical trajectories

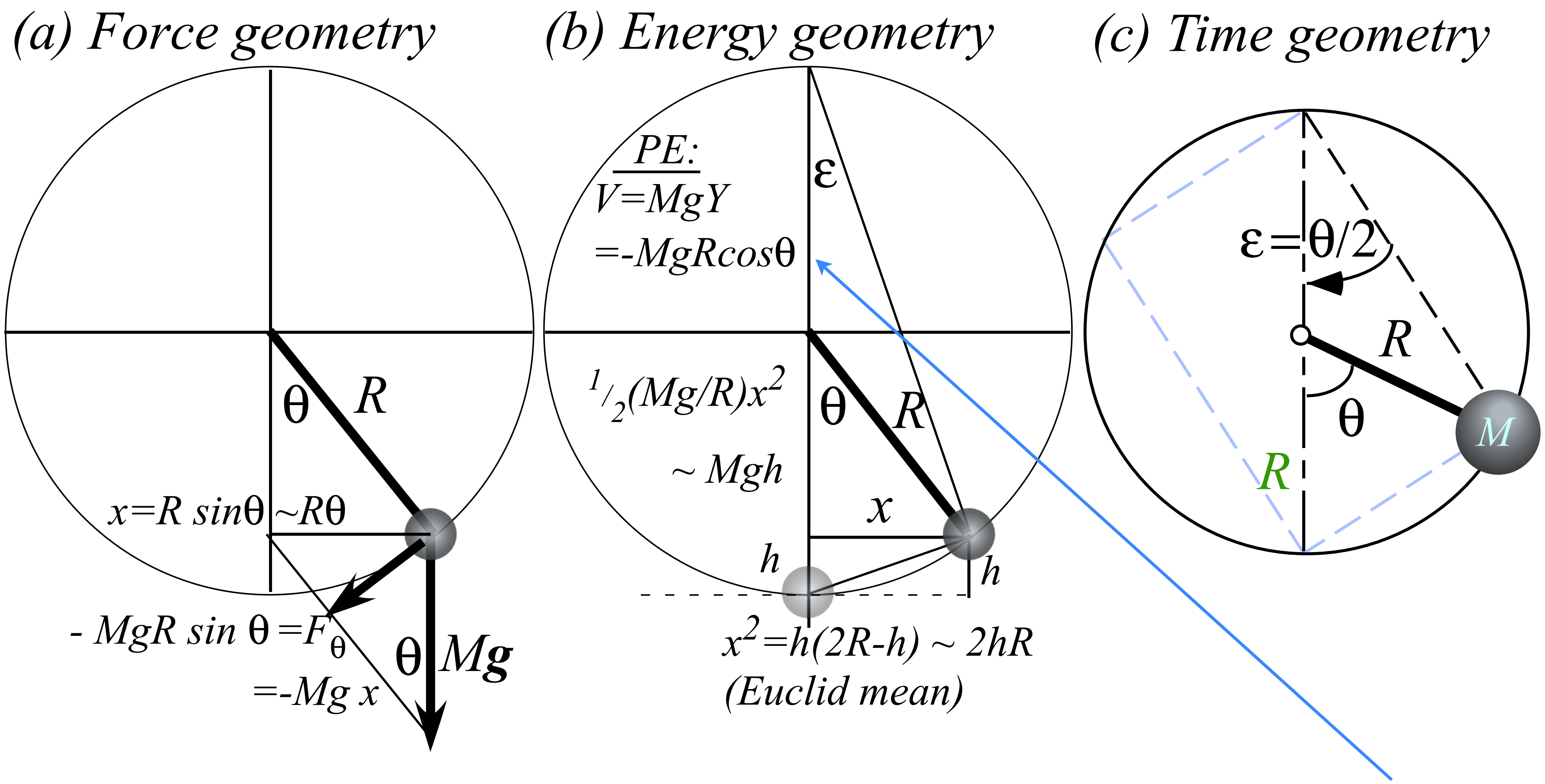


2. Examples of Hamiltonian dynamics and phase plots

→ ***1D Pendulum and phase plot*** *(Simulation)*

Phase space control *(Simulation)*

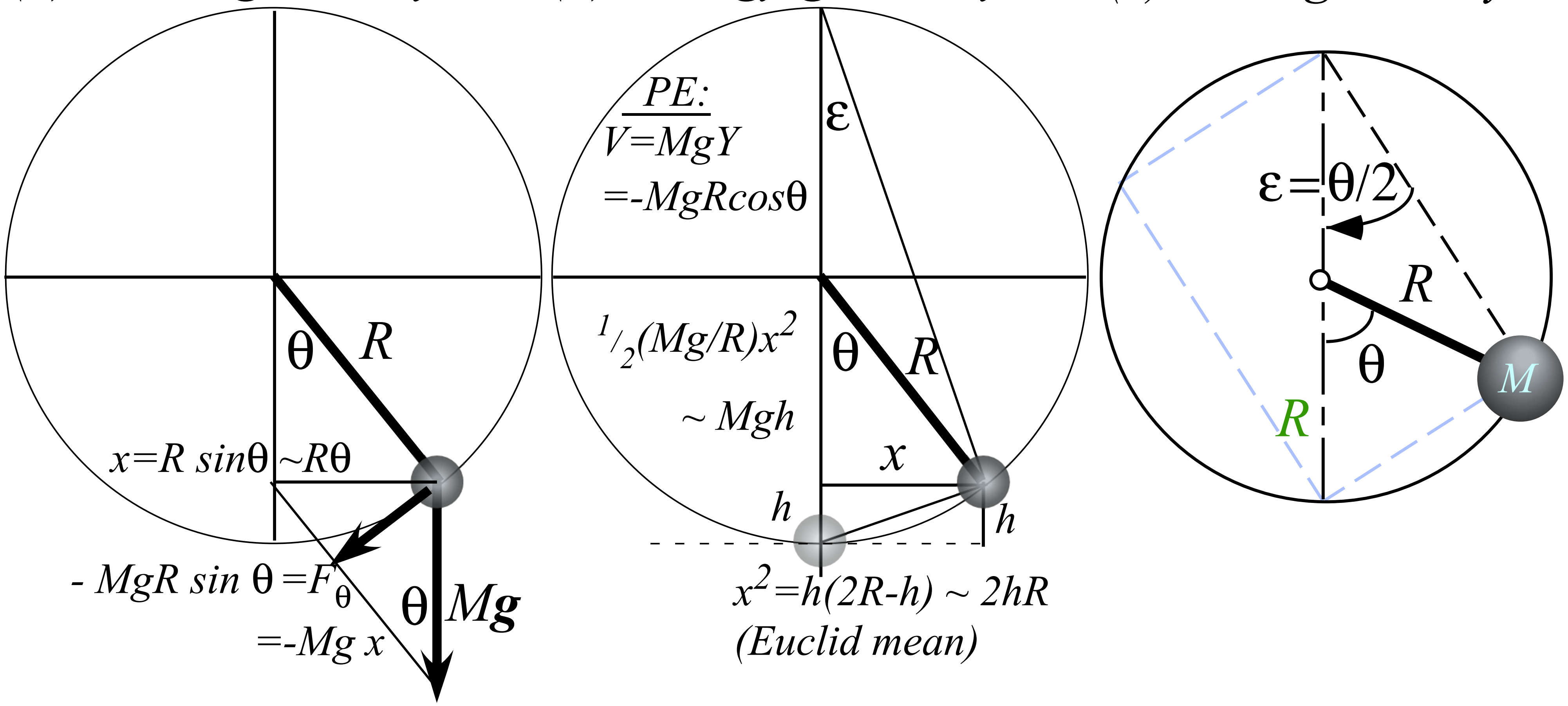
1D Pendulum and phase plot



Lagrangian function L= KE - PE = T - U where potential energy is U(θ) = $-MgR\cos\theta$

$$L(\dot{\theta},\theta)=\frac{1}{2}I\dot{\theta}^2-U(\theta)=\frac{1}{2}I\dot{\theta}^2+MgR\cos\theta$$

1D Pendulum and phase plot



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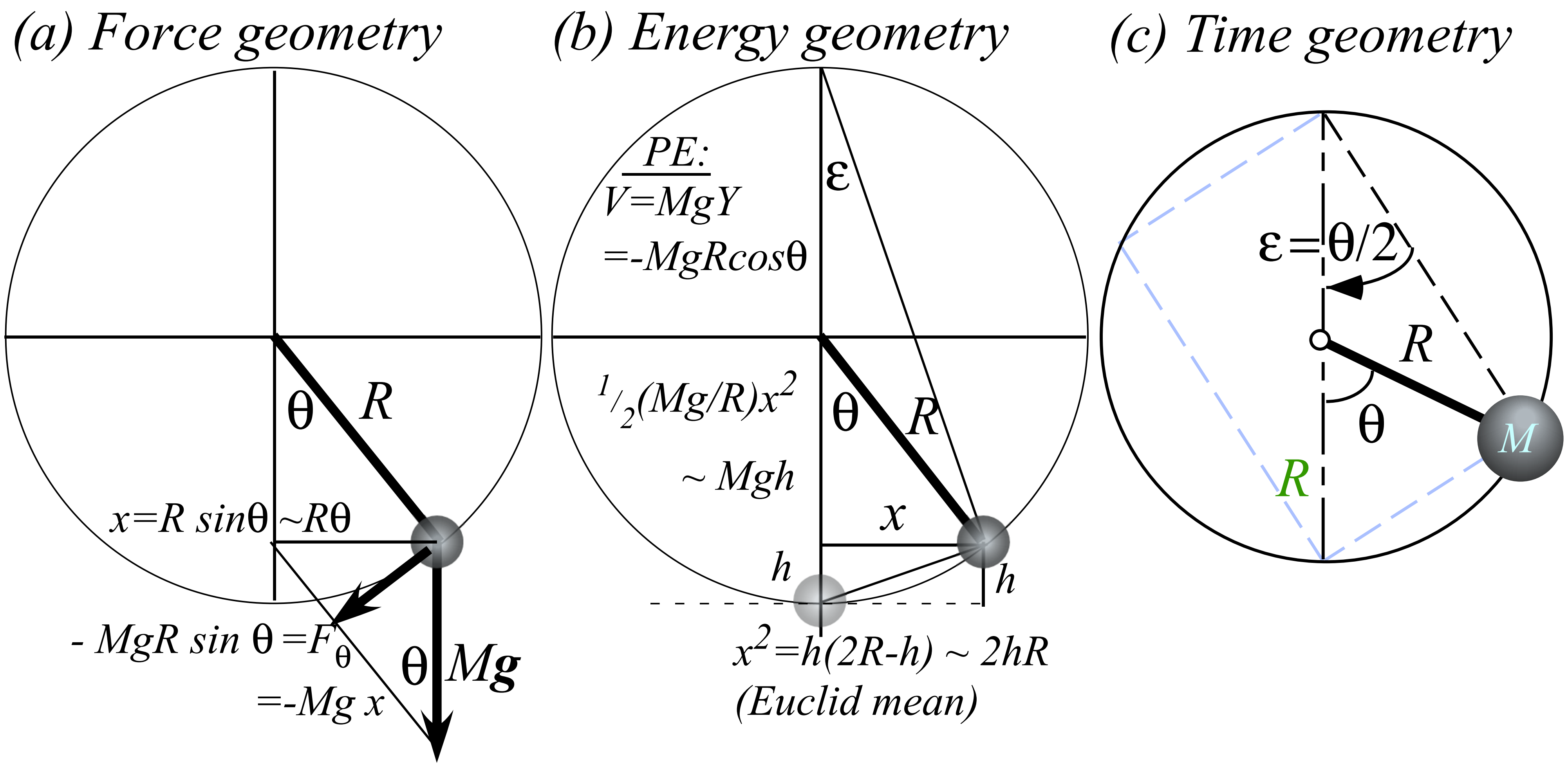
$$L(\dot{\theta},\theta)=\frac{1}{2}I\dot{\theta}^2-U(\theta)=\frac{1}{2}I\dot{\theta}^2+MgR\cos\theta$$

Hamiltonian function H= KE + PE = T +U where potential energy is U(θ) = $-MgR\cos\theta$

$$H(p_\theta,\theta)=\frac{1}{2I}p_\theta{}^2+U(\theta)=\frac{1}{2I}p_\theta{}^2-MgR\cos\theta\ =E\quad =const.$$

1D Pendulum and phase plot



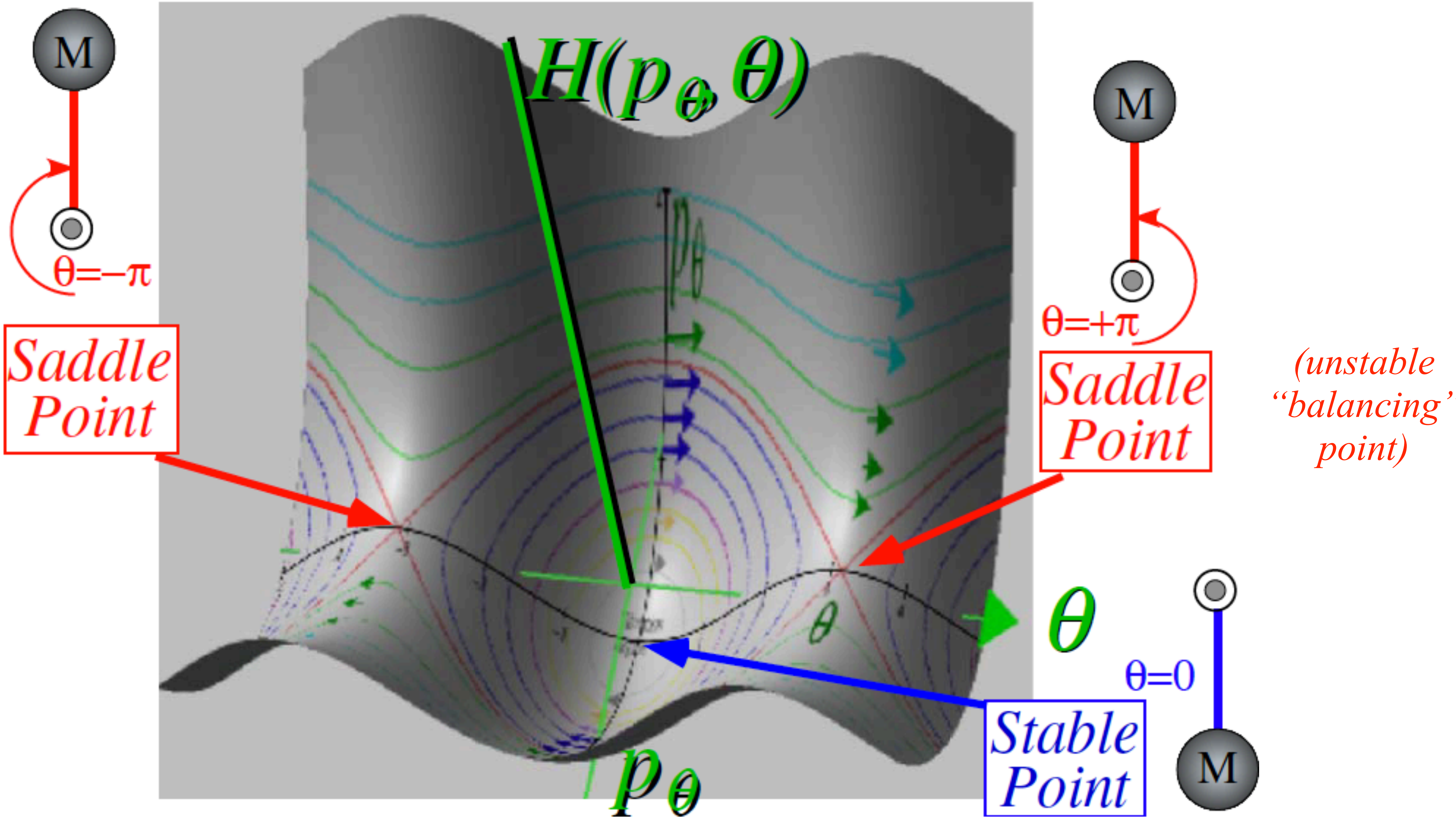
Lagrangian function L= KE - PE = T - U where potential energy is U(θ) = $-MgR\cos\theta$

$$L(\dot{\theta},\theta) = \frac{1}{2}I\dot{\theta}^2 - U(\theta) = \frac{1}{2}I\dot{\theta}^2 + MgR\cos\theta$$

Hamiltonian function H= KE + PE = T +U where potential energy is U(θ) = $-MgR\cos\theta$

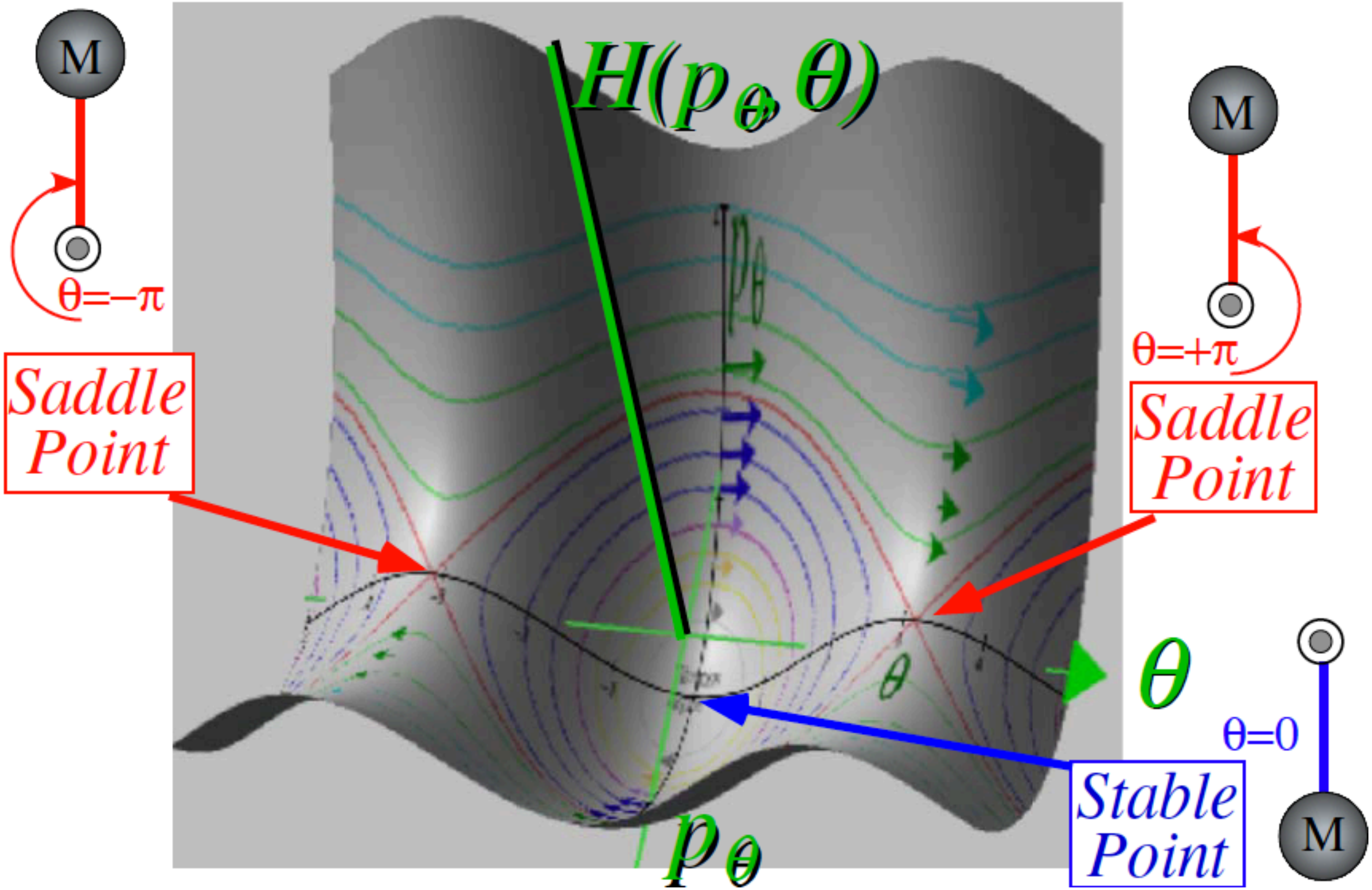
$$H(p_\theta,\theta) = \frac{1}{2I}p_\theta{}^2 + U(\theta) = \frac{1}{2I}p_\theta{}^2 - MgR\cos\theta \;=E \quad =const.$$

implies: $p_\theta = \sqrt{2I\left(E + MgR\cos\theta\right)}$



Example of plot of Hamilton for 1D-solid pendulum in its Phase Space (θ, p_θ)

$$H(p_\theta,\theta) = E = \frac{1}{2I}p_\theta^{\ 2} - MgR\cos\theta\ , \quad \text{or:}\quad p_\theta = \sqrt{2I\left(E + MgR\cos\theta\right)}$$



Example of plot of Hamilton for 1D-solid pendulum in its Phase Space (θ, p_θ)

$$H(p_\theta,\theta) = E = \frac{1}{2I} p_\theta{}^2 - MgR\cos\theta \,, \quad \text{or:} \quad p_\theta = \sqrt{2I\left(E + MgR\cos\theta\right)}$$

Funny way to look at Hamilton's equations:

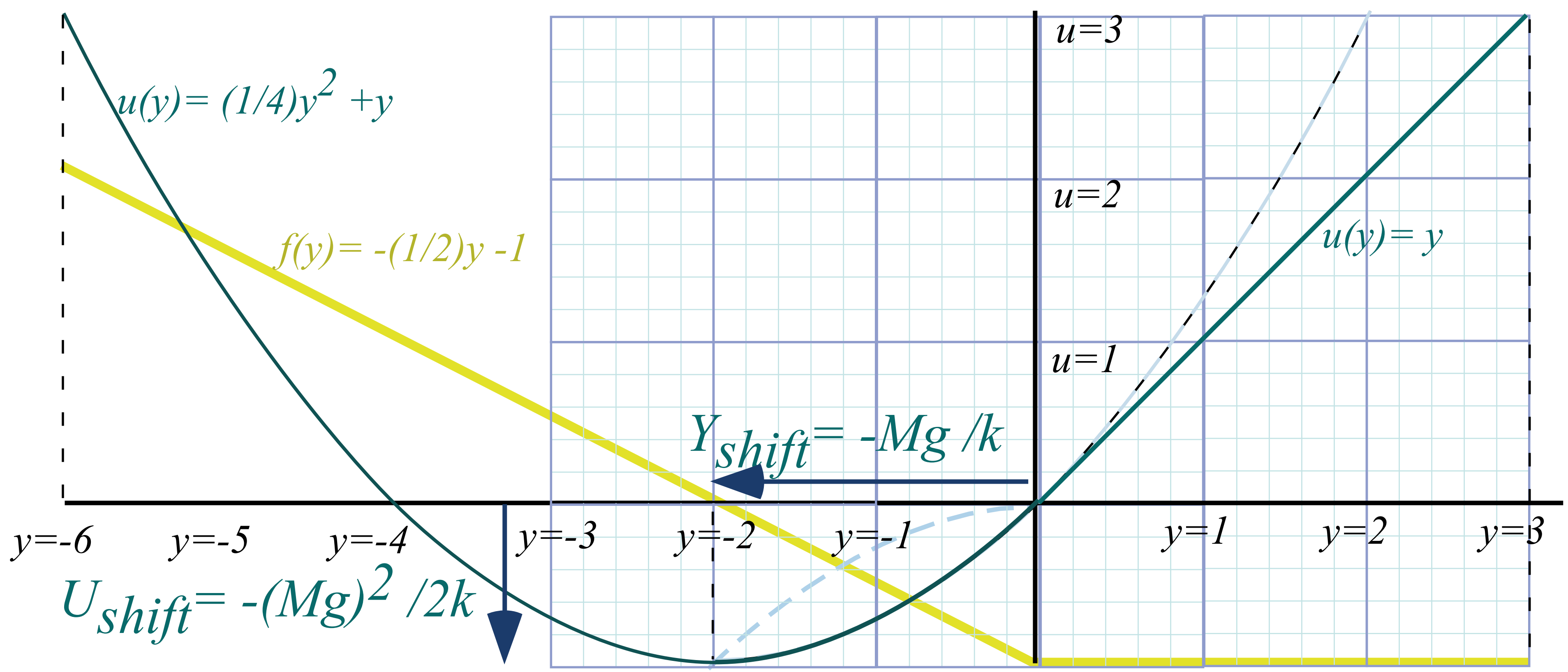
$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \partial_p H \\ -\partial_q H \end{pmatrix} = \mathbf{e_H} \times \left(-\nabla H\right) = (\text{H-axis}) \times (\text{fall line}), \text{ where: } \begin{cases} (\text{H-axis}) = \mathbf{e_H} = \mathbf{e_q} \times \mathbf{e_p} \\ (\text{fall line}) = -\nabla H \end{cases}$$

2. Examples of Hamiltonian dynamics and phase plots

1D Pendulum and phase plot (Simulation)

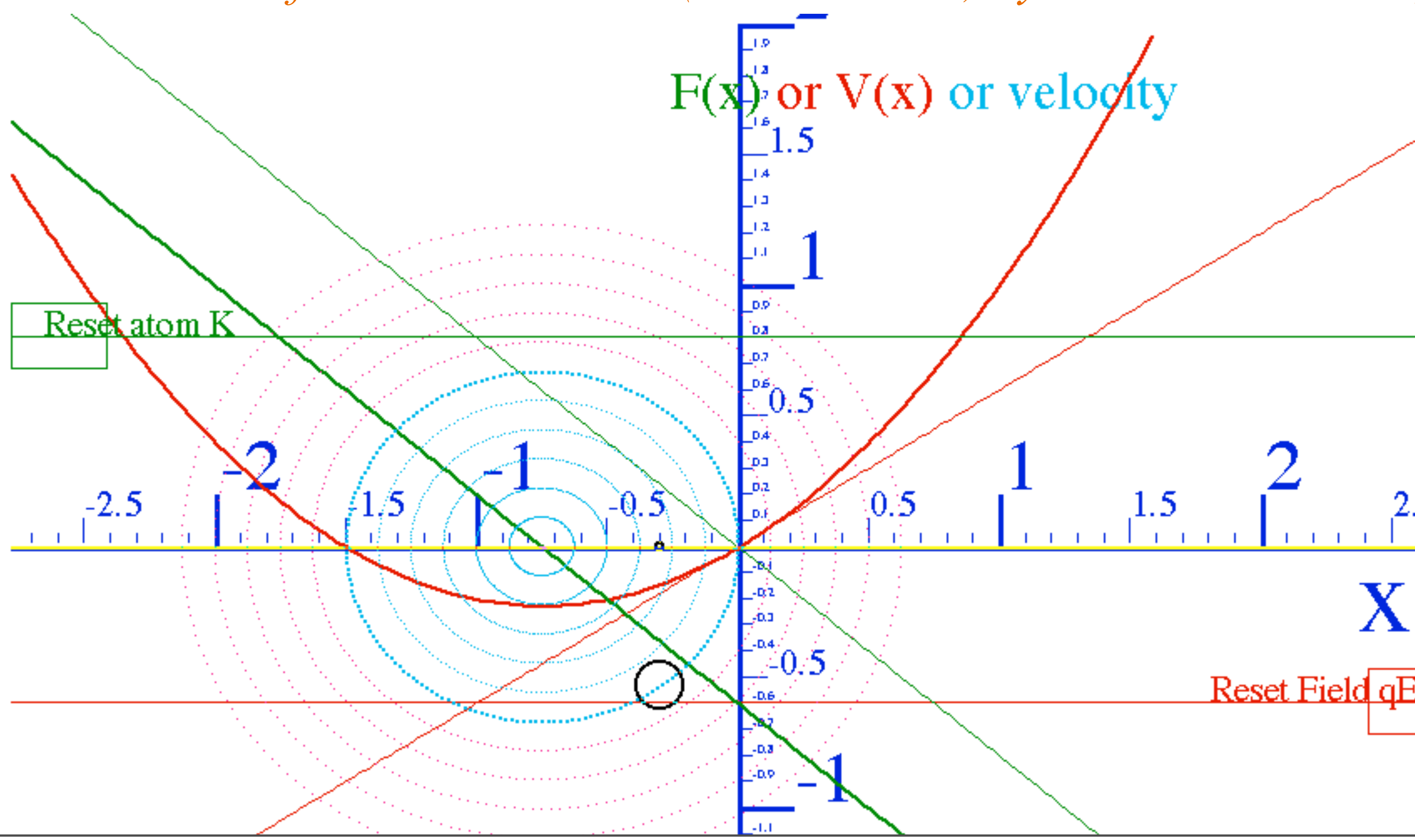
→ ***Phase control*** *(Simulation)*

$F(Y)=-kY-Mg$ $U(Y)=(1/2)kY^2+Mg\ Y$



Unit 1
Fig. 7.4

Simulation of atomic classical (or semi-classical) dynamics under varying phase control



3. Exploring phase space and Lagrangian mechanics more deeply

→ ***A weird "derivation" of Lagrange's equations***

Poincare identity and Action

How Classicists might have "derived" quantum equations

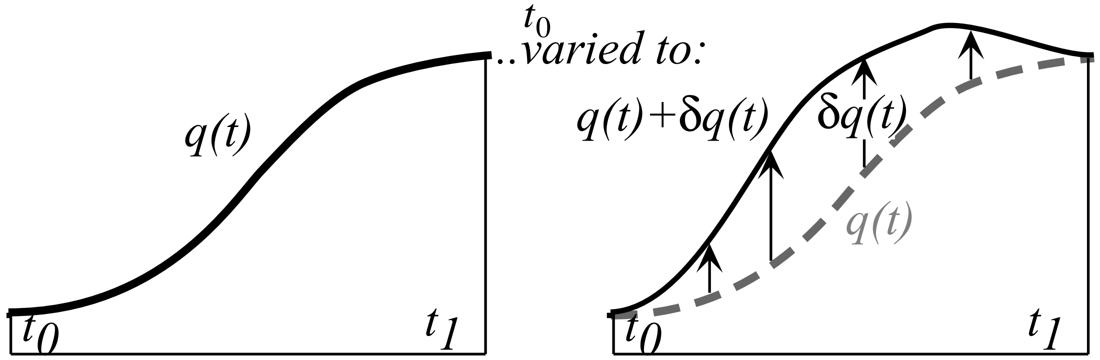
Huygen's contact transformations enforce minimum action

How to do quantum mechanics if you only know classical mechanics

A strange "derivation" of Lagrange's equations by Calculus of Variation

Variational calculus finds extreme (minimum or maximum) values to entire integrals

Minimize (or maximize): $S(q) = \int_{t_0}^{t_1} dt\; L(q(t),\dot{q}(t),t).$



An arbitrary but small variation function $\delta q(t)$ is allowed at every point t in the figure along the curve except at the end points t_0 and t_1. There we demand it not vary at all.(1)

$$\delta q(t_0) = 0 = \delta q(t_1) \qquad (1)$$

1st order L(q+δq) approximate:

$$S(q+\delta q) = \int_{t_0}^{t_1} dt \left[L(q,\dot{q},t) + \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \right] \text{ where: } \delta \dot{q} = \frac{d}{dt}\delta q$$

A weird "derivation" of Lagrange's equations

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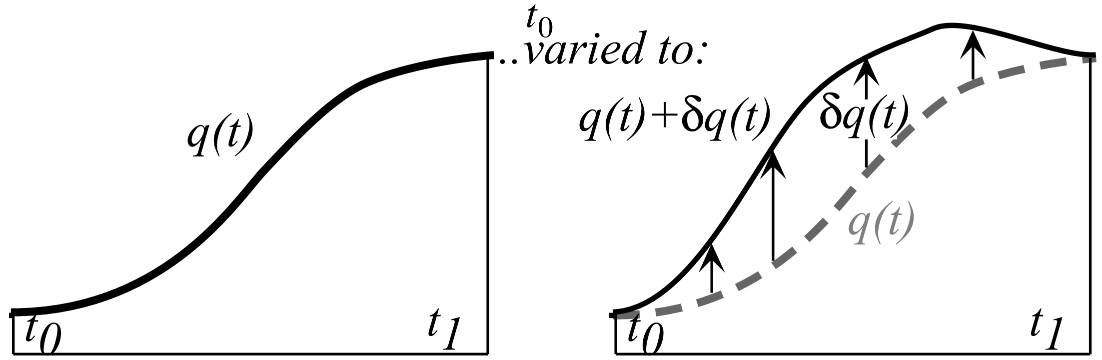

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Replace $\frac{\partial L}{\partial \dot{q}}\delta \dot{q}$ with $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q$

A weird "derivation" of Lagrange's equations

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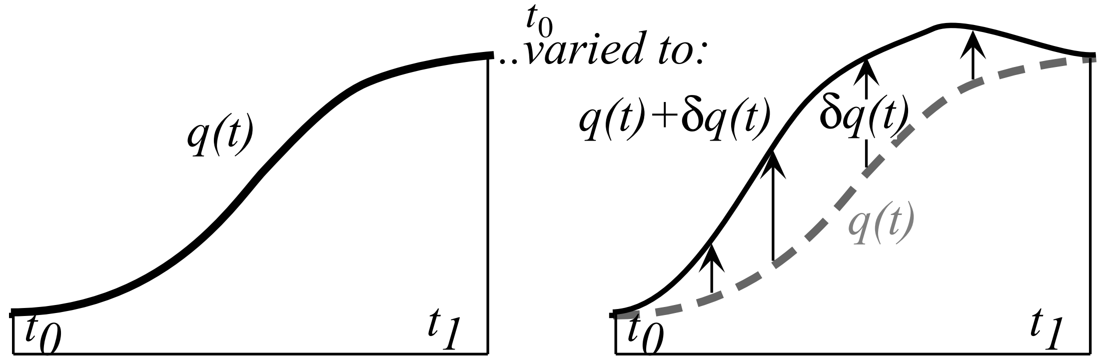

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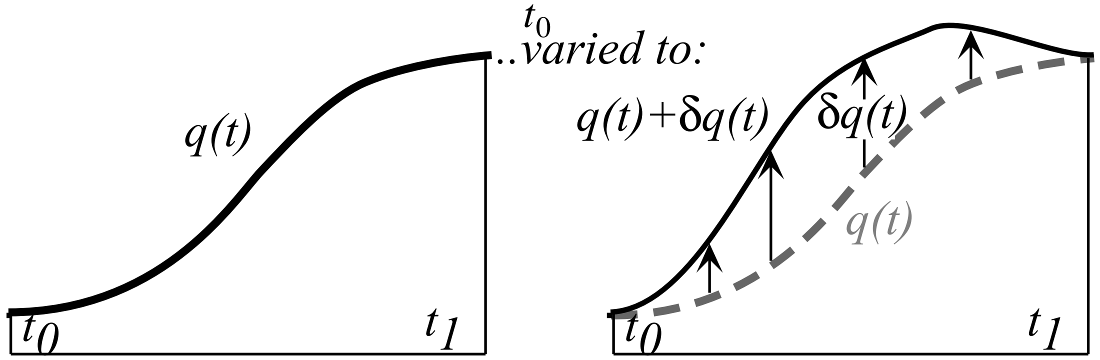

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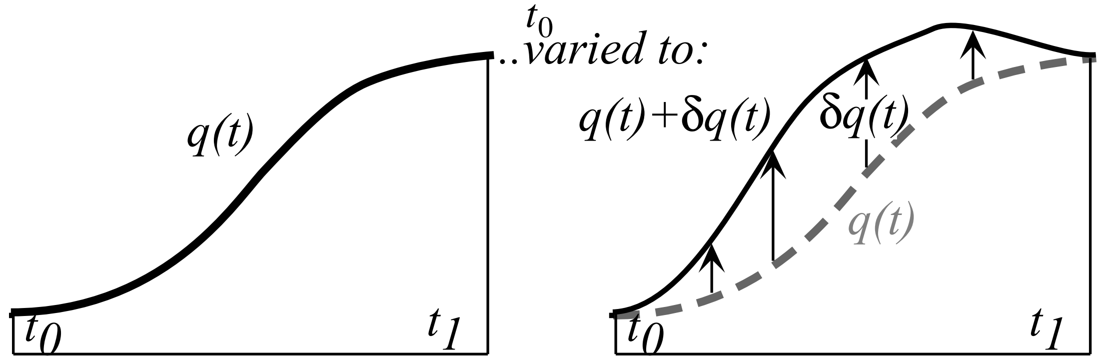
$$= \int_{t_0}^{t_1} dt\ L(q,\dot{q},t) + \int_{t_0}^{t_1} dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \right]\delta q + \left(\frac{\partial L}{\partial \dot{q}}\delta q\right)\Bigg|_{t_0}^{t_1}$$

A weird "derivation" of Lagrange's equations

Variational calculus finds extreme (minimum or maximum) values to entire integrals

$$S(q)=\int_{t_0}^{t_1} dt\ L\left(q(t),\dot q(t),t\right).$$



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$$S(q+\delta q)=\int_{t_0}^{t_1} dt\left[L(q,\dot q,t)+\frac{\partial L}{\partial q}\delta q-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\delta q\right]+\int_{t_0}^{t_1} dt\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\delta q\right)$$

$$=\int_{t_0}^{t_1} dt\ L(q,\dot q,t)+\int_{t_0}^{t_1} dt\left[\frac{\partial L}{\partial q}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\right]\delta q+\left(\frac{\partial L}{\partial \dot q}\delta q\right)\Bigg|_{t_0}^{t_1}$$

Third term vanishes by (1). This leaves first order variation: $\delta S=S(q+\delta q)-S(q)=\int_{t_0}^{t_1} dt\left[\frac{\partial L}{\partial q}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\right]\delta q$

Extreme value (actually *minimum* value) of *S(q)* occurs *if and only if* Lagrange equation is satisfied!

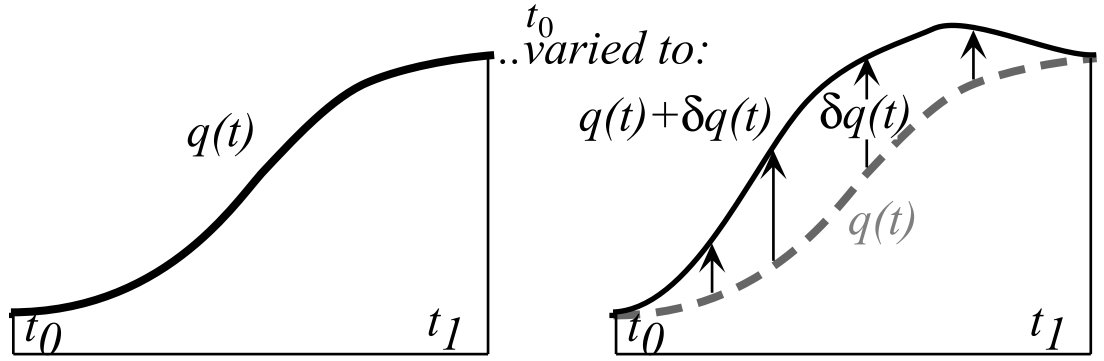
$$\delta S=0\Rightarrow\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)-\frac{\partial L}{\partial q}=0$$

Euler-Lagrange equation(s)

A weird "derivation" of Lagrange's equations

Variational calculus finds extreme (minimum or maximum) values to entire integrals

$$S(q) = \int_{t_0}^{t_1} dt\ L(q(t), \dot{q}(t), t).$$



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$$u \cdot \frac{dv}{dt} = \frac{d}{dt}(uv) - \frac{du}{dt}v$$

Replace $\frac{\partial L}{\partial \dot{q}}\delta \dot{q}$ with $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q$

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$$= \int_{t_0}^{t_1} dt\ L(q,\dot{q},t) + \int_{t_0}^{t_1} dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \right]\delta q + \left. \left(\frac{\partial L}{\partial \dot{q}}\delta q\right) \right|_{t_0}^{t_1}$$

Third term vanishes by (1). This leaves first order variation: $\delta S = S(q+\delta q) - S(q) = \int_{t_0}^{t_1} dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \right]\delta q$

Extreme value (actually *minimum* value) of *S(q)* occurs *if and only if* Lagrange equation is satisfied!

$$\delta S = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

Euler-Lagrange equation(s)

But, WHY is nature so inclined to fly JUST SO as to minimize the Lagrangian L = T - U???

3. Exploring phase space and Lagrangian mechanics more deeply

A weird "derivation" of Lagrange's equations

→ ***Poincare identity and Action***

How Classicists might have "derived" quantum equations

Huygen's contact transformations enforce minimum action

How to do quantum mechanics if you only know classical mechanics

Legendre-Poincare identity and Action

Legendre transform $L(\mathbf{v}) = \mathbf{p}\bullet\mathbf{v} - H(\mathbf{p})$ becomes *Poincare's invariant differential* if *dt* is cleared.

$$L\cdot dt = \mathbf{p}\bullet\mathbf{v}\cdot dt - H\cdot dt = \mathbf{p}\bullet d\mathbf{r} - H\cdot dt \qquad \left(\mathbf{v}=\frac{d\mathbf{r}}{dt}\ \textit{implies: } \mathbf{v}\cdot dt=d\mathbf{r}\right)$$

Legendre-Poincare identity and Action

Legendre transform $L(\mathbf{v}) = \mathbf{p}\bullet\mathbf{v} - H(\mathbf{p})$ becomes *Poincare's invariant differential* if dt is cleared.

$$L\cdot dt = \mathbf{p}\bullet\mathbf{v}\cdot dt - H\cdot dt = \mathbf{p}\bullet d\mathbf{r} - H\cdot dt \qquad \left(\mathbf{v} = \frac{d\mathbf{r}}{dt} \textit{ implies: } \mathbf{v}\cdot dt = d\mathbf{r} \right)$$

This is the time differential dS of *action* $S = \int L\cdot dt$ whose time derivative is rate L of *quantum phase*.

$$dS = L\cdot dt = \mathbf{p}\bullet d\mathbf{r} - H\cdot dt \quad \text{where:} \quad L = \frac{dS}{dt}$$

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A: Differential $dW = f_x(x,y)dx + f_y(x,y)dy$ is *integrable* to a $W(x,y)$ if: $f_x = \frac{\partial W}{\partial x}$ and: $f_y = \frac{\partial W}{\partial y}$

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Similar to conditions for integrating work differential $dW=\mathbf{f}\bullet d\mathbf{r}$ *to get potential* $W(\mathbf{r})$. *That condition is* no curl allowed: $\nabla\times\mathbf{f}=\mathbf{0}$ *or* ∂-symmetry *of W:*

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dS is integrable if: $\frac{\partial S}{\partial \mathbf{r}} = \mathbf{p}$ and: $\frac{\partial S}{\partial t} = -H$

These conditions are known as Jacobi-Hamilton *equations*

3. Exploring phase space and Lagrangian mechanics more deeply

A weird "derivation" of Lagrange's equations

Poincare identity and Action

→ ***How Classicists might have "derived" quantum equations***

Huygen's contact transformations enforce minimum action

How to do quantum mechanics if you only know classical mechanics

How Jacobi-Hamilton *could have "derived" Schrodinger equations*

(Given "quantum wave")

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Try 1st **r**-*derivative of wave* ψ

$$\frac{\partial}{\partial \mathbf{r}}\psi(\mathbf{r},t) = \frac{\partial}{\partial \mathbf{r}} e^{iS/\hbar} = \frac{\partial\left(iS/\hbar\right)}{\partial \mathbf{r}} e^{iS/\hbar} = \left(i/\hbar\right)\frac{\partial S}{\partial \mathbf{r}}\psi(\mathbf{r},t)$$

$$\frac{\partial}{\partial \mathbf{r}}\psi(\mathbf{r},t) = \left(i/\hbar\right)\mathbf{p}\,\psi(\mathbf{r},t) \quad \text{or:} \quad \boxed{\frac{\hbar}{i}\frac{\partial}{\partial \mathbf{r}}\psi(\mathbf{r},t) = \mathbf{p}\,\psi(\mathbf{r},t)}$$

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$$\frac{\partial}{\partial \mathbf{r}}\psi(\mathbf{r},t) = \frac{\partial}{\partial \mathbf{r}}e^{iS/\hbar} = \frac{\partial(iS/\hbar)}{\partial \mathbf{r}}e^{iS/\hbar} = (i/\hbar)\frac{\partial S}{\partial \mathbf{r}}\psi(\mathbf{r},t)$$

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Try 1st t-derivative of wave ψ

$$\frac{\partial}{\partial t}\psi(\mathbf{r},t) = \frac{\partial}{\partial t}e^{iS/\hbar} = \frac{\partial(iS/\hbar)}{\partial t}e^{iS/\hbar} = (i/\hbar)\frac{\partial S}{\partial t}\psi(\mathbf{r},t)$$

$$= (i/\hbar)(-H)\psi(\mathbf{r},t) \quad \text{or:} \quad \boxed{i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = H\psi(\mathbf{r},t)}$$

3. Exploring phase space and Lagrangian mechanics more deeply

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→ ***Huygen's contact transformations enforce minimum action***

How to do quantum mechanics if you only know classical mechanics

Huygen's contact transformations enforce minimum action

Each point $\mathbf{r}_k$ *on a wavefront "broadcasts" in all directions.*
Only ***minimum action*** *path interferes constructively*

Time-independent action $S_H = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{p} \bullet d\mathbf{r}$
(Hamilton's *reduced action*)
is a purely spatial integral .

Time-dependent action $S_p = \int_{\mathbf{r}_0 t_0}^{\mathbf{r}_1 t_1} (\mathbf{p} \bullet d\mathbf{r} - H \cdot dt)$
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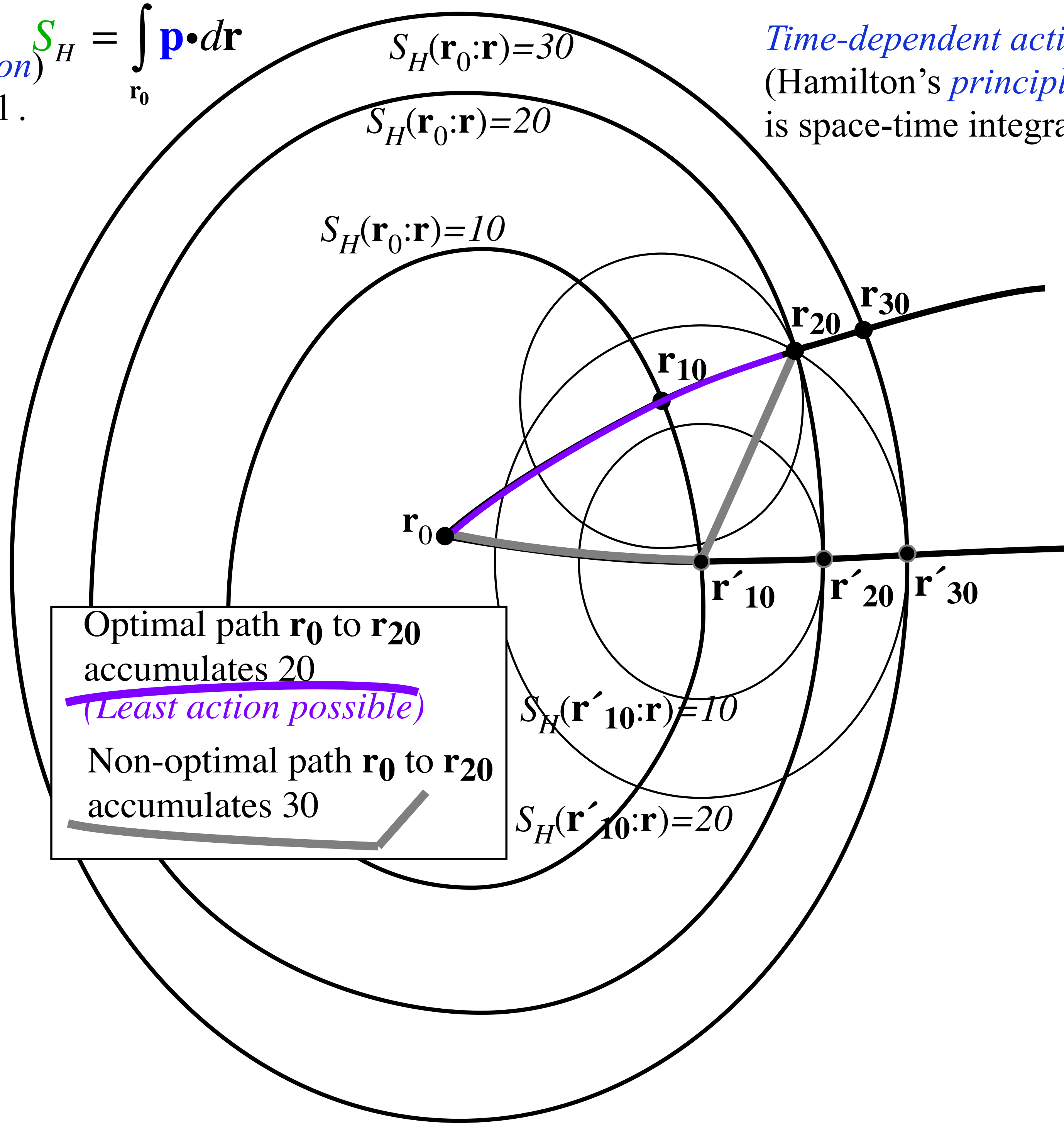

Unit 1
Fig. 12.12

Huygen's contact transformations enforce minimum action

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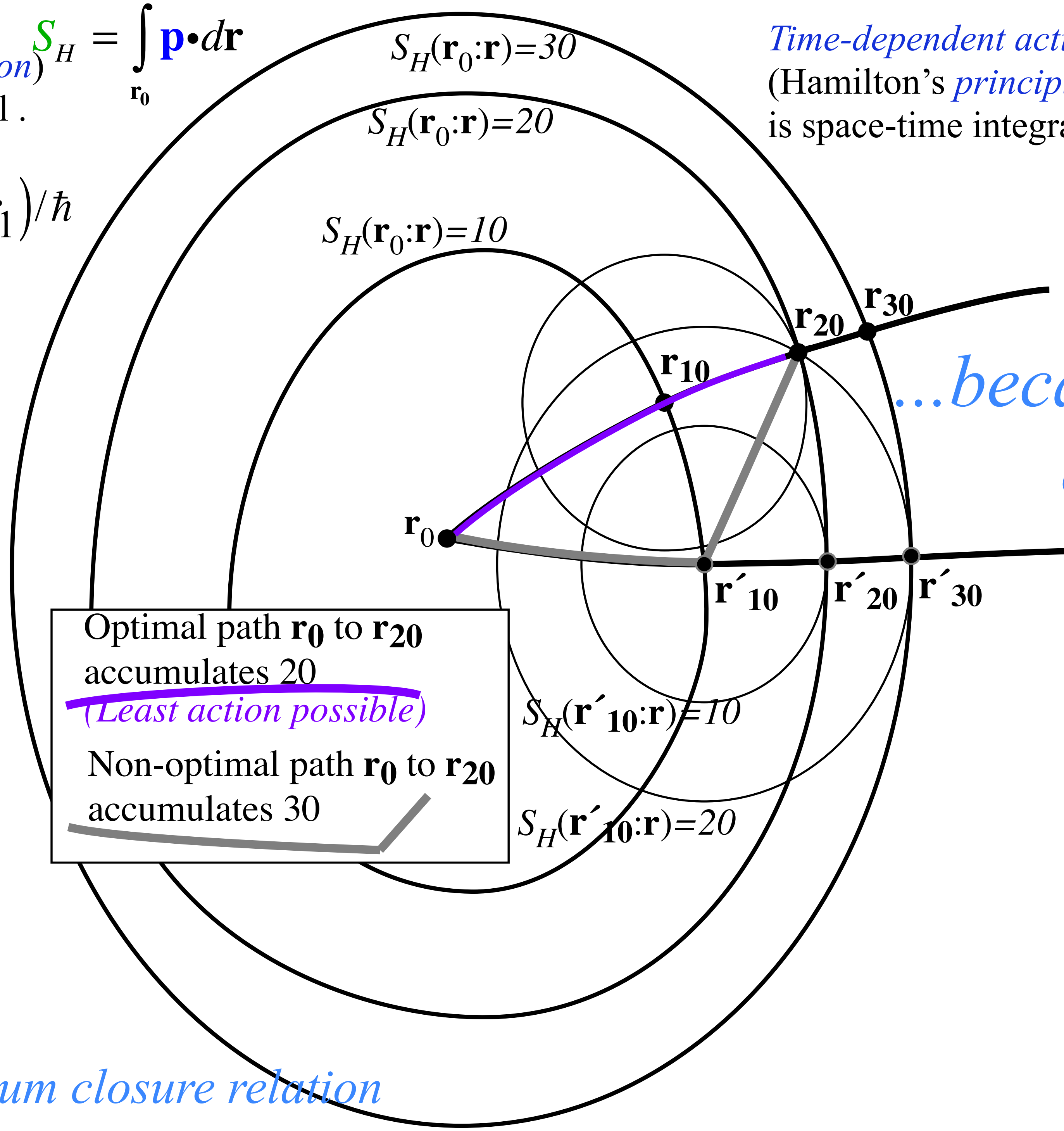
$$\langle \mathbf{r}_1, t_1 | \mathbf{r}_0, t_0 \rangle = e^{i S(\mathbf{r}_0, t_0 : \mathbf{r}_1, t_1)/\hbar}$$



*...because action **is** quantum wave phase*

Unit 1
Fig. 12.12

Feynman's path-sum closure relation

$$\sum_{\mathbf{r}'} \langle \mathbf{r}_1 | \mathbf{r}' \rangle \langle \mathbf{r}' | \mathbf{r}_0 \rangle \cong \sum_{\mathbf{r}'} e^{i\left(S_H(\mathbf{r}_0:\mathbf{r}') + S_H(\mathbf{r}':\mathbf{r}_1)\right)/\hbar} = e^{i S_H(\mathbf{r}_0:\mathbf{r}_1)/\hbar} = \langle \mathbf{r}_1 | \mathbf{r}_0 \rangle$$

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How to do quantum mechanics if you only know classical mechanics

Bohr quantization requires quantum phase $S_H/\hbar$ in amplitude to be an integral multiple n of 2π after a closed loop integral $S_H(\mathbf{r}_0 : \mathbf{r}_0) = \int_{r_0}^{r_0} \mathbf{p} \bullet d\mathbf{r}$. The integer n ($n = 0, 1, 2,...$) is a *quantum number*.

$$1 = \langle \mathbf{r}_0 | \mathbf{r}_0 \rangle = e^{i S_H(\mathbf{r}_0 : \mathbf{r}_0)/\hbar} = e^{i \Sigma_H/\hbar} = 1 \quad \text{for: } \Sigma_H = 2\pi\hbar n = hn$$

Numerically integrate Hamilton's equations and Lagrangian L. Color the trajectory according to the current accumulated value of action $S_H(\mathbf{0} : \mathbf{r})/\hbar$. Adjust energy to quantized pattern (if closed system*)

$$S_H(\mathbf{0} : \mathbf{r}) = S_p(\mathbf{0}, 0 : \mathbf{r}, t) + Ht = \int_0^t L\, dt + Ht \,.$$

How to do quantum mechanics if you only know classical mechanics

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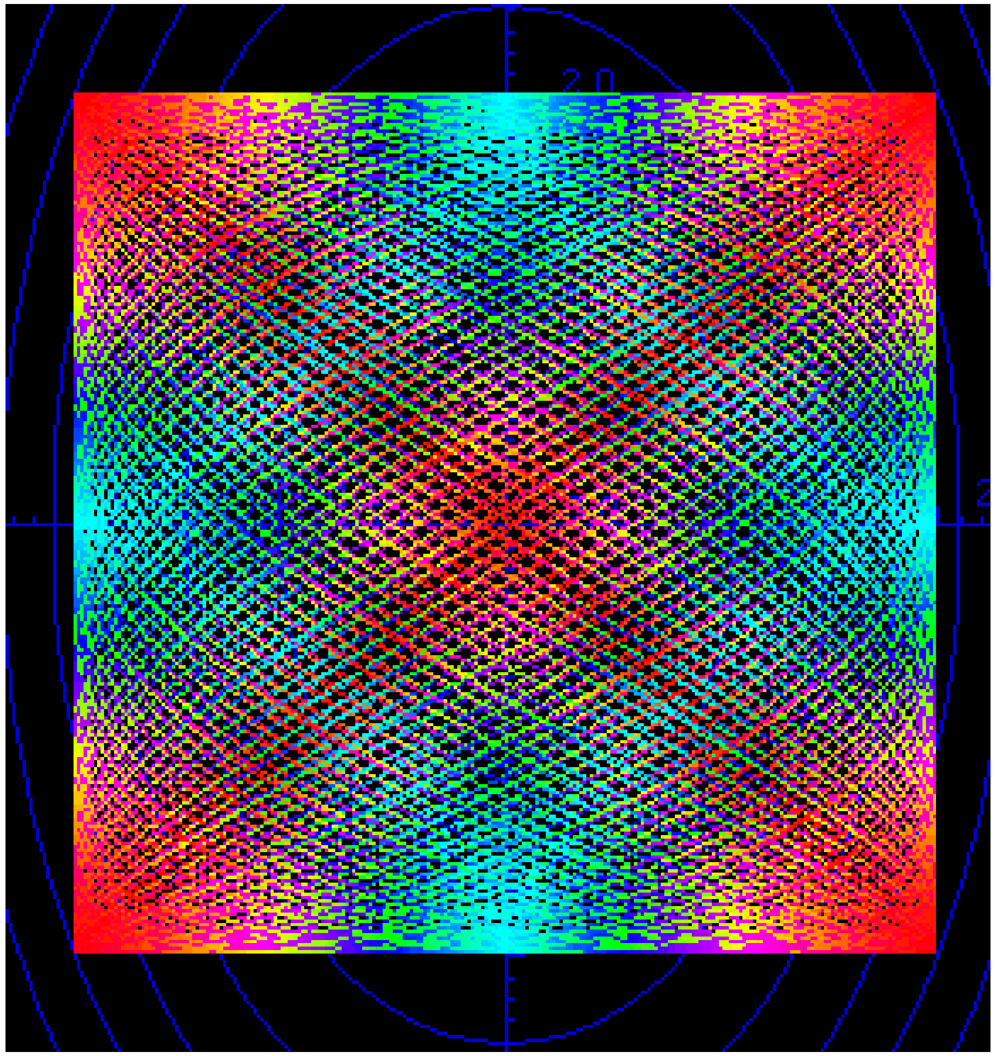
The hue should represent the phase angle $S_H(\mathbf{0} : \mathbf{r})/\hbar$ *modulo* 2π as, for example,
0=red, *π/4=orange*, *π/2=yellow*, *3π/4=green*, *π=cyan* (opposite of red), *5π/4=indigo*, *3π/2=blue*, *7π/4=purple*, *and 2π=red* (full color circle).
Interpolating action on a palette of *32* colors is enough precision for low quanta.



Unit 1 Fig. 12.13

How to do quantum mechanics if you only know classical mechanics

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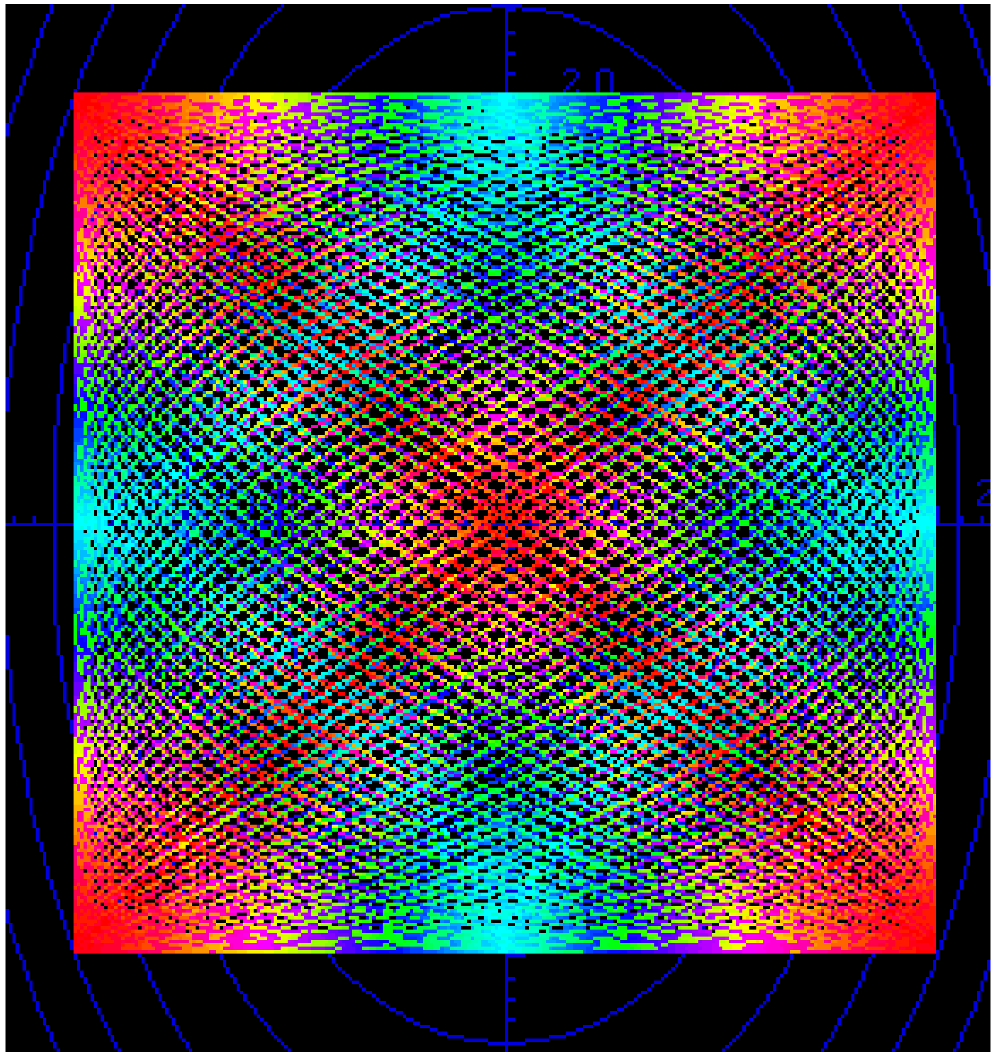
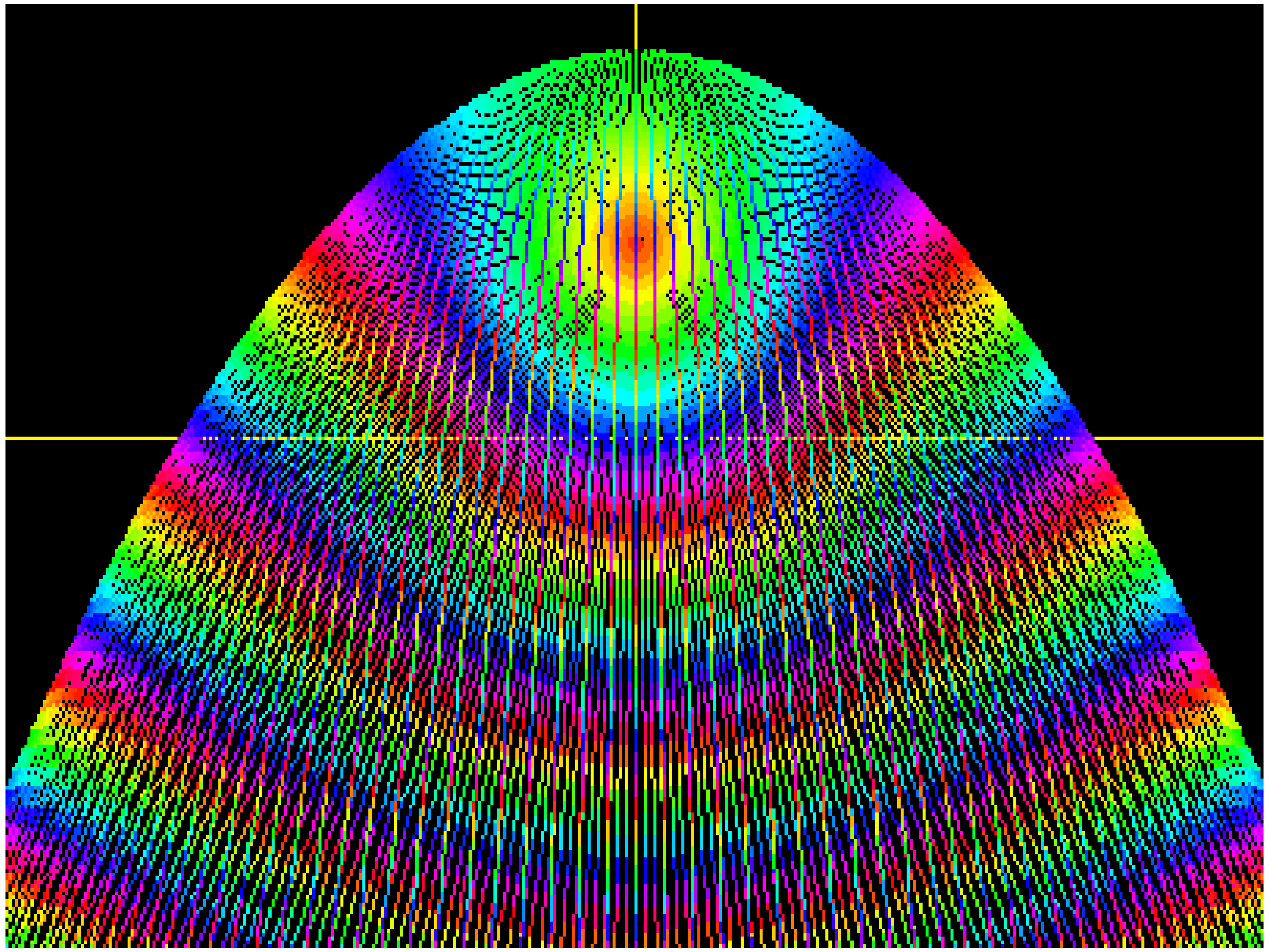


Unit 1 Fig. 12.13

*open system has continuous energy

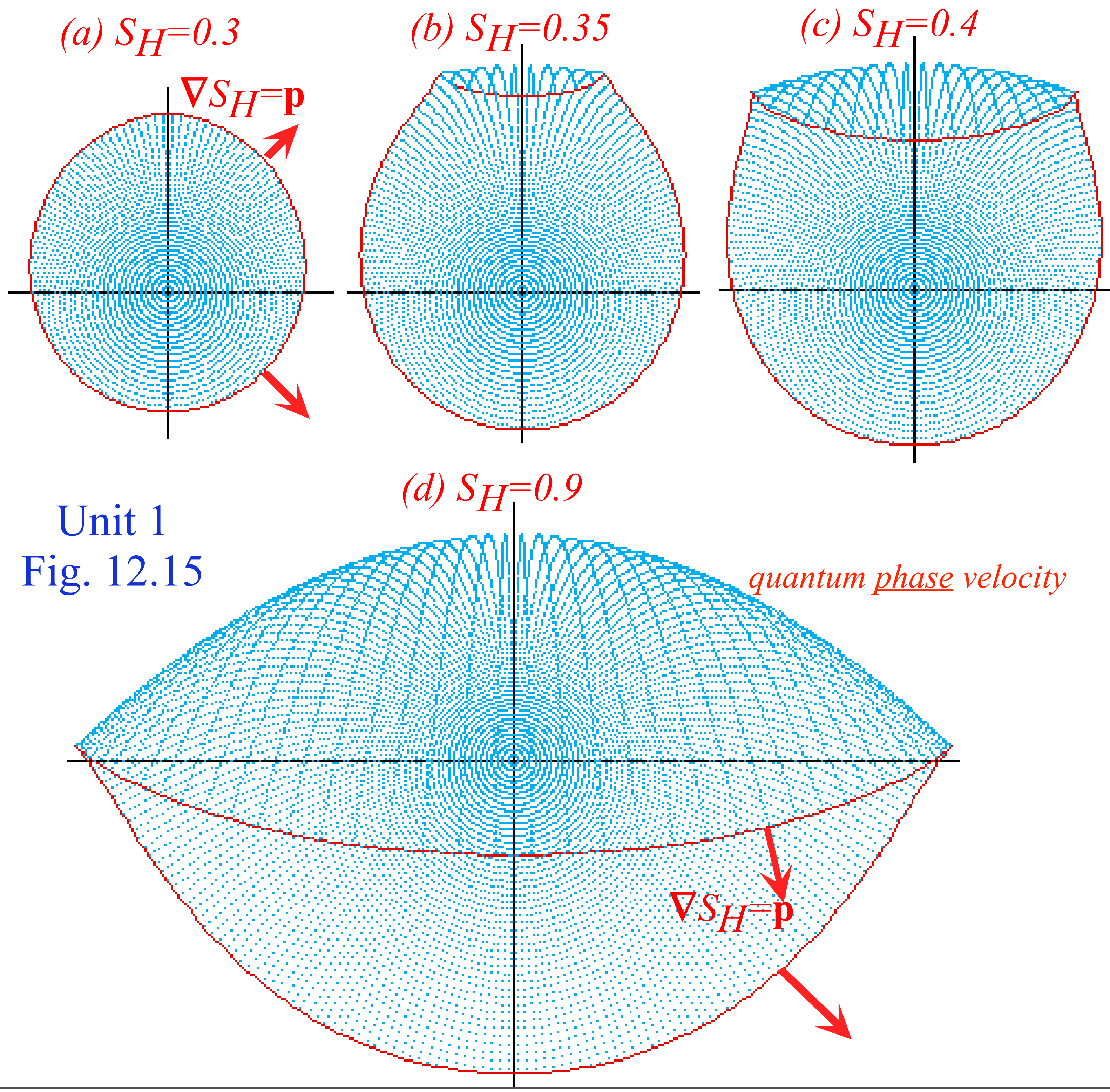


Unit 1 Fig. 12.14

A moving wave has a *quantum phase velocity* found by setting S=*const.* or $dS(0,0{:}r,t)=0=\mathbf{p}\bullet d\mathbf{r}-H dt$.

$$V_{phase}=\frac{d\mathbf{r}}{dt}=\frac{H}{\mathbf{p}}=\frac{\omega}{\mathbf{k}}$$

Quantum "phase wavefronts"



Unit 1
Fig. 12.15

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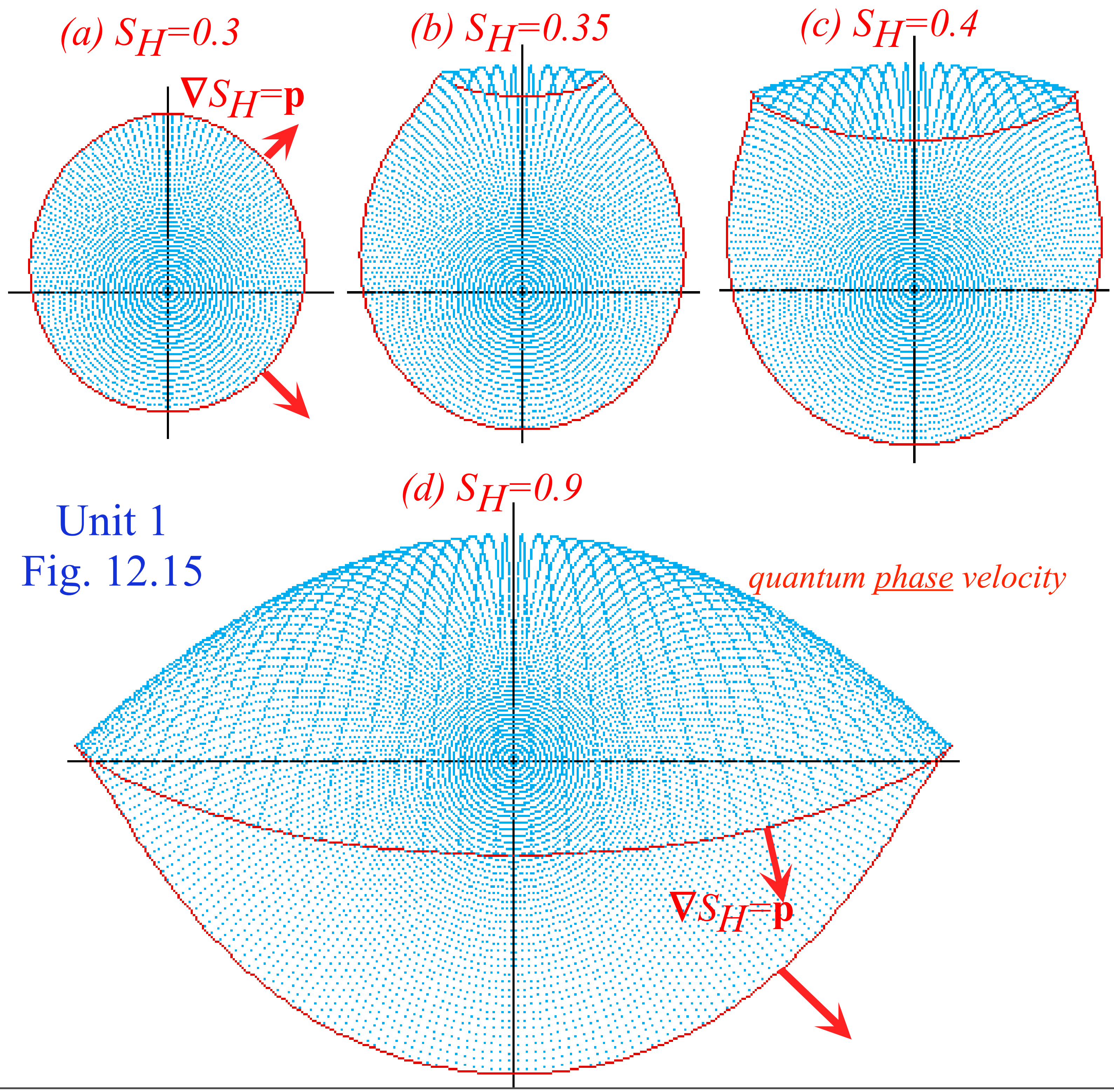
$$\mathbf{V}_{phase}=\frac{d\mathbf{r}}{dt}=\frac{H}{\mathbf{p}}=\frac{\omega}{\mathbf{k}}$$

This is quite the opposite of classical particle velocity which is *quantum group velocity*.

$$\mathbf{V}_{group}=\frac{d\mathbf{r}}{dt}=\dot{\mathbf{r}}=\frac{\partial H}{\partial \mathbf{p}}=\frac{\partial \omega}{\partial \mathbf{k}}$$

Note: This is Hamilton's 1st Equation

Quantum "phase wavefronts"



Unit 1
Fig. 12.15

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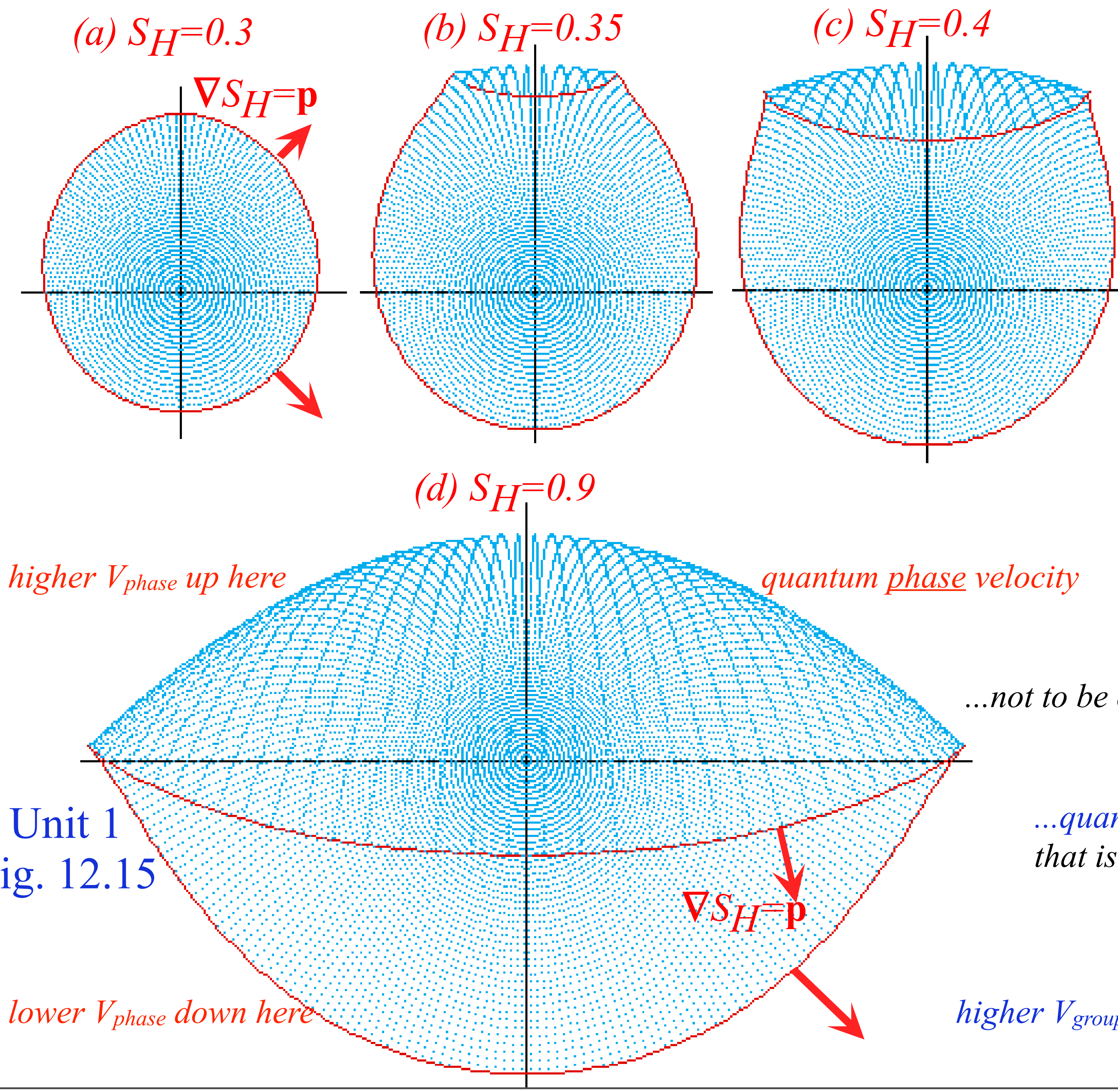
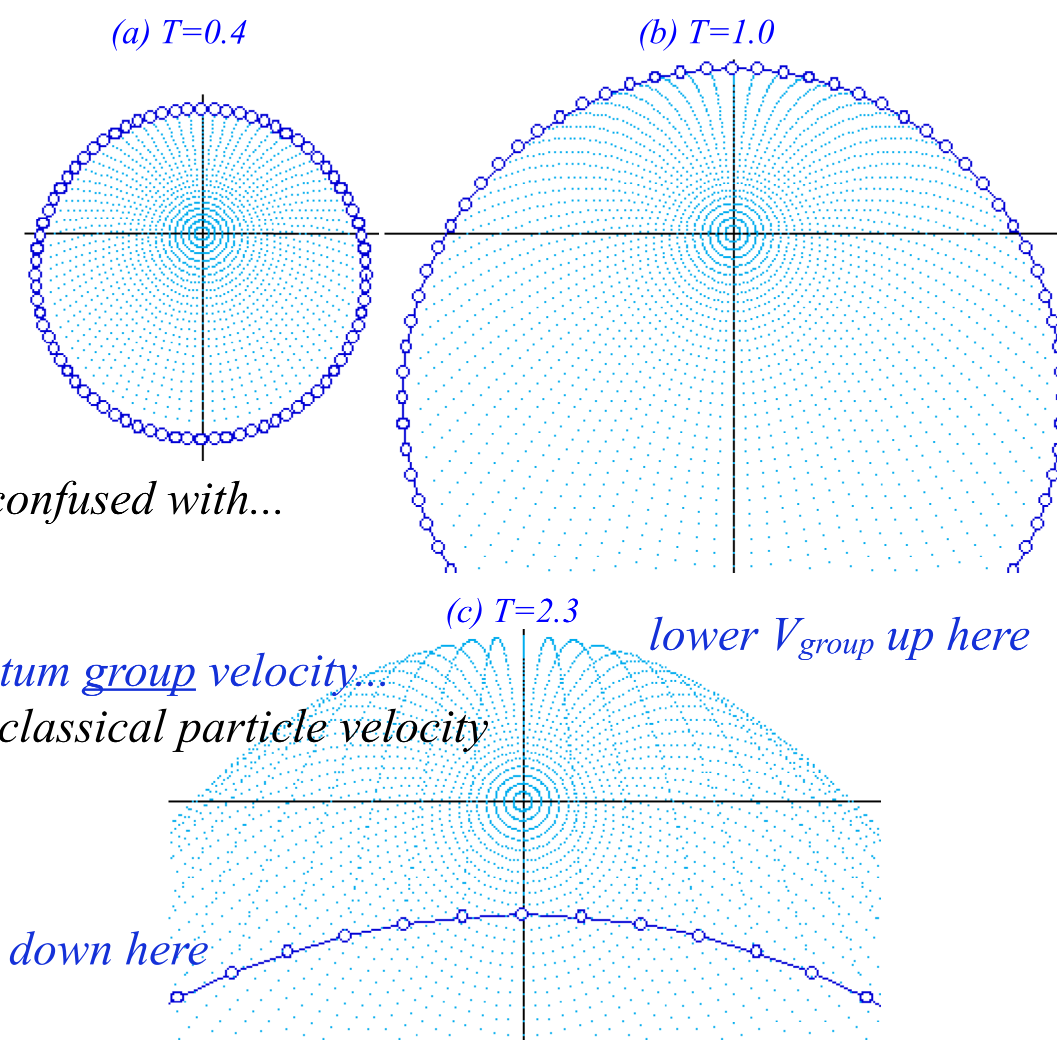


Unit 1
Fig. 12.15

...not to be confused with...

Classical "blast wavefronts"



...quantum group velocity...
that is classical particle velocity