Lecture 17
Tue. 2.28.2012

Lagrangian and Hamiltonian dynamics:
Living with duality in GCC cells and vectors Part II.
(Ch. 12 of Unit 1)

0. Review of Hamilton equations 1 and 2

1. Hamilton prefers Contravariant $g^{mn}$ with Covariant momentum $p_m$
   Deriving Hamilton’s equations in GCC form
   How to finesse centrifugal and Coriolis energy and other things like phase space.

2. Examples of Hamiltonian dynamics and phase plots
   
   Isotropic Harmonic Oscillator in polar coordinates and “effective potential” (Simulation)
   Coulomb orbits in polar coordinates and “effective potential” (Simulation)
   1D Pendulum and phase plot (Simulation)
   Phase control (Simulation)

3. Exploring phase space and Lagrangian mechanics more deeply
   A weird “derivation” of Lagrange’s equations
   Poincare identity and Action
Deriving Hamilton’s equations
Consider total time derivative of Lagrangian \( L = T - U \)
that is explicit function of coordinates and velocity \( \dot{q} \) ...

\[
\dot{L}(q, \dot{q}, t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m} \frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m} \frac{d\dot{q}^m}{dt}
\]

...of coordinates and velocity and time, too. (Imagine Mad Scientist turning \( U \)-dial.)

\[
\dot{L}(q, \dot{q}, t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m} \frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m} \frac{d\dot{q}^m}{dt} + \frac{\partial L}{\partial t}
\]

Recall Lagrange equations:

\[
\begin{align*}
\dot{p}_m &= \frac{\partial L}{\partial q^m} \\
p_m &= \frac{\partial L}{\partial \dot{q}^m}
\end{align*}
\]

Use product rule:

\[
\frac{d}{dt}(uv) = \frac{du}{dt}v + u\frac{dv}{dt}
\]

Define the Hamiltonian function \( H(p) = p \cdot v - L(v) \)

\[
\frac{d}{dt}\left(p_m \dot{q}^m - L\right) = -\frac{\partial L}{\partial t} = \frac{dH}{dt}
\]

where: \( H = p_m \dot{q}^m - L \)

(Recall: \( \frac{\partial L}{\partial p_m} \equiv 0 \)
and: \( \frac{\partial H}{\partial q^m} \equiv 0 \))

Hamilton’s 1\(^{st}\) GCC equation

\[
\frac{\partial H}{\partial p_m} = \dot{q}^m
\]

a most peculiar relation involving partial vs total

Hamilton’s 2\(^{nd}\) GCC equation

\[
\frac{\partial H}{\partial q^m} = -\dot{p}_m
\]
1. Hamilton prefers Contra\textit{variant} $g^{mn}$ with Covariant momentum $p_m$
Hamilton prefers **Contravariant** $g^{mn}$ with **Covariant** momentum $p_m$

Using Legendre transform of Lagrangian $L = T - U$ with covariant metric definitions of $L$ and $p_m$

We already have: $H = p_m \dot{q}^m - L$ and: $L(\dot{q}) = \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n - U$ and: $p_m = \frac{\partial L}{\partial \dot{q}^m} = M g_{mn} \dot{q}^n$

Now we combine all these:
**Hamilton prefers** **Contra**variant $g^{mn}$ with **Covariant** momentum $p_m$

Using Legendre transform of Lagrangian $L=T-U$ with covariant metric definitions of $L$ and $p_m$

We already have: $H = p_m \dot{q}^m - L$ and: $L(q) = \frac{1}{2} M g_{mn} q^m \dot{q}^n - U$ and: $p_m = \frac{\partial L}{\partial \dot{q}^m} = M g_{mn} \dot{q}^n$

Now we combine all these:

$$H = p_m \dot{q}^m - L = \left( M g_{mn} \dot{q}^n \right) \dot{q}^m - \left( \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n - U \right)$$

$$= M g_{mn} \dot{q}^m \dot{q}^n - \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n + U$$
Hamilton prefers **Contravariant** $g^{mn}$ with **Covariant** momentum $p_m$

Using Legendre transform of Lagrangian $L=T-U$ with covariant metric definitions of $L$ and $p_m$

We already have: $H = p_m \dot{q}^m - L$ and: $L(\dot{q}) = \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n - U$ and: $p_m = \frac{\partial L}{\partial \dot{q}^m} = M g_{mn} \dot{q}^n$

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$$= M g_{mn} \dot{q}^m \dot{q}^n - \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n + U$$

This gives an “illegal dependence” for the Hamiltonian (It mustn’t be “explicit” in velocity $\dot{q}^m$.)

$$H = \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n + U = T + U$$

(Numerically correct ONLY!)
Hamilton prefers **contra**variant $g^{mn}$ with **covariant** momentum $p_m$

Using Legendre transform of Lagrangian $L = T - U$ with covariant metric definitions of $L$ and $p_m$

We already have: $H = p_m \dot{q}^m - L$ and: $L(\dot{q}) = \frac{1}{2} Mg_{mn} \dot{q}^m \dot{q}^n - U$ and: $p_m = \frac{\partial L}{\partial \dot{q}^m} = Mg_{mn} \dot{q}^n$

Now we combine all these:

$$H = p_m \dot{q}^m - L = \left( Mg_{mn} \dot{q}^n \right) \dot{q}^m - \left( \frac{1}{2} Mg_{mn} \dot{q}^m \dot{q}^n - U \right)$$

$$= Mg_{mn} \dot{q}^m \dot{q}^n - \frac{1}{2} Mg_{mn} \dot{q}^m \dot{q}^n + U$$

This gives an “illegal dependence” for the Hamiltonian (It musn’t be “explicit” in velocity $\dot{q}^m$.)

$$H = \frac{1}{2} Mg_{mn} \dot{q}^m \dot{q}^n + U = T + U$$

( Numerically correct ONLY! )

An inverse metric relation $\dot{q}^m = \frac{1}{M} g^{mn} p_n$ gives correct form that depends on momentum $p_m$.

$$H = \frac{1}{2M} g^{mn} p_m p_n + U = T + U \equiv E$$

( Formally and Numerically correct )
Hamilton prefers Contra\text{variant} \, g^{mn} \text{ with Covariant momentum } p_m

Using Legendre transform of Lagrangian \( L = T - U \) with covariant metric definitions of \( L \) and \( p_m \)

We already have: \( H = p_m \dot{q}^m - L \) \text{ and: } \( L(\dot{q}) = \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n - U \) \text{ and: } \( p_m = \frac{\partial L}{\partial \dot{q}^m} = M g_{mn} \dot{q}^n \)

Now we combine all these:

\[
H = p_m \dot{q}^m - L = \left( M g_{mn} \dot{q}^n \right) \dot{q}^m - \left( \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n - U \right)
\]

\[
= M g_{mn} \dot{q}^m \dot{q}^n - \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n + U
\]

This gives an “illegal dependence” for the Hamiltonian (It musn’t be “explicit” in velocity \( \dot{q}^m \))

\[
H = \frac{1}{2} M g_{mn} \dot{q}^m \dot{q}^n + U = T + U \quad \text{ (Numerically correct ONLY!)}
\]

An inverse metric relation \( \dot{q}^m = \frac{1}{M} g^{mn} p_n \) gives correct form that depends on momentum \( p_m \).

\[
H = \frac{1}{2 M} g^{mn} p_m p_n + U = T + U \equiv E
\]

(Pol\text{ar coordinate} Lagrangian was given as:

\[
L(\dot{r}, \dot{\phi}, r, \phi) = \frac{1}{2} M \left( g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2 \right) - U(r, \phi) = \frac{1}{2} M \left( \dot{r}^2 + r^2 \dot{\phi}^2 \right) - U(r, \phi)
\]

Pol\text{ar coordinate Hamiltonian is given here:}

\[
H(p_r, p_\phi, r, \phi) = \frac{1}{2 M} \left( g^{rr} p_r^2 + g^{\phi\phi} p_\phi^2 \right) + U(r, \phi) = \frac{1}{2 M} \left( p_r^2 + \frac{1}{r^2} \cdot p_\phi^2 \right) + U(r, \phi)
\]
2. Examples of Hamiltonian dynamics and phase plots

- Isotropic Harmonic Oscillator in polar coordinates and “effective potential” (Simulation)
- Coulomb orbits in polar coordinates and “effective potential”
- 1D Pendulum and phase plot
- Phase control
Effective potential analysis (Reducing 2D-problem to 1D-problem)
Polar coordinate Hamiltonian can take advantage of H-conservation and $p_m$-conservation

Consider polar coordinate Hamiltonian for Isotropic Harmonic Oscillator potential $U(r) = kr^2/2$:

$$H(p_r, p_{\phi}, r, \phi) = \frac{1}{2M} (g_{rr} p_r^2 + g_{\phi\phi} p_{\phi}^2) + \frac{k \cdot r^2}{2} = \frac{1}{2M} (p_r^2 + \frac{1}{r^2} \cdot p_{\phi}^2) + \frac{k \cdot r^2}{2} = E = \text{const.}$$
Effective potential analysis (Reducing 2D-problem to 1D-problem)

Polar coordinate Hamiltonian can take advantage of $H$-conservation and $p_m$-conservation

Consider polar coordinate Hamiltonian for Isotropic Harmonic Oscillator potential $U(r) = kr^2/2$:

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$H$ is not explicit function of $\phi$, and so Hamilton’s 2nd says: $\dot{p}_\phi = - \frac{\partial H}{\partial \phi} = 0$

Thus momentum $p_\phi$ is conserved constant: $p_\phi = \ell = \text{const.}$
Effective potential analysis (Reducing 2D-problem to 1D-problem)
Polar coordinate Hamiltonian can take advantage of $H$-conservation and $p_m$-conservation

Consider polar coordinate Hamiltonian for isotropic Harmonic Oscillator potential $U(r) = kr^2/2$:

$$H(p_r, p_\phi, r, \phi) = \frac{1}{2M} (g^{rr} p_r^2 + g^{\phi\phi} p_\phi^2) + \frac{k \cdot r^2}{2} = \frac{1}{2M} (p_r^2 + \frac{1}{r^2} \cdot p_\phi^2) + \frac{k \cdot r^2}{2} = E = \text{const.}$$

$H$ is not explicit function of $\phi$, and so Hamilton’s 2nd says: $\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$
Thus momentum $p_\phi$ is conserved constant: $p_\phi = \ell = \text{const.}$

$$\frac{p_r^2}{2M} + \frac{p_\phi^2}{2Mr^2} + \frac{k \cdot r^2}{2} = \frac{p_r^2}{2M} + \frac{\ell^2}{2Mr^2} + \frac{k \cdot r^2}{2} = E = \text{const.}$$

$$p_r^2 = 2ME - \frac{\ell^2}{r^2} - Mk \cdot r^2$$
Effective potential analysis (Reducing 2D-problem to 1D-problem)
Polar coordinate Hamiltonian can take advantage of H-conservation and $p_m$-conservation

Consider polar coordinate Hamiltonian for Isotropic Harmonic Oscillator potential $U(r) = kr^2/2$:

\[ H(p_r, p_\phi, r, \phi) = \frac{1}{2M} \left( g^{rr} p_r^2 + g^{\phi\phi} p_\phi^2 \right) + \frac{k \cdot r^2}{2} = \frac{1}{2M} \left( p_r^2 + \frac{1}{r^2} \cdot p_\phi^2 \right) + \frac{k \cdot r^2}{2} = E = \text{const}. \]

$H$ is not explicit function of $\phi$, and so Hamilton’s 2nd says: $\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$ Thus momentum $p_\phi$ is conserved constant: $p_\phi = \ell = \text{const.}$

\[
\frac{p_r^2}{2M} + \frac{p_\phi^2}{2Mr^2} + \frac{k \cdot r^2}{2} = \frac{p_r^2}{2M} + \frac{\ell^2}{2Mr^2} + \frac{k \cdot r^2}{2} = E = \text{const}.
\]

\[
p_r^2 = 2ME - \frac{\ell^2}{r^2} - Mk \cdot r^2
\]

“effective” PE

Same applies to any radial potential $U(r)$

\[
E = \frac{p_r^2}{2M} + \frac{\ell^2}{2Mr^2} + U(r)
\]

“real” PE

“centifugal-barrier” PE
Effective potential analysis (Reducing 2D-problem to 1D-problem)

Polar coordinate Hamiltonian can take advantage of H-conservation and $p_m$-conservation

Consider polar coordinate Hamiltonian for Isotropic Harmonic Oscillator potential $U(r) = kr^2/2$:

$$H(p_r, p_\phi, r, \phi) = \frac{1}{2M}(g^{rr}p_r^2 + g^{\phi\phi}p_\phi^2) + \frac{k}{2}r^2 = \frac{1}{2M}(p_r^2 + \frac{1}{r^2} \cdot p_\phi^2) + \frac{k}{2}r^2 = E = \text{const.}$$

$H$ is not explicit function of $\phi$, and so Hamilton's 2nd says: $\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$

Thus momentum $p_\phi$ is conserved constant: $p_\phi = \ell = \text{const.}$

$$\frac{p_r^2}{2M} + \frac{p_\phi^2}{2Mr^2} + \frac{k}{2} = \frac{p_r^2}{2M} + \frac{\ell^2}{2Mr^2} + \frac{k}{2} = E = \text{const.}$$

$$p_r = 2ME - \frac{\ell^2}{r^2} - Mk\cdot r^2$$

$$p_r = M\dot{r} = \sqrt{2ME - \frac{\ell^2}{r^2} - Mk\cdot r^2} = \sqrt{2M} \sqrt{E - \frac{\ell^2}{2Mr^2} - \frac{k}{2}r^2}$$

Radial KE is

$$\frac{M\dot{r}^2}{2} = E - \frac{\ell^2}{2Mr^2} - \frac{k}{2}r^2$$
Effective potential analysis (Reducing 2D-problem to 1D-problem)

Polar coordinate Hamiltonian can take advantage of \( H \)-conservation and \( p_m \)-conservation

Consider polar coordinate Hamiltonian for Isotropic Harmonic Oscillator potential \( U(r) = kr^2/2 \):

\[
H(p_r, p_\phi, r, \phi) = \frac{1}{2M} \left( g^{rr} p_r^2 + g^{\phi\phi} p_\phi^2 \right) + \frac{k \cdot r^2}{2} = \frac{1}{2M} \left( p_r^2 + \frac{1}{r^2} \cdot p_\phi^2 \right) + \frac{k \cdot r^2}{2} = E = \text{const.}
\]

\( H \) is not explicit function of \( \phi \), and so Hamilton's 2nd says:

\[
\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0
\]

Thus momentum \( p_\phi \) is conserved constant: \( p_\phi = \ell = \text{const.} \)

\[
\frac{p_r^2}{2M} + \frac{p_\phi^2}{2Mr^2} + \frac{k \cdot r^2}{2} = \frac{p_r^2}{2M} + \frac{\ell^2}{2Mr^2} + \frac{k \cdot r^2}{2} = E = \text{const.}
\]

\[
p_r^2 = 2ME - \frac{\ell^2}{r^2} - Mk \cdot r^2
\]

Radial KE is

\[
\frac{M \dot{r}^2}{2} = E - \frac{\ell^2}{2Mr^2} - \frac{k \cdot r^2}{2}
\]

Radial KE is

\[
\frac{dr}{dt} = \sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2 r^2} - \frac{k}{M} \cdot r^2}
\]

Solution: 

\[
t = \int_{r_\text{<}}^{r_\text{>}} \frac{dr}{\sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2 r^2} - \frac{k}{M} \cdot r^2}}
\]
Effective potential analysis (Reducing 2D-problem to 1D-problem)

Polar coordinate Hamiltonian can take advantage of H-conservation and $p_m$-conservation

Consider polar coordinate Hamiltonian for Isotropic Harmonic Oscillator potential $U(r) = kr^2/2$:

$$H(p_r, p_\phi, r, \phi) = \frac{1}{2M} (g^{rr} p_r^2 + g^{\phi\phi} p_\phi^2) + k \cdot r^2 / 2 = \frac{1}{2M} (p_r^2 + \frac{1}{r^2} \cdot p_\phi^2) + \frac{k \cdot r^2}{2} = E = \text{const.}$$

$H$ is not explicit function of $\phi$, and so Hamilton’s 2nd says: $\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$ 

Thus momentum $p_\phi$ is conserved constant: $p_\phi = \ell = \text{const.}$

$$\frac{p_r^2}{2M} + \frac{p_\phi^2}{2Mr^2} + \frac{k \cdot r^2}{2} = \frac{p_r^2}{2M} + \frac{\ell^2}{2Mr^2} + \frac{k \cdot r^2}{2} = E = \text{const.}$$

$$p_r = \sqrt{2ME - \frac{\ell^2}{r^2} - Mk \cdot r^2}$$

$$p_r = M \ddot{r} = \sqrt{2ME - \frac{\ell^2}{r^2} - Mk \cdot r^2} = \sqrt{2M} \sqrt{E - \frac{\ell^2}{2Mr^2} - \frac{k \cdot r^2}{2}}$$

Radial KE is

$$\frac{M \dot{r}^2}{2} = E - \frac{\ell^2}{2Mr^2} - \frac{k \cdot r^2}{2}$$

$$\frac{dr}{dt} = \sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2r^2} - \frac{k}{M} \cdot r^2}$$

Solution:

$$t = \int_{r^2}^{r_2} \frac{dr}{\sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2r^2} - \frac{k}{M} \cdot r^2}}$$

Conclusion: $t = \int_{r^2}^{r_2} \frac{dr}{\sqrt{\frac{2E}{M} - \frac{\ell^2}{M^2r^2} - \frac{2U(r)}{M}}}$
Hamiltonian dynamics for isotropic harmonic oscillator potential $U(r) = kr^2/2$

Energy:
$E = k(a^2 + b^2)/2$

Angular momentum:
$\mu = \sqrt{km} \ a b$

Perigee is faster turning point
Apogee is slower turning point
2. Examples of Hamiltonian dynamics and phase plots

- Isotropic Harmonic Oscillator in polar coordinates and “effective potential”
- Coulomb orbits in polar coordinates and “effective potential” (Simulation)
- 1D Pendulum and phase plot (Simulation)
- Phase control (Simulation)
Hamiltonian dynamics for Coulomb potential $U(r) = -\frac{k}{r}$

Energy:
$E = \frac{p_r^2}{2M} + \frac{\ell^2}{2Mr^2} - \frac{k}{r}$

Angular momentum:
$\ell = \sqrt{|km\lambda|} = b \sqrt{2m|E|}$

Perigee is faster turning point $\rho_-$

Apogee is slower turning point $\rho_+$

"real" PE

"effective" PE

"centrifugal-barrier" PE
Lecture 17 ends here
Tue. 2.28.2012
2. Examples of Hamiltonian dynamics and phase plots

Isotropic Harmonic Oscillator in polar coordinates and “effective potential”
Coulomb orbits in polar coordinates and “effective potential”
1D Pendulum and phase plot *(Simulation)*
Phase control *(Simulation)*
Example of plot of Hamilton for 1D-solid pendulum in its Phase Space \((\theta, p_\theta)\)

\[ H(p_\theta, \theta) = E = \frac{1}{2I} p_\theta^2 - MgR \cos \theta, \text{ or: } p_\theta = \sqrt{2I \left( E + MgR \cos \theta \right)} \]

\[
\begin{pmatrix}
\dot{q} \\
\dot{p}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial}{\partial q} H \\
-\frac{\partial}{\partial q} H
\end{pmatrix} = e_H \times (-\nabla H) = (H\text{-axis}) \times (\text{fall line}), \text{ where: }
\begin{cases}
(H\text{-axis}) = e_H = e_q \times e_p \\
(\text{fall line}) = -\nabla H
\end{cases}
\]