Lagrangian and Hamiltonian dynamics:
Living with duality in GCC cells and vectors,
(Ch. 12 of Unit 1)

1. GCC Cells and base vectors: **Covariant** $E_m$ vs. **Contravariant** $E^m$
   Polar coordinate examples

2. Metric quadratic forms and tensors: **Covariant** $g_{mn}$ vs. **Invariant** $\delta_m^n$ vs. **Contravariant** $g^{mn}$
   Polar coordinate examples

3. Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant velocity** $\dot{q}^m$
   Polar coordinate examples
   How to finesse centrifugal and Coriolis “forces”

   Lecture 16 ends here

4. Hamilton prefers **Contravariant** $g^{mn}$ with **Covariant momentum** $p_m$
   Deriving Hamilton’s equations
   How to finesse centrifugal and Coriolis energy and other things like phase space.
A dual set of quasi-unit vectors show up in Jacobian J and Kajobian K.

J-Columns are covariant vectors $\{E_1 = E_r, E_2 = E_\phi\}$  
K-Rows are contravariant vectors $\{E^1 = E^r, E^2 = E^\phi\}$

\[
\begin{pmatrix}
\frac{\partial x_1}{\partial q_1} & \frac{\partial x_1}{\partial q_2} \\
\frac{\partial x_2}{\partial q_1} & \frac{\partial x_2}{\partial q_2} \\
\frac{\partial x_1}{\partial q_1} & \frac{\partial x_2}{\partial q_2}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial x}{\partial r} = \cos \phi & \frac{\partial x}{\partial \phi} = -r \sin \phi \\
\frac{\partial y}{\partial r} = \sin \phi & \frac{\partial y}{\partial \phi} = r \cos \phi
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{\partial r}{\partial x} = \cos \phi & \frac{\partial r}{\partial y} = \sin \phi \\
\frac{\partial \phi}{\partial x} = -\sin \phi & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{r}
\end{pmatrix} \leftarrow E^r = E_1
\]

\[
\begin{pmatrix}
\frac{\partial r}{\partial x} = \cos \phi & \frac{\partial r}{\partial y} = \sin \phi \\
\frac{\partial \phi}{\partial x} = -\sin \phi & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{r}
\end{pmatrix} \leftarrow E^\phi = E_2
\]

Derived from polar definition: $x = r \cos \phi$ and $y = r \sin \phi$

(a) Polar coordinate bases  
(b) Covariant bases $\{E_1, E_2\}$ (Tangent)  
\[d\mathbf{r} = E_1 dq^1 + E_2 dq^2\]

(c) Contravariant bases $\{E^1, E^2\}$ (Normal)  
\[\mathbf{F} = F_1 E^1 + F_2 E^2\]

Inverse polar definition:  
\[r^2 = x^2 + y^2\quad\text{and}\quad \phi = \text{atan2}(y,x)\]
Comparison: **Covariant** \( E_m = \frac{\partial r}{\partial q^m} \) vs. **Contravariant** \( E^m = \frac{\partial q^m}{\partial r} = \nabla q^m \)

Covariant bases \( \{E_1, E_2\} \) match cell walls

\[
\Delta r = E_1 \Delta q^1 + E_2 \Delta q^2
\]

is based on chain rule:

\[
dr = \frac{\partial r}{\partial q^1} dq^1 + \frac{\partial r}{\partial q^2} dq^2 = E_1 dq^1 + E_2 dq^2
\]

**geometric unit**

(Tangent)
Comparison: **Covariant** $E_m = -\frac{\partial r}{\partial q^m}$ vs. **Contravariant** $E^m = \frac{\partial q^m}{\partial r} = \nabla q^m$

*Covariant bases* $\{E_1, E_2\}$ match cell walls (Tangent)

$\Delta r = E_1 \Delta q^1 + E_2 \Delta q^2$

is based on chain rule: $dr = \frac{\partial r}{\partial q^1} dq^1 + \frac{\partial r}{\partial q^2} dq^2 = E_1 dq^1 + E_2 dq^2$

$E_1$ follows tangent to $q^2 = \text{const.}$ ...

since only $q^1$ varies in $\frac{\partial r}{\partial q^1}$

while $q^2, q^3, ...$ remain constant

$\frac{\partial r}{\partial q^2} = E_2$

$\frac{\partial r}{\partial q^1} = E_1$

$\Delta q^2 = 1.0$

$\Delta q^1 = 1.0$

$q^2 = 20.1$

$q^1 = 10.1$

$q^2 = 200$

$q^1 = 100$

$E_1 = \frac{\partial r}{\partial q^1}$

$E_2 = \frac{\partial r}{\partial q^2}$
Comparison: **Covariant** $E_m^* = \frac{\partial r}{\partial q^m}$ vs. **Contravariant** $E^m = \frac{\partial q^m}{\partial r} = \nabla q^m$

Covariant bases $\{E_1, E_2\}$ match cell walls

$$\Delta r = E_1 \Delta q^1 + E_2 \Delta q^2$$

is based on chain rule: $d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^1} dq^1 + \frac{\partial \mathbf{r}}{\partial q^2} dq^2 = E_1 dq^1 + E_2 dq^2$

$E_1$ follows tangent to $q^2 = \text{const.}$ ...

since only $q^1$ varies in $\frac{\partial \mathbf{r}}{\partial q^1}$

while $q^2, q^3, ...$ remain constant

$E_m$ are convenient bases for extensive quantities like distance and velocity.

$$\mathbf{V} = V^1 E_1 + V^2 E_2 = V^1 \frac{\partial \mathbf{r}}{\partial q^1} + V^2 \frac{\partial \mathbf{r}}{\partial q^2}$$
Comparison: **Covariant** $E_m = \frac{\partial r}{\partial q^m}$ vs. **Contravariant** $E^m = \frac{\partial q^m}{\partial r} = \nabla q^m$

Covariant bases $\{E_1, E_2\}$ match cell walls

$\Delta r = E_1 \Delta q^1 + E_2 \Delta q^2$

is based on chain rule: $dr = \frac{\partial r}{\partial q^1} dq^1 + \frac{\partial r}{\partial q^2} dq^2 = E_1 dq^1 + E_2 dq^2$

$E_1$ follows tangent to $q^2 = \text{const.}$ ... since only $q^1$ varies in $\frac{\partial r}{\partial q^1}$

while $q^2, q^3, \ldots$ remain constant

$E_m$ are convenient bases for extensive quantities like distance and velocity.

$V = V^1 E_1 + V^2 E_2 = V^1 \frac{\partial r}{\partial q^1} + V^2 \frac{\partial r}{\partial q^2}$

Contravariant $\{E^1, E^2\}$ match reciprocal cells

$\frac{\partial q^2}{\partial r} = \nabla q^2 = E^2$

$F = F_1 E^1 + F_2 E^2$

$E^1$ is normal to $q^1 = \text{const.}$ since

gradien of $q^1$ is vector sum $\nabla q^1 = \left( \begin{array}{c} \frac{\partial q^1}{\partial x} \\ \frac{\partial q^1}{\partial y} \end{array} \right)$

of all its partial derivatives
Comparison: **Covariant** \( E_m = \frac{\partial r}{\partial q^m} \) vs. **Contravariant** \( E^m = \frac{\partial q^m}{\partial r} = \nabla q^m \)

**Covariant bases** \( \{E_1, E_2\} \) match cell walls

\[
\Delta r = E_1 \Delta q^1 + E_2 \Delta q^2
\]

is based on chain rule:

\[
dr = \frac{\partial r}{\partial q^1} dq^1 + \frac{\partial r}{\partial q^2} dq^2 = E_1 dq^1 + E_2 dq^2
\]

\( E_1 \) follows **tangent** to \( q^2 = \text{const.} \) ...

since only \( q^1 \) varies in \( \frac{\partial r}{\partial q^1} \)
while \( q^2, q^3, \ldots \) remain constant

\( E_m \) are convenient bases for *extensive* quantities like distance and velocity.

\[
V = V^1 E_1 + V^2 E_2 = V^1 \frac{\partial r}{\partial q^1} + V^2 \frac{\partial r}{\partial q^2}
\]

**Contravariant** \( \{E^1, E^2\} \) match reciprocal cells

\[
\frac{\partial q^2}{\partial r} = \nabla q^2 = E^2
\]

\[
F = F_1 E^1 + F_2 E^2
\]

\( E^1 \) is **normal** to \( q^1 = \text{const.} \) since **gradient** of \( q^1 \) is vector sum \( \nabla q^1 = \begin{pmatrix} \frac{\partial q^1}{\partial x} \\ \frac{\partial q^1}{\partial y} \end{pmatrix} \)

\( E^m \) are convenient bases for *intensive* quantities like force and momentum.

\[
F = F_1 E^1 + F_2 E^2 = F_1 \frac{\partial q^1}{\partial r} + F_2 \frac{\partial q^2}{\partial r} = F_1 \nabla q^1 + F_2 \nabla q^2
\]

\[
E^m \text{ are convenient bases for } \frac{\partial q^m}{\partial r} \text{ as bases for force and momentum.}
\]
Comparison: **Covariant** \( \mathbf{E}_m = -\frac{\partial \mathbf{r}}{\partial q^m} \) vs. **Contravariant** \( \mathbf{E}^n = \frac{\partial q^n}{\partial \mathbf{r}} = \nabla q^n \)

**Covariant bases** \( \{ \mathbf{E}_1, \mathbf{E}_2 \} \) match cell walls

\[
\Delta \mathbf{r} = \mathbf{E}_1 \Delta q^1 + \mathbf{E}_2 \Delta q^2
\]

is based on chain rule: \( \mathbf{d} \mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^1} dq^1 + \frac{\partial \mathbf{r}}{\partial q^2} dq^2 = \mathbf{E}_1 dq^1 + \mathbf{E}_2 dq^2 \)

\( \mathbf{E}_1 \) follows tangent to \( q^2 = \text{const.} \) ...

since only \( q^1 \) varies in \( \frac{\partial \mathbf{r}}{\partial q^1} \)

while \( q^2, q^3, \ldots \) remain constant

\( \mathbf{E}_m \) are convenient bases for extensive quantities like distance and velocity.

\[
\mathbf{v} = V^1 \mathbf{E}_1 + V^2 \mathbf{E}_2 = V^1 \frac{\partial \mathbf{r}}{\partial q^1} + V^2 \frac{\partial \mathbf{r}}{\partial q^2}
\]

**Contravariant** \( \{ \mathbf{E}^1, \mathbf{E}^2 \} \) match reciprocal cells

\[
\frac{\partial q^2}{\partial \mathbf{r}} = \nabla q^2 = \mathbf{E}^2
\]

\[
\mathbf{F} = F_1 \mathbf{E}^1 + F_2 \mathbf{E}^2
\]

\[
\mathbf{E}^1 = \frac{\partial q^1}{\partial \mathbf{r}} = \nabla q^1
\]

\( \mathbf{E}^1 \) is normal to \( q^1 = \text{const.} \) since

**gradient** of \( q^1 \) is vector sum \( \nabla q^1 = \left( \frac{\partial q^1}{\partial x}, \frac{\partial q^1}{\partial y} \right) \)

\( \mathbf{E}^m \) are convenient bases for intensive quantities like force and momentum.

\[
\mathbf{F} = F_1 \mathbf{E}^1 + F_2 \mathbf{E}^2 = F_1 \frac{\partial q^1}{\partial \mathbf{r}} + F_2 \frac{\partial q^2}{\partial \mathbf{r}} = F_1 \nabla q^1 + F_2 \nabla q^2
\]
2. Metric quadratic forms and tensors:

Covariant $g_{mn}$ vs. Invariant $\delta_m^n$ vs. Contravariant $g^{mn}$
Covariant $g_{mn}$ vs. Invariant $\delta^n_m$ vs. Contravariant $g^{mn}$

Covariant metric tensor $g_{mn}$

$$E_m \cdot E_n = \frac{\partial r}{\partial q^m} \cdot \frac{\partial r}{\partial q^n} \equiv g_{mn}$$

Invariant Kronecker unit tensor

$$E_m \cdot E^n = \frac{\partial r}{\partial q^m} \cdot \frac{\partial q^n}{\partial r} = \delta^n_m$$

Contravariant metric tensor $g^{mn}$

$$E^m \cdot E^n = \frac{\partial q^m}{\partial r} \cdot \frac{\partial q^n}{\partial r} \equiv g^{mn}$$

$$\delta^n_m \equiv \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$
**Covariant** \( g_{mn} \) vs. **Invariant** \( \delta_m^n \) vs. **Contravariant** \( g^{mn} \)

\[
E_m \cdot E_n = \frac{\partial r}{\partial q^m} \cdot \frac{\partial r}{\partial q^n} \equiv g_{mn}
\]

\[
E_m \cdot E^m = \frac{\partial r}{\partial q^m} \cdot \frac{\partial q^n}{\partial r} = \delta_m^n
\]

\[
E^m \cdot E^n = \frac{\partial q^m}{\partial r} \cdot \frac{\partial q^n}{\partial r} \equiv g^{mn}
\]

**Covariant** metric tensor \( g_{mn} \)

**Invariant** Kroneker unit tensor \( \delta_m^n \)

\[
\delta_m^n \equiv \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}
\]

**Contravariant** metric tensor \( g^{mn} \)

Polar coordinate examples (again):

\[
\langle J \rangle = \begin{pmatrix}
\frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} \\
\frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial x}{\partial r} = \cos \phi & \frac{\partial x}{\partial \phi} = -r \sin \phi \\
\frac{\partial y}{\partial r} = \sin \phi & \frac{\partial y}{\partial \phi} = r \cos \phi
\end{pmatrix}
\]

\[
\langle K \rangle = \langle J^{-1} \rangle = \begin{pmatrix}
\frac{\partial r}{\partial x} = \cos \phi & \frac{\partial r}{\partial y} = \sin \phi \\
\frac{\partial \phi}{\partial x} = -\sin \phi & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{r}
\end{pmatrix} \leftarrow E^r = E^1
\]

\[
\uparrow E_1 \uparrow E_2 \quad \uparrow E_r \quad \uparrow E_\phi
\]
\[ \text{Covariant } g_{mn} \text{ vs. Invariant } \delta_m^n \text{ vs. Contravariant } g^{mn} \]

\[ E_m \cdot E_n = \frac{\partial r}{\partial q^m} \frac{\partial r}{\partial q^n} \equiv g_{mn} \]

\[ E_m \cdot E^n = \frac{\partial r}{\partial q^m} \frac{\partial q^n}{\partial r} = \delta_m^n \]

\[ E^m \cdot E^n = \frac{\partial q^m}{\partial r} \frac{\partial q^n}{\partial r} \equiv g^{mn} \]

\[ \text{Covariant metric tensor } g_{mn} \]

\[ \text{Invariant Kroneker unit tensor } \delta_m^n \]

\[ \text{Contravariant metric tensor } g^{mn} \]

\[ \delta_m^n \equiv \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \]

\[ \langle J \rangle = \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} = \cos \phi & \frac{\partial x}{\partial \phi} = -r \sin \phi \\ \frac{\partial y}{\partial r} = \sin \phi & \frac{\partial y}{\partial \phi} = r \cos \phi \end{pmatrix} \]

\[ \langle K \rangle = \langle J^{-1} \rangle = \begin{pmatrix} \frac{\partial r}{\partial x} = \cos \phi & \frac{\partial r}{\partial y} = \sin \phi \\ \frac{\partial \phi}{\partial x} = \frac{-\sin \phi}{r} & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{r} \end{pmatrix} \leftrightarrow E^r = E^1 \]

\[ \text{Polar coordinate examples (again):} \]

\[ g_{rr} g_{r\phi} g_{\phi r} g_{\phi\phi} = \begin{pmatrix} E_r \cdot E_r & E_r \cdot E_\phi \\ E_\phi \cdot E_r & E_\phi \cdot E_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \]

\[ \delta^r_r \delta^\phi_\phi = \begin{pmatrix} E_r \cdot E_r & E_r \cdot E_\phi \\ E_\phi \cdot E_r & E_\phi \cdot E_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ \delta^r_\phi \delta^\phi_r = \begin{pmatrix} E_r \cdot E^r & E_r \cdot E^\phi \\ E_\phi \cdot E^r & E_\phi \cdot E^\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix} \]
3. Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant velocity** $\dot{q}^m$
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity

Lagrangian $L=KE-U$ is supposed to be explicit function of velocity.

$L(v)=\frac{1}{2} M v \cdot v - U = \frac{1}{2} M \dot{r} \cdot \dot{r} - U = \frac{1}{2} M (E_m \dot{q}^m) \cdot (E_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})$
Lagrange prefers \textbf{Covariant} $g_{mn}$ with \textbf{Contravariant} velocity

Lagrangian $KE-U$ is supposed to be explicit function of velocity.

$L(v) = \frac{1}{2} M v \cdot v - U = \frac{1}{2} M \ddot{r} \cdot \dot{r} - U = \frac{1}{2} M \left( E_m \dot{q}^m \right) \cdot \left( E_n \dot{q}^n \right) - U = \frac{1}{2} M \left( g_{mn} \dot{q}^m \dot{q}^n \right) - U = L(\dot{q})$

Use polar coordinate \textbf{Covariant} $g_{mn}$ metric (1-page back)

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} E_r \cdot E_r & E_r \cdot E_\phi \\ E_\phi \cdot E_r & E_\phi \cdot E_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity

Lagrangian KE-U is supposed to be explicit function of velocity.

$$L(v) = \frac{1}{2} M v \cdot \dot{v} - U = \frac{1}{2} M \dot{r} \cdot \dot{r} - U = \frac{1}{2} M (E_m \dot{q}^m) \cdot (E_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})$$

Use polar coordinate **Covariant** $g_{mn}$ metric (1-page back)

$$\left[\begin{array}{cc}
g_{rr} & g_{r\phi} \\
g_{\phi r} & g_{\phi \phi}
\end{array}\right] = \left[\begin{array}{cc}
E_r \cdot E_r & E_r \cdot E_\phi \\
E_\phi \cdot E_r & E_\phi \cdot E_\phi
\end{array}\right] = \left[\begin{array}{cc}
1 & 0 \\
0 & r^2
\end{array}\right]$$

This gives polar GCC form (Actually it’s an OCC or Orthogonal Curvilinear Coordinate form)

$$L(\dot{r}, \dot{\phi}) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi \phi} \dot{\phi}^2) - U(r, \phi) = \frac{1}{2} M (1 \cdot \dot{r}^2 + r^2 \dot{\phi}^2) - U(r, \phi)$$
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity

Lagrangian KE-U is supposed to be explicit function of velocity.

$L(v) = \frac{1}{2} M v \cdot v - U = \frac{1}{2} M \dot{r} \cdot \dot{r} - U = \frac{1}{2} M (E_m \dot{q}^m) \cdot (E_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})$

Use polar coordinate **Covariant** $g_{mn}$ metric (1-page back)

$$
\begin{pmatrix}
 g_{rr} & g_{r\phi} \\
 g_{\phi r} & g_{\phi\phi}
\end{pmatrix} = \begin{pmatrix}
 E_r \cdot E_r & E_r \cdot E_\phi \\
 E_\phi \cdot E_r & E_\phi \cdot E_\phi
\end{pmatrix} = \begin{pmatrix}
 1 & 0 \\
 0 & r^2
\end{pmatrix}
$$

This gives polar GCC form (Actually it’s an OCC or Orthogonal Curvilinear Coordinate form)

$$L(\dot{r}, \dot{\phi}) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r, \phi) = \frac{1}{2} M (1 \cdot \dot{r}^2 + r^2 \dot{\phi}^2) - U(r, \phi)$$

GCC Lagrange equations follow. 1st $L$-equation is momentum $p_m$ definition for each coordinate $q^m$:

$$p_r = \frac{\partial L}{\partial \dot{r}} = M \ g_{rr} \dot{r} = M \ \dot{r}$$

Nothing too surprising; radial momentum $p_r$ has the usual linear $M \cdot v$ form
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity

Lagrangian KE-U is supposed to be explicit function of velocity.

$$L(v) = \frac{1}{2} M v \cdot v - U = \frac{1}{2} M \dot{r} \cdot \dot{r} - U = \frac{1}{2} M \left( E_m \dot{q}^m \right) \cdot \left( E_n \dot{q}^n \right) - U = \frac{1}{2} M \left( g_{mn} \dot{q}^m \dot{q}^n \right) - U = L(\dot{q})$$

Use polar coordinate **Covariant** $g_{mn}$ metric (1-page back)

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi \phi} \end{pmatrix} = \begin{pmatrix} E_r \cdot E_r & E_r \cdot E_\phi \\ E_\phi \cdot E_r & E_\phi \cdot E_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

This gives polar GCC form (Actually it's an OCC or Orthogonal Curvilinear Coordinate form)

$$L(\dot{r}, \dot{\phi}) = \frac{1}{2} M \left( g_{rr} \dot{r}^2 + g_{\phi \phi} \dot{\phi}^2 \right) - U(r, \phi) = \frac{1}{2} M \left( 1 \cdot \dot{r}^2 + r^2 \dot{\phi}^2 \right) - U(r, \phi)$$

GCC Lagrange equations follow. 1st $L$-equation is momentum $p_m$ definition for each coordinate $q^m$:

$$p_r = \frac{\partial L}{\partial \dot{r}} = M g_{rr} \dot{r} = M \dot{r}$$

Nothing too surprising; radial momentum $p_r$ has the usual linear $M \cdot v$ form

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = M g_{\phi \phi} \dot{\phi} = M r^2 \dot{\phi}$$

Wow! $g_{\phi \phi}$ gives moment-of-inertia factor $Mr^2$ automatically for the angular momentum $p_\phi = Mr^2 \dot{\phi}$. 
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity

Lagrangian KE-$U$ is supposed to be explicit function of velocity.

$$L(v) = \frac{1}{2} M v \cdot v - U = \frac{1}{2} M (E_m \dot{q}^m) \cdot (E_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})$$

Use polar coordinate **Covariant** $g_{mn}$ metric (1-page back) 

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} E_r \cdot E_r & E_r \cdot E_\phi \\ E_\phi \cdot E_r & E_\phi \cdot E_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

This gives polar GCC form (Actually it's an OCC or Orthogonal Curvilinear Coordinate form)

$$L(\dot{r}, \dot{\phi}) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r, \phi) = \frac{1}{2} M (1 \cdot \dot{r}^2 + r^2 \dot{\phi}^2) - U(r, \phi)$$

GCC Lagrange equations follow. 1st $L$-equation is momentum $p_m$ definition for each coordinate $q^m$:

$$p_r = \frac{\partial L}{\partial \dot{r}} = M g_{rr} \dot{r} = M \dot{r} \text{ Nothing too surprising; radial momentum } p_r \text{ has the usual linear } M \cdot v \text{ form}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = M g_{\phi\phi} \dot{\phi} = M r^2 \dot{\phi} \text{ Wow! } g_{\phi\phi} \text{ gives moment-of-inertia factor } Mr^2 \text{ automatically for the angular momentum } p_\phi = Mr^2 \omega.$$  

2nd $L$-equation involves total time derivative $\dot{p}_m$ for each momentum $p_m$:

$$\dot{p}_r = \frac{\partial L}{\partial r} = \frac{M}{2} \frac{\partial g_{\phi\phi}}{\partial r} \dot{\phi}^2 - \frac{\partial U}{\partial r} = M r \dot{\phi}^2 - \frac{\partial U}{\partial r} \text{ Centrifugal force } Mr\omega^2$$

$$\dot{p}_\phi = \frac{\partial L}{\partial \phi} = 0 - \frac{\partial U}{\partial \phi} \text{ Angular momentum } p_\phi \text{ is conserved if potential } U \text{ has no explicit } \phi \text{-dependence}$$
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity.

Lagrangian KE-$U$ is supposed to be explicit function of velocity.

$$L(v) = \frac{1}{2} M v \cdot v - U = \frac{1}{2} M (E_m \dot{q}^m) \cdot (E_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})$$

Use polar coordinate **Covariant** $g_{mn}$ metric (1-page back)

$$\begin{pmatrix}
g_{rr} & g_{r\phi} \\
g_{\phi r} & g_{\phi\phi}
\end{pmatrix} = \begin{pmatrix}
E_r \cdot E_r & E_r \cdot E_{\phi} \\
E_{\phi} \cdot E_r & E_{\phi} \cdot E_{\phi}
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & r^2
\end{pmatrix}$$

This gives polar GCC form (Actually it’s an OCC or Orthogonal Curvilinear Coordinate form)

$$L(\dot{r}, \dot{\phi}) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r, \phi) = \frac{1}{2} M (1 \cdot \dot{r}^2 + r^2 \dot{\phi}^2) - U(r, \phi)$$

**GCC Lagrange equations follow.** 1st $L$-equation is momentum $p_m$ definition for each coordinate $q^m$:

$$p_r = \frac{\partial L}{\partial \dot{r}} = M g_{rr} \dot{r} = M \dot{r}$$

Nothing too surprising; radial momentum $p_r$ has the usual linear $M \cdot v$ form

$$p_{\phi} = \frac{\partial L}{\partial \dot{\phi}} = Mg_{\phi\phi} \dot{\phi} = Mr^2 \dot{\phi}$$

Wow! $g_{\phi\phi}$ gives moment-of-inertia factor $Mr^2$ automatically for the angular momentum $p_{\phi} = Mr^2 \omega$.

2nd $L$-equation involves total time derivative $\dot{p}_m$ for each momentum $p_m$:

$$\dot{p}_r = \frac{\partial L}{\partial r} = \frac{M}{2} \frac{\partial g_{\phi\phi}}{\partial r} \dot{\phi}^2 - \frac{\partial U}{\partial r} = M r \dot{\phi}^2 - \frac{\partial U}{\partial r}$$

Centrifugal force $M r \omega^2$

$$\dot{p}_{\phi} = \frac{\partial L}{\partial \phi} = 0 - \frac{\partial U}{\partial \phi}$$

Angular momentum $p_{\phi}$ is conserved if potential $U$ has no explicit $\phi$-dependence

Find $\dot{p}_m$ directly from 1st $L$-equation: $\dot{p}_m \equiv \frac{dp_m}{dt} = \frac{d}{dt} M (g_{mn} \dot{q}^n) = M (\dot{g}_{mn} \dot{q}^n + g_{mn} \ddot{q}^n)$ Equate it to $\dot{P}_m$ in 2nd $L$-equation:
Lagrange prefers Covariant \( g_{mn} \) with Contravariant velocity

Lagrangian KE-U is supposed to be explicit function of velocity.

\[
L(\mathbf{v}) = \frac{1}{2} M \mathbf{v} \cdot \mathbf{v} - U = \frac{1}{2} M (\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}) = \frac{1}{2} M (\mathbf{E}_m \dot{q}^m) \cdot (\mathbf{E}_n \dot{q}^n) - U = \frac{1}{2} M \left( g_{mn} \dot{q}^m \dot{q}^n \right) - U = L(\dot{q})
\]

Use polar coordinate Covariant \( g_{mn} \) metric (1-page back)

\[
\begin{pmatrix}
g_{rr} & g_{r\phi} \\
g_{\phi r} & g_{\phi \phi}
\end{pmatrix} = \begin{pmatrix}
E_r \cdot E_r & E_r \cdot E_\phi \\
E_\phi \cdot E_r & E_\phi \cdot E_\phi
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & r^2
\end{pmatrix}
\]

This gives polar GCC form (Actually it’s an OCC or Orthogonal Curvilinear Coordinate form)

\[
L(\dot{r}, \dot{\phi}) = \frac{1}{2} M \left( g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2 \right) - U(r, \phi) = \frac{1}{2} M \left( 1 \cdot \dot{r}^2 + r^2 \dot{\phi}^2 \right) - U(r, \phi)
\]

GCC Lagrange equations follow. 1st \( L \)-equation is momentum \( p_m \) definition for each coordinate \( q^m \):

\[
p_r = \frac{\partial L}{\partial \dot{r}} = M g_{rr} \dot{r} = M \dot{r} \quad \text{Nothing too surprising; radial momentum} \ p_r \ \text{has the usual linear} \ M \cdot \mathbf{v} \ \text{form}
\]

\[
p_\phi = \frac{\partial L}{\partial \dot{\phi}} = M g_{\phi\phi} \dot{\phi} = M r^2 \dot{\phi} \quad \text{Wow!} \ g_{\phi\phi} \ \text{gives moment-of-inertia factor} \ M r^2 \ \text{automatically for the angular momentum} \ p_\phi = M r^2 \omega.
\]

2nd \( L \)-equation involves total time derivative \( \dot{p}_m \) for each momentum \( p_m \):

\[
\dot{p}_r = \frac{\partial L}{\partial r} = \frac{M}{2} \frac{\partial g_{\phi\phi}}{\partial r} \dot{\phi}^2 - \frac{\partial U}{\partial r} = M r \dot{\phi}^2 - \frac{\partial U}{\partial r} \quad \text{Centrifugal force} \ M r \omega^2
\]

\[
\dot{p}_\phi = \frac{\partial L}{\partial \phi} = 0 - \frac{\partial U}{\partial \phi} \quad \text{Angular momentum} \ p_\phi \ \text{is conserved if potential} \ U \ \text{has no explicit} \ \phi \text{-dependence}
\]

Find \( \dot{p}_m \) directly from 1st \( L \)-equation:

\[
\dot{p}_m \equiv \frac{d p_m}{dt} = \frac{d}{dt} M (g_{mn} \dot{q}^n) = M \left( \dot{g}_{mn} \dot{q}^n + g_{mn} \ddot{q}^n \right) \quad \text{Equate it to} \ \dot{p}_m \ \text{in} \ 2nd \ L \text{-equation}:
\]

\[
\dot{p}_r \equiv \frac{d p_r}{dt} = M \ddot{r} \quad \text{Centrifugal (center-fleeing) force equals total}
\]

\[
= M r \dot{\phi}^2 - \frac{\partial U}{\partial r} \quad \text{Centripetal (center-pulling) force}
\]
Lagrange prefers Covariant $g_{mn}$ with Contravariant velocity

Lagrangian KE-U is supposed to be explicit function of velocity.

$$\mathcal{L}(\mathbf{v}) = \frac{1}{2} M \mathbf{v} \cdot \mathbf{v} - U = \frac{1}{2} M \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - U = \frac{1}{2} M (\mathbf{E}_m \dot{q}^m) \cdot (\mathbf{E}_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = \mathcal{L} (\dot{\mathbf{q}})$$

Use polar coordinate Covariant $g_{mn}$ metric (1-page back)

$$\left( \begin{array}{cc} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{array} \right) = \left( \begin{array}{cc} \mathbf{E}_r \cdot \mathbf{E}_r & \mathbf{E}_r \cdot \mathbf{E}_\phi \\ \mathbf{E}_\phi \cdot \mathbf{E}_r & \mathbf{E}_\phi \cdot \mathbf{E}_\phi \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ 0 & r^2 \end{array} \right)$$

This gives polar GCC form (Actually it's an OCC or Orthogonal Curvilinear Coordinate form)

$$\mathcal{L}(\dot{r}, \dot{\phi}) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r, \phi) = \frac{1}{2} M (1 \dot{r}^2 + r^2 \dot{\phi}^2) - U(r, \phi)$$

GCC Lagrange equations follow. 1st L-equation is momentum $p_m$ definition for each coordinate $q^m$:

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = M g_{rr} \dot{r} = M \dot{r}$$

Centrifugal force $Mr^2\omega^2$

$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = M g_{\phi\phi} \dot{\phi} = Mr^2 \dot{\phi}$$

Angular momentum $p_\phi$ is conserved if potential $U$ has no explicit $\phi$-dependence

2nd L-equation involves total time derivative $\dot{p}_m$ for each momentum $p_m$:

$$\dot{p}_r = \frac{\partial \mathcal{L}}{\partial r} = 2 M g_{\phi r} \frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\partial U}{\partial r} = M r \dot{\phi}^2 - \frac{\partial U}{\partial r}$$

Centripetal (center-pulling) force

$$\dot{\phi} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = 0 - \frac{\partial U}{\partial \phi}$$

Angular momentum $p_\phi$ is conserved if potential $U$ has no explicit $\phi$-dependence

Find $\dot{p}_m$ directly from 1st L-equation: $\dot{p}_m \equiv \frac{dp_m}{dt} = \frac{d}{dt} M (g_{mn} \dot{q}^n) = M (g_{mn} \dot{q}^n + g_{mn} \ddot{q}^n)$

Equate it to $\dot{p}_m$ in 2nd L-equation:

$$\dot{p}_r = \frac{d}{dt} M r \dot{\phi}^2 - \frac{\partial U}{\partial r}$$

Centrifugal (center-fleeing) force equals total

$$\dot{\phi} = \frac{d}{dt} 2 Mr \dot{\phi} + Mr^2 \ddot{\phi}$$

Torque relates to two distinct parts: Coriolis and angular acceleration

$$\dot{p}_\phi = 0 - \frac{\partial U}{\partial \phi}$$

Angular momentum $p_\phi$ is conserved if potential $U$ has no explicit $\phi$-dependence
Rewriting GCC Lagrange equations:

\[ \dot{p}_r = \frac{dp_r}{dt} = M \ddot{r} \]

- Centrifugal (center-fleeing) force equals total
- Centripetal (center-pulling) force

\[ = M r \dot{\phi}^2 - \frac{\partial U}{\partial r} \]

Field-free (U=0)

- radial acceleration: \( \ddot{r} = r \dot{\phi}^2 \)

- angular acceleration: \( \ddot{\phi} = -2 \frac{\dot{r} \dot{\phi}}{r} \)

Conventional forms

- radial force: \( M \ddot{r} = M r \dot{\phi}^2 - \frac{\partial U}{\partial r} \)

- angular force or torque: \( M r^2 \ddot{\phi} = -2 M r \dot{\phi} - \frac{\partial U}{\partial \phi} \)

Effects on Northern Hemisphere local weather

- Cyclonic flow around lows
- Makes wind turn to the right
- Northern hemisphere rotation
- Field-free (U=0)
4. *Hamilton prefers Contravariant* $g^{mn}$ *with Covariant momentum* $p_m$

*Deriving Hamilton’s Equations*

*How to finesse centrifugal and Coriolis energy and other things like phase space.*
Deriving Hamilton’s equations
Consider total time derivative of Lagrangian $L=T-U$
that is explicit function of coordinates and velocity $\dot{q}$...

\[ \dot{L}(q, \dot{q}, t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m} \frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m} \frac{d\dot{q}^m}{dt} \]
Deriving Hamilton’s equations
Consider total time derivative of Lagrangian \( L=T-U \) that is explicit function of coordinates and velocity \( \dot{q} \)...

\[
\dot{L}(q,\dot{q},t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m} \frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m} \frac{d\dot{q}^m}{dt}
\]

...of coordinates and velocity and \textbf{time}, too. (You can safely drop last chain-rule factor \([1=dt/dt]\))

\[
\dot{L}(q,\dot{q},t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m} \frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m} \frac{d\dot{q}^m}{dt} + \frac{\partial L}{\partial t} \frac{dt}{dt}
\]
**Deriving Hamilton’s equations**

Consider total time derivative of Lagrangian $L = T - U$ that is explicit function of coordinates and velocity $\dot{q}$...

\[
\dot{L}(q, \dot{q}, t) = \frac{dL}{dt} = \frac{\partial L}{\partial q_m} \frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m} \frac{d\dot{q}^m}{dt}
\]

...of coordinates and velocity and time, too. (Imagine Mad Scientist turning $U(t)$-dial.)

\[
\dot{L}(q, \dot{q}, t) = \frac{dL}{dt} = \frac{\partial L}{\partial q_m} \frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m} \frac{d\dot{q}^m}{dt} + \frac{\partial L}{\partial t}
\]
Deriving Hamilton’s equations
Consider total time derivative of Lagrangian $L=T-U$ that is explicit function of coordinates and velocity $\dot{q}$...

$$\dot{L}(q,\dot{q},t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m} \frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m} \frac{d\dot{q}^m}{dt}$$

...of coordinates and velocity and time, too. (Imagine Mad Scientist turning U-dial.)

Recall Lagrange equations:

Recall Lagrange equations:

$$\dot{L}(q,\dot{q},t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m} \frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m} \frac{d\dot{q}^m}{dt} + \frac{\partial L}{\partial t}$$

$$\dot{\dot{q}}^m = \frac{\partial L}{\partial q^m} \quad \dot{p}_m = \frac{\partial L}{\partial \dot{q}^m}$$
Deriving Hamilton’s equations
Consider total time derivative of Lagrangian $\mathcal{L}=T-U$ that is explicit function of coordinates and velocity $\dot{q}...$

$$\dot{\mathcal{L}}(q,\dot{q},t) = \frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial q^m} \frac{dq^m}{dt} + \frac{\partial \mathcal{L}}{\partial \dot{q}^m} \frac{d\dot{q}^m}{dt}$$

...of coordinates and velocity and time, too. (Imagine Mad Scientist turning U-dial.)

Recall Lagrange equations:

Recall Lagrange equations:

$$\dot{p}_m = \frac{\partial \mathcal{L}}{\partial q^m}$$

$$p_m = \frac{\partial \mathcal{L}}{\partial \dot{q}^m}$$

$$\dot{\mathcal{L}}(q,\dot{q},t) = \frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial q^m} \frac{dq^m}{dt} + \frac{\partial \mathcal{L}}{\partial \dot{q}^m} \frac{d\dot{q}^m}{dt} + \frac{\partial \mathcal{L}}{\partial t}$$

Use product rule:

$$\dot{u} \frac{dy}{dt} + u \frac{d\dot{y}}{dt} = \frac{d}{dt}(uy)$$

$$= \frac{d\mathcal{L}}{dt} = \frac{d}{dt} \left( p_m \dot{q}^m \right) + \frac{\partial \mathcal{L}}{\partial t}$$
Deriving Hamilton’s equations
Consider total time derivative of Lagrangian $L = T - U$ that is explicit function of coordinates and velocity $\dot{q}$...

$$\frac{dL}{dt} = \frac{\partial L}{\partial q_m} \frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m} \frac{d\dot{q}^m}{dt}$$

...of coordinates and velocity and time, too. (Imagine Mad Scientist turning U-dial.)

Recall Lagrange equations:
$$\frac{dL}{dt} = \frac{\partial L}{\partial q_m} \frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m} \frac{d\dot{q}^m}{dt} + \frac{\partial L}{\partial t}$$

Use product rule:
$$\dot{u} \frac{dv}{dt} + u \frac{d\dot{v}}{dt} = \frac{d}{dt}(uv)$$

and switch the $dL/dt$ and $\partial L/\partial t$ to define the Hamiltonian function $H(p) = p \cdot v - L(v)$

$$\frac{d}{dt}\left(p_m \dot{q}^m - L\right) = -\frac{\partial L}{\partial t} = \frac{dH}{dt}$$  where:  $H = p_m \dot{q}^m - L$
Deriving Hamilton’s equations
Consider total time derivative of Lagrangian $L=T-U$ that is explicit function of coordinates and velocity $\dot{q}$ ...

$$\dot{L}(q,\dot{q},t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m} \frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m} \frac{d\dot{q}^m}{dt}$$

...of coordinates and velocity and time, too. (Imagine Mad Scientist turning U-dial.)

Recall Lagrange equations:

$$\dot{p}_m = \frac{\partial L}{\partial \dot{q}^m}$$

$$p_m = \frac{\partial L}{\partial q^m}$$

Use product rule:

$$\frac{d}{dt}(u \dot{v}) = \dot{u} \dot{v} + u \ddot{v} = u \frac{d}{dt}(\dot{v})$$

Define the Hamiltonian function $H(p) = p \cdot \dot{v} - L(v)$ (That’s the old Legendre transform)

$$\frac{d}{dt}(p_m \dot{q}^m) = -\frac{\partial L}{\partial t} = \frac{dH}{dt} \quad \text{where: } H = p_m \dot{q}^m - L \quad \text{(Recall } \frac{\partial L}{\partial p_m} \equiv 0)$$

Hamilton’s 1st GCC equation

$$\frac{\partial H}{\partial p_m} = \dot{q}^m$$
Deriving Hamilton's equations

Consider total time derivative of Lagrangian \( L = T - U \) that is explicit function of coordinates and velocity \( \dot{q} \) ...

\[
\dot{L}(q, \dot{q}, t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m} \frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m} \frac{d\dot{q}^m}{dt}
\]

... of coordinates and velocity and time, too. (Imagine Mad Scientist turning U-dial.)

\[
\dot{L}(q, \dot{q}, t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m} \frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m} \frac{d\dot{q}^m}{dt} + \frac{\partial L}{\partial t}
\]

Recall Lagrange equations:

\[
\dot{p}_m = \frac{\partial L}{\partial \dot{q}^m} \quad \quad p_m = \frac{\partial L}{\partial q^m}
\]

Use product rule:

\[
\frac{d}{dt}(uv) = \frac{du}{dt}v + u \frac{dv}{dt}
\]

Define the Hamiltonian function \( H(p) = p \cdot v - L(v) \) (That's the old Legendre transform)

\[
\frac{d}{dt}(p_m \dot{q}^m - L) = -\frac{\partial L}{\partial t} = \frac{dH}{dt}
\]

where: \( H = p_m \dot{q}^m - L \)

(Recall: \( \frac{\partial L}{\partial p_m} \equiv 0 \) and: \( \frac{\partial H}{\partial q^m} \equiv 0 \))

Hamilton's 1st GCC equation

\[
\frac{\partial H}{\partial p_m} = \dot{q}^m
\]

Hamilton's 2nd GCC equation

\[
\frac{\partial H}{\partial q^m} = -\dot{p}_m
\]
Consider total time derivative of Lagrangian \( L = T - U \)
that is explicit function of coordinates and velocity \( \dot{q} \) ...

\[
\dot{L}(q, \dot{q}, t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m} \frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m} \frac{d\dot{q}^m}{dt}
\]

...of coordinates and velocity and time, too. (Imagine Mad Scientist turning U-dial.)

\[
\dot{L}(q, \dot{q}, t) = \frac{dL}{dt} = \frac{\partial L}{\partial q^m} \frac{dq^m}{dt} + \frac{\partial L}{\partial \dot{q}^m} \frac{d\dot{q}^m}{dt} + \frac{\partial L}{\partial t}
\]

Recall Lagrange equations:

\[
\begin{align*}
\dot{p}_m &= \frac{\partial L}{\partial q^m} \\
P_m &= \frac{\partial L}{\partial \dot{q}^m}
\end{align*}
\]

Use product rule:

\[
\dot{u} \frac{dv}{dt} + u \frac{dv}{dt} = \frac{d}{dt}(uv)
\]

Define the Hamiltonian function \( H(p) = p \cdot v - L(v) \)

(That’s the old Legendre transform)

\[
\frac{d}{dt} \left( P_m \dot{q}^m - L \right) = -\frac{\partial L}{\partial t} = \frac{dH}{dt}
\]

where: \( H = P_m \dot{q}^m - L \)

(Recall: \( \frac{\partial L}{\partial P_m} \equiv 0 \))

\[
\frac{\partial H}{\partial p_m} = \dot{q}^m
\]

\[
\frac{\partial H}{\partial q^m} = -\dot{P}_m
\]

Hamilton’s 1st GCC equation

a most peculiar relation involving partial vs total

Hamilton’s 2nd GCC equation
End of this Lecture