Lecture 15

Revised 12.22.12 from 10.11.2012

Complex Variables, Series, and Field Coordinates II.

(Ch. 10 of Unit 1)

1. The Story of e (A Tale of Great $Interest$)

How good are those power series?

Taylor-Maclaurin series, imaginary interest, and complex exponentials

Lecture 14 Tue. 10.09 starts here

2. What good are complex exponentials?

Easy trig

Easy 2D vector analysis

Easy oscillator phase analysis

Easy rotation and "dot" or "cross" products

3. Easy 2D vector calculus

Easy 2D vector derivatives

Easy 2D source-free field theory

Easy 2D vector field-potential theory

4. Riemann-Cauchy relations (What's analytic? What's not?)

Easy 2D curvilinear coordinate discovery

Easy 2D circulation and flux integrals

Easy 2D monopole, dipole, and 2^n-pole analysis

Lecture 15 Thur. 10.11 starts here

Easy 2^n-multipole field and potential expansion

Easy stereo-projection visualization

Cauchy integrals, Laurent-Maclaurin series

5. Mapping and Non-analytic 2D source field analysis

1. *Complex numbers provide "automatic trigonometry"*
2. *Complex numbers add like vectors.*
3. *Complex exponentials $Ae^{-i\omega t}$ track position and velocity using Phasor Clock.*
4. *Complex products provide 2D rotation operations.*
5. *Complex products provide 2D "dot"(•) and "cross"(x) products.*
6. *Complex derivative contains "divergence"($\nabla\cdot\mathbf{F}$) and "curl"($\nabla\times\mathbf{F}$) of 2D vector field*
7. *Invent source-free 2D vector fields [$\nabla\cdot\mathbf{F}=0$ and $\nabla\times\mathbf{F}=0$]*
8. *Complex potential ϕ contains "scalar"($\mathbf{F}=\nabla\Phi$) and "vector"($\mathbf{F}=\nabla\times\mathbf{A}$) potentials*

   The half-n'-half results: (Riemann-Cauchy Derivative Relations)
9. *Complex potentials define 2D Orthogonal Curvilinear Coordinates (OCC) of field*
10. *Complex integrals $\int f(z)dz$ count 2D "circulation"($\int\mathbf{F}\cdot d\mathbf{r}$) and "flux"($\int\mathbf{F}\times d\mathbf{r}$)*
11. *Complex integrals define 2D monopole fields and potentials*
12. *Complex derivatives give 2D dipole fields*
13. *More derivatives give 2D 2^N-pole fields…*
14. *...and 2^N-pole multipole expansions of fields and potentials...*
15. *...and Laurent Series...*
16. *...and non-analytic source analysis.*

6. Complex derivative contains "divergence"($\nabla\cdot\mathbf{F}$) and "curl"($\nabla\times\mathbf{F}$) of 2D vector field

Relation of (z,z^*) to $(x=\mathrm{Re}\,z, y=\mathrm{Im}\,z)$ defines a z-derivative $\frac{df}{dz}$ and "star" z^*-derivative. $\frac{df}{dz^*}$

$$z = x + iy \qquad x = \tfrac{1}{2}(z + z^*)$$

$$z^* = x - iy \qquad y = \tfrac{1}{2i}(z - z^*)$$

Applying chain-rule

$$\frac{df}{dz} = \frac{\partial x}{\partial z}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{i}{2}\frac{\partial f}{\partial y}$$

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Derivative chain-rule shows real part of $\frac{df}{dz}$ has 2D divergence $\nabla\bullet\mathbf{f}$ and imaginary part has curl $\nabla\times\mathbf{f}$.

$$\frac{df}{dz}=\frac{d}{dz}(f_x+if_y)=\frac{1}{2}\left(\frac{\partial f}{\partial x}-i\frac{\partial f}{\partial y}\right)(f_x+if_y)=\frac{1}{2}\overbrace{\left(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}\right)}+\frac{i}{2}\overbrace{\left(\frac{\partial f_y}{\partial x}-\frac{\partial f_x}{\partial y}\right)}=\frac{1}{2}\nabla\bullet\mathbf{f}+\frac{i}{2}|\nabla\times\mathbf{f}|_{Z\perp(x,y)}$$

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7. Invent source-free 2D vector fields [$\nabla\cdot\mathbf{F}=0$ and $\nabla\times\mathbf{F}=0$]

We can invent *source-free 2D vector fields* that are both *zero-divergence* and *zero-curl*.
Take any function *f(z)*, conjugate it (change all *i*'s to –*i*) to give $f^*(z^*)$ for which $\boxed{\frac{df^*}{dz}=0}$.

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For example: if *f(z)=a·z* then $f^*(z^*)=a\cdot z^*=a(x-iy)$ is not function of *z* so it has *zero z-derivative*.

$\mathbf{F}=(F_x,F_y)=(f^*_x,f^*_y)=(a\cdot x,-a\cdot y)$ *has zero divergence*: $\nabla\bullet\mathbf{F}=0$ *and has zero curl*: $|\nabla\times\mathbf{F}|=0$.

$$\nabla\bullet\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} = \frac{\partial(ax)}{\partial x} + \frac{\partial F(-ay)}{\partial y} = 0 \qquad |\nabla\times\mathbf{F}|_{Z\perp(x,y)} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \frac{\partial(-ay)}{\partial x} - \frac{\partial F(ax)}{\partial y} = 0$$

A *DFL* field **F** (*Divergence-Free-Laminar*)

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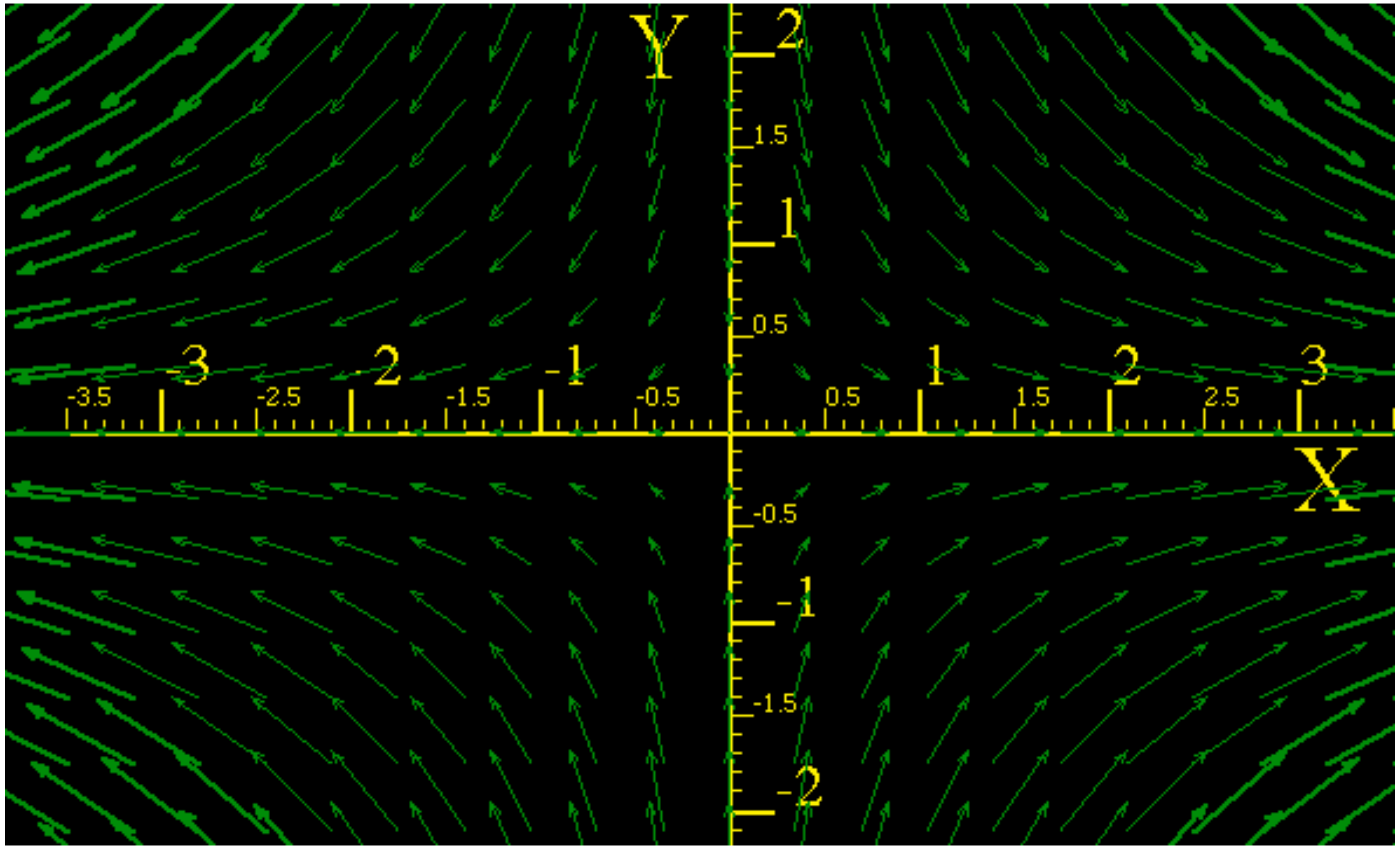

precursor to Unit 1 Fig. 10.7

$\mathbf{F}=(f^*_x,f^*_y)=(a\cdot x,-a\cdot y)$ is a *divergence-free laminar (DFL)* field.

What Good are complex variables?

Easy 2D vector calculus

Easy 2D vector derivatives

Easy 2D source-free field theory

→ *Easy 2D vector field-potential theory*

8. Complex potential ϕ contains "scalar"($\mathbf{F}=\nabla\Phi$) and "vector"($\mathbf{F}=\nabla x\mathbf{A}$) potentials

Any *DFL* field **F** is a gradient of a *scalar potential field* $\mathbf{\Phi}$ or a curl of a *vector potential field* **A**.

$$\mathbf{F}=\nabla\mathbf{\Phi} \qquad\qquad \mathbf{F}=\nabla\times\mathbf{A}$$

A *complex potential* $\phi(z)=\mathbf{\Phi}(x,y)+i\mathbf{A}(x,y)$ exists whose z-derivative is $f(z)=d\phi/dz$.

Its complex conjugate $\phi^*(z^*)=\mathbf{\Phi}(x,y)-i\mathbf{A}(x,y)$ has z^*-derivative $f^*(z^*)=d\phi^*/dz^*$ giving *DFL* field **F**.

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To find $\phi=\mathbf{\Phi}+i\mathbf{A}$ integrate $f(z)=a\cdot z$ to get ϕ and isolate real ($\mathrm{Re}\,\phi=\mathbf{\Phi}$) and imaginary ($\mathrm{Im}\,\phi=\mathbf{A}$) parts.

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$$f(z)=\tfrac{d\phi}{dz} \quad\Rightarrow\quad \phi= \qquad \mathbf{\Phi} \qquad +i\ \mathbf{A}=\int f\cdot dz=\int az\cdot dz=\tfrac{1}{2}\,az^2$$

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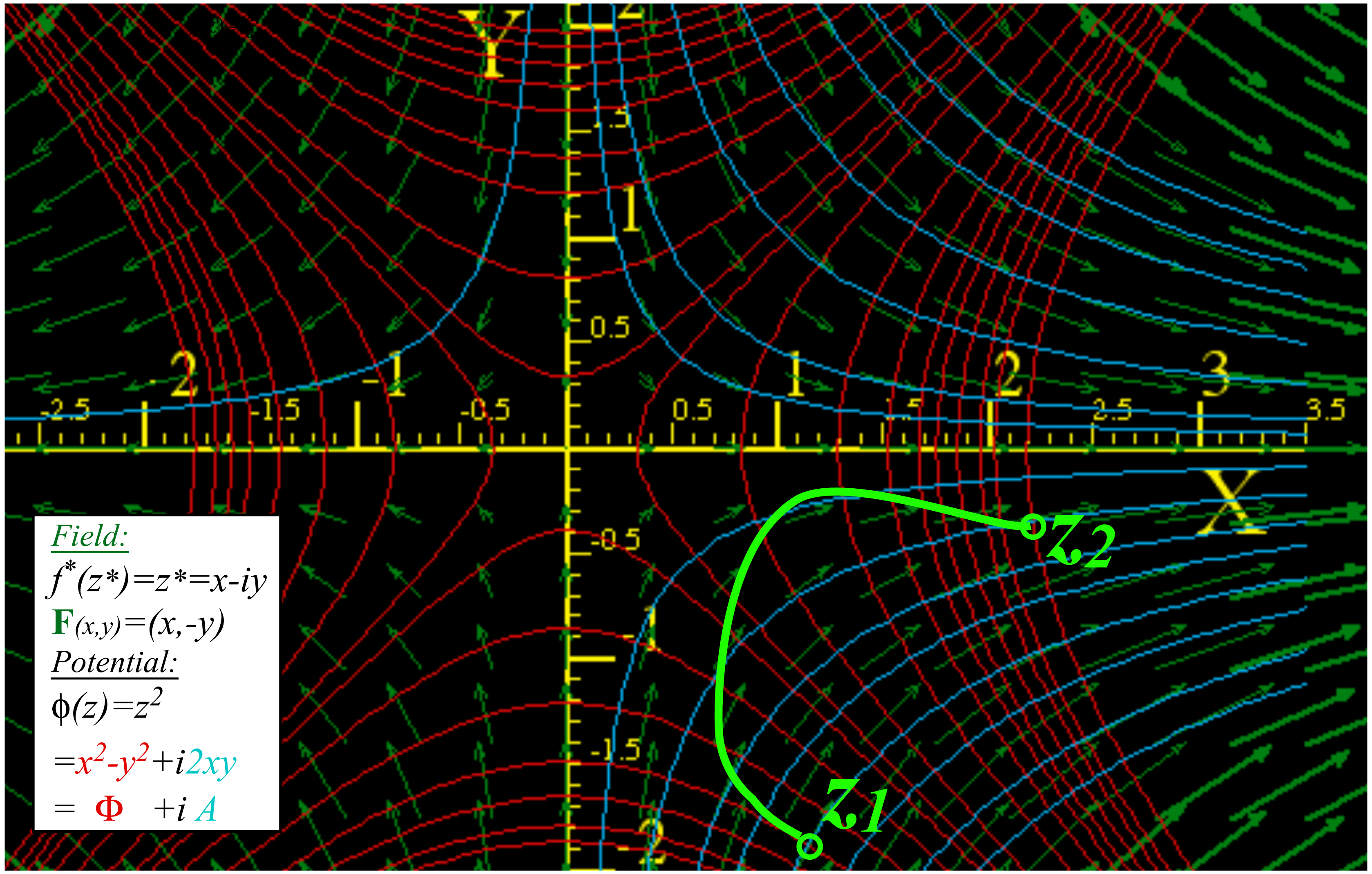
Unit 1
Fig. 10.7



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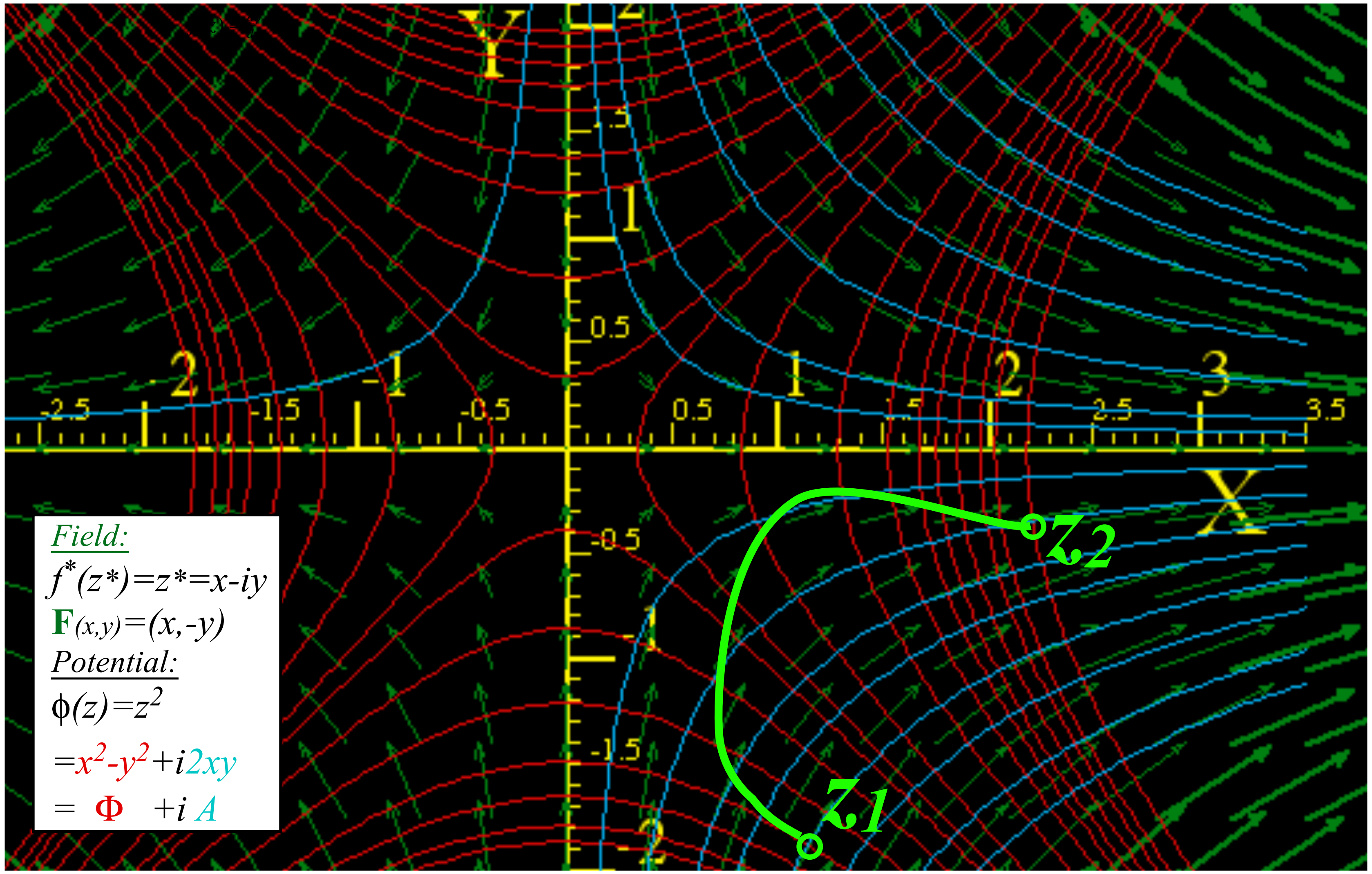
BONUS!
Get a free coordinate system!



Unit 1 Fig. 10.7

The (Φ,$\mathbf{A}$) grid is a GCC coordinate system*:

$q^1=\Phi=(x^2-y^2)/2=$ *const.*

$q^2=\mathbf{A}=(xy)=$ *const.*

*Actually it's OCC.

What Good are complex variables?

Easy 2D vector calculus

Easy 2D vector derivatives

Easy 2D source-free field theory

→ *Easy 2D vector field-potential theory*

→ *The half-n'-half results: (Riemann-Cauchy Derivative Relations)*

8. (contd.) Complex potential ϕ contains "scalar"($\mathbf{F}=\nabla\Phi$) and "vector"($\mathbf{F}=\nabla\times\mathbf{A}$) potentials

...and either one (or half-n'-half!) works just as well.

Derivative $\frac{d\phi^*}{dz^*}$ has 2D gradient $\nabla\Phi = \begin{pmatrix}\frac{\partial\Phi}{\partial x}\\ \frac{\partial\Phi}{\partial y}\end{pmatrix}$ of scalar Φ and curl $\nabla\times\mathbf{A} = \begin{pmatrix}\frac{\partial \mathbf{A}}{\partial y}\\ -\frac{\partial \mathbf{A}}{\partial y}\end{pmatrix}$ of vector $\mathbf{A}$ (*and they're* ***equal!***)

$f(z)=\frac{d\phi}{dz} \quad \Rightarrow$

$$\frac{d}{dz^*}\phi^* = \frac{d}{dz^*}(\Phi - i\mathbf{A}) = \frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(\Phi - i\mathbf{A}) = \frac{1}{2}\underbrace{\left(\frac{\partial\Phi}{\partial x}+i\frac{\partial\Phi}{\partial y}\right)} + \frac{1}{2}\underbrace{\left(\frac{\partial\mathbf{A}}{\partial y}-i\frac{\partial\mathbf{A}}{\partial x}\right)} = \frac{1}{2}\nabla\Phi + \frac{1}{2}\nabla\times\mathbf{A}$$

What Good Are Complex Exponentials? (contd.)

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Note, *mathematician definition* of force field **F**=+$\nabla\Phi$ replaces usual physicist's definition **F**=-$\nabla\Phi$

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Note, *mathematician definition* of force field **F**=+$\nabla\Phi$ replaces usual physicist's definition **F**=-$\nabla\Phi$

Given ϕ:

$$\boxed{\begin{aligned}\phi &= \Phi + i\,\mathbf{A} \\ &= \tfrac{1}{2}a(x^2-y^2) + i\,axy\end{aligned}}$$

The half-n'-half result

find:

$$\nabla\Phi = \begin{pmatrix} \frac{\partial\Phi}{\partial x} \\ \frac{\partial\Phi}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x}\frac{a}{2}(x^2-y^2) \\ \frac{\partial}{\partial y}\frac{a}{2}(x^2-y^2) \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

or find:

$$\nabla\times\mathbf{A} = \begin{pmatrix} \frac{\partial \mathbf{A}}{\partial y} \\ -\frac{\partial \mathbf{A}}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y}axy \\ -\frac{\partial}{\partial x}axy \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

What Good Are Complex Exponentials? (contd.)

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Note, *mathematician definition* of force field $\mathbf{F}=+\nabla\Phi$ replaces usual physicist's definition $\mathbf{F}=-\nabla\Phi$

Given ϕ: $\boxed{\begin{aligned}\phi &= \Phi + i\,\mathbf{A}\\ &=\tfrac{1}{2}a(x^2-y^2) + i\,axy\end{aligned}}$

The half-n'-half result

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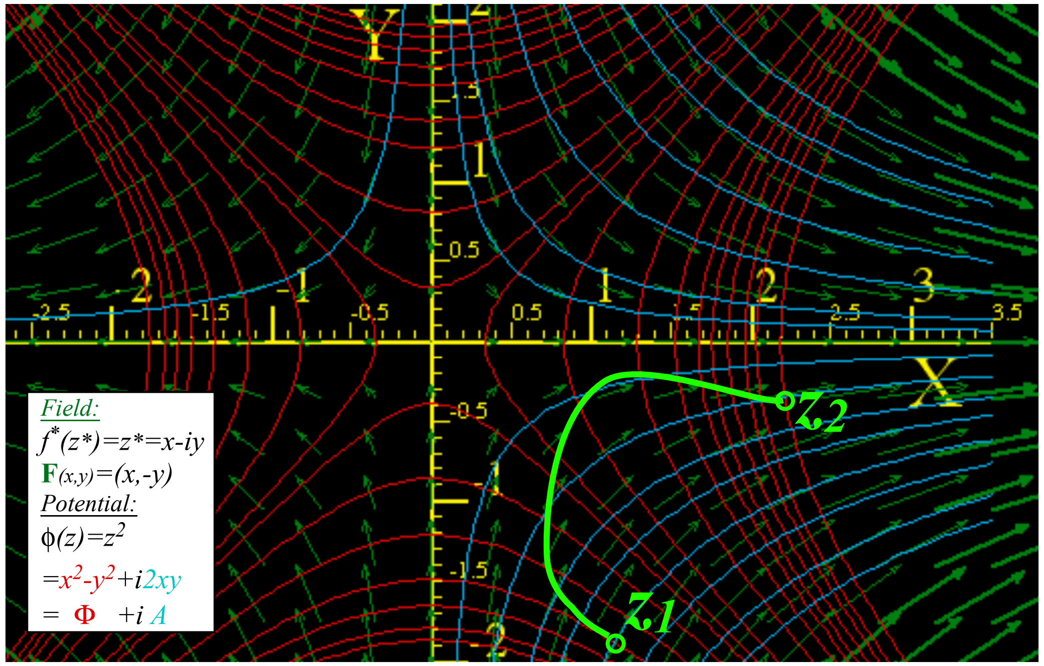
Scalar *static potential lines* Φ=*const.* and vector *flux potential lines* $\mathbf{A}$=*const.* define *DFL field-net*.



8. (contd.) Complex potential ϕ contains "scalar"($\mathbf{F}=\nabla\Phi$) and "vector"($\mathbf{F}=\nabla\times\mathbf{A}$) potentials

...and either one (or half-n'-half!) works just as well.

Derivative $\frac{d\phi^*}{dz^*}$ has 2D gradient $\nabla\Phi=\begin{pmatrix}\frac{\partial\Phi}{\partial x}\\ \frac{\partial\Phi}{\partial y}\end{pmatrix}$ of scalar Φ and curl $\nabla\times\mathbf{A}=\begin{pmatrix}\frac{\partial \mathbf{A}}{\partial y}\\ -\frac{\partial \mathbf{A}}{\partial y}\end{pmatrix}$ of vector $\mathbf{A}$ (*and they're equal!*)

The half-n'-half result

$$\frac{d}{dz^*}\phi^*=\frac{d}{dz^*}(\Phi-i\mathbf{A})=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(\Phi-i\mathbf{A})=\frac{1}{2}\left(\frac{\partial\Phi}{\partial x}+i\frac{\partial\Phi}{\partial y}\right)+\frac{1}{2}\left(\frac{\partial\mathbf{A}}{\partial y}-i\frac{\partial\mathbf{A}}{\partial x}\right)=\frac{1}{2}\nabla\Phi+\frac{1}{2}\nabla\times\mathbf{A}$$

Note, *mathematician definition* of force field $\mathbf{F}=+\nabla\Phi$ replaces usual physicist's definition $\mathbf{F}=-\nabla\Phi$

Given ϕ: $\boxed{\begin{aligned}\phi&=\Phi+i\,\mathbf{A}\\ &=\tfrac{1}{2}a(x^2-y^2)+i\,axy\end{aligned}}$

The half-n'-half result

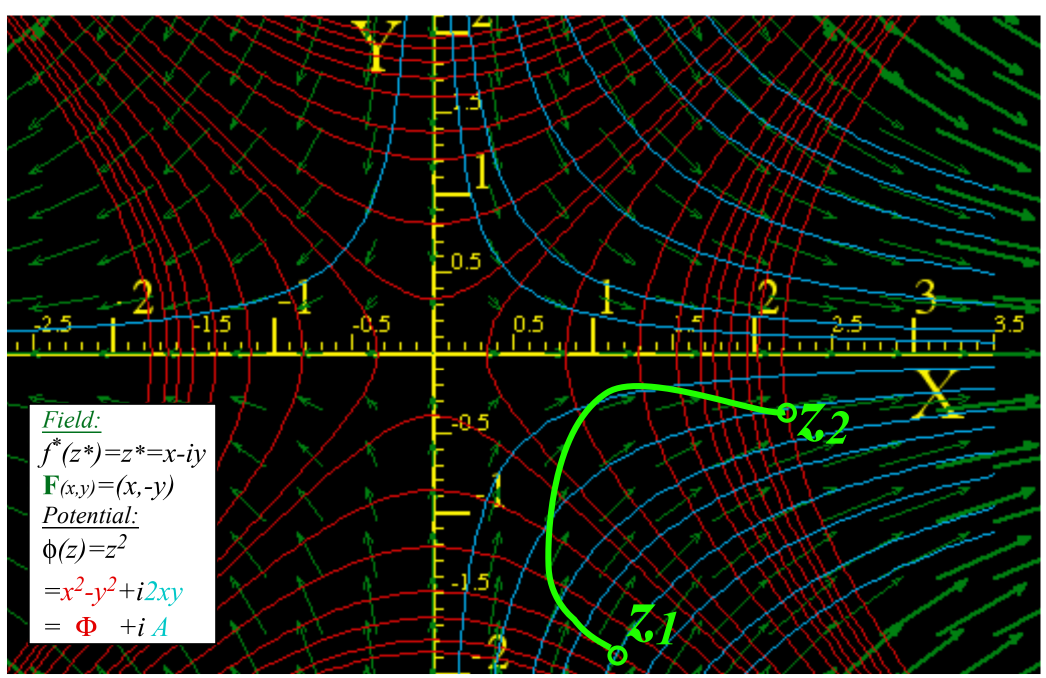
find:

$$\nabla\Phi=\begin{pmatrix}\frac{\partial\Phi}{\partial x}\\ \frac{\partial\Phi}{\partial y}\end{pmatrix}=\begin{pmatrix}\frac{\partial}{\partial x}\frac{a}{2}(x^2-y^2)\\ \frac{\partial}{\partial y}\frac{a}{2}(x^2-y^2)\end{pmatrix}=\begin{pmatrix}ax\\ -ay\end{pmatrix}=\mathbf{F}$$

or find:

$$\nabla\times\mathbf{A}=\begin{pmatrix}\frac{\partial\mathbf{A}}{\partial y}\\ -\frac{\partial\mathbf{A}}{\partial x}\end{pmatrix}=\begin{pmatrix}\frac{\partial}{\partial y}axy\\ -\frac{\partial}{\partial x}axy\end{pmatrix}=\begin{pmatrix}ax\\ -ay\end{pmatrix}=\mathbf{F}$$

Scalar *static potential lines* Φ=*const.* and vector *flux potential lines* $\mathbf{A}$=*const.* define *DFL field-net*.



The half-n'-half results are called Riemann-Cauchy Derivative Relations

$$\frac{\partial\Phi}{\partial x}=\frac{\partial\mathbf{A}}{\partial y}\quad\text{is:}\quad\frac{\partial\,\mathrm{Re}f(z)}{\partial x}=\frac{\partial\,\mathrm{Im}f(z)}{\partial y}$$

$$\frac{\partial\Phi}{\partial y}=-\frac{\partial\mathbf{A}}{\partial x}\quad\text{is:}\quad\frac{\partial\,\mathrm{Re}f(z)}{\partial y}=-\frac{\partial\,\mathrm{Im}f(z)}{\partial x}$$

4. Riemann-Cauchy conditions *What's analytic? (...and what's <u>not</u>?)*

Review (z,z) to (x,y) transformation relations*

$$z = x + iy \qquad x = \tfrac{1}{2}(z + z^*)$$

$$z^* = x - iy \qquad y = \tfrac{1}{2i}(z - z^*)$$

$$\frac{df}{dz} = \frac{\partial x}{\partial z}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2i}\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)f$$

$$\frac{df}{dz^*} = \frac{\partial x}{\partial z^*}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z^*}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2i}\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)f$$

Criteria for a field function $f = f_x(x,y) + i f_y(x,y)$ to be an ***analytic function f(z)*** *of $z=x+iy$:*

First, f(z) must not be a function of $z^*=x-iy$, that is: $\frac{df}{dz^*} = 0$

This implies f(z) satisfies differential equations known as the ***Riemann-Cauchy conditions***

$$\frac{df}{dz^*} = 0 = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} - \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} + \frac{\partial f_x}{\partial y}\right) \text{ implies: } \boxed{\frac{\partial f_x}{\partial x} = \frac{\partial f_y}{\partial y} \quad \text{and:} \quad \frac{\partial f_y}{\partial x} = -\frac{\partial f_x}{\partial y}}$$

$$\frac{df}{dz} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) = \frac{\partial f_x}{\partial x} + i\frac{\partial f_y}{\partial x} = \frac{\partial f_y}{\partial y} - i\frac{\partial f_x}{\partial y} = \frac{\partial}{\partial x}(f_x + i f_y) = \frac{\partial}{\partial iy}(f_x + i f_y)$$

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First, f(z) must <u>not</u> be a function of $z=x+iy$, that is:* $\frac{df}{dz} = 0$

This implies f(z) satisfies differential equations we call **Anti-Riemann-Cauchy conditions***

$$\frac{df}{dz} = 0 = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) = \text{implies : } \boxed{\frac{\partial f_x}{\partial x} = -\frac{\partial f_y}{\partial y} \quad and : \quad \frac{\partial f_y}{\partial x} = \frac{\partial f_x}{\partial y}}$$

$$\frac{df}{dz^*} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} - \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} + \frac{\partial f_x}{\partial y}\right) = \frac{\partial f_x}{\partial x} + i\frac{\partial f_y}{\partial x} = -\frac{\partial f_y}{\partial y} + i\frac{\partial f_x}{\partial y} = \frac{\partial}{\partial x}(f_x + i f_y) = -\frac{\partial}{\partial iy}(f_x + i f_y)$$

What's analytic? (...and what's <u>not</u>?)

Example: Is $f(x,y) = 2x + iy$ an analytic function of $z=z+iy$?

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Example: Q: Is $f(x,y) = 2x + i4y$ an analytic function of $z=z+iy$?

Well, test it using definitions: $z = x + iy$ and: $z^* = x - iy$

or: $x = (z+z^*)/2$ and: $y = -i(z-z^*)/2$

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A: *NO!* *It's a function of z <u>and</u> z* so not analytic for <u>either</u>.*

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Example 2: Q: Is $r(x,y) = x^2 + y^2$ an analytic function of $z=z+iy$?

A: *NO!* *$r(xy)=z^*z$ is a function of z <u>and</u> z* so not analytic for <u>either</u>.*

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Example 3: Q: Is $s(x,y) = x^2-y^2 + 2ixy$ an analytic function of $z=z+iy$?

A: *YES!* *$s(xy)=(x+iy)^2 =z^2$ is analytic function of z.* *(Yay!)*

4. Riemann-Cauchy conditions *What's analytic? (...and what's not?)*

Easy 2D circulation and flux integrals

Easy 2D curvilinear coordinate discovery

Easy 2D monopole, dipole, and 2^n-pole analysis

Easy 2^n-multipole field and potential expansion

Easy stereo-projection visualization

9. Complex integrals ∫ f(z)dz count 2D "circulation"(∫F•dr) and "flux"(∫Fxdr)

Integral of $f(z)$ between point z_1 and point z_2 is potential difference $\Delta\phi = \phi(z_2) - \phi(z_1)$

$$\Delta\phi = \phi(z_2) - \phi(z_1) = \int_{z_1}^{z_2} f(z)dz = \underbrace{\mathbf{\Phi}(x_2, y_2) - \mathbf{\Phi}(x_1, y_1)}_{\Delta\mathbf{\Phi}} + i\,\underbrace{[\mathbf{A}(x_2, y_2) - \mathbf{A}(x_1, y_1)]}_{\Delta\mathbf{A}}$$

$$\Delta\phi = \Delta\mathbf{\Phi} + i\,\Delta\mathbf{A}$$

In *DFL*-field **F**, $\Delta\phi$ is independent of the integration path $z(t)$ connecting z_1 and z_2.

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$$\Delta\phi = \quad \Delta\mathbf{\Phi} \quad + i \quad \Delta\mathbf{A}$$

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$$\int f(z)dz = \int \left(f^*(z^*) \right)^* dz = \int \left(f^*(z^*) \right)^* \left(dx + i\,dy \right) = \int \left(f_x^* + i\,f_y^* \right)^* \left(dx + i\,dy \right) = \int \left(f_x^* - i\,f_y^* \right)\left(dx + i\,dy \right)$$

$$= \int (f_x^* dx + f_y^* dy) + i \int (f_x^* dy - f_y^* dx)$$

$$= \quad \int \mathbf{F} \bullet d\mathbf{r} \quad + i \int \mathbf{F} \times d\mathbf{r} \bullet \hat{\mathbf{e}}_Z$$

$$= \quad \int \mathbf{F} \bullet d\mathbf{r} \quad + i \int \mathbf{F} \bullet d\mathbf{r} \times \hat{\mathbf{e}}_Z$$

$$= \quad \int \mathbf{F} \bullet d\mathbf{r} \quad + i \int \mathbf{F} \bullet d\mathbf{S} \qquad \text{where:} \quad d\mathbf{S} = d\mathbf{r} \times \hat{\mathbf{e}}_Z$$

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$$= \int \mathbf{F}\bullet d\mathbf{r} \qquad + i\int \mathbf{F}\times d\mathbf{r}\bullet\hat{\mathbf{e}}_Z$$

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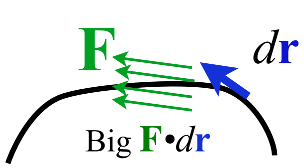
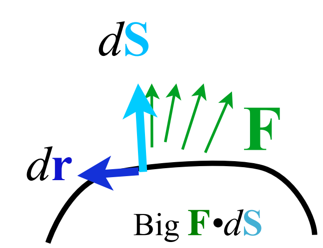


Real part $\int_1^2 \mathbf{F}\bullet d\mathbf{r} = \Delta\mathbf{\Phi}$
sums **F** projections *along* path $d\mathbf{r}$ that is, *circulation* on path to get $\Delta\mathbf{\Phi}$.

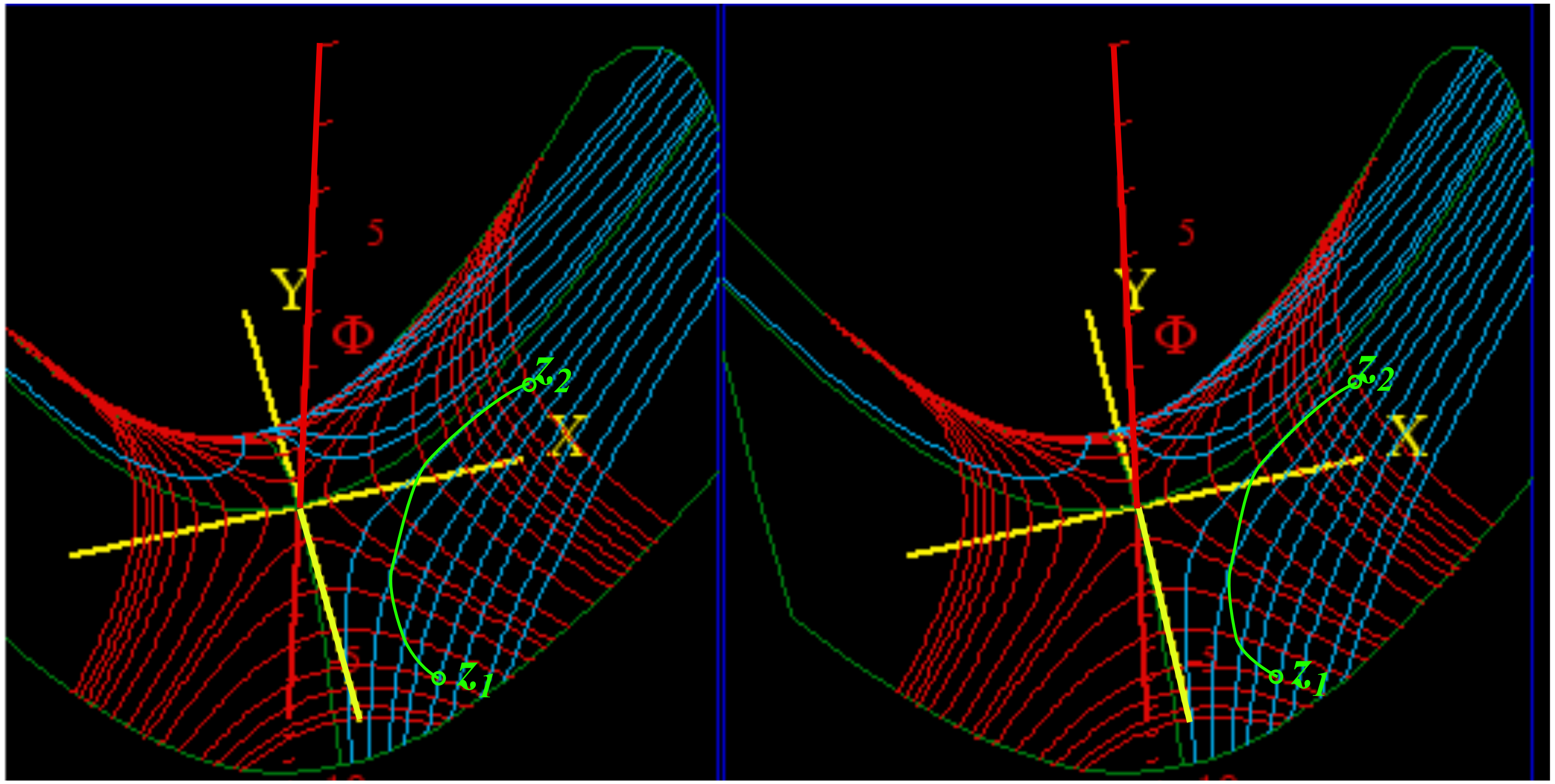


Imaginary part $\int_1^2 \mathbf{F}\bullet d\mathbf{S} = \Delta\mathbf{A}$
sums **F** projection *across* path $d\mathbf{r}$ that is, *flux* thru surface elements $d\mathbf{S}=d\mathbf{r}\times\mathbf{e}_Z$ normal to $d\mathbf{r}$ to get $\Delta\mathbf{A}$.

Here the scalar potential $\Phi=(x^2-y^2)/2$ is stereo-plotted vs. (x,y)

The $\Phi=(x^2-y^2)/2=const.$ curves are topography lines

The $A=(xy)=const.$ curves are streamlines normal to topography lines



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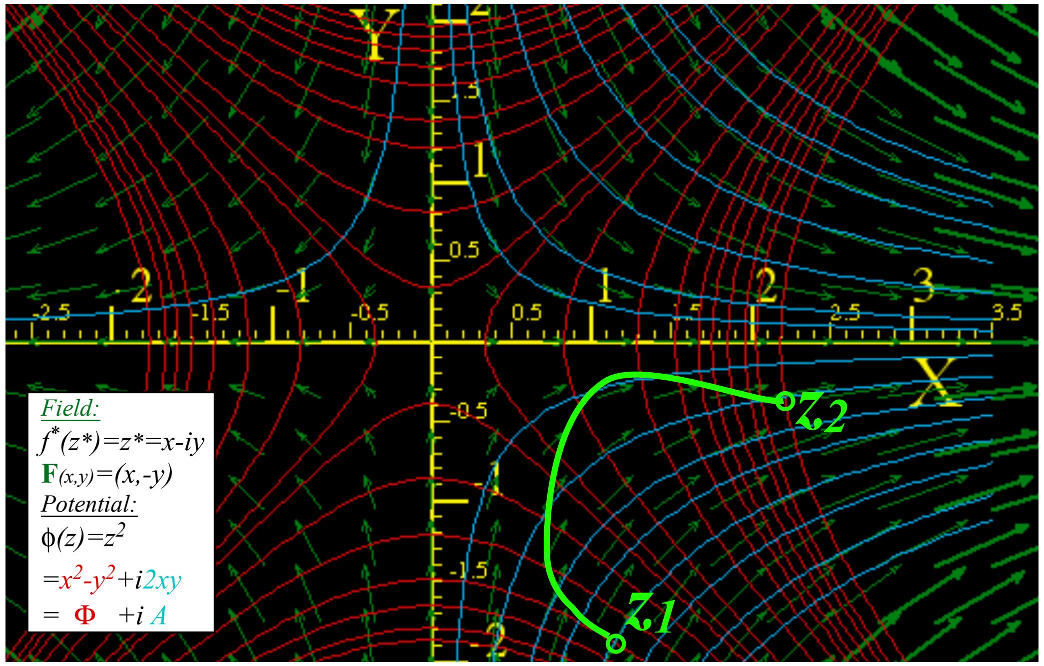
10. Complex potentials define 2D Orthogonal Curvilinear Coordinates (OCC) of field

The (Φ,A) grid is a GCC coordinate system*:

$q^1 = \Phi = (x^2-y^2)/2 = const.$

$q^2 = A = (xy) = const.$

*Actually it's OCC.



$$Kajobian = \begin{pmatrix} \frac{\partial q^1}{\partial x} & \frac{\partial q^1}{\partial y} \\ \frac{\partial q^2}{\partial x} & \frac{\partial q^2}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial \Phi}{\partial x} & \frac{\partial \Phi}{\partial y} \\ \frac{\partial A}{\partial x} & \frac{\partial A}{\partial y} \end{pmatrix} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{array}{l} \leftarrow \mathbf{E}^{\Phi} \\ \leftarrow \mathbf{E}^{A} \end{array}$$

$$Jacobian = \begin{pmatrix} \frac{\partial x}{\partial q^1} & \frac{\partial x}{\partial q^2} \\ \frac{\partial y}{\partial q^1} & \frac{\partial y}{\partial q^2} \end{pmatrix} = \begin{array}{c} \begin{pmatrix} \frac{\partial x}{\partial \Phi} & \frac{\partial x}{\partial A} \\ \frac{\partial y}{\partial \Phi} & \frac{\partial y}{\partial A} \end{pmatrix} \\ \begin{array}{cc} \uparrow & \uparrow \\ \mathbf{E}_{\Phi} & \mathbf{E}_{A} \end{array} \end{array} = \frac{1}{r^2} \begin{array}{c} \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \\ \begin{array}{cc} \uparrow & \uparrow \\ \mathbf{E}_{\Phi} & \mathbf{E}_{A} \end{array} \end{array}$$

$$Metric\,tensor = \begin{pmatrix} g_{\Phi\Phi} & g_{\Phi A} \\ g_{A\Phi} & g_{AA} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_{\Phi}\bullet\mathbf{E}_{\Phi} & \mathbf{E}_{\Phi}\bullet\mathbf{E}_{A} \\ \mathbf{E}_{A}\bullet\mathbf{E}_{\Phi} & \mathbf{E}_{A}\bullet\mathbf{E}_{A} \end{pmatrix} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \end{pmatrix} \text{ where: } r^2 = x^2 + y^2$$

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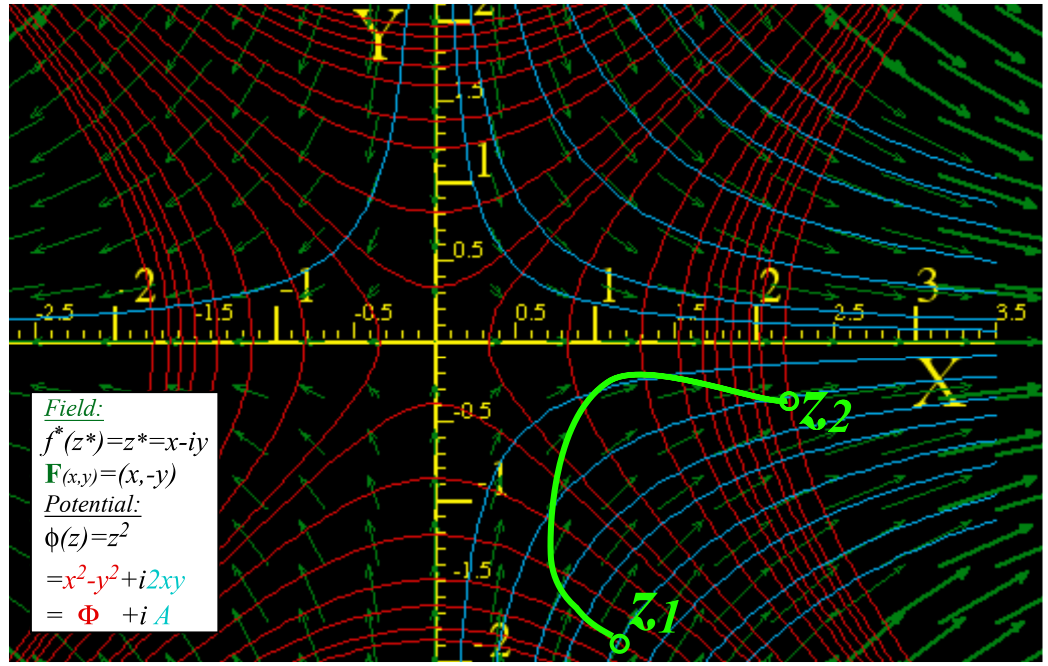

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$$Metric\,tensor = \begin{pmatrix} g_{\Phi\Phi} & g_{\Phi A} \\ g_{A\Phi} & g_{AA} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_{\Phi}\bullet\mathbf{E}_{\Phi} & \mathbf{E}_{\Phi}\bullet\mathbf{E}_{A} \\ \mathbf{E}_{A}\bullet\mathbf{E}_{\Phi} & \mathbf{E}_{A}\bullet\mathbf{E}_{A} \end{pmatrix} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \end{pmatrix} \text{ where: } r^2 = x^2 + y^2$$

Riemann-Cauchy Derivative Relations make coordinates orthogonal

$$\nabla \Phi = \begin{pmatrix} \frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} \frac{a}{2}(x^2 - y^2) \\ \frac{\partial}{\partial y} \frac{a}{2}(x^2 - y^2) \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

The half-n'-half results assure

$$\mathbf{E}_{\Phi}\bullet\mathbf{E}_{A} = \frac{\partial \Phi}{\partial x}\frac{\partial A}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{\partial A}{\partial y} = -\frac{\partial \Phi}{\partial x}\frac{\partial \Phi}{\partial y} + \frac{\partial \Phi}{\partial y}\frac{\partial \Phi}{\partial x} = 0$$

$$\nabla \times \mathbf{A} = \begin{pmatrix} \frac{\partial A}{\partial y} \\ -\frac{\partial A}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y} axy \\ -\frac{\partial}{\partial x} axy \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

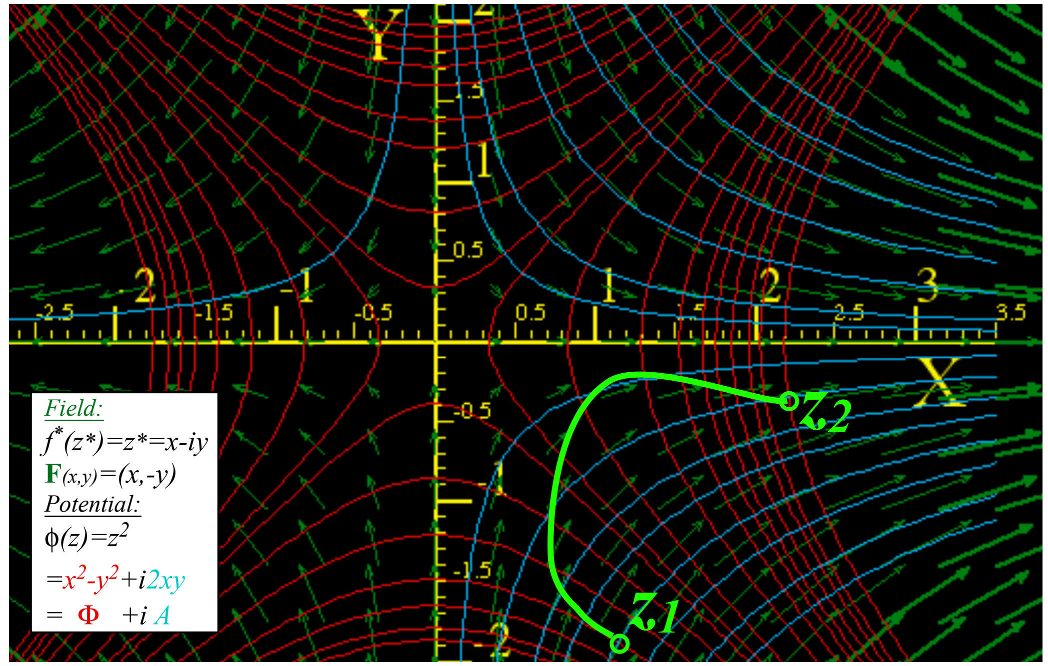
10. Complex potentials define 2D Orthogonal Curvilinear Coordinates (OCC) of field

The (Φ,A) grid is a GCC coordinate system*:

$q^1 = \Phi = (x^2-y^2)/2 =$ const.

$q^2 = A = (xy) =$ const.

*Actually it's OCC.



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The half-n'-half results assure

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$$\nabla\times\mathbf{A} = \begin{pmatrix} \frac{\partial A}{\partial y} \\ -\frac{\partial A}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y}axy \\ -\frac{\partial}{\partial x}axy \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

or Riemann-Cauchy

Zero divergence requirement: $0 = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} = \frac{\partial}{\partial x}\frac{\partial \Phi}{\partial x} + \frac{\partial}{\partial y}\frac{\partial \Phi}{\partial y} = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$ *potential Φ obeys Laplace equation*

and so does $\mathbf{A}$

4. Riemann-Cauchy conditions *What's analytic? (...and what's not?)*

Easy 2D circulation and flux integrals

Easy 2D curvilinear coordinate discovery

Easy 2D monopole, dipole, and 2^n-pole analysis

Easy 2^n-multipole field and potential expansion

Easy stereo-projection visualization

11. Complex integrals define 2D monopole fields and potentials

Of all power-law fields $f(z)=az^n$ one lacks a power-law potential $\phi(z)=\frac{a}{n+1}z^{n+1}$. It is the $n = -1$ case.

Unit monopole field: $f(z)=\frac{1}{z}= z^{-1}$ $\qquad$ $f(z)=\frac{a}{z}= az^{-1}$ Source-a monopole

It has a *logarithmic potential* $\phi(z)=a\cdot\ln(z)=a\cdot\ln(x+iy)$.

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$$\phi(z)=\quad \Phi \quad + \quad i\mathbf{A}=\int f(z)dz=\int \frac{a}{z}dz=a\ln(z)$$

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It has a *logarithmic potential* $\phi(z)=a\cdot\ln(z)=a\cdot\ln(x+iy)$. Note: $\ln(a\cdot b)=\ln(a)+\ln(b)$, $\ln(e^{i\theta})=i\theta$, and $z=re^{i\theta}$.

$$\phi(z) = \underbrace{\Phi}_{=a\ln(r)} + i\underbrace{A}_{a\theta} = \int f(z)dz = \int \frac{a}{z}dz = a\ln(z) = a\ln(re^{i\theta})$$

$$= a\ln(r) + i\,a\theta$$

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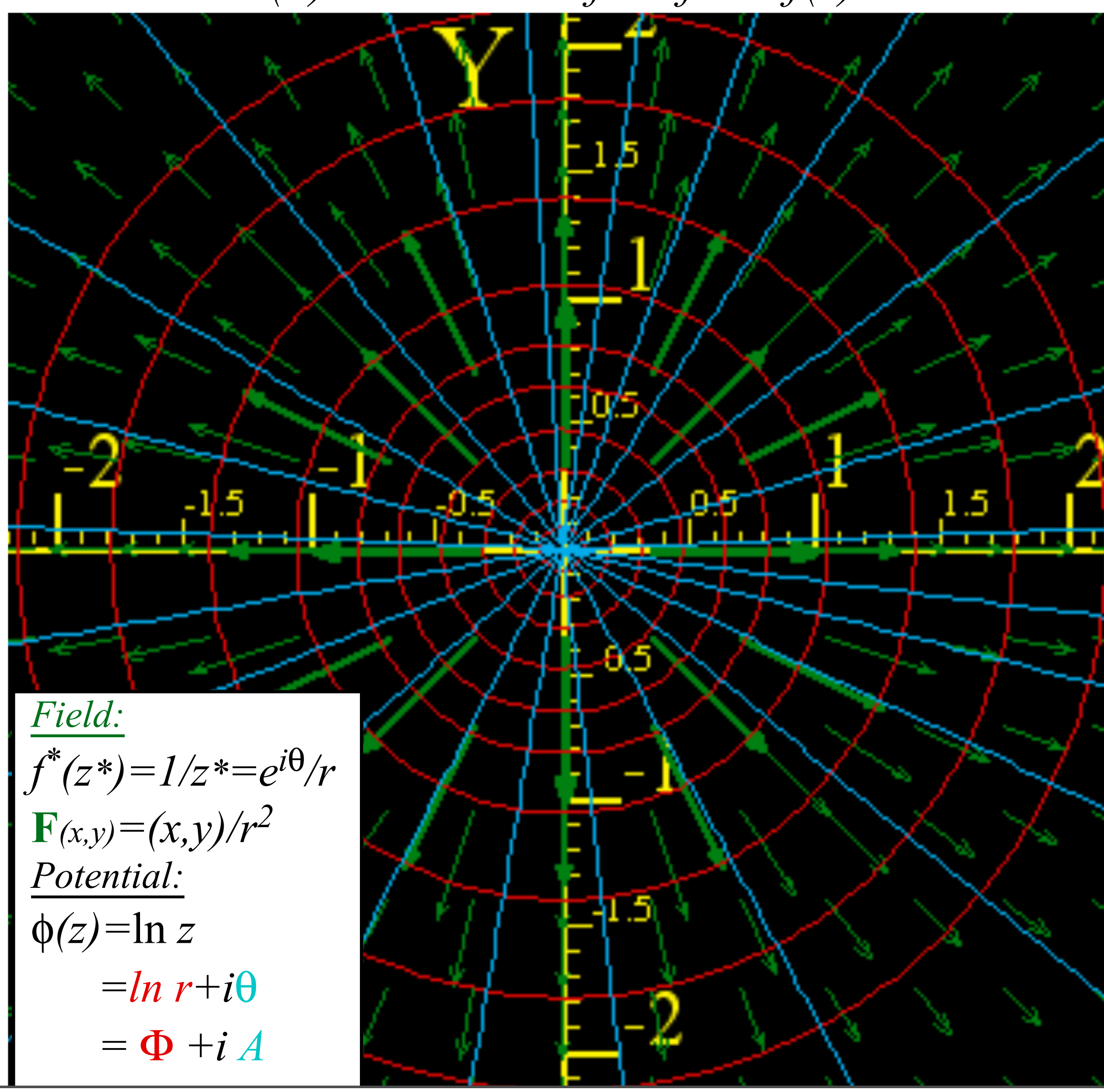
$$\phi(z) = \Phi + i\mathbf{A} = \int f(z)dz = \int \frac{a}{z}dz = a\ln(z) = a\ln(re^{i\theta})$$

$$= \overbrace{a\ln(r)} + i\,\overbrace{a\theta}$$

(a) Unit Z-line-flux field f(z)=1/z



Lecture 14 Thur. 10.9
ends here

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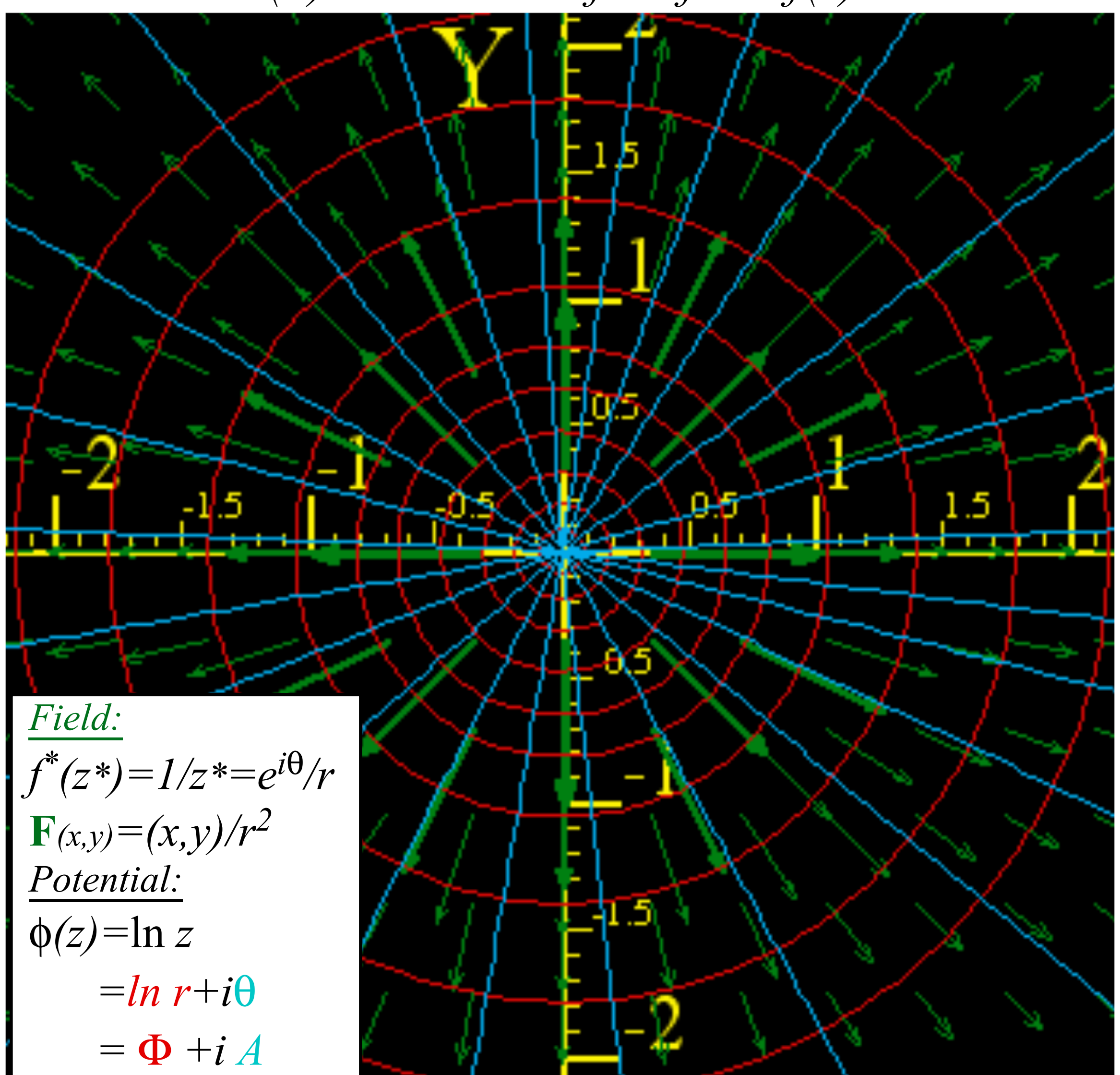
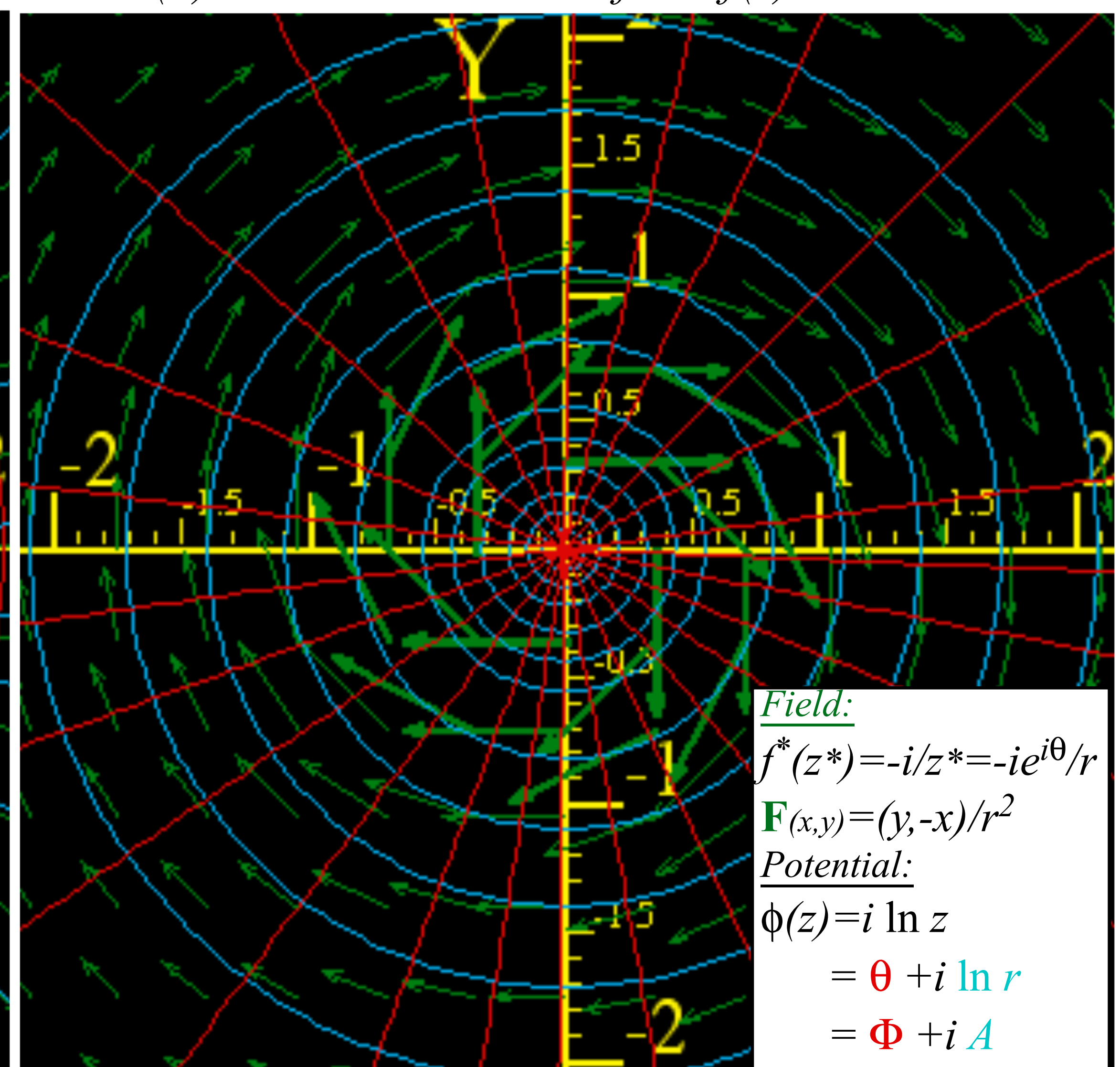
$$= \underbrace{a\ln(r)}_{\Phi} + i\,\underbrace{a\theta}_{A}$$

(a) Unit Z-line-flux field f(z)=1/z



(b) Unit Z-line-vortex field f(z)=i/z



11. Complex integrals define 2D monopole fields and potentials

Of all power-law fields $f(z)=az^n$ one lacks a power-law potential $\phi(z)=\frac{a}{n+1}z^{n+1}$. It is the $n = -1$ case.

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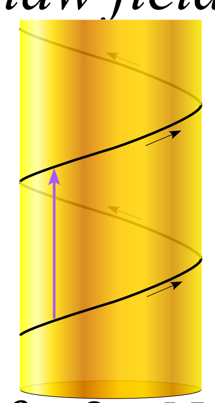
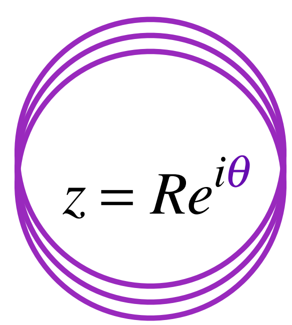
$$\phi(z)=\underbrace{\Phi}_{=a\ln(r)}+i\underbrace{A}_{a\theta}=\int f(z)dz=\int\frac{a}{z}dz=a\ln(z)=a\ln(re^{i\theta})$$

$$=a\ln(r)+i\,a\theta$$

A monopole field is the only power-law field whose integral (potential) depends on path of integration.

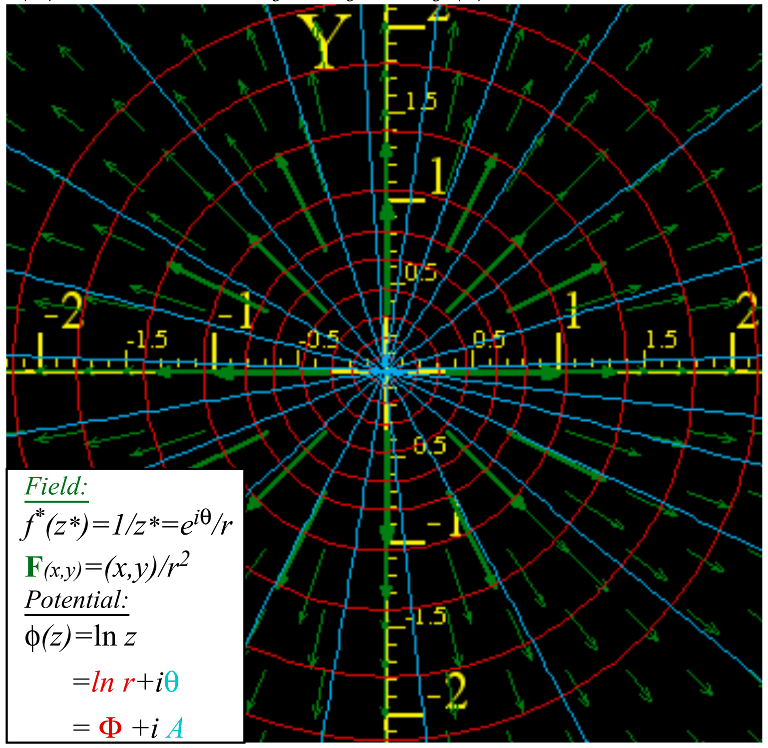




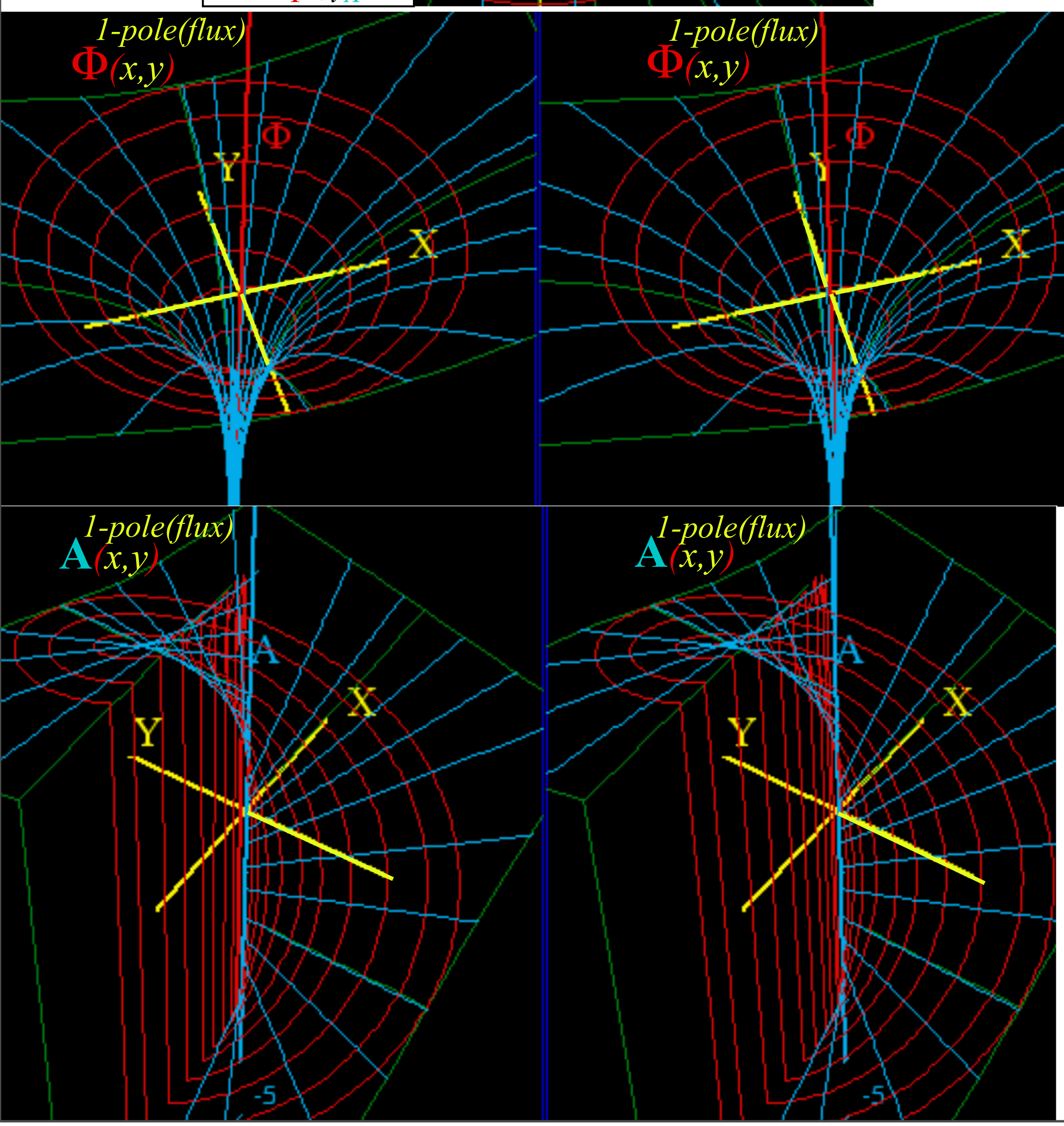
path that goes N times around origin (r=0) at constant r = R.

$$\Delta\phi=\oint f(z)dz=a\oint\frac{dz}{z}=a\int_{\theta=0}^{\theta=2\pi N}\frac{d(Re^{i\theta})}{Re^{i\theta}}=a\int_{\theta=0}^{\theta=2\pi N}id\theta=ai\,\theta\Big|_0^{2\pi N}=2a\pi iN$$

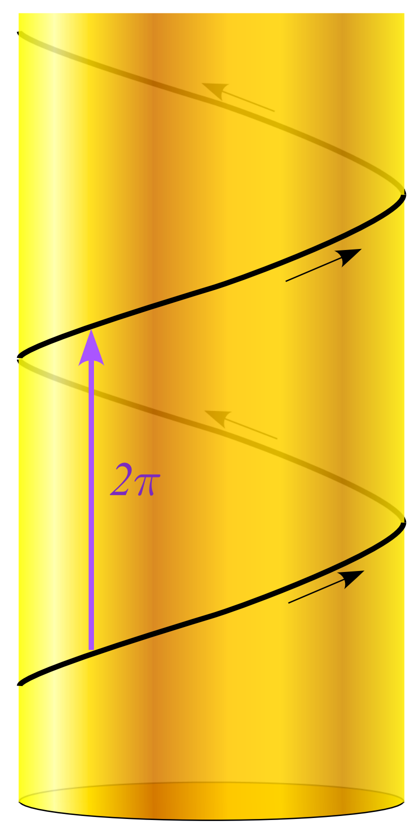
(a) Unit Z-line-flux field f(z)=1/z



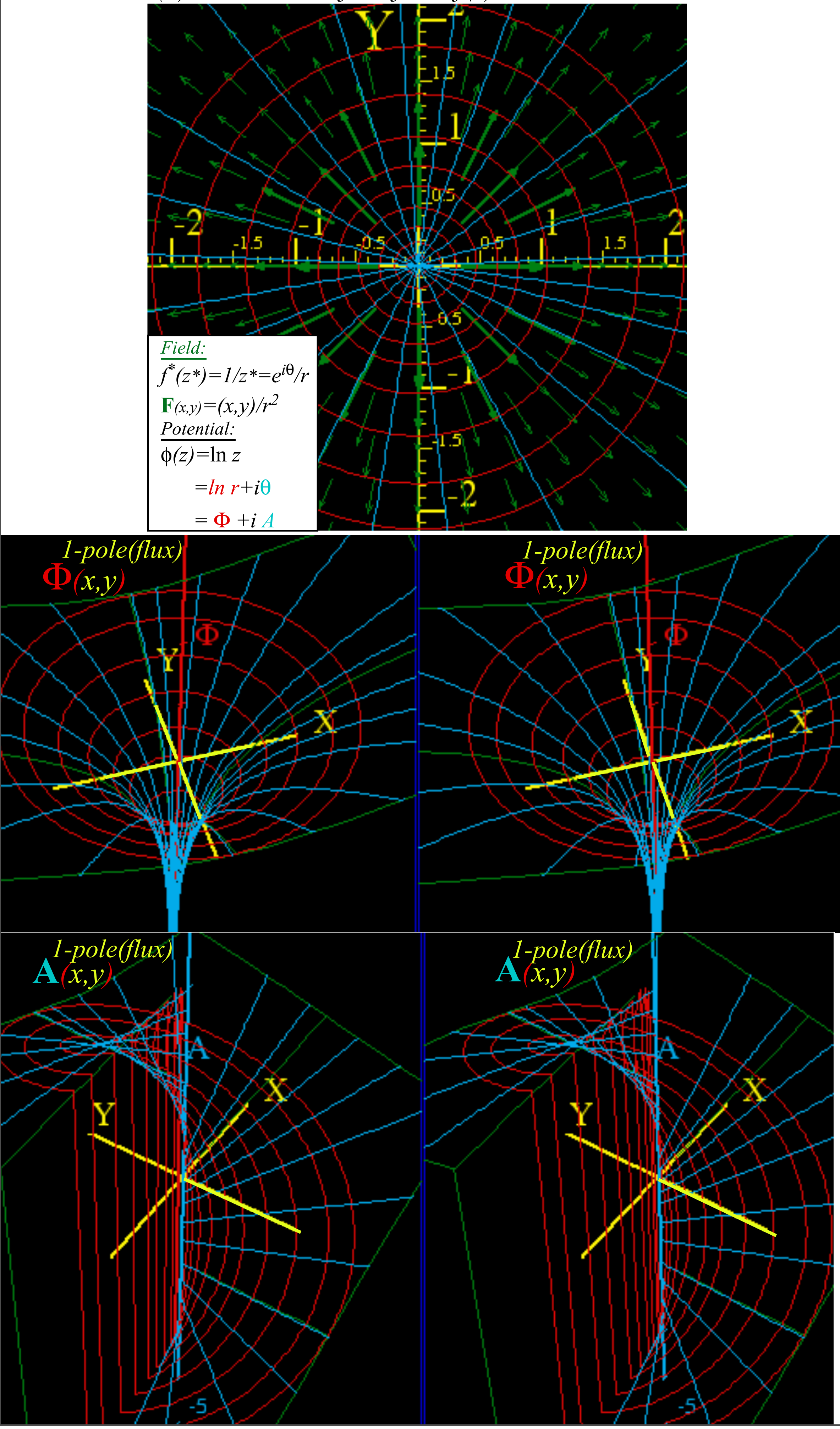
$$\phi(z) = \Phi + i\mathbf{A} = \int f(z)dz = \int \frac{a}{z}dz = a\ln(re^{i\theta})$$
$$= \ln(r) + i\theta$$



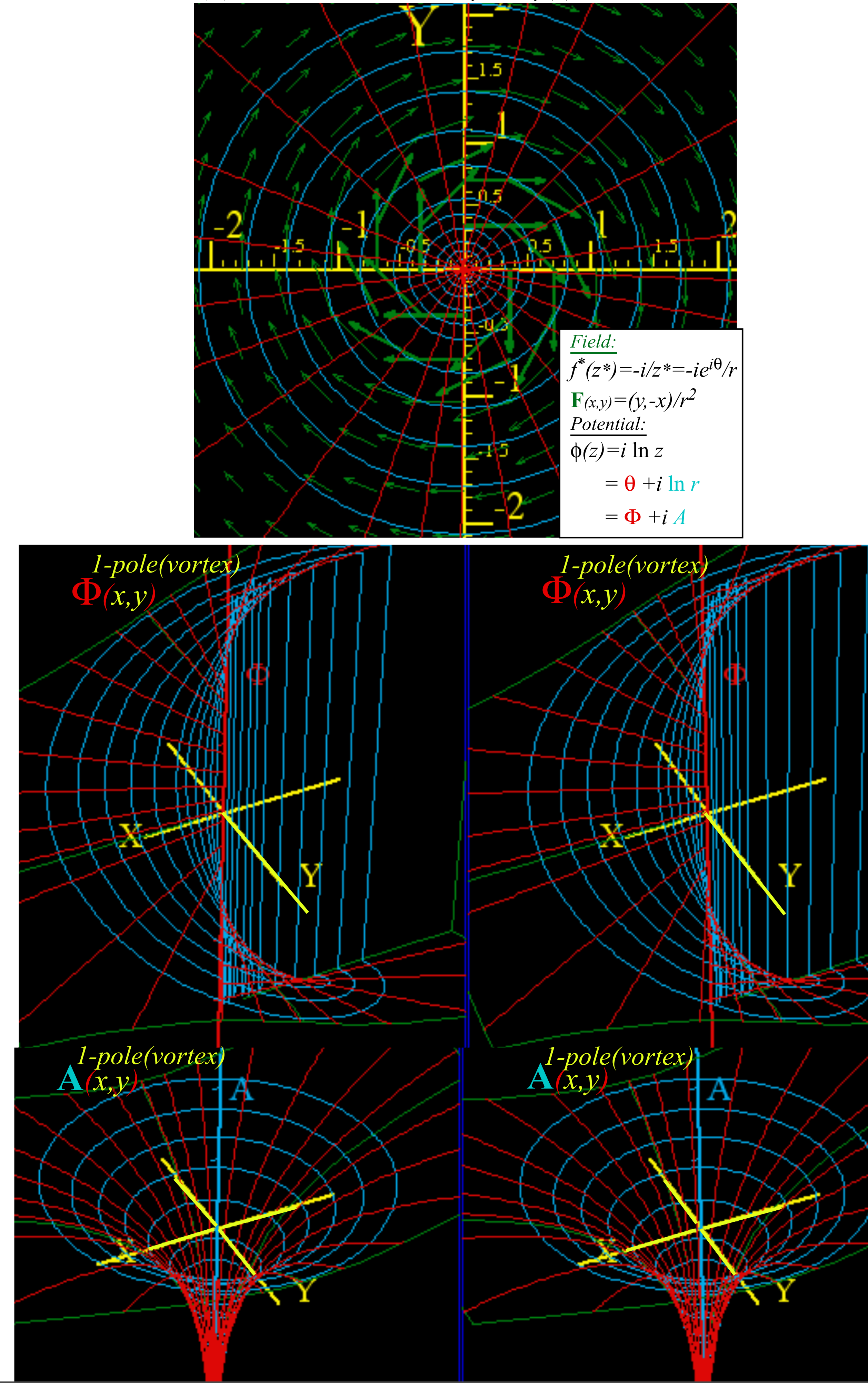
*Each turn around origin adds 2πi to vector potential i**A***



(a) Unit Z-line-flux field f(z)=1/z



(b) Unit Z-line-vortex field f(z)=i/z

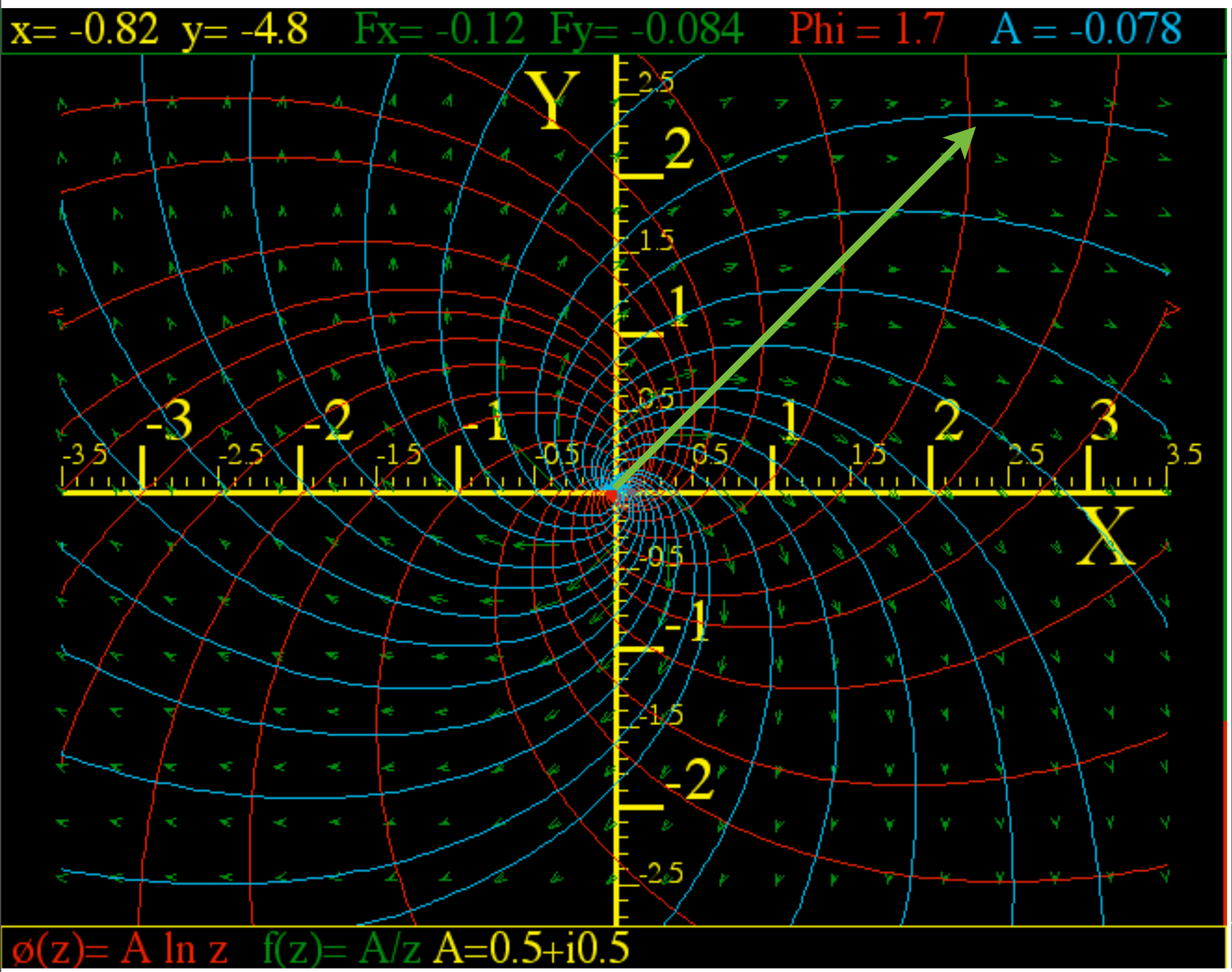


What Good Are Complex Exponentials? (contd.)

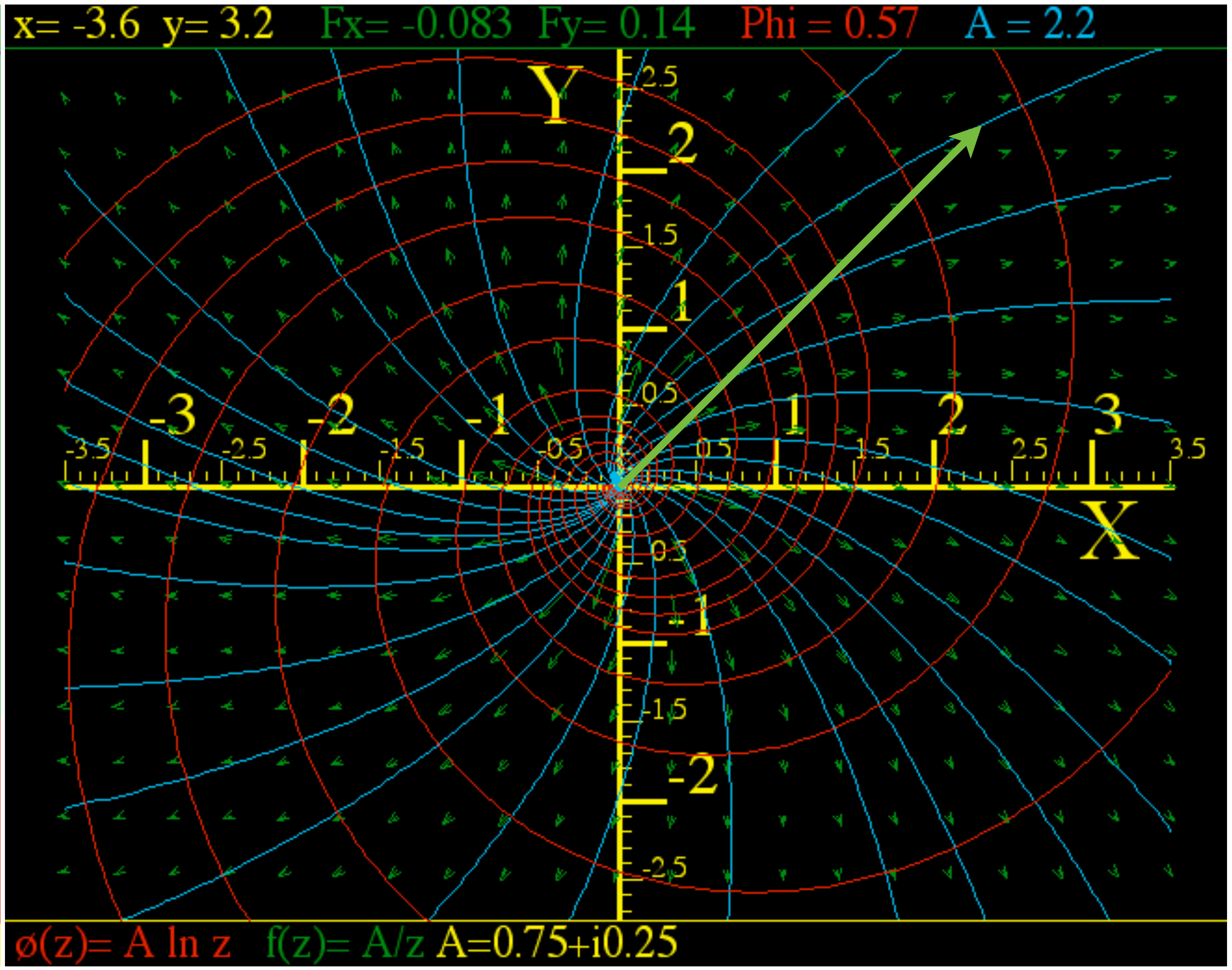
$f(z) = (0.5 + i0.5)/z = e^{i\pi/4}/z\sqrt{2}$

$f(z) = (0.75 + i0.25)/z = e^{i18^{\circ}}/z\sqrt{n}$

“Vortex”



“Hurricane”



4. Riemann-Cauchy conditions *What's analytic? (...and what's <u>not</u>?)*

Easy 2D circulation and flux integrals

Easy 2D curvilinear coordinate discovery

→ *Easy 2D monopole, dipole, and 2^n-pole analysis*

Easy 2^n-multipole field and potential expansion

Easy stereo-projection visualization

12. Complex derivatives give 2D dipole fields

Start with $f(z)=az^{-1}$: 2D line *monopole field* and is its *monopole potential* $\phi(z)=a\ln z$ of source strength a.

$$f^{1\text{-}pole}(z)=\frac{a}{z}=\frac{d\phi^{1\text{-}pole}}{dz} \qquad \phi^{1\text{-}pole}(z)=a\ln z$$

Now let these two line-sources of equal but opposite source constants $+a$ and $-a$ be located at $z=\pm\Delta/2$ separated by a small interval Δ. This sum (actually difference) of $f^{1\text{-}pole}$-fields is called a *dipole field*.

$$f^{dipole}(z)=\frac{a}{z+\frac{\Delta}{2}}-\frac{a}{z-\frac{\Delta}{2}}=\frac{-a\cdot\Delta}{z^2-\frac{\Delta^2}{4}} \qquad \phi^{dipole}(z)=a\ln(z-\tfrac{\Delta}{2})-a\ln(z+\tfrac{\Delta}{2})=a\ln\frac{z-\frac{\Delta}{2}}{z+\frac{\Delta}{2}}$$

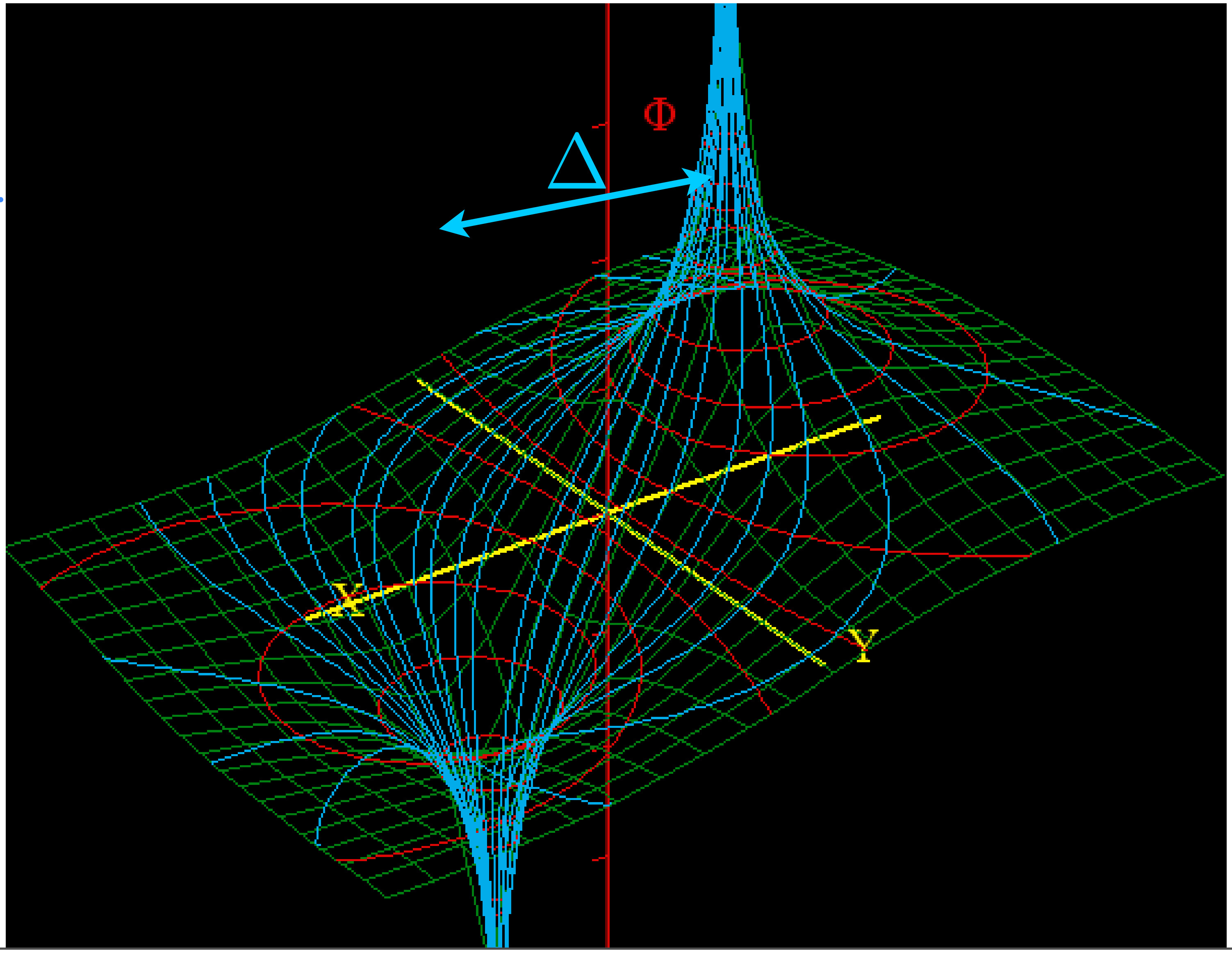
This is like the derivative definition:

$$\frac{df}{dz}=\frac{f(z+\Delta)-f(z)}{\Delta}$$

or:

$$\frac{df}{dz}=\frac{f(z+\frac{\Delta}{2})-f(z-\frac{\Delta}{2})}{\Delta}$$

if Δ is infinitesimal ($\Delta\rightarrow 0$)



So-called "physical dipole" has finite Δ

(+)(−) separation

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If interval Δ is *tiny* and is divided out we get a *point-dipole field* $f^{2\text{-}pole}$ that is the z-derivative of $f^{1\text{-}pole}$.

$$f^{2\text{-}pole}=\frac{-a}{z^2}=\frac{df^{1\text{-}pole}}{dz}=\frac{d\phi^{2\text{-}pole}}{dz} \qquad \phi^{2\text{-}pole}=\frac{a}{z}=\frac{d\phi^{1\text{-}pole}}{dz}$$

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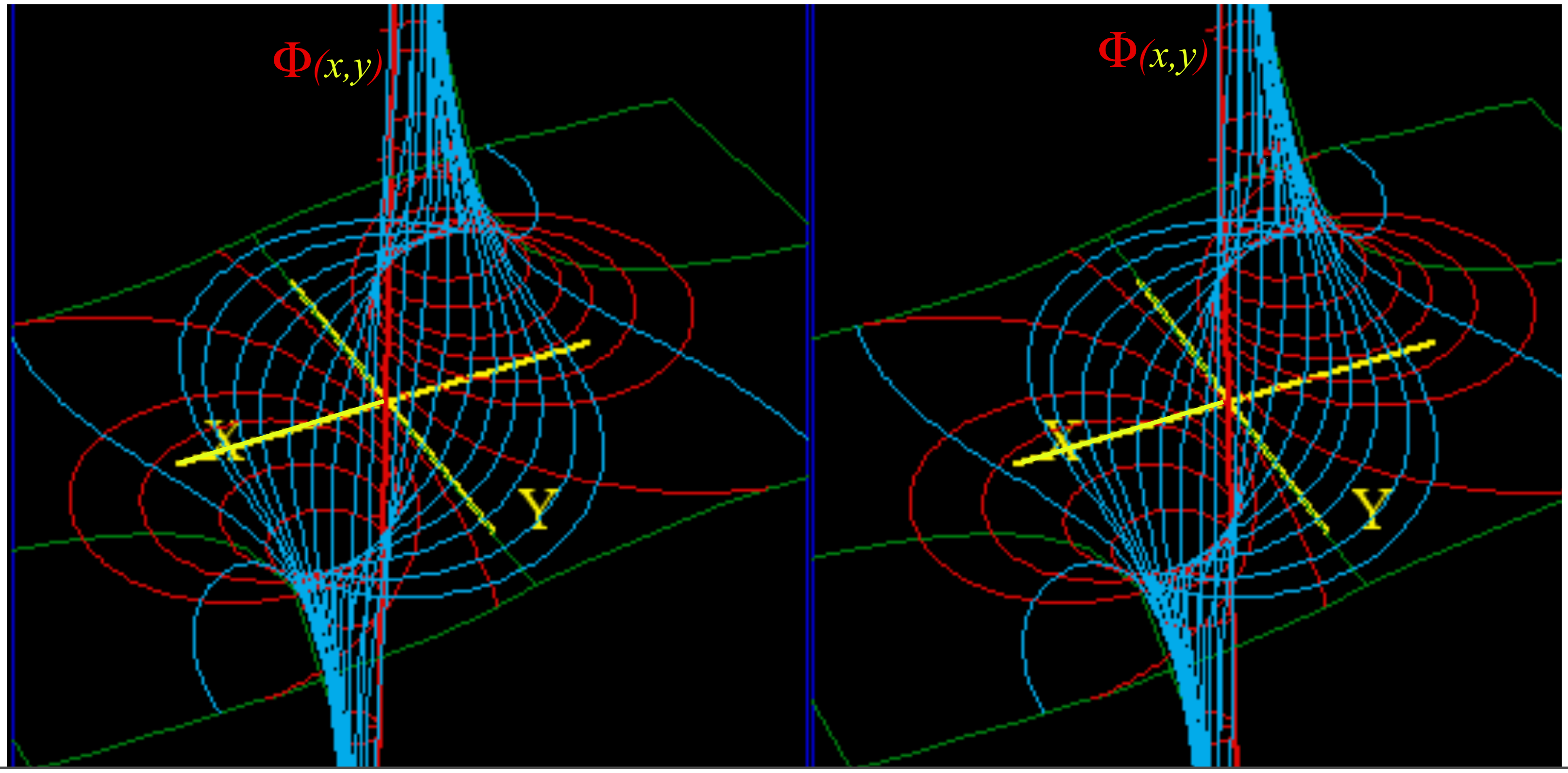

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A *point-dipole potential* $\phi^{2\text{-}pole}$ (whose z-derivative is $f^{2\text{-}pole}$) is a z-derivative of $\phi^{1\text{-}pole}$.

$$\phi^{2\text{-}pole}=\frac{a}{z}=\frac{a}{x+iy}=\frac{a}{x+iy}\frac{x-iy}{x-iy}=\frac{ax}{x^2+y^2}+i\frac{-ay}{x^2+y^2}=\frac{a}{r}\cos\theta-i\frac{a}{r}\sin\theta$$

$$=\Phi^{2\text{-}pole}+i\,\mathbf{A}^{2\text{-}pole}$$

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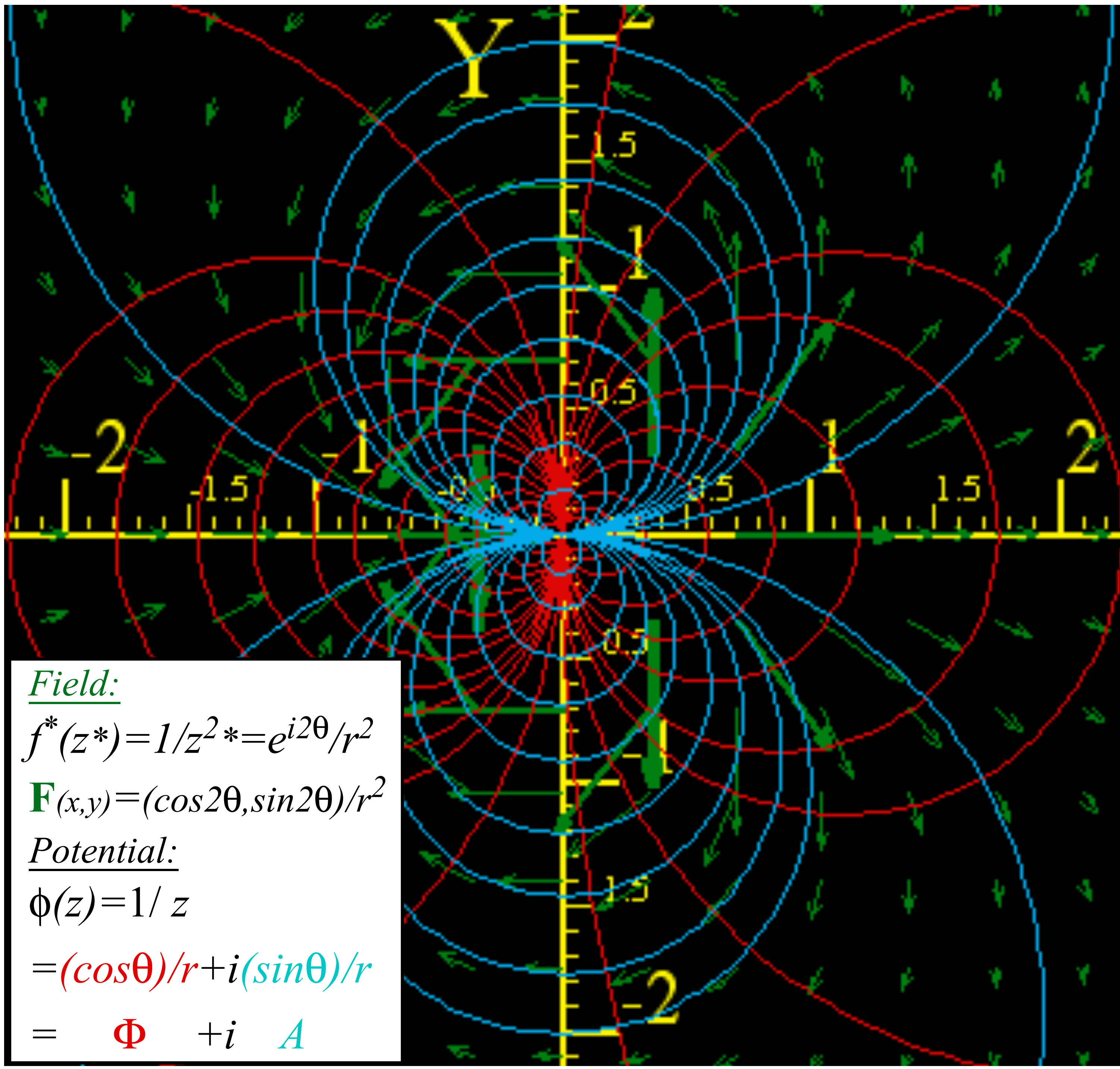
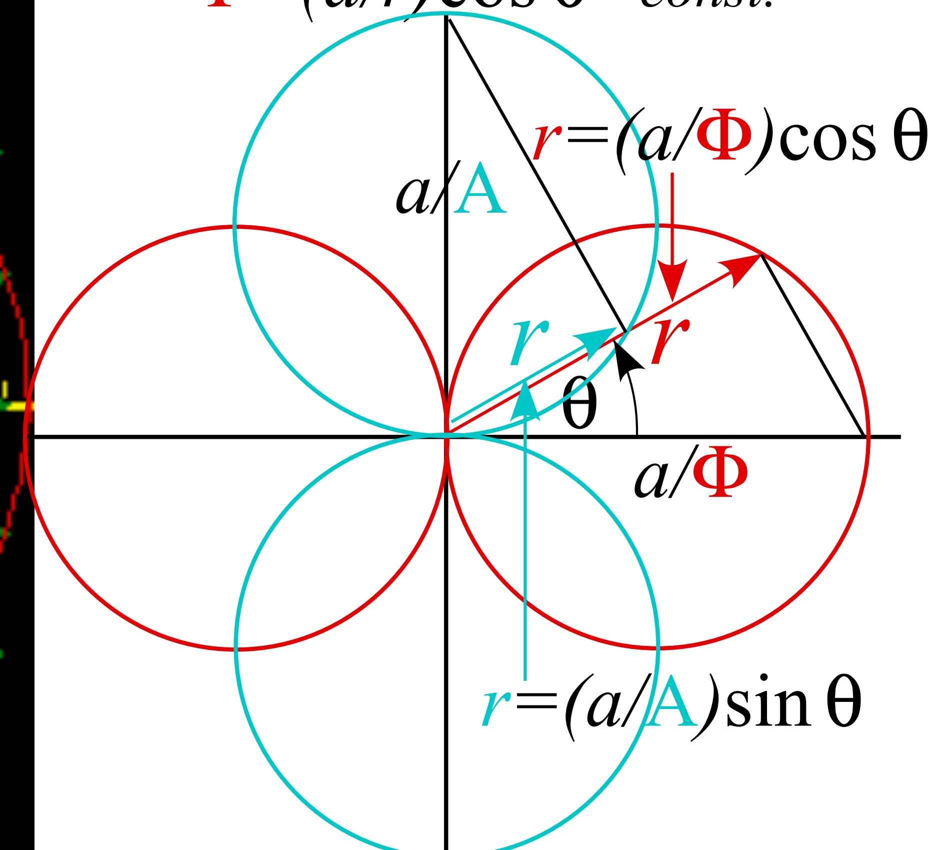


Scalar potentials

$\Phi=(a/r)\cos\theta=const.$



Vector potentials

$A=(a/r)\sin\theta=const.$

2^n*-pole analysis* *(quadrupole:* 2^2*=4-pole, octapole:* 2^3*=8-pole,…, pole dancer,*

What if we put a *(-)*copy of a *2-pole* near its original?

Well, the result is *4-pole* or *quadrupole* field $f^{\text{4-pole}}$ and potential $\phi^{\text{4-pole}}$.

Each a z-derivative of $f^{\text{2-pole}}$ and $\phi^{\text{2-pole}}$.

$$f^{4\text{-}pole} = \frac{a}{z^3} = \frac{1}{2}\frac{df^{2\text{-}pole}}{dz} = \frac{d\phi^{4\text{-}pole}}{dz}$$

$$\phi^{4\text{-}pole} = -\frac{a}{2z^2} = \frac{1}{2}\frac{d\phi^{2\text{-}pole}}{dz}$$

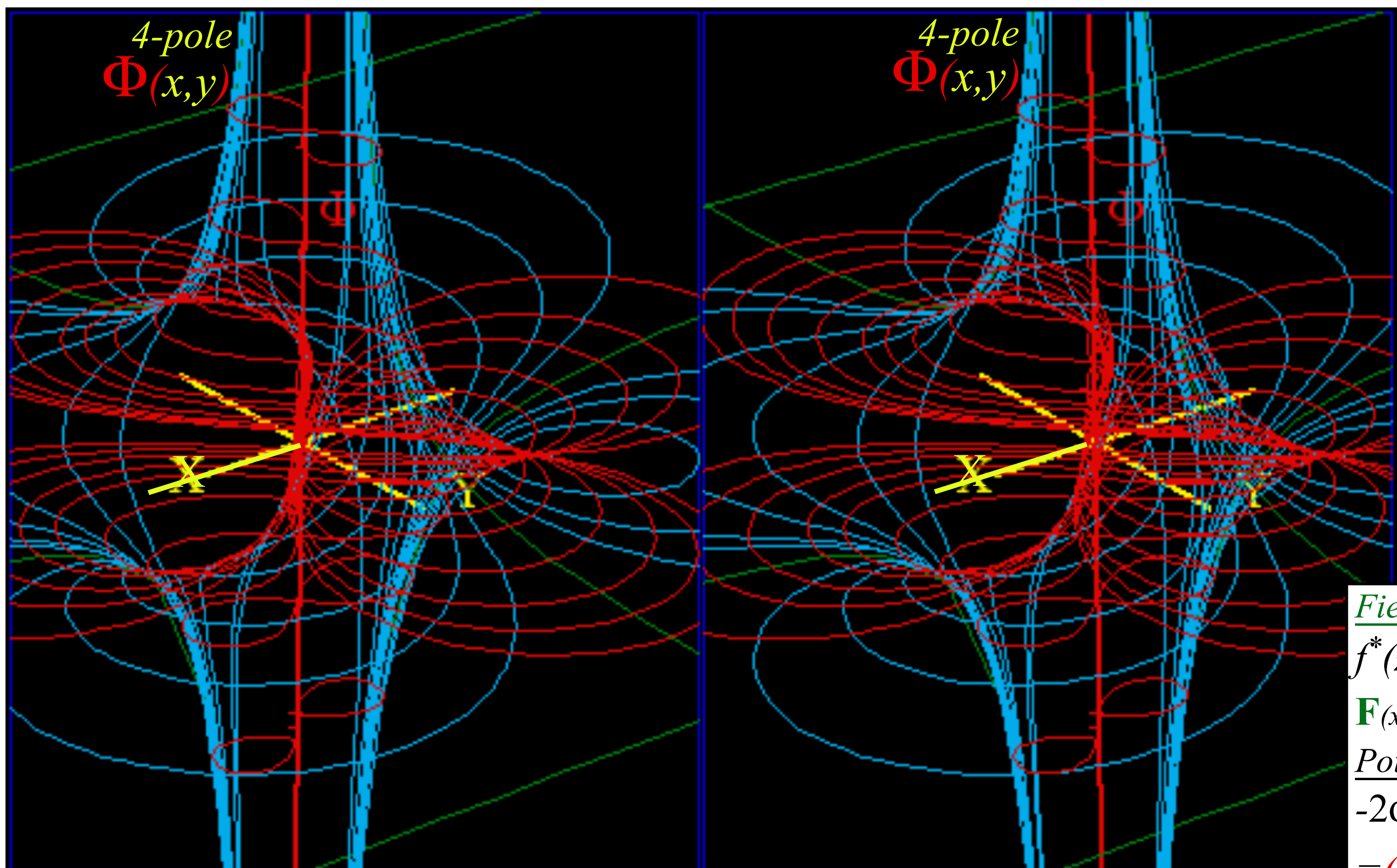
2^n-pole analysis *(quadrupole:2^2=4-pole, octapole:2^3=8-pole,…, pole dancer,*

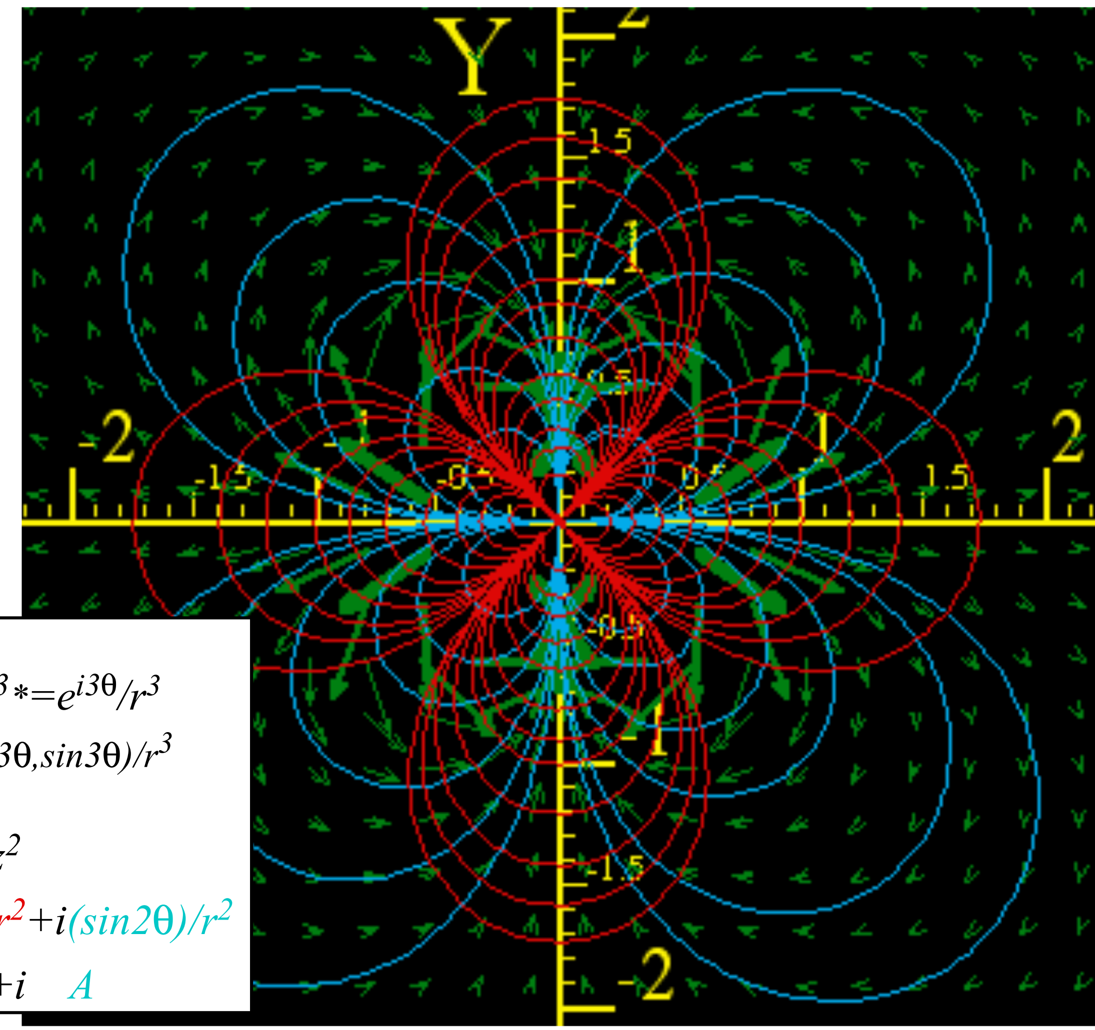
What if we put a *(-)*copy of a *2-pole* near its original?

Well, the result is *4-pole* or *quadrupole* field $f^{\,4\text{-}pole}$ and potential $\phi^{4\text{-}pole}$.

Each a z-derivative of $f^{\,2\text{-}pole}$ and $\phi^{2\text{-}pole}$.

$$f^{4\text{-}pole} = \frac{a}{z^3} = \frac{1}{2}\frac{df^{2\text{-}pole}}{dz} = \frac{d\phi^{4\text{-}pole}}{dz} \qquad\qquad \phi^{4\text{-}pole} = -\frac{a}{2z^2} = \frac{1}{2}\frac{d\phi^{2\text{-}pole}}{dz}$$





Field:
$f^*(z^*)=1/z^{3*}=e^{i3\theta}/r^3$
$\mathbf{F}_{(x,y)}=(\cos 3\theta,\sin 3\theta)/r^3$
Potential:
$-2\phi(z)=1/z^2$
$=(\cos 2\theta)/r^2+i(\sin 2\theta)/r^2$
$=\quad \Phi \quad +i \quad A$

4. Riemann-Cauchy conditions *What's analytic? (...and what's <u>not</u>?)*

Easy 2D circulation and flux integrals

Easy 2D curvilinear coordinate discovery

Easy 2D monopole, dipole, and 2^n-pole analysis

→ *Easy 2^n-multipole field and potential expansion*

Easy stereo-projection visualization

2^n-pole analysis: Laurent series *(Generalization of Maclaurin-Taylor series)*

Laurent series or *multipole expansion* of a given complex field function *f(z)* around *z=0*.

$$\frac{d\phi}{dz}=f(z)=\ldots a_{-3}z^{-3}+a_{-2}z^{-2}+a_{-1}z^{-1}+a_0+a_1 z+a_2 z^2+a_3 z^3+a_4 z^4+a_5 z^5+\ldots$$

$\cdots 2^2$-pole	2^1-pole	2^0-pole	2^1-pole	2^2-pole	2^3-pole	2^4-pole	2^5-pole	2^6-pole $\cdots$
(quadrupole)	*(dipole)*	*(monopole)*	*(dipole)*	*(quadrupole)*	*(octapole)*	*(hexadecapole)*		
at $z=0$	at $z=0$	at $z=0$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$

$$\int f\,dz=$$

$$\phi(z)=\ldots\frac{a_{-3}}{-2}z^{-2}+\frac{a_{-2}}{-1}z^{-1}+a_{-1}\ln z+a_0 z+\frac{a_1}{2}z^2+\frac{a_2}{3}z^3+\frac{a_3}{4}z^4+\frac{a_4}{5}z^5+\frac{a_5}{6}z^6+\ldots$$

All field terms $a_{m-1}z^{m-1}$ except *1-pole* $\frac{a_{-1}}{z}$ have potential term $a_{m-1}z^m/m$ of a 2^m*-pole*.

These are located at $z=0$ for $m<0$ and at $z=\infty$ for $m>0$.

(octapole)$_0$ *(quadrupole)*$_0$ *(dipole)*$_0$ *(monopole)* *(dipole)*$_\infty$ *(quadrupole)*$_\infty$ *(octapole)*$_\infty$

$$\phi(z)=\ldots\frac{a_{-4}}{-3}z^{-3}+\frac{a_{-3}}{-2}z^{-2}+\frac{a_{-2}}{-1}z^{-1}+a_{-1}\ln z+a_0 z+\frac{a_1}{2}z^2+\frac{a_2}{3}z^3+\ldots$$

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at $z=0$	at $z=0$	at $z=0$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$

$\int f\, dz =$

$$\phi(z)=\ldots\frac{a_{-3}}{-2}z^{-2} + \frac{a_{-2}}{-1}z^{-1} + a_{-1}\ln z + a_0 z + \frac{a_1}{2}z^2 + \frac{a_2}{3}z^3 + \frac{a_3}{4}z^4 + \frac{a_4}{5}z^5 + \frac{a_5}{6}z^6 + \ldots$$

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$$\phi(w)=\ldots\frac{a_{-4}}{-3}w^{-3} + \frac{a_{-3}}{-2}w^{-2} + \frac{a_{-2}}{-1}w^{-1} + a_{-1}\ln w + a_0 w + \frac{a_1}{2}w^2 + \frac{a_2}{3}w^3 + \ldots$$

(*with* $z=w^{-1}$)

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(quadrupole)	*(dipole)*	*(monopole)*	*(dipole)*	*(quadrupole)*	*(octapole)*	*(hexadecapole)*		
at $z=0$	at $z=0$	at $z=0$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$

$$\int f\,dz=\phi(z)=\ldots\frac{a_{-3}}{-2}z^{-2}+\frac{a_{-2}}{-1}z^{-1}+a_{-1}\ln z+a_0z+\frac{a_1}{2}z^2+\frac{a_2}{3}z^3+\frac{a_3}{4}z^4+\frac{a_4}{5}z^5+\frac{a_5}{6}z^6+\ldots$$

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(*with* $z\rightarrow w$)

$$=\ldots\frac{a_2}{3}z^{-3}+\frac{a_1}{2}z^{-2}+a_0z^{-1}-a_{-1}\ln z+\frac{a_{-2}}{-1}z+\frac{a_{-3}}{-2}z^2+\frac{a_{-4}}{-3}z^3+\ldots$$

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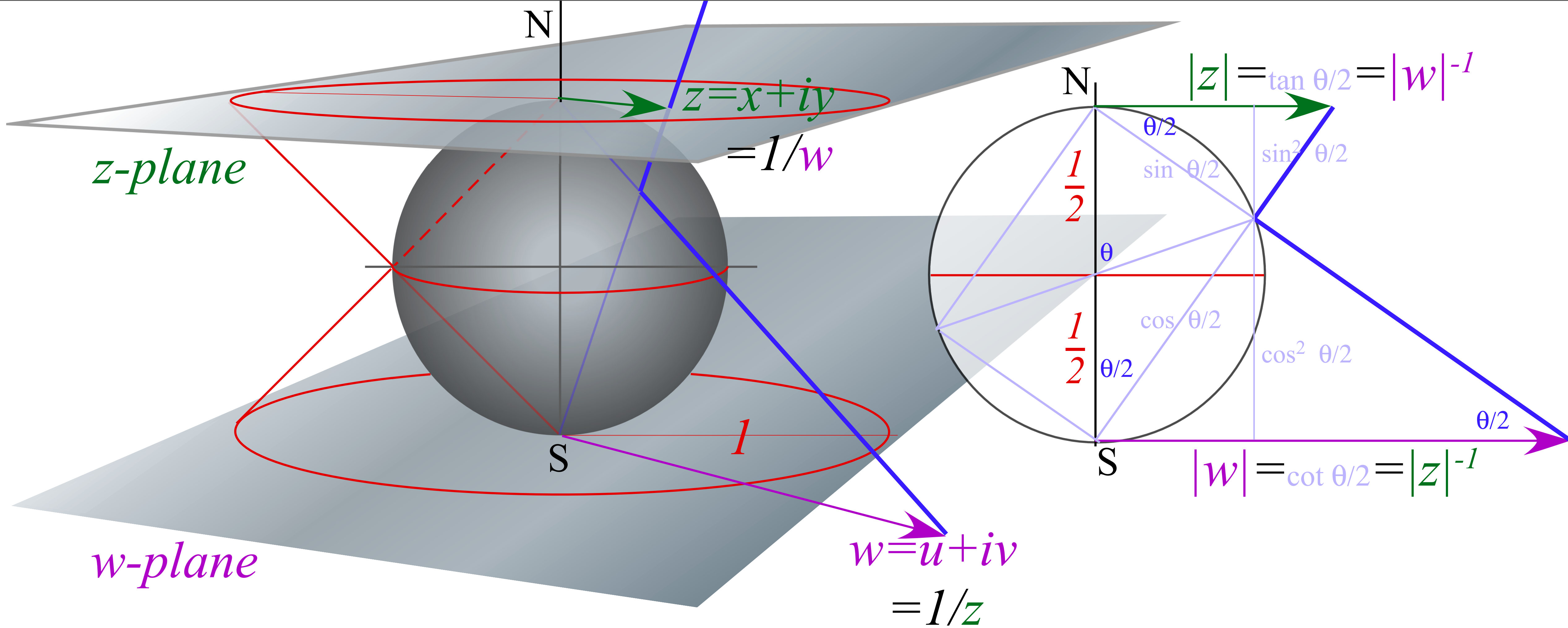

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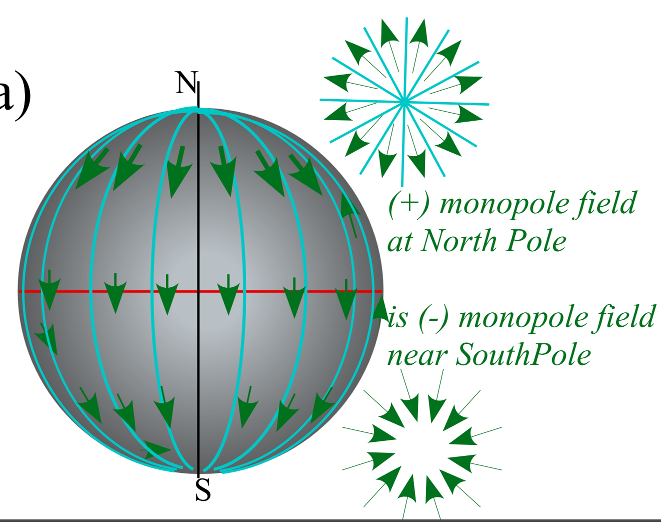
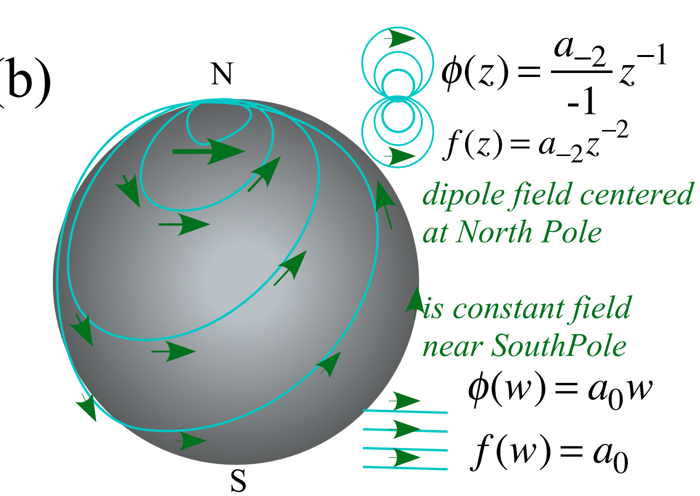
(a)



(b)



$\phi(z)=\frac{a_{-3}}{-2}z^{-2}$

$f(z)=a_{-3}z^{-3}$

quadrupole field centered at North Pole

is quadratic field near South Pole

$\phi(w)=a_0 w^2$

$f(w)=a_1 w$

$$f(z) = \dots a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \dots$$

Of all *2^m-pole* field terms $a_{m-1}z^{m-1}$, only the $m=0$ monopole $a_{-1}z^{-1}$ has a non-zero loop integral (10.39).

$$\oint f(z)dz = \oint a_{-1}z^{-1}dz = 2\pi i a_{-1} \qquad a_{-1} = \frac{1}{2\pi i}\oint f(z)dz$$

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(but any contour that doesn't "touch a gives same answer)

The $f(a)$ result is called a *Cauchy integral*. Then repeated a-derivatives gives a sequence of them.

$$\frac{df(a)}{da} = \frac{1}{2\pi i}\oint \frac{f(z)}{(z-a)^2}dz \;,\; \frac{d^2 f(a)}{da^2} = \frac{2}{2\pi i}\oint \frac{f(z)}{(z-a)^3}dz \;,\; \frac{d^3 f(a)}{da^3} = \frac{3!}{2\pi i}\oint \frac{f(z)}{(z-a)^4}dz, \cdots, \frac{d^n f(a)}{da^n} = \frac{n!}{2\pi i}\oint \frac{f(z)}{(z-a)^{n+1}}dz$$

$$f(z) = \dots a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \dots$$

Of all 2^m-*pole* field terms $a_{m-1}z^{m-1}$, only the $m=0$ monopole $a_{-1}z^{-1}$ has a non-zero loop integral (10.39).

$$\oint f(z)dz = \oint a_{-1}z^{-1}dz = 2\pi i a_{-1} \qquad a_{-1} = \tfrac{1}{2\pi i}\oint f(z)dz$$

This $m=1$-*pole* constant-a_{-1} formula is just the first in a series of *Laurent coefficient expressions*.

$$\cdots a_{-3} = \tfrac{1}{2\pi i}\oint z^2 f(z)dz\ ,\ a_{-2} = \tfrac{1}{2\pi i}\oint z^1 f(z)dz\ ,\ a_{-1} = \tfrac{1}{2\pi i}\oint f(z)dz\ ,\ a_0 = \tfrac{1}{2\pi i}\oint \frac{f(z)}{z}dz\ ,\ a_1 = \tfrac{1}{2\pi i}\oint \frac{f(z)}{z^2}dz\ ,\cdots$$

Source analysis starts with 1-pole loop integrals $\oint z^{-1}dz = 2\pi i$ or, with origin shifted $\oint (z-a)^{-1}dz = 2\pi i$.

They hold for any loop about point-a. Function $f(z)$ is just $f(a)$ on a *tiny* circle around point-a.

(assume *tiny* circle around $z=a$)

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$$f(z)=\dots a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \dots$$

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$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n \qquad where: \ a_n = \frac{1}{2\pi i}\oint \frac{f(z)}{(z-a)^{n+1}}dz \left(= \frac{1}{n!}\frac{d^n f(a)}{da^n} \quad for: \ n \geq 0 \right)$$

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(quadrupole)₀ (dipole)₀ (monopole) (dipole)∞ (quadrupole)∞ (octapole)∞ (hexadecapole)∞ ...

$$f(z) = ...a_{-3}z^{-3} + \underbrace{a_{-2}z^{-2}}_{\text{dipole moment}} + \underbrace{a_{-1}z^{-1}}_{\text{monopole moment}} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + ...$$

5. Mapping and Non-analytic 2D source field analysis

The half-n'-half results

are called

Riemann-Cauchy

Derivative Relations

$$\boxed{\begin{aligned}\frac{\partial \Phi}{\partial x} &= \frac{\partial A}{\partial y} \\ \frac{\partial \Phi}{\partial y} &= -\frac{\partial A}{\partial x}\end{aligned}} \begin{aligned}&\text{is:}\\&\text{is:}\end{aligned} \boxed{\begin{aligned}\frac{\partial \mathrm{Re}\,\phi(z)}{\partial x} &= \frac{\partial \mathrm{Im}\,\phi(z)}{\partial y} \\ \frac{\partial \mathrm{Re}\,\phi(z)}{\partial y} &= -\frac{\partial \mathrm{Im}\,\phi(z)}{\partial x}\end{aligned}} \begin{aligned}&\text{or:}\\&\text{or:}\end{aligned} \boxed{\begin{aligned}\frac{\partial \mathrm{Re}\,f(z)}{\partial x} &= \frac{\partial \mathrm{Im}\,f(z)}{\partial y} \\ \frac{\partial \mathrm{Re}\,f(z)}{\partial y} &= -\frac{\partial \mathrm{Im}\,f(z)}{\partial x}\end{aligned}} \begin{aligned}&\text{is:}\\&\text{is:}\end{aligned} \boxed{\begin{aligned}\frac{\partial f_x(z)}{\partial x} &= \frac{\partial f_y(z)}{\partial y} \\ \frac{\partial f_x(z)}{\partial y} &= -\frac{\partial f_y(z)}{\partial x}\end{aligned}}$$

RC applies to analytic potential $\phi(z) = \Phi + iA$ *and analytic field* $f(z) = f_x + if_y$ *and any analytic function*

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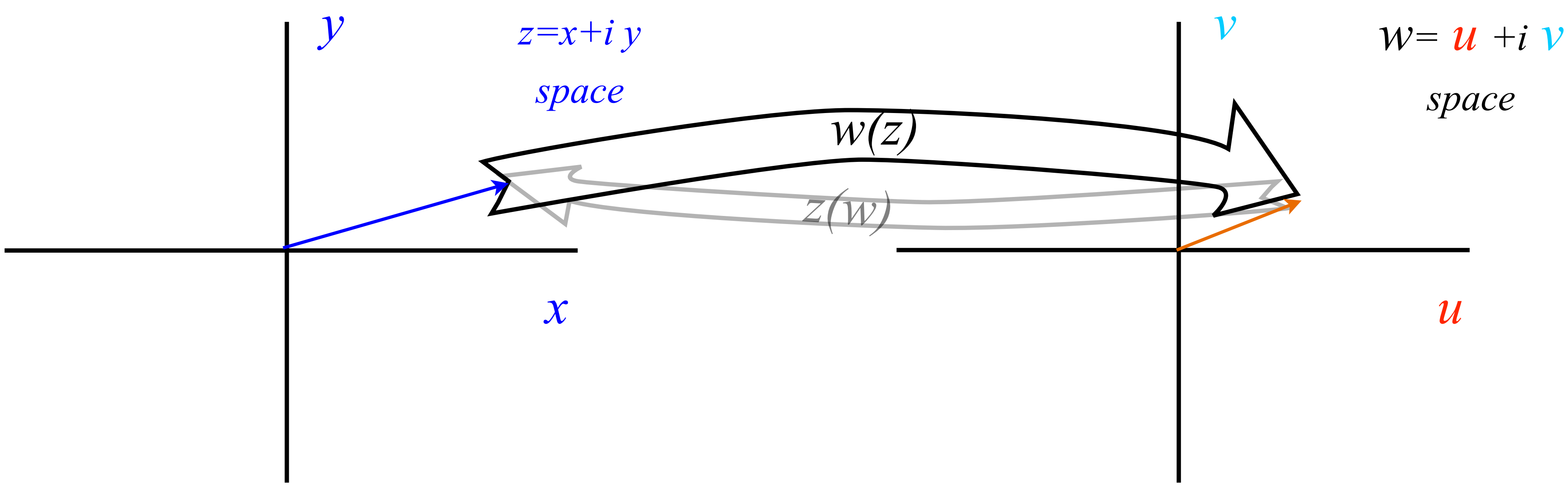
Derivative Relations

$\frac{\partial \Phi}{\partial x}=\frac{\partial A}{\partial y}$	is:	$\frac{\partial \mathrm{Re}\,\phi(z)}{\partial x}=\frac{\partial \mathrm{Im}\,\phi(z)}{\partial y}$	or:	$\frac{\partial \mathrm{Re}\,f(z)}{\partial x}=\frac{\partial \mathrm{Im}\,f(z)}{\partial y}$	is:	$\frac{\partial f_x(z)}{\partial x}=\frac{\partial f_y(z)}{\partial y}$
$\frac{\partial \Phi}{\partial y}=-\frac{\partial A}{\partial x}$	is:	$\frac{\partial \mathrm{Re}\,\phi(z)}{\partial y}=-\frac{\partial \mathrm{Im}\,\phi(z)}{\partial x}$	or:	$\frac{\partial \mathrm{Re}\,f(z)}{\partial y}=-\frac{\partial \mathrm{Im}\,f(z)}{\partial x}$	is:	$\frac{\partial f_x(z)}{\partial y}=-\frac{\partial f_y(z)}{\partial x}$

RC applies to analytic potential $\phi(z)=\Phi+i\mathbf{A}$ *and analytic field* $f(z)=f_x+if_y$ *and any analytic function*

Common notation for mapping: $w(z)=u+iv$



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Derivative Relations

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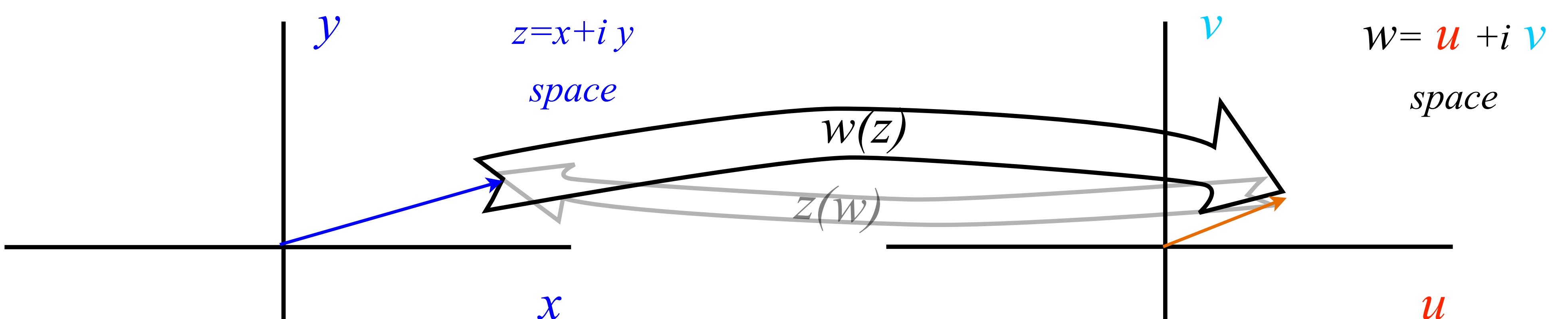

Jacobian for mapping:

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$$

$$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy$$

$$\begin{pmatrix} du\\ dv\end{pmatrix}=\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}\begin{pmatrix} dx\\ dy\end{pmatrix}$$

Complex derivative for mapping:

$$\frac{dw}{dz}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)(u+iv)=\frac{1}{2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{i}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

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Derivative Relations

$$\boxed{\frac{\partial \Phi}{\partial x}=\frac{\partial A}{\partial y}} \text{ is: } \boxed{\frac{\partial \mathrm{Re}\,\phi(z)}{\partial x}=\frac{\partial \mathrm{Im}\,\phi(z)}{\partial y}} \text{ or: } \boxed{\frac{\partial \mathrm{Re}\,f(z)}{\partial x}=\frac{\partial \mathrm{Im}\,f(z)}{\partial y}} \text{ is: } \boxed{\frac{\partial f_x(z)}{\partial x}=\frac{\partial f_y(z)}{\partial y}}$$

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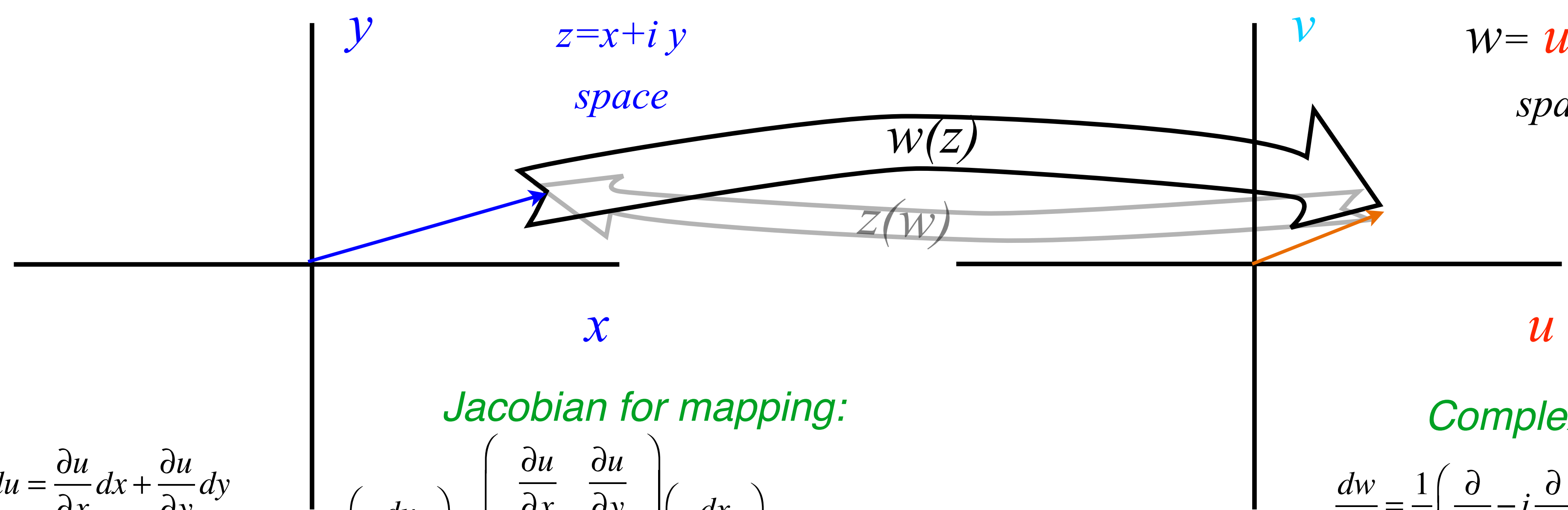

Jacobian for mapping:

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$$
$$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy$$

$$\begin{pmatrix} du \\ dv \end{pmatrix}=\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix}$$

$$=\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix}=\begin{pmatrix} \frac{\partial v}{\partial y} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix}$$

Complex derivative for mapping:

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$$=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y} \qquad =\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x}$$

Complex derivative abs-square:

$$\left|\frac{dw}{dz}\right|^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2$$

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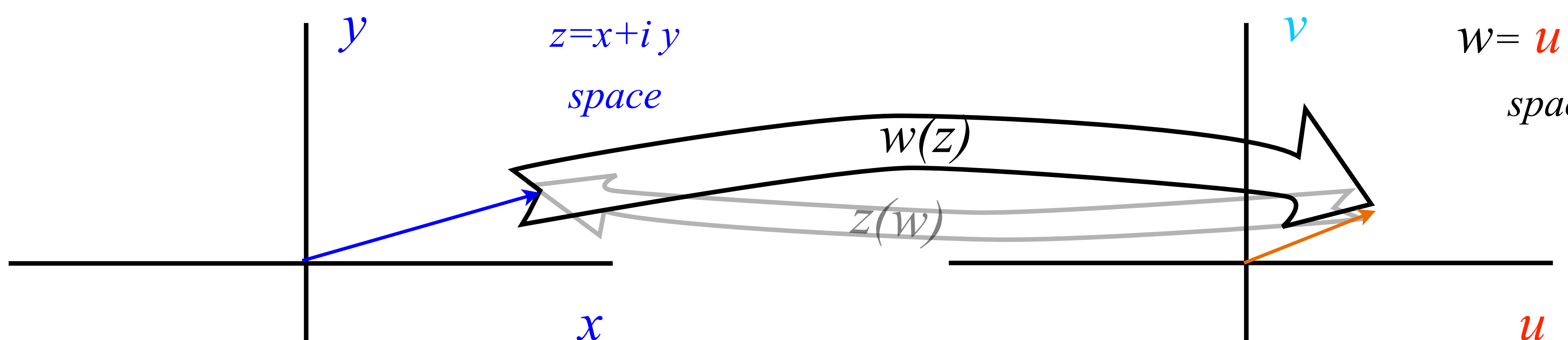
RC applies to analytic potential $\phi(z)=\Phi+iA$ *and analytic field* $f(z)=f_x+if_y$ *and any analytic function*

Common notation for mapping: $w(z)=u+iv$



$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$$
$$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy$$

Jacobian for mapping:

$$\begin{pmatrix}du\\dv\end{pmatrix}=\begin{pmatrix}\frac{\partial u}{\partial x}&\frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x}&\frac{\partial v}{\partial y}\end{pmatrix}\begin{pmatrix}dx\\dy\end{pmatrix}$$
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Complex derivative for mapping:

$$\frac{dw}{dz}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)(u+iv)=\frac{1}{2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{i}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$
$$=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}\qquad=\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x}$$

Complex derivative abs-square:

$$\left|\frac{dw}{dz}\right|^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2=\det|J|$$

...equals Jacobian Determinant

The *half*-n'-*half* results

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$$\boxed{\frac{\partial \Phi}{\partial x}=\frac{\partial \mathbf{A}}{\partial y}} \text{ is: } \boxed{\frac{\partial \mathrm{Re}\,\phi(z)}{\partial x}=\frac{\partial \mathrm{Im}\,\phi(z)}{\partial y}} \text{ or: } \boxed{\frac{\partial \mathrm{Re}\,f(z)}{\partial x}=\frac{\partial \mathrm{Im}\,f(z)}{\partial y}} \text{ is: } \boxed{\frac{\partial f_x(z)}{\partial x}=\frac{\partial f_y(z)}{\partial y}}$$

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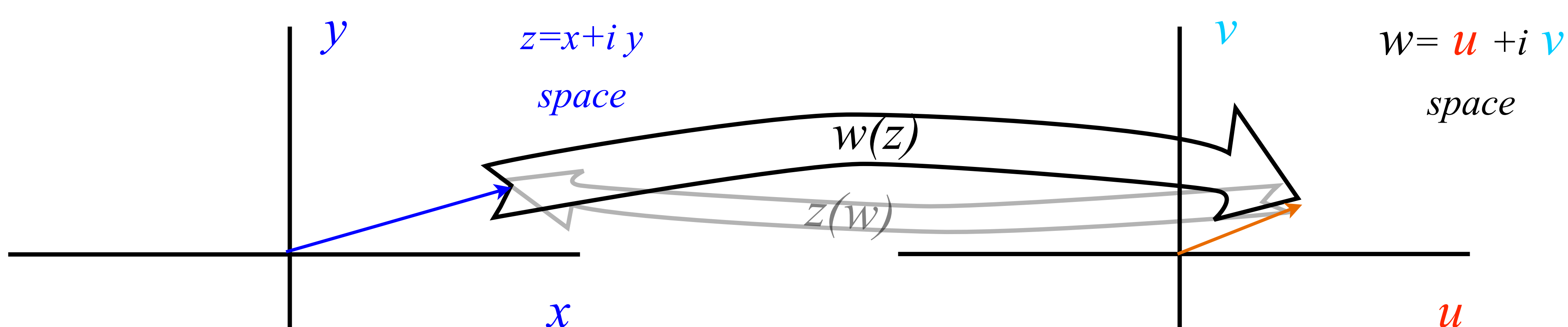
RC applies to analytic potential $\phi(z)=\Phi+i\mathbf{A}$ *and analytic field* $f(z)=f_x+if_y$ *and any analytic function*

Common notation for mapping: $w(z)=u+iv$

Important result:

$$dw=\sqrt{J}\cdot e^{i\theta}\cdot dz$$

is scaled rotation of dz.



Jacobian for mapping is scaled rotation:

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$$

$$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy$$

$$\begin{pmatrix} du \\ dv \end{pmatrix}=\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix}=\sqrt{\det J}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix}$$

$$=\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix}=\begin{pmatrix} \frac{\partial v}{\partial y} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix}$$

Complex derivative for mapping:

$$\frac{dw}{dz}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)(u+iv)=\frac{1}{2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{i}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

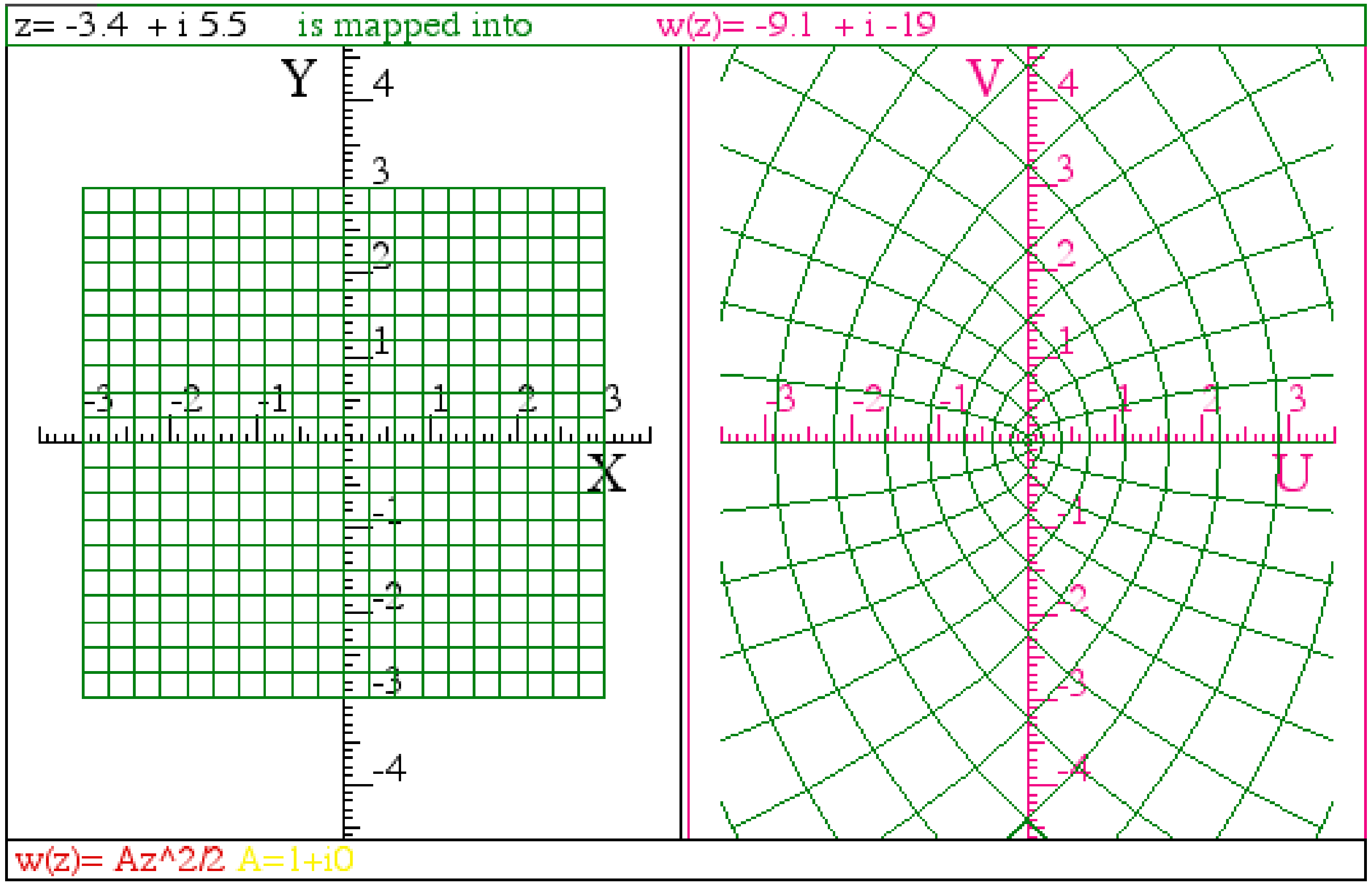
$$=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y} \qquad =\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x}$$

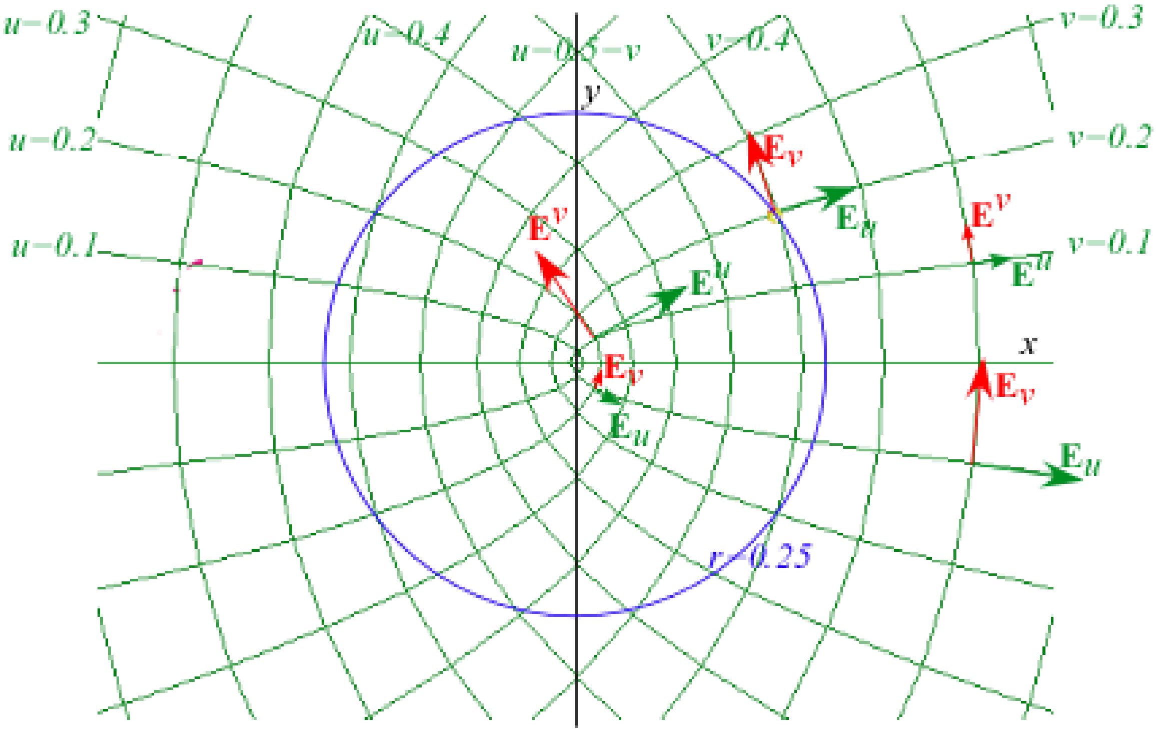
Complex derivative abs-square:

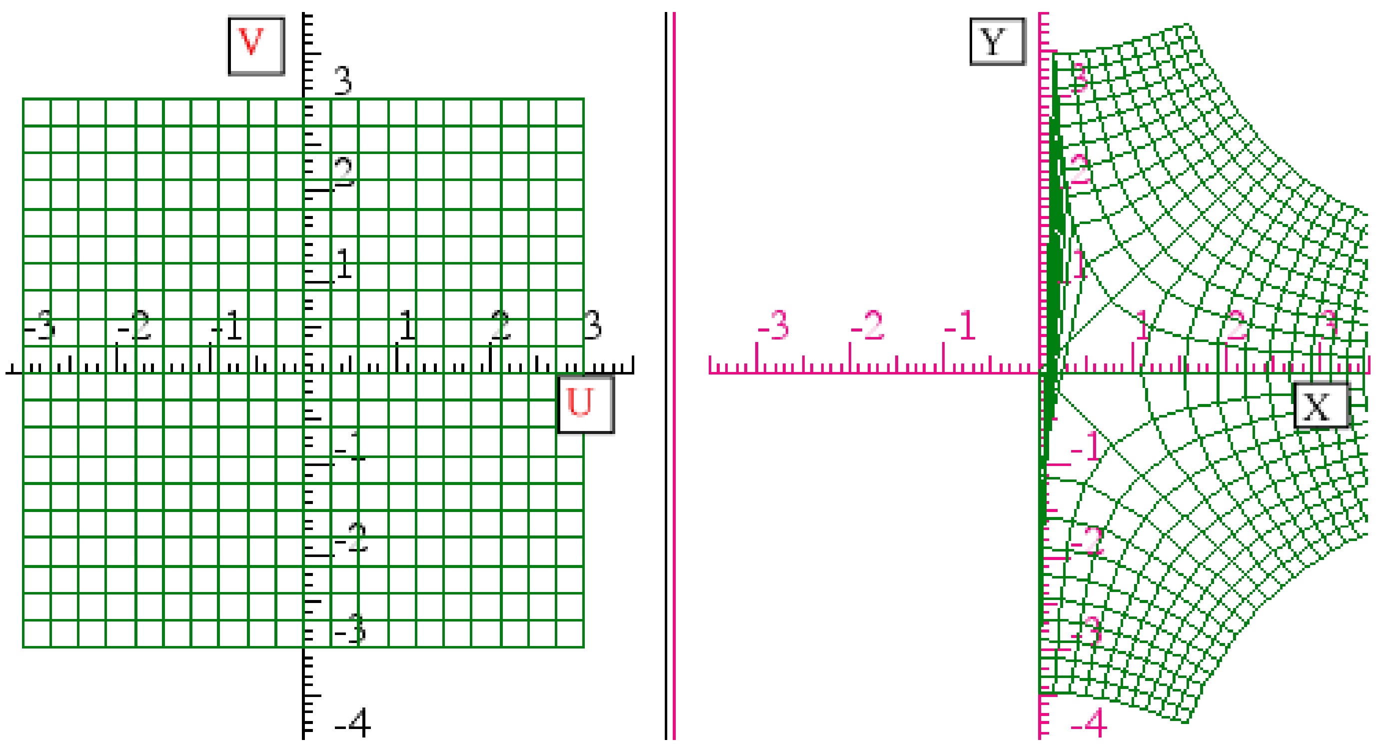
$$\left|\frac{dw}{dz}\right|^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2=\det|J|$$

...equals Jacobian Determinant

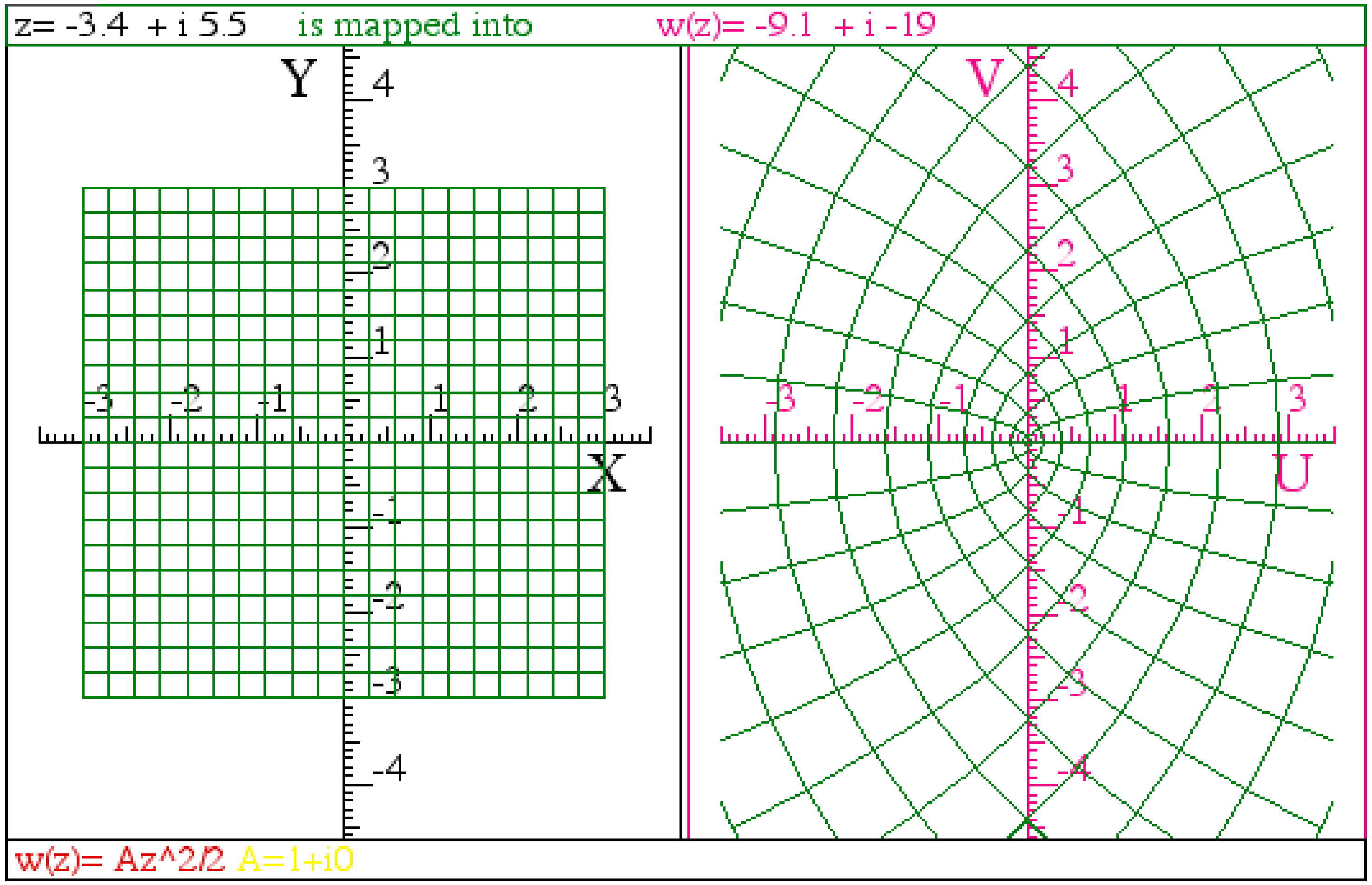
$w(z) = z^2$ gives parabolic OCC

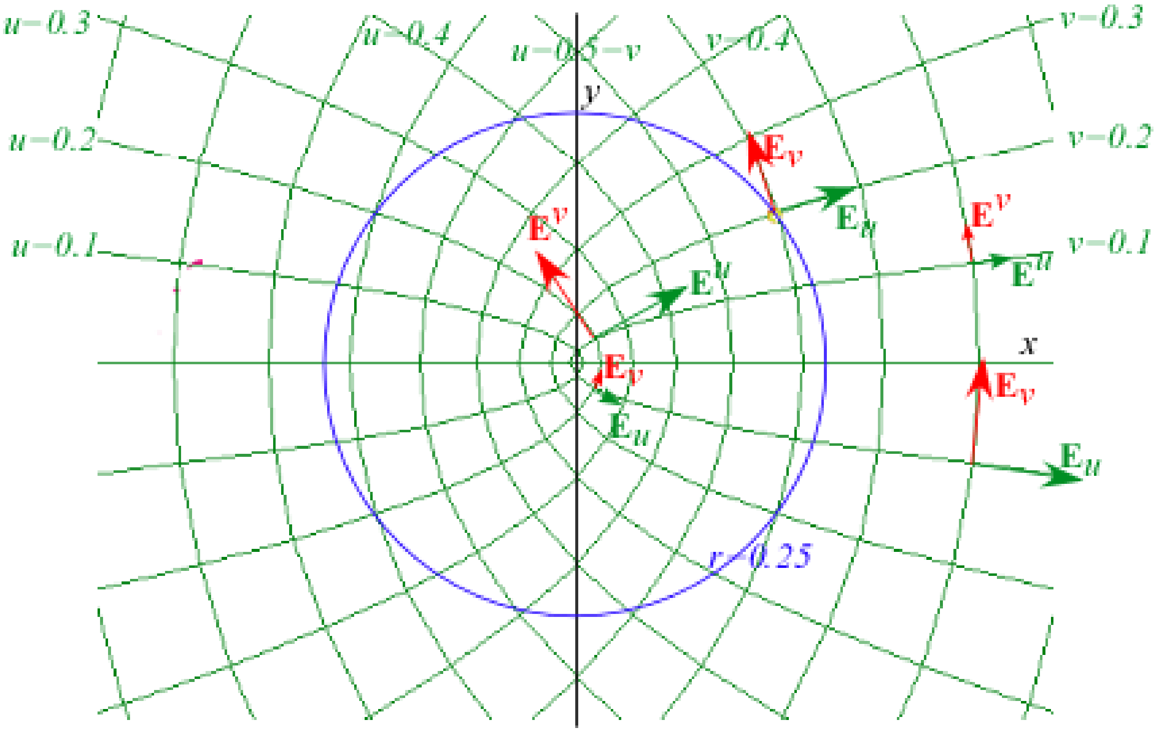




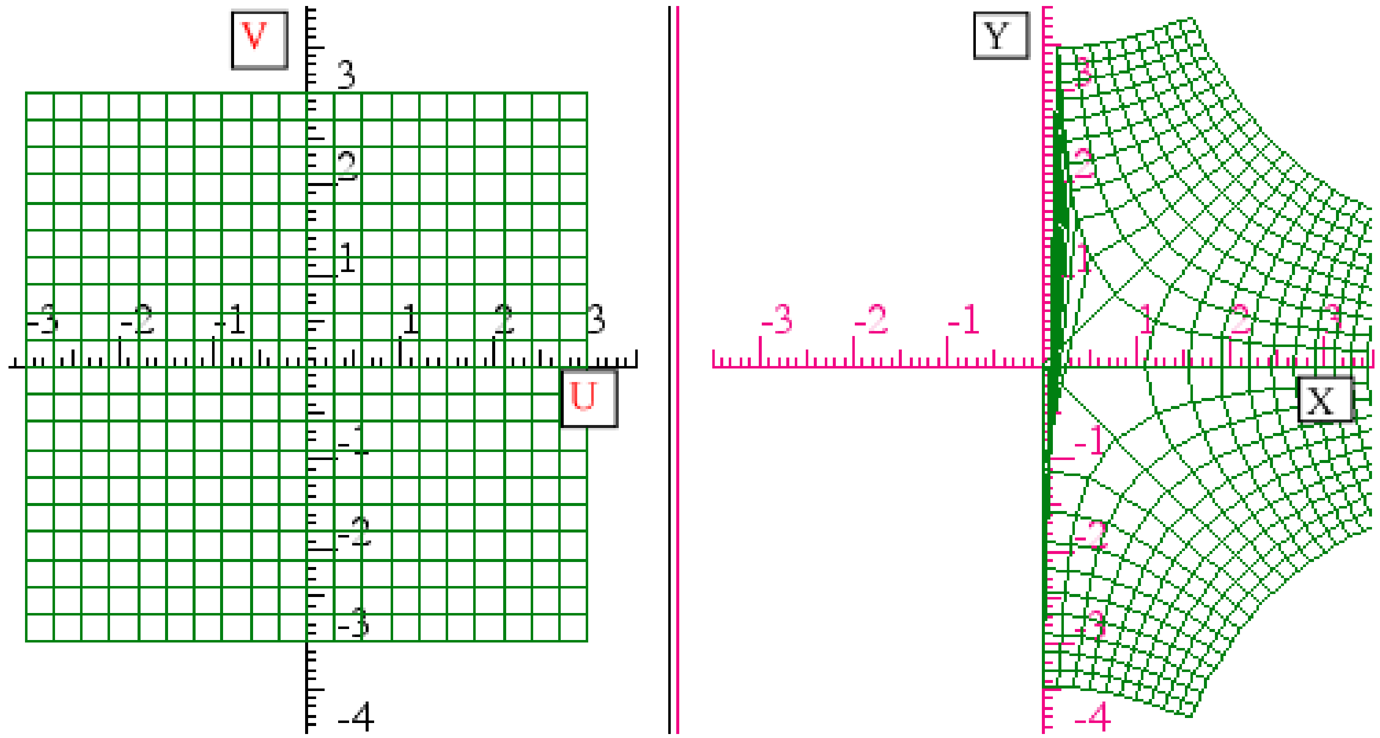


$w(z) = z^2$ gives parabolic OCC

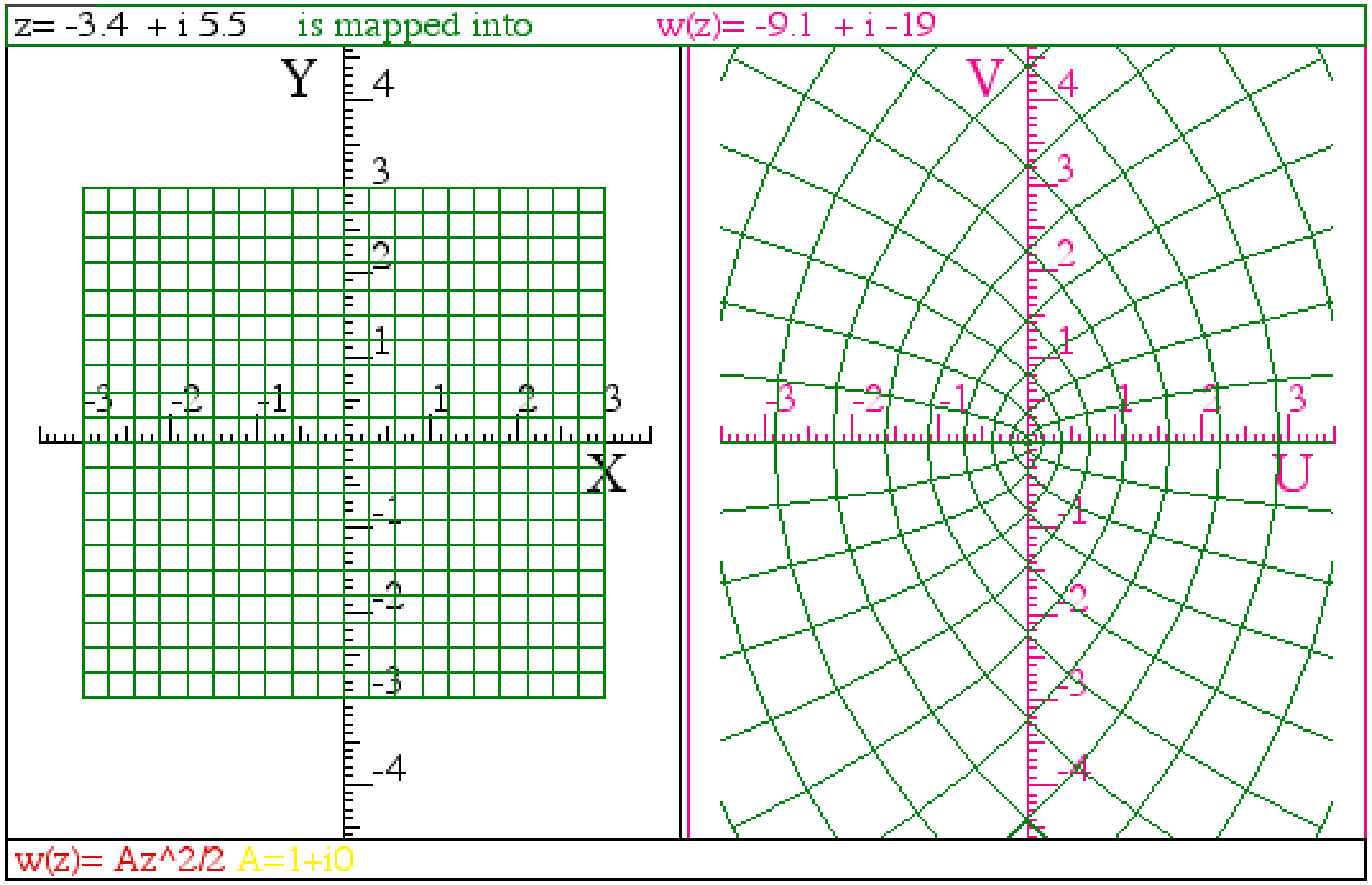


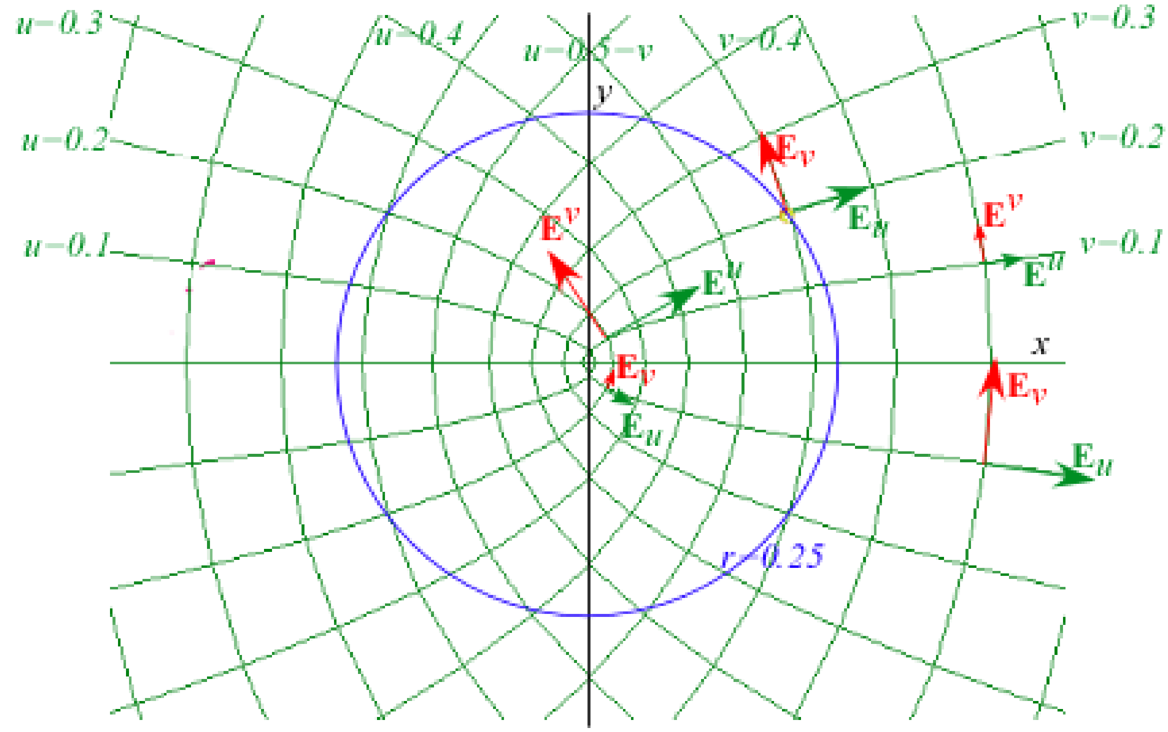


Inverse: $z(w) = w^{1/2}$ gives hyperbolic OCC

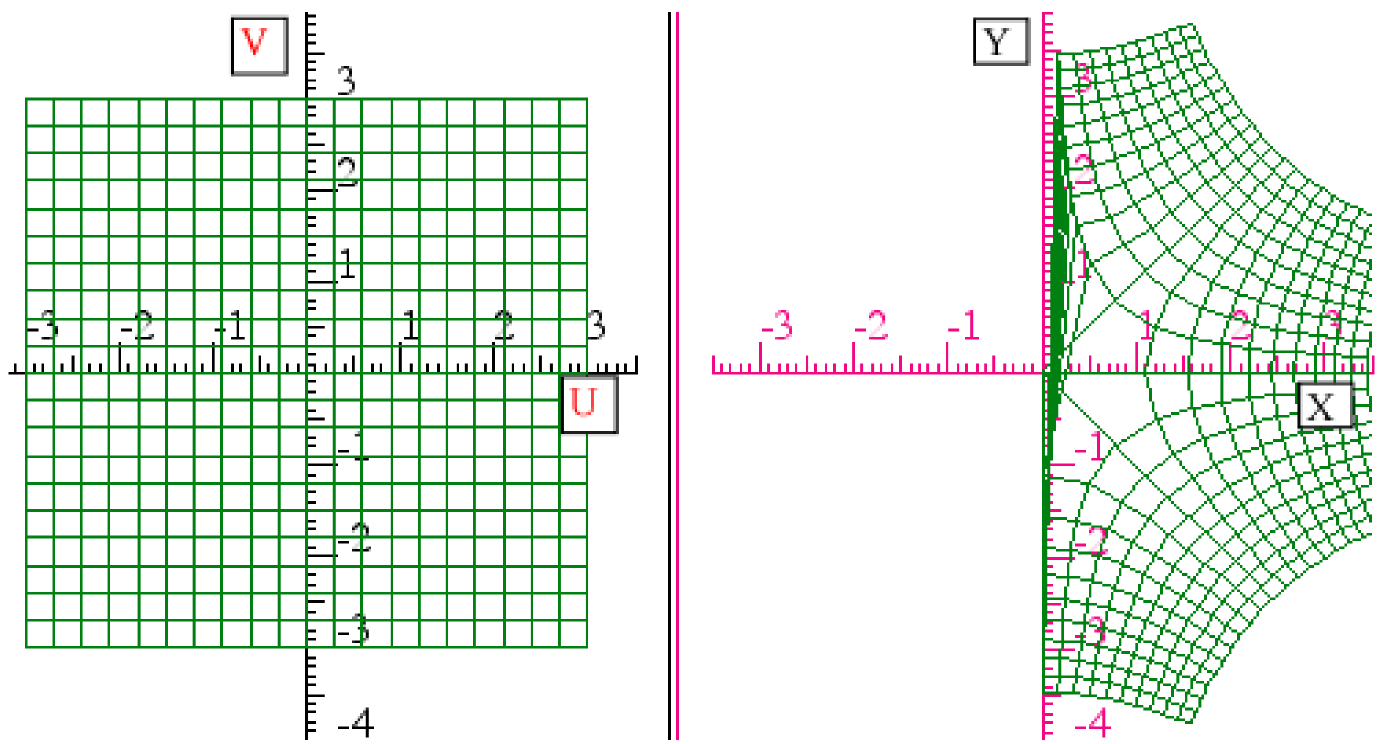


$w(z)=z^2$ gives parabolic OCC





Inverse: $z(w)=w^{1/2}$ gives hyperbolic OCC



$w=(u+iv)=z^2=(x+iy)^2$ is analytic function of z and w

Expansion: $u=x^2-y^2$ and $v=2xy$ may be solved using $|w|=|z^2|=|z|^2$

Expansion: $|w|=\sqrt{u^2+v^2}=x^2+y^2=|z|^2$

Solution: $x^2=\dfrac{u+\sqrt{u^2+v^2}}{2}$ $\quad y^2=\dfrac{-u+\sqrt{u^2+v^2}}{2}$

$$\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}=\begin{pmatrix}\bar{\mathbf{E}}^u\\ \bar{\mathbf{E}}^v\end{pmatrix}=\begin{pmatrix}2x & -2y\\ +2y & 2x\end{pmatrix}$$

$$\begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{pmatrix}=\begin{pmatrix}\bar{\mathbf{E}}_u & \bar{\mathbf{E}}_v\end{pmatrix}=\frac{\begin{pmatrix}2x & +2y\\ -2y & 2x\end{pmatrix}}{4\left(x^2+y^2\right)}$$

Non-analytic potential, force, and source field functions

A general 2D complex field may have:

1. non-analytic *potential field function* $\phi(z,z^*)=\Phi(x,y)+iA(x,y)$,
2. non-analytic *force field function* $f(z,z^*) = fx(x,y) + ify(x,y)$,
3. non-analytic *source distribution function* $s(z,z^*) = \rho(x,y) + i\, I(x,y)$.

Source definitions are made to generalize the **f*** field equations (10.33) based on relations (10.31) and (10.32).

$$2\frac{df^*}{dz} = s^*(z,z^*) \qquad\qquad 2\frac{df}{dz^*} = s(z,z^*)$$

Field equations for the potentials are like (10.33) with an extra factor of *2*.

$$2\frac{d\phi}{dz} = f(z,z^*) \qquad\qquad 2\frac{d\phi^*}{dz^*} = f^*(z,z^*)$$

Source equations (10.46) expand like (10.32) into a real and imaginary parts of divergence and curl terms.

$$s^*(z,z^*) = 2\frac{df^*}{dz} = \left[\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right]\left[f_x^*(x,y) + if_y^*(x,y)\right] \quad = \rho - i\, I\ , \quad \text{where: } f_x^* = f_x\ , \ \text{ and: } f_y^* = -f_y$$

$$= \left[\frac{\partial f_x^*}{\partial x} + \frac{\partial f_y^*}{\partial y}\right] + i\left[\frac{\partial f_y^*}{\partial x} - \frac{\partial f_x^*}{\partial y}\right] \quad = \left[\nabla \bullet \mathbf{f}^*\right] + i\left[\nabla \times \mathbf{f}^*\right]_Z$$

Real part: *Poisson scalar source equation* (*charge density* ρ). Imaginary part: *Biot-Savart vector source equation*(*current density I*)

$$\nabla \bullet \mathbf{f}^* = \rho \qquad\qquad \nabla \times \mathbf{f}^* = -I$$

Field equations (10.47) expand into Re and Im parts; *x* and *y* components of grad Φ and curlA_Z from potential $\phi = \Phi + iA$ or $\phi^* = \Phi - iA$.

$$f^*(z,z^*) = 2\frac{d\phi^*}{dz^*} = \left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right](\Phi - iA) = f_x^* + if_y^*$$

$$= \left[\frac{\partial \Phi}{\partial x} + i\frac{\partial \Phi}{\partial y}\right] + \left[\frac{\partial A}{\partial y} - i\frac{\partial A}{\partial x}\right] = \left[\nabla\Phi\right] + \left[\nabla \times \mathbf{A}_Z\right]$$

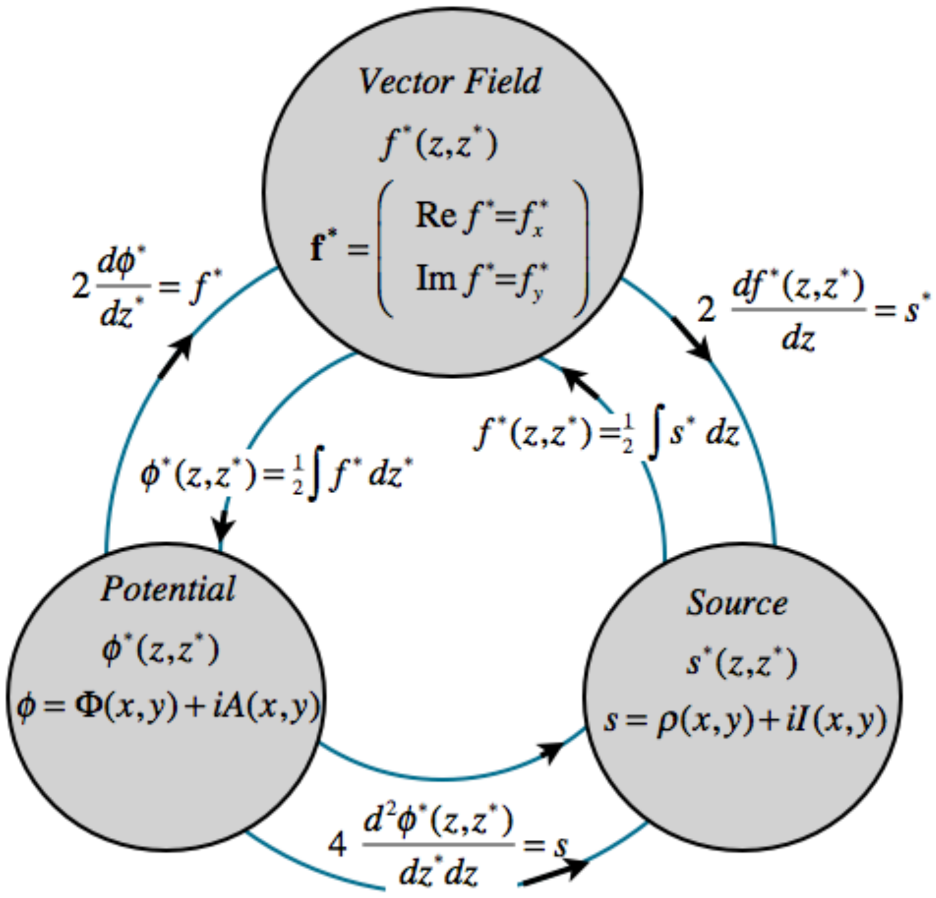
Two parts: gradient of scalar potential called the *longitudinal field* $\mathbf{f}_\mathbf{L}^*$ and curl of a vector potential called the *transverse field* $\mathbf{f}_\mathbf{T}^*$.

$$\mathbf{f}^* = \mathbf{f}_\mathbf{L}^* + \mathbf{f}_\mathbf{T}^* \qquad\qquad \mathbf{f}_\mathbf{L}^* = \nabla\Phi \qquad\qquad \mathbf{f}_\mathbf{T}^* = \nabla \times \mathbf{A}$$

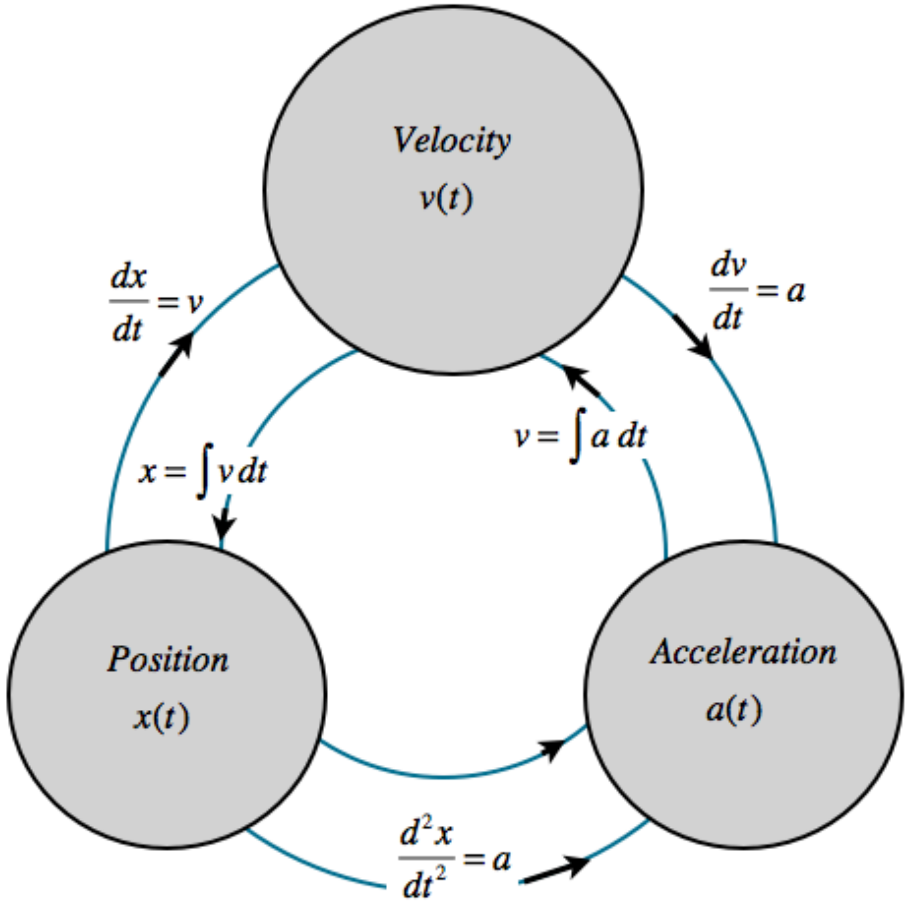
(For source-free analytic functions these two fields are identical.)

Potential, force, and source field equations vs. *position, velocity, and acceleration equations*

Field equations



Newton equations



Example 1

Consider a non-analytic field $f(z) = (z^*)^2$ or $f^*(z) = z^2$.

The non-analytic potential function follows by integrating

$$s^*(z,z^*) = 2\frac{df^*}{dz} = 4z = 4x + i4y,$$
$$or: \quad \rho = 4x, \quad and: \quad I = -4y.$$

$$\phi(z,z^*) = \frac{1}{2}\int f(z)\,dz = \frac{1}{2}\int (z^*)^2\,dz = \frac{z(z^*)^2}{2} = \frac{(x+iy)(x^2-y^2-i2xy)}{2},$$
$$or: \quad \Phi = \frac{x^3+xy^2}{2}, \quad and: \ A = \frac{-y^3-yx^2}{2}.$$

The longitudinal field $\mathbf{f}_T^*$ is quite different from the transverse field $\mathbf{f}_L^*$.

$$\mathbf{f}_L^*+\mathbf{f}_T^*$$

$$\mathbf{f}_L^* = \nabla\Phi = \nabla\left(\frac{x^3+xy^2}{2}\right) = \begin{pmatrix} \frac{3x^2+y^2}{2} \\ xy \end{pmatrix}, \quad \mathbf{f}_T^* = \nabla\times\mathbf{A} = \nabla\times\left(\frac{-y^3-yx^2}{2}\mathbf{e}_z\right) = \begin{pmatrix} \frac{\partial A}{\partial y} \\ -\frac{\partial A}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{-3y^2-x^2}{2} \\ xy \end{pmatrix}.$$

The longitudinal field $\mathbf{f}_L^*$ has no curl and the transverse field $\mathbf{f}_T^*$ has no divergence. The sum field has both making a violent storm, indeed, as shown by a plot of in Fig. 10.17.

$$\mathbf{f}^* = \mathbf{f}_L^*+\mathbf{f}_T^* = \begin{pmatrix} \frac{3x^2+y^2}{2} \\ xy \end{pmatrix} + \begin{pmatrix} \frac{-3y^2-x^2}{2} \\ xy \end{pmatrix} = \begin{pmatrix} x^2-y^2 \\ 2xy \end{pmatrix}, \quad \nabla\bullet\mathbf{f}^* = \nabla\bullet\mathbf{f}_L^* = 4x = \rho, \quad \nabla\times\mathbf{f}^* = \nabla\times\mathbf{f}_T^* = 4y = -I.$$