Lecture 13

Revised 12.22.12 from 10.4.2012

Poincare, Lagrange, Hamiltonian, and Jacobi mechanics

(Unit 1 Ch. 12, Unit 2 Ch. 2-7, Unit 3 Ch. 1-3)

Review of Lecture 12 relations:

Examples of Hamiltonian mechanics in phase plots

1D Pendulum and phase plot (Simulation)

1D-HO phase-space control (Simulation of "Catcher in the Eye")

Exploring phase space and Lagrangian mechanics more deeply

A weird "derivation" of Lagrange's equations

Poincare identity and Action, Jacobi-Hamilton *equations*

How Classicists might have "derived" quantum equations

Huygen's contact transformations enforce minimum action

How to do quantum mechanics if you only know classical mechanics

Examples of Hamiltonian mechanics in phase plots

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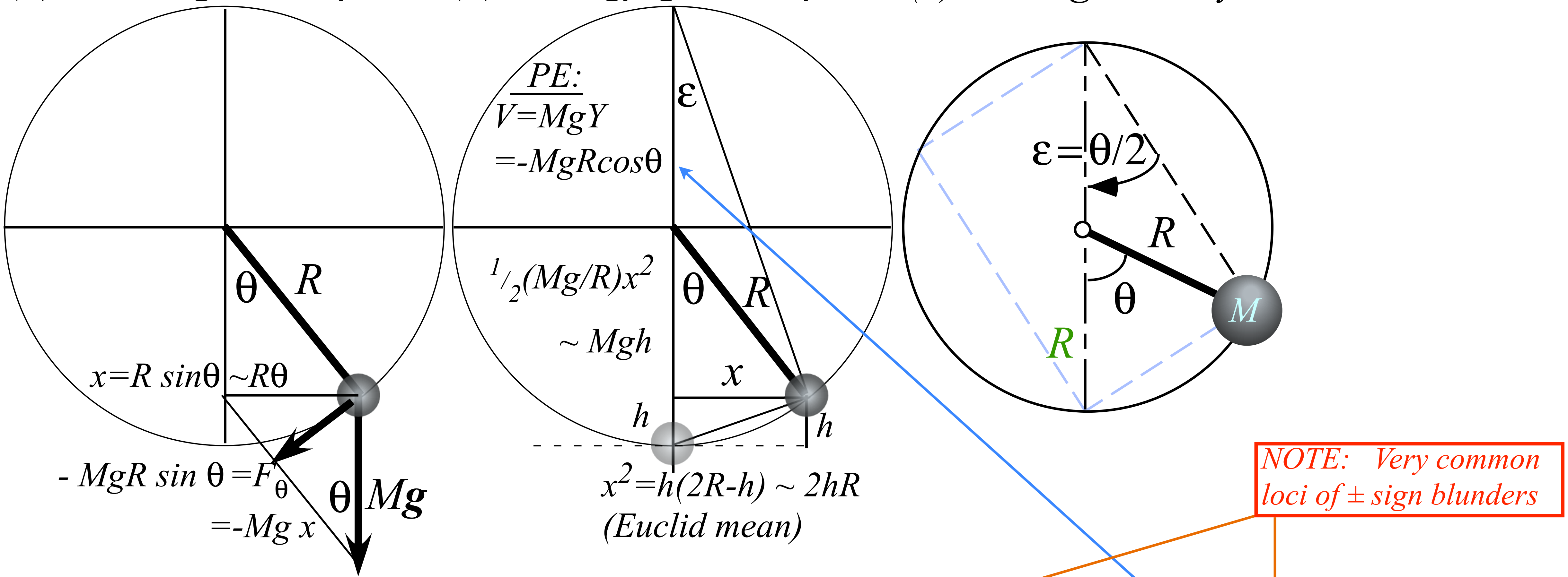
1D Pendulum and phase plot

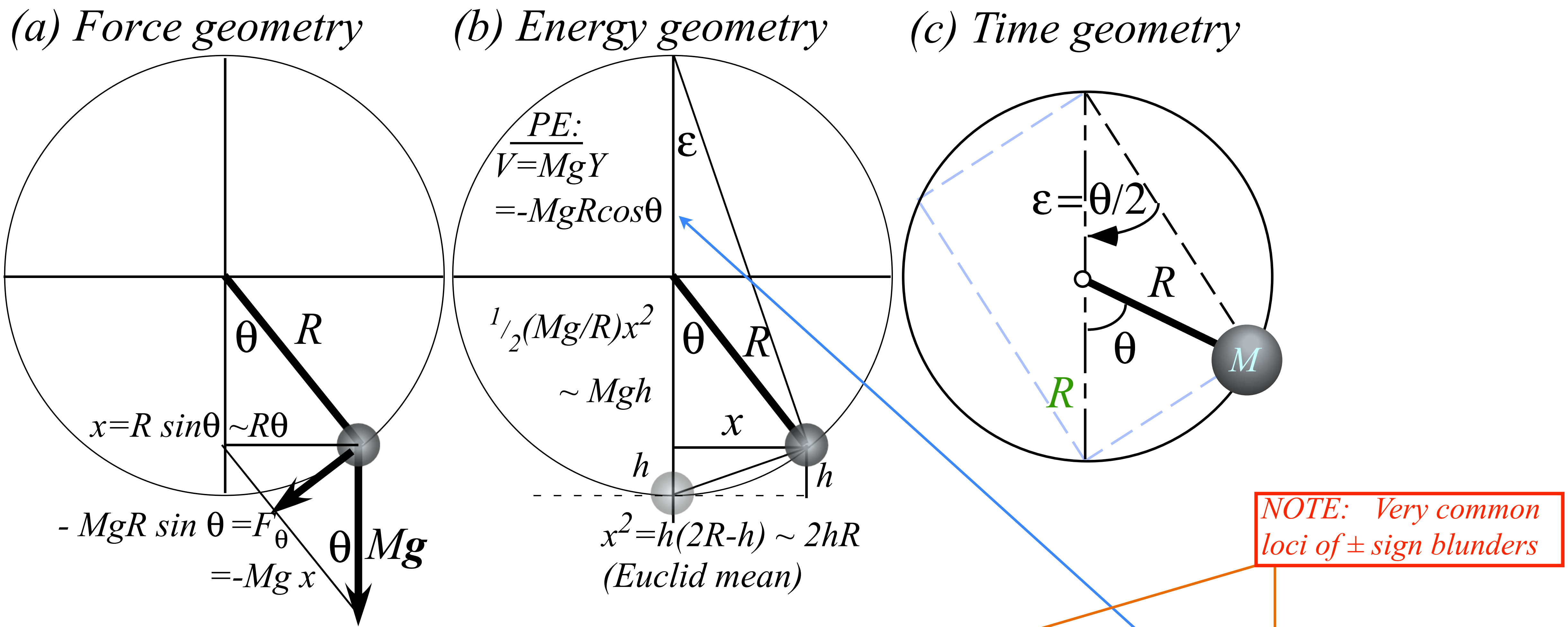
(a) Force geometry *(b) Energy geometry* *(c) Time geometry*



Lagrangian function L= KE - PE = T - U where potential energy is U(θ) = $-MgR\cos\theta$

$$L(\dot{\theta},\theta)=\frac{1}{2}I\dot{\theta}^2-U(\theta)=\frac{1}{2}I\dot{\theta}^2+MgR\cos\theta$$

1D Pendulum and phase plot



Lagrangian function L= KE - PE = T - U where potential energy is U(θ) = $-MgR\cos\theta$

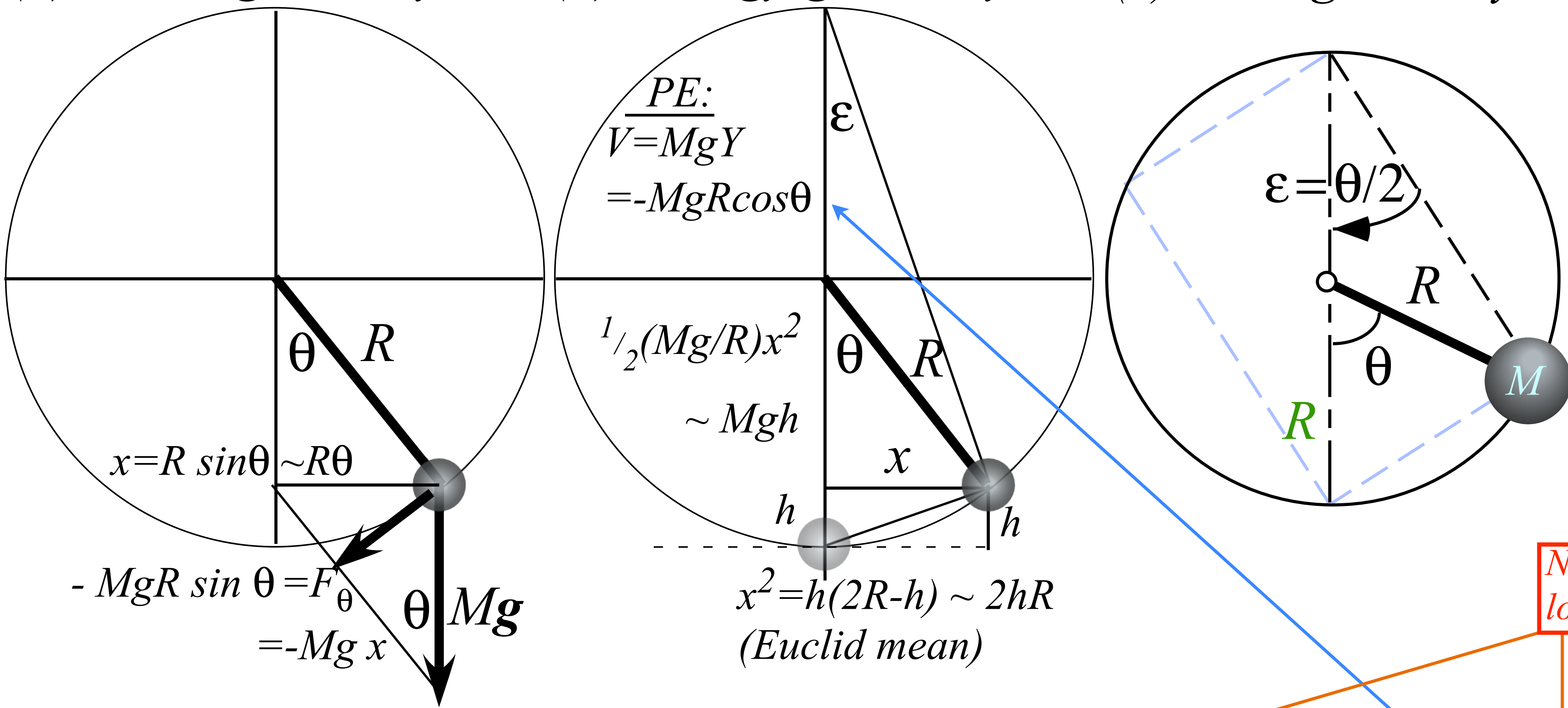
$$L(\dot\theta,\theta) = \frac{1}{2} I\dot\theta^2 - U(\theta) = \frac{1}{2} I\dot\theta^2 + MgR\cos\theta$$

Hamiltonian function H= KE + PE = T +U where potential energy is U(θ) = $-MgR\cos\theta$

$$H(p_\theta,\theta) = \frac{1}{2I} p_\theta{}^2 + U(\theta) = \frac{1}{2I} p_\theta{}^2 - MgR\cos\theta \;=E \quad =const.$$

1D Pendulum and phase plot

(a) Force geometry *(b) Energy geometry* *(c) Time geometry*



NOTE: Very common loci of ± sign blunders

Lagrangian function L= KE - PE = T - U where potential energy is U(θ) = $-MgR\cos\theta$

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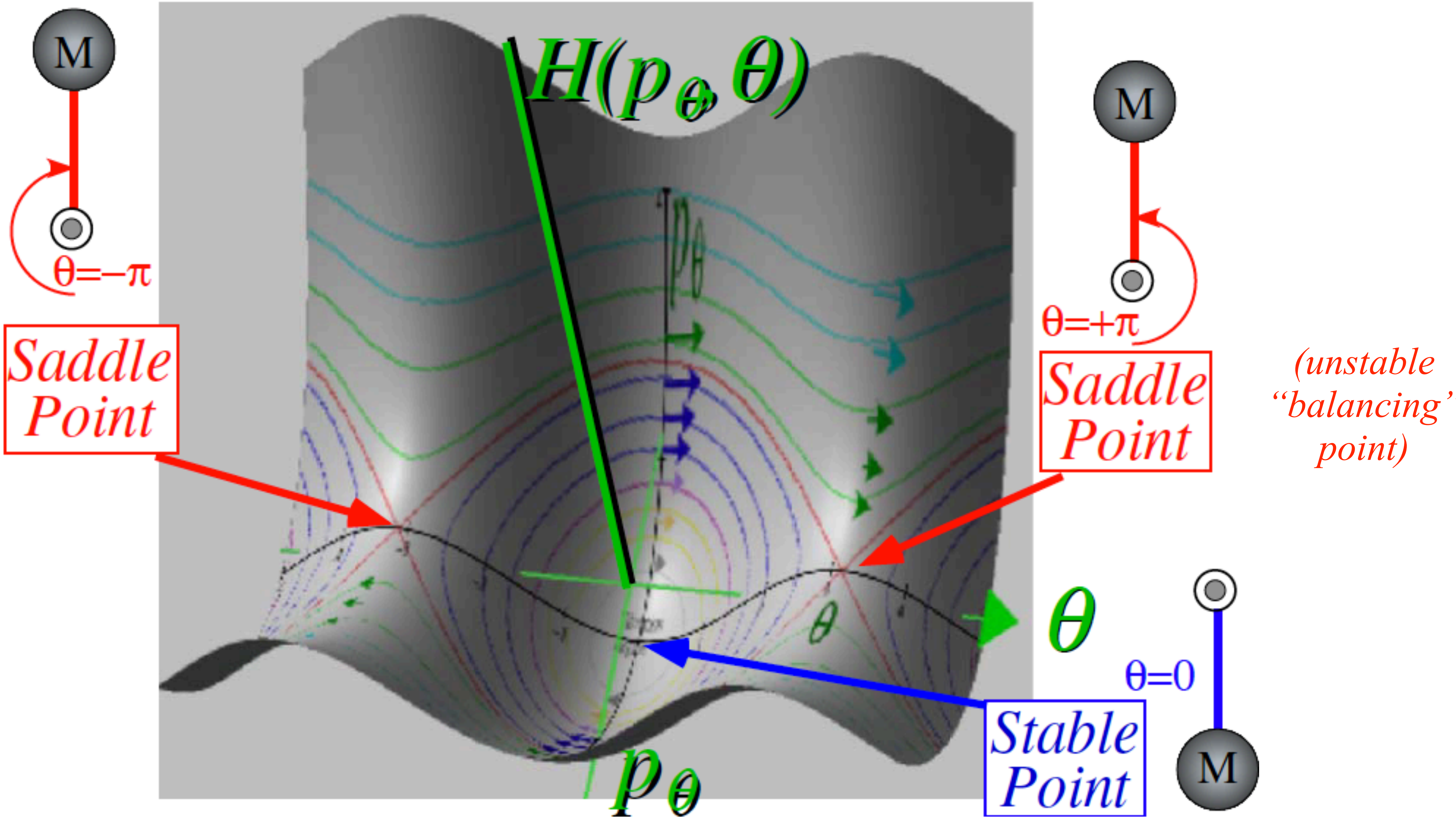
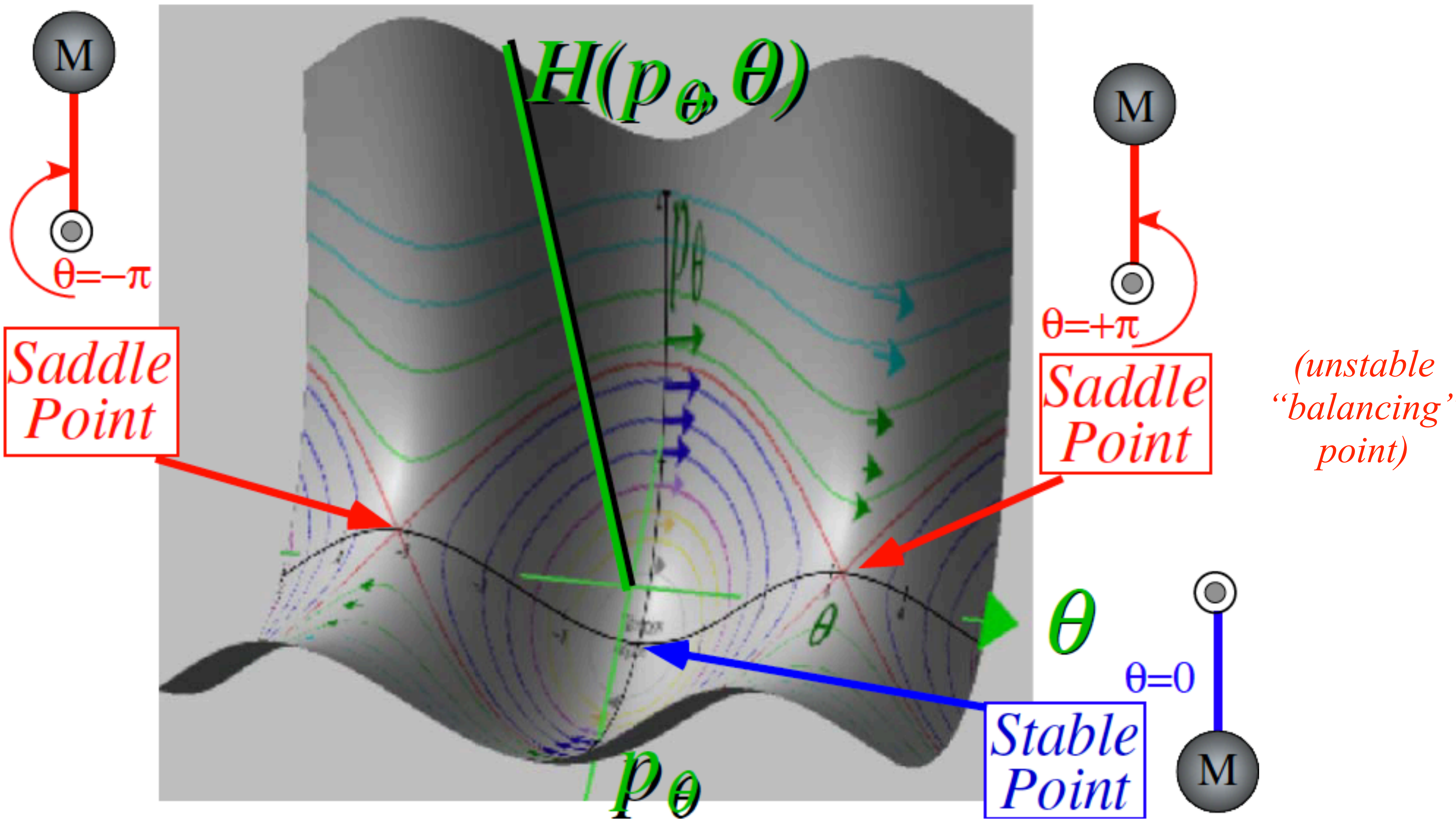
implies: $p_\theta=\sqrt{2I\left(E+MgR\cos\theta\right)}$



Example of plot of Hamilton for 1D-solid pendulum in its Phase Space (θ,pθ)

$$H(p_\theta,\theta) = E = \frac{1}{2I} p_\theta^{\,2} - MgR\cos\theta\ , \quad \text{or:} \quad p_\theta = \sqrt{2I\left(E + MgR\cos\theta\right)}$$



Example of plot of Hamilton for 1D-solid pendulum in its Phase Space (θ,p_θ)

$$H(p_\theta,\theta)=E=\frac{1}{2I}p_\theta^{\;2}-MgR\cos\theta\ ,\quad \text{or:}\quad p_\theta=\sqrt{2I\left(E+MgR\cos\theta\right)}$$

Funny way to look at Hamilton's equations:

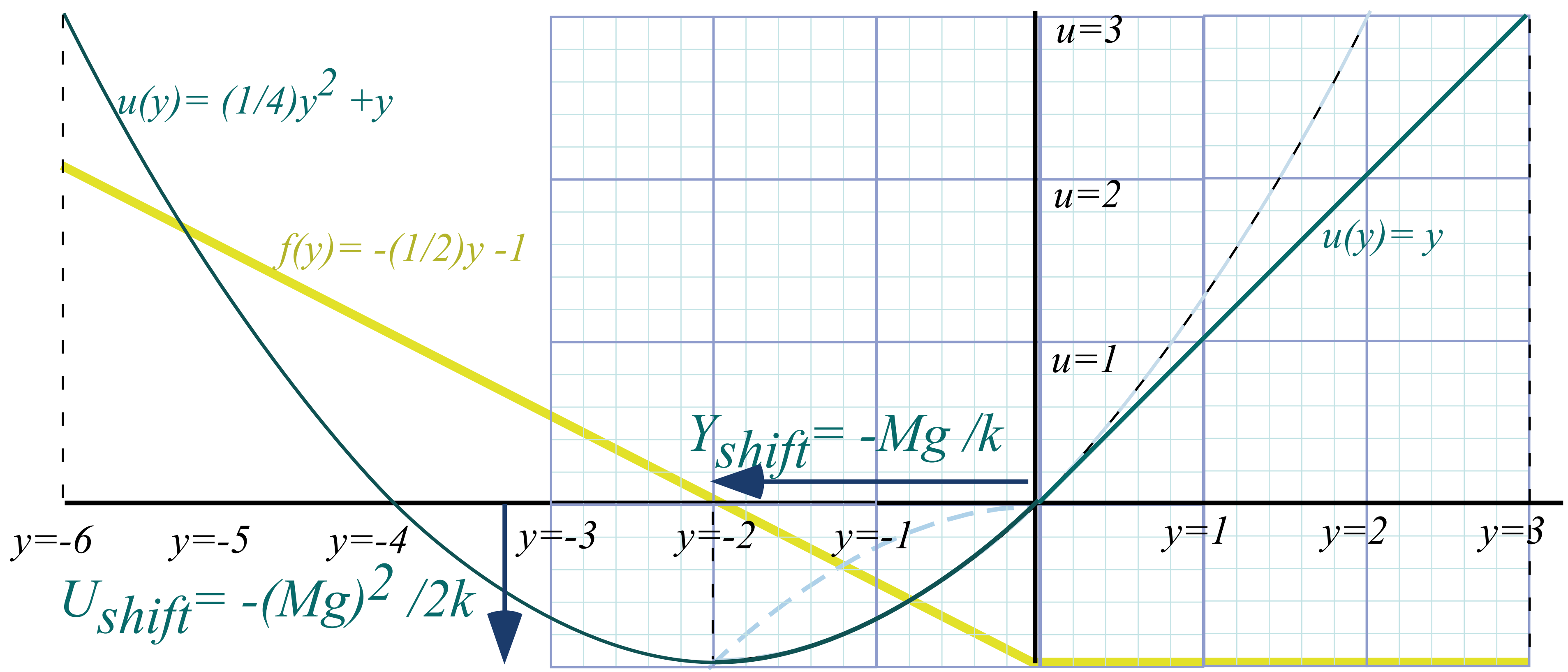
$$\begin{pmatrix}\dot{q}\\ \dot{p}\end{pmatrix}=\begin{pmatrix}\partial_p H\\ -\partial_q H\end{pmatrix}=\mathbf{e_H}\times\left(-\nabla H\right)=(\text{H-axis})\times(\text{fall line}),\ \text{where:}\begin{cases}(\text{H-axis})=\mathbf{e_H}=\mathbf{e_q}\times\mathbf{e_p}\\ (\text{fall line})=-\nabla H\end{cases}$$

Examples of Hamiltonian dynamics and phase plots

1D Pendulum and phase plot (Simulation)

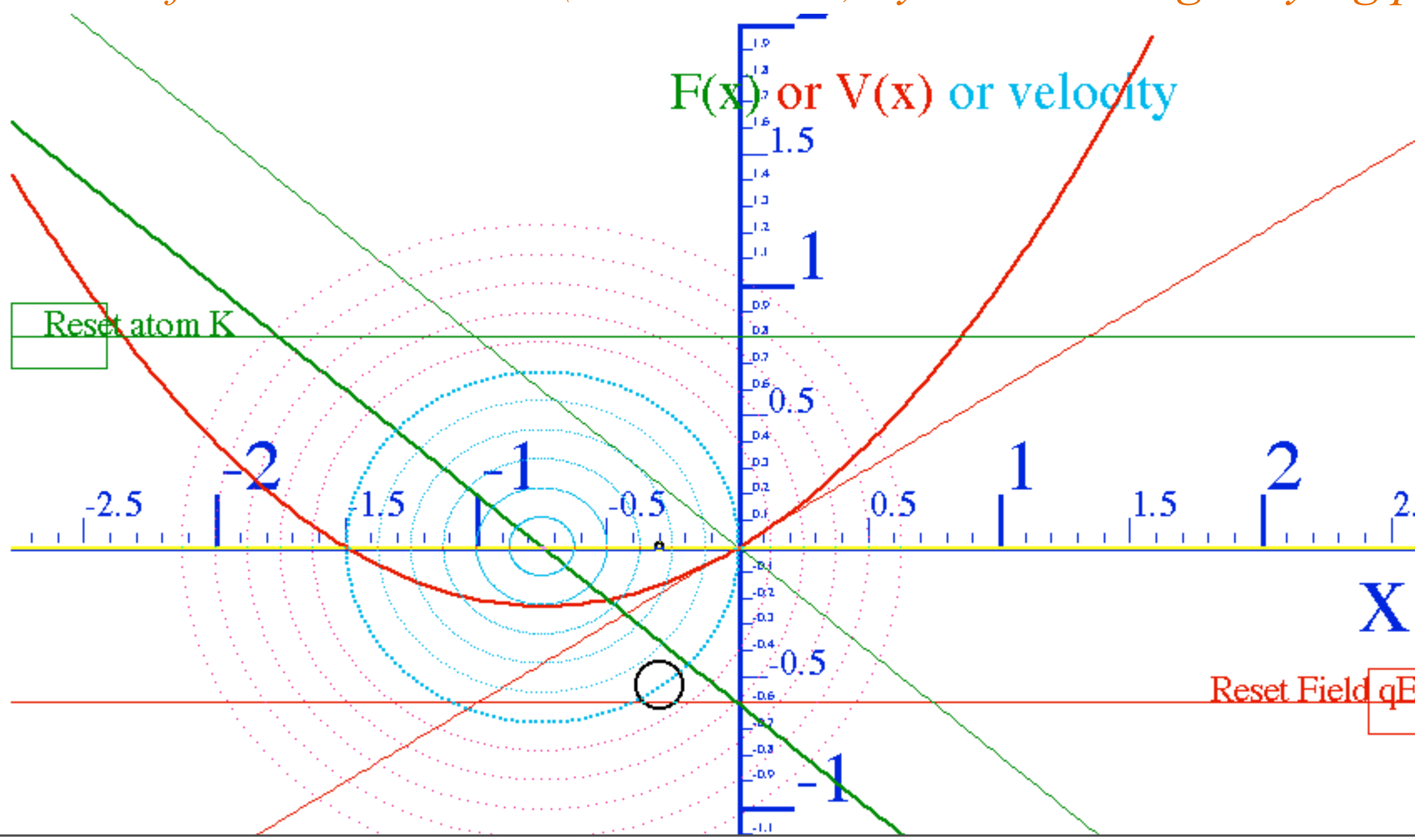
→ ***Phase control*** *(Simulation of "Catcher in the Eye")*

$F(Y)=-kY-Mg$ $\qquad$ $U(Y)=(1/2)kY^2+Mg\,Y$



Unit 1
Fig. 7.4

Simulation of atomic classical (or semi-classical) dynamics using varying phase control



Exploring phase space and Lagrangian mechanics more deeply

→ ***A weird "derivation" of Lagrange's equations***

Poincare identity and Action, Jacobi-Hamilton *equations*

How Classicists might have "derived" quantum equations

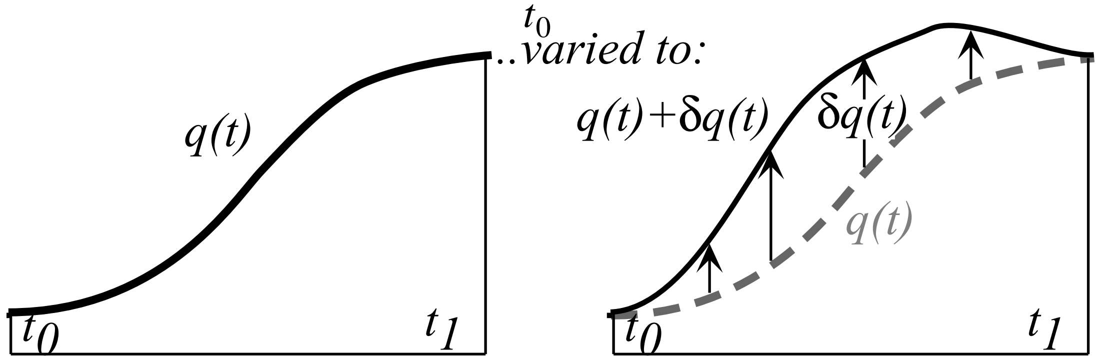
Huygen's contact transformations enforce minimum action

How to do quantum mechanics if you only know classical mechanics

A strange "derivation" of Lagrange's equations by Calculus of Variation

Variational calculus finds extreme (minimum or maximum) values to entire integrals

Minimize (or maximize): $S(q) = \int_{t_0}^{t_1} dt\ L\big(q(t),\dot{q}(t),t\big).$



An arbitrary but small variation function $\delta q(t)$ is allowed at every point t in the figure along the curve except at the end points t_0 and t_1. There we demand it not vary at all.(1)

$$\delta q(t_0) = 0 = \delta q(t_1) \qquad (1)$$

1st order L(q+δq) approximate:

$$S(q+\delta q) = \int_{t_0}^{t_1} dt \left[L(q,\dot{q},t) + \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \right] \text{ where: } \delta \dot{q} = \frac{d}{dt}\delta q$$

A weird "derivation" of Lagrange's equations

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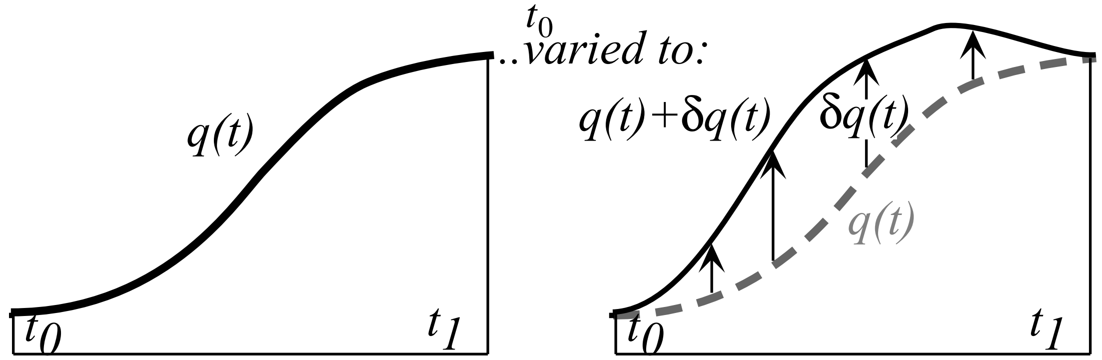

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Replace $\frac{\partial L}{\partial \dot{q}}\delta\dot{q}$ with $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right)-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q$

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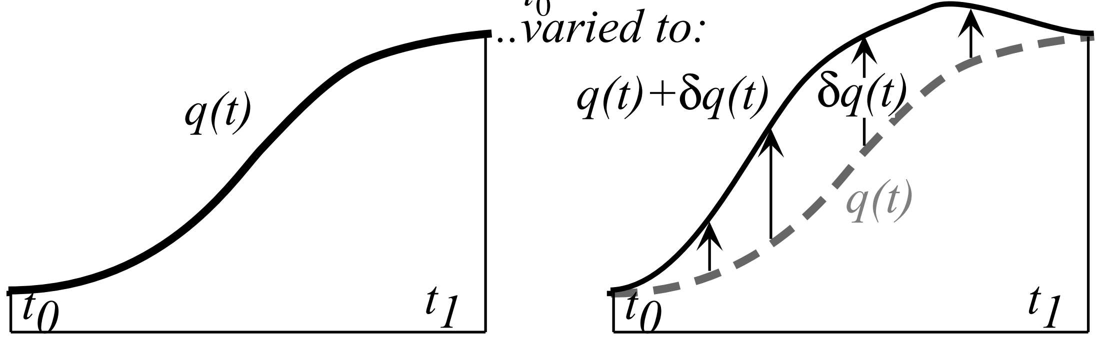

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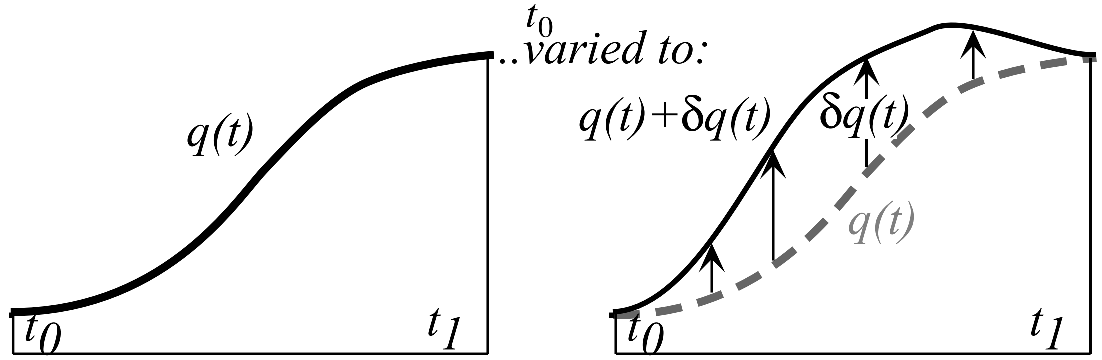

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$$=\int_{t_0}^{t_1} dt\ L(q,\dot{q},t)+\int_{t_0}^{t_1} dt\left[\frac{\partial L}{\partial q}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right]\delta q+\left(\frac{\partial L}{\partial \dot{q}}\delta q\right)\Bigg|_{t_0}^{t_1}$$

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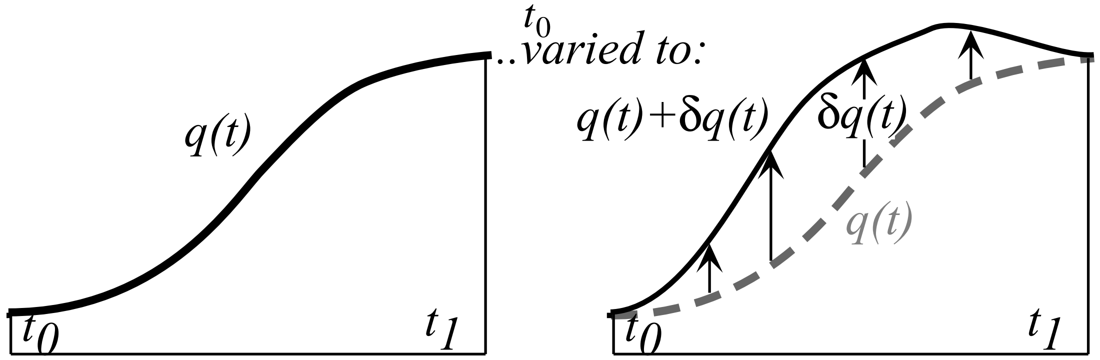
$$S(q) = \int_{t_0}^{t_1} dt\ L(q(t), \dot{q}(t), t).$$



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$$= \int_{t_0}^{t_1} dt\ L(q,\dot{q},t) + \int_{t_0}^{t_1} dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \right]\delta q + \left(\frac{\partial L}{\partial q}\delta q\right)\Big|_{t_0}^{t_1}$$

Third term vanishes by (1). This leaves first order variation: $\delta S = S(q+\delta q) - S(q) = \int_{t_0}^{t_1} dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \right]\delta q$

Extreme value (actually *minimum* value) of *S(q)* occurs *if and only if* Lagrange equation is satisfied!

$$\delta S = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

Euler-Lagrange equation(s)

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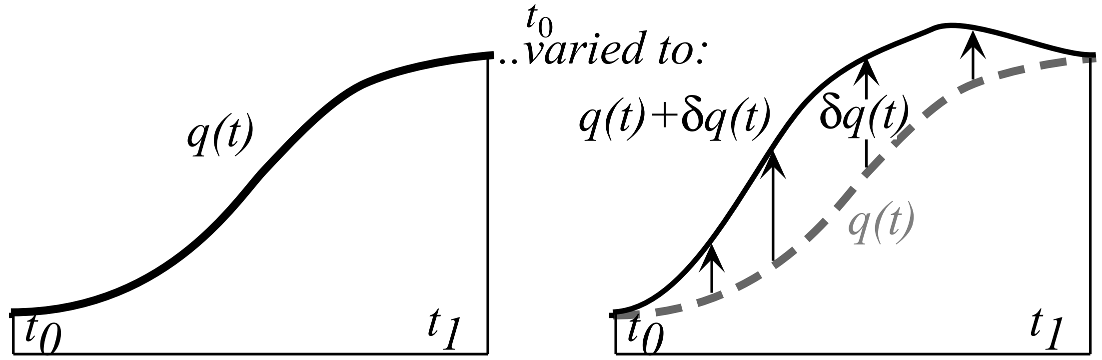

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$$\delta S=0\Rightarrow\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}=0$$

Euler-Lagrange equation(s)

But, WHY is nature so inclined to fly JUST SO as to minimize the Lagrangian L = T - U???

Exploring phase space and Lagrangian mechanics more deeply

A weird "derivation" of Lagrange's equations

→ ***Poincare identity and Action,*** Jacobi-Hamilton *equations*

How Classicists might have "derived" quantum equations

Huygen's contact transformations enforce minimum action

How to do quantum mechanics if you only know classical mechanics

Legendre-Poincare identity and Action

Legendre transform $L(\mathbf{v}) = \mathbf{p} \bullet \mathbf{v} - H(\mathbf{p})$ becomes *Poincare's invariant differential* if dt is cleared.

$$L \cdot dt = \mathbf{p} \bullet \mathbf{v} \cdot dt - H \cdot dt = \mathbf{p} \bullet d\mathbf{r} - H \cdot dt \qquad \left(\mathbf{v} = \frac{d\mathbf{r}}{dt} \textit{ implies: } \mathbf{v} \cdot dt = d\mathbf{r} \right)$$

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This is the time differential dS of *action* $S = \int L \cdot dt$ whose time derivative is rate L of *quantum phase*.

$$dS = L\cdot dt = \mathbf{p}\bullet d\mathbf{r} - H\cdot dt \quad \text{where:} \quad L = \frac{dS}{dt}$$

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Unit 8 shows *DeBroglie law* $\mathbf{p} = \hbar \mathbf{k}$ and *Planck law* $H = \hbar \omega$ make *quantum plane wave phase* Φ:

$$\Phi = S/\hbar = \int L \cdot dt / \hbar$$

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$$\Phi = S/\hbar = \int L\cdot dt/\hbar$$

$$\psi(\mathbf{r},t) = e^{iS/\hbar} = e^{i(\mathbf{p}\bullet\mathbf{r}-H\cdot t)/\hbar} = e^{i(\mathbf{k}\bullet\mathbf{r}-\omega\cdot t)}$$

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Q:When is the *Action*-differential dS integrable?

A: A differential $dW=f_x(x,y)dx+f_y(x,y)dy$ is *integrable* to a $W(x,y)$ if: $f_x = \frac{\partial W}{\partial x}$ and: $f_y = \frac{\partial W}{\partial y}$

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Similar to conditions for integrating work differential $dW=\mathbf{f}\bullet d\mathbf{r}$ *to get potential* $W(\mathbf{r})$. *That condition is* no curl allowed: $\nabla\times\mathbf{f}=\mathbf{0}$ *or* ∂-symmetry *of W:*

$$\frac{\partial f_x}{\partial y} = \frac{\partial^2 W}{\partial y\partial x} = \frac{\partial^2 W}{\partial x\partial y} = \frac{\partial f_y}{\partial x}$$

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dS is integrable if: $\frac{\partial S}{\partial \mathbf{r}} = \mathbf{p}$ and: $\frac{\partial S}{\partial t} = -H$

Similar to conditions for integrating work differential $dW=\mathbf{f}\bullet d\mathbf{r}$ *to get potential* $W(\mathbf{r})$*. That condition is* no curl allowed: $\nabla\times\mathbf{f}=\mathbf{0}$ *or* ∂*-symmetry of* W:

$$\frac{\partial f_x}{\partial y} = \frac{\partial^2 W}{\partial y \partial x} = \frac{\partial^2 W}{\partial x \partial y} = \frac{\partial f_y}{\partial x}$$

These conditions are known as Jacobi-Hamilton *equations*

Exploring phase space and Lagrangian mechanics more deeply

A weird "derivation" of Lagrange's equations

Poincare identity and Action, Jacobi-Hamilton *equations*

→ ***How Classicists might have "derived" quantum equations***

Huygen's contact transformations enforce minimum action

How to do quantum mechanics if you only know classical mechanics

How Jacobi-Hamilton *could have "derived" Schrodinger equations*

(Given "quantum wave")

$$\psi(\mathbf{r},t) = e^{iS/\hbar} = e^{i(\mathbf{p}\bullet\mathbf{r} - H\cdot t)/\hbar} = e^{i(\mathbf{k}\bullet\mathbf{r} - \omega\cdot t)}$$

dS is integrable if: $\frac{\partial S}{\partial \mathbf{r}} = \mathbf{p}$ and: $\frac{\partial S}{\partial t} = -H$

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How Jacobi-Hamilton *could have "derived" Schrodinger equations*

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These conditions are known as Jacobi-Hamilton *equations*

Try 1st **r**-*derivative of wave* ψ

$$\frac{\partial}{\partial \mathbf{r}}\psi(\mathbf{r},t) = \frac{\partial}{\partial \mathbf{r}} e^{iS/\hbar} = \frac{\partial (iS/\hbar)}{\partial \mathbf{r}} e^{iS/\hbar} = (i/\hbar)\frac{\partial S}{\partial \mathbf{r}}\psi(\mathbf{r},t)$$

$$\frac{\partial}{\partial \mathbf{r}}\psi(\mathbf{r},t) = (i/\hbar)\mathbf{p}\psi(\mathbf{r},t) \quad \text{or:} \quad \boxed{\frac{\hbar}{i}\frac{\partial}{\partial \mathbf{r}}\psi(\mathbf{r},t) = \mathbf{p}\psi(\mathbf{r},t)}$$

How Jacobi-Hamilton *could have "derived" Schrodinger equations*

(Given "quantum wave")

$$\psi(\mathbf{r},t) = e^{iS/\hbar} = e^{i(\mathbf{p}\bullet\mathbf{r}-H\cdot t)/\hbar} = e^{i(\mathbf{k}\bullet\mathbf{r}-\omega\cdot t)}$$

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Try 1st **r**-*derivative of wave* ψ

$$\frac{\partial}{\partial \mathbf{r}}\psi(\mathbf{r},t) = \frac{\partial}{\partial \mathbf{r}}e^{iS/\hbar} = \frac{\partial(iS/\hbar)}{\partial \mathbf{r}}e^{iS/\hbar} = (i/\hbar)\frac{\partial S}{\partial \mathbf{r}}\psi(\mathbf{r},t)$$

$$\frac{\partial}{\partial \mathbf{r}}\psi(\mathbf{r},t) = (i/\hbar)\mathbf{p}\psi(\mathbf{r},t) \quad \text{or:} \quad \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial \mathbf{r}}\psi(\mathbf{r},t) = \mathbf{p}\psi(\mathbf{r},t)$$

Try 1st t-derivative of wave ψ

$$\frac{\partial}{\partial t}\psi(\mathbf{r},t) = \frac{\partial}{\partial t}e^{iS/\hbar} = \frac{\partial(iS/\hbar)}{\partial t}e^{iS/\hbar} = (i/\hbar)\frac{\partial S}{\partial t}\psi(\mathbf{r},t)$$

$$= (i/\hbar)(-H)\psi(\mathbf{r},t) \quad \text{or:} \quad \mathrm{i}\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = H\psi(\mathbf{r},t)$$

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→ ***Huygen's contact transformations enforce minimum action***

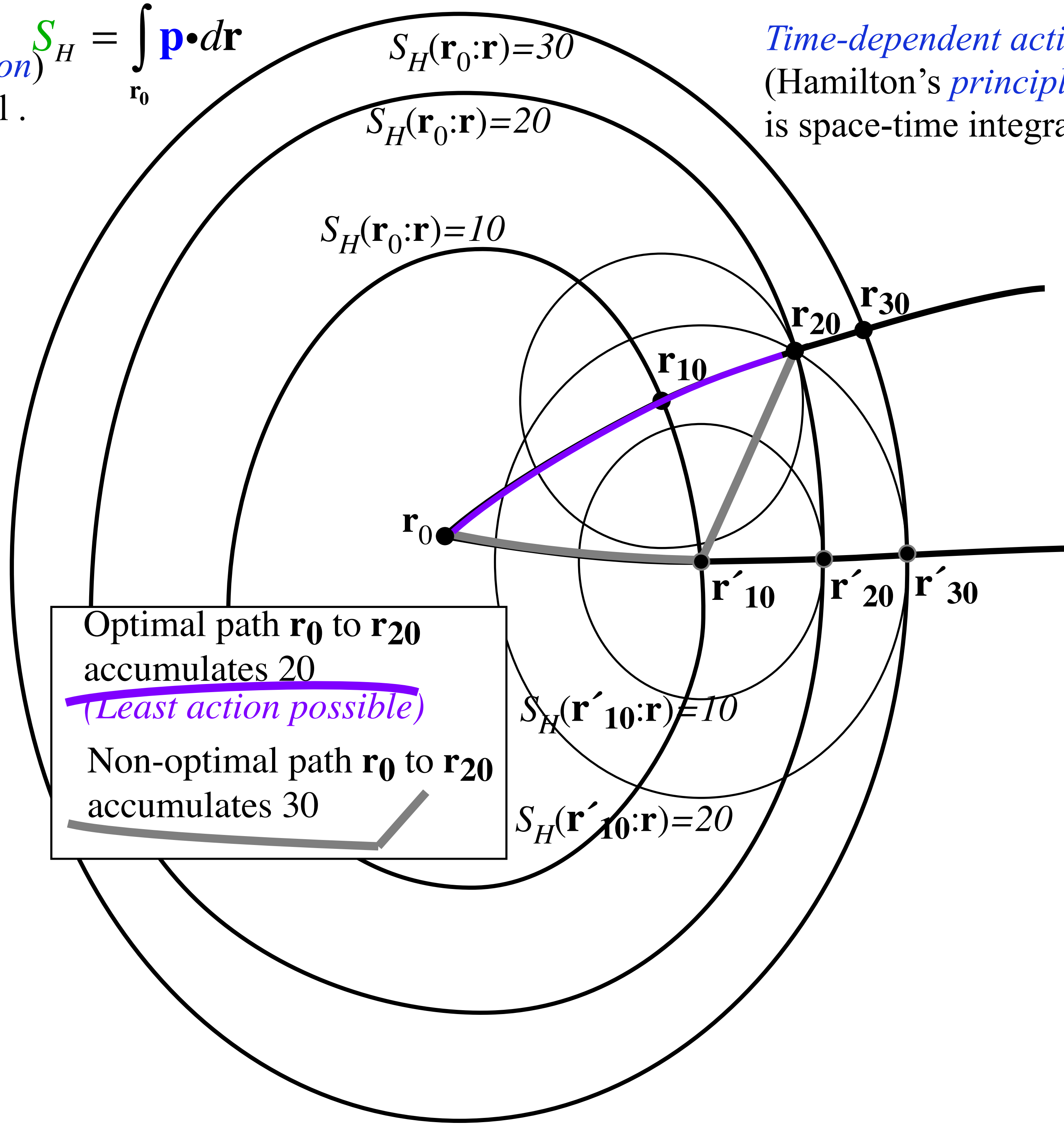
How to do quantum mechanics if you only know classical mechanics

Huygen's contact transformations enforce minimum action

Each point $\mathbf{r}_k$ on a wavefront "broadcasts" in all directions.
*Only **minimum action** path interferes constructively*

Time-independent action $S_H = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{p} \bullet d\mathbf{r}$
(Hamilton's *reduced action*)
is a purely spatial integral .

Time-dependent action $S_p = \int_{\mathbf{r}_0 t_0}^{\mathbf{r}_1 t_1} (\mathbf{p} \bullet d\mathbf{r} - H \cdot dt)$
(Hamilton's *principle action*)
is space-time integral .



Unit 1
Fig. 12.12

Huygen's contact transformations enforce minimum action

Each point $\mathbf{r}_k$ *on a wavefront "broadcasts" in all directions.*
Only ***minimum action*** *path interferes constructively*

Time-independent action $S_H = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{p}\bullet d\mathbf{r}$
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(Hamilton's *principle action*)
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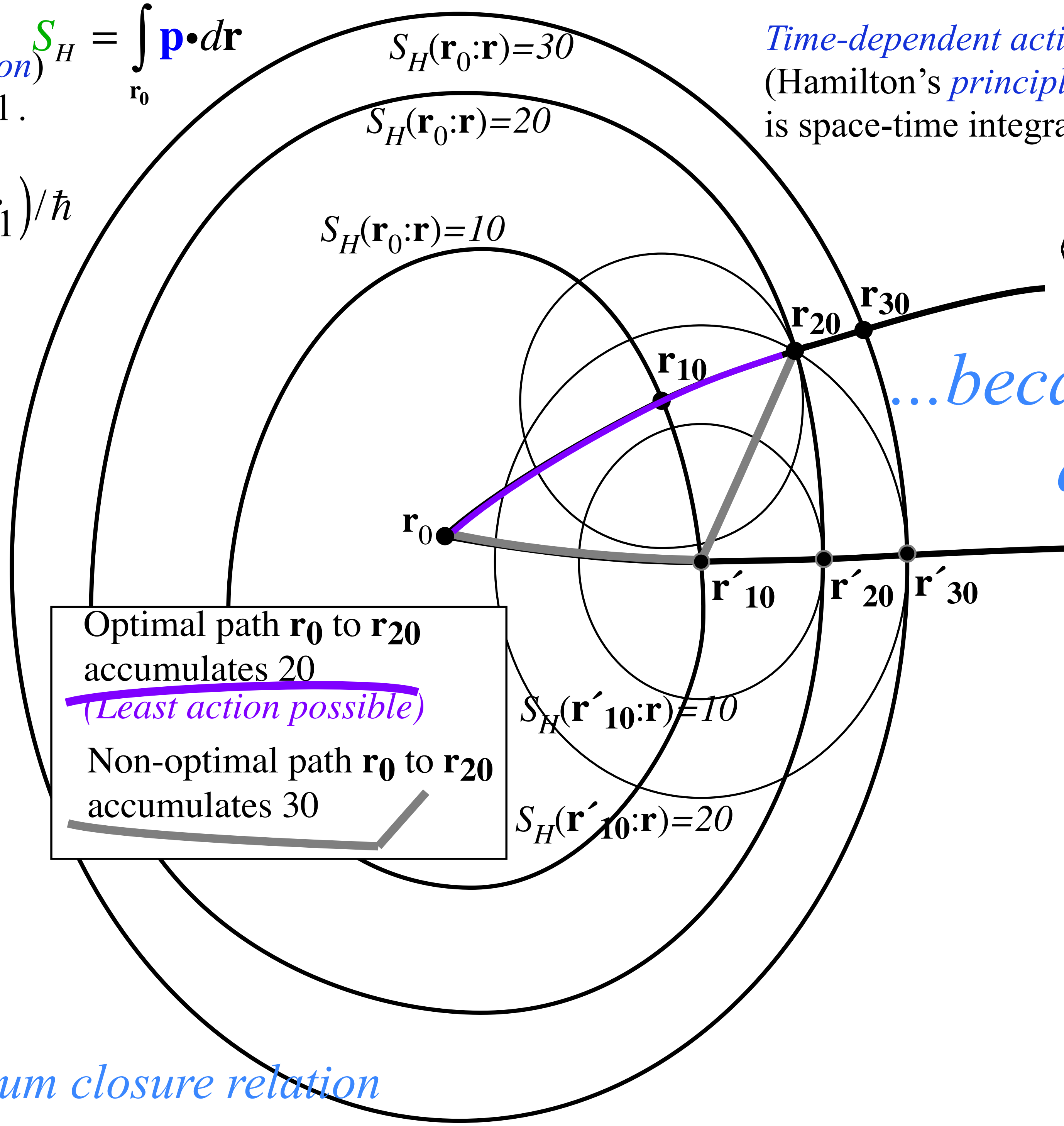
$$\langle \mathbf{r}_1 | \mathbf{r}_0 \rangle = e^{i S_H(\mathbf{r}_0 : \mathbf{r}_1)/\hbar}$$

$$\langle \mathbf{r}_1, t_1 | \mathbf{r}_0, t_0 \rangle = e^{i S(\mathbf{r}_0, t_0 : \mathbf{r}_1, t_1)/\hbar}$$



...because action *is*
quantum wave phase

Unit 1
Fig. 12.12

Feynman's path-sum closure relation

$$\sum_{\mathbf{r}'} \langle \mathbf{r}_1 | \mathbf{r}' \rangle \langle \mathbf{r}' | \mathbf{r}_0 \rangle \cong \sum_{\mathbf{r}'} e^{i\left(S_H(\mathbf{r}_0:\mathbf{r}') + S_H(\mathbf{r}':\mathbf{r}_1)\right)/\hbar} = e^{i S_H(\mathbf{r}_0:\mathbf{r}_1)/\hbar} = \langle \mathbf{r}_1 | \mathbf{r}_0 \rangle$$

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→ ***How to do quantum mechanics if you only know classical mechanics***

How to do quantum mechanics if you only know classical mechanics

Bohr quantization requires quantum phase $S_H/\hbar$ in amplitude to be an integral multiple n of 2π after a closed loop integral $S_H(\mathbf{r}_0:\mathbf{r}_0)=\int_{r_0}^{r_0}\mathbf{p}\bullet d\mathbf{r}$. The integer n (n = 0, 1, 2,...) is a *quantum number*.

$$1=\langle\mathbf{r}_0|\mathbf{r}_0\rangle=e^{iS_H(\mathbf{r}_0:\mathbf{r}_0)/\hbar}=e^{i\Sigma_H/\hbar}=1 \text{ for: } \Sigma_H=2\pi\hbar n=hn$$

Numerically integrate Hamilton's equations and Lagrangian L. Color the trajectory according to the current accumulated value of action $S_H(\mathbf{0}:\mathbf{r})/\hbar$. Adjust energy to quantized pattern (if closed system*)

$$S_H(\mathbf{0}:\mathbf{r})=S_p(\mathbf{0},0:\mathbf{r},t)+Ht=\int_0^t L\,dt+Ht\,.$$

How to do quantum mechanics if you only know classical mechanics

Bohr quantization requires quantum phase $S_H/\hbar$ in amplitude to be an integral multiple n of 2π after a closed loop integral $S_H(\mathbf{r}_0:\mathbf{r}_0)=\int_{r_0}^{r_0}\mathbf{p}\bullet d\mathbf{r}$. The integer n ($n = 0, 1, 2,...$) is a *quantum number*.

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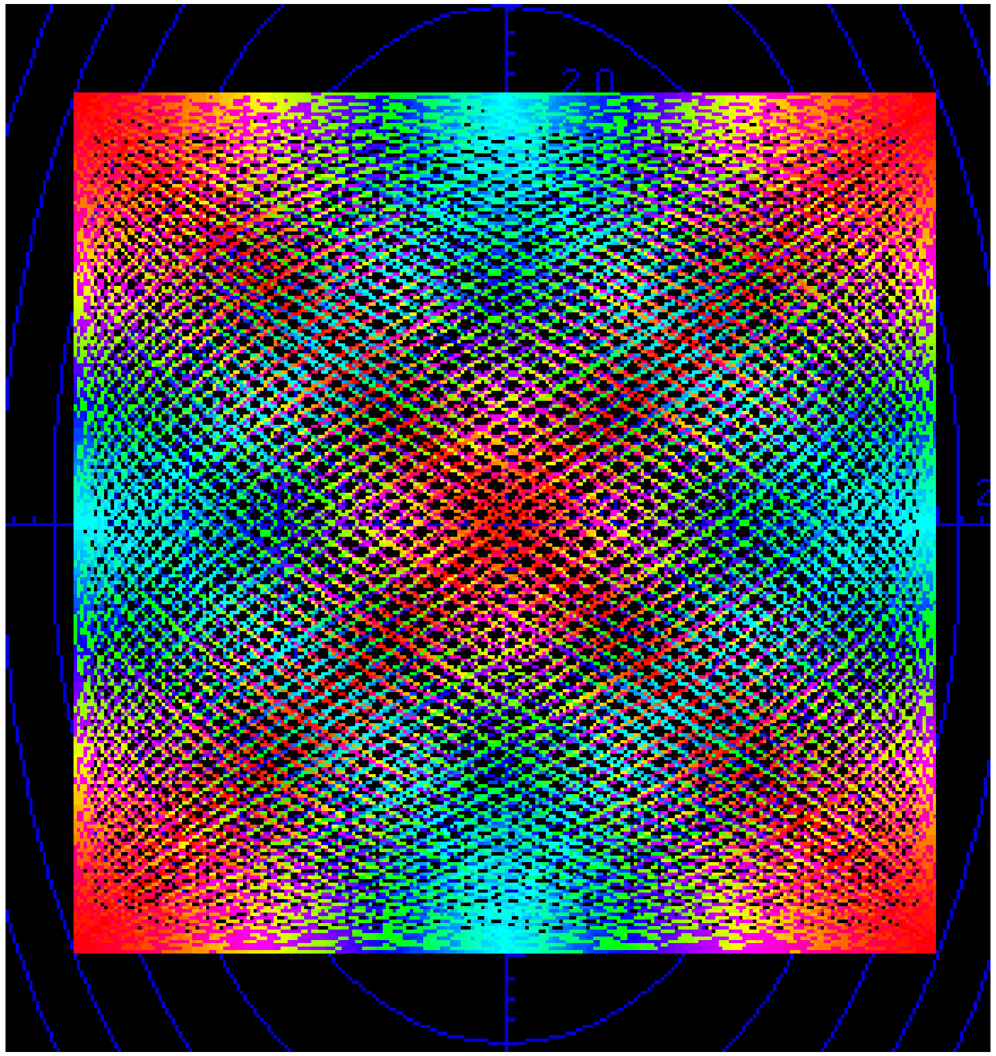
$$S_H(\mathbf{0}:\mathbf{r})=S_p(\mathbf{0},0:\mathbf{r},t)+Ht=\int_0^t L\,dt+Ht\,.$$

The hue should represent the phase angle $S_H(\mathbf{0}:\mathbf{r})/\hbar$ *modulo* 2π as, for example,
0=red, π/4=orange, π/2=yellow, 3π/4=green, π=cyan (opposite of red), *5π/4=indigo, 3π/2=blue, 7π/4=purple, and 2π=red* (full color circle).
Interpolating action on a palette of *32* colors is enough precision for low quanta.



Unit 1 Fig. 12.13

How to do quantum mechanics if you only know classical mechanics

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$$S_H(\mathbf{0}:\mathbf{r}) = S_p(\mathbf{0},0:\mathbf{r},t) + Ht = \int_0^t L\,dt + Ht\,.$$

The hue should represent the phase angle $S_H(\mathbf{0}:\mathbf{r})/\hbar$ *modulo* 2π as, for example,
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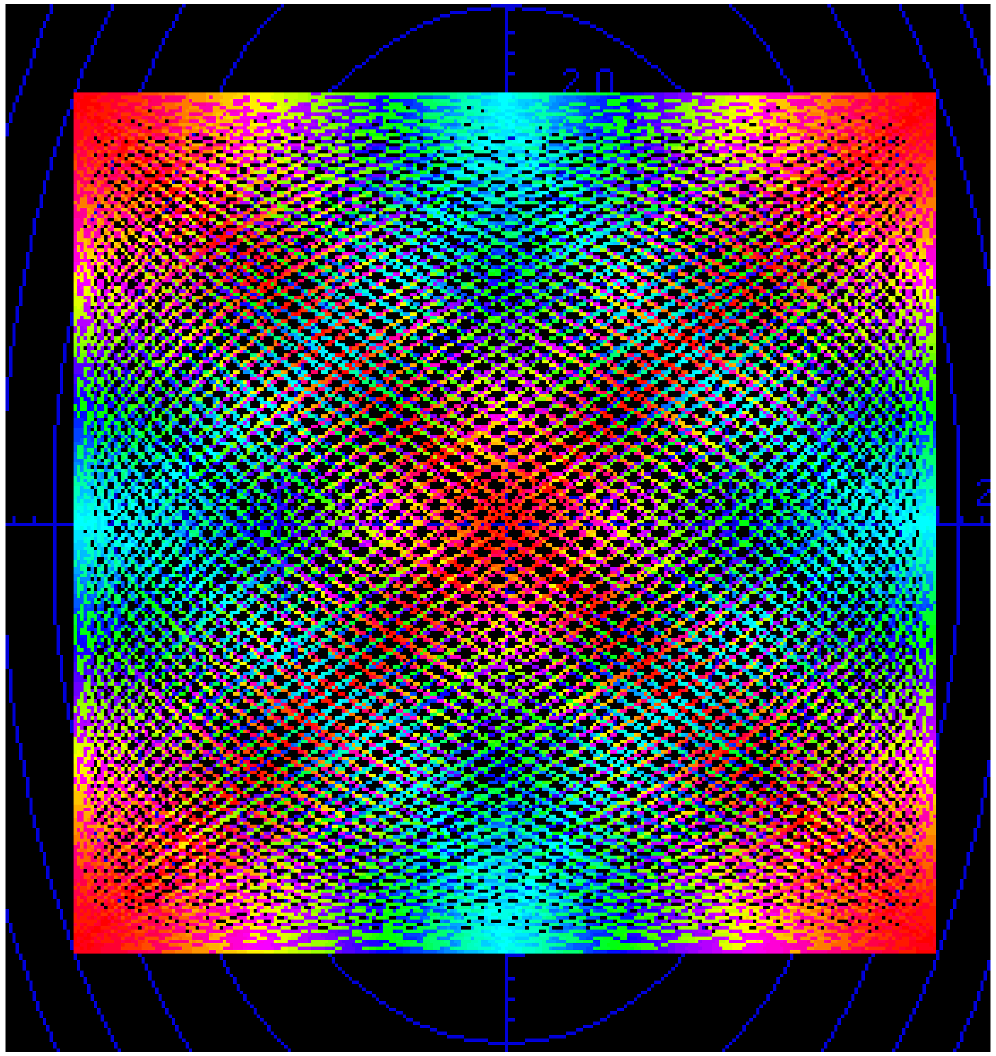
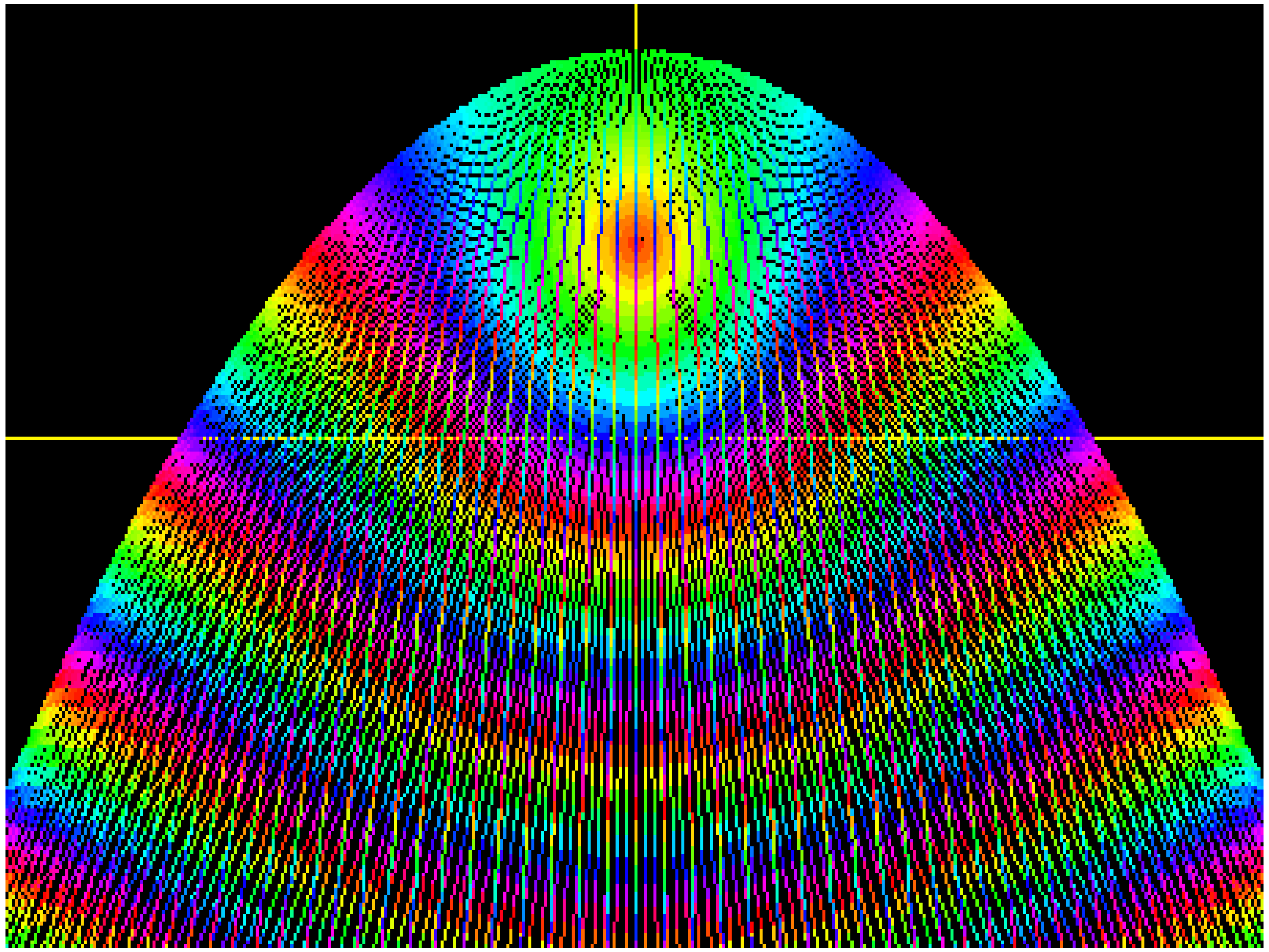


Unit 1
Fig.
12.13

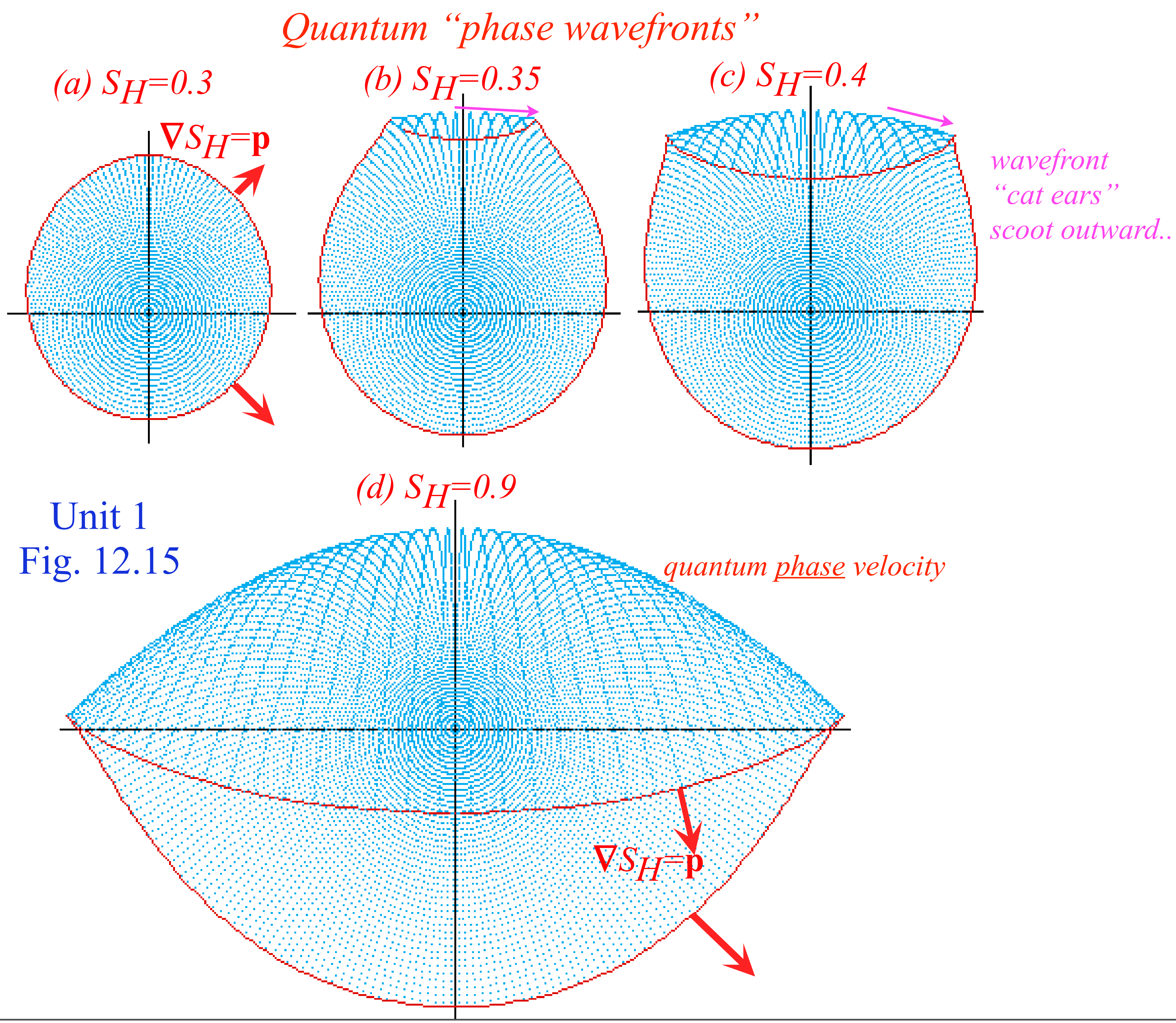
*open system has continuous energy



Unit 1
Fig.
12.14

A moving wave has a *quantum phase velocity* found by setting S=*const.* or $dS(0,0:r,t)=0=\mathbf{p}\bullet d\mathbf{r}-H dt$.

$$\mathbf{V}_{phase} = \frac{d\mathbf{r}}{dt} = \frac{H}{\mathbf{p}} = \frac{\omega}{\mathbf{k}}$$



Unit 1
Fig. 12.15

A moving wave has a *quantum phase velocity* found by setting $S=const.$ or $dS(0,0:r,t)=0=\mathbf{p}\cdot d\mathbf{r}-Hdt$.

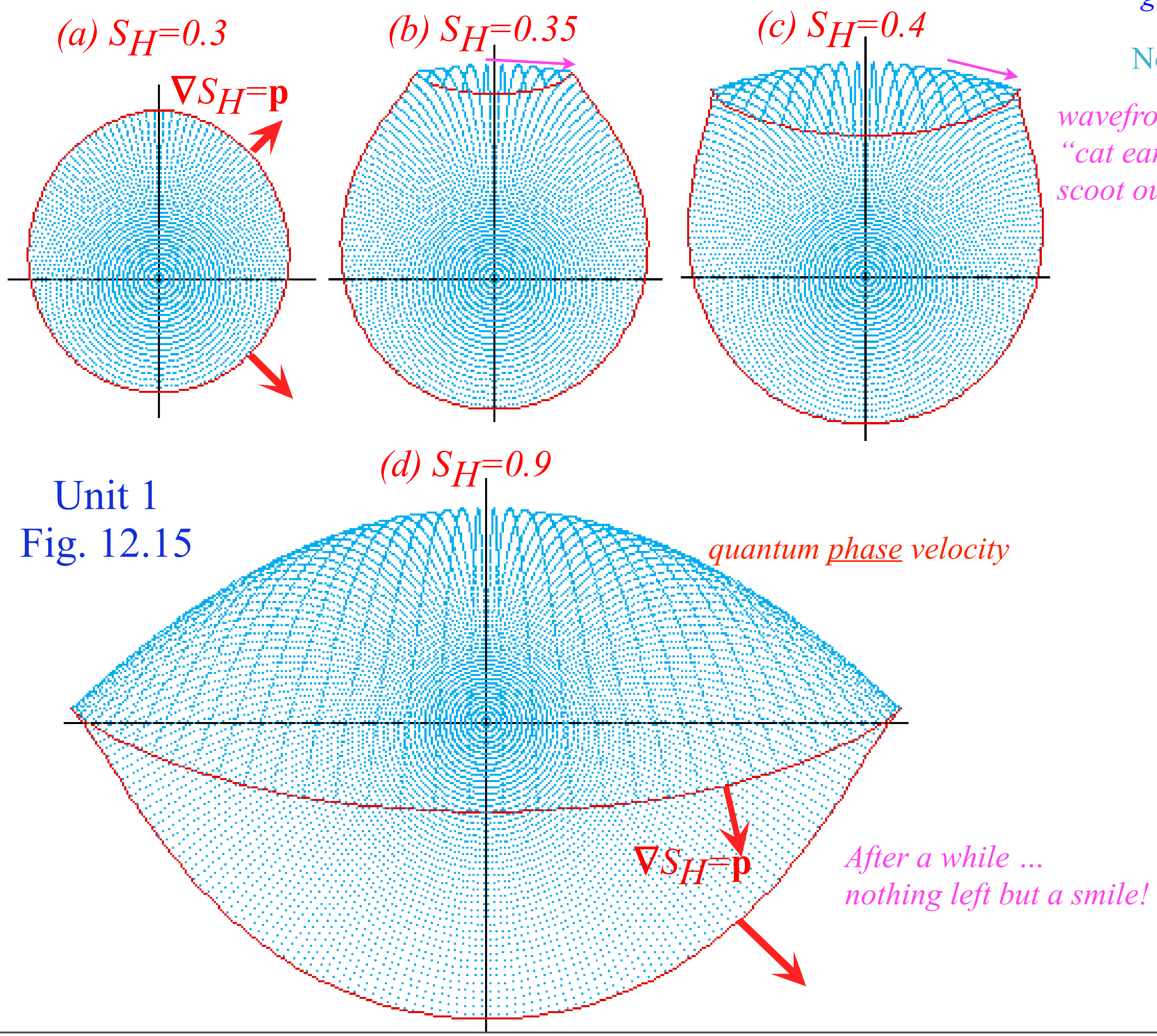
$$\mathbf{V}_{phase} = \frac{d\mathbf{r}}{dt} = \frac{H}{\mathbf{p}} = \frac{\omega}{\mathbf{k}}$$

This is quite the opposite of classical particle velocity which is *quantum group velocity*.

$$\mathbf{V}_{group} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\partial \omega}{\partial \mathbf{k}}$$

Note: This is Hamilton's 1st Equation

Quantum "phase wavefronts"



Unit 1
Fig. 12.15

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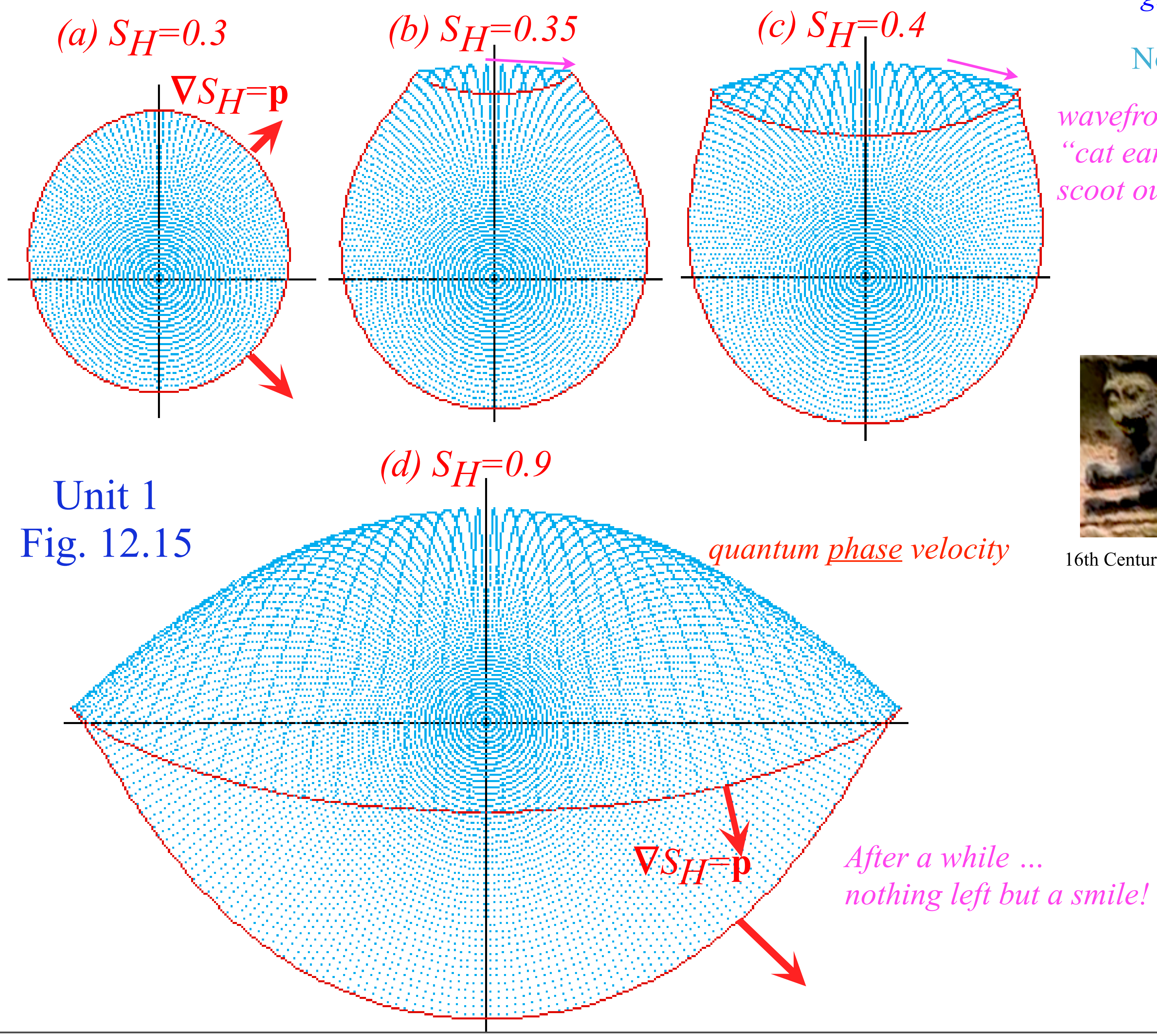

Unit 1
Fig. 12.15

16th Century carving on St. Wifred's in Grappenhall

...on St. Nicolas

From *Alice's Adventures in Wonderland* by Lewis Carrol (1865)

A moving wave has a *quantum phase velocity* found by setting S=*const.* or $dS(0,0:r,t)=0=\mathbf{p}\bullet d\mathbf{r}-Hdt$.

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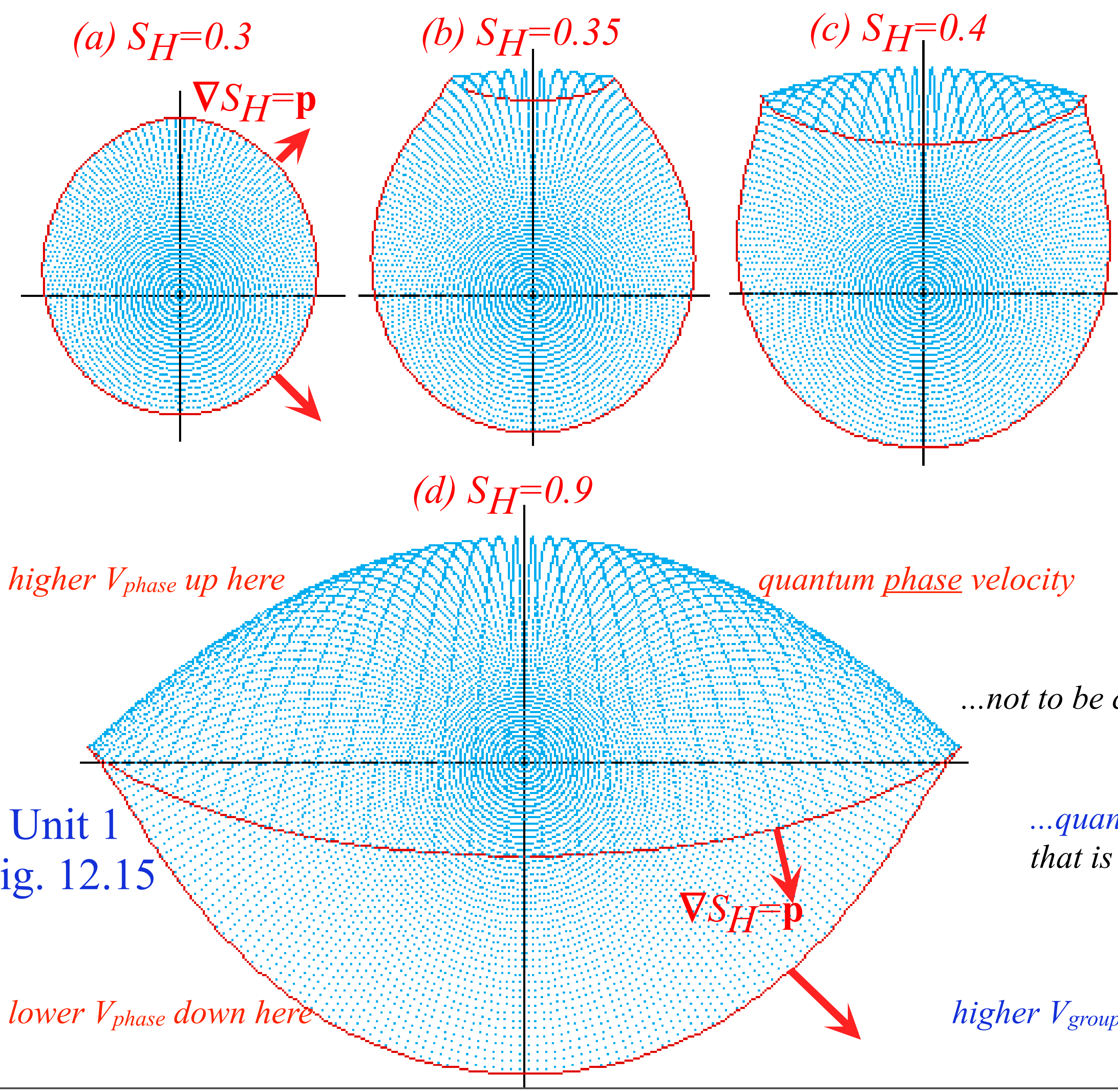

Unit 1
Fig. 12.15

Classical "blast wavefronts"

...not to be confused with...