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Some Geometrical Aspects of Classical Coulomb Scattering

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We derive a number of seemingly forgotten facts concerning Coulomb orbits that make the subject simpler than it is as found in the texts.

I. INTRODUCTION

There exists a beautifully simple geometrical construction for hyperbolic orbits of a mass in a Coulomb field that does not appear to be well known. In fact at least three of the more well-known classical mechanics texts give impossible trajectory[1–3] diagrams that are misleading and fail to show the simplicity of Coulomb scattering geometry and the Rutherford cross section formula.

We give here this construction for a single particle scattering orbit in a fixed Coulomb field and generalize it for the case of two different particles scattering from each other in the center of mass rest frame. Then the orbits in the so-called "lab frame" are drawn in order to exhibit some other interesting points of geometry.

II. CONSTRUCTING SINGLE PARTICLE ORBITS OF POSITIVE ENERGY

Suppose a number of alpha particles are each to be sent, one at a time, from infinity, down different parallel paths [let these be the dotted lines in Fig. 1(a)] toward the neighborhood of a fixed or infinitely massive nucleus N. The nucleus is assumed to give a repulsive Coulomb force field k/r^2 that acts upon all of the alpha particles provided they remain outside the nuclear radius. Let particle 1 be directed along the line that intersects the center of the nucleus, while the original path of the jth particle will be assumed to lie parallel to and a distance b_j. (This is the impact parameter) above the path of the first one. ($b_1=0$, but $b_j\neq 0$ for $j\neq 1$.)

The Coulomb force on particle 1 is therefore always tangent to its path of motion. We shall assume that each particle, including particle 1, has kinetic energy E at infinity. Hence particle 1 approaches on a straight line, slowing until it stops at point A_1, which is an assumed distance $2a=|k/E|$ from N, then returns to infinity along the line whence it came. [Fig. 1(a)].

Assuming we know location of A_1 the following construction gives the orbits of the other particles. To find the orbit of particle 2 which started from infinity along path 2 one bisects segment NA_2, to obtain two segments of length a [Fig. 1(a)], and then draws a circle of radius a centered on the line called path 2, directly above the midpoint of NA_1. [Fig. 1(b)]. Another line is then drawn from the nucleus N through the center C_2 of the circle, to point A_2. Finally, the acute angle ω which NA_2 makes with path 2 is copied on the arc of the circle to give point A_2', and a line is drawn through A_2' from the center of the circle to infinity.

The orbit of particle 2 is a hyperbola that passes through A_2 tangent to the circle. (A_2 is point of closest approach.) The asymptotes are "path 2" and line C_2A_2' as shown in Fig. 1(c). The center of this hyperbola is at C_2 and the focal distance is NC_2 so the exact orbit can easily be constructed.

The proof of this follows easily from the standard formulas[4]

$$a = k/2\,|E| \qquad \text{(II.1)}$$

for the semimajor axis and

$$b = L(2m\,|E|)^{-1/2} \qquad \text{(II.2)}$$

for the semiminor axis of hyperbolic orbit. The quantity a is clearly the same for each alpha particle in our discussion, while b is the impact parameter b_j. (L is angular momentum measured at N, different for each particle but constant in time). Now the geometry of the hyperbola re-

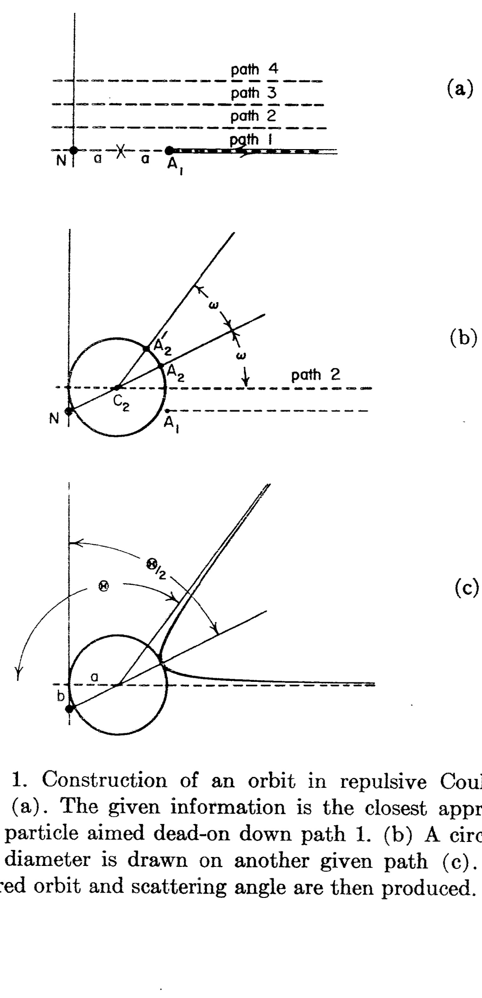


FIG. 1. Construction of an orbit in repulsive Coulomb field (a). The given information is the closest approach of a particle aimed dead-on down path 1. (b) A circle of this diameter is drawn on another given path (c). The desired orbit and scattering angle are then produced.

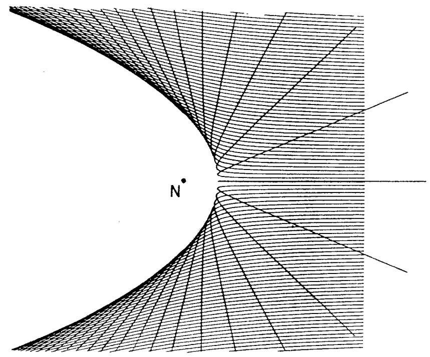


FIG. 2. A family of orbits for particles having identical linear momentum and energy, but varying angular momentum.

quires that the focal distance NC_2 should be

$$a\epsilon = (a^2 + b^2)^{1/2},$$

thus proving Fig. 1(c). Furthermore the angle between the vertical segment labeled b and line NC_2 is half the scattering angle Θ, whence

$$b = a \cot(\Theta/2) = (k/2E)\cot(\Theta/2).$$

Then the differential scattering cross section follows immediately.

$$\begin{aligned} d\sigma/d\Omega &= (b/\sin\Theta)(db/d\Theta) \\ &= (k^2/16E^2)\sin^{-4}(\Theta/2) \qquad \text{(II.3)} \end{aligned}$$

The total cross section is infinite since the integral of (II.3) diverges, and this can be visualized by constructing the family of orbits for various b, as is done in Fig. 2. Applying standard methods[5] for deriving family envelopes we obtain the following equation for the boundary parabola in Fig. 2:

$$x = (-y^2/8a) + 2a$$

Finally we note that if the sign of k in Eq. (II.1) is changed with all else the same, (the Coulomb field becomes attractive) the orbit is constructed in an analogous way by facing the approach circle *away* from incoming beam. This is demonstrated along with the generalized construction of the following section.

III. TWO PARTICLE ORBITS IN CENTER OF MASS SYSTEM

The standard procedure for describing the orbit $\mathbf{r}_1$ of a particle of mass m_1 interacting via Coulomb force with the orbit $\mathbf{r}$ of a particle of mass m_2 is to solve the differential equation

$$\mu(d^2\mathbf{r}/dt^2) = (k\mathbf{r}/r^3) = (k/r^2)\hat{\mathbf{r}}, \qquad \text{(III.1)}$$

for the relative coordinate $\mathbf{r}$

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \qquad \text{(III.2)}$$

in terms of the reduced mass μ, and coupling k:

$$\mu = m_1 m_2/(m_1+m_2) \qquad \text{(III.3)}$$

However, to show that the construction given in Sec. II can be used, we write separate equations in terms of m_1 and m_2, in a coordinate system which has the center of mass $\mathbf{r}^{\mathrm{CM}}$ fixed at origin.

$$\mathbf{r}^{\mathrm{CM}} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/(m_1+m_2) = 0. \qquad \text{(III.4)}$$

Using this Eq. (III.4) and Eq. (III.2) we have the following:

$$\mathbf{r}_1 = [m_2/(m_1+m_2)]\mathbf{r}, \qquad \text{(III.5)}$$

$$\mathbf{r}_2 = [-m_1/(m_1+m_2)]\mathbf{r}. \qquad \text{(III.5)}$$

Substituting this in Eq. (III.1), using Eq. (III.3), two equations are obtained, one for each mass:

$$m_1(d^2\mathbf{r}_1/dt^2) = (k\mu^2/m_1^2)(\mathbf{r}_1/r_1^3)$$

$$\equiv k_1\mathbf{r}_1/r_1^3, \qquad \text{(III.6a)}$$

$$m_2(d^2\mathbf{r}_2/dt^2) = (k\mu^2/m_2^2)(\mathbf{r}_2/r_2^3)$$

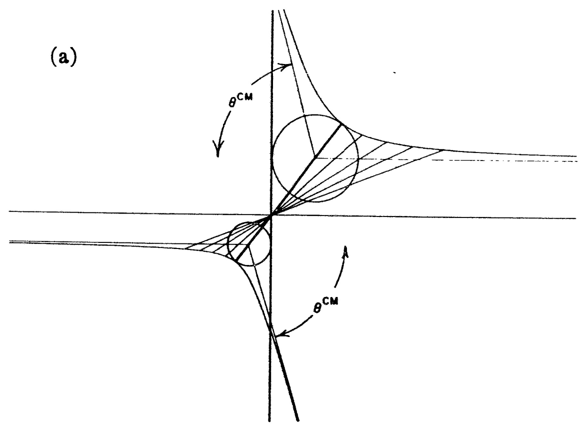
$$\equiv k_2\mathbf{r}_2/r_2^3. \qquad \text{(III.6b)}$$

Now it is seen that each mass m_j behaves as though it was orbiting in a fixed Coulomb field whose origin is the center of mass, but has a "reduced coupling constant" k_j.

Suppose the force is repulsive ($k>0$). Then in a head-on collision, the potential energy of the particles when they have approached each other to the least distance $r_<$, where

$$r_< = 2a_1 + 2a_2 \qquad \text{(III.7)}$$

must equal the sum of the kinetic energies E_1 and E_2 which they have at infinity in center of mass coordinates. [In (III.7), $2a_1$ and $2a_2$ are the



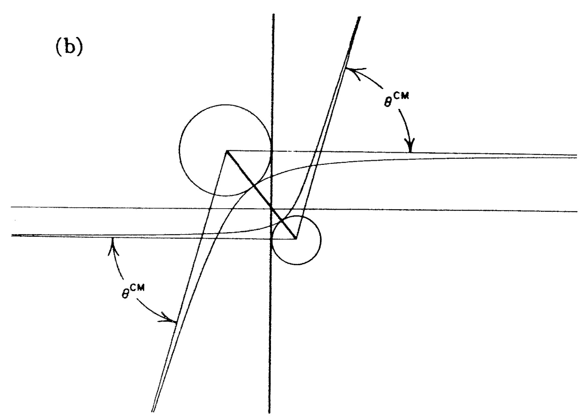


FIG. 3. Construction of orbits in center of mass system of two particles whose mass ratio is 2:1 for the repulsive interaction (a) and the attractive interaction (b).

smallest values attained by r_1 and r_2, respectively, in a head-on collision.]

$$k/r_< = k\mu/2a_1m_1 = E_1+E_2 = (m_1/\mu)E_1,$$

$$k/r_< = k\mu/2a_2m_2 = E_1+E_2 = (m_2/\mu)E_2. \quad \text{(III.8)}$$

To obtain (III.8), we use Eq. (III.3), and the fact that momenta $|m_1v_1|$ and $|m_2v_2|$ are equal magnitudes in center of mass. It is now clear that two sets of Eqs. (III.9) analogous to Eq. (II.1) and Eq. (II.2) can be written

$$a_1 = k_1/2E_1,$$

$$a_2 = k_2/2E_2,$$

$$b_1 = L_1(2m_1E_1)^{-1/2},$$

$$b_2 = L_2(2m_2E_2)^{-1/2}, \quad \text{(III.9)}$$

where

$$a_1/a_2 = b_1/b_2 = m_2/m_1$$

and that orbits can be constructed exactly as they were in Sec. II. The hyperbola traced by m_2 is a copy of the one traced in m_1, but scaled down by factor m_2/m_1, as shown in Fig. 3(a). Figure 3(b) shows the construction when the sign of k is reversed.

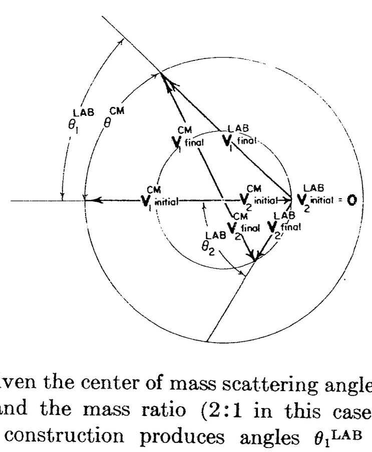


FIG. 4. Given the center of mass scattering angle θ^{CM} (from Fig. 3) and the mass ratio (2:1 in this case) a vector addition construction produces angles θ_1^{LAB} and θ_2^{LAB} shown here.

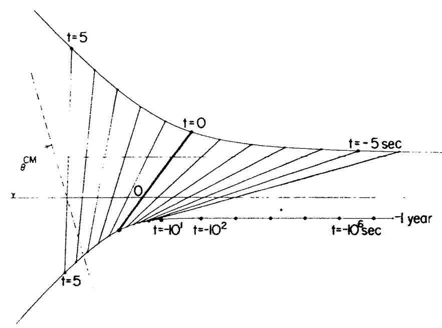


FIG. 5. The laboratory picture of Fig. 3. The scattering begins with both particles infinitely far to the right. The heavier particle is at rest and the lighter particle is moving left about 0.3 mile per day in the scale of this drawing. When the heavier particle first appears on this picture, one or two years before the "collision," it is creeping extremely slowly leftward, while the lighter particle is still over a hundred miles off to the right. The heavier particle continues creeping until finally the lighter particle arrives in the picture and moves through in about 12 sec. Most of the momentum is transferred in 3 or 4 sec.

IV. TWO PARTICLE ORBITS IN LABORATORY SYSTEM

Having obtained the scattering angle θ^{CM} and the orbits in the center of mass system (Fig. 3) it is interesting to see if the same thing can be done in a so-called "laboratory" coordinate system which is defined so that the second mass m_2 is originally at rest.

It is shown in a number of texts that lab scattering angles θ_1^{LAB} and θ_2^{LAB} are given by the construction shown in Fig. 4, once θ^{CM} is known. This gives the slopes of the final asymptotes of the two particles in the lab system, which while conserving total angular momentum, must intersect somewhere on the original path line of first mass m_1. But, we had hoped to be able to construct the lab coordinates of the two orbits including, if possible, the starting position of the second mass m_2, the point of closest approach, and the location of the final asymptotes.

However, the construction of the orbits is beyond the reach of simple geometry. Furthermore it is most interesting to note that, with respect to the point of closest approach in the lab system, *neither the starting position of m_2, nor the final asymptotes exist at all!*

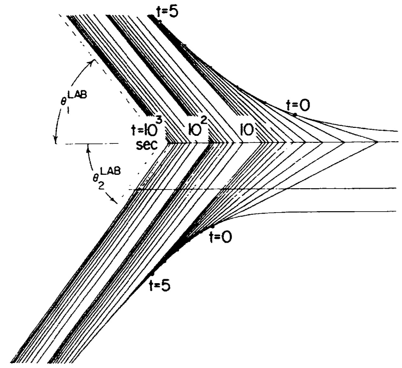


FIG. 6. Logarithmic recession of tangents demonstrates the nonexistence of asymptotes, for pure Coulomb scattering in laboratory system. At $t=10^3$ the slopes of the tangents are shy of θ_1^{LAB} and θ_2^{LAB} by only 0.02° and 0.04°, respectively.

As we shall now show, both the latter quantities exist in a potential field $1/r^n (n>2)$ or a Yukawa field $e^{-\alpha r}/r$, but not in a pure unscreened Coulomb field.

To demonstrate the problem we produce by direct calculation[6] a number of points on the lab system orbits. Taking Fig. 3(a) where $m_2=2m_1$, we compute the times for a number of points on the orbit of m_1, using (IV.1),[7]

$$v_1^{CM}(\text{initial})\times t=\pm\{\epsilon[(x'-a\epsilon)^2-a^2]^{1/2} + a\cosh^{-1}(x'-a\epsilon)/a\} \quad \text{(IV.1)}$$

These points are then transcribed onto the lab system plot by placing the center of gravity for each time at its correct position along its line of motion. The result is drawn in Fig. 5. In this drawing origin O in space and time is the location of the center of mass when the particles are closest to each other. Note that the lab particle m_2 has been creeping up from the right, and was actually never completely at rest at any finite time in the past. In fact if you run time backwards the velocity of particle 2 is given approximately by

$$\begin{aligned} v_2^{LAB}(t) &= \int (|F|/m_2)dt \\ &\cong \int k\,dt/m_2[v_1^{CM}(\text{initial})t]^2 \\ &\cong [-k/m_2 v_1^{CM}(\text{initial})^2]t^{-1} \quad \text{(IV.2)} \end{aligned}$$

and the position of this particle diverges logarithmically. However, if a proper screening is inserted, this divergence may be removed.

The location of the asymptotes in the lab system could be found if the final angular momentum of just one of the particles, say L_1^{LAB} of m_1, could be found. Denoting by $\mathbf{V}$ the velocity of the center of mass in the lab system, we have the following:

$$\begin{aligned} \mathbf{L}_1^{LAB} &= \mathbf{r}_1^{LAB} \times \mathbf{v}_1^{LAB} \\ &= (\mathbf{r}_1^{CM}+\mathbf{V}t)\times(\mathbf{v}_1^{CM}+\mathbf{V}) \\ &= \mathbf{L}_1^{CM}-\mathbf{V}\times(\mathbf{r}_1^{CM}-\mathbf{v}_1^{CM}t). \quad \text{(IV.3)} \end{aligned}$$

Since $\mathbf{L}_1^{CM}$ is known, we must determine the term on the right of (IV.3):

$$\begin{aligned} \mathbf{r}_1^{CM}-\mathbf{v}_1^{CM}t &= \int \mathbf{v}_1^{CM}dt-\mathbf{v}_1^{CM}t \\ &= \int t\,d\mathbf{v}_1^{CM} \\ &= \int (t\,d\mathbf{v}_1^{CM}/dt)dt \\ &= \int (t\mathbf{F}/m_1)dt. \quad \text{(IV.4)} \end{aligned}$$

For an unscreened Coulomb force, the integral (IV.4) must diverge as a logarithm and with it the position of the asymptotes.

This is shown on the Fig. 6. Again very small changes in spatial dependence of the force can be sufficient to eliminate the divergence.

V. FINAL COMMENTS

Any paper on classical Coulomb orbits that appears in the late twentieth century may seem to be a bit late. Indeed, it would be preposterous

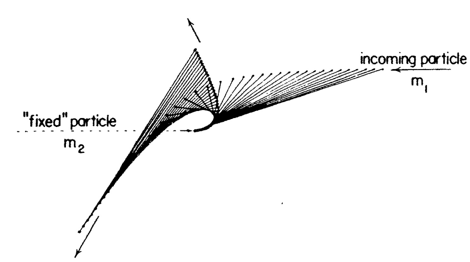


FIG. 7. Attractive Coulomb scattering in laboratory system. This has the same "anomalies" as the repulsive case.

to propose that the points brought up were new since hardly anyone could claim to have examined all literature on classical mechanics postdating the *Principia*.[8] However, in old and new texts that were available to us, including those previously mentioned, we found these points had been missed.

Since our colleagues found these points surprising and intriguing, we decided to share them with others through this paper. But, we felt it would be more appropriate to introduce these points of view as "forgotten" and "rediscovered" even though we cannot presently say by whom, if anyone, they were previously discovered or where, if anywhere, they were forgotten.

[1] H. Goldstein, *Classical Mechanics* (Addison–Wesley, Reading, Mass., 1965), p. 84.

[2] K. Symon, *Mechanics* (Addison–Wesley, Reading, Mass., 1971), p. 134.

[3] R. Lindsay, *Physical Mechanics* (Van Nostrand, New York, 1950), p. 87.

[4] Reference 2, p. 134.

[5] E. Kreyszig, *Differential Geometry* (Univ. of Toronto, Toronto, Canada, 1959), p. 262.

[6] Calculations and plots were performed on a Hewlitt–Packard desk plotter.

[7] Equation (IV.1) is derived using Eq. (3-214) of Ref. 2, p. 125. The distance X' is measured along the symmetry axis of the hyperbola.

[8] I. Newton, *Philosophie Naturalis Principia Mathematica* (1687), transl. by A. Motte (Univ. of Calif. Press, Berkeley, 1946).