**Analysis of 1D 2-Body Collisions**  
(Ch. 2 to Ch. 4 of Unit 1)

- Review of elastic Kinetic Energy ellipse geometry
- The X2 Superball pen launcher  
  Perfectly elastic “ka-bong” velocity amplification effects (Faux-Flubber)
- Geometry of X2 launcher bouncing in box  
  Independent Bounce Model (IBM)
  Geometric optimization and range-of-motion calculation(s)
  Integration of \((V_1,V_2)\) data to space-time plots \((y_1(t),t)\) and \((y_2(t),t)\) plots
  Integration of \((V_1,V_2)\) data to space-space plots \((y_1, y_2)\)  
  Examples \((M_1=7, M_2=1)\) and \((M_1=49, M_2=1)\)

- Multiple collisions calculated by matrix operator products  
  Matrix or tensor algebra of 1-D 2-body collisions

**Ellipse rescaling-geometry and reflection-symmetry analysis**

- Rescaling KE ellipse to circle
  How this relates to Lagrangian, l’Etrangian, and Hamiltonian mechanics later on

- Reflections in the clothing store: “It’s all done with mirrors!”

- Introducing hexagonal symmetry \(D_6 \sim C_{6v}\) (Resulting for \(m_1/m_2=3\))
  Group multiplication and product table
  Classical collision paths with \(D_6 \sim C_{6v}\) (Resulting from \(m_1/m_2=3\))
  Other not-so-symmetric examples: \(m_1/m_2=4\) and \(m_1/m_2=7\) and \((M_1=100, M_2=1)\)
Review of elastic Kinetic Energy ellipse geometry

**Elastic Kinetic Energy ellipse**

- Ellipse equation: $b = \sqrt{\frac{2 \cdot KE}{m_{VW}}} = 120.42$
- Initial point: $(60, 10)$
- Final point: $(40, 90)$

**Inelastic Kinetic Energy ellipse**

- Ellipse equation: $b = \sqrt{\frac{2 \cdot IE}{m_{VW}}} = 111.8$
- Initial point: $(50, 50)$
- Final point: $(40, 90)$

- Momentum: $p_{Total} = 250$
- Total energy: $E_{Total} = 7,250$
- Inelastic energy: $E_{COM} = 1,000$

**Fig. 2.3a** (Unit 1)

**Fig. 2.3b** (Unit 1)
The X-2 Pen launcher and Superball Collision Simulator*

ballpoint pen

$M_2 = 10\text{gm}$

The X-2 pen-launcher

Superball penetration depth

$d = \frac{r^2}{2R}$

bounce plate

$M_0 = 10\text{kg}$

Fig. 3.1 (Unit 1)

$M_1 = 70\text{gm}$

$m_1 = 70.000 \text{ g}$

$m_2 = 10.000 \text{ g}$

$V_1 = -1.000 \text{ cm/s}$

$V_2 = -1.000 \text{ cm/s}$

*Launch Generic Superball Collision Web Simulator

http://www.uark.edu/ua/modphys/markup/BounceItWeb.html?scenario=1007

(With $g=0$ and 70:10 mass ratio)
The X-2 Pen launcher and Superball Collision Simulator*  

Caution: Product Liability Disclaimer  
This ballpoint pen could be hazardous to your health! The experiments which are the subject of this discussion are both spectacular and potentially dangerous, and care to protect one’s eyes should be taken. The simplest experiment involves sticking a ball point pen into a superball or other hard rubber ball and dropping the two onto a hard floor. If done correctly the pen will eject the ball with such force it may stick in the ceiling of the room. Obviously you want to be careful with this weapon. And, this goes doubly and triply for the more advanced models that may be developed in the course of studying this stuff. It is recommended that experimenters wear safety glasses when doing these experiments with pens. (We could just say don’t use pens, but that’s an easy way to do this experiment and probably the way most people will go about it.) Some of the tangential experiments associated with this development are less hazardous. To measure the potential force function of a ball one may simply paint the ball and measure the spot size as a function of drop height \( h \).

The saggital approximation \( d = r^2/2R \) allows one to quickly convert spot radius \( r \) to penetration depth \( x \) for a superball of radius \( R \) as shown in the figure. Equating this to \( Mgh \) gives the ball potential energy function \( V(x) \).

Fig. 3.1 (Unit 1)  

*Launch Generic Superball Collision Web Simulator  
http://www.uark.edu/ua/modphys/markup/BounceItWeb.html?scenario=1007
The X-2 pen-launcher

Superball penetration depth

Superball mass

Graafassador

Ballpoint pen

Superball mass

1st bang: mass \( M_0 \) vs. mass \( M_1 \)

1st bang: \( M_1 \) off floor

This 1st bang is a floor-bounce of \( M_1 \) off very massive plate/Earth \( M_0 \)

*Launch Generic Superball Collision Web Simulator*  
[http://www.uark.edu/ua/modphys/markup/BounceItWeb.html](http://www.uark.edu/ua/modphys/markup/BounceItWeb.html)

(With \( g=0 \) and 70:10 mass ratio)
The X-2 pen-launcher

Superball penetration depth
\[ d = \frac{r^2}{2R} \]

Superball
\[ M_1 = 70 \text{gm} \]

With \( g = 0 \) and 70:10 mass ratio

(a) Super-elastic 2nd-body bounce

(b) 2-Bang Model

(c) n-Body Supernova Superballs

(Still Bigger BANG!)

(Bigger BANG!)

1st bang:
\[ M_1 \text{ off floor} \]

This 1st bang is a floor-bounce of \( M_1 \) off very massive plate/Earth \( M_0 \)

(With \( g = 0 \) and 70:10 mass ratio)

http://www.uark.edu/ua/modphys/markup/BounceItWeb.html?scenario=1007

(With \( g = 0 \) and 70:35 mass ratio)

http://www.uark.edu/ua/modphys/testing/markup/BounceItWeb.html
(a) 1st bang of $M_1$ off
floor plate $M_\oplus = 100 \ M_1$ along
$(V_1, V_\oplus)$-momentum line of slope
$-M_1/M_\oplus = -1/100$
from IN-end to COM to FIN-end
of $(a/b = \sqrt{M_\oplus}/\sqrt{M_1}=10)$ ellipse
(With g=0 and 70:10 mass ratio)

http://www.uark.edu/ua/modphys/markup/BounceItWeb.html?scenario=1007
Figure 3.3 (Unit 1)
(a) Super-elastic 2nd-body bounce
(b) 2-Bang Model
(c) n-Body Supernova Superballs

Figure 3.4 (Unit 1)

1st bang: 
$M_1$ off floor
2nd bang: 
m_2 off $M_1$

This 1st bang is a floor-bounce of 
$M_1$ off very massive plate/Earth $M_0$

(With $g=0$ and 70:10 mass ratio)
http://www.uark.edu/ua/modphys/markup/BounceItWeb.html?scenario=1007

(With $g$ and 70:35 mass ratio)
http://www.uark.edu/ua/modphys/testing/markup/BounceItWeb.html
**Fig. 3.3 (Unit 1)**

(a) Super-elastic 2nd-body bounce

(b) 2-Bang Model

(c) n-Body Supernova Superballs

(Still Bigger BANG!)

(Bigger BANG!)

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**Fig. 3.4 (Unit 1)**

(a) Mirror reflection thru m2 axis

This 1st bang is a floor-bounce of $M_1$ off very massive plate/Earth $M_0$

(With $g=0$ and 70:10 mass ratio)

http://www.uark.edu/ua/modphys/markup/BounceItWeb.html?scenario=1007

---

3rd bang: $m_2$ off ceiling

1st bang: $M_1$ off floor

2nd bang: $m_2$ off $M_1$

---

(With $g$ and 70:35 mass ratio)

http://www.uark.edu/ua/modphys/testing/markup/BounceItWeb.html
Geometry of X2 launcher bouncing in box

Independent Bounce Model (IBM)
Geometric optimization and range-of-motion calculation(t)
Integration of \((V_1, V_2)\) data to space-time plots \((y_1(t), t)\) and \((y_2(t), t)\) plots
Integration of \((V_1, V_2)\) data to space-space plots \((y_1, y_2)\)
The X-2 pen-launcher

Superball penetration depth
\[ d = \frac{r^2}{2R} \]

(b) 2-Bang Model

(c) n-Body Supernova Superballs

(Bigger BANG!)

This 1st bang is a floor-bounce of \( M_1 \) off very massive plate/Earth \( M_0 \)
Geometry of X2 launcher bouncing in box

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(With g=0 and 70:35 mass ratio)

http://www.uark.edu/ua/modphys/testing/markup/BounceItWeb.html

Fig. 3.3
(With g=0 and 70:10 mass ratio)

http://www.uark.edu/ua/modphys/markup/BounceItWeb.html?scenario=1007

(b) 2-Bang Model

(c) n-Body Supernova Superballs

(Still Bigger BANG!)
The X-2 pen-launcher

Superball penetration depth
\[ d = \frac{r^2}{2R} \]

Fig. 3.3 (Unit 1)

(a) Super-elastic 2nd-body bounce
(b) 2-Bang Model
(c) n-Body Supernova Superballs

(With g and 70:10 mass ratio)

http://www.uark.edu/ua/modphys/markup/BounceItWeb.html?

http://www.uark.edu/ua/modphys/testing/markup/BounceItWeb.html

Fig. 4.5a-b

With g=0 and 70:10 mass ratio (1007)
http://www.uark.edu/ua/modphys/markup/BounceItWeb.html?

With g and 70:35 mass ratio
http://www.uark.edu/ua/modphys/testing/markup/BounceItWeb.html
Geometry of X2 launcher bouncing in box

Independent Bounce Model (IBM)
Geometric optimization and range-of-motion calculation(s)
Integration of \((V_1, V_2)\) data to space-time plots \((y_1(t), t)\) and \((y_2(t), t)\) plots
Integration of \((V_1, V_2)\) data to space-space plots \((y_1, y_2)\)

Examples \((M_1=7, M_2=1)\) and \((M_1=49, M_2=1)\)
Geometric “Integration” (Converting Velocity data to Spacetime)

Velocity $V_{\text{y}2}$ vs. $V_{\text{y}1}$ Plot

Position $y$ vs. Time $t$ Plot

$V_{\text{y}2} = -0.5$
means $M_2$ is

someplace on some path of slope -0.5

$V_{\text{y}1} = +1.0$
means $M_1$ is

someplace on some path of slope +1.0
Geometric “Integration” (Converting Velocity data to Spacetime)

**Velocity** $V_{y2}$ vs. $V_{y1}$ Plot

- $V_{y2} = -0.5$ means $M_2$ is somewhere on some path of slope -0.5
- $V_{y1} = +1.0$ means $M_1$ is somewhere on some path of slope +1.0

**Position** $y$ vs. **Time** $t$ Plot

- Ceiling at $y = 7.1$
- Floor at $y = 0$

**Height** $y$-axis

**Time** $t$-axis

Tuesday, February 2, 2016
Geometric “Integration” (Converting Velocity data to Spacetime)

- **Velocity** $V_{y2}$ vs. $V_{y1}$ Plot
  - $V_{y2} = -0.5$ means $M_2$ is somewhere on some path of slope -0.5
  - $V_{y1} = +1.0$ means $M_1$ is somewhere on some path of slope +1.0

- **Position** $y$ vs. **Time** $t$ Plot
  - Ceiling at $y = 7.1$
  - Floor at $y = 0$
  - Slope $-0.5/1 = -0.5$
  - Slope $1/1 = +1$

- Until you specify initial conditions $y_0(t_0)$...
  - ...you don’t know what $v_y$-line to use
Geometric “Integration” (Converting Velocity data to Spacetime)

Velocity $V_{y2}$ vs. $V_{y1}$ Plot

$V_{y2} = -0.5$ means $M_2$ is somewhere on some path of slope -0.5

$V_{y1} = +1.0$ means $M_1$ is somewhere on some path of slope +1.0

Position $y$ vs. Time $t$ Plot

Until you specify initial conditions $y_0(t_0)$...

...you don’t know which $v_y$-lines to use

(a) $V_{y2}$ vs. $V_{y1}$ Plot of Bang-1 (01)

$(V_{y1}, V_{y2}) = (-1.0, -1.0)$

Bang-1 (01) Bounces (-1,-1) to (+1,-1)

(b) $y$ vs. $t$ Plot of Bang-1 (01)

$y_2(0) = 3$

$y_1(0) = 1$
Geometric “Integration” (Converting Velocity data to Spacetime)

(a) \( V_y2 \) vs. \( V_y1 \) Plot of Bang-1 \( (01) \)

- \( V_y2 = -0.5 \) means \( M_2 \) is somewhere on some path of slope -0.5
- \( V_y1 = +1.0 \) means \( M_1 \) is somewhere on some path of slope +1.0

(b) \( y \) vs. \( t \) Plot of Bang-1 \( (01) \)

- Initial conditions \( y_1(0) \) and \( y_2(0) \)
- \( y_2(0) = 3 \) and \( y_1(0) = 1 \)

Until you specify initial conditions \( y_0(t_0) \)...
...you don’t know which \( v_y \)-lines to use
Geometric “Integration” (Converting Velocity data to Spacetime)

Velocity $V_{y2}$ vs. $V_{y1}$ Plot

- $V_{y2} = -0.5$ means $M_2$ is somewhere on some path of slope $-0.5$
- $V_{y1} = +1.0$ means $M_1$ is somewhere on some path of slope $+1.0$

Position $y$ vs. Time $t$ Plot

- Until you specify initial conditions $y_0(t_0)$...
- ...you don't know which $v_y$-lines to use

(a) $V_{y2}$ vs. $V_{y1}$ Plot of Bang-1 $(01)$

- $(V_{y1}, V_{y2}) = (-1.0, -1.0)$
- $(V_{y1}, V_{y2}) = (+1.0, -1.0)$

(b) $y$ vs. $t$ Plot of Bang-1 $(01)$

- $y_2(0) = 3$
- $y_1(0) = 1$
- $y_0(0) = 1$
- Bang-1 $(01)$ Bounces (-1,-1) to (+1,-1)

Fig. 3.6
(Unit 1)
Geometric “Integration” (Converting Velocity data to Spacetime)

(a) $V_{y2}$ vs. $V_{y1}$ Plot of Bang-1 $(01)$

(b) $y$ vs. $t$ Plot of Bang-1 $(01)$

Initial conditions $y_1(0)$ and $y_2(0)$

Position $y$ vs. Time $t$ Plot

Until you specify initial conditions $y_0(t_0)$...

...you don’t know which $v_y$-lines to use

Fig. 3.6 (Unit 1)
Geometric “Integration” (Converting Velocity data to Spacetime)

Until you specify initial conditions \(y_0(t_0)\)...
...you don’t know which \(v_y\)-lines to use

![Velocity vs. Position Graphs](image)

**Fig. 3.6**
(Units 1)

(a) \(V_{y2} vs. V_{y1}\) Plot of Bang-1 \(_{01}\)

\(V_{y2} = -0.5\) means \(M_2\) is somewhere on some path of slope -0.5

\(V_{y1} = +1.0\) means \(M_1\) is somewhere on some path of slope +1.0

(b) \(y vs. t\) Plot of Bang-1 \(_{01}\)

Initial conditions \(y_1(0)\)...
...and \(y_2(0)\)

\(y_2(0) = -1.0\)

\(y_1(0) = 1.0\)

Bang-1 \(_{01}\) Bounces (-1,-1) to (+1,-1)

Position \((y=0, t=1)\)

Ceiling at \(y=7.1\)

Floor at \(y=0\)
Geometric “Integration” (Converting Velocity data to Spacetime)

Velocity $V_{y2}$ vs. $V_{y1}$ Plot

$V_{y2} = 0.5$ means $M_2$ is somewhere on some path of slope -0.5

$V_{y1} = +1.0$ means $M_1$ is somewhere on some path of slope +1.0

Position $y$ vs. Time $t$ Plot

Until you specify initial conditions $y_0(t_0)$...

...you don’t know which $v_y$-lines to use

Fig. 3.6 (Unit 1)

(a) $V_{y2}$ vs. $V_{y1}$ Plot of Bang-1$_{(01)}$

initial conditions $y_1(0)$...and $y_2(0)$

(B) $y$ vs. $t$ Plot of Bang-1$_{(01)}$

Height

$y$-axis

Ceiling at $y=7$

Bang-1$_{(01)}$ Bounces (-1,-1) to (+1,-1)

Bang-2$_{(12)}$ position

$y_1(0)=3$

$y_2(0)=1$

$y(0,t)=0$

$y(1,t)=2$

Floor at $y=0$

Bang-1$_{(01)}$ position

$y(0,t)=1$

Position ($y=0$, $t=1$) ($y=1$, $t=2$)
Geometric “Integration” (Converting Velocity data to Spacetime)

(a)

(b)

Height y

Floor at y=0

Ceiling at y=7

M_1 slope: +0.5

M_2 slope: +2.5

Bang-2(12)

Bang-3(20)

Bang-4(12)

Bang-1(01)

Time t
**Kinetic Energy Ellipse**

\[
KE = \frac{1}{2} M_1 V_1^2 + \frac{1}{2} M_2 V_2^2 = \frac{1}{2} + \frac{7}{2} = 4
\]

\[
1 = \frac{V_1^2}{2KE / M_1} + \frac{V_2^2}{2KE / M_2} = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2}
\]
**Geometric “Integration” (Converting Velocity data to Spacetime)**

- **Diagram (a)**: Diagram showing the integration of velocity data over time. Points marked as Bang-1(01), Bang-3(20), etc., with corresponding velocities and time markers.

- **Diagram (b)**: Diagram illustrating height y over time t, with markers for Bang-1(01), Bang-2(12), etc., and slope details for M1 and M2.

- **Diagram (c)**: Diagram for the Kinetic Energy Ellipse, showing the relationship between velocity and energy. Formula: 
  \[
  KE = \frac{1}{2} M_1 V_1^2 + \frac{1}{2} M_2 V_2^2 = \frac{7}{2} + \frac{1}{2} = 4
  \]
  
  \[
  1 = \frac{V_1^2}{2KE / M_1} + \frac{V_2^2}{2KE / M_2} = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2}
  \]

- **Diagram (d)**: Diagram showing the relationship between time t and velocity, with markers for Bang-1(01), Bang-2(12), etc., and ceiling and floor markers.

**Fig. 3.7**
(Unit 1)

**Kinetic Energy Ellipse**

- **Ellipse radius 1**: 
  \[
  a_1 = \sqrt{2KE / M_1}
  \]

- **Ellipse radius 2**: 
  \[
  a_2 = \sqrt{2KE / M_1}
  \]
**Kinetic Energy Ellipse**

\[ KE = \frac{1}{2} M_1 V_1^2 + \frac{1}{2} M_2 V_2^2 = \frac{7}{2} + \frac{1}{2} = 4 \]

\[ 1 = \frac{V_1^2}{2KE / M_1} + \frac{V_2^2}{2KE / M_2} = \frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} \]

**Ellipse radius 1**

\[ a_1 = \sqrt{2KE / M_1} = \sqrt{2KE / 7} = \sqrt{8 / 7} = 1.07 \]

**Ellipse radius 2**

\[ a_2 = \sqrt{2KE / M_1} = \sqrt{2KE / 1} = \sqrt{8 / 1} = 2.83 \]
Geometric “Integration” (Converting Velocity data to Spacetime)

\[ \text{BounceIt Superball Collision Web Simulator: } M_1=70, M_2=10 \text{ with Newtonian time plot} \]

\[ \text{BounceIt Superball Collision Web Simulator: } M_1=70, M_2=10 \text{ with } V_2 \text{ vs } V_1 \text{ plot} \]

\( t = 10.2 \quad E = 3.999930 \quad \text{Fig. 4.8 in Unit 1} \)

\( t = \infty \text{?} \quad E = 4 \quad \text{Fig. 4.9 in Unit 1} \)

\( t = 25 \quad E = 3.9999964 \quad \text{Pen} \)

\( t = 278 \quad E = 3.99995 \quad \text{Ball} \)

Ergodic Fill-in at \( t=\infty \)?
Geometry of X2 launcher bouncing in box

Independent Bounce Model (IBM)
Geometric optimization and range-of-motion calculation(t)
Integration of \((V_1,V_2)\) data to space-time plots \((y_1(t),t)\) and \((y_2(t),t)\) plots
Integration of \((V_1,V_2)\) data to space-space plots \((y_1, y_2)\) Examples \((M_1=7, M_2=1)\) and \((M_1=49, M_2=1)\)
**Geometric “Integration” (Converting Velocity data to Space-space trajectory)**

**Fig. 3.8 (Unit 1)**

**Step-0:** At starting position \( y(0) = (1,3) \) draw initial velocity \( v(0) = (-1,-1) \) line.

**Step-1:** Extend \( v(0) \) line to floor point \( y(0) = (0,?) \) and draw Bang-1 \((01)\) velocity \( v(1) = (1,-1) \) line. (Find \( v(1) \) using V-V plot)

**Step-2:** Extend \( v(1) \) line to collision point \( y(0) = (?,?) \) and draw Bang-2 \((12)\) velocity \( v(2) = (0.5,2.5) \). (Find \( v(2) \) using V-V plot)

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**Ellipse radius 1**

\[
a_1 = \sqrt{\frac{2KE}{M_1}} = \sqrt{\frac{2KE}{7}} = \sqrt{8/7} = 1.07
\]

**Ellipse radius 2**

\[
a_2 = \sqrt{\frac{2KE}{M_1}} = \sqrt{\frac{2KE}{1}} = \sqrt{8/1} = 2.83
\]
Geometric “Integration” (Converting Velocity data to Space-space trajectory)

**Fig. 3.9 (Unit 1)**

Step-2: Extend \( \mathbf{v}(2) \) line to ceiling point \( y(3)=(-7, 1) \) and draw Bang-3\(_{(20)}\) velocity \( \mathbf{v}(3)=(1, -1) \) line. (Find \( \mathbf{v}(3) \) using V-V plot.)

Step-3: Extend \( \mathbf{v}(3) \) line to collision point \( y(4)=(-7, 7) \) and draw Bang-4\(_{(12)}\) velocity \( \mathbf{v}(4)=(0.5, 2.5) \). (Find \( \mathbf{v}(4) \) using V-V plot.)

Step-4: Extend \( \mathbf{v}(4) \) line to ceiling point \( y(4)=(7, 7.1) \) and draw Bang-5\(_{(20)}\) velocity \( \mathbf{v}(5)=(1, -1) \) line. (Find \( \mathbf{v}(5) \) using V-V plot.)

Step-5: Extend \( \mathbf{v}(5) \) line to collision point \( y(6)=(7, 7) \) and draw Bang-6\(_{(12)}\) velocity \( \mathbf{v}(6)=(0.5, 2.5) \). (Find \( \mathbf{v}(6) \) using V-V plot.)

**Ellipse radius 1**

\[ a_1 = \sqrt{\frac{2KE}{M_1}} \]

\[ = \frac{2KE}{7} \]

\[ = \sqrt{8/7} \]

\[ = 1.07 \]

**Ellipse radius 2**

\[ a_2 = \sqrt{\frac{2KE}{M_1}} \]

\[ = \frac{2KE}{1} \]

\[ = \sqrt{8/1} \]

\[ = 2.83 \]
\[
\begin{align*}
\text{Ellipse radius 1} & \quad \text{Ellipse radius 2} \\
a_1 &= \sqrt{\frac{2KE}{M_1}} & a_2 &= \sqrt{\frac{2KE}{M_1}} \\
&= \sqrt{\frac{2KE}{7}} & = \sqrt{\frac{2KE}{1}} \\
&= \frac{\sqrt{8}}{7} & = \frac{\sqrt{8}}{1} \\
&= 1.07 & = 2.83
\end{align*}
\]
Example with masses: \( m_1 = 49 \) and \( m_2 = 1 \)
Example with masses: $m_1=49$ and $m_2=1$
**Geometric “Integration” (Converting Velocity data to Space-time trajectory)**

**Example with masses: \( m_1=49 \) and \( m_2=1 \)**

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**BounceIt Superball Collision Web Simulator:**

\[ M_1=49, M_2=1 \] with Newtonian time plot

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**Fig. 4.1 (Unit 1)**

**BounceIt Superball Collision Web Simulator:**

\[ M_1=49, M_2=1 \] with \( V_2 \) vs \( V_1 \) plot
Example with masses: $m_1=49$ and $m_2=1$

**Kinetic Energy Ellipse**

$$ KE = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2 = \frac{49}{2} + \frac{1}{2} = 25 $$

$$ 1 = \frac{V_1^2}{2KE/m_1} + \frac{V_2^2}{2KE/m_2} = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} $$

**Ellipse radius 1**

$$ a_1 = \sqrt{2KE/m_1} = \sqrt{2\times25/49} = \sqrt{50/49} = 1.01 $$

**Ellipse radius 2**

$$ a_2 = \sqrt{2KE/m_2} = \sqrt{2\times25/1} = \sqrt{50/1} = 7.07 $$
Multiple collisions calculated by matrix operator products

Matrix or tensor algebra of 1-D 2-body collisions

“Mass-bang” matrix $\mathbf{M}$, “Floor-bang” matrix $\mathbf{F}$, “Ceiling-bang” matrix $\mathbf{C}$.

Geometry and algebra of “ellipse-Rotation” group product: $\mathbf{R} = \mathbf{C} \cdot \mathbf{M}$
Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give:

$$V_{COM} = \frac{V_{FIN} + V_{IN}}{2} = \frac{m_1v_1 + m_2v_2}{m_1 + m_2}$$
Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give:

\[ V_{\text{COM}} = \frac{V_{\text{FIN}} + V_{\text{IN}}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \]

Gives \( v_{\text{FIN}} \) in terms of \( v_{\text{IN}} \)...

\[
\begin{pmatrix}
  v_{1 \text{FIN}} \\
  v_{2 \text{FIN}}
\end{pmatrix} = \begin{pmatrix}
  2V_{\text{COM}} - v_{1 \text{IN}} \\
  2V_{\text{COM}} - v_{2 \text{IN}}
\end{pmatrix}
\]

Matrix operations include...

Floor bounce \( F \) of \( m_1 \): Mass collision \( M \) of \( m_1 \) and \( m_2 \): Ceiling bounce \( C \) of \( m_2 \):
Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give:

\[ V_{COM} = \frac{V_{FIN} + V_{IN}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \]

Gives \( v^{FIN} \) in terms of \( v^{IN} \):

\[
\begin{pmatrix}
  v_1^{FIN} \\
  v_2^{FIN}
\end{pmatrix}
= \begin{pmatrix}
  2V_{COM} - v_1^{IN} \\
  2V_{COM} - v_2^{IN}
\end{pmatrix}
= \begin{pmatrix}
  \frac{2}{m_1 + m_2} \left( 2 m_1 v_1^{IN} + m_2 v_2^{IN} - v_1^{IN} \right) \\
  \frac{2}{m_1 + m_2} \left( 2 m_1 v_1^{IN} + m_2 v_2^{IN} - v_2^{IN} \right)
\end{pmatrix}
\]
Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give:

\[ v^{\text{COM}} = \frac{v^{\text{FIN}} + v^{\text{IN}}}{2} = \frac{m_1v_1 + m_2v_2}{m_1 + m_2} \]

Gives \( v^{\text{FIN}} \) in terms of \( v^{\text{IN}} \)... Finally as a matrix operation:

\[
\begin{pmatrix}
    v_1^{\text{FIN}} \\
    v_2^{\text{FIN}}
\end{pmatrix}
= \left( \begin{array}{c}
    2v^{\text{COM}} - v_1^{\text{IN}} \\
    2v^{\text{COM}} - v_2^{\text{IN}}
\end{array} \right)
= \frac{2}{m_1 + m_2} \begin{pmatrix}
    m_1v_1^{\text{IN}} + m_2v_2^{\text{IN}} - v_1^{\text{IN}} \\
    m_1v_1^{\text{IN}} + m_2v_2^{\text{IN}} - v_2^{\text{IN}}
\end{pmatrix}
\]

\[
\begin{pmatrix}
    m_1 - m_2 \\
    2m_1 - m_2
\end{pmatrix}
\]

Define a "rotation" \( R \) as group product:

\[ R = C \cdot M \]

Let: \( m_1 = 49 \) and \( m_2 = 1 \)

\[ M = \begin{pmatrix}
    0.96 & 0.04 \\
    1.96 & -0.96
\end{pmatrix} \]

\[ v^{\text{FIN}} = M \cdot v^{\text{IN}} \]
Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give:

\[ v_{\text{COM}} = \frac{v^{\text{FIN}} + v^{\text{IN}}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \]

Gives \( v^{\text{FIN}} \) in terms of \( v^{\text{IN}} \):

\[
\begin{pmatrix}
  v_1^{\text{FIN}} \\
  v_2^{\text{FIN}} \\
\end{pmatrix} = \frac{2 v_{\text{COM}} - v_1^{\text{IN}}}{m_1 + m_2} - \frac{2 m_1 v_1^{\text{IN}} + m_2 v_2^{\text{IN}}}{m_1 + m_2} \]

Finally as a matrix operation:

\[ v^{\text{FIN}} = M \cdot v^{\text{IN}} \]

Quiz question about linear solution

Linear formula \( v^{\text{FIN}} = M \cdot v^{\text{IN}} \) gives just one solution to quadratic collision equations.
Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give:

\[ v_{\text{COM}} = \frac{V_{\text{FIN}} + V_{\text{IN}}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \]

Gives \( v_{\text{FIN}} \) in terms of \( v_{\text{IN}} \)...

\[
\begin{pmatrix}
  v_{1,\text{FIN}} \\
  v_{2,\text{FIN}}
\end{pmatrix}
= \begin{pmatrix}
  2V_{\text{COM}} - v_{1,\text{IN}} \\
  2V_{\text{COM}} - v_{2,\text{IN}}
\end{pmatrix}
= \begin{pmatrix}
  2 \frac{m_1 v_{1,\text{IN}} + m_2 v_{2,\text{IN}}}{m_1 + m_2} - v_{1,\text{IN}} \\
  2 \frac{m_1 v_{1,\text{IN}} + m_2 v_{2,\text{IN}}}{m_1 + m_2} - v_{2,\text{IN}}
\end{pmatrix}
= \begin{pmatrix}
  m_1 \frac{v_{1,\text{IN}} - m_2 v_{1,\text{IN}} + 2 m_2 v_{2,\text{IN}}}{m_1 + m_2} \\
  m_1 \frac{2 m_1 v_{1,\text{IN}} + m_2 v_{2,\text{IN}} - m_1 v_{2,\text{IN}}}{m_1 + m_2}
\end{pmatrix}
= \begin{pmatrix}
  m_1 - m_2 & 2 m_2 \\
  2 m_1 & m_2 - m_1
\end{pmatrix}
\begin{pmatrix}
  v_{1,\text{IN}} \\
  v_{2,\text{IN}}
\end{pmatrix}
\]

Finally as a matrix operation:

\[ v_{\text{FIN}} = M \cdot v_{\text{IN}} \]

Quiz question about linear solution

Linear formula \( v_{\text{FIN}} = M \cdot v_{\text{IN}} \) gives just one solution to quadratic collision equations.

Q: What is the second solution and to what simple process would it correspond?

Example with friction
Multiple collisions calculated by matrix operator products

Matrix or tensor algebra of 1-D 2-body collisions


Geometry and algebra of “ellipse-Rotation” group product: $R = C \cdot M$
Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give:

\[ V^{COM} = \frac{V^{FIN} + V^{IN}}{2} = \frac{m_1v_1 + m_2v_2}{m_1 + m_2} \]

Gives \( v^{FIN} \) in terms of \( v^{IN} \):

\[
\begin{pmatrix}
 v_1^{FIN} \\
 v_2^{FIN}
\end{pmatrix} = \begin{pmatrix}
 2V^{COM} - v_1^{IN} \\
 2V^{COM} - v_2^{IN}
\end{pmatrix} = \begin{pmatrix}
 2\frac{m_1v_1^{IN} + m_2v_2^{IN}}{m_1 + m_2} - v_1^{IN} \\
 2\frac{m_1v_1^{IN} + m_2v_2^{IN}}{m_1 + m_2} - v_2^{IN}
\end{pmatrix}
\]

Finally as a matrix operation:

\[ v^{FIN} = M \cdot v^{IN} \]

Matrix operations include...

Floor-bang \( F \) of \( m_1 \):

\[
F = \begin{pmatrix}
 -1 & 0 \\
 0 & 1
\end{pmatrix}
\]

Define a "rotation" \( R \) as group product:

\[ R = C \cdot M \]

Tuesday, February 2, 2016
Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give:

\[ v_{\text{COM}} = \frac{v_{\text{FIN}} + v_{\text{IN}}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \]

Gives \( v_{\text{FIN}} \) in terms of \( v_{\text{IN}} \):

\[
\begin{pmatrix}
v_1^{\text{FIN}} \\
v_2^{\text{FIN}}
\end{pmatrix}
= \begin{pmatrix}
2v_{\text{COM}} - v_1^{\text{IN}} \\
2v_{\text{COM}} - v_2^{\text{IN}}
\end{pmatrix}
= \begin{pmatrix}
\frac{2m_1 v_1^{\text{IN}} + m_2 v_2^{\text{IN}}}{m_1 + m_2} - v_1^{\text{IN}} \\
\frac{2m_1 v_1^{\text{IN}} + m_2 v_2^{\text{IN}}}{m_1 + m_2} - v_2^{\text{IN}}
\end{pmatrix}
\]

Finally as a matrix operation:

\[ v_{\text{FIN}} = M \cdot v_{\text{IN}} \]

Matrix operations include...

**Floor-bang** \( F \) of \( m_1 \):

\[
F = \begin{pmatrix}-1 & 0 \\ 0 & 1\end{pmatrix}
\]

**Ceiling-bang** \( C \) of \( m_2 \):

\[
C = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}
\]
Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give: 

\[ V_{\text{com}} = \frac{V_{\text{FIN}} + V_{\text{IN}}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \]

Gives \( V_{\text{FIN}} \) in terms of \( V_{\text{IN}} \)... 

\[
\begin{pmatrix}
  v_{1}^{\text{FIN}} \\
  v_{2}^{\text{FIN}} 
\end{pmatrix} = \begin{pmatrix}
  2V_{\text{COM}} - v_{1}^{\text{IN}} \\
  2V_{\text{COM}} - v_{2}^{\text{IN}} 
\end{pmatrix} = \begin{pmatrix}
  2 \frac{m_1 v_1^{\text{IN}} + m_2 v_2^{\text{IN}}}{m_1 + m_2} - v_{1}^{\text{IN}} \\
  2 \frac{m_1 v_1^{\text{IN}} + m_2 v_2^{\text{IN}}}{m_1 + m_2} - v_{2}^{\text{IN}} 
\end{pmatrix}
\]

Finally as a matrix operation: 

\[ v_{\text{FIN}} = M \cdot v_{\text{IN}} ... \]

Matrix operations include... 

Floor-bang \( F \) of \( m_1 \): 

\[ F = \begin{pmatrix}
  -1 & 0 \\
  0 & 1 
\end{pmatrix} \]

Mass-bang \( M \) of \( m_1 \) and \( m_2 \): 

\[ M = \begin{pmatrix}
  m_1 - m_2 & 2m_2 \\
  m_1 + m_2 & m_1 + m_2 \\
  2m_1 & m_2 - m_1 \\
  m_1 + m_2 & m_1 + m_2 
\end{pmatrix} \]

Ceiling-bang \( C \) of \( m_2 \): 

\[ C = \begin{pmatrix}
  1 & 0 \\
  0 & -1 
\end{pmatrix} \]
Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give:

\[ v_{\text{COM}} = \frac{v_{\text{FIN}} + v_{\text{IN}}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \]

Gives \( v_{\text{FIN}} \) in terms of \( v_{\text{IN}} \)... \[
\begin{pmatrix}
v_{1,\text{FIN}} \\
v_{2,\text{FIN}}
\end{pmatrix} =
\begin{pmatrix}
2v_{\text{COM}} - v_{1,\text{IN}} \\
2v_{\text{COM}} - v_{2,\text{IN}}
\end{pmatrix} =
\begin{pmatrix}
\frac{2m_1 v_{1,\text{IN}} + m_2 v_{2,\text{IN}}}{m_1 + m_2} - v_{1,\text{IN}} \\
\frac{2m_1 v_{1,\text{IN}} + m_2 v_{2,\text{IN}}}{m_1 + m_2} - v_{2,\text{IN}}
\end{pmatrix}
\]

Finally as a matrix operation:

\[ v_{\text{FIN}} = M \cdot v_{\text{IN}} \]

Matrix operations include...

**Floor-bang** \( F \) of \( m_1 \):

\[ F = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \]

**Mass-bang** \( M \) of \( m_1 \) and \( m_2 \):

\[ M = \begin{pmatrix} \frac{m_1 - m_2}{m_1 + m_2} & \frac{2m_2}{m_1 + m_2} \\ \frac{m_1}{m_1 + m_2} & \frac{m_2 - m_1}{m_1 + m_2} \end{pmatrix} \]

**Ceiling-bang** \( C \) of \( m_2 \):

\[ C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

Let: \( m_1 = 49 \) and \( m_2 = 1 \)

\[ M = \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \]
Multiple collisions calculated by matrix operator products

Matrix or tensor algebra of 1-D 2-body collisions
“Mass-bang” matrix $\mathbf{M}$, “Floor-bang” matrix $\mathbf{F}$, “Ceiling-bang” matrix $\mathbf{C}$.

Geometry and algebra of “ellipse-Rotation” group product: $\mathbf{R} = \mathbf{C} \cdot \mathbf{M}$
Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give:

\[ v_{\text{COM}} = \frac{v_{\text{FIN}} + v_{\text{IN}}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \]

Gives \( v_{\text{FIN}} \) in terms of \( v_{\text{IN}} \)... Finally as a matrix operation:

\[ v_{\text{FIN}} = M \cdot v_{\text{IN}} \]

Matrix operations include...

Floor-bang \( F \) of \( m_1 \):

\[
F = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
\]

Mass-bang \( M \) of \( m_1 \) and \( m_2 \):

\[
M = \begin{pmatrix} \frac{m_1 - m_2}{m_1 + m_2} & \frac{2m_2}{m_1 + m_2} \\ \frac{2m_1}{m_1 + m_2} & \frac{m_2 - m_1}{m_1 + m_2} \end{pmatrix}
\]

Ceiling-bang \( C \) of \( m_2 \):

\[
C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

Let: \( m_1 = 49 \) and \( m_2 = 1 \)

\[
M = \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}
\]

Define “ellipse-Rotation” \( R \) as group product:

\[
R = C \cdot M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} = \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}
\]
\[
\begin{pmatrix}
v_{1}^{FIN-9} \\
v_{2}^{FIN-9}
\end{pmatrix} =
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
0.96 & 0.4 \\
1.96 & -0.96
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
0.96 & 0.4 \\
1.96 & -0.96
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
0.96 & 0.4 \\
1.96 & -0.96
\end{pmatrix}
\begin{pmatrix}
-1 & 0 \\
0 & +1
\end{pmatrix}
\begin{pmatrix}
1 \\
0
\end{pmatrix}
\begin{pmatrix}
\left| IN^{0} \right> \\
\left| IN^{0} \right>
\end{pmatrix}
\left( \text{INITIAL (0)} \right)
\]
“ellipse-Rotation” group product: $R = C \cdot M$
\[
\begin{pmatrix}
\begin{bmatrix}
1 & 0 \\
0 & -1 \\
\end{bmatrix}
\end{pmatrix}
\begin{bmatrix}
0.96 & 0.04 \\
1.96 & -0.96 \\
\end{bmatrix}
\cdot
\begin{bmatrix}
1 & 0 \\
0 & -1 \\
\end{bmatrix}
\begin{bmatrix}
0.96 & 0.04 \\
1.96 & -0.96 \\
\end{bmatrix}
\cdot
\begin{bmatrix}
1 & 0 \\
0 & -1 \\
\end{bmatrix}
\begin{bmatrix}
0.96 & 0.04 \\
1.96 & -0.96 \\
\end{bmatrix}
\cdot
\begin{bmatrix}
1 & 0 \\
0 & -1 \\
\end{bmatrix}
\begin{bmatrix}
0.96 & 0.04 \\
1.96 & -0.96 \\
\end{bmatrix}
\cdot
\begin{bmatrix}
-1 & 0 \\
0 & +1 \\
\end{bmatrix}
\begin{bmatrix}
v_1^{IN} = -1 \\
v_2^{IN} = -1 \\
\end{bmatrix}
\end{pmatrix}
= 
\begin{pmatrix}
\begin{bmatrix}
v_1 = 0.2925 \\
v_2 = -6.768 \\
\end{bmatrix}
\end{pmatrix}
\text{(after Bang-9)}
\]

“ellipse-Rotation” group product: \( \mathbf{R} = \mathbf{C} \cdot \mathbf{M} \)
\[
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
0.96 & 0.04 \\
1.96 & -0.96
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
0.96 & 0.04 \\
1.96 & -0.96
\end{bmatrix}
= \begin{bmatrix}
-1 & 0 \\
0 & +1
\end{bmatrix}
\begin{bmatrix}
v_1 \in [-1] \\
v_2 \in [-1]
\end{bmatrix}
\text{(INITIAL)}
\]

\[
\begin{bmatrix}
v_1 = 0.2925 \\
v_2 = -6.768
\end{bmatrix}
\text{(after Bang-9)}
\]

“ellipse-Rotation” group product: \( \mathbf{R} = \mathbf{C} \cdot \mathbf{M} \)

Collisions for mass ratio \( m_1 : m_2 = 49 : 1 \)
\[
\begin{bmatrix}
F_{IN-9}^1 \\
F_{IN-9}^2
\end{bmatrix} =
\begin{bmatrix}
\mathbf{C} & \mathbf{M} & \mathbf{C} & \mathbf{M} & \mathbf{C} & \mathbf{M} & \mathbf{C} & \mathbf{M} & \mathbf{F}
\end{bmatrix} \cdot
\begin{bmatrix}
0.96 & 0.04 \\
1.96 & -0.96 \\
0.96 & 0.04 \\
1.96 & -0.96 \\
0.96 & 0.04 \\
1.96 & -0.96 \\
0.96 & 0.04 \\
1.96 & -0.96 \\
\end{bmatrix} \cdot
\begin{bmatrix}
-1 & 0 \\
0 & +1
\end{bmatrix}
\]

\[
\begin{cases}
v_1 = 0.2925 \\
v_2 = -6.768
\end{cases}
\]

\text{“ellipse-Rotation” group product: } \mathbf{R} = \mathbf{C} \cdot \mathbf{M}

\text{Collisions for mass ratio } m_1:m_2 = 49:1

\begin{align*}
\begin{bmatrix}
F_{IN-11}^1 \\
F_{IN-11}^2
\end{bmatrix} &=
\begin{bmatrix}
0.96 & 0.04 \\
-1.96 & 0.96
\end{bmatrix} \cdot
\begin{bmatrix}
F_{IN-9}^1 \\
F_{IN-9}^2
\end{bmatrix} \\
&=\begin{bmatrix}
v_1 = 0.0100 \\
v_2 = -7.071
\end{bmatrix}
\end{align*}

\text{BounceIt Superball Collision Web Simulator: } M_1=49, M_2=1 \text{ with Newtonian time plot}

\text{Fig. 4.1a-b (revised)}
Ellipse rescaling-geometry and reflection-symmetry analysis

Rescaling KE ellipse to circle

How this relates to Lagrangian, l’Etrangian, and Hamiltonian mechanics later on
Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: \( V_1 = v_1 \cdot \sqrt{m_1} \), \( V_2 = v_2 \cdot \sqrt{m_1} \), symmetrize: \( KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2 \)

Collisions for mass ratio \( m_1:m_2 = 49:1 \)
**Ellipse rescaling geometry and reflection symmetry analysis**

Convert to rescaled velocity: \( V_1 = v_1 \cdot \sqrt{m_1}, \quad V_2 = v_2 \cdot \sqrt{m_1}, \) symmetrize:

\[
KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2
\]

\[
\begin{pmatrix}
v_1^{FIN} \\ v_2^{FIN}
\end{pmatrix} = \frac{1}{M} \begin{pmatrix}
m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1
\end{pmatrix} \begin{pmatrix}
v_1 \\ v_2
\end{pmatrix}
\]

becomes:

\[
\begin{pmatrix}
v_1^{FIN} / \sqrt{m_1} \\ v_2^{FIN} / \sqrt{m_2}
\end{pmatrix} = \frac{1}{M} \begin{pmatrix}
m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1
\end{pmatrix} \begin{pmatrix}
v_1 / \sqrt{m_1} \\ v_2 / \sqrt{m_2}
\end{pmatrix}
\]

Collisions for mass ratio \( m_1 : m_2 = 49 : 1 \)
**Ellipse rescaling geometry and reflection symmetry analysis**

Convert to rescaled velocity: \( V_1 = v_1 \cdot \sqrt{m_1} \) \, , \, \( V_2 = v_2 \cdot \sqrt{m_1} \) \, , \, symmetrize: \( KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2 \)

or:

\[
\begin{pmatrix}
V_{1,\text{FIN}} \\
V_{2,\text{FIN}}
\end{pmatrix} = \frac{1}{M} \begin{pmatrix}
m_1 - m_2 & 2m_2 \\
2m_1 & m_2 - m_1
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2
\end{pmatrix}
\]

becomes:

\[
\begin{pmatrix}
V_{1,\text{FIN}} / \sqrt{m_1} \\
V_{2,\text{FIN}} / \sqrt{m_2}
\end{pmatrix} = \frac{1}{M} \begin{pmatrix}
m_1 - m_2 & 2m_2 \\
2m_1 & m_2 - m_1
\end{pmatrix}
\begin{pmatrix}
V_1 / \sqrt{m_1} \\
V_2 / \sqrt{m_2}
\end{pmatrix}
\]

Collisions for mass ratio \( m_1:m_2 = 49:1 \)

\[
\begin{pmatrix}
v_1 \\
v_2
\end{pmatrix} = \mathbf{M} \cdot \mathbf{\ddot{v}}
\]

\[
\begin{pmatrix}
V_1 / \sqrt{m_1} \\
V_2 / \sqrt{m_2}
\end{pmatrix} = \frac{1}{M} \begin{pmatrix}
m_1 - m_2 & 2\sqrt{m_1 m_2} \\
-2\sqrt{m_1 m_2} & m_1 - m_2
\end{pmatrix}
\begin{pmatrix}
V_1 \\
V_2
\end{pmatrix} = \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{\ddot{v}}
\]
Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: \( V_1 = v_1 \cdot \sqrt{m_1} \) , \( V_2 = v_2 \cdot \sqrt{m_1} \), symmetrize:\( KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2 \)

\[
\begin{bmatrix}
 v_{1}^{FIN} \\
 v_{2}^{FIN}
\end{bmatrix} = \frac{1}{M} \begin{bmatrix}
 m_1 - m_2 & 2m_2 \\
 2m_1 & m_2 - m_1 
\end{bmatrix} \begin{bmatrix}
 v_1 \\
 v_2
\end{bmatrix}
\]

becomes:

\[
\begin{bmatrix}
 v_{1}^{FIN} / \sqrt{m_1} \\
 v_{2}^{FIN} / \sqrt{m_2}
\end{bmatrix} = \frac{1}{M} \begin{bmatrix}
 m_1 - m_2 & 2m_2 \\
 2m_1 & m_2 - m_1 
\end{bmatrix} \begin{bmatrix}
 v_1 / \sqrt{m_1} \\
 v_2 / \sqrt{m_2}
\end{bmatrix}
\]

or:

\[
\begin{bmatrix}
 v_{1}^{FIN} \\
 v_{2}^{FIN}
\end{bmatrix} = \frac{1}{M} \begin{bmatrix}
 m_1 - m_2 & 2\sqrt{m_1 m_2} \\
 2\sqrt{m_1 m_2} & m_2 - m_1 
\end{bmatrix} \begin{bmatrix}
 v_1 \\
 v_2
\end{bmatrix}
\]

= \mathbf{M} \cdot \tilde{v} \text{, or:} \begin{bmatrix}
 v_{1}^{FIN} \\
 v_{2}^{FIN}
\end{bmatrix} = \frac{1}{M} \begin{bmatrix}
 m_1 - m_2 & 2\sqrt{m_1 m_2} \\
 -2\sqrt{m_1 m_2} & m_1 - m_2 
\end{bmatrix} \begin{bmatrix}
 v_1 \\
 v_2
\end{bmatrix} = \mathbf{C} \cdot \mathbf{M} \cdot \tilde{v}
\]

Then collisions become reflections \( \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix} \) and double-collisions become rotations \( \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \)

where:

\[ \cos \theta \equiv \left( \frac{m_1 - m_2}{m_1 + m_2} \right) \quad \text{and:} \quad \sin \theta \equiv \left( \frac{2\sqrt{m_1 m_2}}{m_1 + m_2} \right) \quad \text{with:} \quad \left( \frac{m_1 - m_2}{m_1 + m_2} \right)^2 + \left( \frac{2\sqrt{m_1 m_2}}{m_1 + m_2} \right)^2 = 1 \]

\[ \text{Collisions for mass ratio} \quad m_1:m_2 = 49:1 \]
Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: \( V_1 = v_1 \cdot \sqrt{m_1} \), \( V_2 = v_2 \cdot \sqrt{m_1} \), symmetrize: \( KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2 \)

\[
\begin{pmatrix}
    v_{1}^{FIN} \\
    v_{2}^{FIN}
\end{pmatrix} = \frac{1}{M} \begin{pmatrix}
    m_1 - m_2 & 2m_2 \\
    2m_1 & m_2 - m_1
\end{pmatrix} \begin{pmatrix}
    v_1 \\
    v_2
\end{pmatrix}
\]

becomes:

\[
\begin{pmatrix}
    v_{1}^{FIN} / \sqrt{m_1} \\
    v_{2}^{FIN} / \sqrt{m_2}
\end{pmatrix} = \frac{1}{M} \begin{pmatrix}
    m_1 - m_2 & 2m_2 \\
    2m_1 & m_2 - m_1
\end{pmatrix} \begin{pmatrix}
    v_1 / \sqrt{m_1} \\
    v_2 / \sqrt{m_2}
\end{pmatrix}
\]

or:

\[
\begin{pmatrix}
    v_{1}^{FIN} \\
    v_{2}^{FIN}
\end{pmatrix} = \frac{1}{M} \begin{pmatrix}
    m_1 - m_2 & 2\sqrt{m_1 m_2} \\
    2\sqrt{m_1 m_2} & m_2 - m_1
\end{pmatrix} \begin{pmatrix}
    v_1 \\
    v_2
\end{pmatrix} = M \cdot \vec{V}, \quad \text{or:} \quad \begin{pmatrix}
    v_{1}^{FIN} \\
    v_{2}^{FIN}
\end{pmatrix} = \frac{1}{M} \begin{pmatrix}
    m_1 - m_2 & 2\sqrt{m_1 m_2} \\
    -2\sqrt{m_1 m_2} & m_1 - m_2
\end{pmatrix} \begin{pmatrix}
    v_1 \\
    v_2
\end{pmatrix} = C \cdot M \cdot \vec{V}
\]

Then collisions become \textit{reflections} \( \begin{pmatrix}
    \cos \theta & \sin \theta \\
    \sin \theta & -\cos \theta
\end{pmatrix} \) and double-collisions become \textit{rotations} \( \begin{pmatrix}
    \cos \theta & \sin \theta \\
    -\sin \theta & \cos \theta
\end{pmatrix} \)

where:

\[
\cos \theta = \left( \frac{m_1 - m_2}{m_1 + m_2} \right) \quad \text{and:} \quad \sin \theta = \left( \frac{2\sqrt{m_1 m_2}}{m_1 + m_2} \right)
\]

with:

\[
\left( \frac{m_1 - m_2}{m_1 + m_2} \right)^2 + \left( \frac{2\sqrt{m_1 m_2}}{m_1 + m_2} \right)^2 = 1
\]

\[
\frac{m_1 - m_2}{m_1 + m_2} = \frac{48}{50}
\]

\[
\frac{2\sqrt{m_1 m_2}}{m_1 + m_2} = \frac{14}{50}
\]

\[
\theta = 16.26^\circ
\]

\[
\frac{m_1 - m_2}{m_1 + m_2} = 0.49
\]

\( m_1 : m_2 = 49 : 1 \)

Collisions for mass ratio \( m_1 : m_2 = 49 : 1 \)

Fig. 4.2a-c (revised)
Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: \( V_1 = v_1 \cdot \sqrt{m_1} \), \( V_2 = v_2 \cdot \sqrt{m_1} \), symmetrize: \( KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2 \)

\[
\begin{bmatrix}
v_1^{FIN} \\
v_2^{FIN}
\end{bmatrix} = \frac{1}{M} \begin{bmatrix}
m_1 - m_2 & 2m_2 \\
2m_1 & m_2 - m_1
\end{bmatrix} \begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
\]

becomes:

\[
\begin{bmatrix}
v_1^{FIN} / \sqrt{m_1} \\
v_2^{FIN} / \sqrt{m_2}
\end{bmatrix} = \frac{1}{M} \begin{bmatrix}
m_1 - m_2 & 2m_2 \\
2m_1 & m_2 - m_1
\end{bmatrix} \begin{bmatrix}
v_1 / \sqrt{m_1} \\
v_2 / \sqrt{m_2}
\end{bmatrix}
\]

or:

\[
\begin{bmatrix}
v_1^{FIN} \\
v_2^{FIN}
\end{bmatrix} = \frac{1}{M} \begin{bmatrix}
m_1 - m_2 & 2\sqrt{m_1 m_2} \\
2\sqrt{m_1 m_2} & m_2 - m_1
\end{bmatrix} \begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} = M \cdot \vec{v}, \quad \text{or:} \quad \begin{bmatrix}
v_1^{FIN} \\
v_2^{FIN}
\end{bmatrix} = \frac{1}{M} \begin{bmatrix}
m_1 - m_2 & 2\sqrt{m_1 m_2} \\
-2\sqrt{m_1 m_2} & m_1 - m_2
\end{bmatrix} \begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} = C \cdot M \cdot \vec{v}
\]

Then collisions become reflections \( \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix} \) and double-collisions become rotations \( \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \)

where:

\[
\cos \theta \equiv \left( \frac{m_1 - m_2}{m_1 + m_2} \right) \quad \text{and:} \quad \sin \theta \equiv \left( \frac{2\sqrt{m_1 m_2}}{m_1 + m_2} \right)
\]

with:

\[
\left( \frac{m_1 - m_2}{m_1 + m_2} \right)^2 + \left( \frac{2\sqrt{m_1 m_2}}{m_1 + m_2} \right)^2 = 1
\]

\[
\frac{m_1 - m_2}{m_1 + m_2} = \frac{48}{50} \quad \text{and} \quad \frac{2\sqrt{m_1 m_2}}{m_1 + m_2} = \frac{14}{50}
\]

\[\theta = 16.26^\circ\]

Collisions for mass ratio \( m_1 : m_2 = 49 : 1 \)

Fig. 4.2a-c (revised)

**Note:** If \( m_1 \cdot m_2 \) is perfect-square, then \( \theta \)-triangle is rational \( (3^2 + 4^2 = 5^2, \text{etc.)} \)

Tuesday, February 2, 2016
Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: \[ V_1 = v_1 \cdot \sqrt{m_1}, \quad V_2 = v_2 \cdot \sqrt{m_1}, \] symmetrize: \[ KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2 \]

Then collisions become *reflections* \( \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \) and double-collisions become *rotations* where:

\[
\cos \theta \equiv \left( \frac{m_1 - m_2}{m_1 + m_2} \right) \quad \text{and} \quad \sin \theta \equiv \left( \frac{2\sqrt{m_1 m_2}}{m_1 + m_2} \right)
\]

with:

\[
\left( \frac{m_1 - m_2}{m_1 + m_2} \right)^2 + \left( \frac{2\sqrt{m_1 m_2}}{m_1 + m_2} \right)^2 = 1
\]
\[ \theta = 16.26^\circ \]

\[ \frac{m_1 - m_2}{m_1 + m_2} = \frac{48}{50} \]

\[ \frac{2\sqrt{m_1 m_2}}{m_1 + m_2} = \frac{14}{50} \]

\[ m_1 - m_2 = \sqrt{m_1} \sqrt{m_2} = -0.49 \]

\[ \frac{\sqrt{m_2}}{\sqrt{m_1}} = \frac{1}{7} \]

\[ \text{slope: } \frac{\sqrt{m_2}}{\sqrt{m_1}} = \frac{1}{7} \]

\[ \text{slope: } \frac{\sqrt{m_2}}{\sqrt{m_1}} = -\frac{1}{7} \]

Fig. 4.2a-c (revised)
**Ellipse rescaling-geometry and reflection-symmetry analysis**

Rescaling KE ellipse to circle

How this relates to Lagrangian, l’Etrangian, and Hamiltonian mechanics later on

Reflections in the clothing store: “It’s all done with mirrors!”

Introducing hexagonal symmetry $D_6\sim C_{6v}$ (Resulting for $m_1/m_2=3$)

Group multiplication and product table

Classical collision paths with $D_6\sim C_{6v}$ (Resulting from $m_1/m_2=3$)

Other not-so-symmetric examples: $m_1/m_2=4$ and $m_1/m_2=7$
What ellipse rescaling leads to...

How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics later on

(a) Lagrangian \( L = L(v_1, v_2) \)

Collision line and COM tangent slope \( = -\frac{m_1}{m_2} = -16 \)

COM Bisector slope \( = \frac{1}{1} \)

velocity \( v_1 \) rescaled to momentum: \( p_1 = m_1 v_1 \)

velocity \( v_2 \) rescaled to momentum: \( p_2 = m_2 v_2 \)

(b) Hamiltonian \( H = H(p_1, p_2) \)

COM Bisector slope \( = \frac{m_2}{m_1} = 1/16 \)

Collision line and COM tangent slope \( = -\frac{1}{1} \)

\( p_1 = m_1 v_1 \)

\( p_2 = m_2 v_2 \)
What ellipse rescaling leads to...

How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics later on

(a) Lagrangian \( L = L(v_1, v_2) \)

Collision line and COM tangent slope
\[ \text{slope} = \frac{-m_1}{m_2} = -16 \]

COM Bisector
\[ \text{slope} = 1/1 \]

(b) l'Etrangian \( E = E(V_1, V_2) \)

velocity \( v_1 \) rescaled to momentum: \( p_1 = m_1 v_1 \)
velocity \( v_2 \) rescaled to momentum: \( p_2 = m_2 v_2 \)

Lagrangian \( L(v_1, v_2) = KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \)
rescaled to

Hamiltonian \( H(p_1, p_2) = KE = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} \)

(c) Hamiltonian \( H = H(p_1, p_2) \)

Collision line and COM tangent slope
\[ \text{slope} = \frac{m_2}{m_1} = 1/16 \]

COM Bisector slope
\[ \text{slope} = -1/1 \]

\[ p_1 = m_1 v_1 \]
\[ p_2 = m_2 v_2 \]
What ellipse rescaling leads to...

How this relates to Lagrangian, l’Etrangian, and Hamiltonian mechanics later on

(a) Lagrangian \( L = L(v_1, v_2) \)

Collision line and COM tangent slope
\( \frac{-m_1}{m_2} = -16 \)

COM Bisector slope
\( \sqrt{\frac{m_1}{m_2}} = 4 \)

(b) Estrangian \( E = E(V_1, V_2) \)

Collision line and COM tangent slope
\( \frac{-\sqrt{m_1}}{\sqrt{m_2}} = -4 \)

COM Bisector slope
\( \sqrt{\frac{m_2}{m_1}} = 1/4 \)

(c) Hamiltonian \( H = H(p_1, p_2) \)

Collision line and COM tangent slope
\( \frac{m_2}{m_1} = 1/16 \)

COM Bisector slope
\( \sqrt{\frac{m_2}{m_1}} = 1/4 \)

Fig. 10.1? (Unit 1)
Ellipse rescaling-geometry and reflection-symmetry analysis

- Rescaling KE ellipse to circle
- How this relates to Lagrangian, l’Etrangian, and Hamiltonian mechanics later on

Reflections in the clothing store: “It’s all done with mirrors!”

- Introducing hexagonal symmetry $D_6 \sim C_{6v}$ (Resulting for $m_1/m_2 = 3$)
- Group multiplication and product table
- Classical collision paths with $D_6 \sim C_{6v}$ (Resulting from $m_1/m_2 = 3$)
- Other not-so-symmetric examples: $m_1/m_2 = 4$ and $m_1/m_2 = 7$
Reflections in clothing store mirrors

Fig. 5.4a-b
Symmetry: It’s all done with mirrors!

(a) Reflections $\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $-\sigma_A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$x = \sigma_A \cdot x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$-y = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$= \sigma_A \cdot y$

(b) Reflections $\sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $-\sigma_B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$x = \sigma_B \cdot x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$-x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= -\sigma_B \cdot x$

$y = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$= -\sigma_B \cdot y$

$-y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$= -\sigma_B \cdot x$

Fig. 4.3 (Unit 1)
Symmetry: It’s all done with mirrors!

(a) Reflections \( \sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \), \( -\sigma_A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \)

\[
\begin{align*}
y &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
x &= \sigma_A \cdot x = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
-\sigma_A \cdot y &= \begin{pmatrix} 0 \\ -1 \end{pmatrix}
\end{align*}
\]

(c) \( \sigma_\phi \) reflection

of \( x \)-vector:

\[
\sigma_\phi \cdot x = \begin{pmatrix} \cos \phi & \sin \phi \\ \sin \phi & -\cos \phi \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos \phi \cdot x_1 + \sin \phi \cdot x_2 \\ \sin \phi \cdot x_1 - \cos \phi \cdot x_2 \end{pmatrix}
\]

(b) Reflections \( \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( -\sigma_B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \)

\[
\begin{align*}
y &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
x &= \sigma_B \cdot x = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
-\sigma_B \cdot y &= \begin{pmatrix} 0 \\ -1 \end{pmatrix}
\end{align*}
\]

...of \( y \)-vector:

\[
\sigma_\phi \cdot y = \begin{pmatrix} \sin \phi \\ -\cos \phi \end{pmatrix}
\]

Fig. 4.3 (Unit 1)
Symmetry: It’s all done with mirrors!

(a) Reflections $\sigma_A = (1, 0)$, $-\sigma_A = (0, 1)$

$$\begin{align*}
y &= (0) \\
x &= \sigma_A \cdot x = (0) \\
y &= -\sigma_A \cdot y
\end{align*}$$

(b) Reflections $\sigma_B = (0, 1)$, $-\sigma_B = (1, 0)$

$$\begin{align*}
y &= (0) \\
x &= \sigma_B \cdot x = (1) \\
y &= -\sigma_B \cdot y
\end{align*}$$

(c) $\sigma_\phi$ reflection

$$\begin{align*}
\sigma_\phi \cdot x &= \left( \begin{array}{cc} \cos \phi & \sin \phi \\ \sin \phi & -\cos \phi \end{array} \right) \cdot x \\
\sigma_\phi \cdot y &= \left( \begin{array}{cc} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{array} \right) \cdot y
\end{align*}$$

(d) Rotation: $R_+\phi = \sigma_\phi \sigma_A = \left( \begin{array}{cc} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{array} \right)$

(e) Rotation: $R_-\phi = \sigma_A \sigma_\phi = \left( \begin{array}{cc} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{array} \right)$

[Fig. 4.3 (Unit 1)]
Why reflections underlie all symmetry analyses

They work in 1D, 2D, 3D, ... , ND

Product of odd number of reflections is a reflection

... even number of reflections is a rotation (or unit-op 1)

Product of rotations just give rotations

Classical objects are semi-rigid and rotate easily
Waves patterns are non-rigid and reflect easily
Why reflections underlie all symmetry analyses

They work in 1D, 2D, 3D,......,ND

Product of odd number of reflections is a reflection

... even number of reflections is a rotation (or unit-op 1)

Product of rotations just give rotations

Classical objects are semi-rigid and rotate easily
Waves patterns are non-rigid and reflect easily

.. wave reflections underlie modern physics
Ellipse rescaling-geometry and reflection-symmetry analysis

Rescaling KE ellipse to circle
How this relates to Lagrangian, l’Etrangian, and Hamiltonian mechanics later on

Reflections in the clothing store: “It’s all done with mirrors!”

Introducing hexagonal symmetry \( D_6 \sim C_{6v} \) (Resulting for \( m_1/m_2 = 3 \))
Group multiplication and product table
Classical collision paths with \( D_6 \sim C_{6v} \) (Resulting from \( m_1/m_2 = 3 \))
Other not-so-symmetric examples: \( m_1/m_2 = 4 \) and \( m_1/m_2 = 7 \)
Introducing Symmetry Operators

To make KE ellipses into circles l’Etrangian plot reduces $v_1$ scale by $1/\sqrt{m_1}$, etc.

Here:

$1/\sqrt{m_1} = 1/\sqrt{3} = 0.577$

$1/\sqrt{m_2} = 1/\sqrt{1} = 1.0$

$V_1 = v_1 \sqrt{m_1} = v_1 \sqrt{3}$

Collisions for mass ratio $m_1:m_2 = 3:1$
Introducing Symmetry Operators

To make KE ellipses into circles l’Etrangian plot reduces \( v_1 \) scale by \( 1/\sqrt{m_1} \), etc.

Here:

\[
1/\sqrt{m_1} = 1/\sqrt{3} = 0.577
\]
\[
1/\sqrt{m_2} = 1/\sqrt{1} = 1.0
\]

\( v_1 = V_1/\sqrt{m_1} = V_1\sqrt{3} \)

\[
\cos \theta = \frac{m_1 - m_2}{m_1 + m_2} = \frac{m_2}{m_1 + 1} = \frac{2}{4} = \frac{1}{2}
\]

Collisions for mass ratio \( m_1 : m_2 = 3:1 \)

\[ \begin{align*}
\theta &= 60° \\
\cos \theta &= \frac{m_1 - m_2}{m_1 + m_2} = \frac{m_2}{m_1 + 1} = \frac{2}{4} = \frac{1}{2}
\end{align*} \]
Introducing Symmetry Operators

Collisions for mass ratio $m_1:m_2 = 3:1$

To make KE ellipses into circles l’Etrangian plot reduces $v_1$ scale by $1/\sqrt{m_1}$, etc.

Here:

\[ \frac{1}{\sqrt{m_1}} = \frac{1}{\sqrt{3}} = 0.577 \]
\[ \frac{1}{\sqrt{m_2}} = \frac{1}{\sqrt{1}} = 1.0 \]

\[ V_1 = v_1 \sqrt{m_1} = v_1 \sqrt{3} \]

\[ \cos \theta = \frac{m_1 - m_2}{m_1 + m_2} = \frac{m_1 - 1}{m_1 + 1} = \frac{2}{4} = \frac{1}{2} \]

\[ \frac{m_1}{m_2} = \frac{1 + \cos \theta}{1 - \cos \theta} = \frac{3}{2} = 3 \]

\[ \theta = 60^\circ \]
Introducing Symmetry Operators

To make KE ellipses into circles l’Etrangian plot reduces $v_1$ scale by $1/\sqrt{m_1}$, etc.

Here:

$1/\sqrt{m_1} = 1/\sqrt{3} = 0.577$

$1/\sqrt{m_2} = 1/\sqrt{1} = 1.0$

$V_1 = v_1 \sqrt{m_1} = v_1 \sqrt{3}$

$m_1 = \frac{1 + \cos \theta}{1 - \cos \theta} = \frac{\sin^2 \alpha}{\cos^2 \alpha}$

$\alpha = \theta/2$

$\cos \theta = \frac{m_1 - m_2}{m_1 + m_2} = \frac{m_1 - 1}{m_2} = \frac{2}{4} = \frac{1}{2}$

$\theta = 60^\circ$

Collisions for mass ratio $m_1:m_2 = 3:1$
Collisions for mass ratio $m_1:m_2 = 3:1$

Reduce $v_1$ scale by $1/\sqrt{m_1} = 1/\sqrt{3} = 0.577$

$m_1/m_2=(3)/(1)$

"Generic" initial velocity $(v_1=1.0, v_2=0.1)$

"Symmetric" initial velocity $(v_1=1, v_2=0)$ or $(v_1=1, v_2=-1)$

$KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2$
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Other not-so-symmetric examples: $m_1/m_2 = 4$ and $m_1/m_2 = 7$
Effects of Ceiling Bang Matrix \( \mathbf{C} = \sigma_z \mathbf{e} = \begin{pmatrix} e_1 \cdot \mathbf{C} \cdot e_1 & e_1 \cdot \mathbf{C} \cdot e_2 \\ e_2 \cdot \mathbf{C} \cdot e_1 & e_2 \cdot \mathbf{C} \cdot e_2 \end{pmatrix} = \begin{pmatrix} e_1 \cdot \bar{e}_1 & e_1 \cdot \bar{e}_2 \\ e_2 \cdot \bar{e}_1 & e_2 \cdot \bar{e}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \).
Effects of Ceiling Bang Matrix

\[ C = \sigma_z = \begin{pmatrix} e_1 \cdot C \cdot e_1 & e_1 \cdot C \cdot e_2 \\ e_2 \cdot C \cdot e_1 & e_2 \cdot C \cdot e_2 \end{pmatrix} = \begin{pmatrix} e_1 \cdot e_1 & e_1 \cdot e_2 \\ e_2 \cdot e_1 & e_2 \cdot e_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

\[ e_1 = \sigma_z \cdot e_1 = e_1 \]

\[ e_2 = C \cdot e_2 = \sigma_z \cdot e_2 = -e_2 \]

Known as matrix elements or components

Known as relative direction cosines

Effects of Mass Bang Matrix

\[ M = \sigma_{30^\circ} = \begin{pmatrix} e \cdot M \cdot e_1 & e \cdot M \cdot e_2 \\ e_2 \cdot M \cdot e_1 & e_2 \cdot M \cdot e_2 \end{pmatrix} = \begin{pmatrix} e_1 \cdot e_1 & e_1 \cdot e_2 \\ e_2 \cdot e_1 & e_2 \cdot e_2 \end{pmatrix} = \begin{pmatrix} \cos 60^\circ & \sin 60^\circ \\ \sin 60^\circ & -\cos 60^\circ \end{pmatrix} \]

\[ e_1 = \sigma_{30^\circ} \cdot e_1 \]

\[ e_2 = M \cdot e_2 = \sigma_{30^\circ} \cdot e_2 \]

Known as matrix elements or components

Known as relative direction cosines
Effects of Floor Bang Matrix

\[
\mathbf{F} = -\sigma_z = \begin{pmatrix} e_1 \cdot \mathbf{F} \cdot e_1 & e_1 \cdot \mathbf{F} \cdot e_2 \\ e_2 \cdot \mathbf{F} \cdot e_1 & e_2 \cdot \mathbf{F} \cdot e_2 \end{pmatrix} = \begin{pmatrix} e_1 \cdot \mathbf{e}_1 & e_1 \cdot \mathbf{e}_2 \\ e_2 \cdot \mathbf{e}_1 & e_2 \cdot \mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[\mathbf{e}_2 = \mathbf{F} \cdot \mathbf{e}_2 = -\sigma_z \cdot \mathbf{e}_2 = +\mathbf{e}_2\]

\[\mathbf{e}_1 = \mathbf{F} \cdot \mathbf{e}_1 = -\sigma_z \cdot \mathbf{e}_1 = -\mathbf{e}_1\]

-\(\sigma_z\)-reflected state

\[|F\rangle = \mathbf{F} |1\rangle\]

Generic Initial state
Effects of Floor Bang Matrix

\[
F = -\sigma_z = \begin{pmatrix} e_1 \cdot F \cdot e_1 & e_1 \cdot F \cdot e_2 \\ e_2 \cdot F \cdot e_1 & e_2 \cdot F \cdot e_2 \end{pmatrix} = \begin{pmatrix} e_1 \cdot e_1 & e_1 \cdot e_2 \\ e_2 \cdot e_1 & e_2 \cdot e_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
\]

Initial state

\[e_1 = \bar{e}_1 = \begin{pmatrix} e_1 \cdot e_1 \\ e_2 \cdot e_1 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_1 \end{pmatrix}
\]

- \(\sigma_z\)-reflected state

\[|F\rangle = F|1\rangle\]

Generic Initial state

Effects of Ceiling \(C\) after Bang \(M\):

\[r_{-60^\circ} = C \cdot M = \sigma_z \cdot \sigma_{30^\circ}\]

\[e_2 = \bar{e}_2 = \begin{pmatrix} e_2 \\ -e_2 \end{pmatrix} = \begin{pmatrix} e_2 \\ -e_2 \end{pmatrix}
\]

\[e_1 = \bar{e}_1 = \begin{pmatrix} e_1 \\ -e_1 \end{pmatrix} = \begin{pmatrix} e_1 \\ -e_1 \end{pmatrix}
\]

\(\sigma_{30^\circ}\)-reflected state

\[|M\rangle = M|1\rangle\]

Generic Initial state

\[|r_{-60^\circ}\rangle = C \cdot M|1\rangle = r_{-60^\circ}|1\rangle\]

\(\sigma_{30^\circ}\ \sigma_{30^\circ}\)-reflected state

is a \(r_{-60^\circ}\)-rotated state
Ellipse rescaling-geometry and reflection-symmetry analysis

Rescaling KE ellipse to circle
How this relates to Lagrangian, l’Etrangian, and Hamiltonian mechanics later on

Reflections in the clothing store: “It’s all done with mirrors!”

Introducing hexagonal symmetry $D_6\sim C_{6v}$ (Resulting for $m_1/m_2=3$)

Group multiplication and product table

Classical collision paths with $D_6\sim C_{6v}$ (Resulting from $m_1/m_2=3$)

Other not-so-symmetric examples: $m_1/m_2=4$ and $m_1/m_2=7$
<table>
<thead>
<tr>
<th>$D_6$</th>
<th>1</th>
<th>$r_{120}$</th>
<th>$\bar{r}_{120}$</th>
<th>$\sigma_{60}$</th>
<th>$\bar{\sigma}_{60}$</th>
<th>$\sigma_z$</th>
<th>$I$</th>
<th>$\bar{r}_{60}$</th>
<th>$r_{60}$</th>
<th>$\sigma_{30}$</th>
<th>$\bar{\sigma}_{30}$</th>
<th>$\bar{\sigma}_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Easy to make hexagonal ($D_6$) symmetry group table:

**Example 1:** Find $\sigma_{30^\circ} \cdot \sigma_{-60^\circ} = \_\_\_\_\_\_\_\_\_?$

**Solution:** Find $\sigma_{30^\circ}$-plane and state- $|\sigma_{-60^\circ}\rangle$

Operate former on latter to get: $\sigma_{30^\circ} |\sigma_{-60^\circ}\rangle = |I\rangle$

That gives answer: $\sigma_{30^\circ} \cdot \sigma_{-60^\circ} = |I\rangle$.

Rest of $\sigma_{30^\circ}$ row follows:

<table>
<thead>
<tr>
<th>$\tau_{30^\circ}$-row</th>
<th>1</th>
<th>$r_{30^\circ}$</th>
<th>$\bar{r}_{30^\circ}$</th>
<th>$\sigma_{30^\circ}$</th>
<th>$\bar{\sigma}_{30^\circ}$</th>
<th>$\sigma_z$</th>
<th>$I$</th>
<th>$\bar{r}_{60}$</th>
<th>$r_{60}$</th>
<th>$\sigma_{30}$</th>
<th>$\bar{\sigma}_{30}$</th>
<th>$\bar{\sigma}_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_{30^\circ}$</td>
<td>1</td>
<td>$\sigma_{30^\circ}$</td>
<td>$\sigma_{30^\circ}$</td>
<td>$\sigma_z$</td>
<td>$I$</td>
<td>$\bar{r}_{60}$</td>
<td>$r_{60}$</td>
<td>$\sigma_{30}$</td>
<td>$\bar{\sigma}_{30}$</td>
<td>$\sigma_z$</td>
<td>$I$</td>
<td>$\bar{r}_{120}$</td>
</tr>
</tbody>
</table>

**Example 2:** Find $r_{60^\circ} \cdot \sigma_{-60^\circ} = \_\_\_\_\_\_\_\_\_?$

**Solution:** Do $r_{60^\circ}$-rotation $r_{60^\circ} |\sigma_{-60^\circ}\rangle = |\sigma_{-30^\circ}\rangle$

That gives answer: $r_{60^\circ} \cdot \sigma_{-60^\circ} = |\sigma_{-30^\circ}\rangle$
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Collisions for mass ratio $m_1:m_2 = 3:1$

“Generic” initial velocity
$(v_1=1.0, v_2=0.1)$

“Symmetric” initial velocity
$(v_1=1, v_2=0)$ or $(v_1=1, v_2=-1)$

Corresponding space-space $(y_1, y_2)$ paths
Scaled $y$ down by $\frac{1}{\sqrt{3}} = 0.577$

.. or could have scaled $x$ up by $\sqrt{3} = 1.732$

Collisions for mass ratio $m_1:m_2 = 3:1$

Space-space $(y_1, y_2)$ paths

Space-space $(y_1, y_2)$ paths scaled down by $\frac{1}{\sqrt{3}}$...

..and reflected...

..and transformed by the rest of $D_6$...
Collisions for
mass ratio
$m_1:m_2 = 3:1$

...they’re just straight lines going forever.

and translated by space-group $D_6$...
Initial velocity $v_1=1$, $v_2=0$

$M_1/m_2=2/1$

$\phi = \arccos(M_1-m_2)/(M_1+m_2) = \arccos(1/3) = 70.53^\circ$

Initial velocity $v_1=1$, $v_2=1$

Initial velocity $v_1=0$, $v_2=1$

$Collisions for mass ratio\ m_1:m_2=3:1$

$\phi = 60^\circ = \arccos(1/2)$

$\phi = 53.13^\circ = \arccos(3/5)$
**Ellipse rescaling-geometry and reflection-symmetry analysis**

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Other not-so-symmetric examples: $m_1/m_2=4$ and $m_1/m_2=7$
Kinetic Energy Ellipse

\[ KE = \frac{1}{2} M_1 v_1^2 + \frac{1}{2} M_2 v_2^2 = \frac{7}{2} \frac{1}{2} = 4 \]

\[ 1 = \frac{v_1^2}{2 KE / M_1} + \frac{v_2^2}{2 KE / M_2} = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} \]

Ellipse radius 1

\[ a_1 = \sqrt{2 KE / M_1} = \sqrt{2 KE / 7} = \sqrt{8 / 7} = 1.07 \]

Ellipse radius 2

\[ a_2 = \sqrt{2 KE / M_1} = \sqrt{2 KE / 1} = \sqrt{8 / 1} = 2.83 \]

Fig. 4.7a-d in Unit 1

Collisions for mass ratio

\[ m_1 : m_2 = 7 : 1 \]
Collisions for mass ratio $m_1:m_2 = 7:1$

Step 2: Extend $v(2)$ line to ceiling $y(3) = (?, 7, 1)$ and draw Bang-3(20) velocity $v(3) = (1, 1)$ line. (Find $v(3)$ using V-V plot.)

Step 3: Extend $v(3)$ line to collision point $y(4) = (?, ?)$ and draw Bang-4(12) velocity $v(4) = (0, 5, 2.5)$. (Find $v(4)$ using V-V plot.)

Step 4: Extend $v(4)$ line to ceiling point $y(5) = (?, 7, 1)$ and draw Bang-5(20) velocity $v(5) = (1, -1)$ line. (Find $v(5)$ using V-V plot.)

Step 5: Extend $v(5)$ line to collision point $y(6) = (?, ?)$ and draw Bang-6(12) velocity $v(6) = (0, 5, 2.5)$. (Find $v(6)$ using V-V plot.)

(Dash lines show velocity if $m_1$ does 2nd bounce)

$v(7)$ only possible if floor is penetrated by $m_1$...

... and later $m_2$ penetrates the floor, too...

“Gameover collision” occurs way down here!

$y(8)$

$y_1$-axis

$y_2$-axis
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Other not-so-symmetric examples: $m_1/m_2=4$ and $m_1/m_2=7$
First part of Exercise 1.4.1 has pen-ball initial values \( v_1(0) = -1 = v_2(0) \)

Collisions for mass ratio \( m_1 : m_2 = 4 : 1 \)
Collisions for mass ratio $m_1:m_2 = 4:1$
The $V_1$-$V_2$-path for this problem repeats steps 0 to 12, (more or less!)

2nd part of Exercise 1.4.1 has pen-ball initial velocity values $v_1(0)=1$ and $v_2(0)=0$

at: $x_1(0)=1.5$ and $x_2(0)=3.0$

Collisions for mass ratio $m_1:m_2=4:1$
The $V_1$-$V_2$-path for this problem repeats steps 0 to 12 (more or less!)

$Ceiling (y_2 = 7.0)$

Simulations by BounceIt
The $V_1, V_2$-path for this problem repeats steps 0 to 12 (more or less!).

Ceiling ($y_2 = 7.0$)

Start

Floor ($y_1 = 0$)

0,13,...

11,24,...

6,32,...

12,25,...

5,18,...

19,45,...

0

Simulations by BounceIt
The $v_1$-$v_2$-path for this problem repeats steps 0 to 12 (more or less!)

The $v_1$-$v_2$-path for this problem repeats steps 0 to 12 (more or less!)

The $v_1$-$v_2$-path for this problem repeats steps 0 to 12 (more or less!)
The $v_1$-$v_2$-path for this problem repeats steps 0 to 12 (more or less!)

The diagrams illustrate the classical no-man's land and simulations by BounceIt.
The $V_1$-$V_2$-path for this problem repeats steps 0 to 12 (more or less!)

$\begin{align*}
\text{Ceiling (} y_2 = 7.0 \text{)} & \quad \text{Simulations by BounceIt}
\end{align*}$
The $v_1$-$v_2$-path for this problem repeats steps 0 to 12 (more or less!).

- **Ceiling** ($v_2 = 7.0$)
  - Start 0
  - Floor ($y_1 = 0$)
  - $y_1$ axis ($y_2 = 0$)

- **Classical No-Man's Land**
  - Simulations by BounceIt

- **No-**Ma'n"s Land
  - Classical No-Man's Land

- **Simulations by BounceIt**

- **Start**

- **Floor ($y_1 = 0$)**
  - **Ceiling ($y_2 = 7.0$)**

- **$v_1$ axis ($v_2 = 0$)**
The $V_1$-$V_2$-path for this problem repeats steps 0 to 12 (more or less!)

$\text{Ceiling } (y_2 = 7.0)$

$\text{Floor } (y_1 = 0)$

Simulations by BounceIt

Tuesday, February 2, 2016
The $V_1$-$V_2$-path for this problem repeats steps 0 to 12 (more or less!).

Ceiling ($v_2 = 7.0$)

Classical No-Man’s Land

Simulations by BounceIt
The $V_1$-$V_2$-path for this problem repeats steps 0 to 12 (more or less!)

$\text{Floor (}y_1 = 0\text{)}$

$\text{Ceiling (}y_2 = 7.0\text{)}$

Simulations by BounceIt
The $V_1$-$V_2$-path for this problem repeats steps 0 to 12 (more or less!)

$Ceiling (y_2 = 7.0)$

$Floor (y_1 = 0)$

Simulations by BounceIt

Classical No-Man's Land
The \( V_1, V_2 \)-path for this problem repeats steps 0 to 12 (more or less!)
The $V_1,V_2$-path for this problem repeats steps 0 to 12 (more or less)

Ceiling ($y_2 = 7.0$)

Simulations by BounceIt
The $V_1$-$V_2$-path for this problem repeats steps 0 to 12 (more or less!).

Ceiling ($y_2 = 7.0$)

Classical
No-Man's
Land

Simulations by BounceIt
The $V_1$-$V_2$-path for this problem repeats steps 0 to 12 (more or less!)}
The \( V_1-V_2 \)-path for this problem repeats steps 0 to 12 (more or less!)

**Ceiling \( (y_2 = 7.0) \)**

**Classical No-Man's Land**

Simulations by BounceIt
The $V_1$-$V_2$-path for this problem repeats steps 0 to 12 (more or less!)

Ceiling ($y_2 = 7.0$)

Floor ($y_1 = 0$)

Simulations by BounceIt

Classical No-Man’s Land

Classical No-Man’s Land
The $V_1-V_2$-path for this problem repeats steps 0 to 12 (more or less!)

Ceiling ($y_2 = 7.0$)

Simulations by BounceIt

Classical No-Man's Land
The \( y_2 \)-path for this problem repeats steps 0 to 12 (or less!) repeatedly.

Simulations by Bennett

Floor \((y_1 = 0)\)

Ceiling \((y_2 = 0)\)
The $V_1-V_2$-path for this problem repeats steps 0 to 12 (more or less!)

$V_1$ axis ($v_2 = 0$)

Ceiling ($y_2 = 7.0$)

$V_2$ axis ($v_2 = 1.14$)

Start

Floor ($y_1 = 0$)

Simulations by BounceIt

Classical No-Man's Land

Tuesday, February 2, 2016
The $V_1$-$V_2$-path for this problem repeats steps 0 to 12 (more or less!).

Ceiling ($y_2 = 7.0$)

Floor ($y_1 = 0$)

Simulations by BounceIt
The $V_1$-$V_2$-path for this problem repeats steps 0 to 12 (more or less!)

Ceiling ($y_2 = 7.0$)

Floor ($y_1 = 0$)

Simulations by BounceIt
The $V_1-V_2$-path for this problem repeats steps 0 to 12 (more or less!)

The problem is divided into two parts:

1. **Classical No-Man's Land**

   - Start at 0
   - Move along the line $y_1 = 0$
   - Reach the Ceiling at $y_2 = 7.0$

2. **Simulations by BounceIt**

   - Start at 0
   - Move along the line $y_1 = 0$
   - Reach the Classical No-Man's Land

---

Tuesday, February 2, 2016
The $v_1$-$v_2$-path for this problem repeats steps 0 to 12 (more or less!)
The $V_1$-$V_2$-path for this problem repeats steps 0 to 12 (more or less!)

The $y_1$-$y_2$-axis (not shown).

Ceiling ($y_2 = 7.0$)

Floor ($y_1 = 0$)

Simulations by BounceIt
The $V_1$-$V_2$-path for this problem repeats steps 0 to 12 (more or less!)

**Simulations by BounceIt**
Estrangian plot of $m_1/m_2=4/1$
collision sequence shows symmetry (sort of)

$\text{COM line}$ has slope $\sqrt{m_2/m_1}=1/2$

$c.o.m. \text{ lines (cons. of mom.) have slope } -\sqrt{m_2/m_1}=-2/1$

$\text{COM line}$ has slope $\sqrt{m_2/m_1}=1/2$
Collisions for mass ratio $m_1:m_2 = 100:1$

$\phi = \arccos \left( \frac{m_1 - m_2}{m_1 + m_2} \right) = \arccos \left( \frac{99}{101} \right) = 11.4^\circ$

$2\phi = 22.8^\circ$
Collisions for mass ratio $m_1:m_2 = 100:1$

- Slope of initial $(V_1,V_2)$ line: $\phi = \cos^{-1}\left(\frac{m_1 - m_2}{m_1 + m_2}\right) = \cos^{-1}\left(\frac{99}{101}\right) = 11.4^\circ$

- Slope of all momentum-conservation $(V_1,V_2)$-lines: $= -\sqrt{\frac{m_1}{m_2}} = -10$

- Slope of COM bisector line: $\frac{m_2}{m_1} = 1/10 = \tan \alpha$

- $\tan^2 \alpha = (1/\cos^2 \alpha) - 1 = m_2/m_1$

- $\cos \alpha = m_1/(m_1 + m_2)$

- $\sin \alpha = m_2/(m_1 + m_2)$

- $\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = (m_1 - m_2)(m_1 + m_2) = \cos \phi$

- $\phi = 2\alpha = 11.4^\circ$

- $2\phi = 22.8^\circ$

- $\alpha = 5.7^\circ$

- $72.8^\circ + 5.7^\circ = 165.3^\circ$

- $72.8^\circ + 11.4^\circ = 84.2^\circ$

- $-72.8^\circ + 5.7^\circ = -67.1^\circ$

- $-72.8^\circ + 11.4^\circ = -61.4^\circ$

- Tan values:
  - $\tan 2\alpha = (m_1 - m_2)^2/(m_1 + m_2)^2$
  - $\tan \alpha = (m_1 - m_2)/m_1$

- $\tan 2\alpha = -0.053$

- $\tan 2\alpha = (m_1 - m_2)/(m_1 + m_2) = \cos \phi$

- $\cos \phi = 0.957$

- $\phi = 16.1^\circ$

- $\cos \theta = \cos \phi$

- $\cos \theta = 1 - \sin^2 \theta$

- $\sin \theta = \frac{-m_1}{m_1 + m_2}$