Lecture 21xtra C_N Wave Modes
Tue. 3.31.2016

C_N-Symmetric Wave Modes

(Ch. 5 of Unit 4   3.29.15)

C_{12} and higher symmetry mode models: Archetypes of dispersion functions and 1-CW phase velocity
½-Sum-½-Diff-theory of 2-CW group and phase velocity

Algebra and geometry of resonant revivals: Farey Sums and Ford Circles

Relating C_N symmetric H and K matrices to differential wave operators

Wave resonance in cyclic C_n symmetry

Harmonic oscillator with cyclic C_2 symmetry

C_2 symmetric (B-type) modes

Projector analysis of 2D-HO modes and mixed mode dynamics

½-Sum-½-Diff-Identity for resonant beat analysis

Mode frequency ratios and continued fractions

Geometry of that 90°-phase lag (again)

Harmonic oscillator with cyclic C_3 symmetry

C_3 symmetric spectral decomposition by 3rd roots of unity

Deriving C_3 projectors

Deriving and labeling moving wave modes

Deriving dispersion functions and degenerate standing waves

Examples by WaveIt *animation*

C_6 symmetric mode model:Distant neighbor coupling

C_6 moving waves and degenerate standing waves

C_6 dispersion functions for 1^{st}, 2^{nd}, and 3^{rd}-neighbor coupling

C_6 dispersion functions split by C-type symmetry(complex, chiral, ...)*

➔ *C_{12} and higher symmetry mode models: Archetypes of dispersion functions and 1-CW phase velocity*

½-Sum-½-Diff-theory of 2-CW group and phase velocity

C_N Symmetric Mode Models:

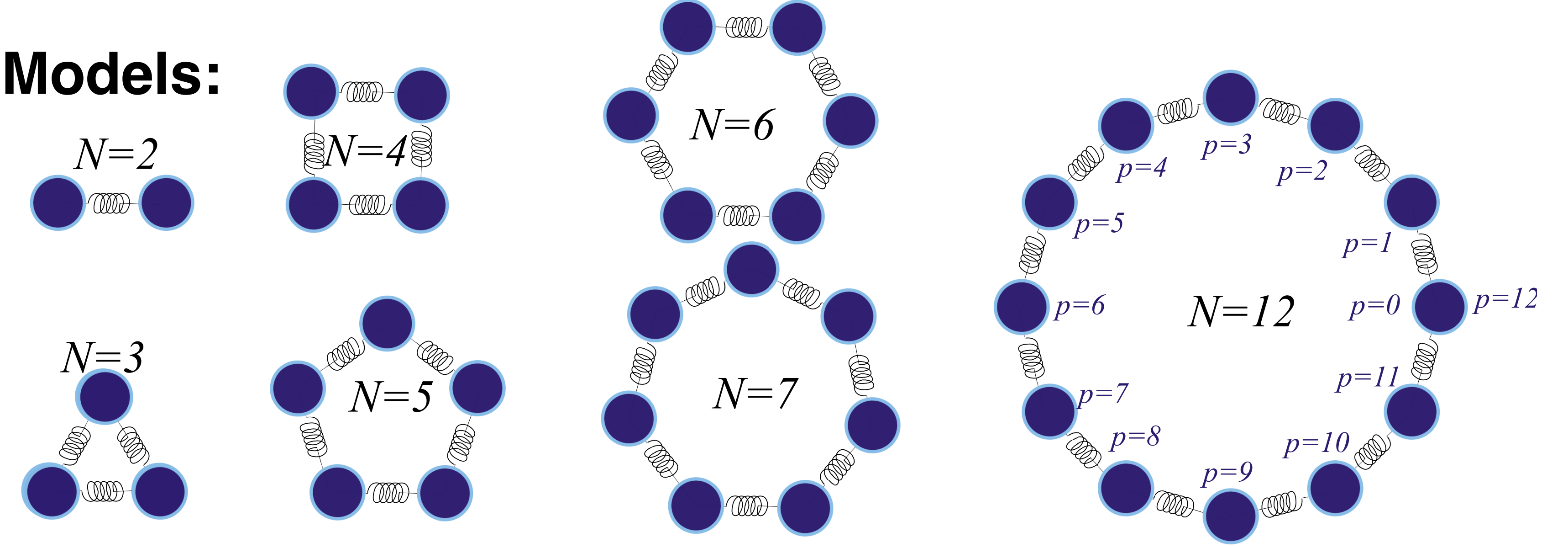


Fig. 4.8.4
Unit 4
CMwBang

C_N Symmetric Mode Models:

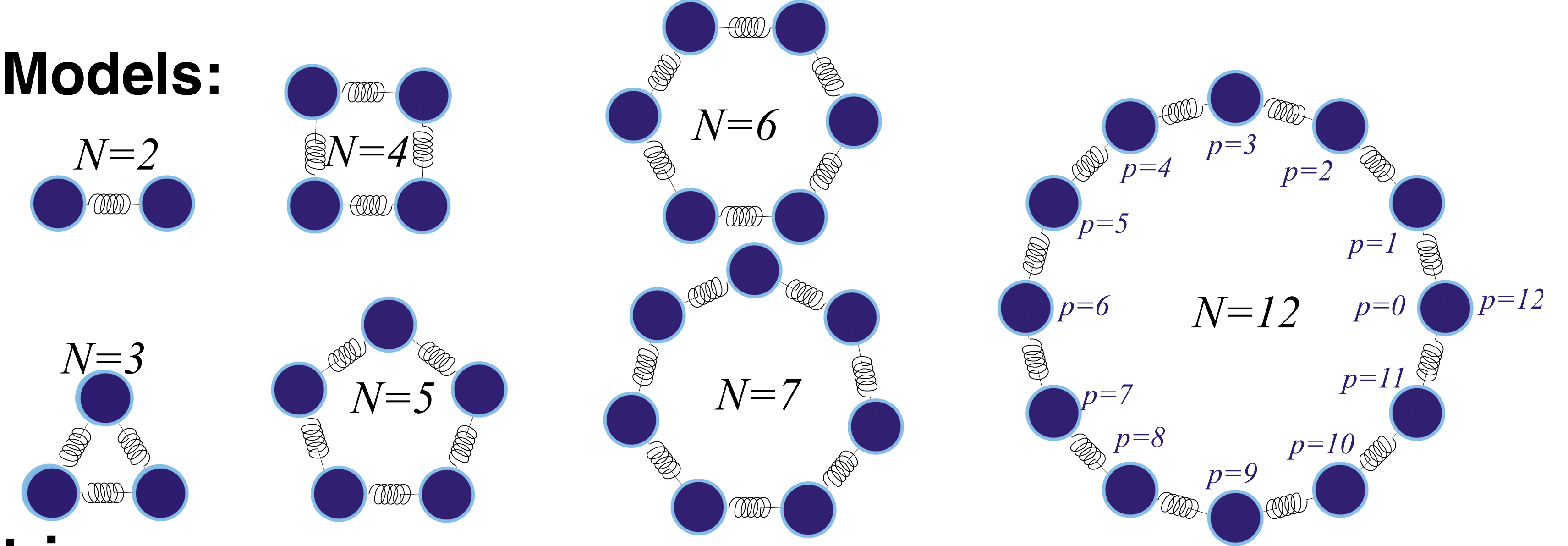


Fig. 4.8.4
Unit 4
CMwBang

1st Neighbor K-matrix

$$
-\begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ \vdots \\ F_{N-1} \end{pmatrix}
=
\begin{pmatrix}
K & -k_{12} & . & . & . & \cdots & -k_{12} \\
-k_{12} & K & -k_{12} & . & . & \cdots & . \\
. & -k_{12} & K & -k_{12} & . & \cdots & . \\
. & . & -k_{12} & K & -k_{12} & \cdots & . \\
. & . & . & -k_{12} & K & \cdots & . \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & -k_{12} \\
-k_{12} & . & . & . & . & -k_{12} & K
\end{pmatrix}
\bullet
\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{N-1} \end{pmatrix}
$$

where:

$$K = k + 2k_{12}$$

$$k = \frac{Mg}{\ell}$$

$$(\cdot) = 0$$

C_N Symmetric Mode Models:

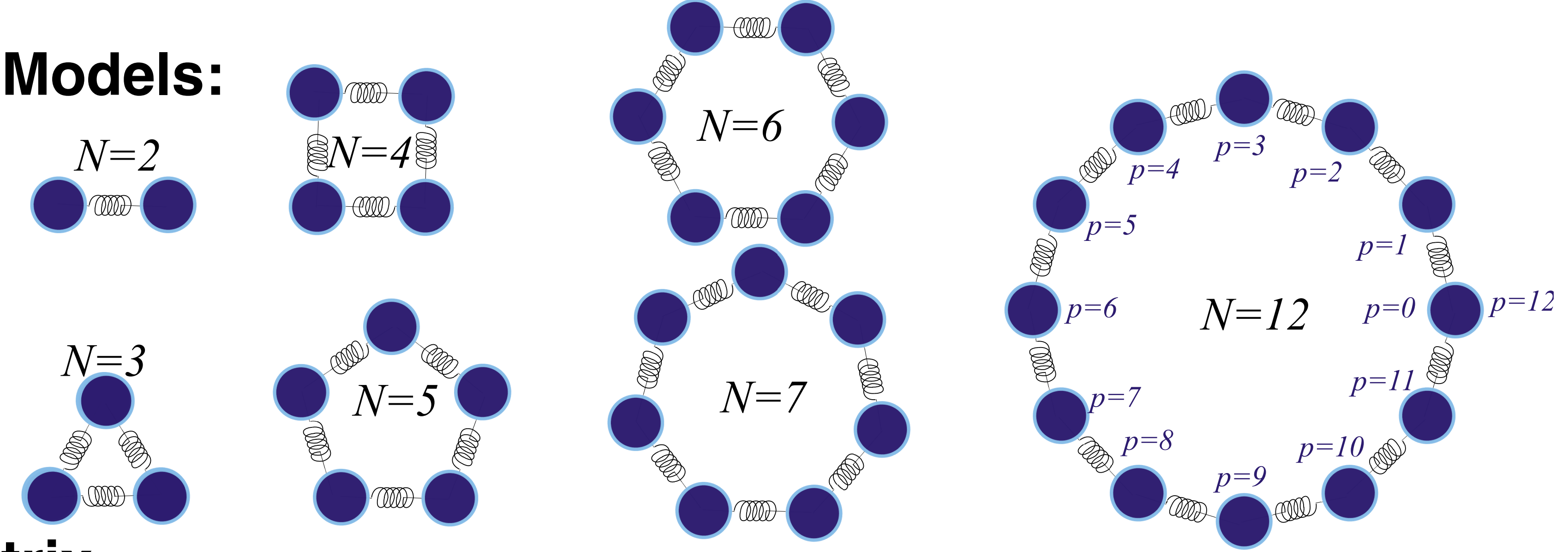


Fig. 4.8.4
Unit 4
CMwBang

1st Neighbor K-matrix

$$
-\begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ \vdots \\ F_{N-1} \end{pmatrix} = \begin{pmatrix} K & -k_{12} & . & . & . & \cdots & -k_{12} \\ -k_{12} & K & -k_{12} & . & . & \cdots & . \\ . & -k_{12} & K & -k_{12} & . & \cdots & . \\ . & . & -k_{12} & K & -k_{12} & \cdots & . \\ . & . & . & -k_{12} & K & \cdots & . \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & -k_{12} \\ -k_{12} & . & . & . & . & -k_{12} & K \end{pmatrix} \bullet \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{N-1} \end{pmatrix}
$$

where: $K = k + 2k_{12}$, $k = \frac{Mg}{\ell}$, $(\cdot) = 0$

Nth roots of 1 $e^{i\, m\cdot p\, 2\pi/N} = \langle m | \mathbf{r}^p | m \rangle$ *serving as e-values, eigenfunctions, transformation matrices, dispersion relations, Group reps. etc.*

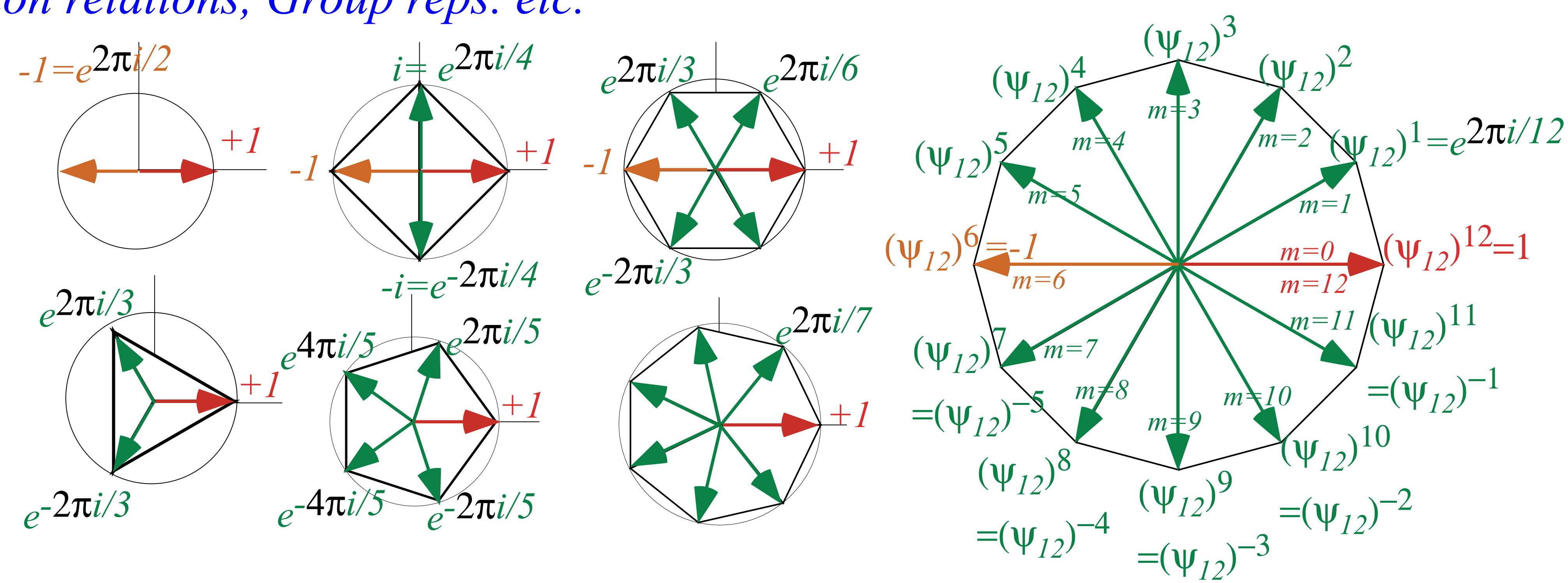


Fig. 4.8.5
Unit 4
CMwBang

C_N Symmetric Mode Models:

N^{th} roots of 1 $e^{i\, m\cdot p\, 2\pi/N} = \langle m|\, \mathbf{r}^p\, |m\rangle$ *serving as e-values, eigenfunctions, transformation matrices, dispersion relations, Group reps. etc.*

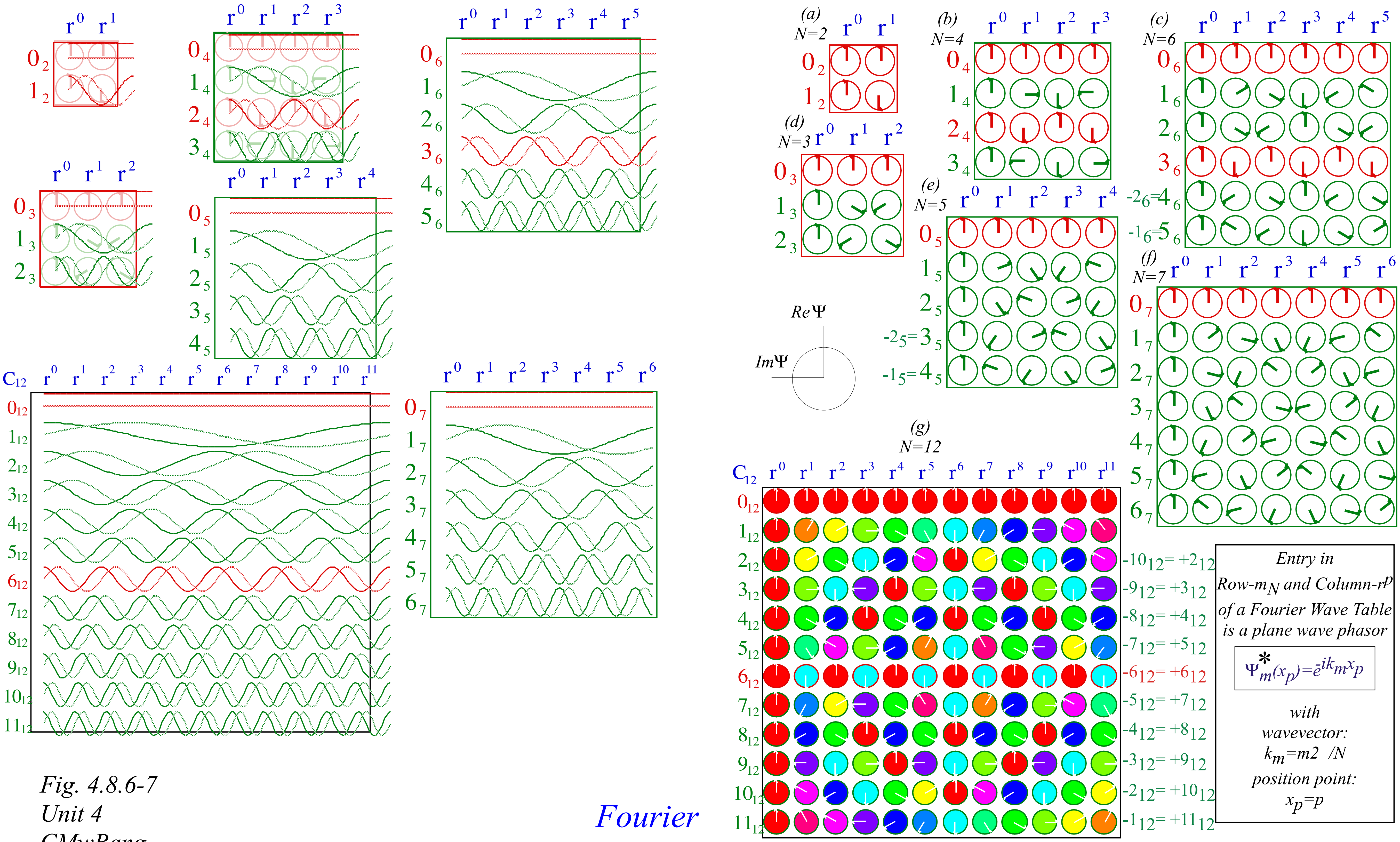
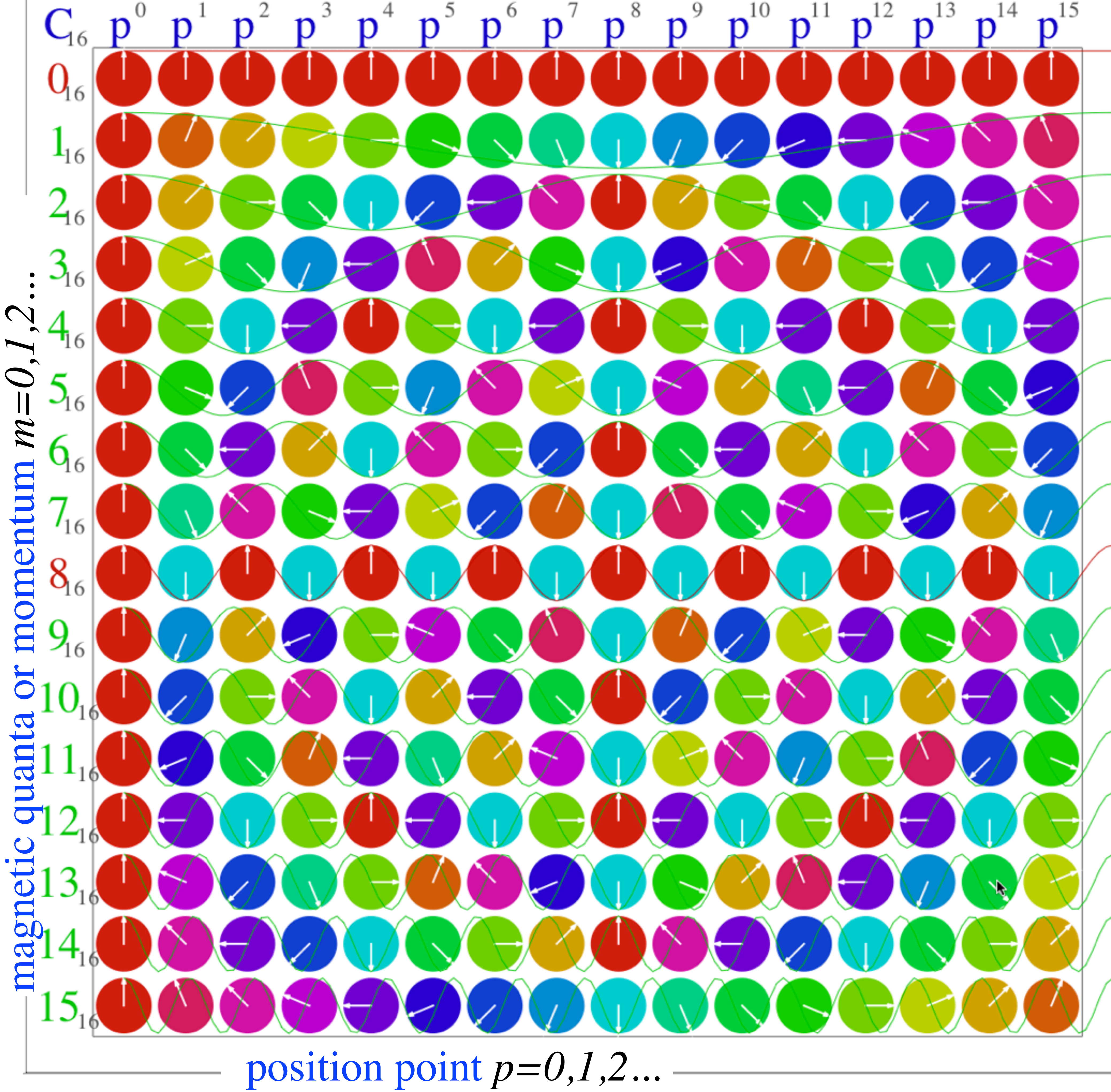


Fig. 4.8.6-7
Unit 4
CMwBang

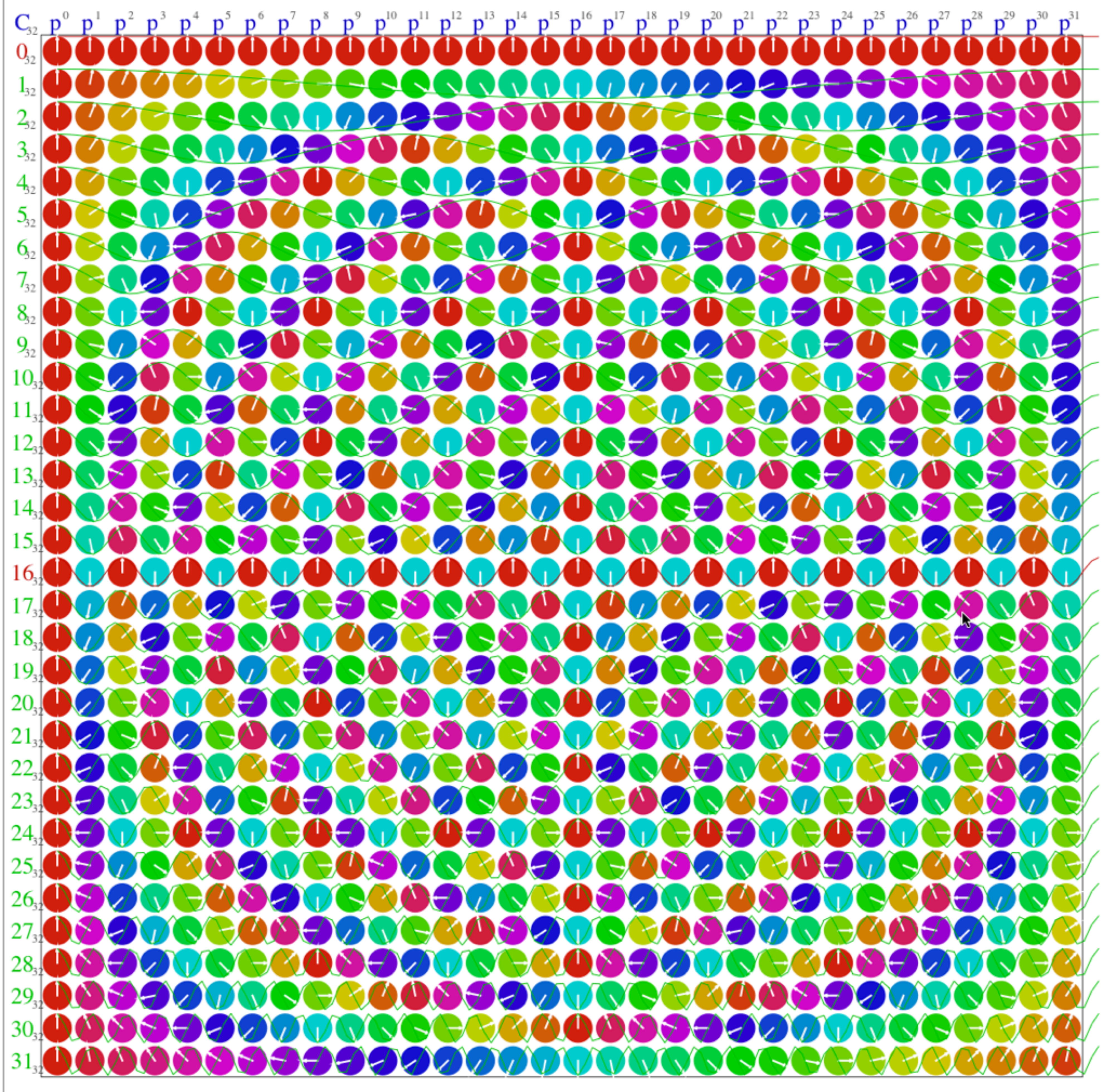
Fourier transformation matrices



C_{16}

phasor character table

$$\chi_p^m = e^{ik_m r^p} = e^{\frac{2\pi imp}{16}}$$



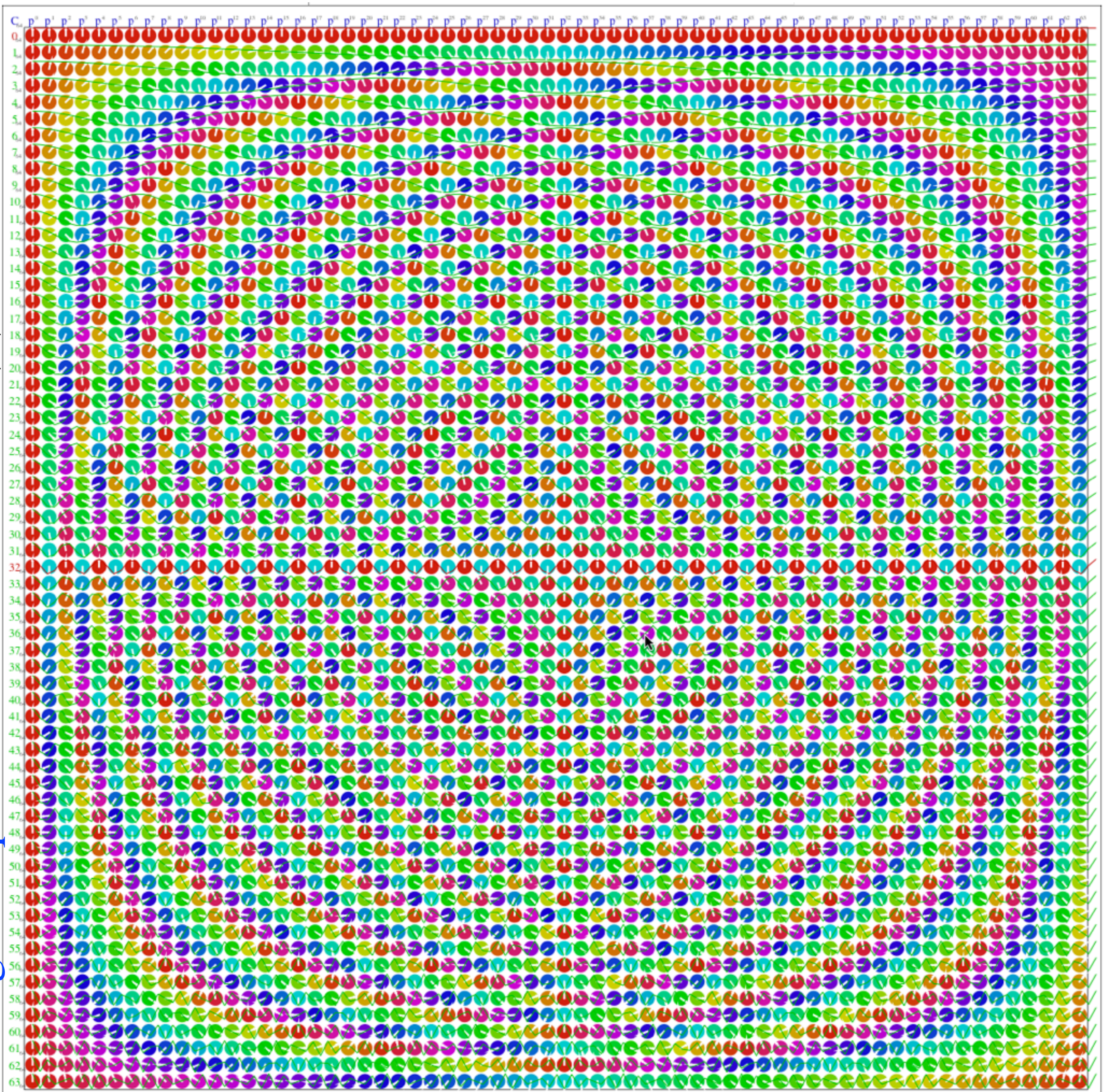
magnetic quanta or momentum $m=0,1,2\ldots$

position point $p=0,1,2\ldots$

C_{32}

phasor
character
table

$$\chi_p^m = e^{ik_m r^p} = e^{\frac{2\pi imp}{32}}$$



magnetic quanta or momentum *m=0,1,2…*

position point *p=0,1,2…*

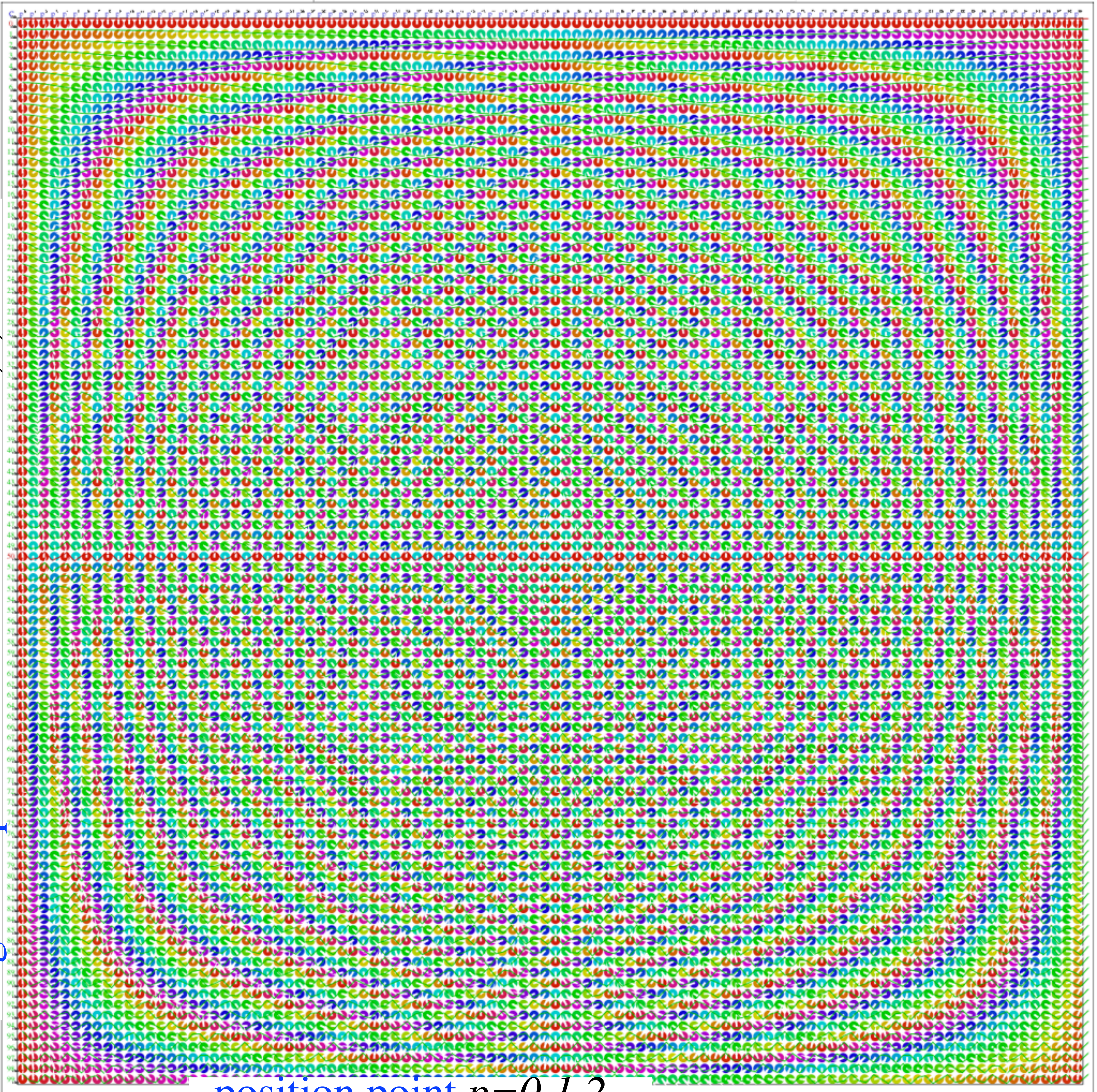
C_{64}

phasor character table

$$\chi_p^m = e^{ik_m r^p}$$

$$= e^{\frac{2\pi i m p}{64}}$$

Invariant phase "Uncertainty" hyperbolas: $m \cdot p = const.$



magnetic quanta or momentum $m=0,1,2\ldots$

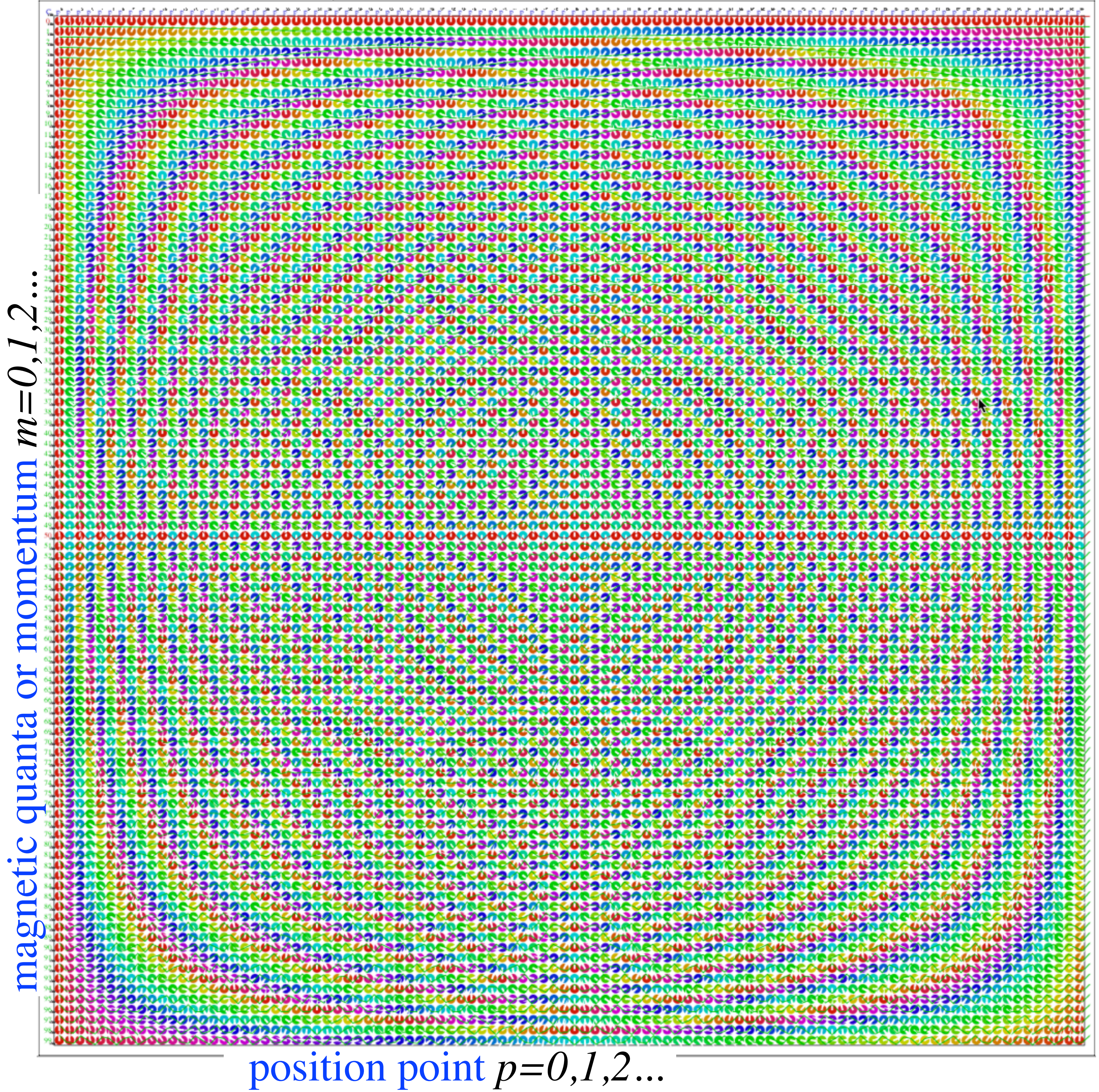
position point $p=0,1,2\ldots$

C_{100}

phasor character table

$$\chi_p^m = e^{ik_m r^p} = e^{\frac{2\pi imp}{100}}$$

Invariant phase "Uncertainty" hyperbolas: $m \cdot p = const.$



C_{256}

phasor character table

$$\chi_p^m = e^{ik_m r^p}$$

$$= e^{\frac{2\pi imp}{256}}$$

Invariant phase "Uncertainty" hyperbolas: $m \cdot p = const.$

Wave resonance in cyclic C_n symmetry

Harmonic oscillator with cyclic C_2 symmetry

C_2 symmetric (B-type) modes

Projector analysis of 2D-HO modes and mixed mode dynamics

½-Sum-½-Diff-Identity for resonant beat analysis

Mode frequency ratios and continued fractions

Geometry of that 90°-phase lag (again)

Harmonic oscillator with cyclic C_3 symmetry

C_3 symmetric spectral decomposition by 3rd roots of unity

Deriving C_3 projectors

Deriving and labeling moving wave modes

Deriving dispersion functions and degenerate standing waves

Examples by WaveIt *animation*

C_6 symmetric mode model:Distant neighbor coupling

C_6 moving waves and degenerate standing waves

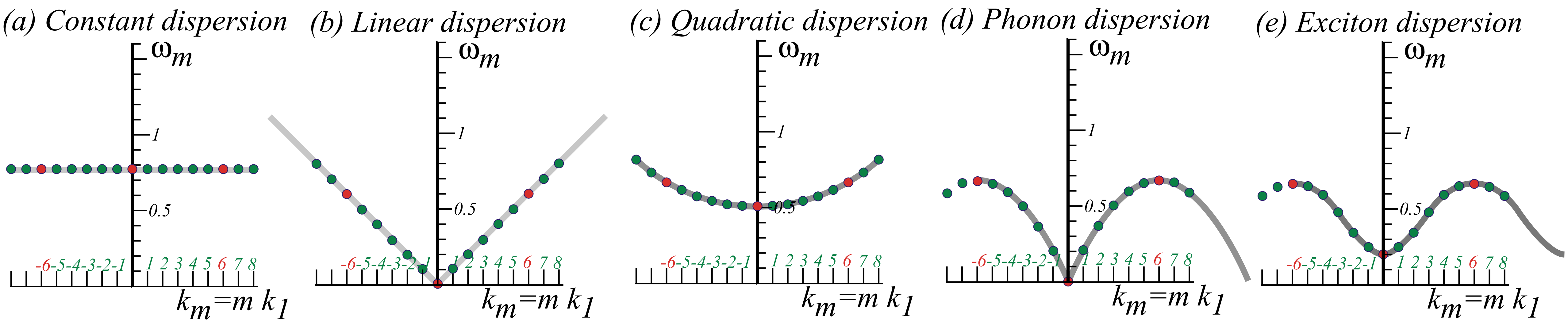
C_6 dispersion functions for 1st, 2nd, and 3rd-neighbor coupling

C_6 dispersion functions split by C-type symmetry(complex, chiral, ...)

➤ *C_{12} and higher symmetry mode models: Archetypes of dispersion functions and 1-CW phase velocity* ➤

½-Sum-½-Diff-theory of 2-CW group and phase velocity

Archetypical Examples of C_{12} Dispersion Functions

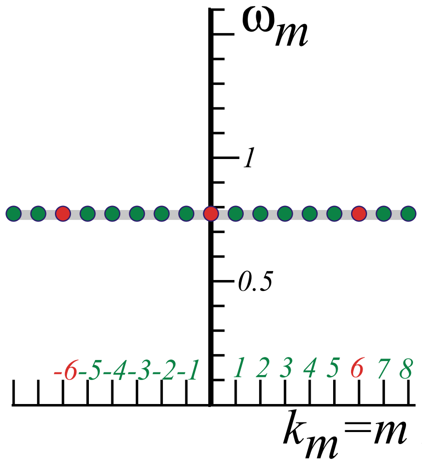


Applications:

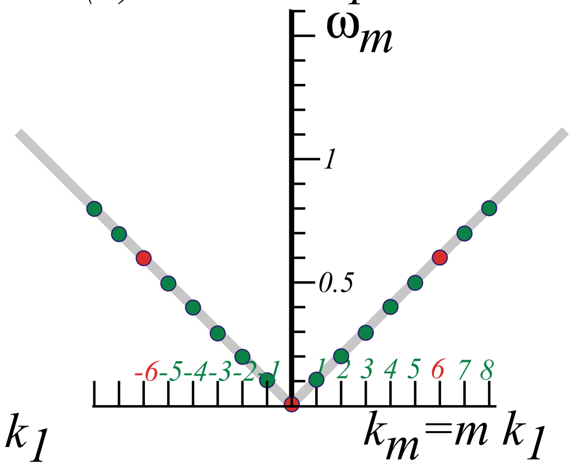
(a) Constant dispersion	(b) Linear dispersion	(c) Quadratic dispersion	(d) Phonon dispersion	(e) Exciton dispersion
Uncoupled pendulums	Weakly coupled pendulums (No gravity)	Weakly coupled pendulums (With gravity)	Strongly coupled pendulums (No gravity)	Strongly coupled pendulums (With gravity)
Movie marquis Xmas lights	Light in vacuum (Exactly) Sound (Approximately)	Light in fiber (Approx) Non-relativistic Schrodinger matter wave	Acoustic mode in solids	Optical mode in solids Relativistic matter (If exact hyperbola)

Archetypical Examples of C_{12} Dispersion Functions

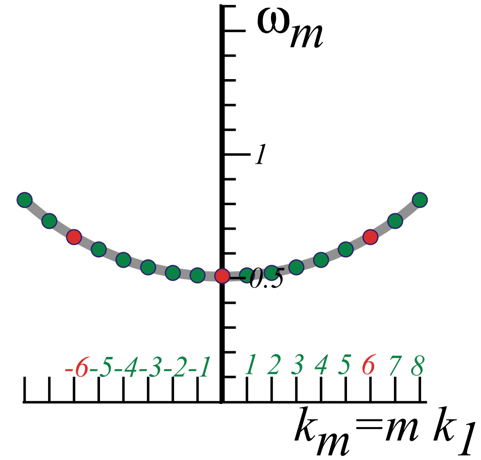
(a) Constant dispersion



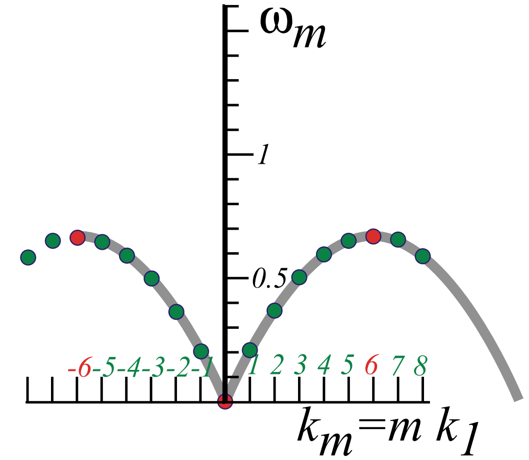
(b) Linear dispersion



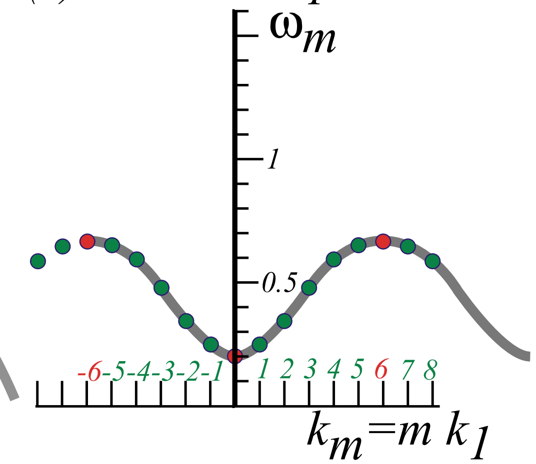
(c) Quadratic dispersion



(d) Phonon dispersion



(e) Exciton dispersion



Applications:

(a) *Uncoupled pendulums*

Movie marquis
Xmas lights

(b) *Weakly coupled pendulums (No gravity)*

Light in vacuum (Exactly)
Sound (Approximately)

(c) *Weakly coupled pendulums (With gravity)*

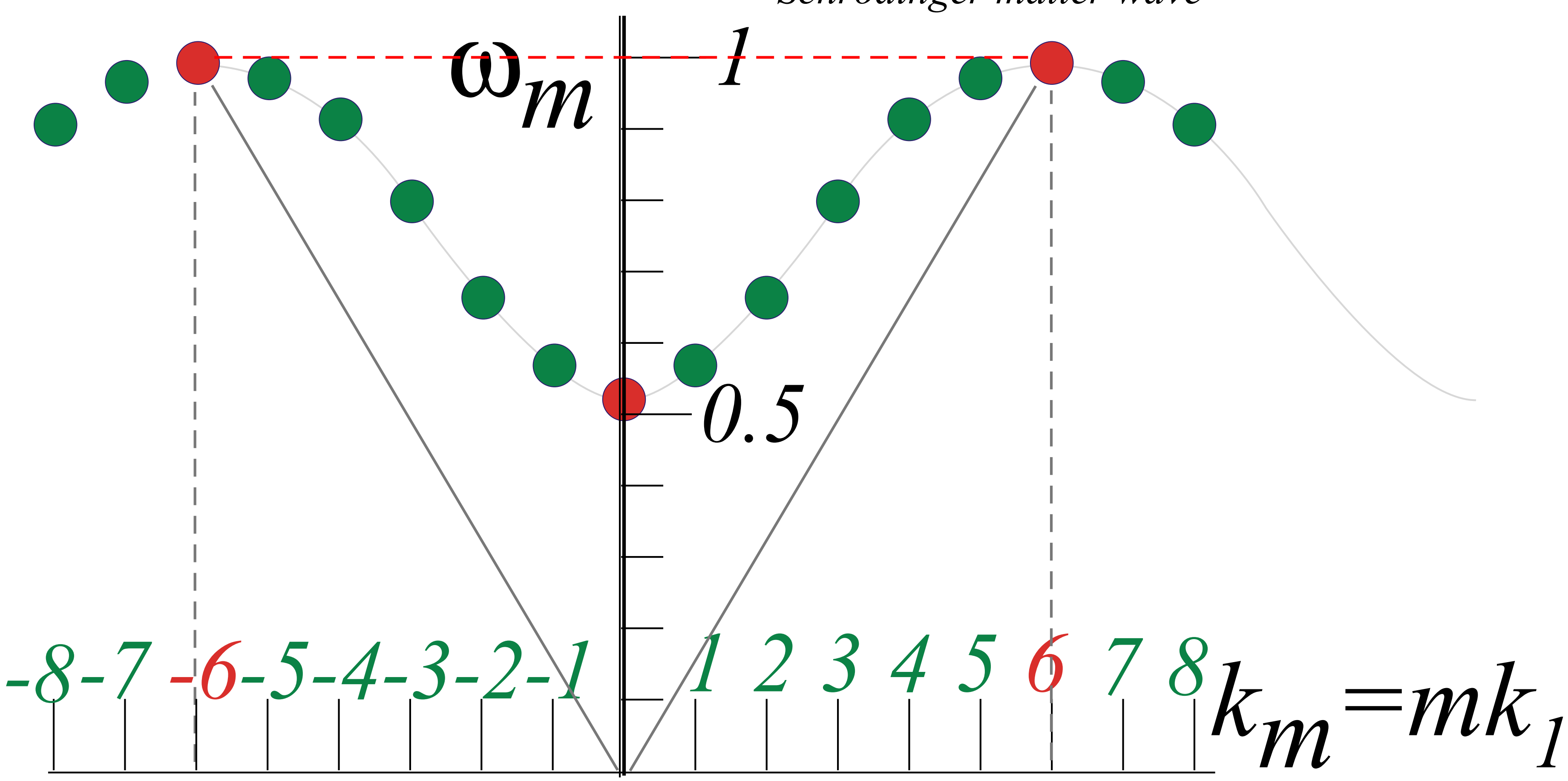
Light in fiber (Approx)
Non-relativistic Schrodinger matter wave

(d) *Strongly coupled pendulums (No gravity)*

Acoustic mode in solids

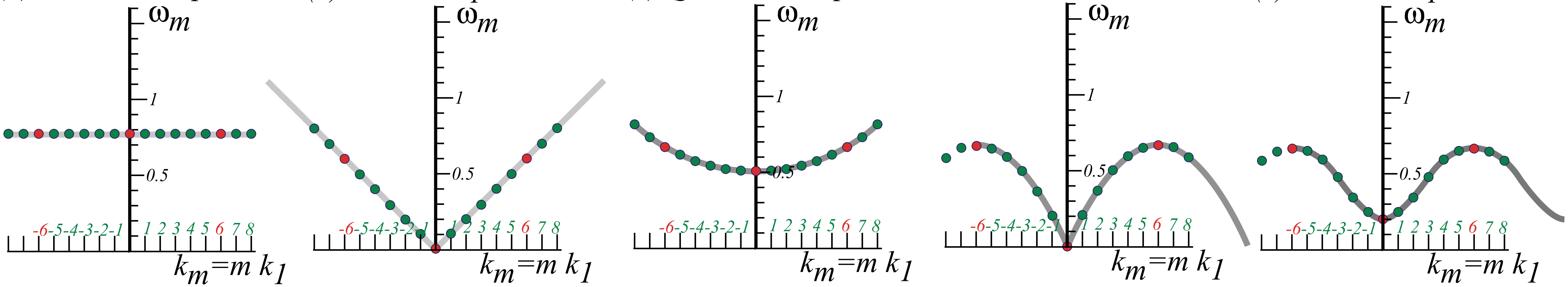
(e) *Strongly coupled pendulums (With gravity)*

Optical mode in solids
Relativistic matter
(If exact hyperbola)



Archetypical Examples of C_{12} Dispersion Functions

(a) Constant dispersion | *(b) Linear dispersion* | *(c) Quadratic dispersion* | *(d) Phonon dispersion* | *(e) Exciton dispersion*



Applications:

(a) *Uncoupled pendulums*

Movie marquis
Xmas lights

(b) *Weakly coupled pendulums (No gravity)*

Light in vacuum (Exactly)
Sound (Approximately)

(c) *Weakly coupled pendulums (With gravity)*

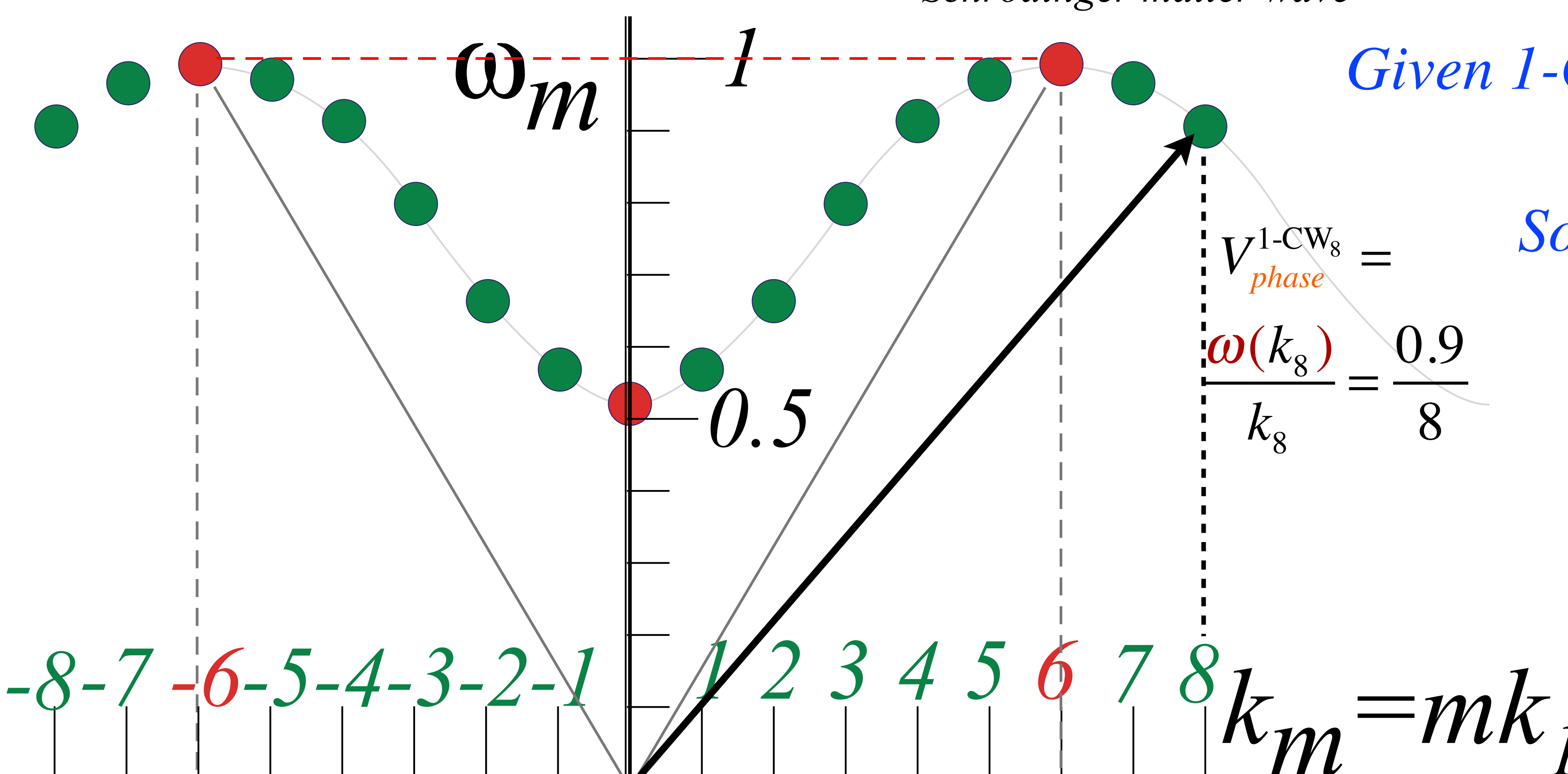
Light in fiber (Approx)
Non-relativistic
Schrodinger matter wave

(d) *Strongly coupled pendulums (No gravity)*

Acoustic mode in solids

(e) *Strongly coupled pendulums (With gravity)*

Optical mode in solids
Relativistic matter
(If exact hyperbola)



$$V_{phase}^{1\text{-}CW_8} = \frac{\omega(k_8)}{k_8} = \frac{0.9}{8}$$

Given 1-CW phase of wave $e^{i(kx-\omega t)}$:

$$a = k\cdot x - \omega\cdot t$$

Solve for 1-CW phase velocity

$$x = \frac{\omega}{k}\cdot t + \frac{a}{k}$$

Wave velocities depend on
Dispersion function
$\omega = \omega(k)$

(a) 1-CW *phase* velocity:

$$V_{phase}^{1\text{-}CW} = \frac{\omega(k)}{k}$$

- *Wave resonance in cyclic C_n symmetry*
  - *Harmonic oscillator with cyclic C_2 symmetry*
    - *C_2 symmetric (B-type) modes*
  - *Projector analysis of 2D-HO modes and mixed mode dynamics*
    - *½-Sum-½-Diff-Identity for resonant beat analysis*
      - *Mode frequency ratios and continued fractions*
      - *Geometry of that 90°-phase lag (again)*
  - *Harmonic oscillator with cyclic C_3 symmetry*
    - *C_3 symmetric spectral decomposition by 3rd roots of unity*
    - *Deriving C_3 projectors*
      - *Deriving and labeling moving wave modes*
      - *Deriving dispersion functions and degenerate standing waves*
        - *Examples by* WaveIt *animation*
  - *C_6 symmetric mode model:Distant neighbor coupling*
    - *C_6 moving waves and degenerate standing waves*
    - *C_6 dispersion functions for 1^{st}, 2^{nd}, and 3^{rd}-neighbor coupling*
    - *C_6 dispersion functions split by C-type* symmetry*(complex, chiral, ...)*

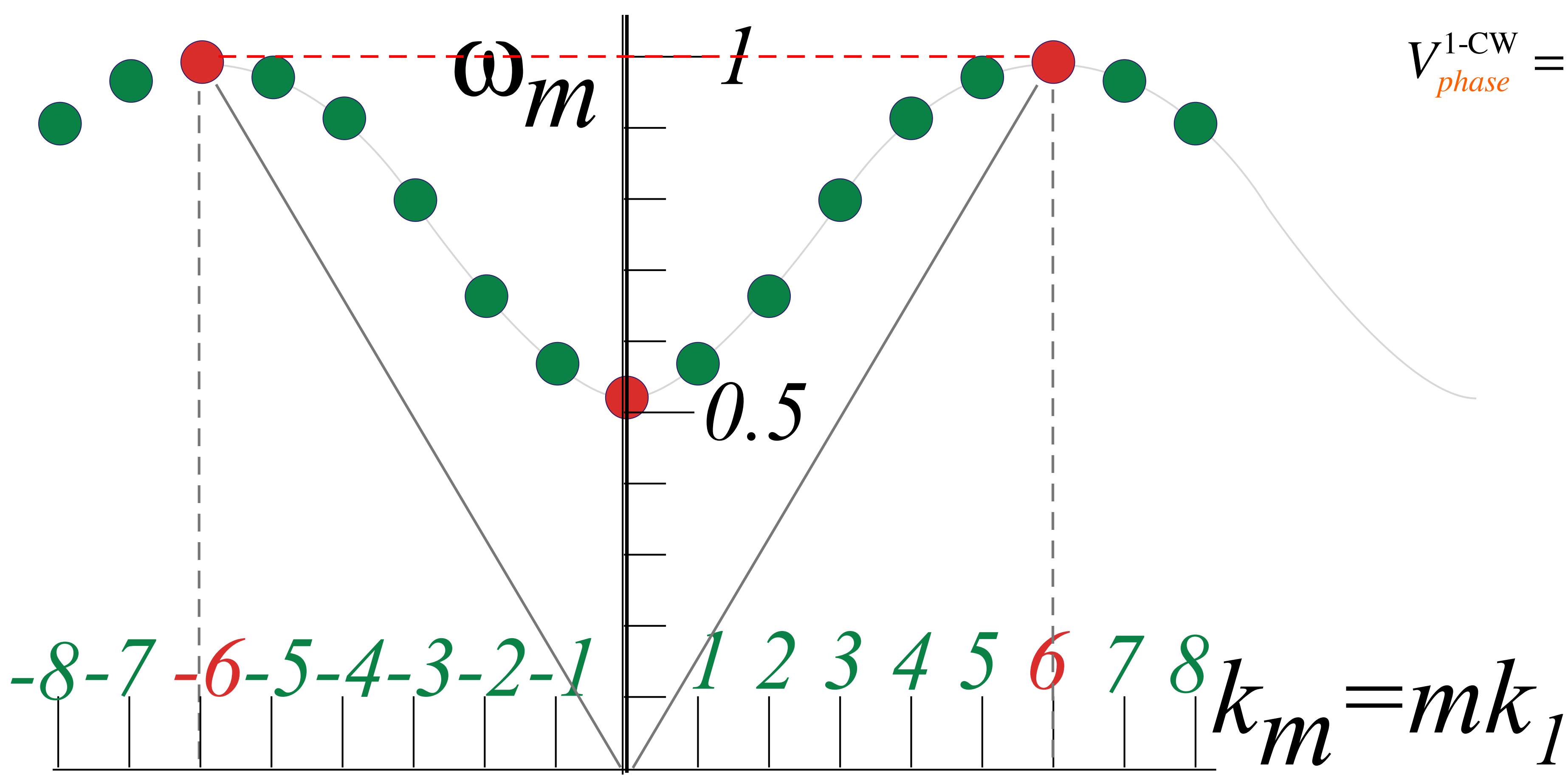
- *C_{12} and higher symmetry mode models: Archetypes of dispersion functions and 1-CW phase velocity*
  - ➔ *½-Sum-½-Diff-theory of 2-CW group and phase velocity* ⬅

The ½-Sum-½-Diff-Identity and 2-CW phase and group velocity

Given 2-CW phases:

$$a = k_a \cdot x - \omega_a \cdot t \quad \text{and} \quad b = k_b \cdot x - \omega_b \cdot t$$

...find 2-CW phase velocity $V^{2\text{-CW}}_{phase}$ *and group velocity* $V^{2\text{-CW}}_{group}$



Velocities depend upon
Dispersion function
$\omega = \omega(k)$

(a) 1-CW *phase* velocity:

$$V^{1\text{-CW}}_{phase} = \frac{\omega(k)}{k}$$

The ½-Sum-½-Diff-Identity and 2-CW phase and group velocity

Given 2-CW phases: ...find 2-CW phase velocity $V^{2\text{-CW}}_{phase}$ and group velocity $V^{2\text{-CW}}_{group}$

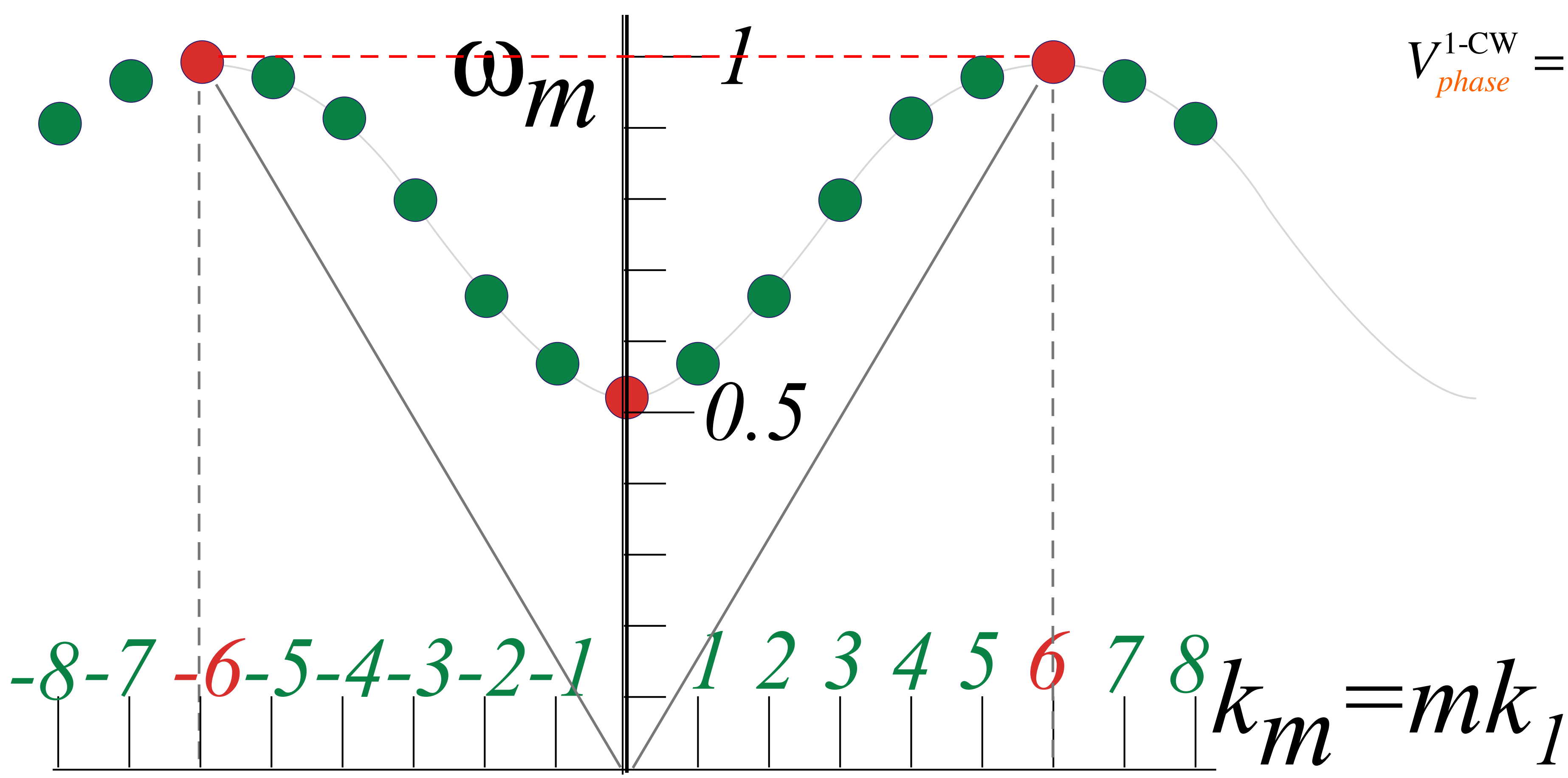
$$a = k_a \cdot x - \omega_a \cdot t \quad \text{and} \quad b = k_b \cdot x - \omega_b \cdot t$$

$$\frac{e^{ia} + e^{ib}}{2} = e^{i\frac{a+b}{2}} \left(\frac{e^{i\frac{a-b}{2}} + e^{-i\frac{a-b}{2}}}{2} \right) = e^{i\frac{a+b}{2}} \cos\left(\frac{a-b}{2}\right)$$

Velocities depend upon
Dispersion function
$\omega = \omega(k)$

(a) 1-CW *phase* velocity:

$$V^{1\text{-CW}}_{phase} = \frac{\omega(k)}{k}$$



The ½-Sum-½-Diff-Identity and 2-CW phase and group velocity

Given 2-CW phases: *...find 2-CW phase velocity* $V^{2\text{-CW}}_{phase}$ *and group velocity* $V^{2\text{-CW}}_{group}$

$$a = k_a \cdot x - \omega_a \cdot t \quad \text{and} \quad b = k_b \cdot x - \omega_b \cdot t$$

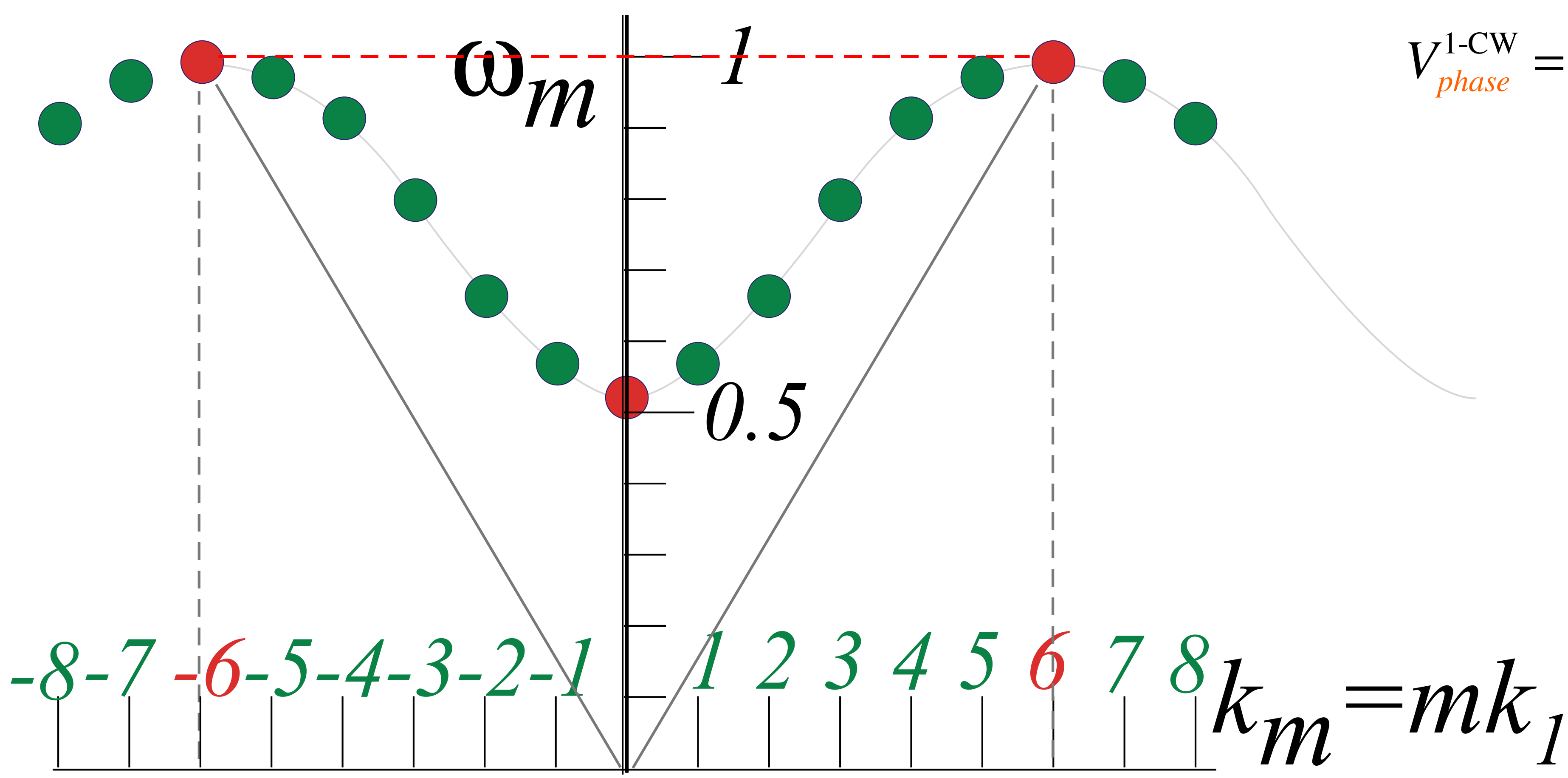
$$\frac{e^{ia}+e^{ib}}{2} = e^{i\frac{a+b}{2}}\left(\frac{e^{i\frac{a-b}{2}}+e^{-i\frac{a-b}{2}}}{2}\right) = e^{i\frac{a+b}{2}}\cos\left(\frac{a-b}{2}\right)$$

$$= e^{i\frac{(k_a+k_b)}{2}x-\frac{(\omega_a+\omega_b)}{2}t}\cos\left(\frac{(k_a-k_b)}{2}x-\frac{(\omega_a-\omega_b)}{2}t\right)$$

Velocities depend upon
Dispersion function
$\omega = \omega(k)$

(a) 1-CW *phase* velocity:

$$V^{1\text{-CW}}_{phase} = \frac{\omega(k)}{k}$$



The ½-Sum-½-Diff-Identity and 2-CW phase and group velocity

Given 2-CW phases: ...*find 2-CW phase velocity* $V^{2\text{-CW}}_{phase}$ *and group velocity* $V^{2\text{-CW}}_{group}$

$$a = k_a \cdot x - \omega_a \cdot t \quad \text{and} \quad b = k_b \cdot x - \omega_b \cdot t$$

$$\frac{e^{ia}+e^{ib}}{2} = e^{i\frac{a+b}{2}}\left(\frac{e^{i\frac{a-b}{2}}+e^{-i\frac{a-b}{2}}}{2}\right) = e^{i\frac{a+b}{2}}\cos\left(\frac{a-b}{2}\right)$$

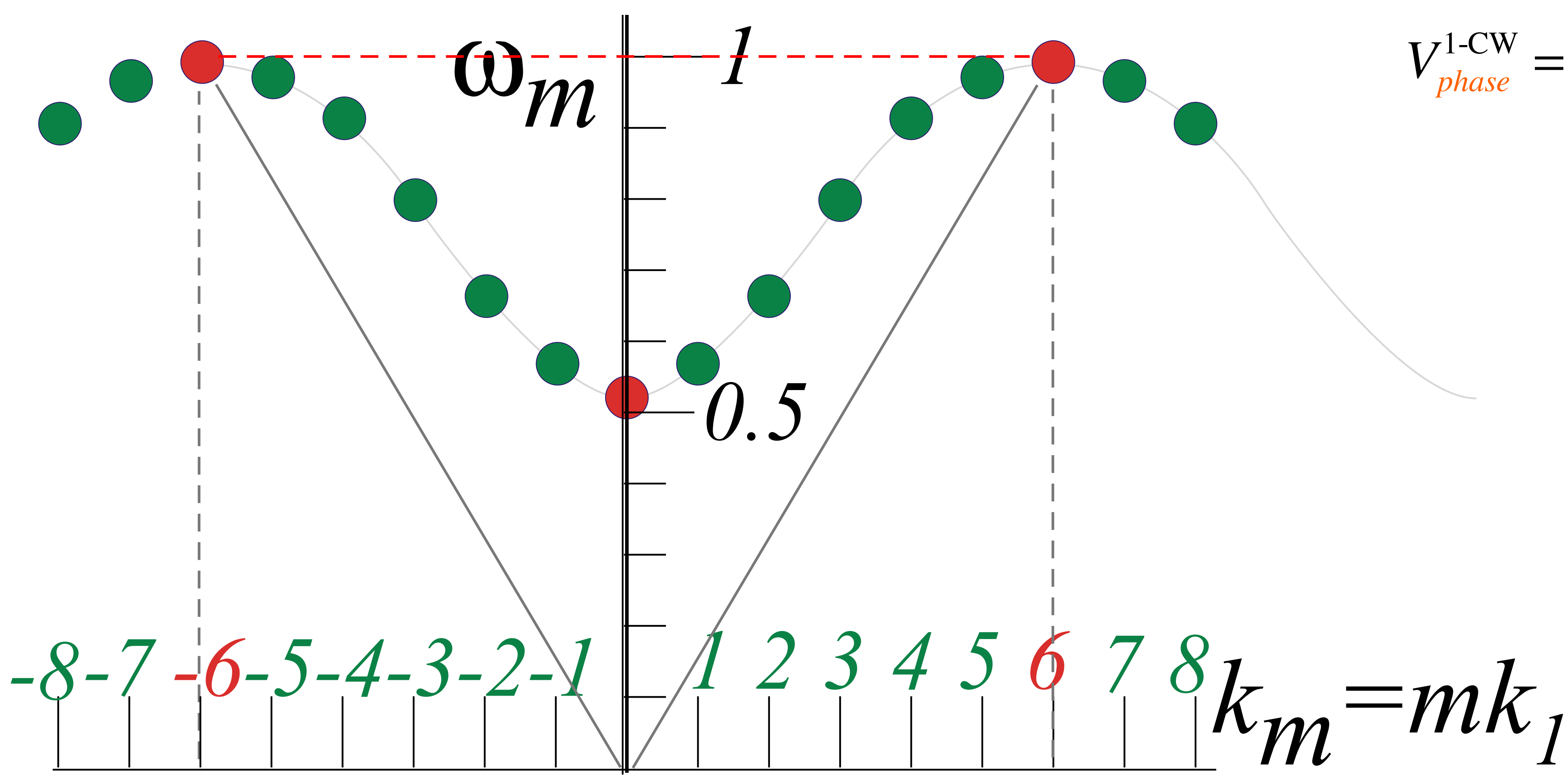
$$= e^{i\frac{(k_a+k_b)}{2}x-\frac{(\omega_a+\omega_b)}{2}t}\cos\left(\frac{(k_a-k_b)}{2}x-\frac{(\omega_a-\omega_b)}{2}t\right)$$

$$V^{2\text{-CW}}_{phase} = \frac{(\omega_a+\omega_b)}{(k_a+k_b)} \qquad V^{2\text{-CW}}_{group} = \frac{(\omega_a-\omega_b)}{(k_a-k_b)}$$

Velocities depend upon
Dispersion function
$\omega = \omega(k)$

(*a*) 1-CW *phase* velocity:

$$V^{1\text{-CW}}_{phase} = \frac{\omega(k)}{k}$$



The ½-Sum-½-Diff-Identity and 2-CW phase and group velocity

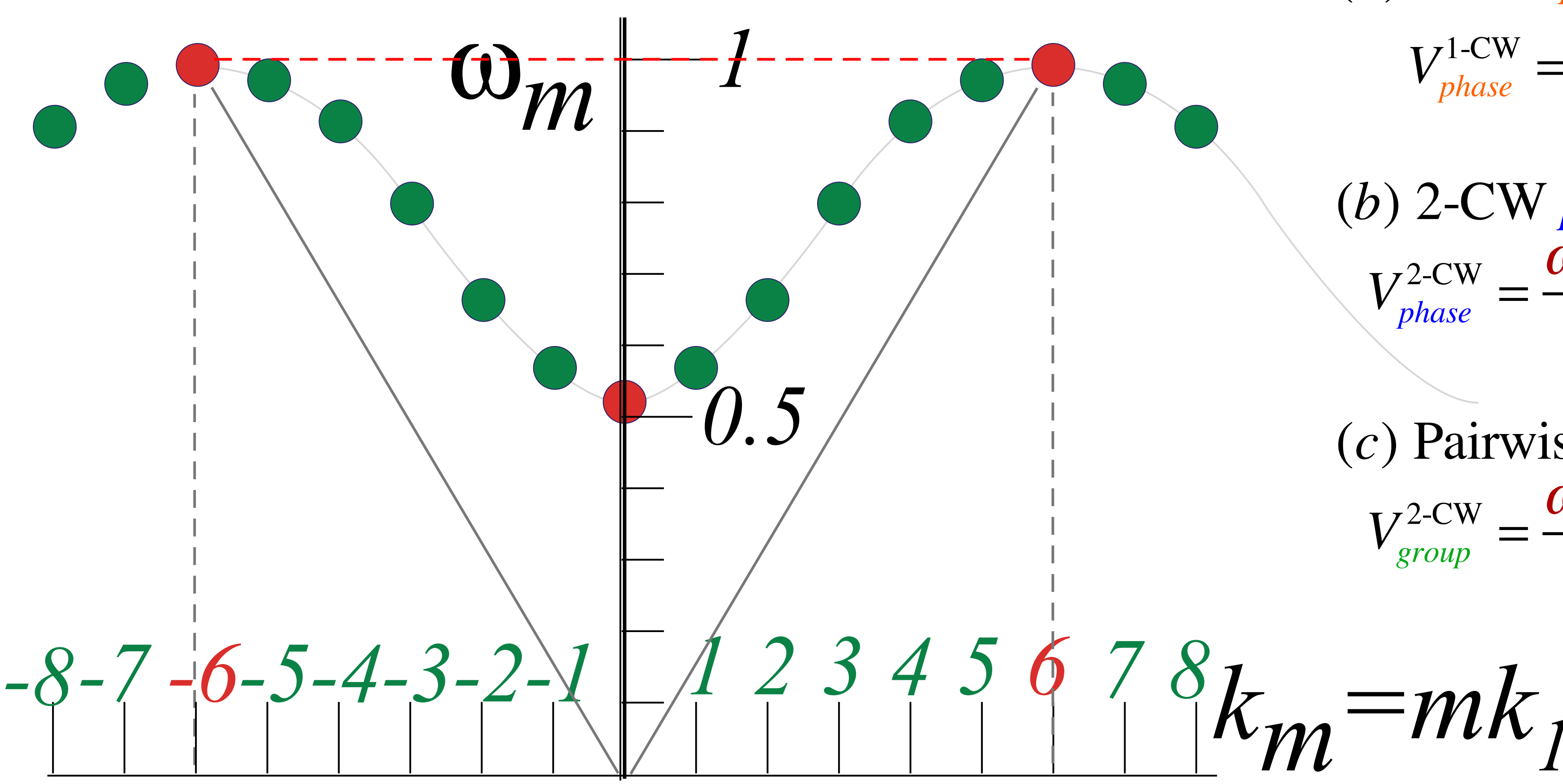
Given 2-CW phases: ...find 2-CW phase velocity $V^{2\text{-CW}}_{phase}$ and group velocity $V^{2\text{-CW}}_{group}$

$$a = k_a \cdot x - \omega_a \cdot t \quad \text{and} \quad b = k_b \cdot x - \omega_b \cdot t$$

$$\frac{e^{ia}+e^{ib}}{2} = e^{i\frac{a+b}{2}}\left(\frac{e^{i\frac{a-b}{2}}+e^{-i\frac{a-b}{2}}}{2}\right) = e^{i\frac{a+b}{2}}\cos\left(\frac{a-b}{2}\right)$$

$$= e^{i\frac{(k_a+k_b)}{2}x-\frac{(\omega_a+\omega_b)}{2}t}\cos\left(\frac{(k_a-k_b)}{2}x-\frac{(\omega_a-\omega_b)}{2}t\right)$$

$$V^{2\text{-CW}}_{phase} = \frac{(\omega_a+\omega_b)}{(k_a+k_b)} \qquad V^{2\text{-CW}}_{group} = \frac{(\omega_a-\omega_b)}{(k_a-k_b)}$$



Velocities depend upon
Dispersion function
$\omega = \omega(k)$

(a) 1-CW *phase* velocity:

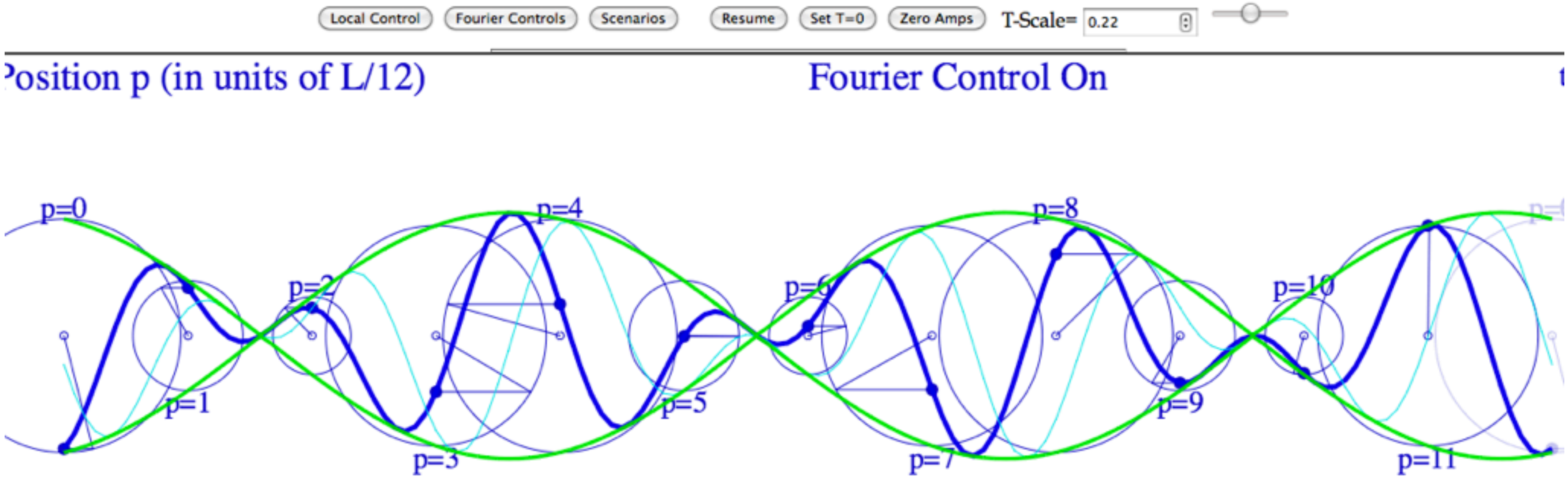
$$V^{1\text{-CW}}_{phase} = \frac{\omega(k)}{k}$$

(b) 2-CW *phase* velocity:

$$V^{2\text{-CW}}_{phase} = \frac{\omega(k_1)+\omega(k_2)}{k_1+k_2}$$

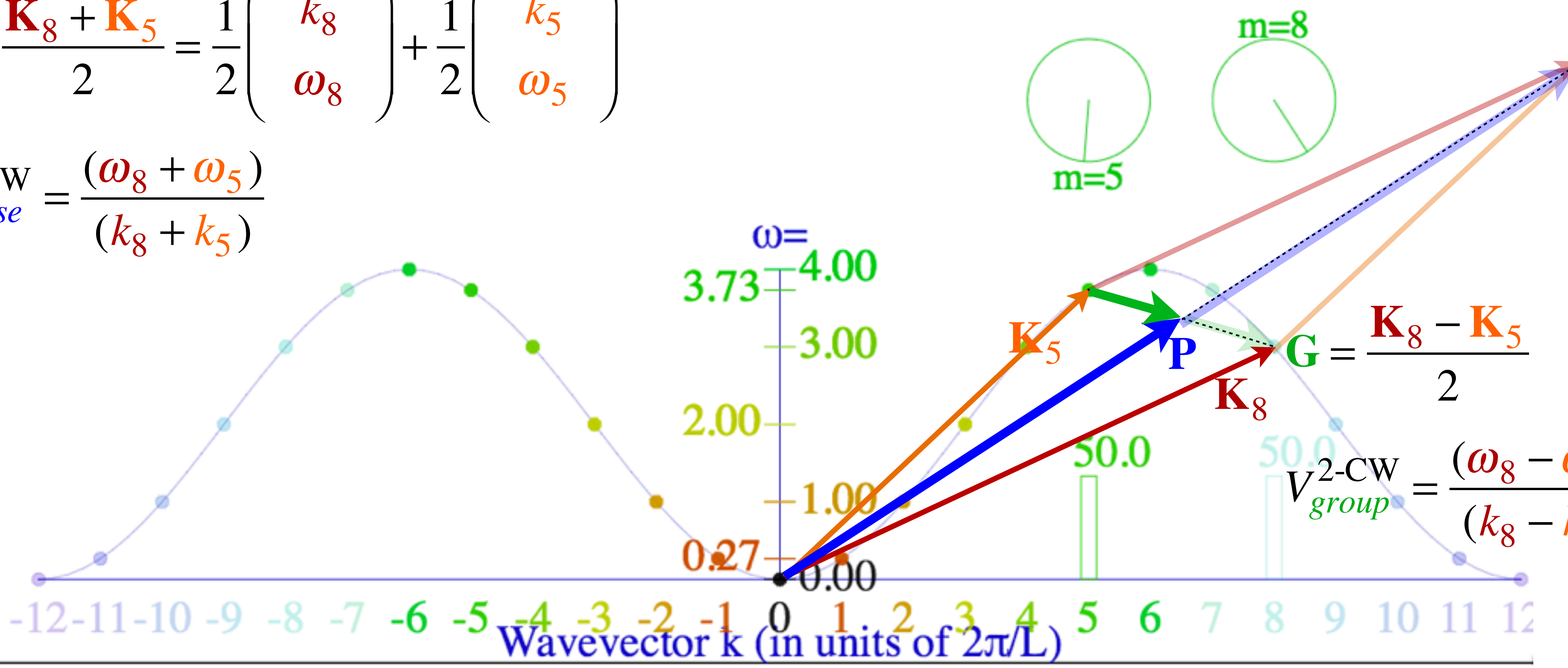
(c) Pairwise *group* velocity:

$$V^{2\text{-CW}}_{group} = \frac{\omega(k_1)-\omega(k_2)}{k_1-k_2}$$



$$\mathbf{P} = \frac{\mathbf{K}_8 + \mathbf{K}_5}{2} = \frac{1}{2}\begin{pmatrix} k_8 \\ \omega_8 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} k_5 \\ \omega_5 \end{pmatrix}$$

$$V^{2\text{-CW}}_{phase} = \frac{(\omega_8 + \omega_5)}{(k_8 + k_5)}$$



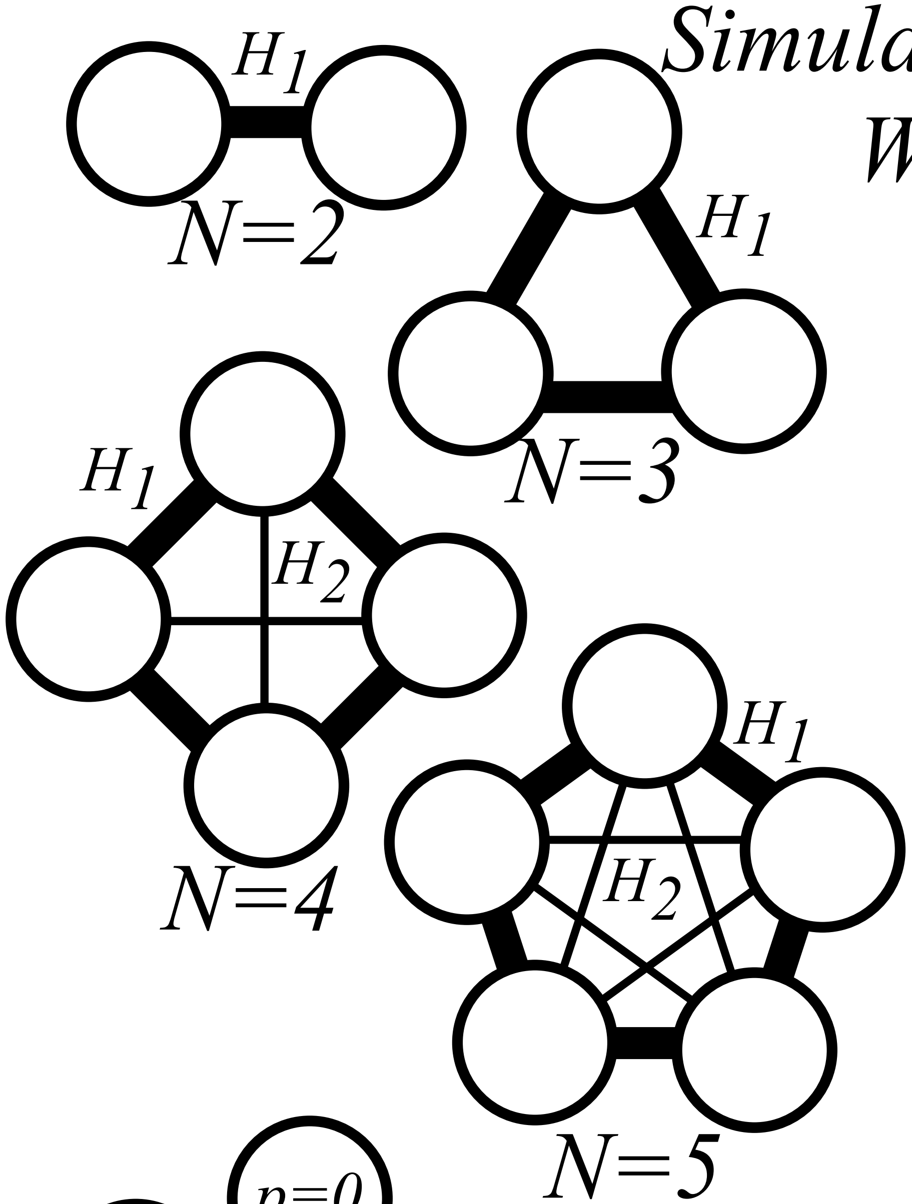
$$\mathbf{G} = \frac{\mathbf{K}_8 - \mathbf{K}_5}{2}$$

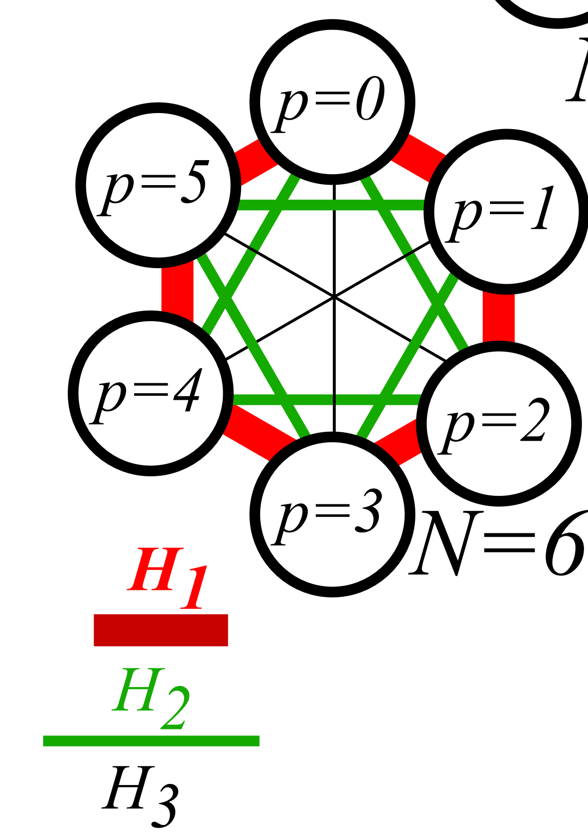
$$V^{2\text{-CW}}_{group} = \frac{(\omega_8 - \omega_5)}{(k_8 - k_5)}$$

Simulating Complex Systems With Simpler Ones

Made of Quantum Dots

[Harter, J. Mol. Spec. 210, 166-182 (2001)]





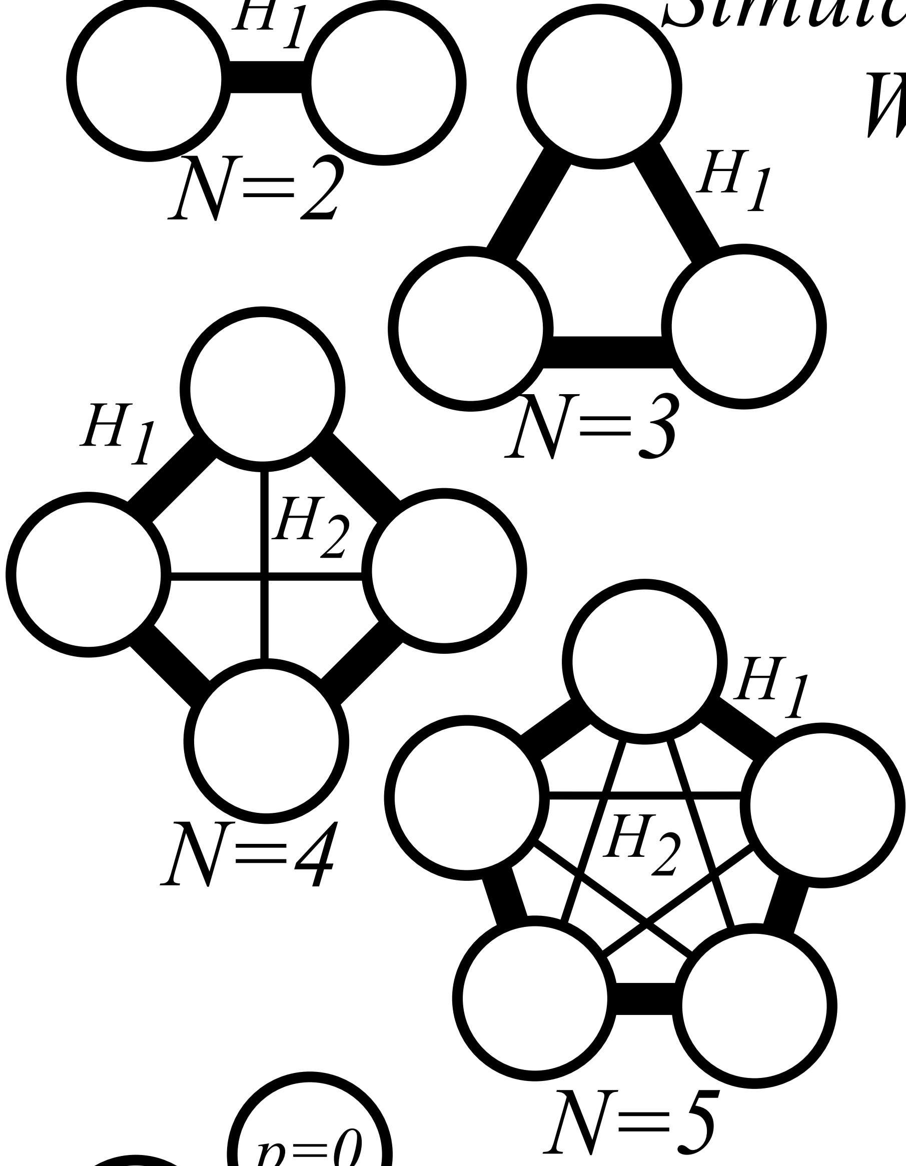
Hexagonal 2D Rotor

$$\begin{pmatrix} H_0 & H_1 & H_2 & H_3 & H_2 & H_1 \\ H1 & H_0 & H_1 & H_2 & H_3 & H_2 \\ H_2 & H_1 & H_0 & H_1 & H_2 & H_3 \\ H_3 & H_2 & H_1 & H_0 & H_1 & H_2 \\ H_2 & H_3 & H_2 & H_1 & H_0 & H_1 \\ H_1 & H_2 & H_3 & H_2 & H_1 & H_0 \end{pmatrix}$$

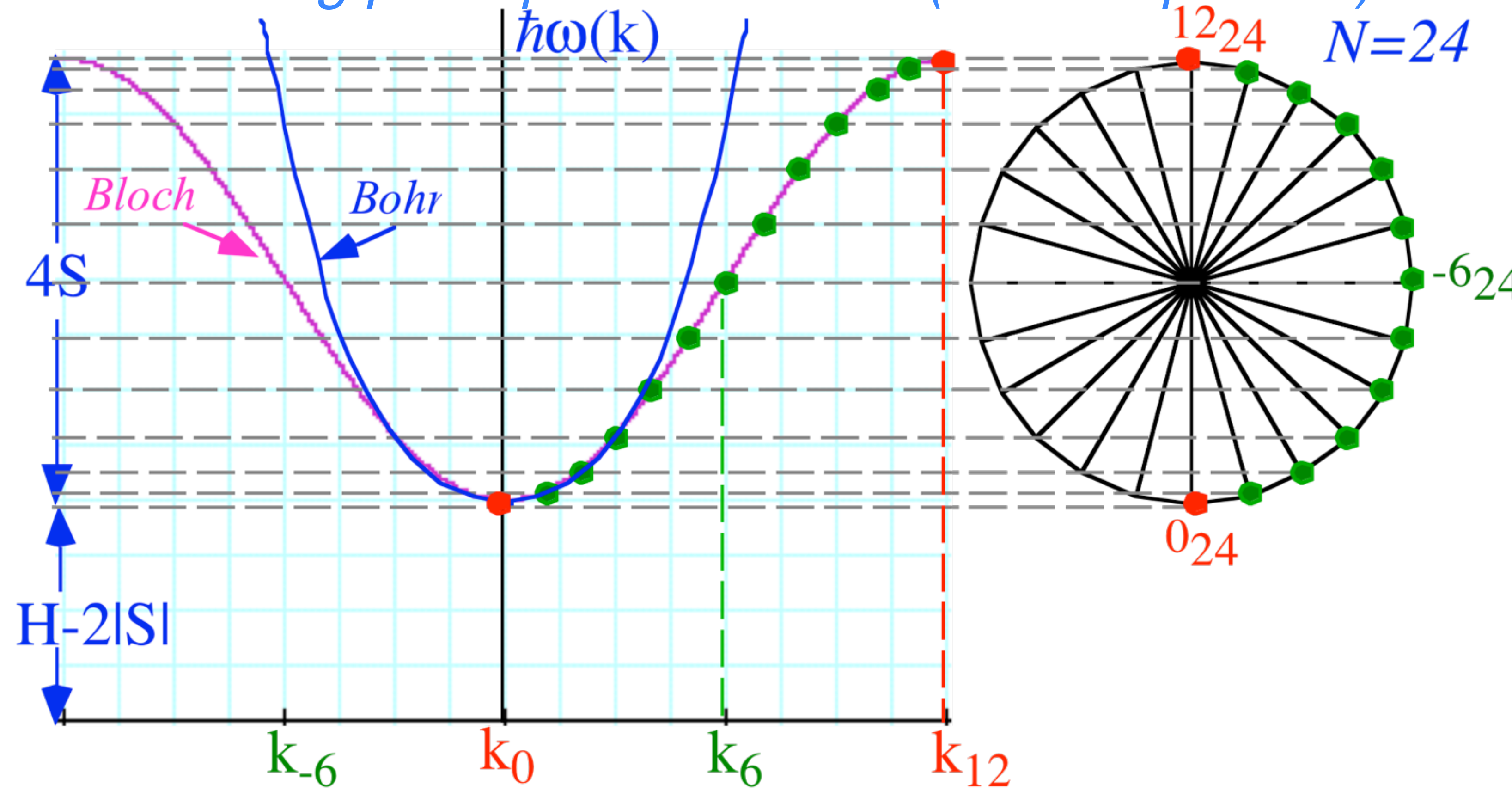
Simulating Complex Systems With Simpler Ones

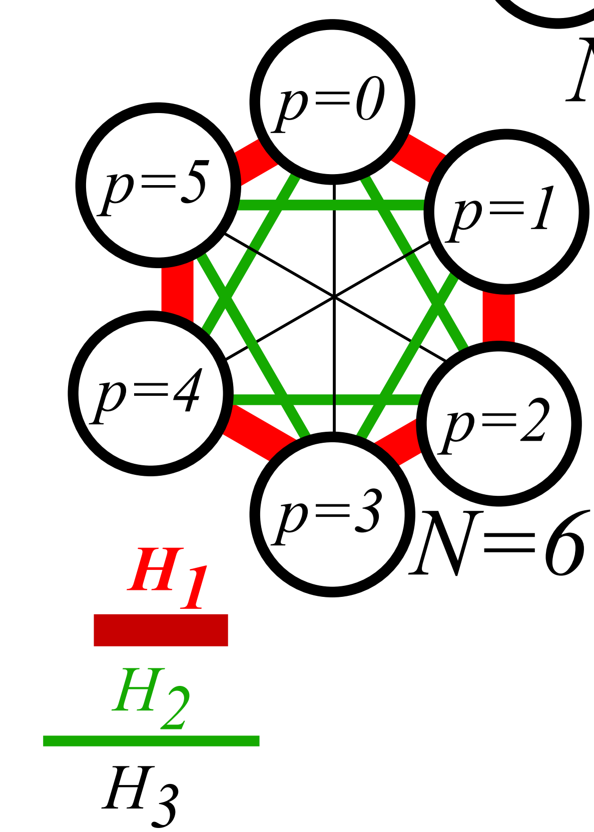
Made of Quantum Dots

[Harter, J. Mol. Spec. 210, 166-182 (2001)]



Making pure quadratic $\omega = ck^2$ (Bohr dispersion)





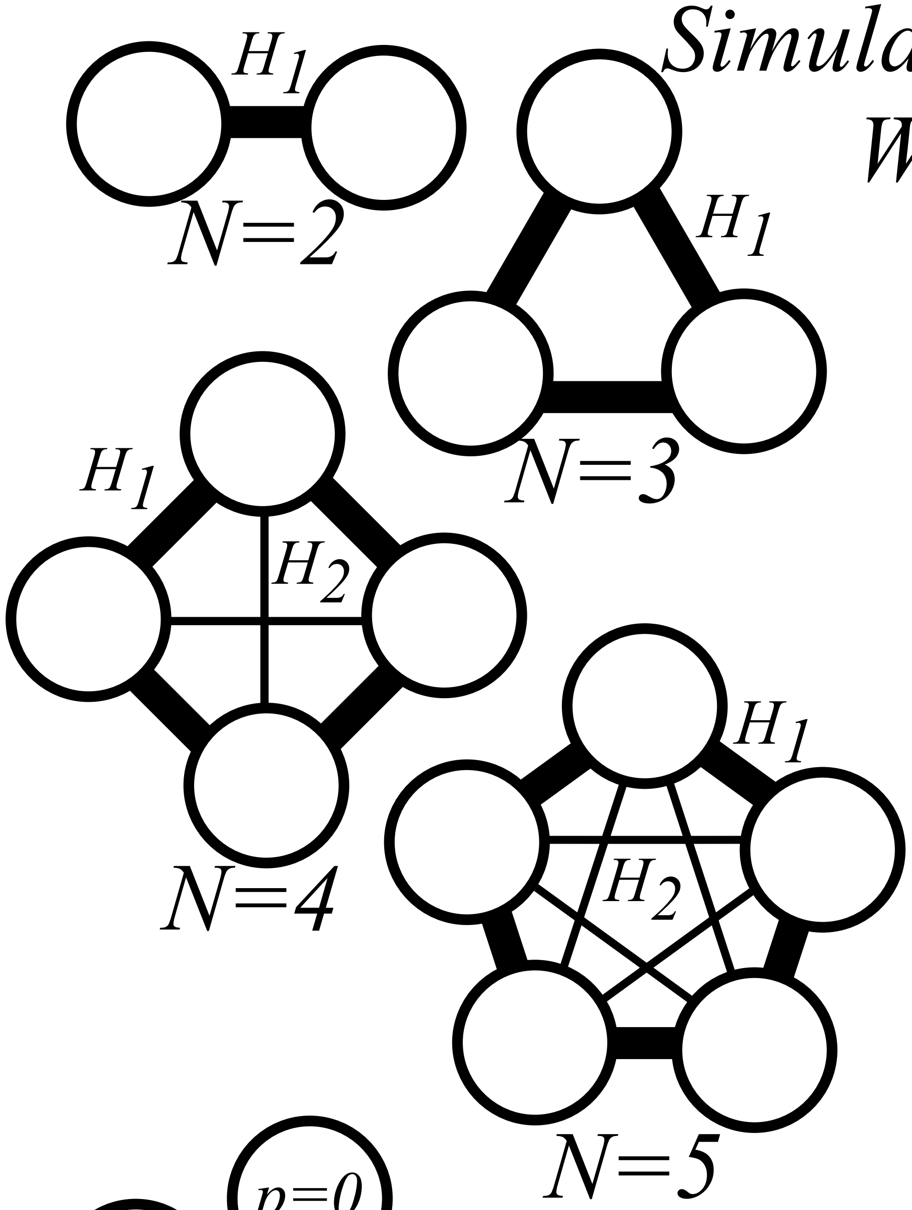
Hexagonal 2D Rotor

$$\begin{pmatrix} H_0 & \mathbf{H_1} & H_2 & H_3 & H_2 & \mathbf{H_1} \\ \mathbf{H1} & H_0 & \mathbf{H_1} & H_2 & H_3 & H_2 \\ H_2 & \mathbf{H_1} & H_0 & \mathbf{H_1} & H_2 & H_3 \\ H_3 & H_2 & \mathbf{H_1} & H_0 & \mathbf{H_1} & H_2 \\ H_2 & H_3 & H_2 & \mathbf{H_1} & H_0 & \mathbf{H_1} \\ \mathbf{H_1} & H_2 & H_3 & H_2 & \mathbf{H_1} & H_0 \end{pmatrix}$$

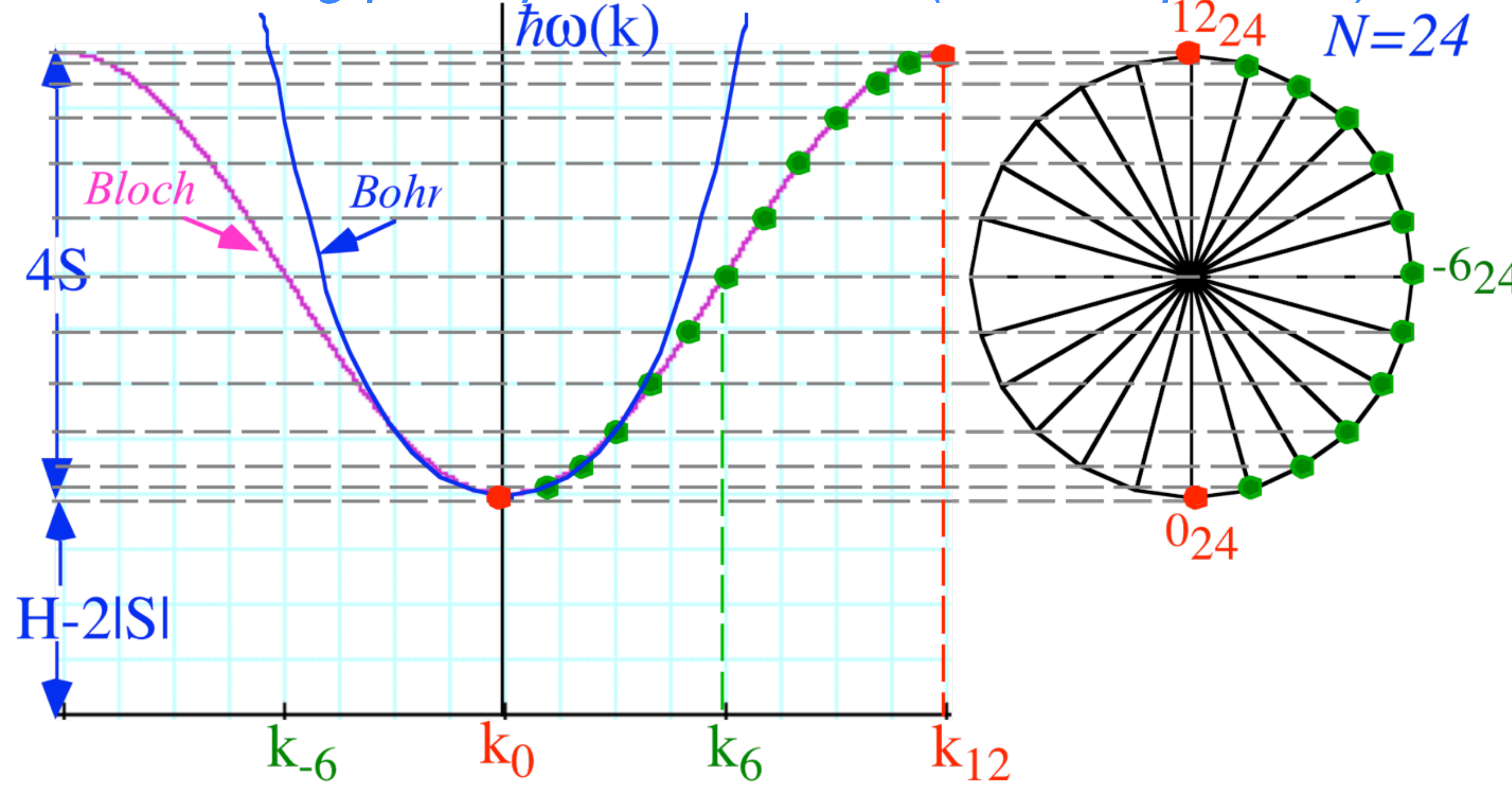
Simulating Complex Systems With Simpler Ones

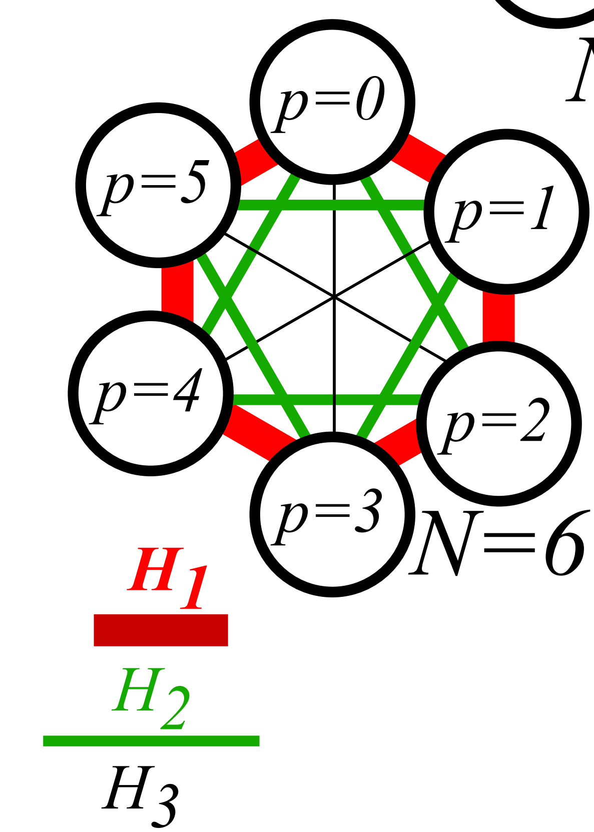
Made of Quantum Dots

[Harter, J. Mol. Spec. 210, 166-182 (2001)]



Making pure quadratic $\omega = ck^2$ (Bohr dispersion)





Hexagonal 2D Rotor

$$\begin{pmatrix} H_0 & H_1 & H_2 & H_3 & H_2 & H_1 \\ H1 & H_0 & H_1 & H_2 & H_3 & H_2 \\ H_2 & H_1 & H_0 & H_1 & H_2 & H_3 \\ H_3 & H_2 & H_1 & H_0 & H_1 & H_2 \\ H_2 & H_3 & H_2 & H_1 & H_0 & H_1 \\ H_1 & H_2 & H_3 & H_2 & H_1 & H_0 \end{pmatrix}$$

	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8
$N=2$	1/2	-1/2							
$N=3$	2/3	-1/3							
$N=4$	3/2	-1	1/2						
$N=5$	2	-1.1708	0.1708						
$N=6$	19/6	-2	2/3	-1/2					
$N=7$	4	-2.393	0.51	-0.1171					
$N=8$	11/2	-3.4142	1	-0.5858	1/2				
$N=9$	20/3	-4.0165	0.9270	-1/3	0.0895				
$N=10$	17/2	-5.2361	1.4472	-0.7639	0.5528	-1/2			
$N=11$	10	-6.0442	1.4391	-0.5733	0.2510	-0.0726			
$N=12$	73/6	-7.4641	2	-1	2/3	-0.5359	1/2		
$N=13$	14	-8.4766	2.0500	-0.8511	0.4194	-0.2028	0.06116		
$N=14$	33/2	-10.098	2.6560	-1.2862	0.8180	-0.6160	0.5260	-1/2	
$N=15$	57/3	-11.314	2.7611	-1.1708	0.6058	-1/3	0.1708	-0.0528	
$N=16$	43/2	-13.137	3.4142	-1.6199	1	-0.7232	0.5858	-0.5198	1/2
$N=17$	24	-14.557	3.5728	-1.5340	0.81413	-0.4732	0.2781	-0.1479	0.0465

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry

C_2 symmetric (B-type) modes

Harmonic oscillator with cyclic C_3 symmetry

C_3 symmetric spectral decomposition by 3rd roots of unity

Resolving C_3 projectors and moving wave modes

Dispersion functions and standing waves

C_6 symmetric mode model:Distant neighbor coupling

C_6 spectra of gauge splitting by C-type symmetry(complex, chiral, coriolis, current, ..

C_N symmetric mode models: Made-to order dispersion functions

Quadratic dispersion models: Super-beats and fractional revivals

➔ *Phase arithmetic*

2-level-system and C_2 symmetry phase dynamics

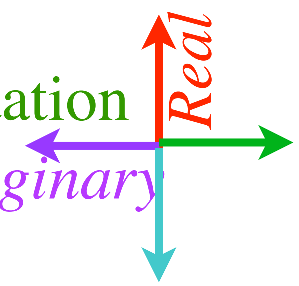
C_2 Character Table describes eigenstates

symmetric A_1

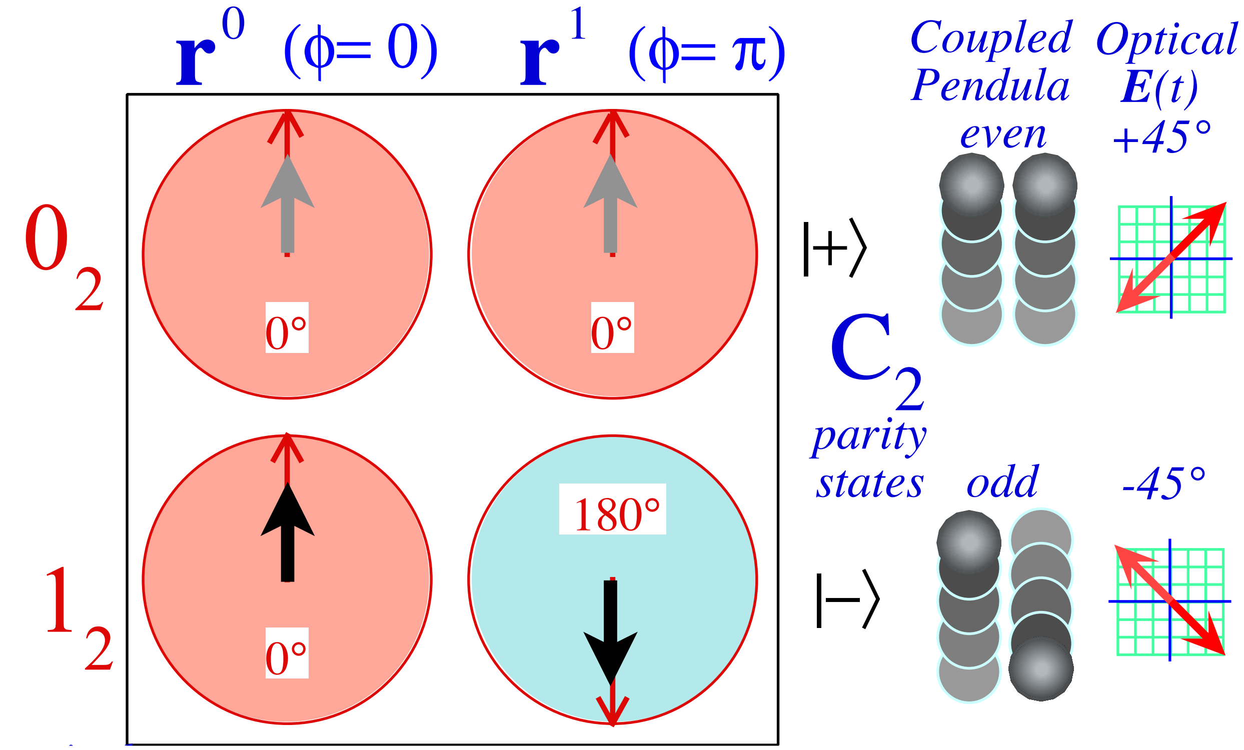
vs.

antisymmetric A_2

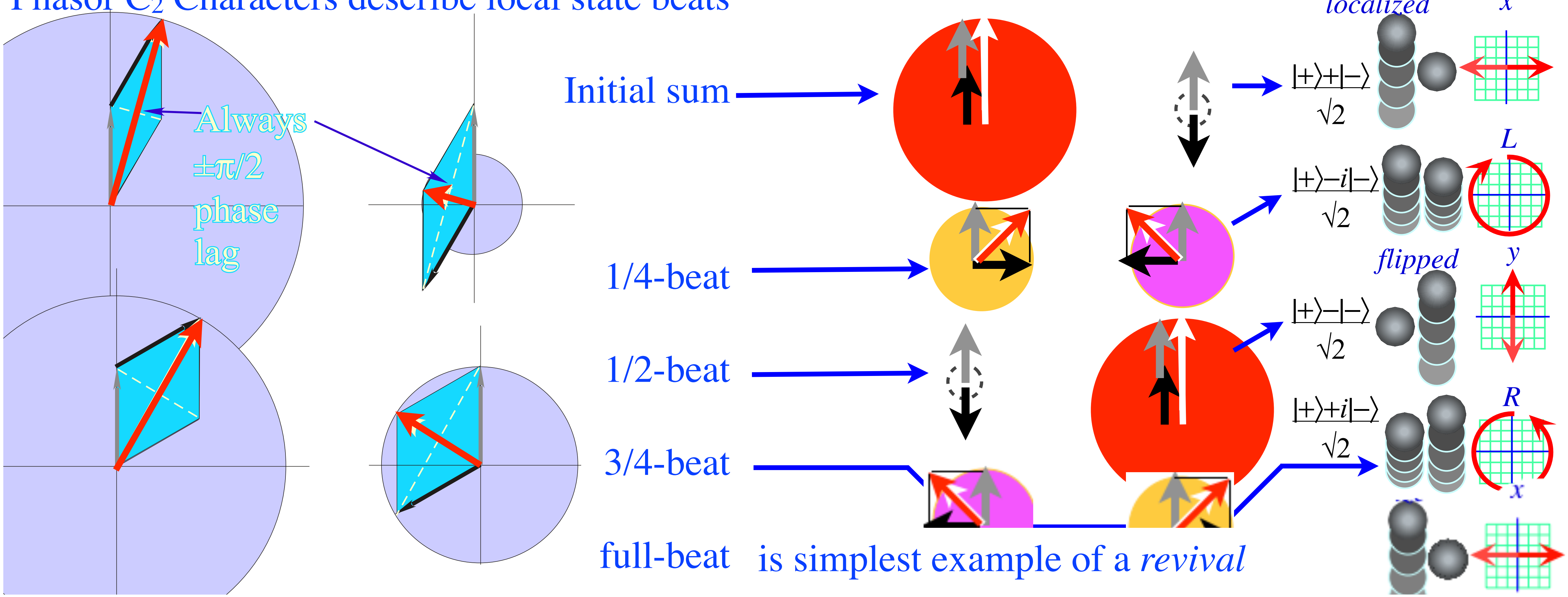
	$1=r^0$	$r=r^1$
$0 \bmod 2$	1	1
$\pm 1 \bmod 2$	1	-1



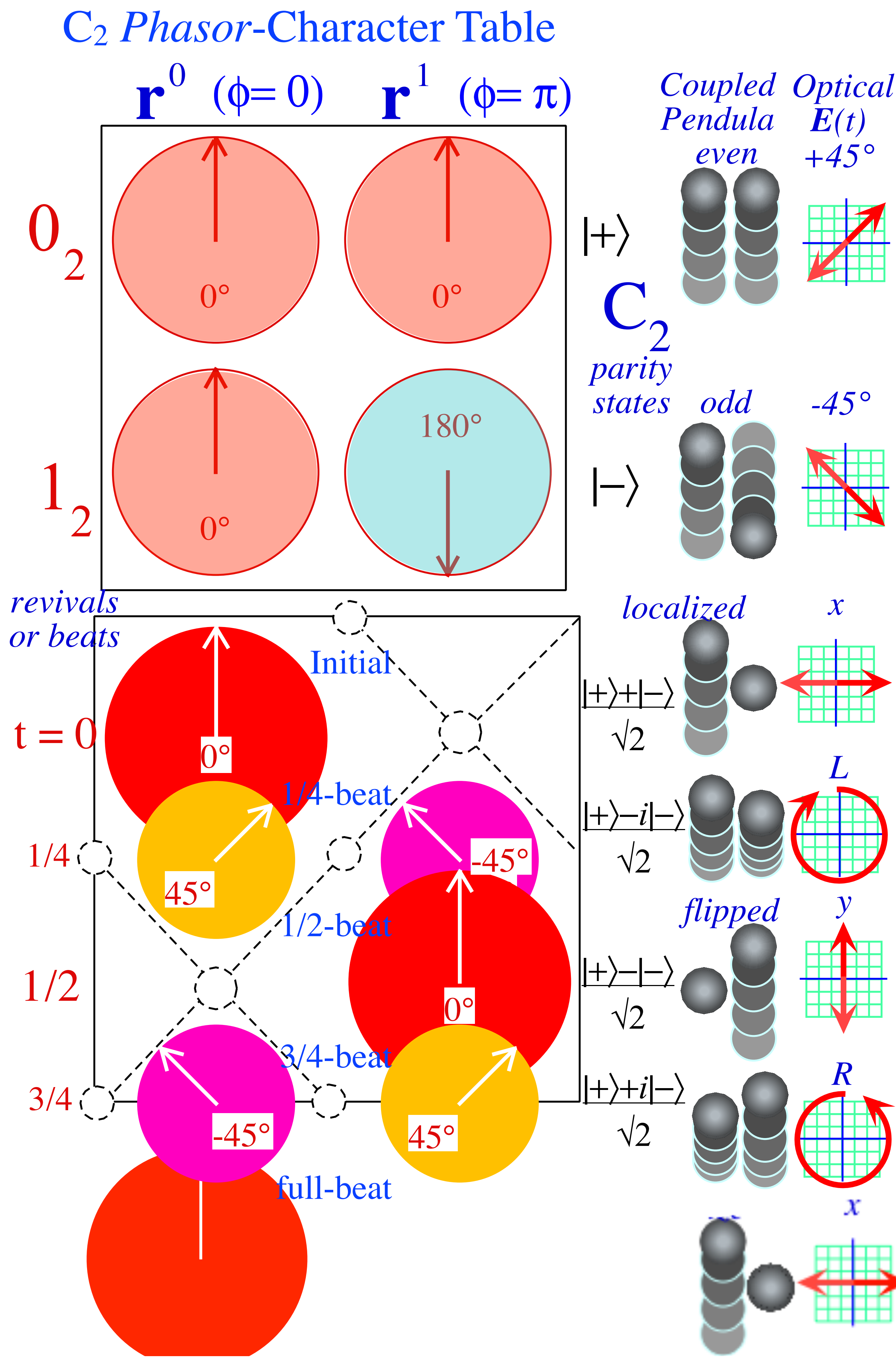
C_2 *Phasor*-Character Table

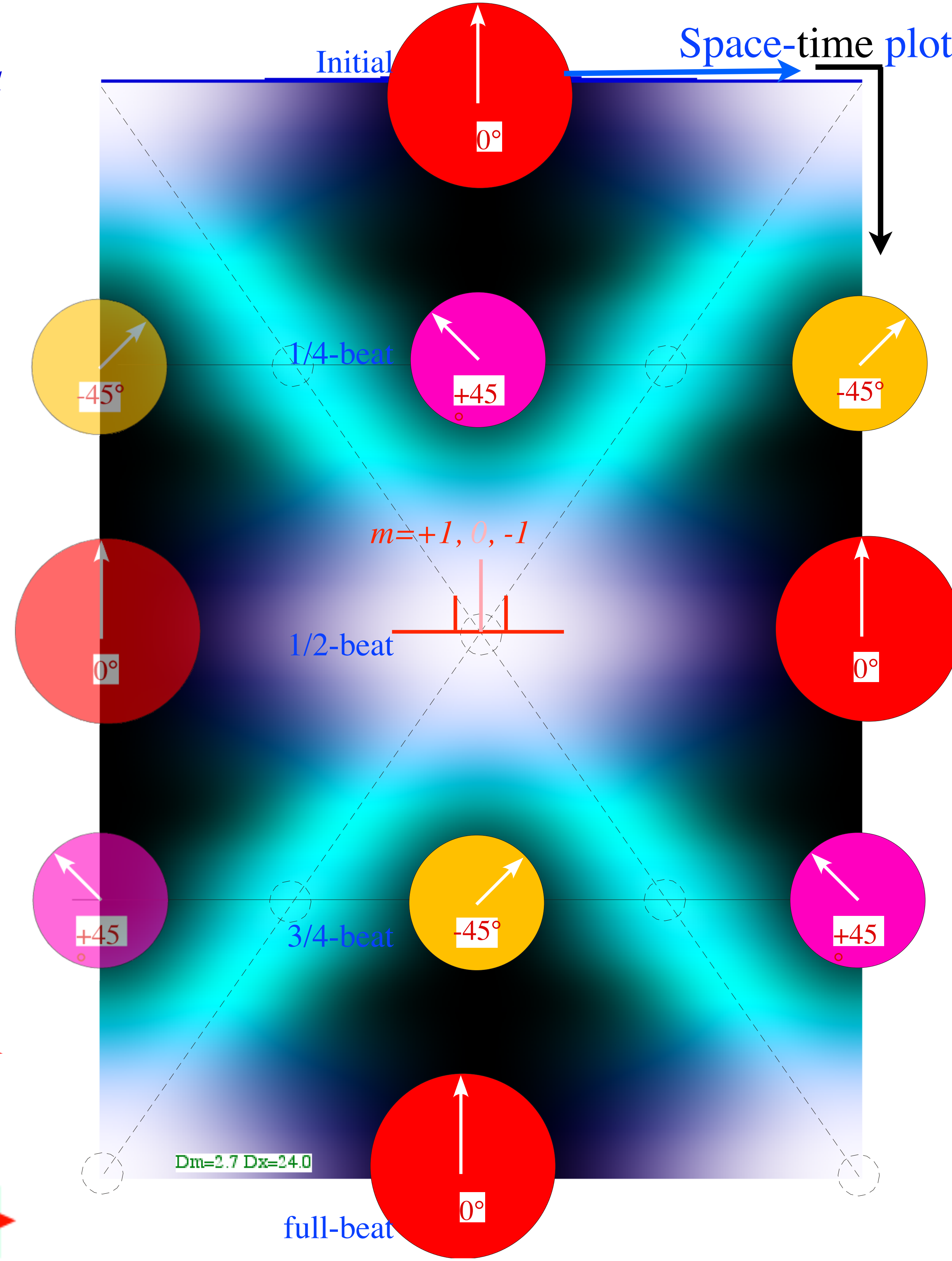


Phasor C_2 Characters describe local state beats

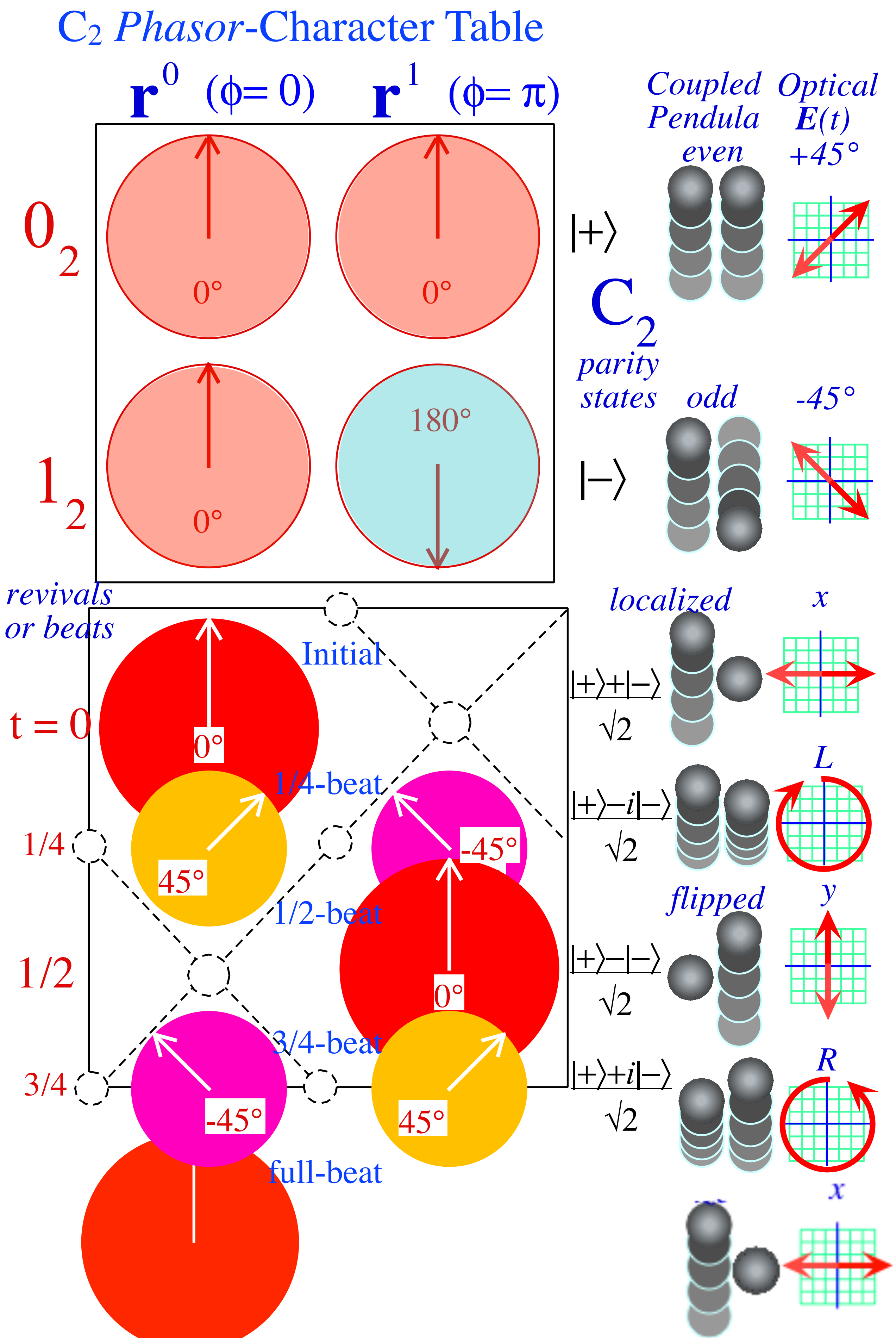


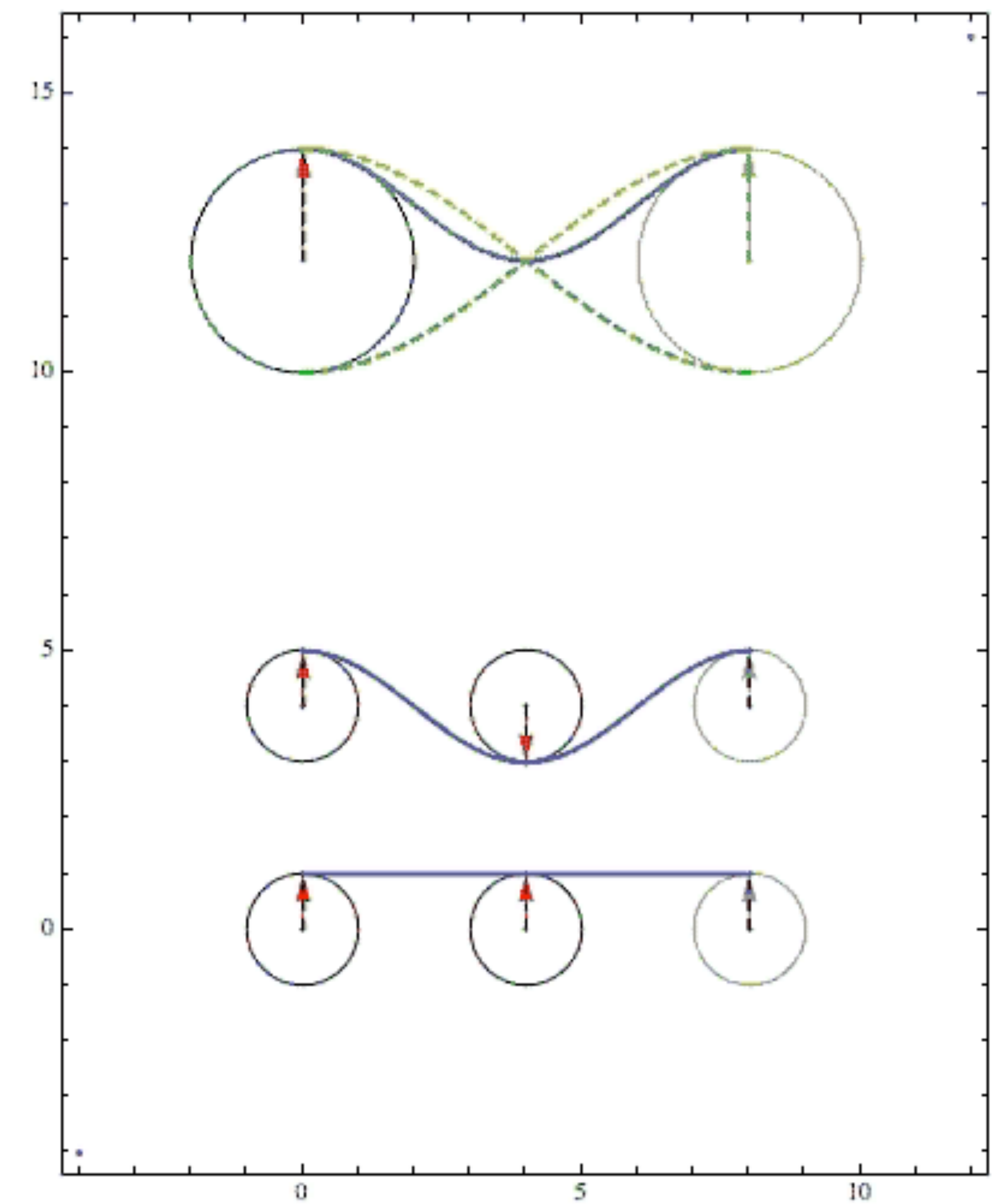
2-level-system and C_2 symmetry phase dynamics



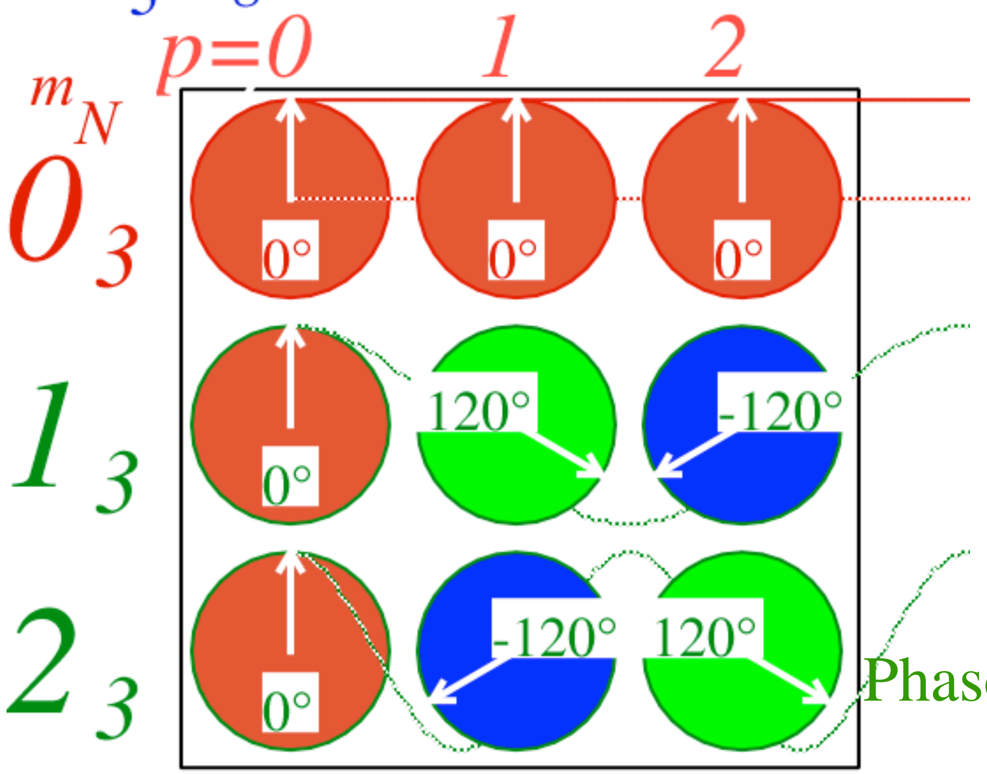


2-level-system and C_2 symmetry phase dynamics



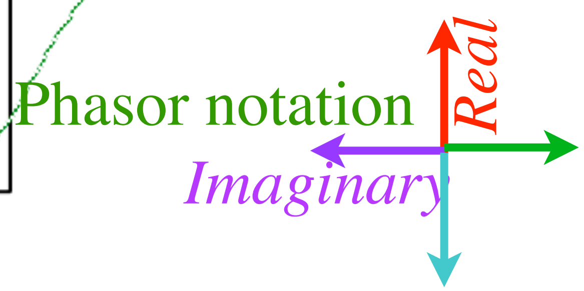


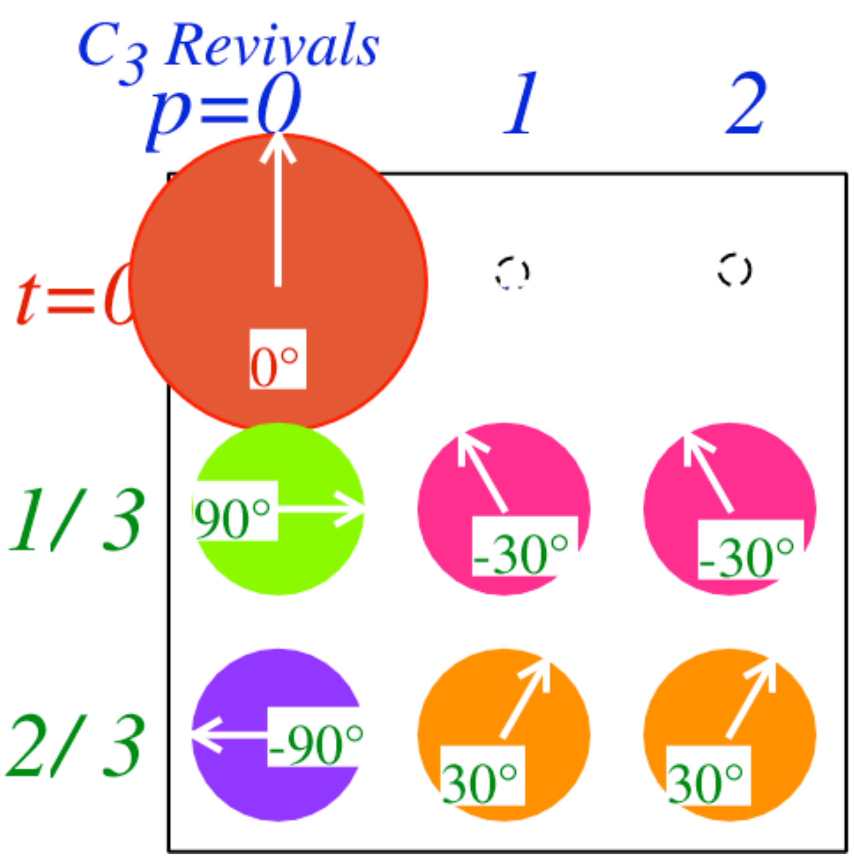
C_3 symmetry phase in *1*, *2*, *or 3*-level-systems

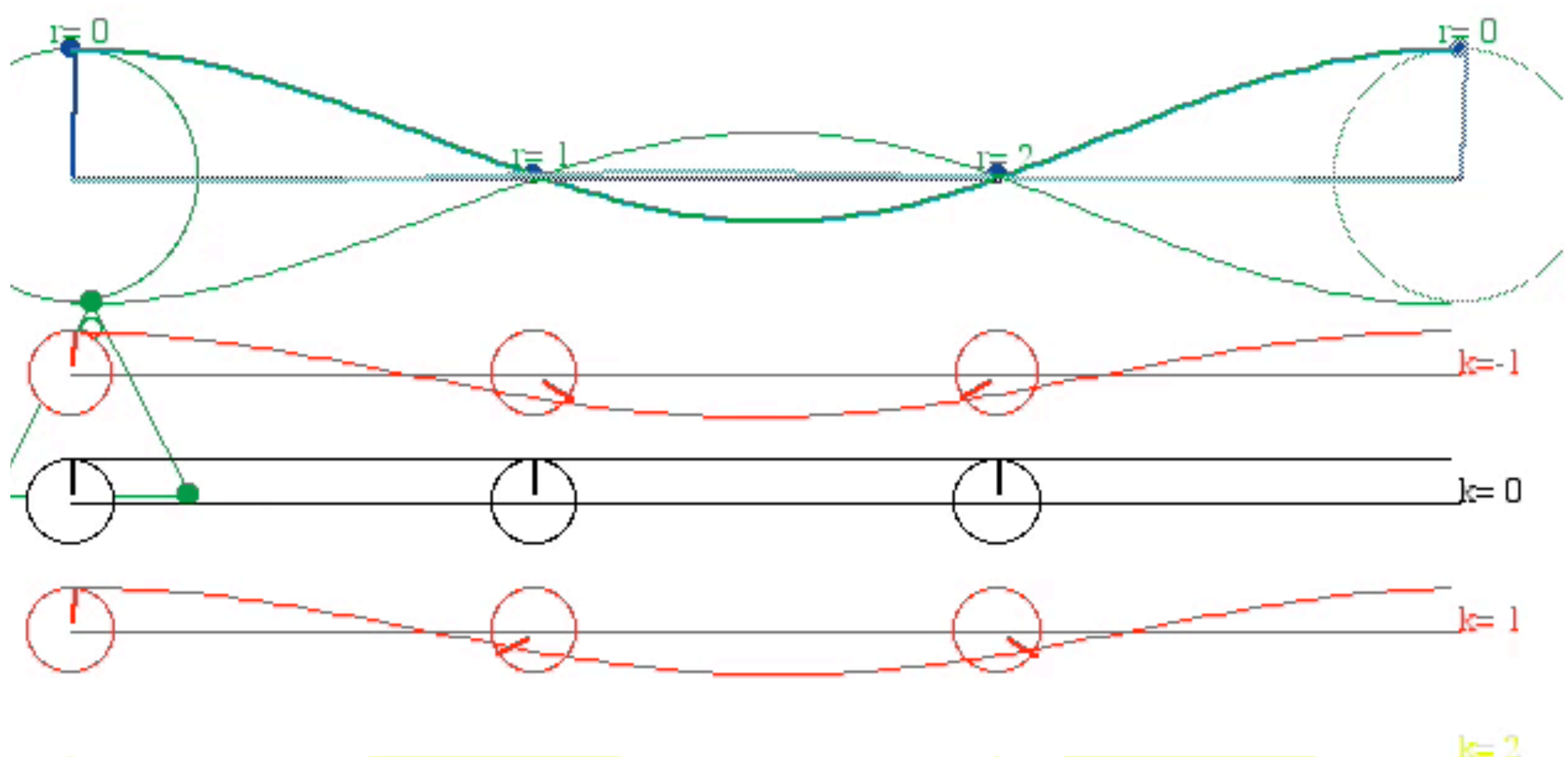


Non - chiral
C_{3v} system

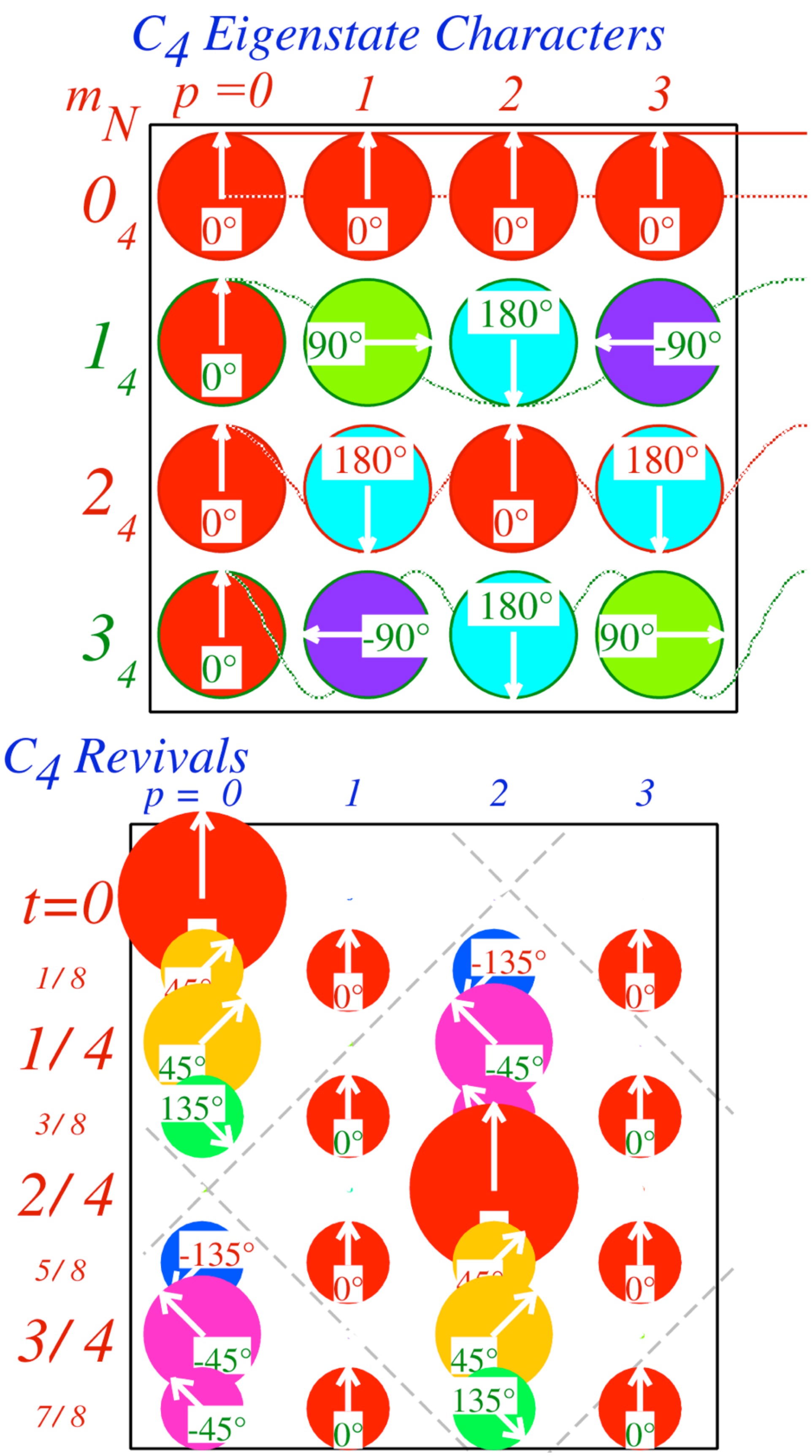
Chiral
"quantum-Hall-like"
systems
deserve special treatment



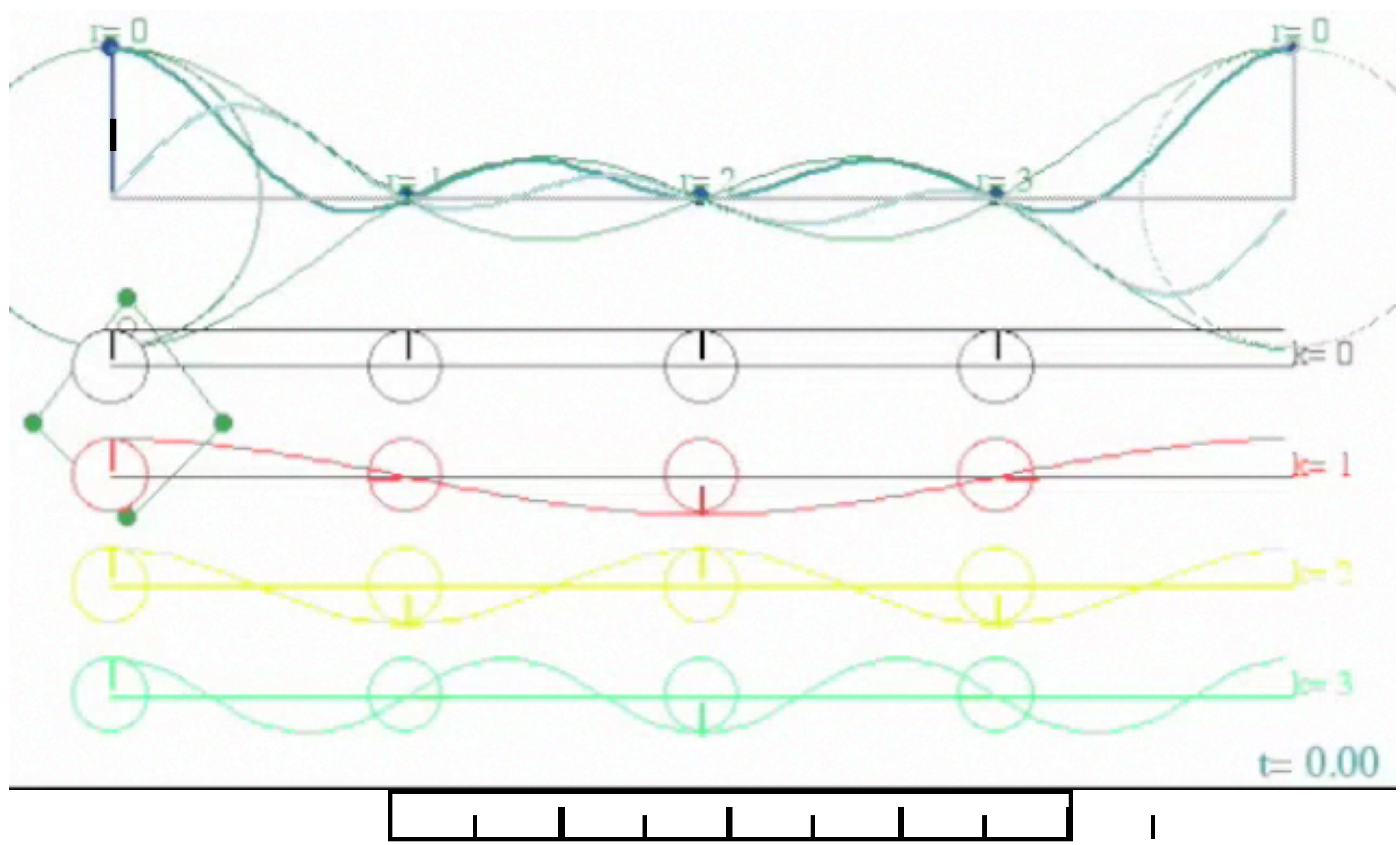




C_4 symmetry phase in 1, 2, 3 , or 4 level-systems

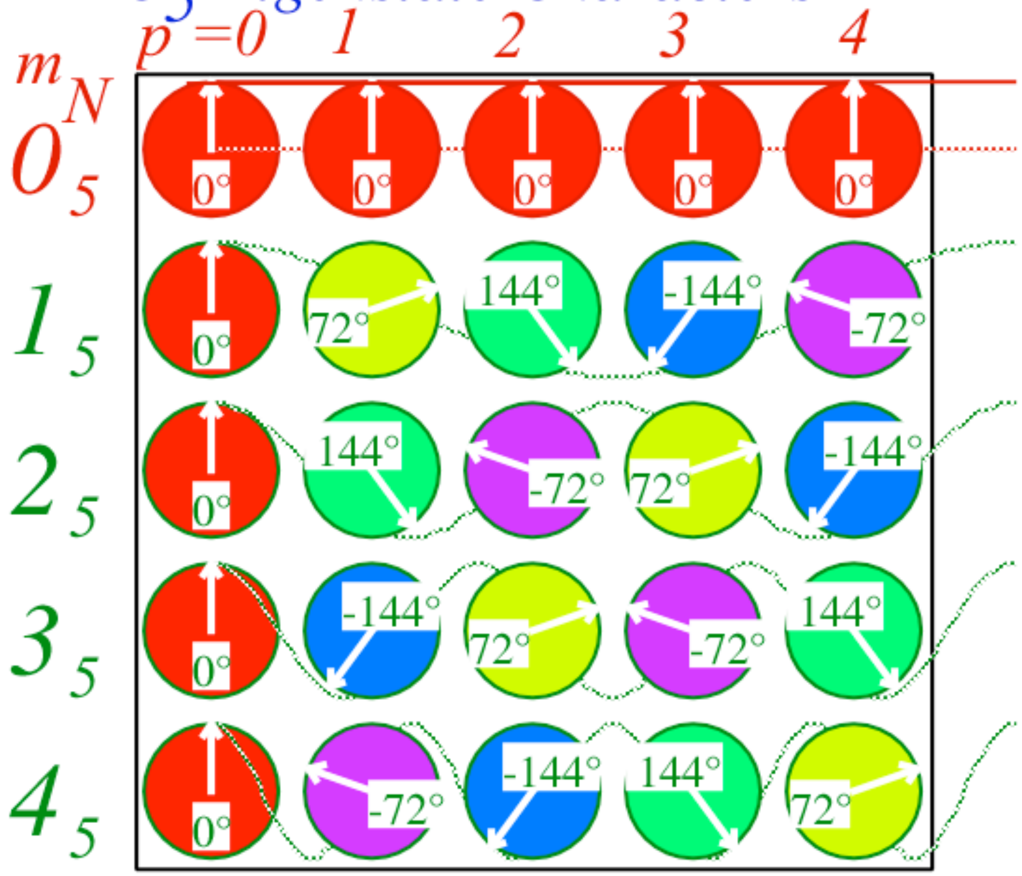


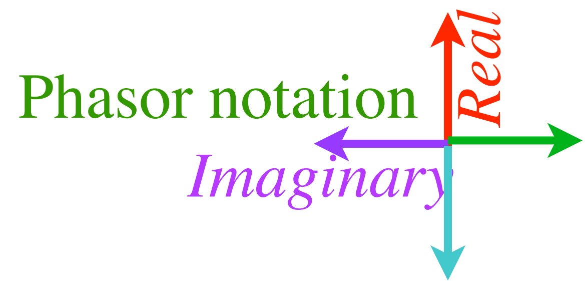
Non - chiral C_{4v} system



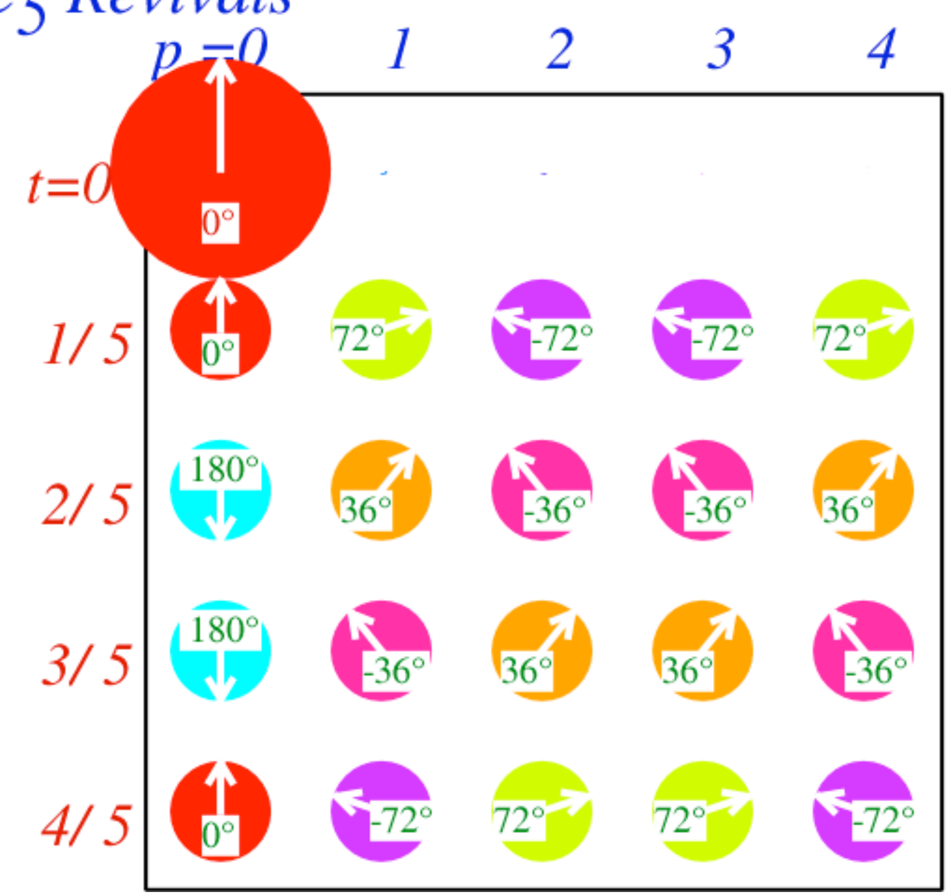
C_5 symmetry phase in *1*, *2*,...*5* level-systems

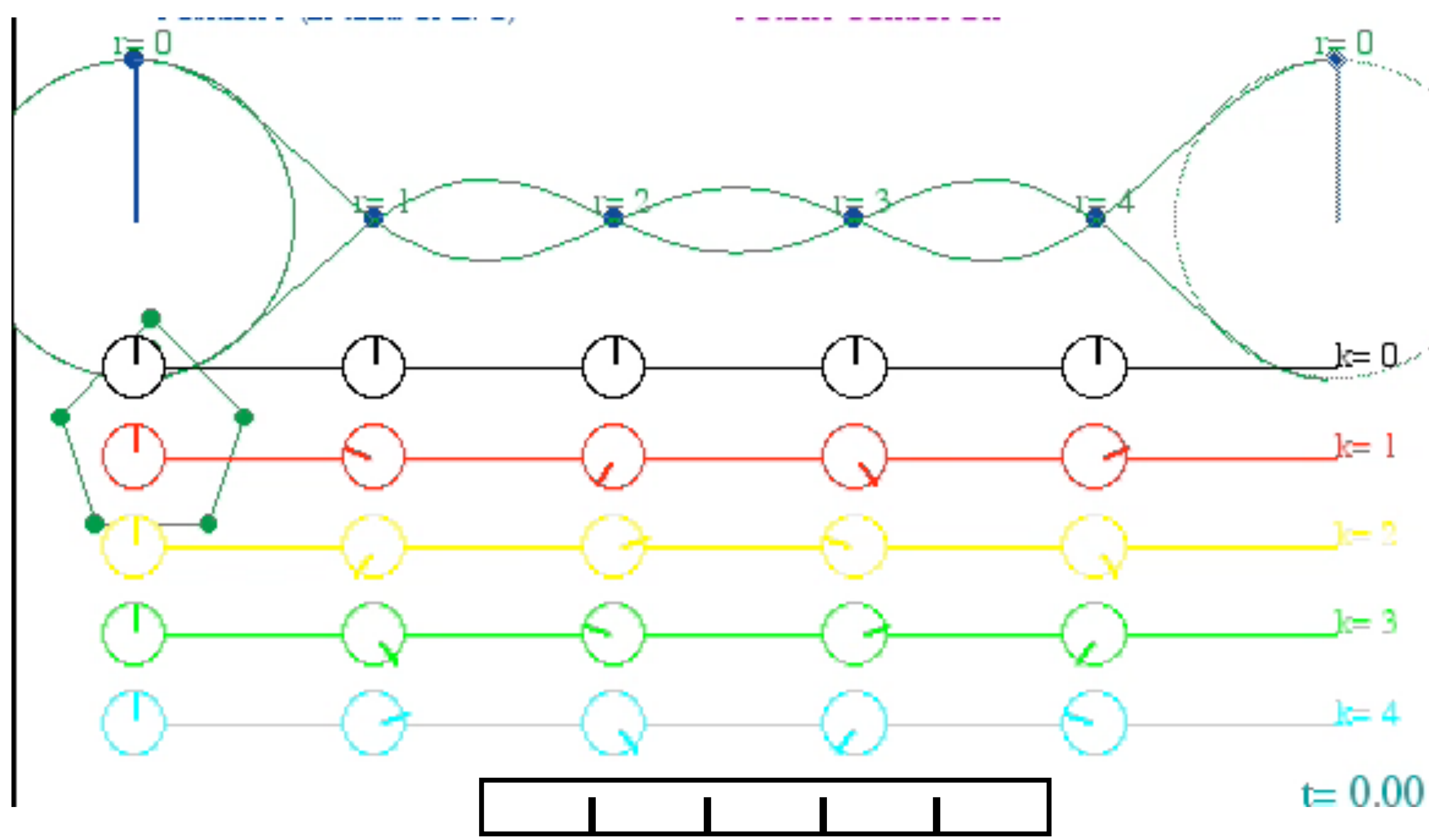
C_5 Eigenstate Characters



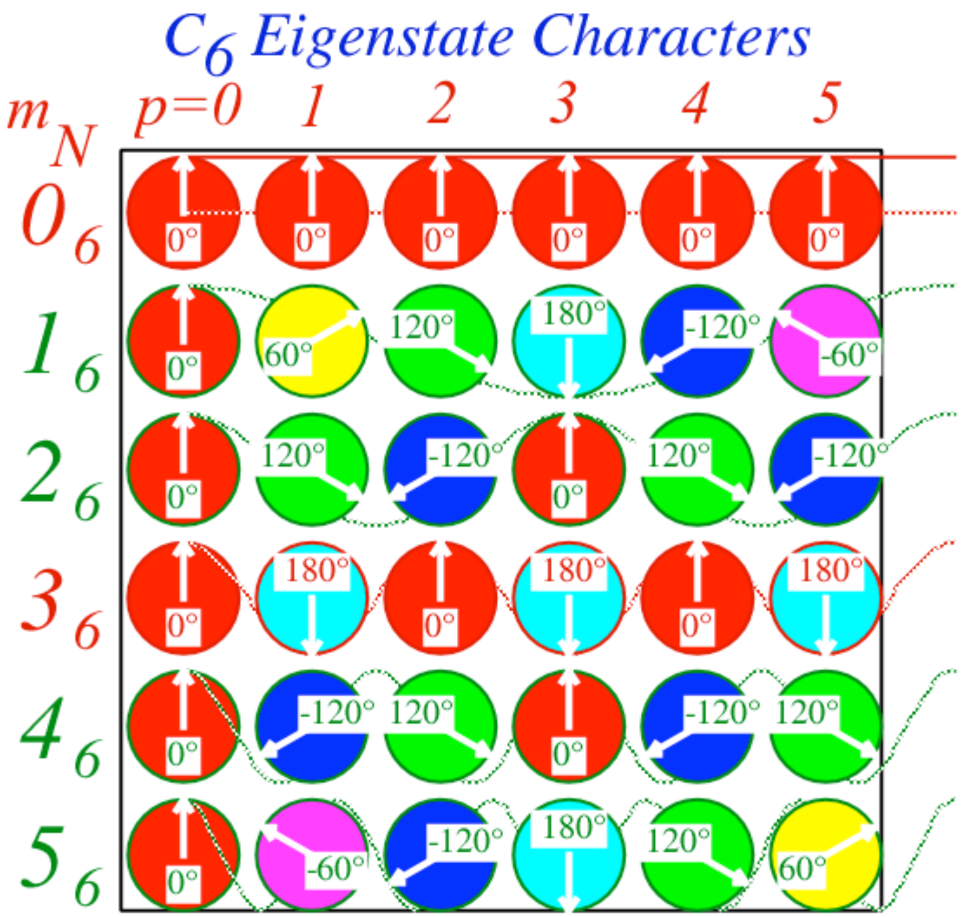


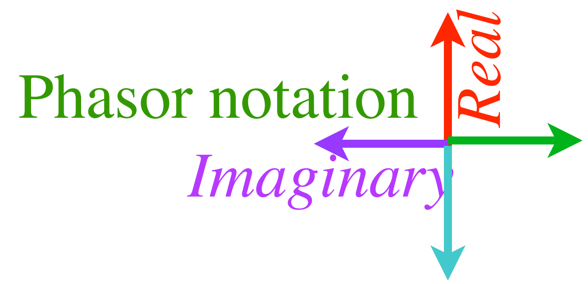
C_5 Revivals

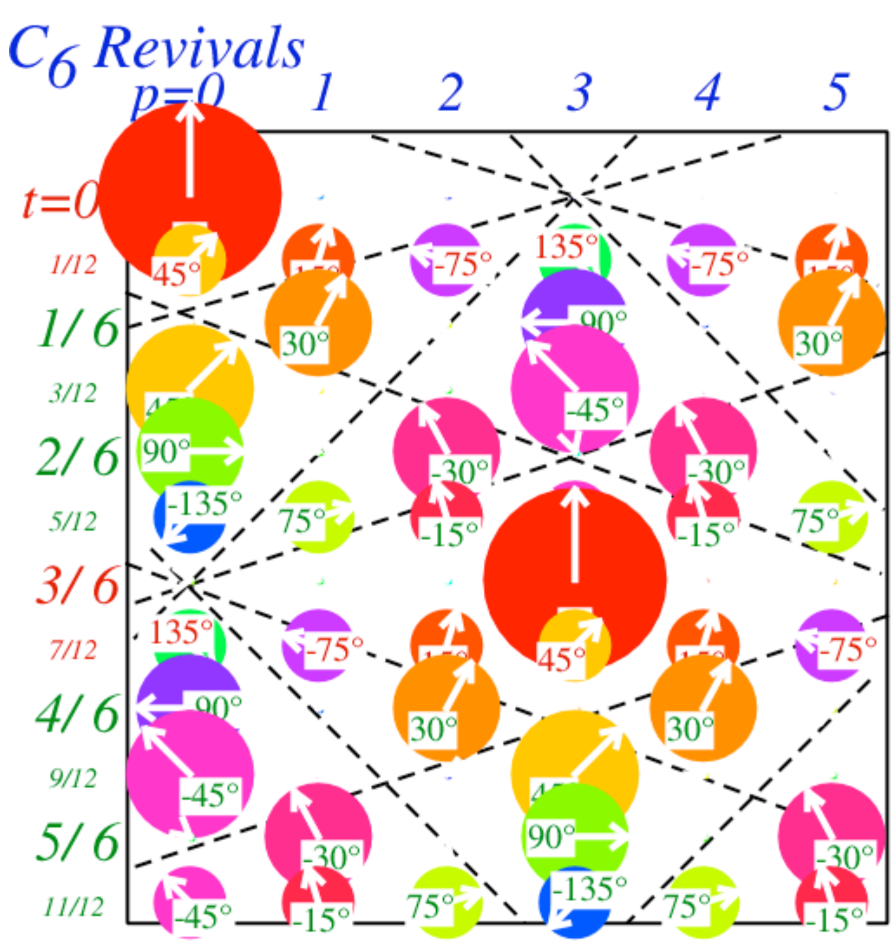


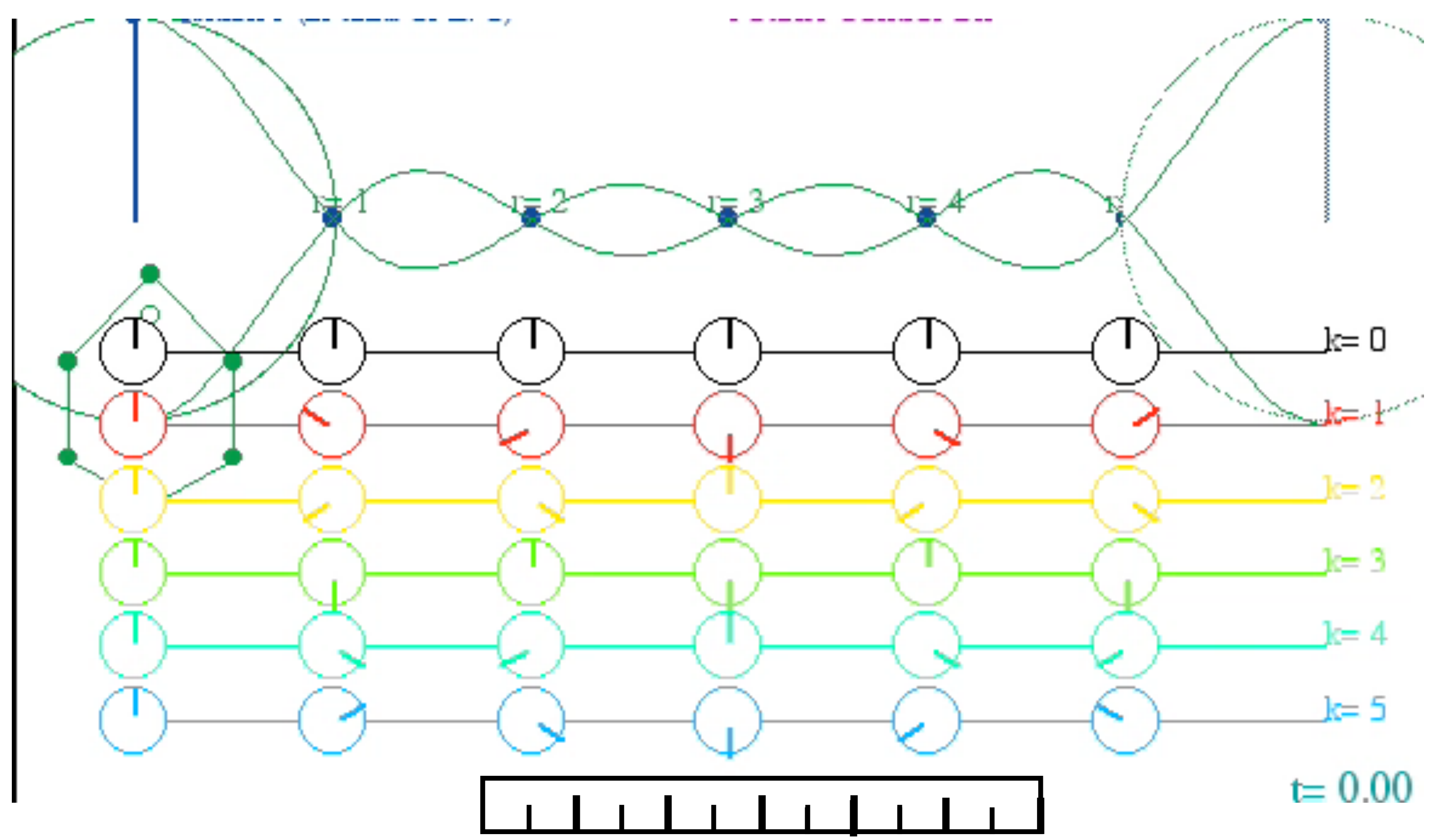


C_6 symmetry phase in *1, …6* level-systems





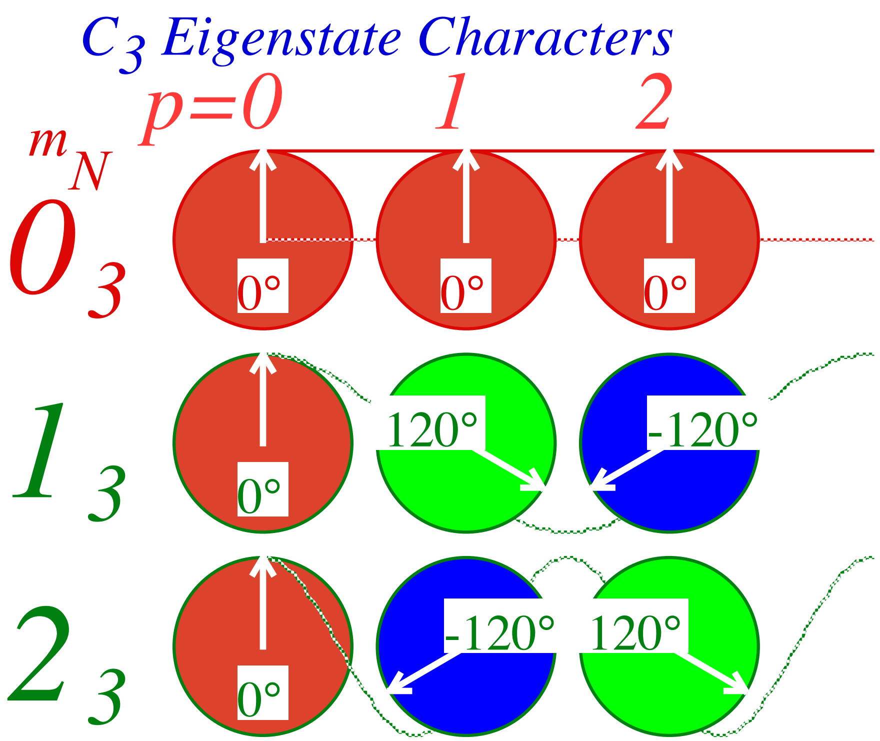


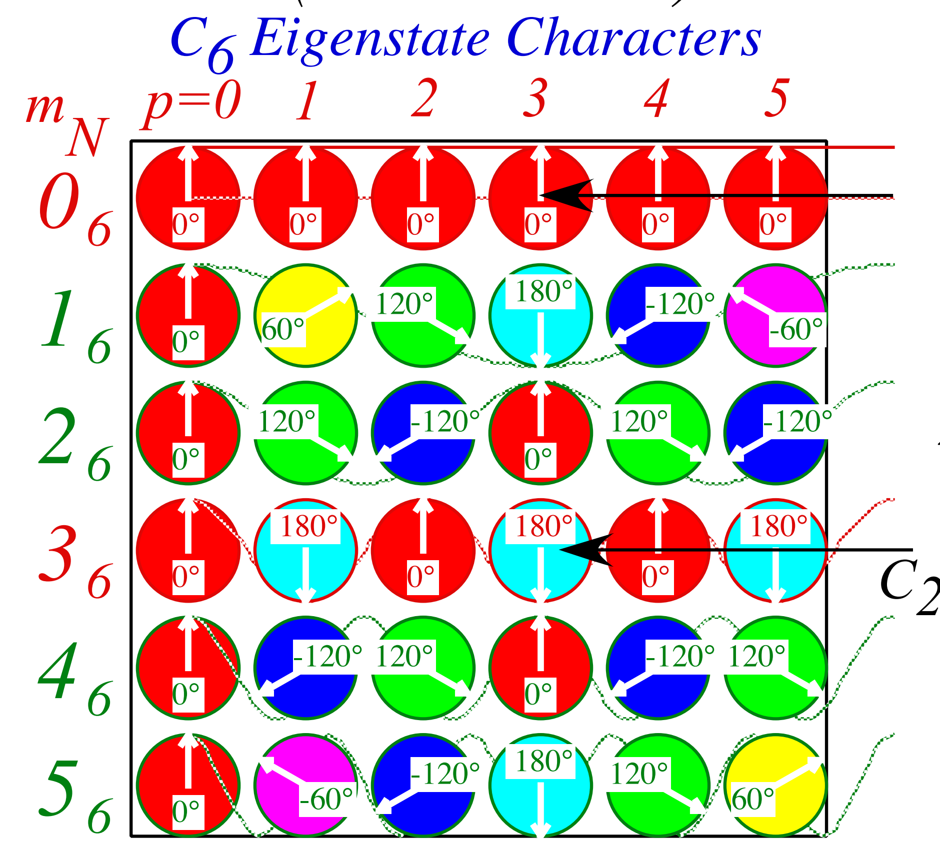


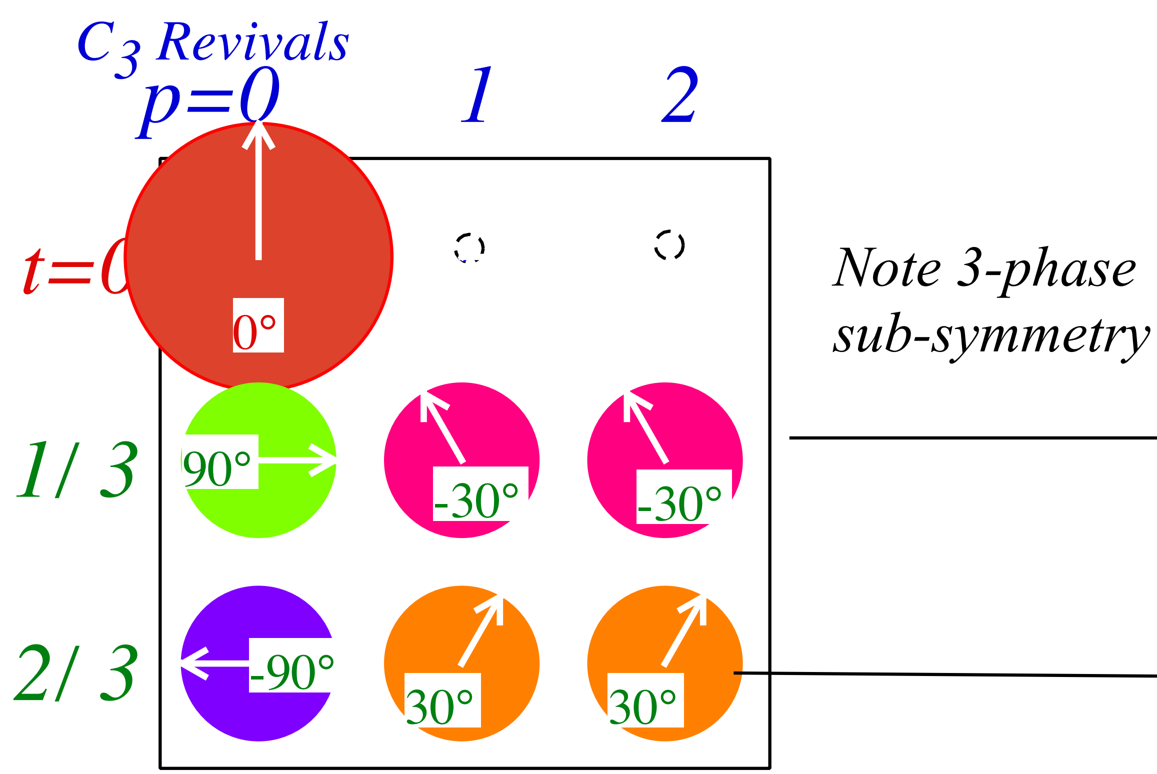
C_m algebra of revival-phase dynamics

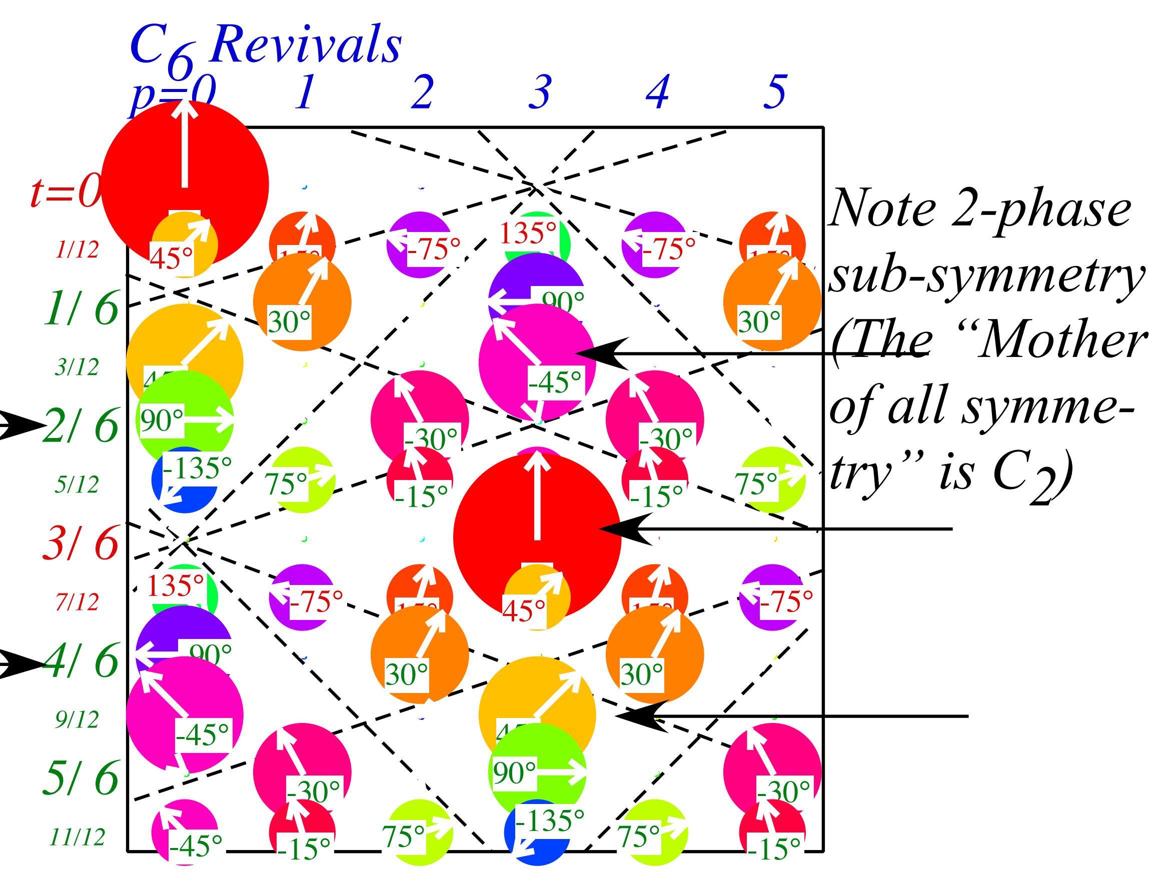
Discrete 3-State or Trigonal System (Tesla's 3-Phase AC)

Discrete 6-State or Hexagonal System (6-Phase AC)



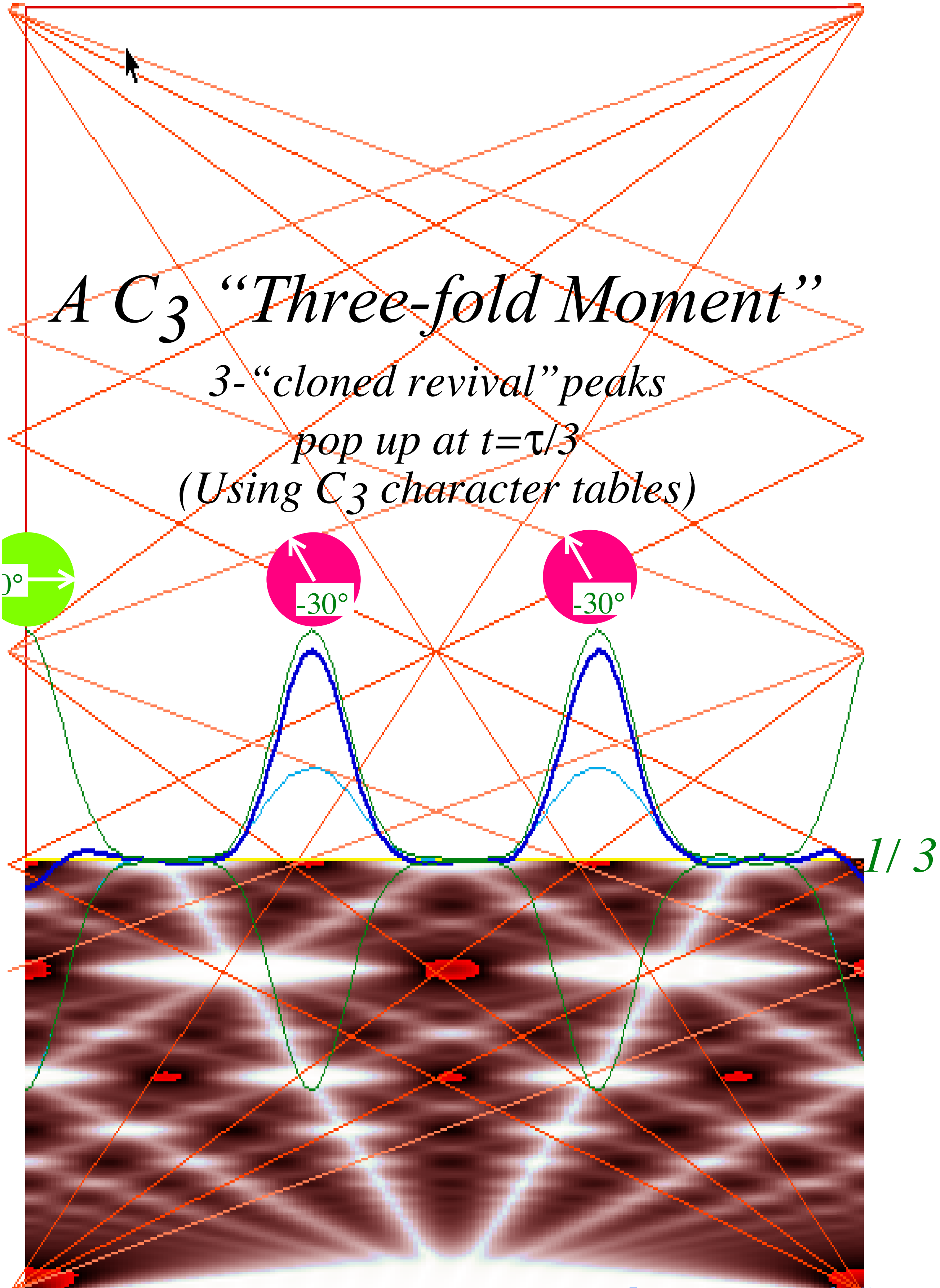


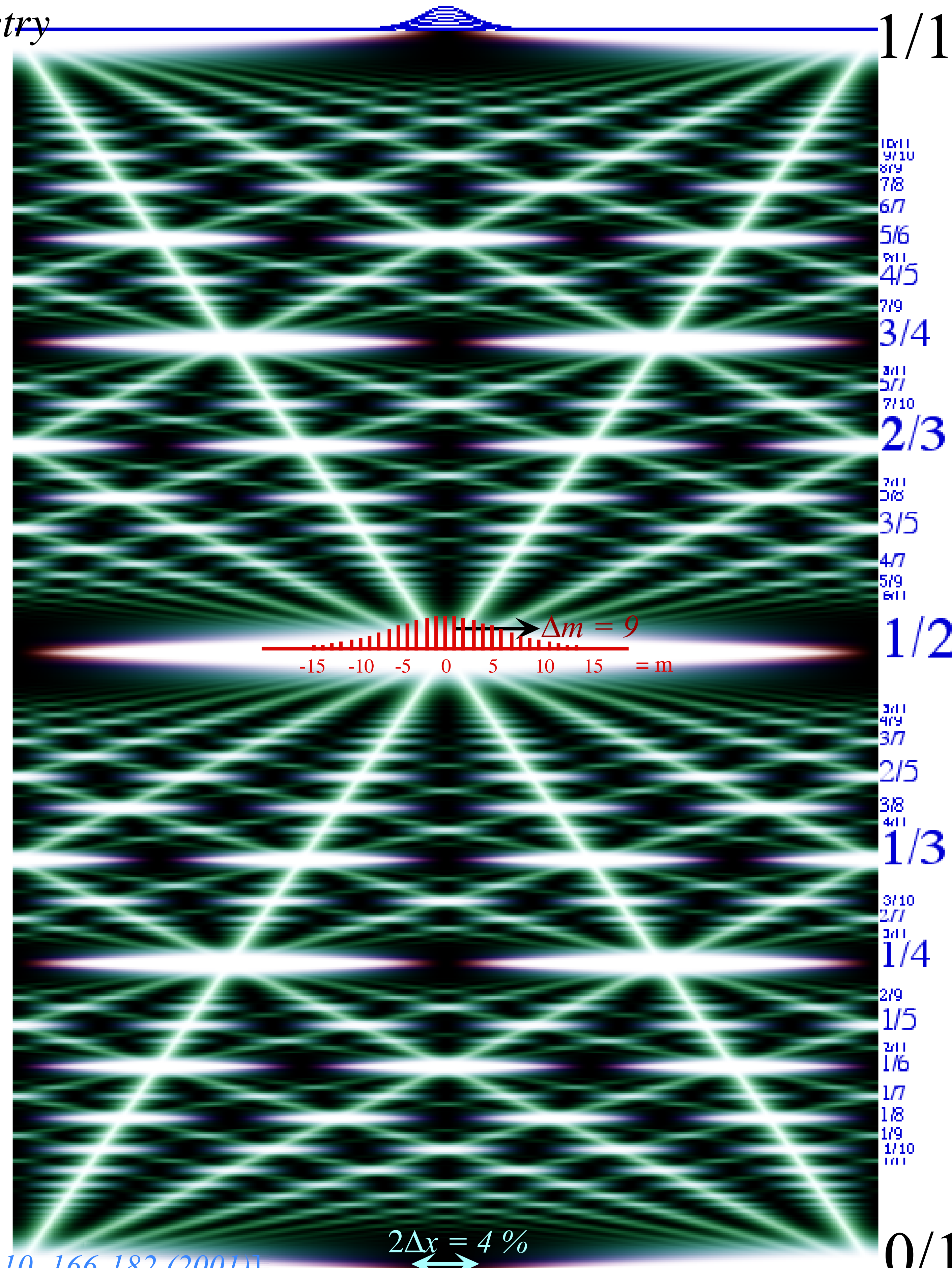




C_m algebra of revival-phase dynamics

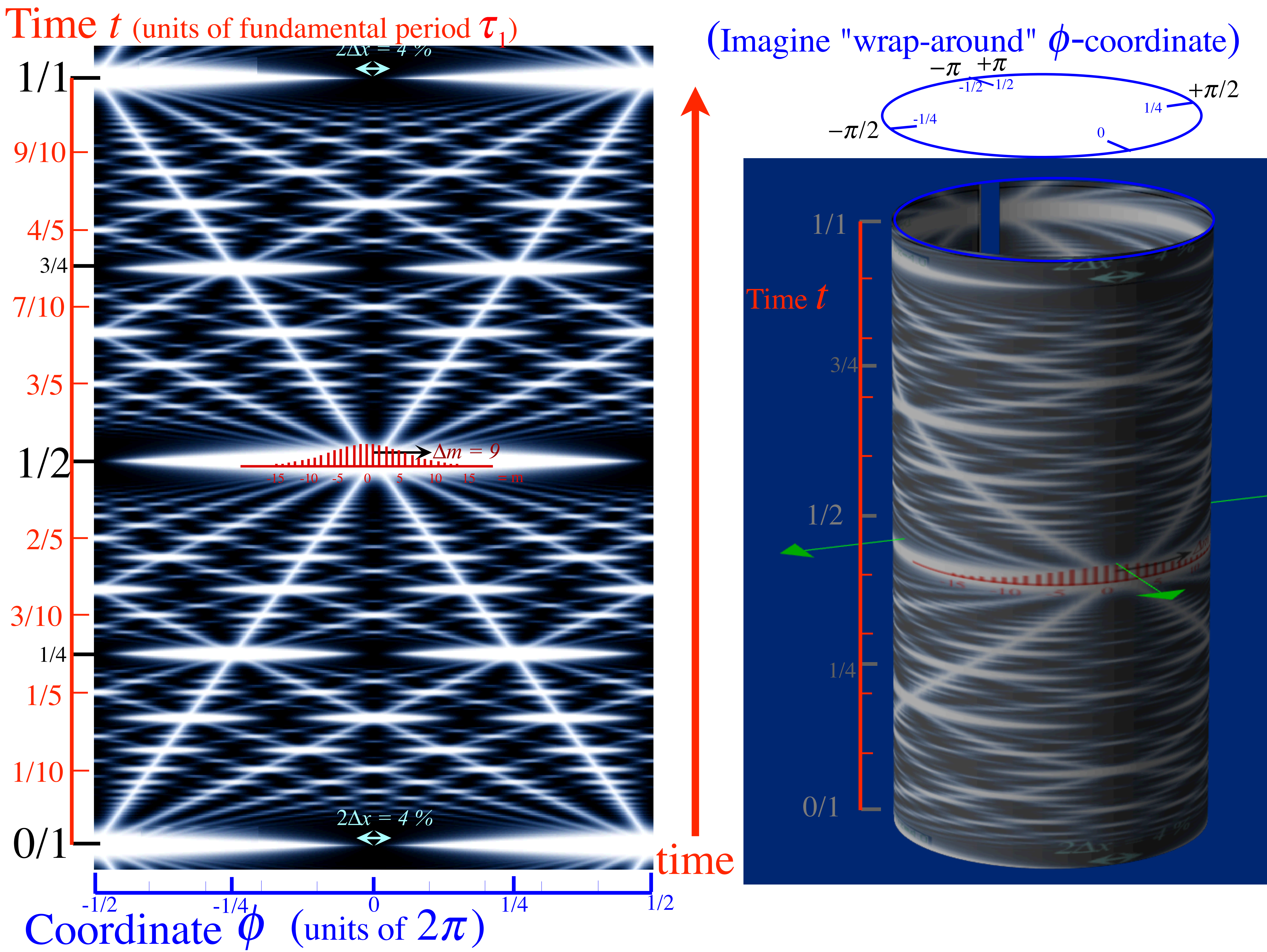
Quantum rotor fractional take turns at Cn symmetry





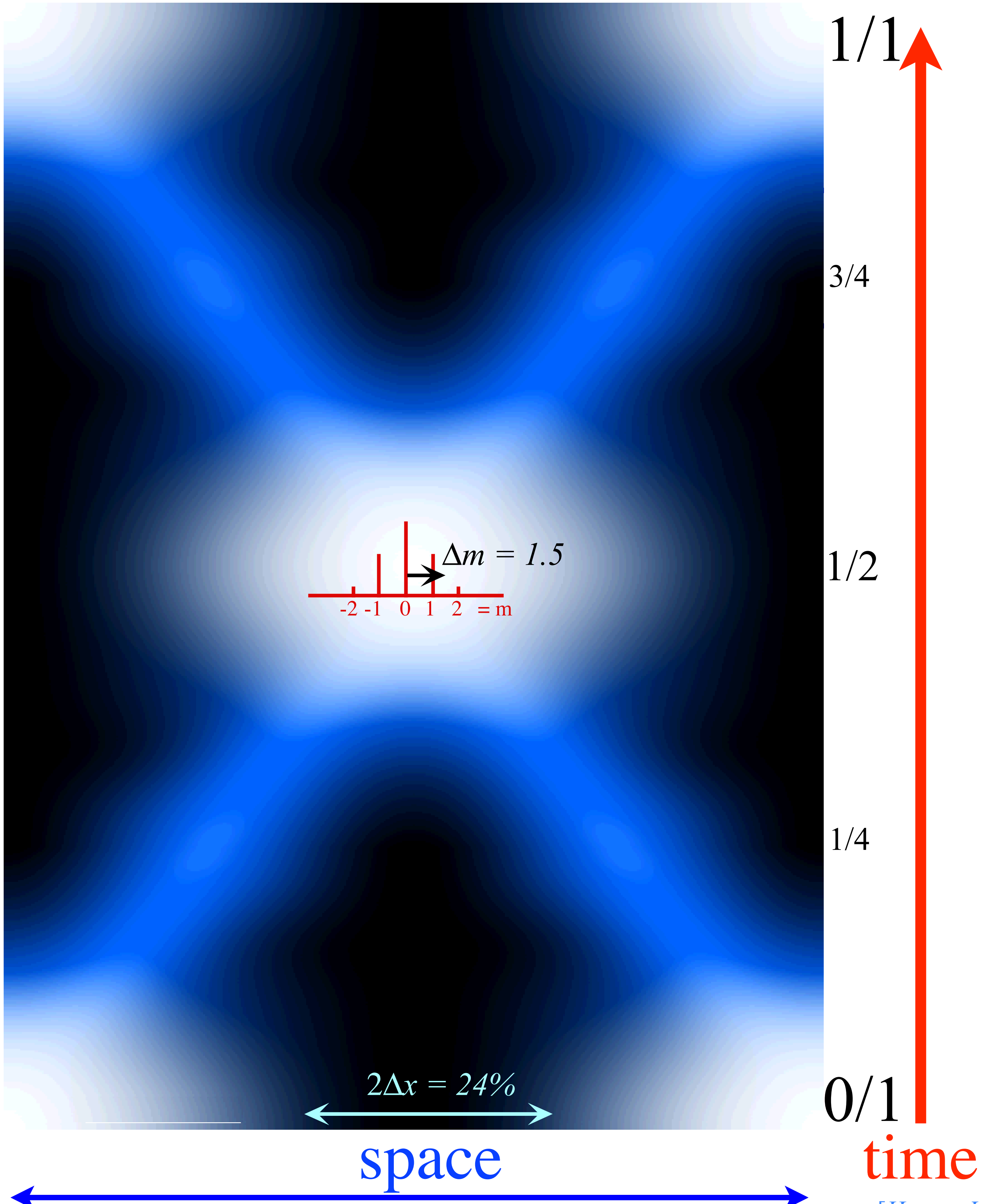
[Harter, J. Mol. Spec. 210, 166-182 (2001)]

Algebra and geometry of resonant revivals: Farey Sums and Ford Circles


Time t (units of fundamental period τ1)
1/1
9/10
4/5
3/4
7/10
3/5
1/2
2/5
3/10
1/4
1/5
1/10
0/1
2Δx = 4 %
Δm = 9
-15 -10 -5 0 5 10 15 = m
time
-1/2
-1/4
0
1/4
1/2
Coordinate ϕ (units of 2π)
(Imagine "wrap-around" ϕ-coordinate)
−π +π
-1/2 1/2
+π/2
1/4
−π/2
-1/4
0
Time t
1/1
3/4
1/2
1/4
0/1

N-level-rotor system revival-beat wave dynamics

(Just 2-levels (0, ±1) (and some ±2) excited)



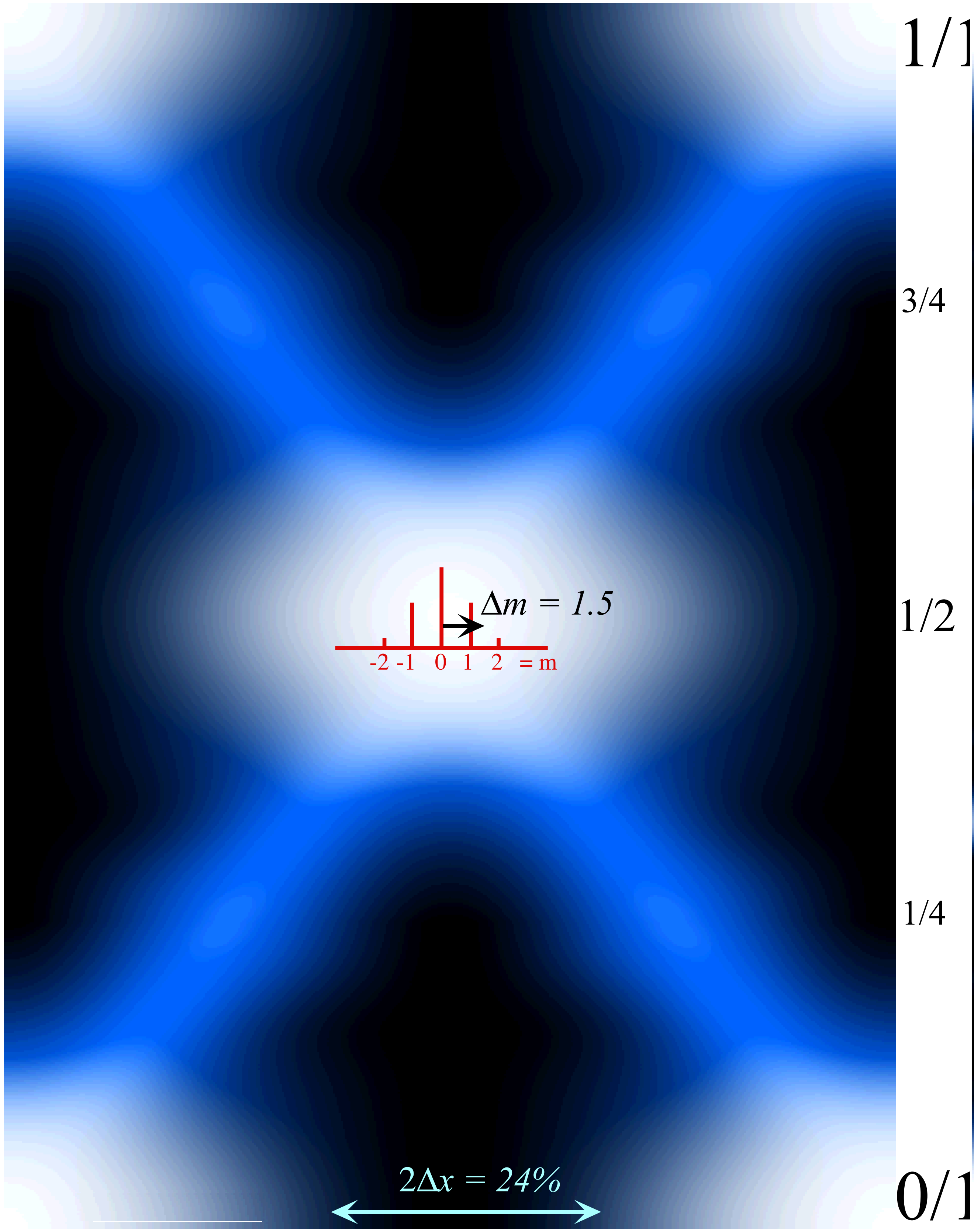
$|\Psi(x,t)|$ in space-time

Simplest quantum revival:
Exciting first two levels
(ℓ=0 and ℓ=±1)
is like a
2-level system quantum beat
in space-time

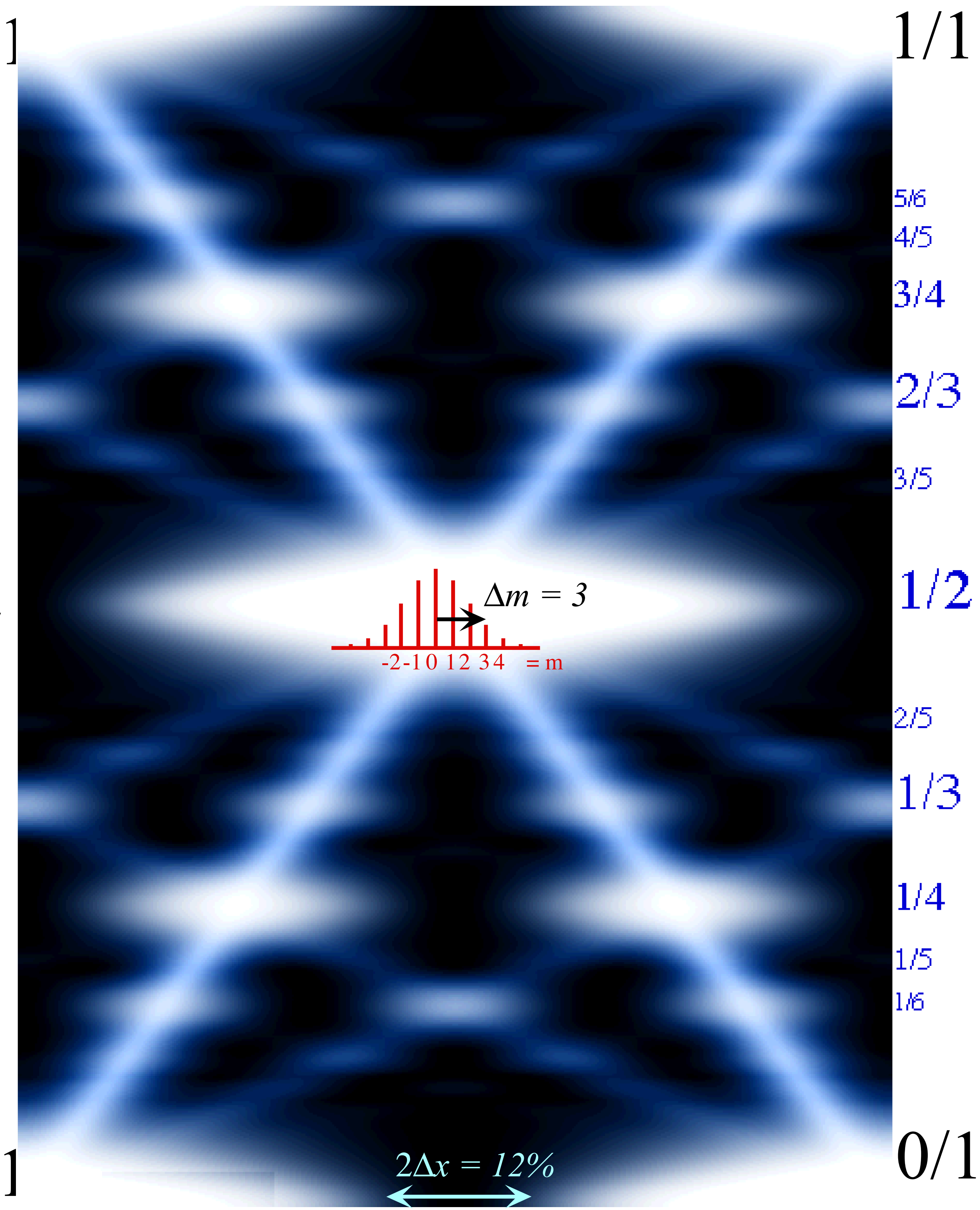
[Harter, J. Mol. Spec. 210, 166-182 (2001)]

N-level-rotor system revival-beat wave dynamics

(Just 2-levels (0, ±1) (and some ±2) excited)



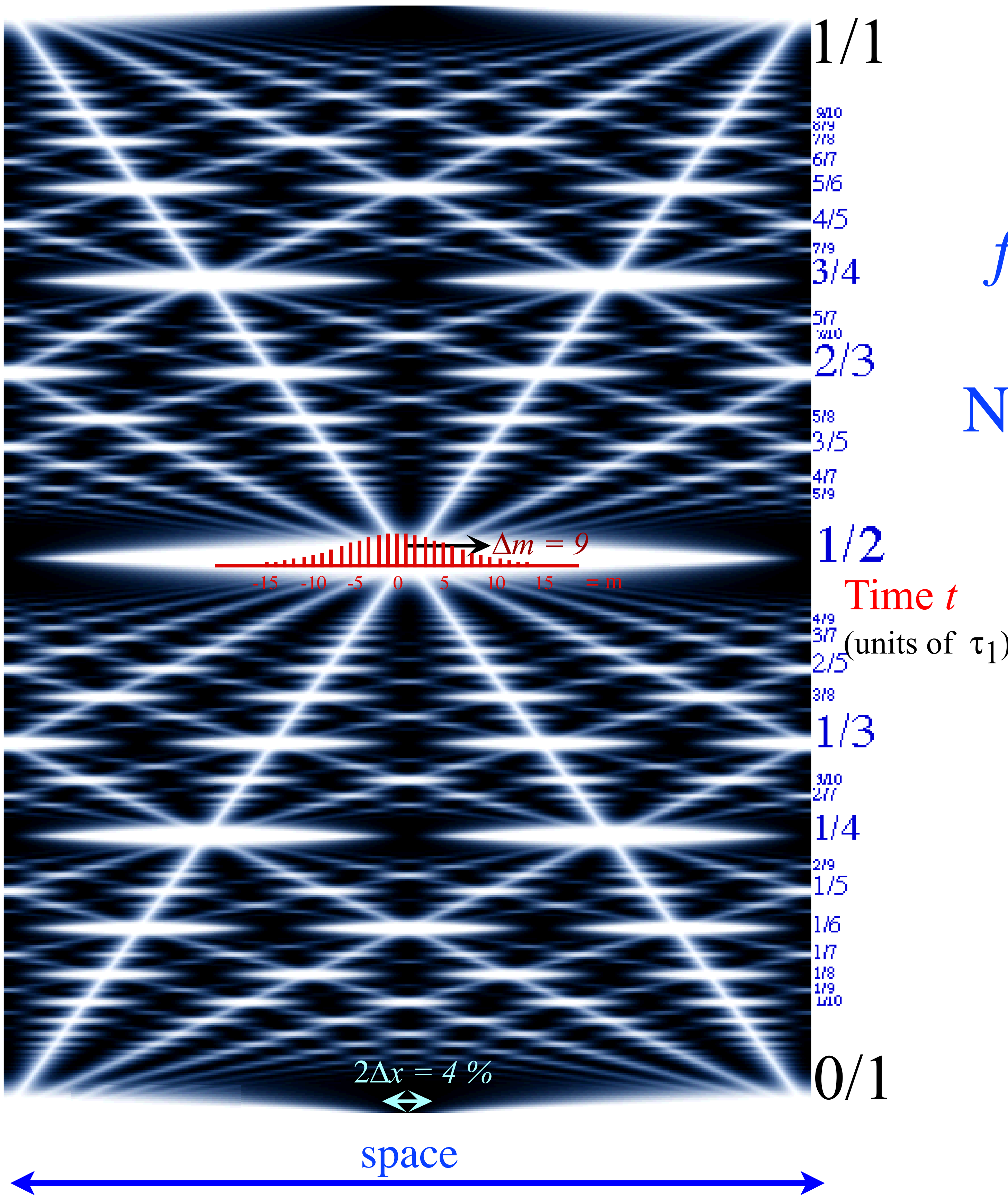
(4-levels (0, ±1,±2,±3) (and some ±4) excited)



Simplest *fractional* quantum revivals: 3,4,5-level systems

N-level-rotor system revival-beat wave dynamics

(9 or10-levels (0, ±1, ±2, ±3, ±4,..., ±9, ±10, ±11...) excited)



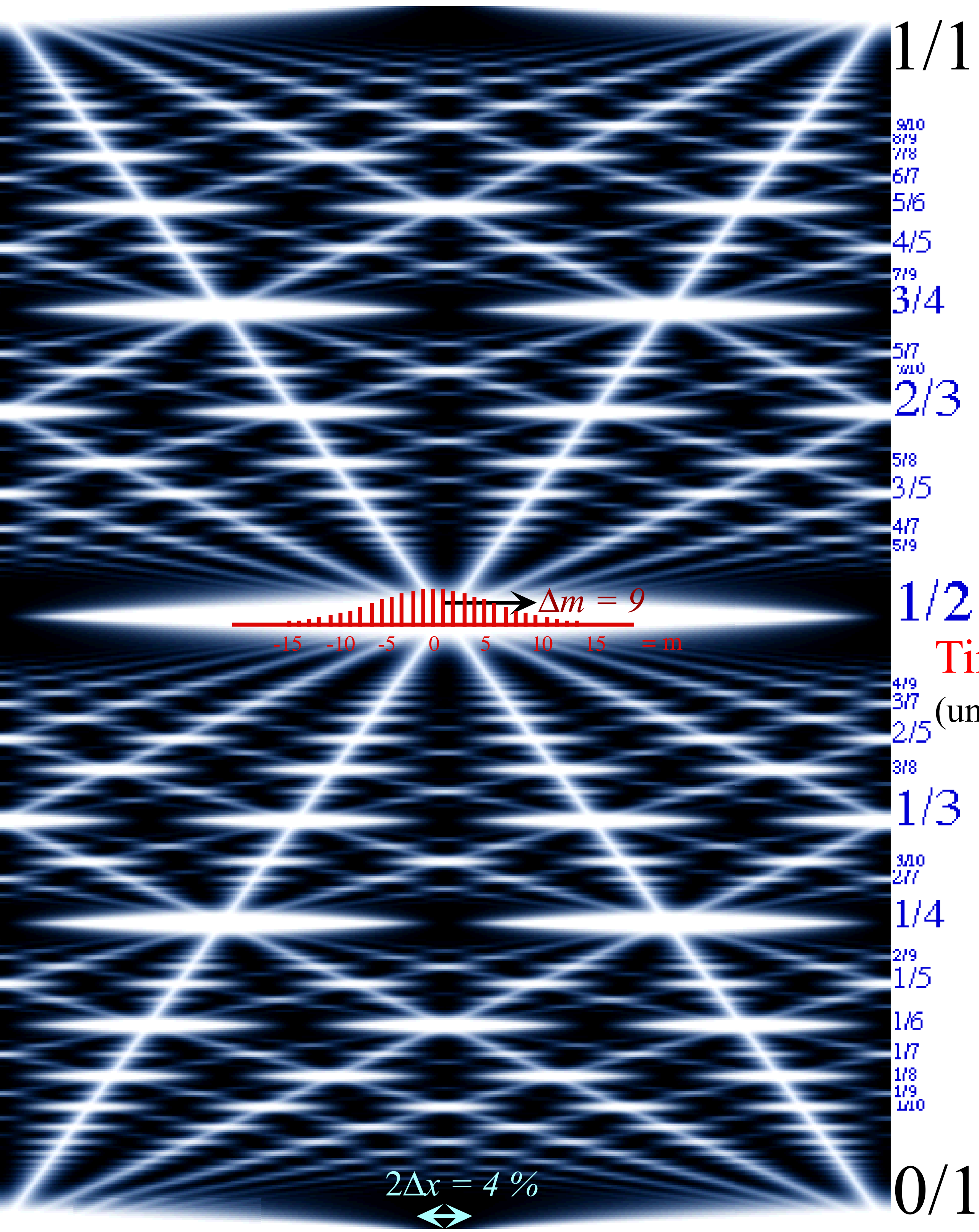
fractional quantum revivals:
in 3,4,…, N-level systems
Number increases rapidly with
number of levels
and/or bandwidth
of excitation

[Harter, J. Mol. Spec. 210, 166-182 (2001)]

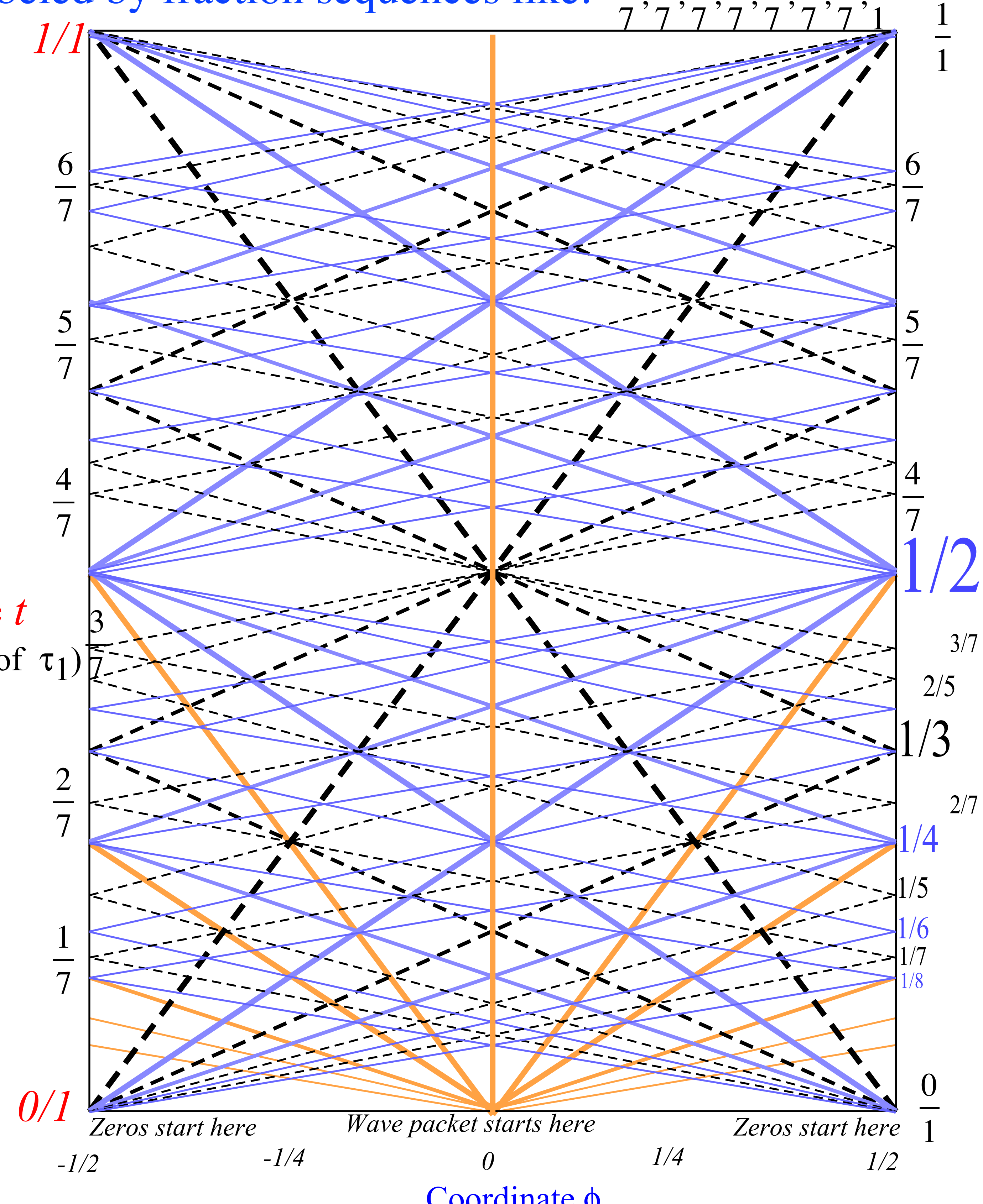
N-level-rotor system revival-beat wave dynamics

(9 or10-levels (0, ±1, ±2, ±3, ±4,..., ±9, ±10, ±11...) excited)

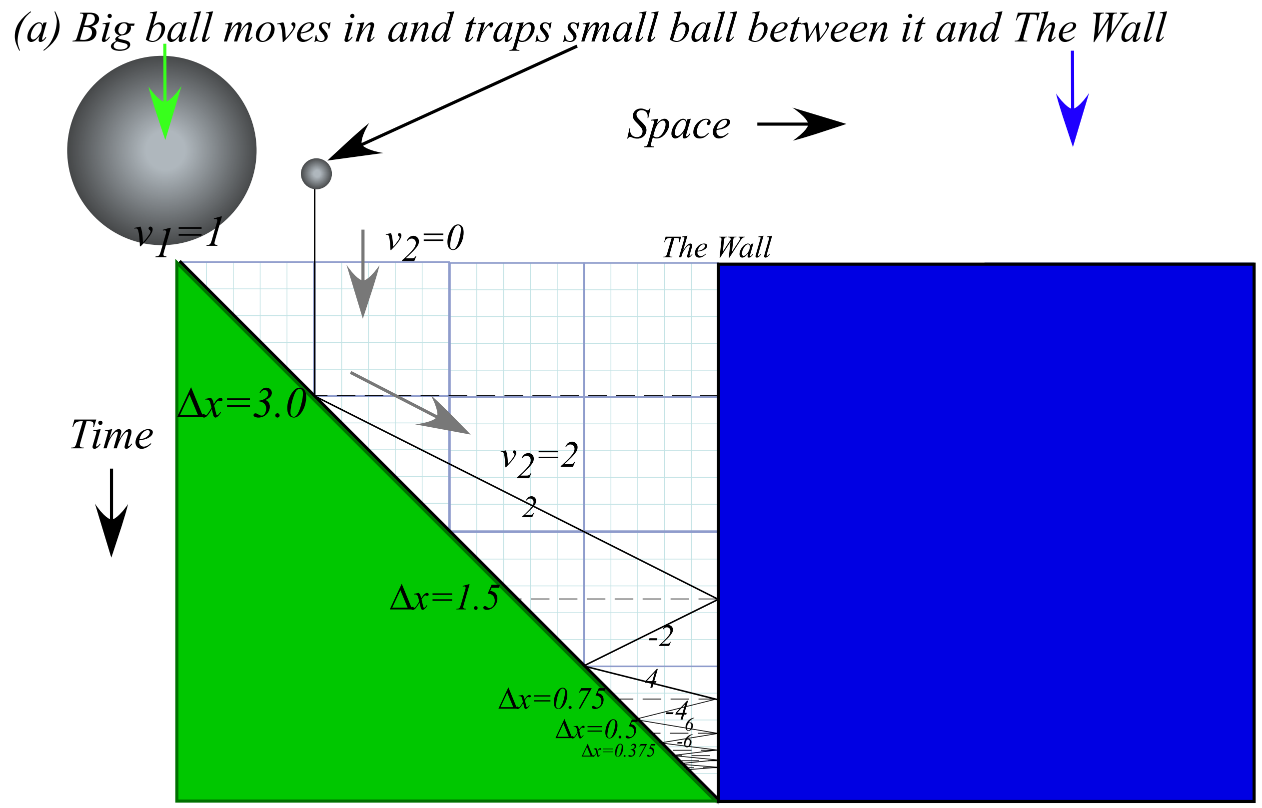
Zeros (clearly) and "particle-packets" (faintly) have paths labeled by fraction sequences like: $\frac{0}{7}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \frac{1}{1}$



Time *t* (units of τ_1)



[Harter, J. Mol. Spec. 210, 166-182 (2001)]



Lect. 5 (9.11.14)

The Classical "Monster Mash"

Classical introduction to

Heisenberg "Uncertainty" Relations

$v_2 = \frac{const.}{Y}$ or: $Y \cdot v_2 = const.$

is analogous to: $\Delta x \cdot \Delta p = N \cdot \hbar$

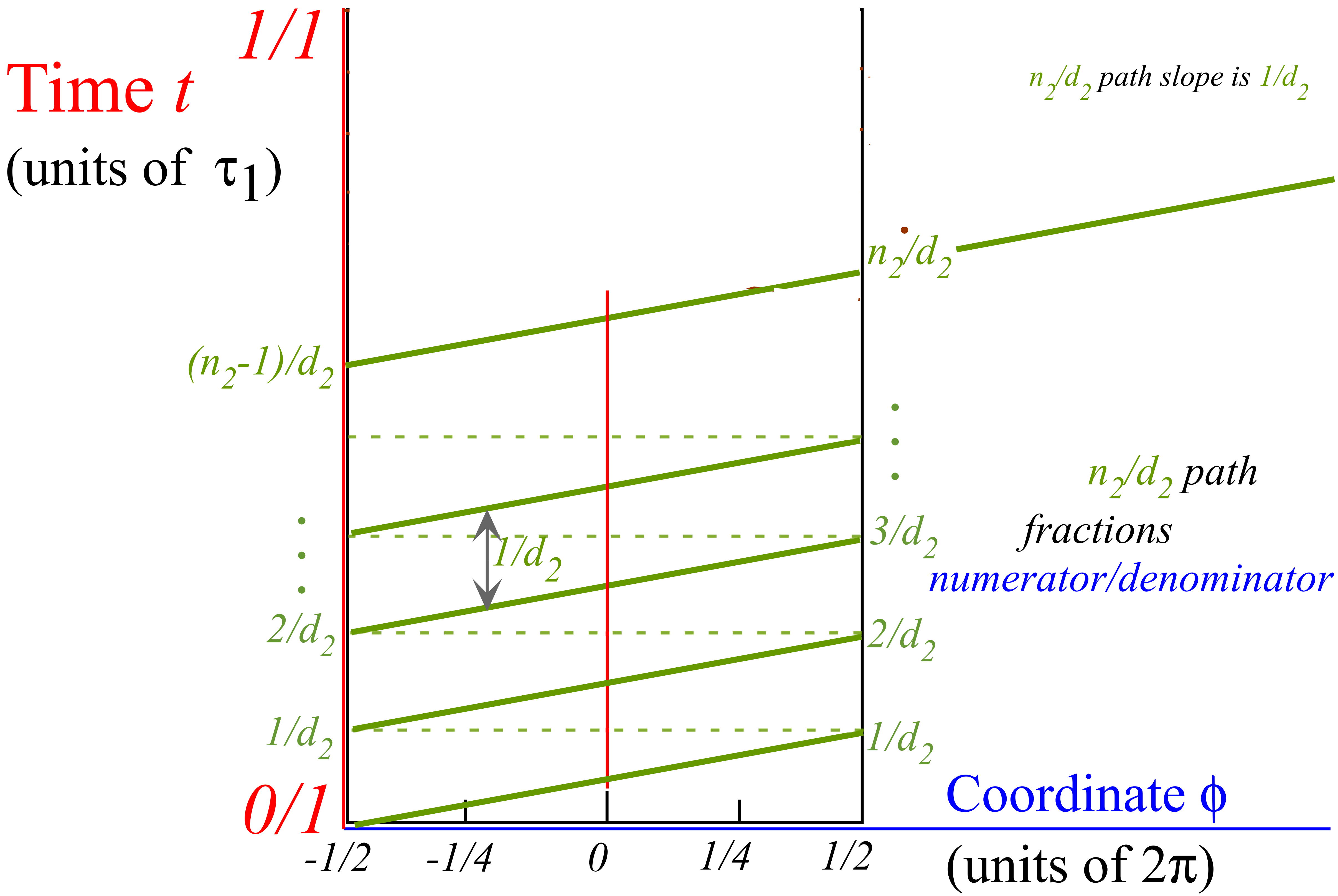
Recall classical "Monster Mash" in Lecture 5

with small-ball trajectory paths having same geometry
as revival beat wave-zero paths

Farey-Sum arithmetic of revival wave-zero paths
(How *Rational Fractions N/D* occupy real space-time)

Farey Sum algebra of revival-beat wave dynamics

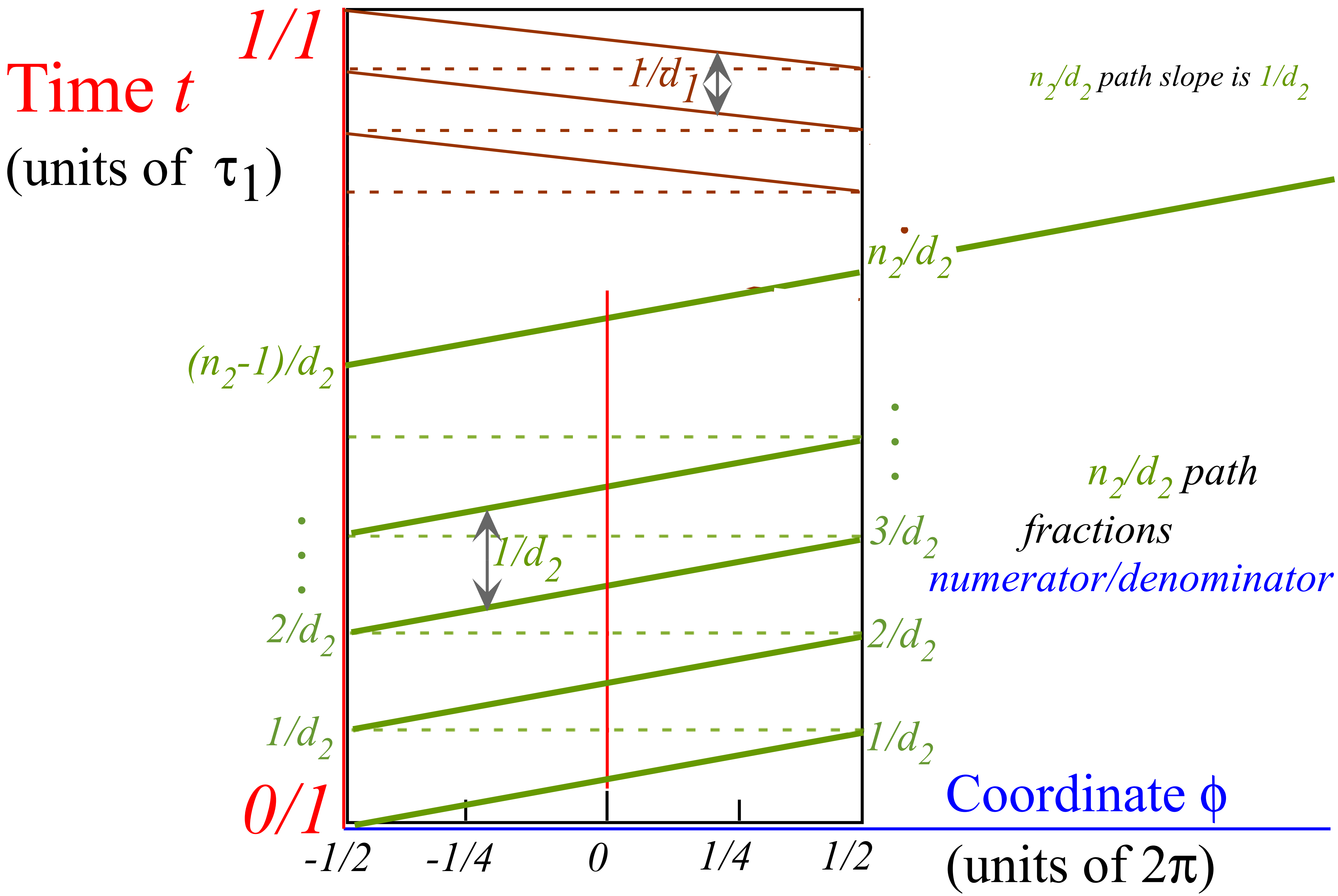
Label by *numerators N* and *denominators D* of rational fractions *N/D*



Harter, J. Mol. Spec. 210, 166-182 (2001) and ISMS (2013)

Farey Sum algebra of revival-beat wave dynamics

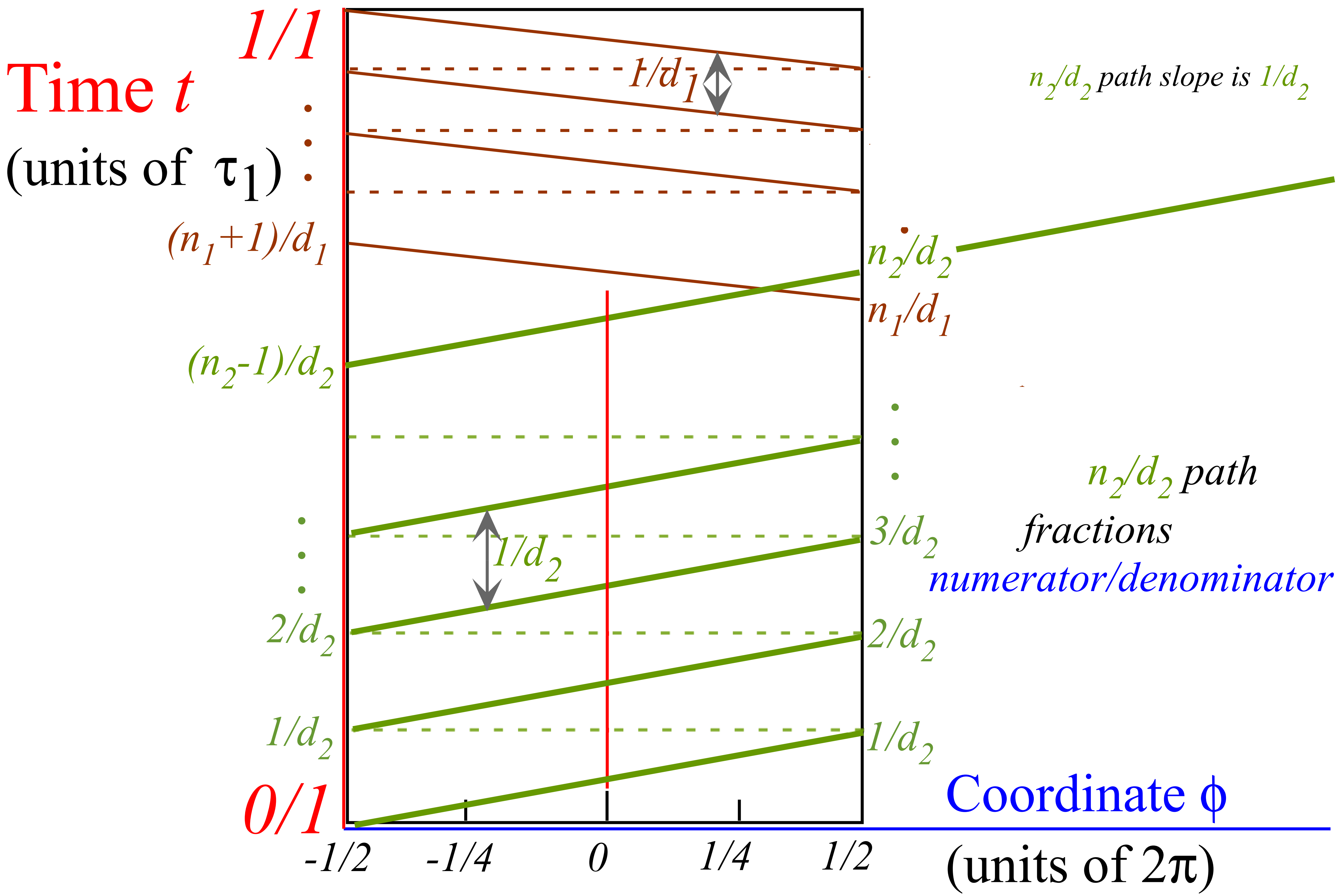
Label by *numerators N* and *denominators D* of rational fractions *N/D*



Harter, J. Mol. Spec. 210, 166-182 (2001) and ISMS (2013)

Farey Sum algebra of revival-beat wave dynamics

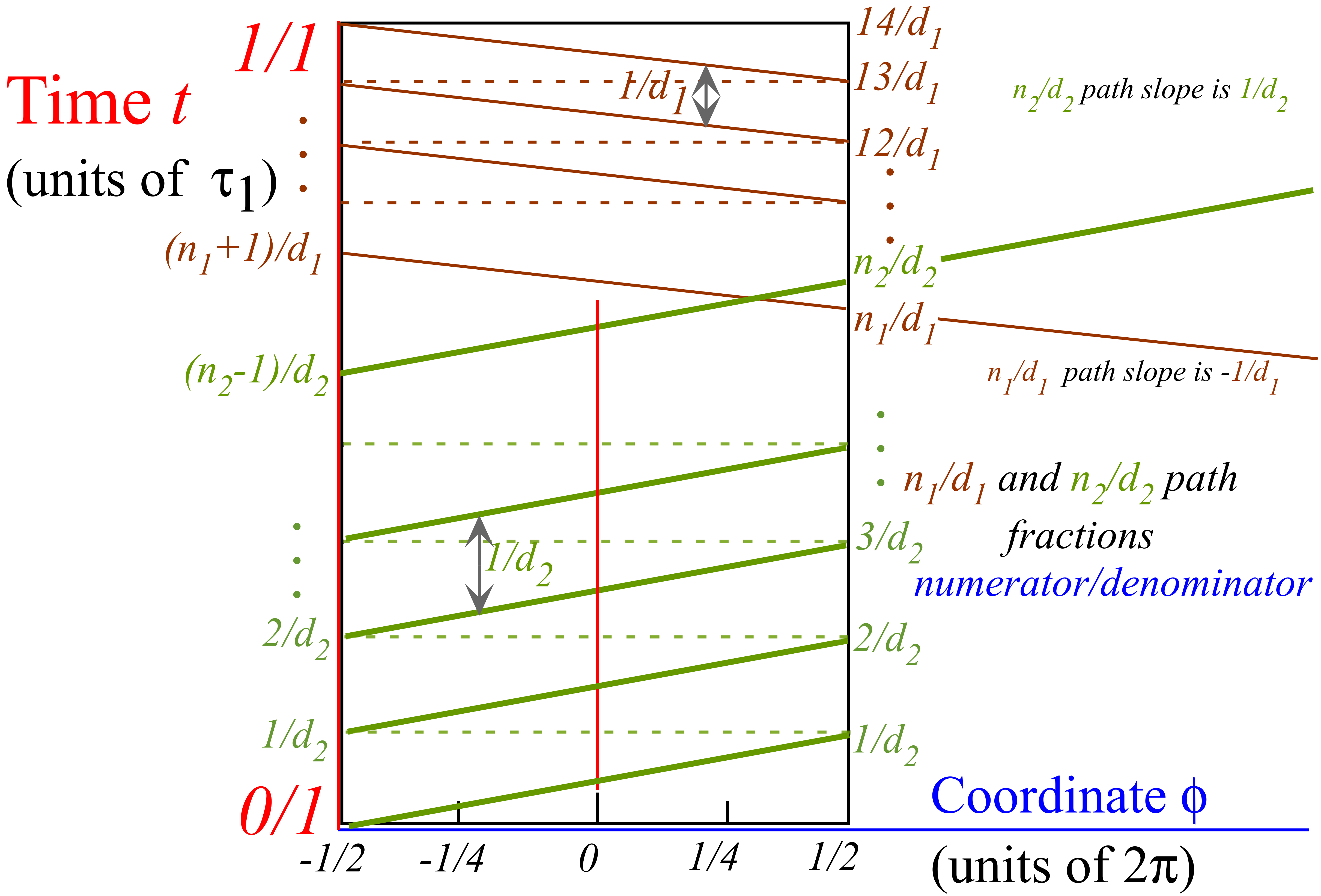
Label by *numerators N* and *denominators D* of rational fractions *N/D*



Harter, J. Mol. Spec. 210, 166-182 (2001) and ISMS (2013)

Farey Sum algebra of revival-beat wave dynamics

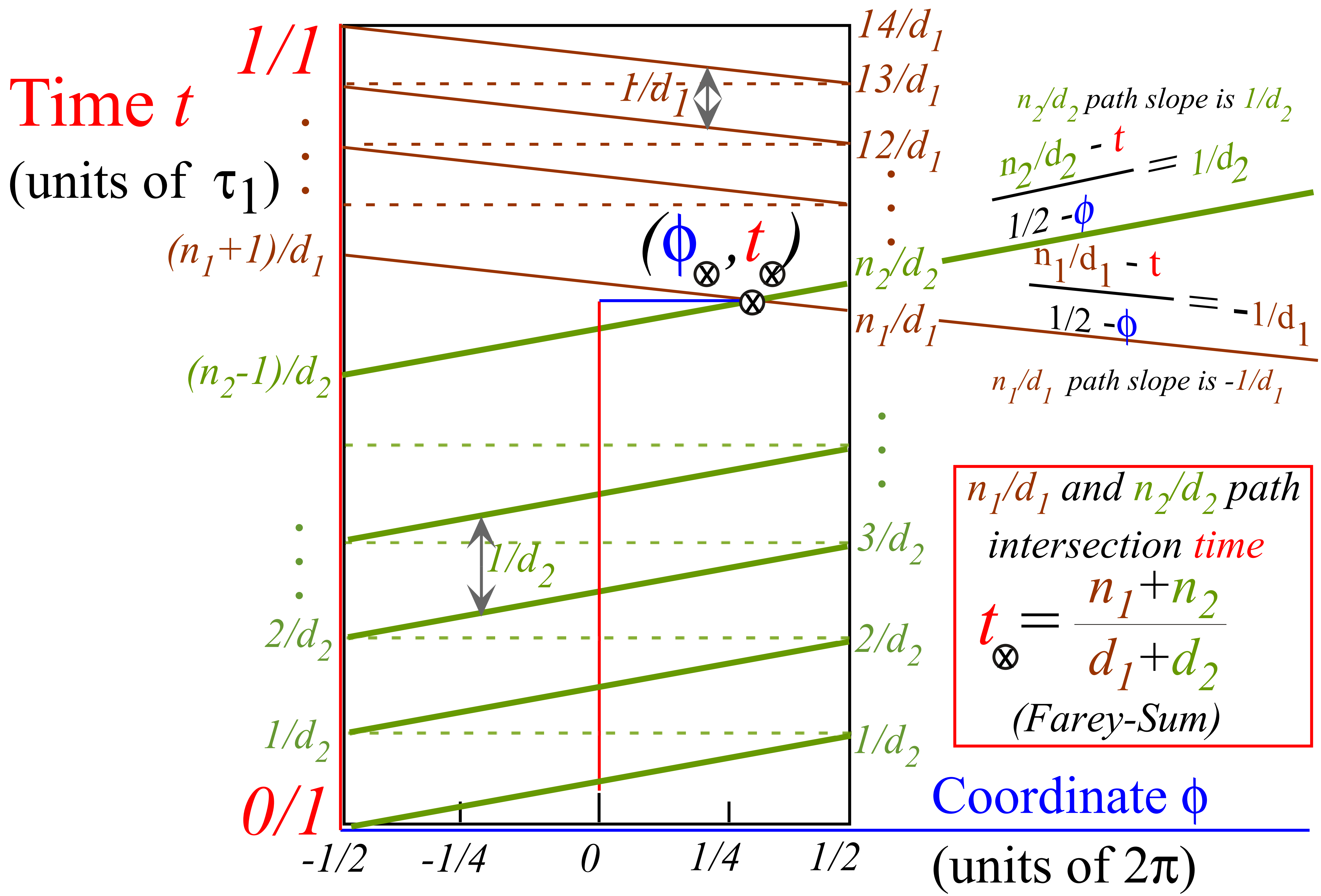
Label by *numerators N* and *denominators D* of rational fractions *N/D*



Harter, J. Mol. Spec. 210, 166-182 (2001) and ISMS (2013)

Farey Sum algebra of revival-beat wave dynamics

Label by *numerators N* and *denominators D* of rational fractions *N/D*

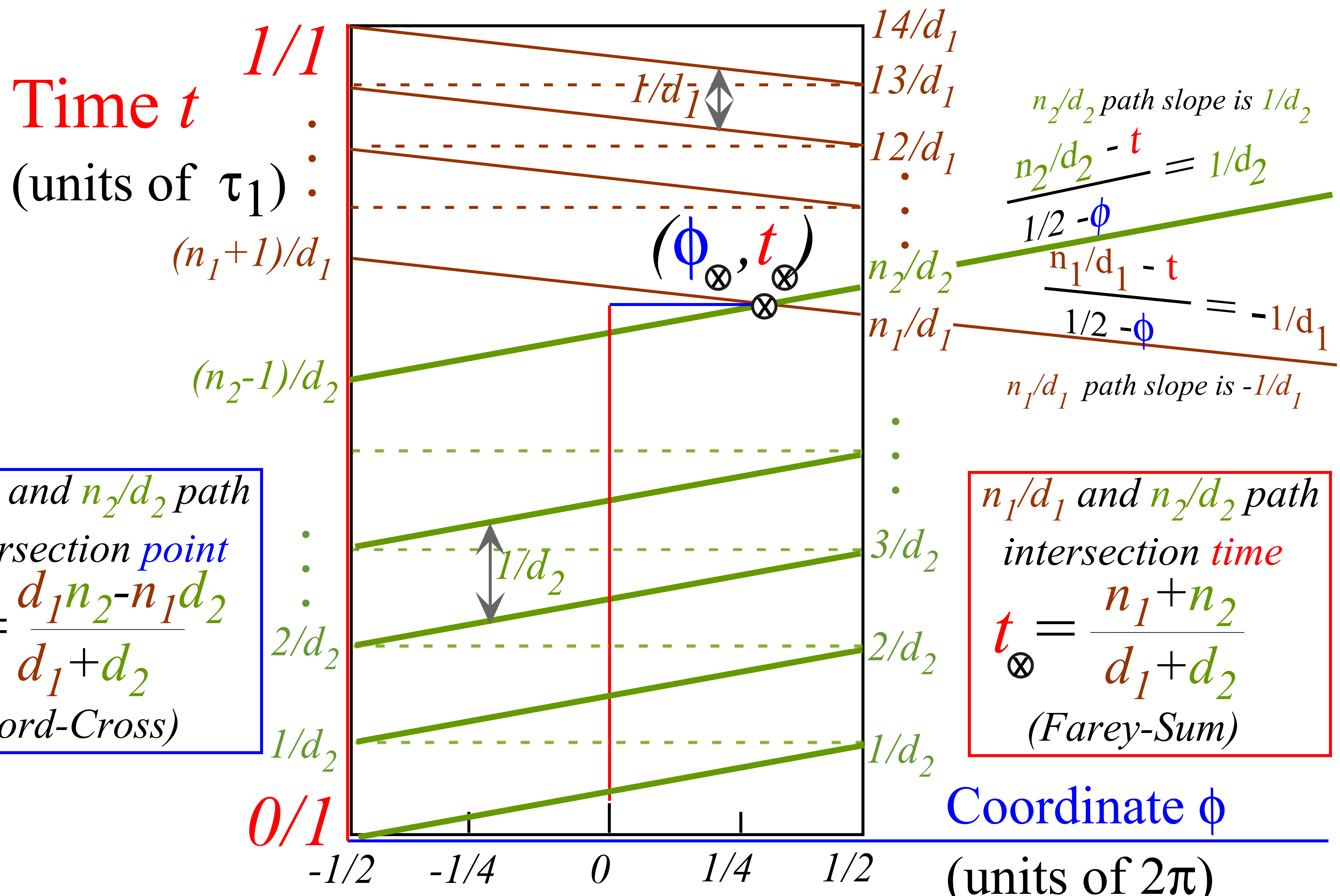


[John Farey, Phil. Mag.(1816)]

Harter, J. Mol. Spec. 210, 166-182 (2001) and ISMS (2013)

Farey Sum algebra of revival-beat wave dynamics

Label by *numerators N* and *denominators D* of rational fractions *N/D*



n_1/d_1 and n_2/d_2 path intersection point

$$\phi_\otimes = \frac{d_1 n_2 - n_1 d_2}{d_1 + d_2}$$

(Ford-Cross)

n_1/d_1 and n_2/d_2 path intersection time

$$t_\otimes = \frac{n_1 + n_2}{d_1 + d_2}$$

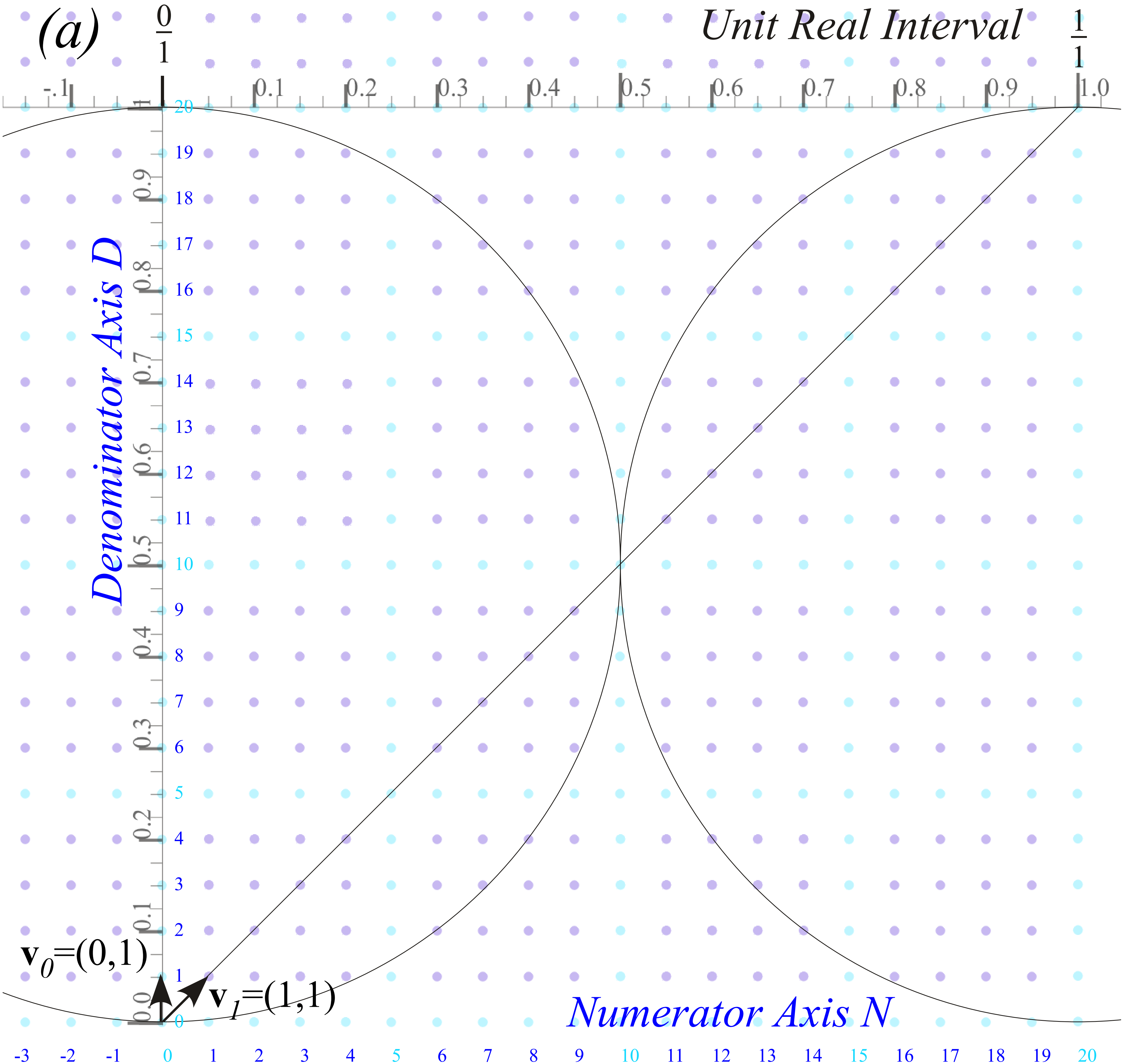
(Farey-Sum)

[Lester. R. Ford, Am. Math. Monthly 45,586(1938)]

Harter, J. Mol. Spec. 210, 166-182 (2001) and ISMS (2013)

[John Farey, Phil. Mag.(1816)]

Ford-Circle geometry of revival paths
(How *Rational Fractions N/D* occupy real space-time)

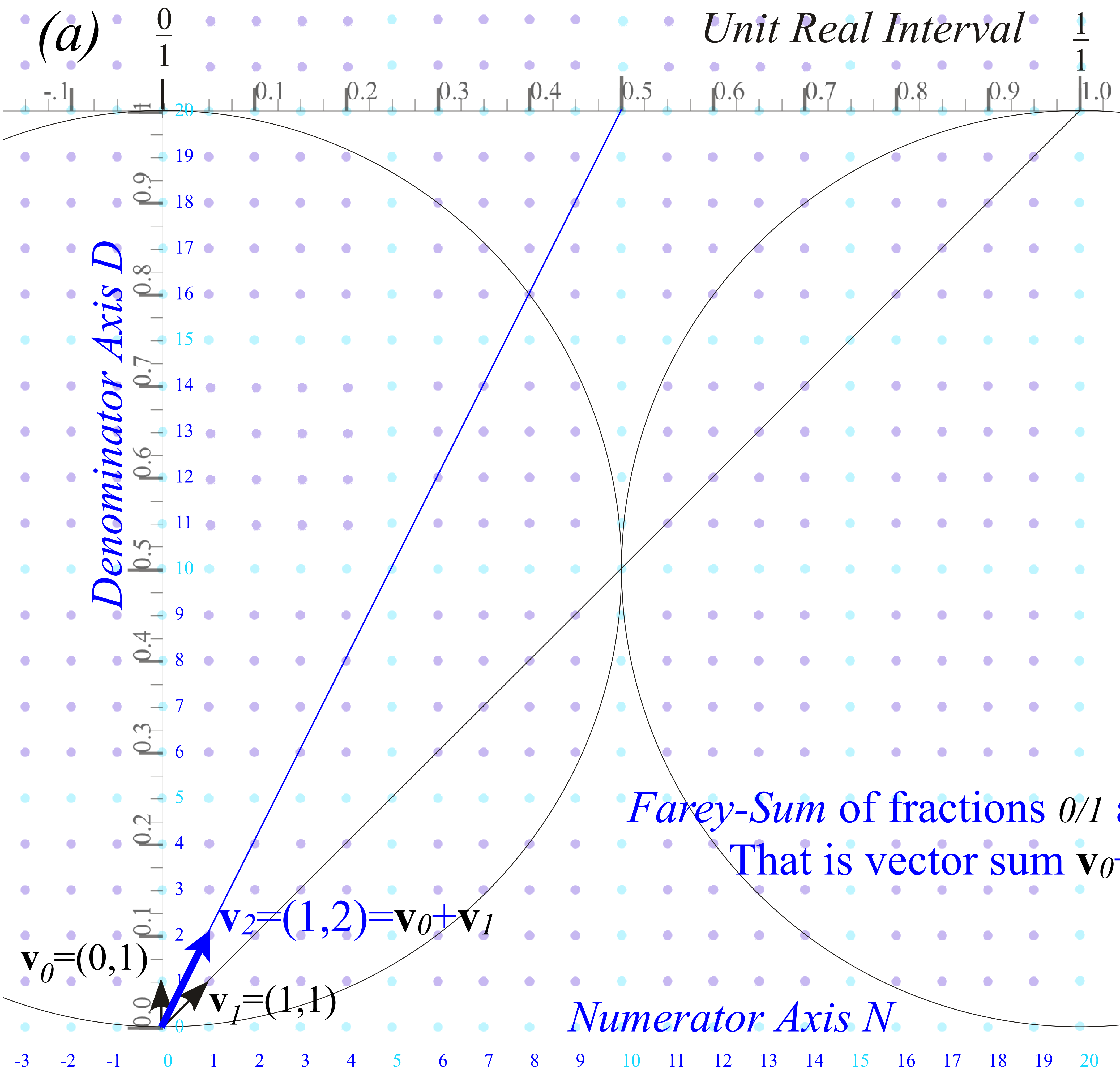


Farey Sum related to vector sum and *Ford Circles*

1/1-circle has diameter *1*

A. Li and W. Harter, Chem. Phys. Letters, 633, 208-213 (2015)

Harter and Alvason Li Int. Symposium on Molecular Spectroscopy OSU Columbus (2013)



Farey Sum related to vector sum and *Ford Circles*

1/1-circle has diameter *1*

Farey-Sum of fractions *0/1* and *1/1* is *1/2*

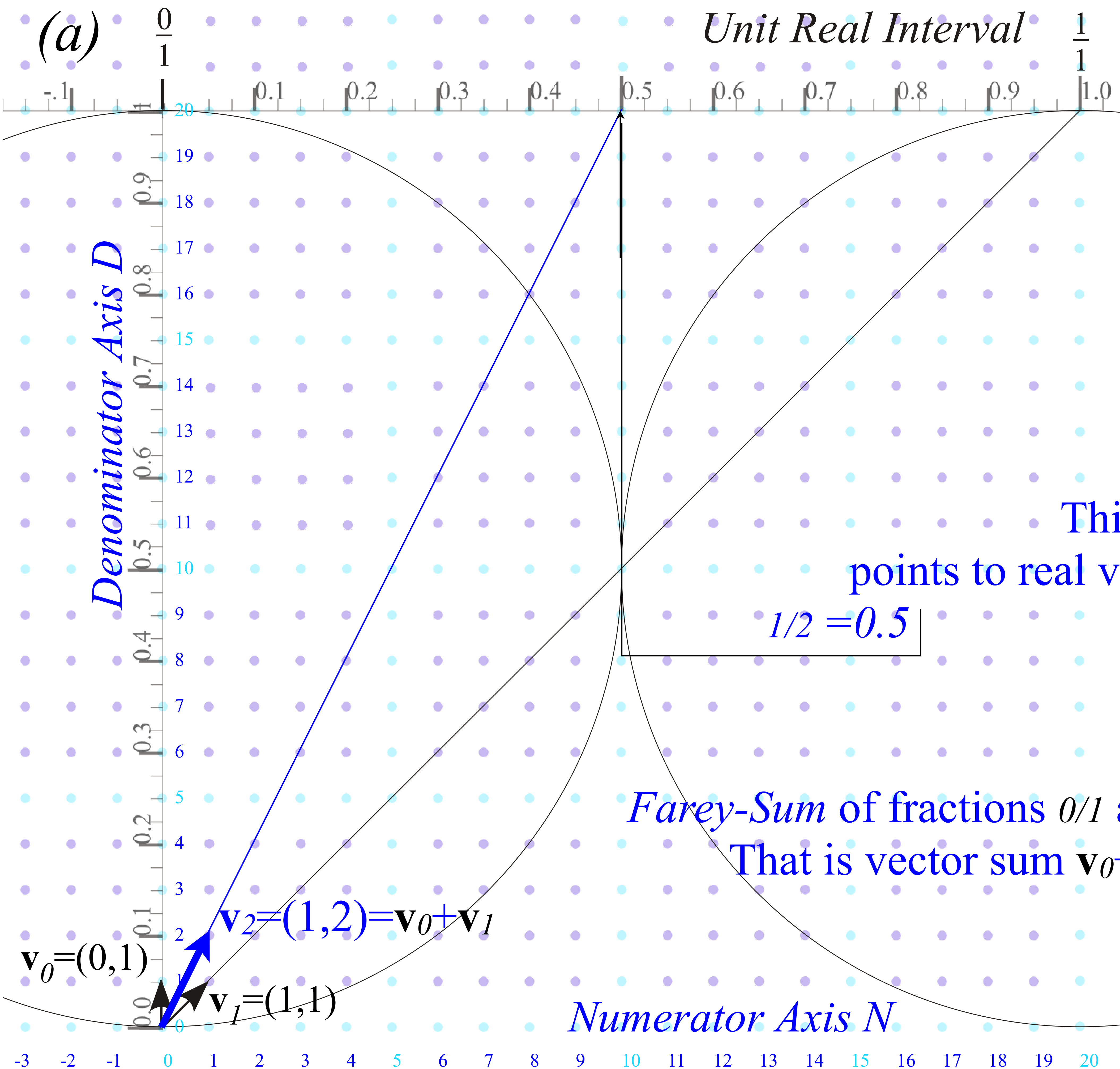
That is vector sum $\mathbf{v}_0+\mathbf{v}_1 = (1,2) = \mathbf{v}_2$

A. Li and W. Harter, Chem. Phys. Letters, 633, 208-213 (2015)

Harter and Alvason Li Int. Symposium on Molecular Spectroscopy OSU Columbus (2013)

Farey Sum related to vector sum and *Ford Circles*

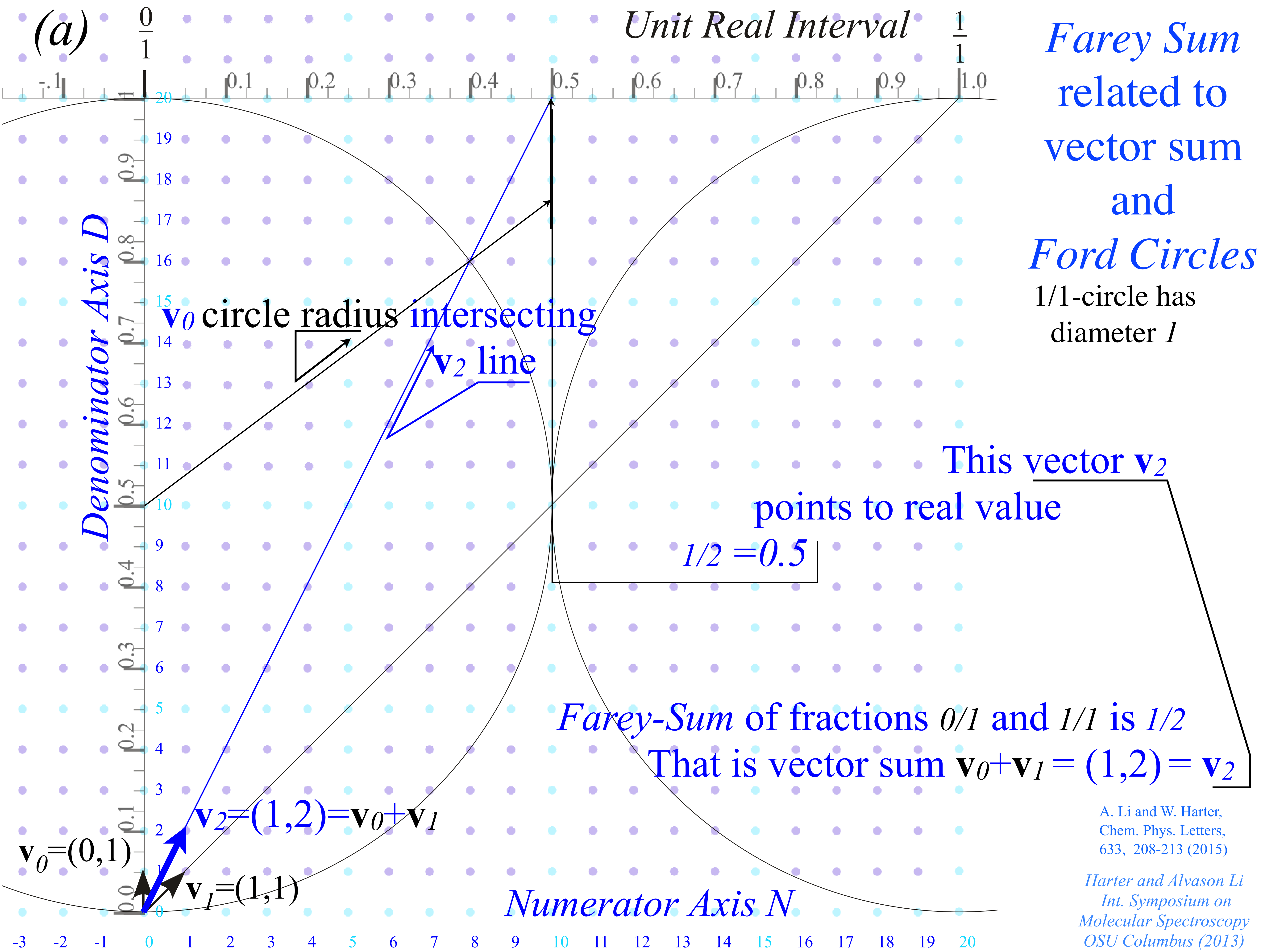
1/1-circle has diameter *1*

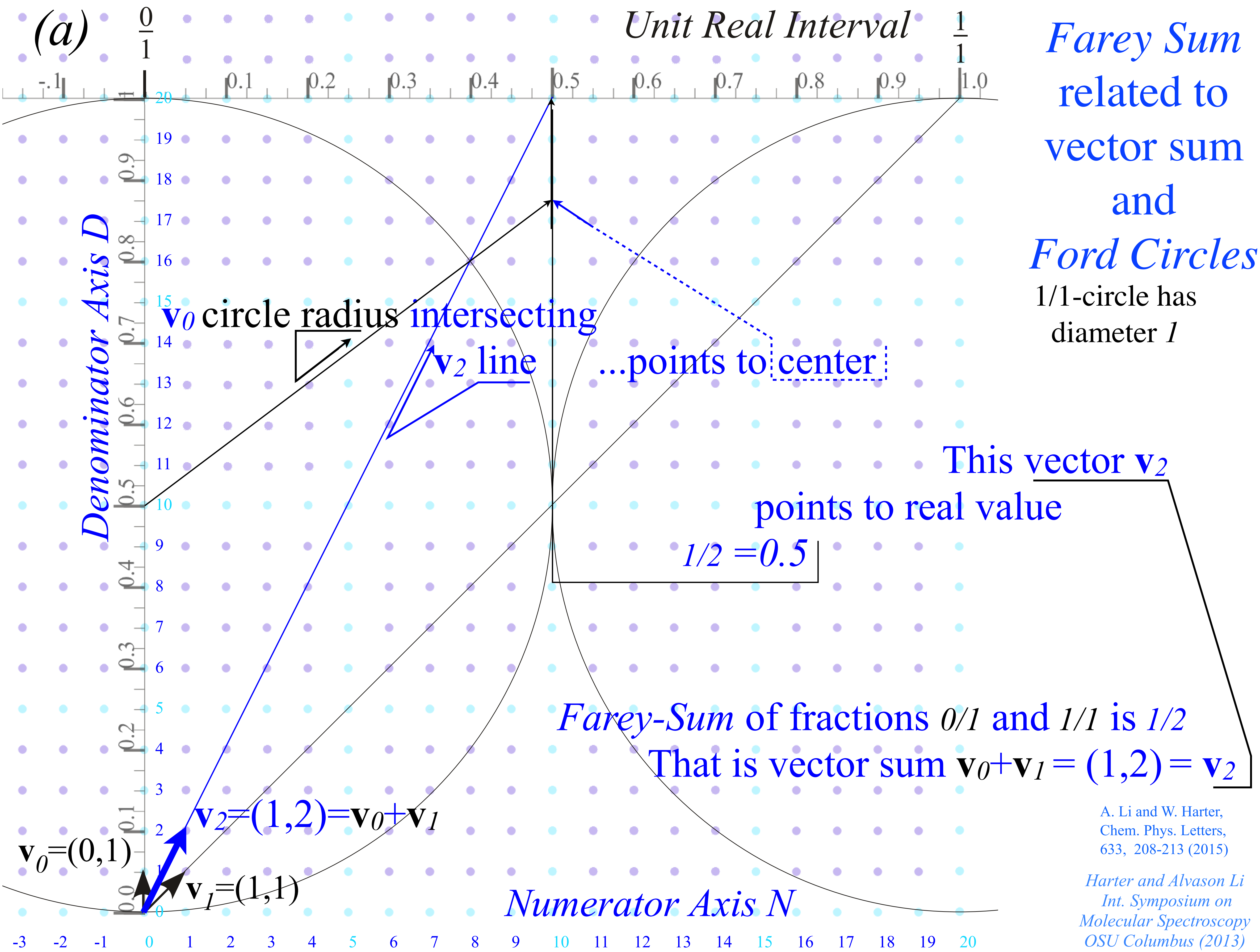


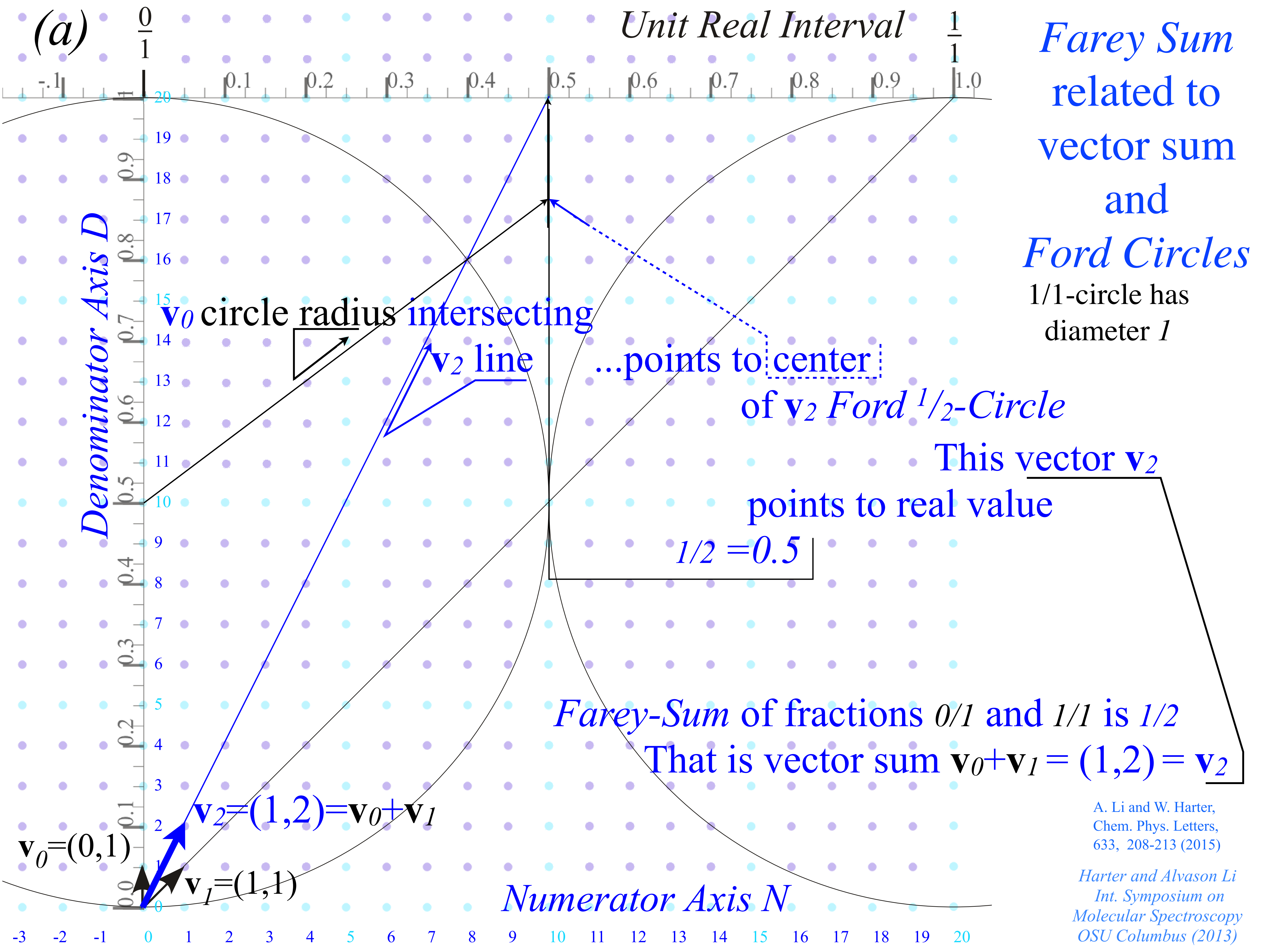
Farey-Sum of fractions *0/1* and *1/1* is *1/2*
That is vector sum $\mathbf{v}_0+\mathbf{v}_1 = (1,2) = \mathbf{v}_2$

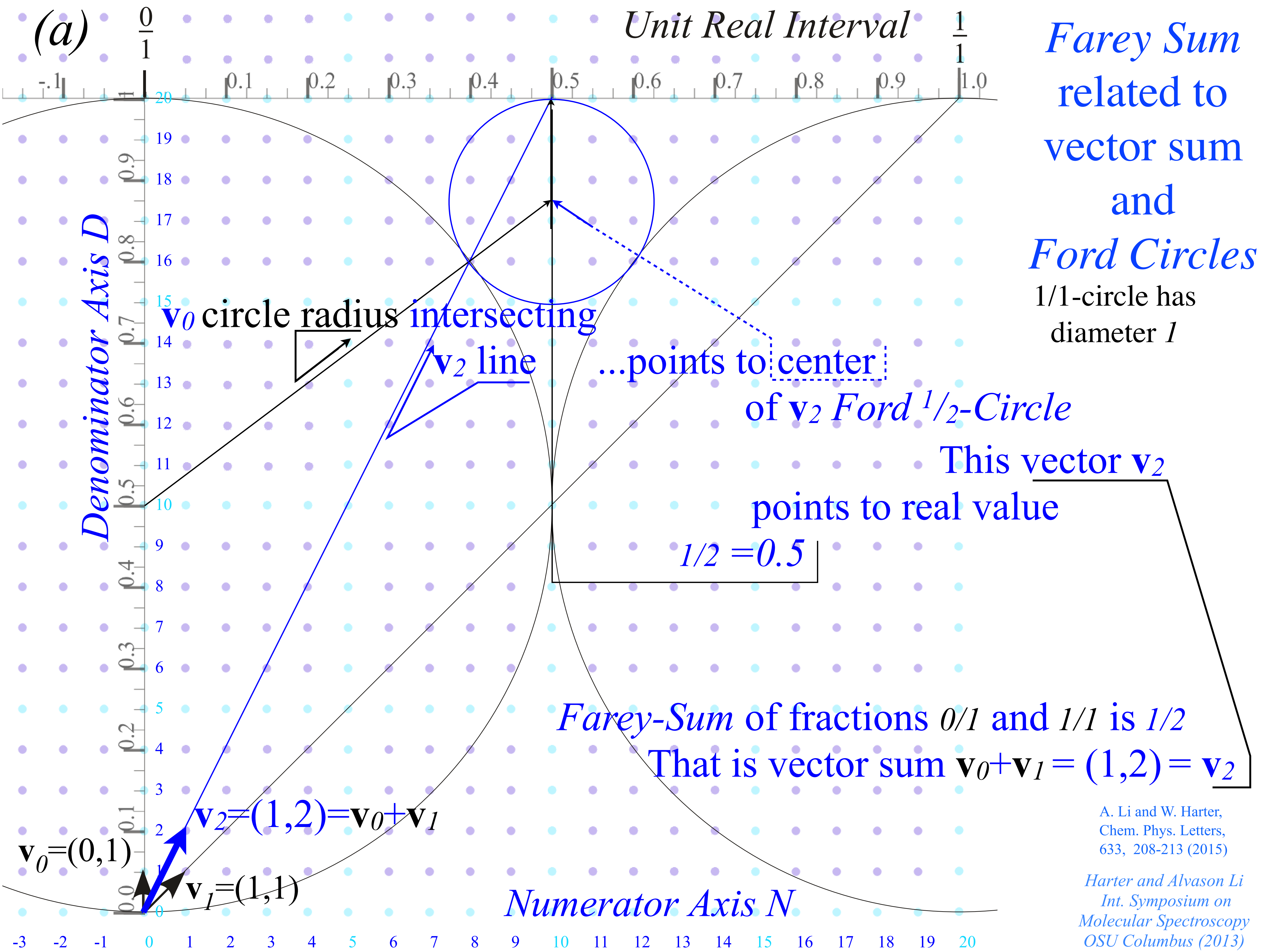
A. Li and W. Harter, Chem. Phys. Letters, 633, 208-213 (2015)

Harter and Alvason Li Int. Symposium on Molecular Spectroscopy OSU Columbus (2013)


(a)
Unit Real Interval
0/1
1/1
Farey Sum
related to
vector sum
and
Ford Circles
1/1-circle has
diameter 1
Denominator Axis D
$\mathbf{v}_0$ circle radius intersecting
$\mathbf{v}_2$ line
This vector $\mathbf{v}_2$
points to real value
$1/2 = 0.5$
Farey-Sum of fractions $0/1$ and $1/1$ is $1/2$
That is vector sum $\mathbf{v}_0+\mathbf{v}_1 = (1,2) = \mathbf{v}_2$
$\mathbf{v}_2=(1,2)=\mathbf{v}_0+\mathbf{v}_1$
$\mathbf{v}_0=(0,1)$
$\mathbf{v}_1=(1,1)$
Numerator Axis N
A. Li and W. Harter,
Chem. Phys. Letters,
633, 208-213 (2015)
Harter and Alvason Li
Int. Symposium on
Molecular Spectroscopy
OSU Columbus (2013)


(a)
0/1
Unit Real Interval
1/1
Farey Sum related to vector sum and Ford Circles
1/1-circle has diameter 1
-.1
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
Denominator Axis D
$\mathbf{v}_0$ circle radius intersecting $\mathbf{v}_2$ line
...points to center
This vector $\mathbf{v}_2$ points to real value 1/2 =0.5
Farey-Sum of fractions 0/1 and 1/1 is 1/2
That is vector sum $\mathbf{v}_0+\mathbf{v}_1 = (1,2) = \mathbf{v}_2$
$\mathbf{v}_2=(1,2)=\mathbf{v}_0+\mathbf{v}_1$
$\mathbf{v}_0=(0,1)$
$\mathbf{v}_1=(1,1)$
Numerator Axis N
-3 -2 -1 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
A. Li and W. Harter, Chem. Phys. Letters, 633, 208-213 (2015)
Harter and Alvason Li Int. Symposium on Molecular Spectroscopy OSU Columbus (2013)

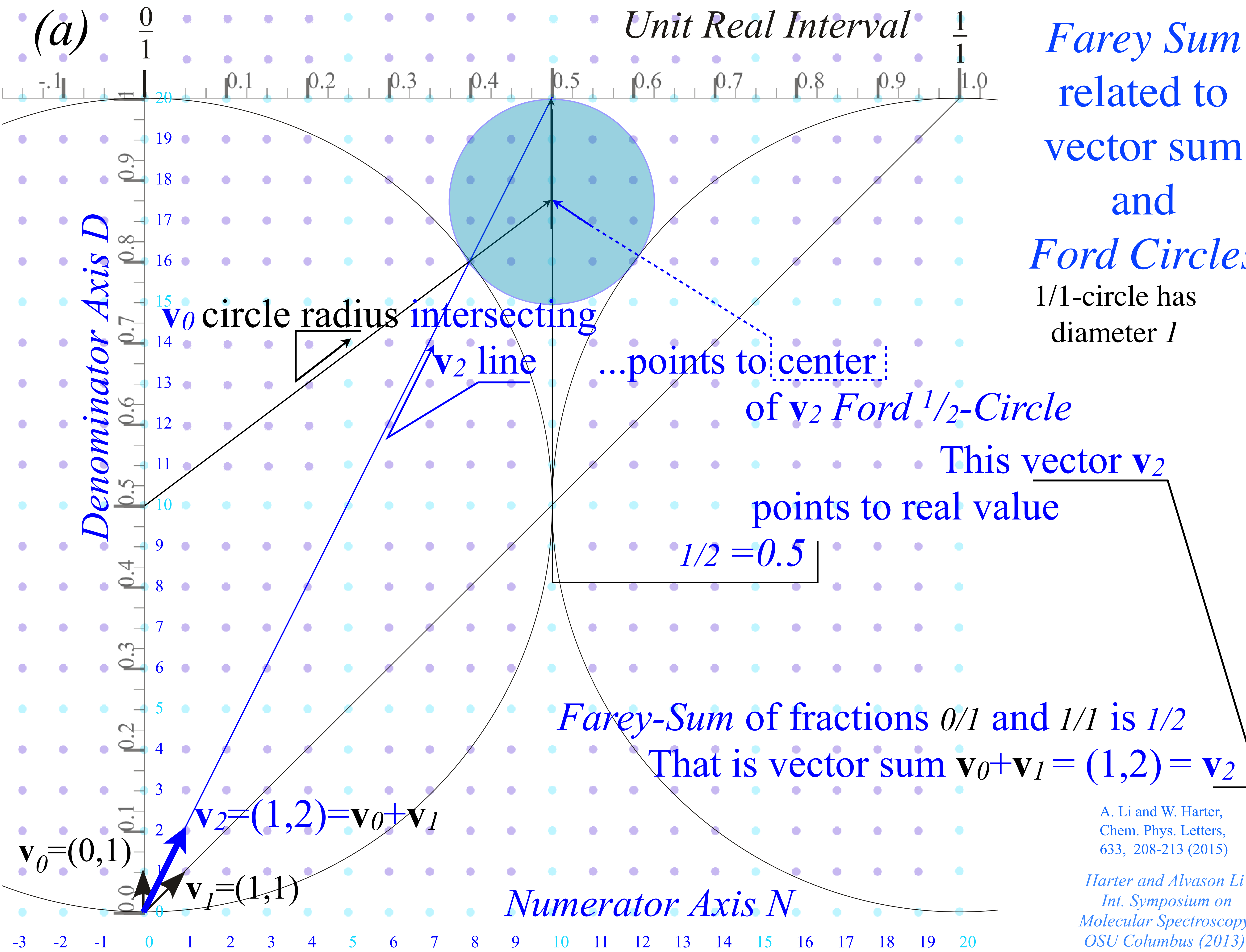

(a)
Unit Real Interval
Farey Sum related to vector sum and Ford Circles
1/1-circle has diameter 1
v0 circle radius intersecting
v2 line
...points to center of v2 Ford 1/2-Circle
This vector v2 points to real value 1/2 =0.5
Denominator Axis D
Numerator Axis N
Farey-Sum of fractions 0/1 and 1/1 is 1/2
That is vector sum v0+v1 = (1,2) = v2
v2=(1,2)=v0+v1
v0=(0,1)
v1=(1,1)
A. Li and W. Harter, Chem. Phys. Letters, 633, 208-213 (2015)
Harter and Alvason Li Int. Symposium on Molecular Spectroscopy OSU Columbus (2013)



(a)
Unit Real Interval
Farey Sum related to vector sum and Ford Circles
1/1-circle has diameter 1
Denominator Axis D
Numerator Axis N
v0 circle radius intersecting v2 line
...points to center of v2 Ford 1/2-Circle
This vector v2 points to real value 1/2 =0.5
Farey-Sum of fractions 0/1 and 1/1 is 1/2
That is vector sum v0+v1 = (1,2) = v2
v2=(1,2)=v0+v1
v0=(0,1)
v1=(1,1)
A. Li and W. Harter, Chem. Phys. Letters, 633, 208-213 (2015)
Harter and Alvason Li Int. Symposium on Molecular Spectroscopy OSU Columbus (2013)

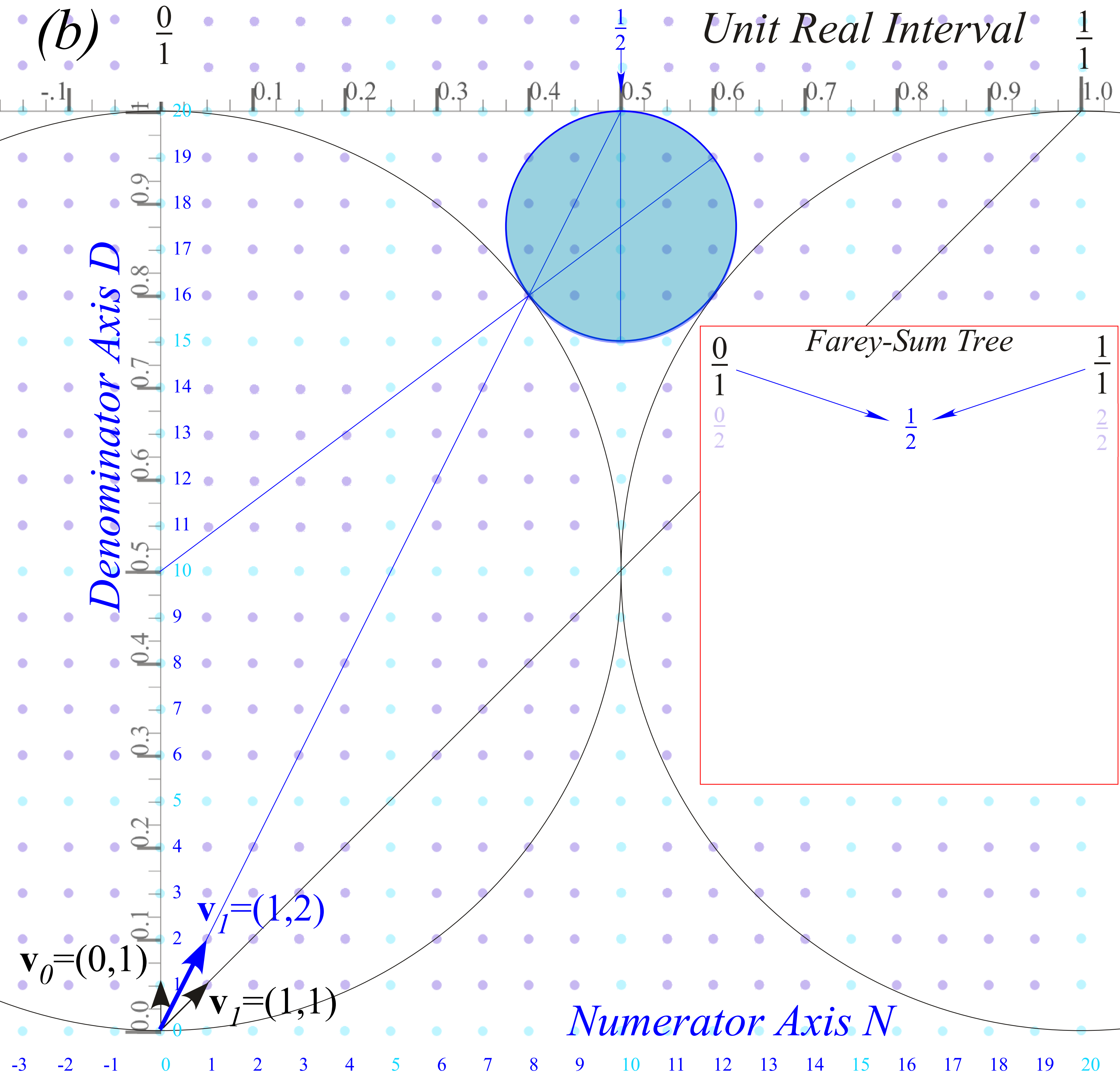
Farey Sum related to vector sum and *Ford Circles*

1/1-circle has diameter *1*



A. Li and W. Harter, Chem. Phys. Letters, 633, 208-213 (2015)

Harter and Alvason Li Int. Symposium on Molecular Spectroscopy OSU Columbus (2013)



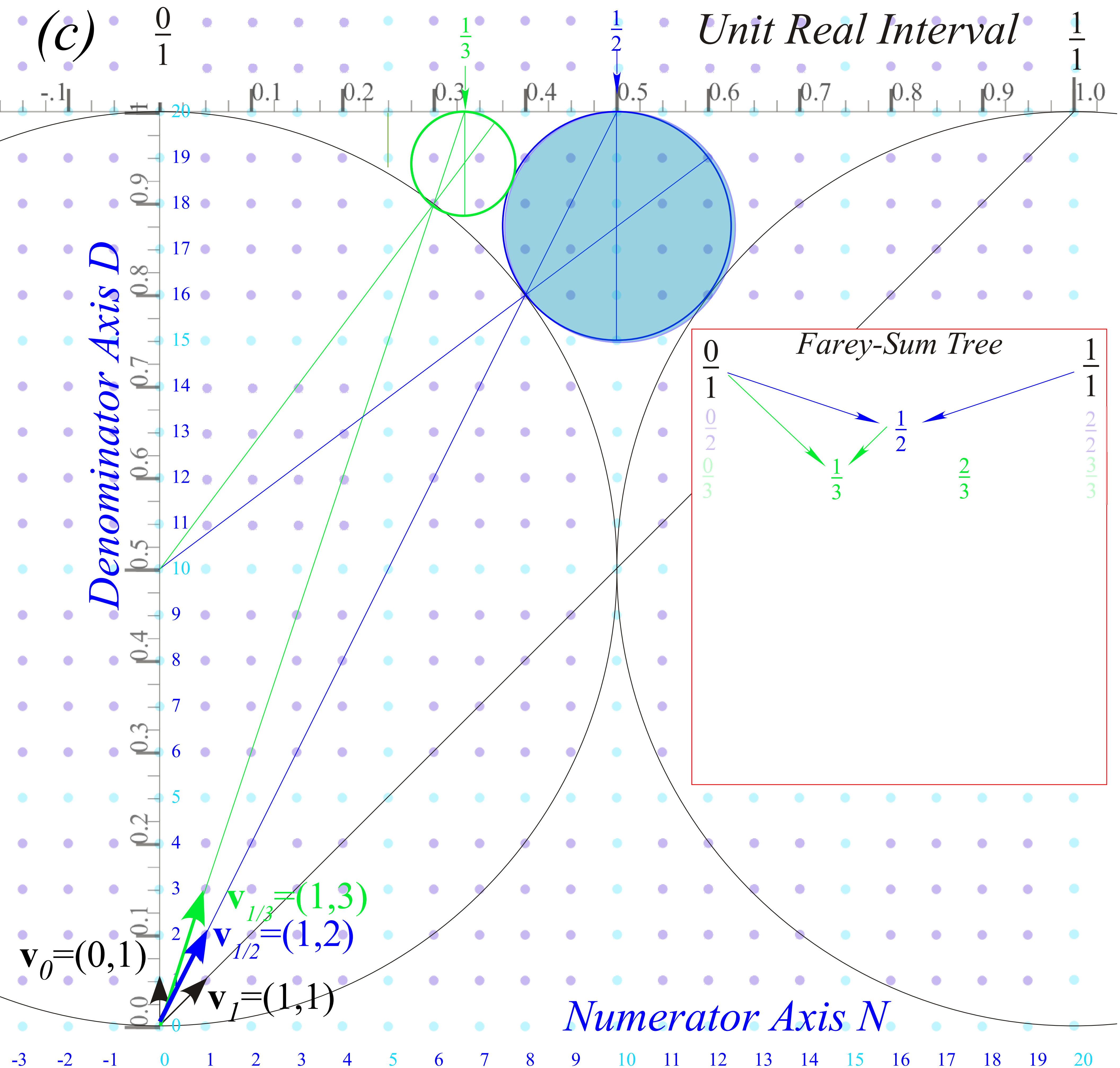
Farey Sum related to vector sum and *Ford Circles*

1/1-circle has diameter *1*

1/2-circle has diameter $1/2^2=1/4$

A. Li and W. Harter, Chem. Phys. Letters, 633, 208-213 (2015)

Harter and Alvason Li Int. Symposium on Molecular Spectroscopy OSU Columbus (2013)



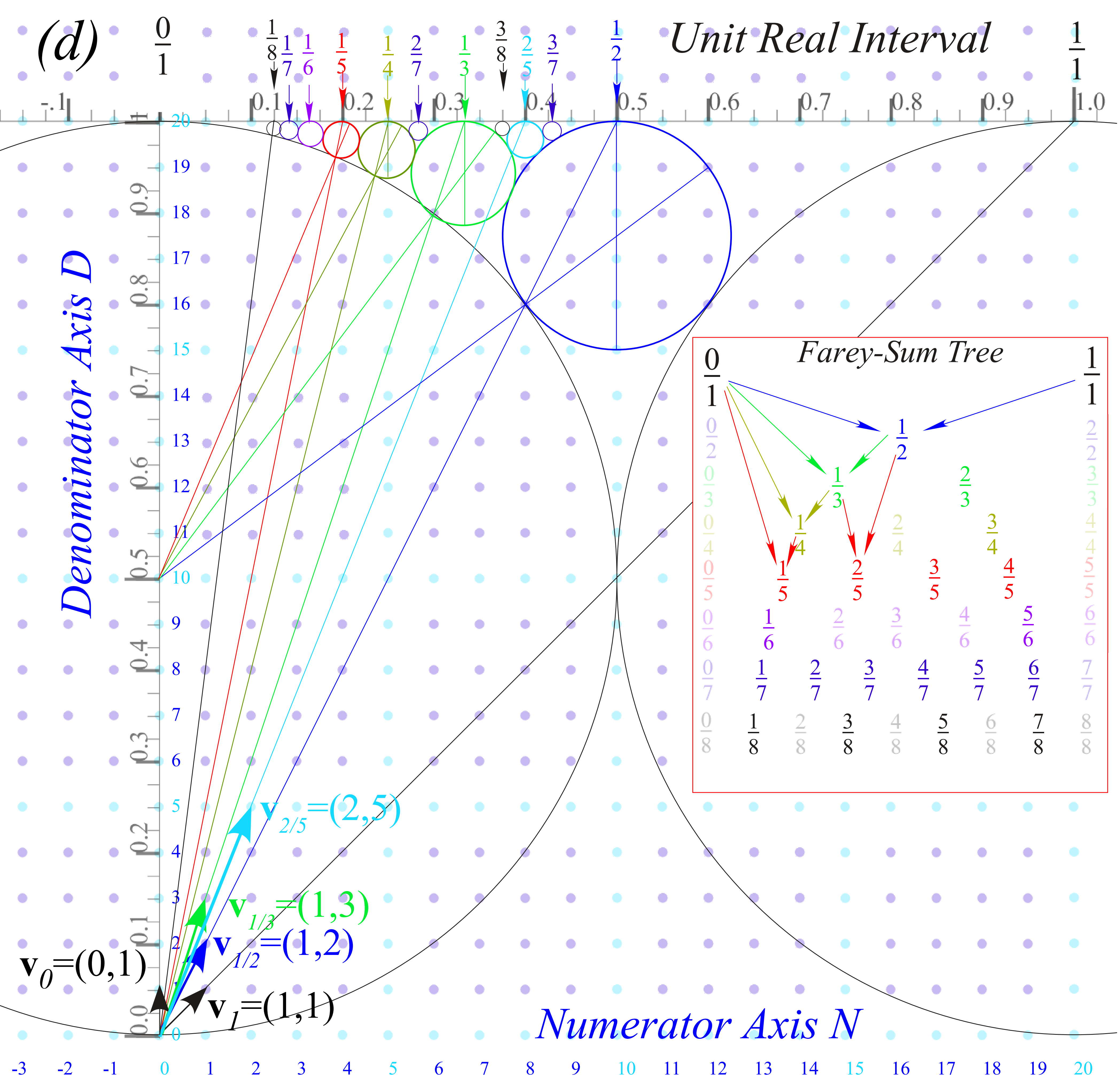
Farey Sum related to vector sum and *Ford Circles*

1/2-circle has diameter $1/2^2=1/4$

1/3-circles have diameter $1/3^2=1/9$

A. Li and W. Harter, Chem. Phys. Letters, 633, 208-213 (2015)

Harter and Alvason Li Int. Symposium on Molecular Spectroscopy OSU Columbus (2013)



Farey Sum related to vector sum and *Ford Circles*

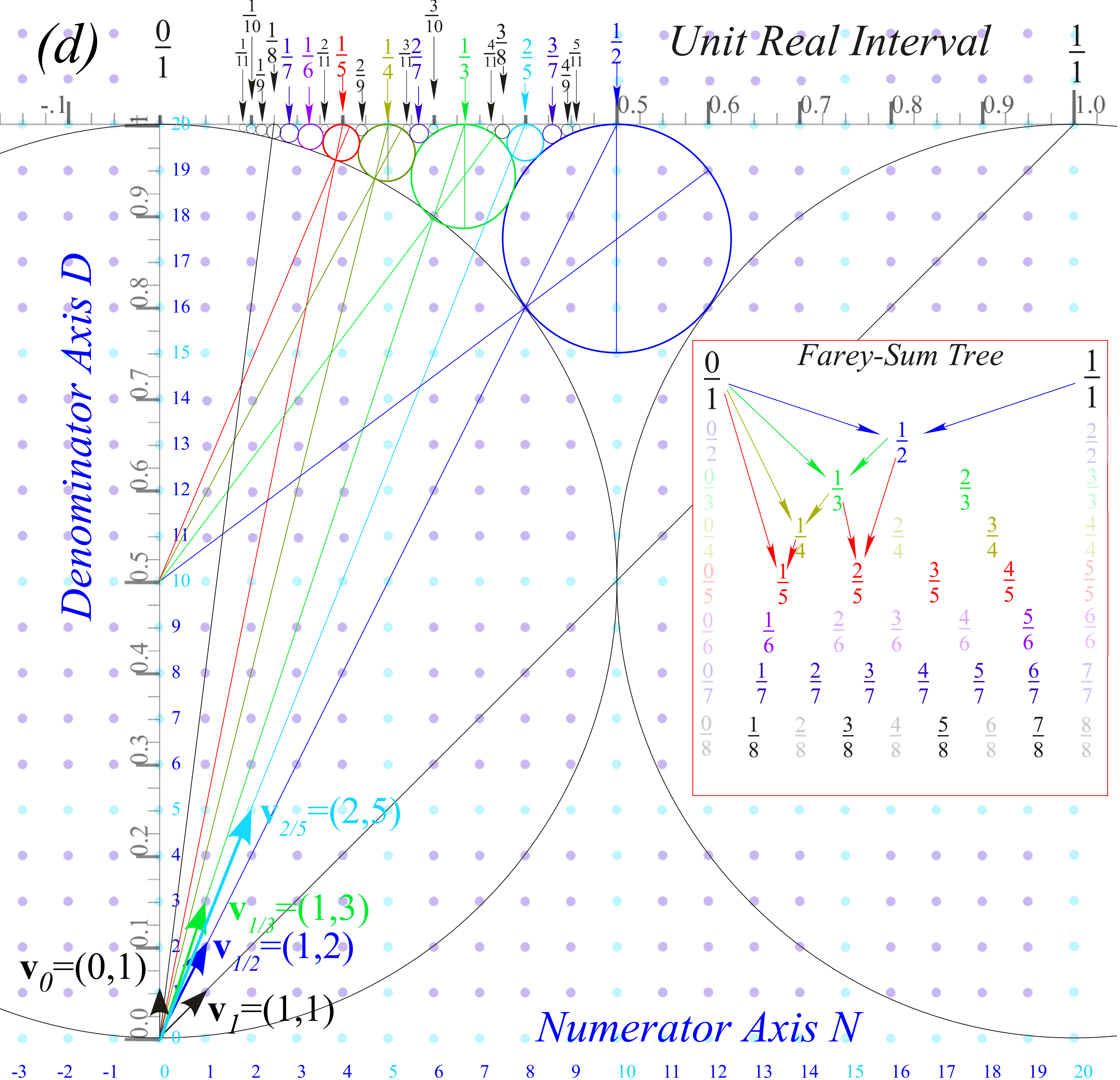
1/2-circle has diameter $1/2^2=1/4$

1/3-circles have diameter $1/3^2=1/9$

n/d-circles have diameter $1/d^2$

A. Li and W. Harter, Chem. Phys. Letters, 633, 208-213 (2015)

Harter and Alvason Li Int. Symposium on Molecular Spectroscopy OSU Columbus (2013)



Farey Sum related to vector sum and *Ford Circles*

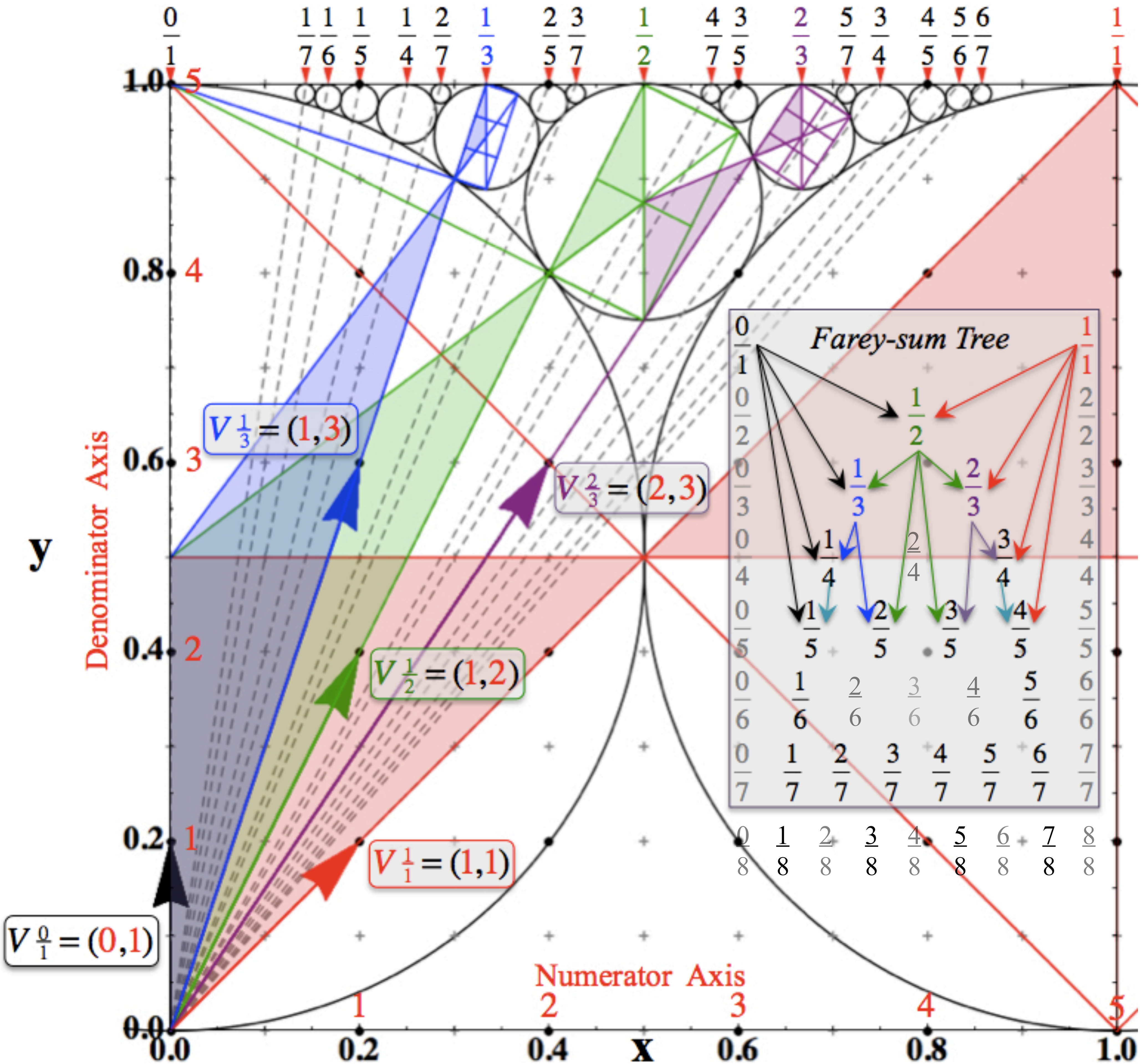
1/2-circle has diameter $1/2^2=1/4$

1/3-circles have diameter $1/3^2=1/9$

n/d-circles have diameter $1/d^2$

A. Li and W. Harter, Chem. Phys. Letters, 633, 208-213 (2015)

Harter and Alvason Li Int. Symposium on Molecular Spectroscopy OSU Columbus (2013)

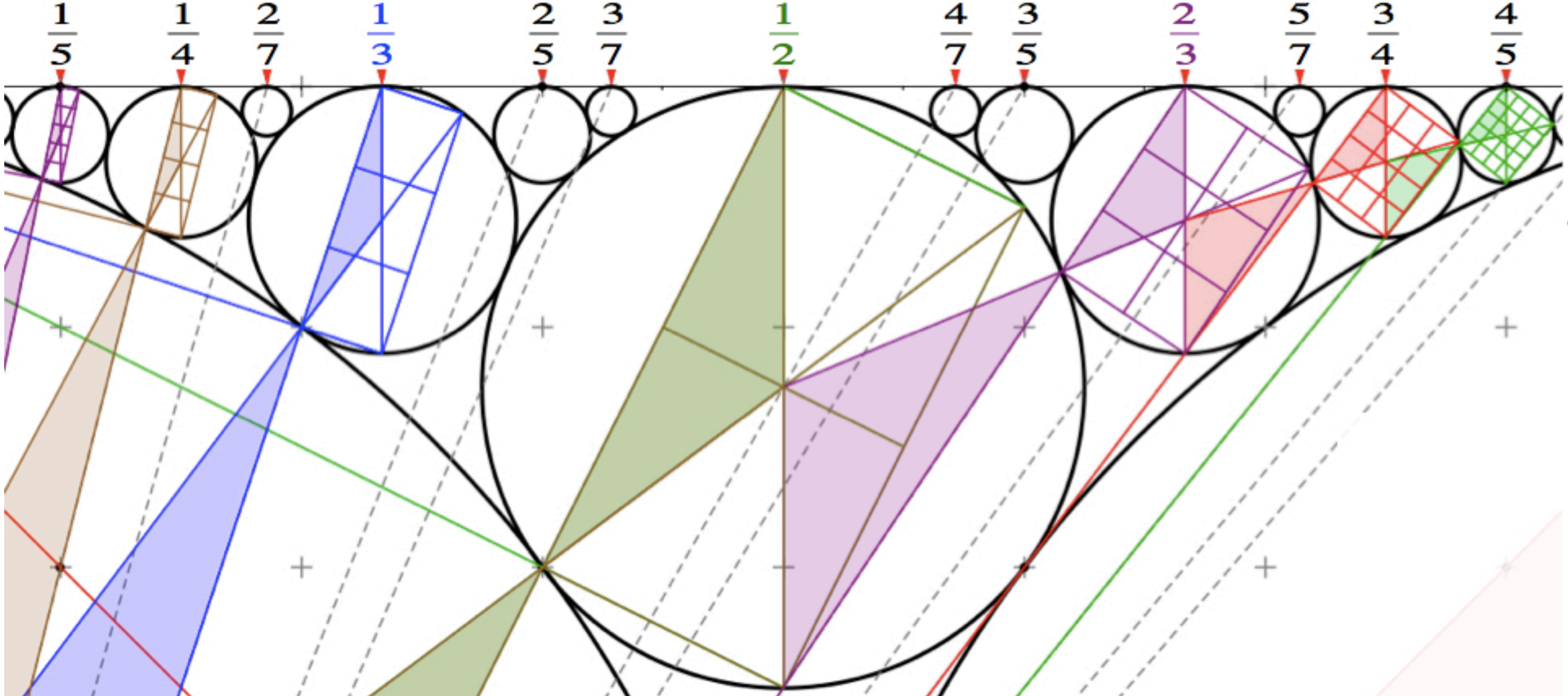


Thales Rectangles

provide analytic geometry of fractal structure

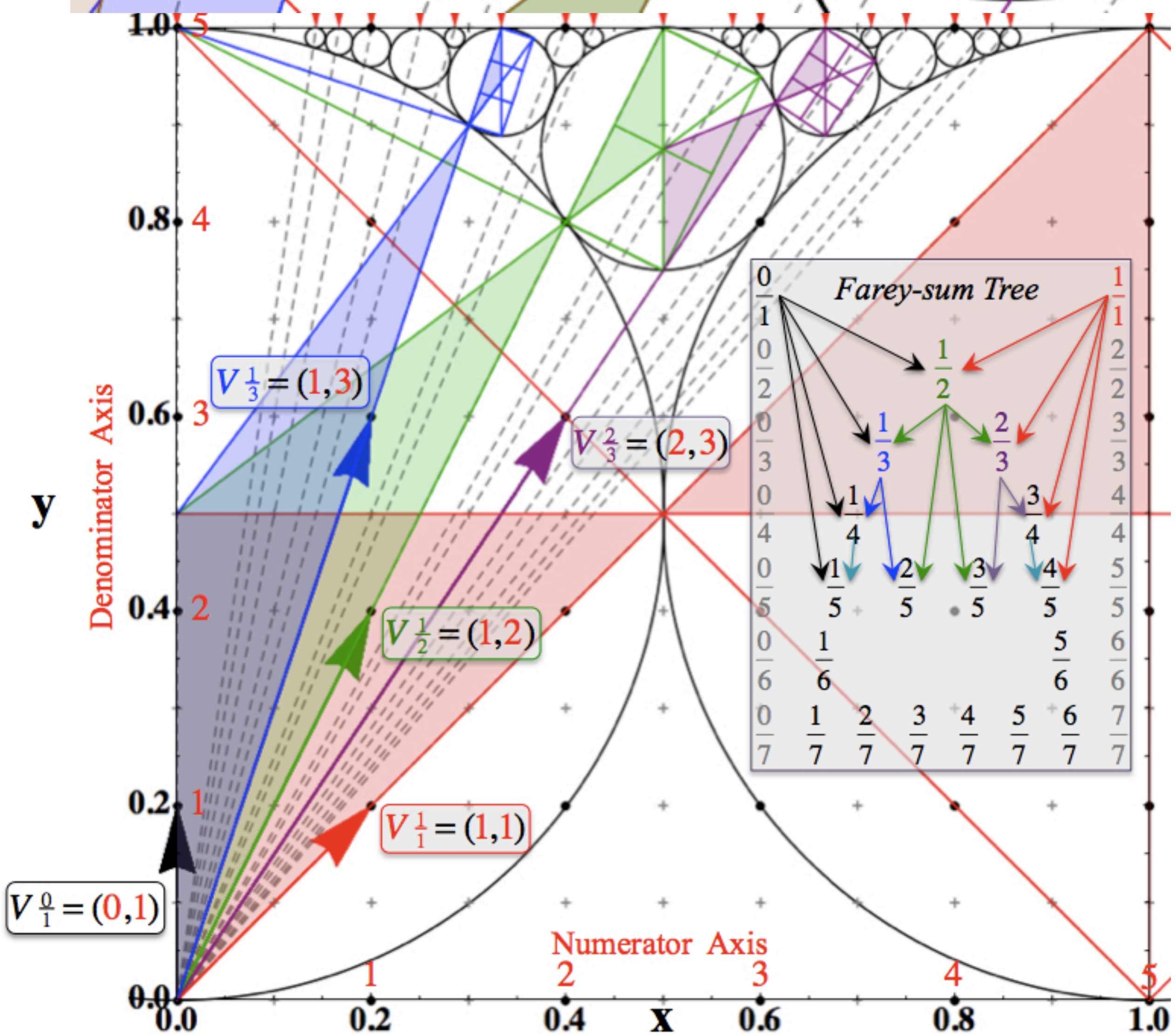
A. Li and W. Harter, Chem. Phys. Letters, 633, 208-213 (2015)

Harter and Alvason Li
Int. Symposium on Molecular Spectroscopy
OSU Columbus (2013)



“Quantized” Thales Rectangles

provide analytic geometry of fractal structure



A. Li and W. Harter,
Chem. Phys. Letters,
633, 208-213 (2015)

Harter and Alvason Li
Int. Symposium on
Molecular Spectroscopy
OSU Columbus (2013)

Relating C_N symmetric H and K matrices to differential wave operators

Relating C_N symmetric H and K matrices to wave differential operators

The 1st neighbor **K** matrix relates to a 2nd *finite-difference* matrix of 2nd x-derivative for high C_N.

$$\mathbf{K} = k(2\mathbf{1} - \mathbf{r} - \mathbf{r}^{-1}) \text{ analogous to:} -k\frac{\partial^2}{\partial x^2}$$

1st derivative momentum: $p = \frac{\hbar}{i}\frac{\partial y}{\partial x} \approx \frac{\hbar}{i}\frac{y(x+\Delta x) - y(x)}{(\Delta x)}$

$$\frac{\hbar}{i}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}\begin{pmatrix} \cdot \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ \cdot \end{pmatrix} = \frac{\hbar}{i}\begin{pmatrix} \cdot \\ y_1 - y_0 \\ y_2 - y_1 \\ y_3 - y_2 \\ y_4 - y_3 \\ \cdot \end{pmatrix}$$

2nd derivative KE: $2mE = -\hbar^2\frac{\partial^2 y}{\partial x^2} \approx \frac{y(x+\Delta x) - 2y(x) + y(x-\Delta x)}{(\Delta x)^2}$

$$-\hbar^2\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & 2 & -1 & \cdot & \cdot & \cdot \\ \cdot & -1 & 2 & -1 & \cdot & \cdot \\ \cdot & \cdot & -1 & 2 & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 & 2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}\begin{pmatrix} \cdot \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ \cdot \end{pmatrix} = \hbar^2\begin{pmatrix} \cdot \\ y_0 - 2y_1 + y_2 \\ y_1 - 2y_2 + y_3 \\ y_2 - 2y_3 + y_4 \\ y_3 - 2y_4 + y_5 \\ \cdot \end{pmatrix}$$

H and **K** matrix equations are finite-difference versions of quantum and classical wave equations.

$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \mathbf{H}|\psi\rangle$ *(**H**-matrix equation)*

$-\frac{\partial^2}{\partial t^2}|y\rangle = \mathbf{K}|y\rangle$ *(**K**-matrix equation)*

$i\hbar\frac{\partial}{\partial t}|\psi\rangle = (-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V)|\psi\rangle$ *(Scrodinger equation)*

$-\frac{\partial^2}{\partial t^2}|y\rangle = -k\frac{\partial^2}{\partial x^2}|y\rangle$ *(Classical wave equation)*

Square p^2 gives 1st neighbor **K** matrix.

$$\frac{\hbar}{i}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}\frac{\hbar}{i}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} = \hbar^2\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & 2 & -1 & \cdot & \cdot & \cdot \\ \cdot & -1 & 2 & -1 & \cdot & \cdot \\ \cdot & \cdot & -1 & 2 & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 & 2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

Higher order p^3, p^4,.. involve 2nd, 3rd, 4th..neighbor **H**

$$p^4 \cong \begin{pmatrix} \ddots & \vdots & 1 & \cdot & \cdot & \cdot \\ \cdots & 6 & -4 & 1 & \cdot & \cdot \\ 1 & -4 & 6 & -4 & 1 & \cdot \\ \cdot & 1 & -4 & 6 & -4 & 1 \\ \cdot & \cdot & 1 & -4 & 6 & -4 \\ \cdot & \cdot & \cdot & 1 & -4 & 6 \end{pmatrix}$$

Symmetrized finite-difference operators

$$\bar{\Delta} = \frac{1}{2}\begin{pmatrix} \ddots & \vdots & & & & \\ \cdots & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & -1 & 0 & 1 & \\ & & & -1 & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}, \quad \bar{\Delta}^3 = \frac{1}{2^3}\begin{pmatrix} \ddots & \vdots & 0 & -1 & & \\ \cdots & 0 & 3 & 0 & -1 & \\ 0 & -3 & 0 & 3 & 0 & -1 \\ 1 & 0 & -3 & 0 & 3 & 0 \\ & 1 & 0 & -3 & 0 & 3 \\ & & 1 & 0 & -3 & 0 \end{pmatrix}$$

$$\bar{\Delta}^2 = \frac{1}{2^2}\begin{pmatrix} \ddots & \vdots & 1 & & & \\ \cdots & -2 & 0 & 1 & & \\ 1 & 0 & -2 & 0 & 1 & \\ & 1 & 0 & -2 & 0 & 1 \\ & & 1 & 0 & -2 & 0 \\ & & & 1 & 0 & -2 \end{pmatrix}, \quad \bar{\Delta}^4 = \frac{1}{2^4}\begin{pmatrix} \ddots & \vdots & -4 & 0 & 1 & \\ \cdots & 6 & 0 & -4 & 0 & 1 \\ -4 & 0 & 6 & 0 & -4 & 0 \\ 0 & -4 & 0 & 6 & 0 & -4 \\ 1 & 0 & -4 & 0 & 6 & 0 \\ & 1 & 0 & -4 & 0 & 6 \end{pmatrix}$$