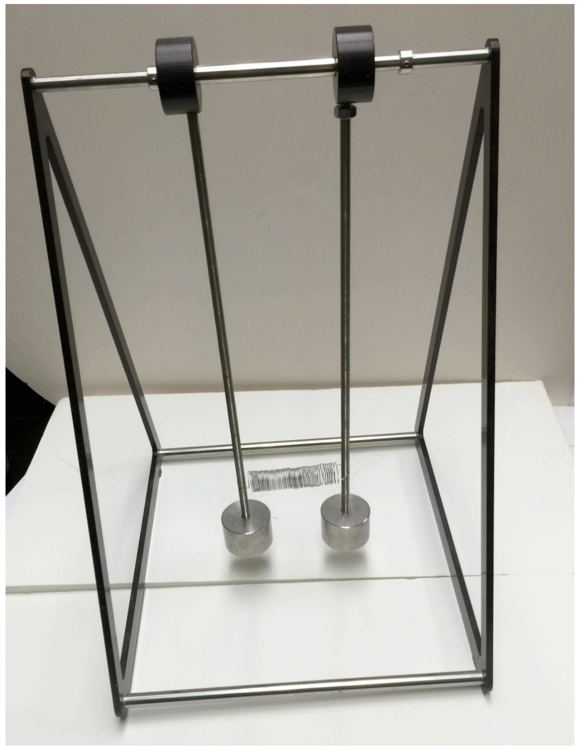
Lecture 16 thru 17
Thur. 3.10. to Tue. 3.15.2016

Introduction to coupled oscillation and eigenmodes

(Ch. 3-4 of Unit 2)



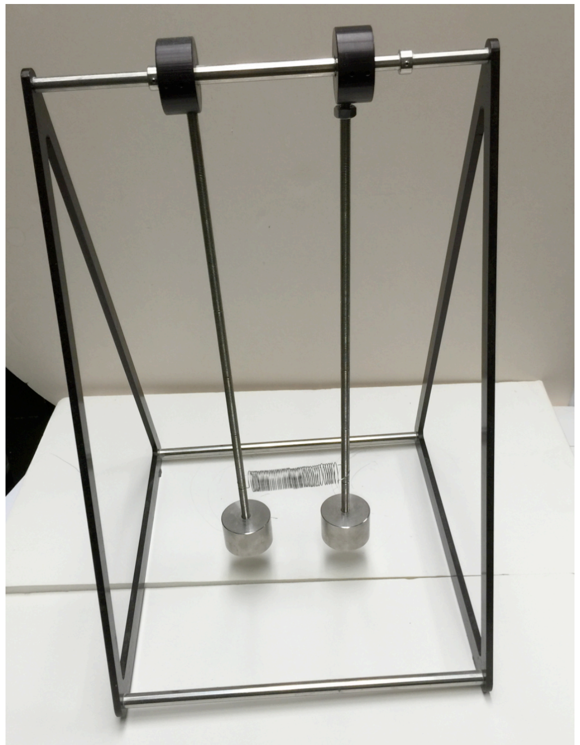
Review of 1D FDHO (Forced-Damped-Harmonic Oscillator) response
Amplitude and phase variation due to resonance

2D harmonic oscillator (2D-HO) equations of motion
Lagrangian and matrix forms

2D harmonic oscillator equation eigensolutions (normal modes)
2D classical HO compared to U(2) quantum 2-state system
Introducing ABCD Hamilton Pauli spinor symmetry expansion

Eigensolutions by geometry for 2D-HO with bilateral (B-Type) symmetry
Symmetric (low frequency) mode versus antisymmetric (high frequency) mode
Mixed mode beat dynamics (with constant $\pi/2$ phase-lag)
Geometry of phase and polarization

Eigensolutions by matrix-algebra with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$
Secular equation
Hamilton-Cayley equation and projectors
Idempotent projectors (how eigenvalues⇒eigenvectors)
Operator orthonormality and Completeness (Idempotent means: **P·P=P***)*
Spectral Decompositions
Functional spectral decomposition



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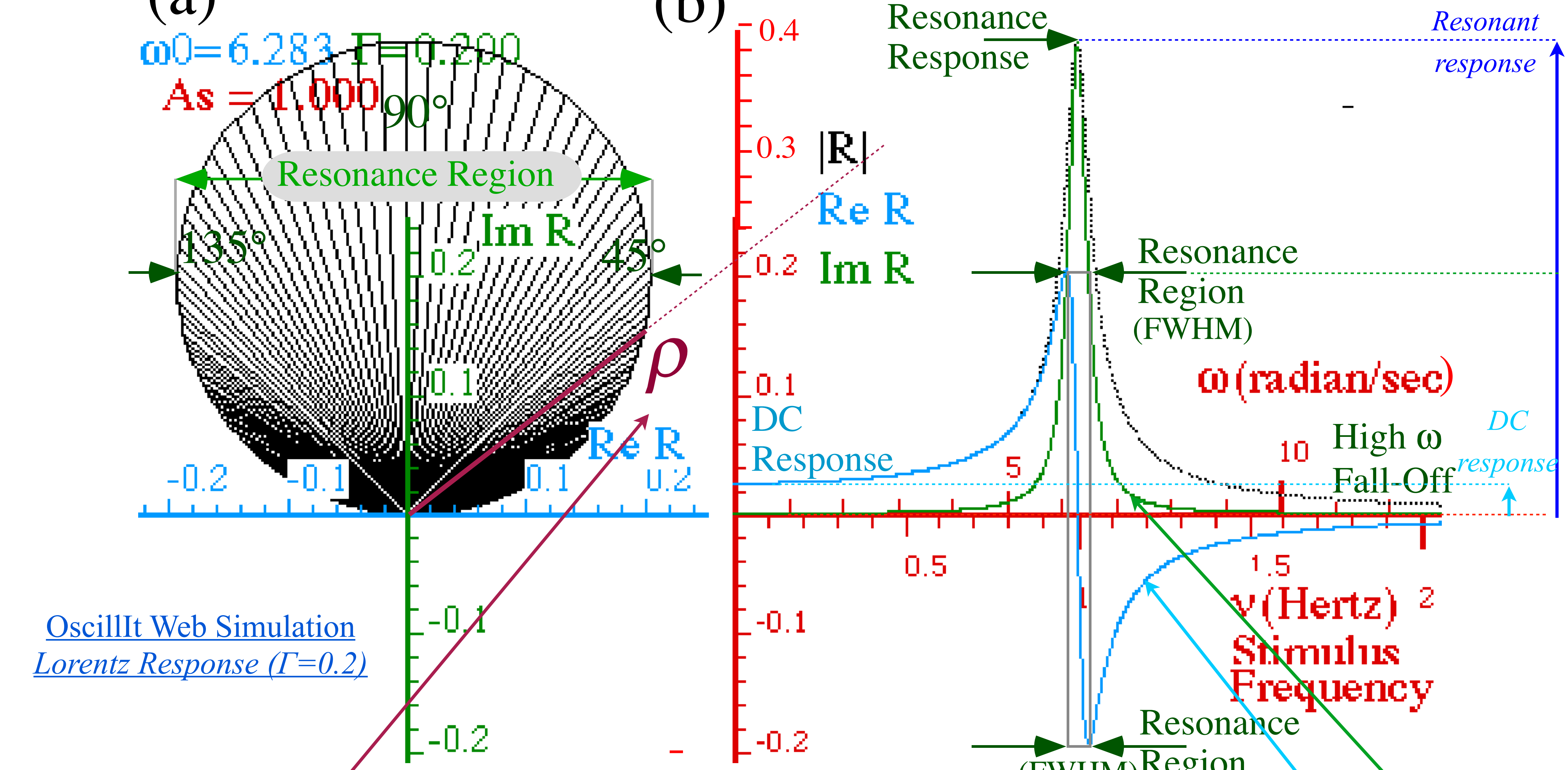


Fig. 2.2.6 Anatomy of oscillator Green-Lorentz response function plots

Phase lag angle

$$\rho = \tan^{-1}\left(\frac{2\Gamma\omega_s}{\omega_0^2 - \omega_s^2}\right)$$

$$\mathrm{Re}\, G_{\omega_0}(\omega_s) = \frac{\omega_0^2 - \omega_s^2}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

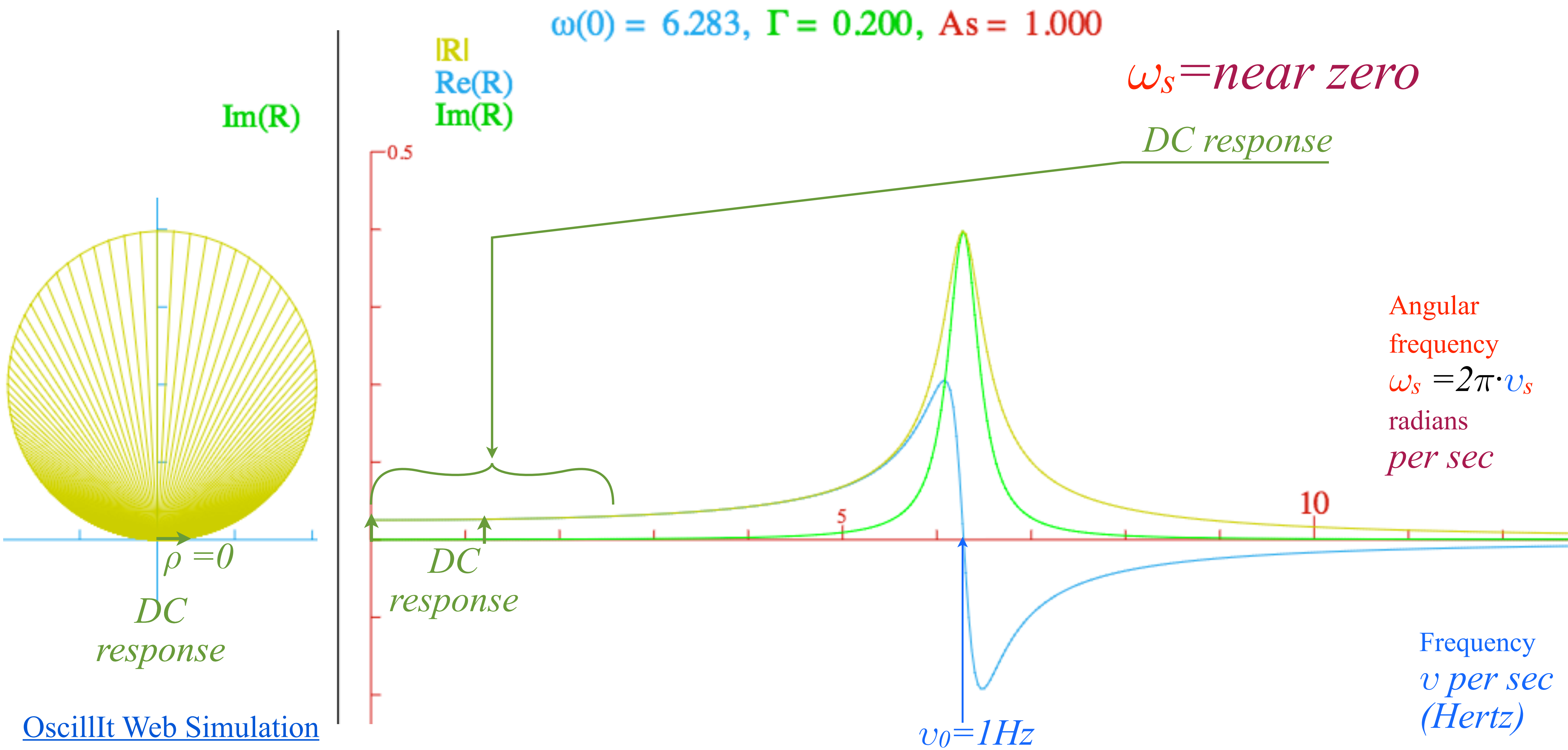
Real part

$$\mathrm{Im}\, G_{\omega_0}(\omega_s) = \frac{2\Gamma\omega_s}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

Imaginary part

$$AAF = \frac{\text{Resonant response}}{\text{DC response}} = \frac{\left|G_{\omega_0}\left(\omega_s = \omega_0\right)\right|}{\left|G_{\omega_0}(0)\right|} = \frac{1/\left(2\Gamma\omega_0\right)}{1/\omega_0^2} = \frac{\omega_0}{2\Gamma} \equiv q \quad (\textit{angular quality factor})$$

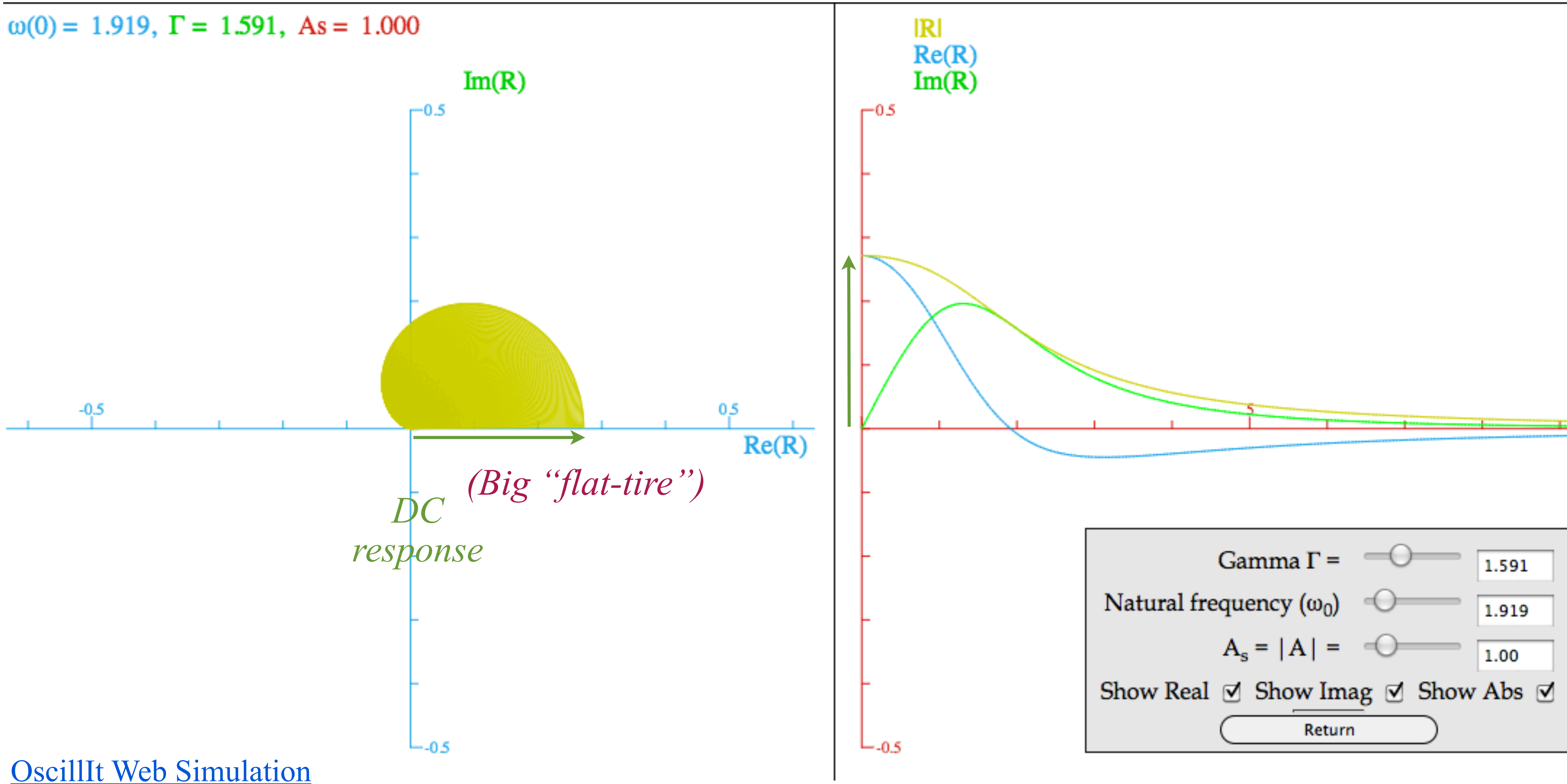
http://www.uark.edu/ua/modphys/markup/OscillItWeb.html



[OscillIt Web Simulation](#)
[*Lorentz Response (Γ=0.2)*](#)

link to time plot

Response functions for very low q oscillator



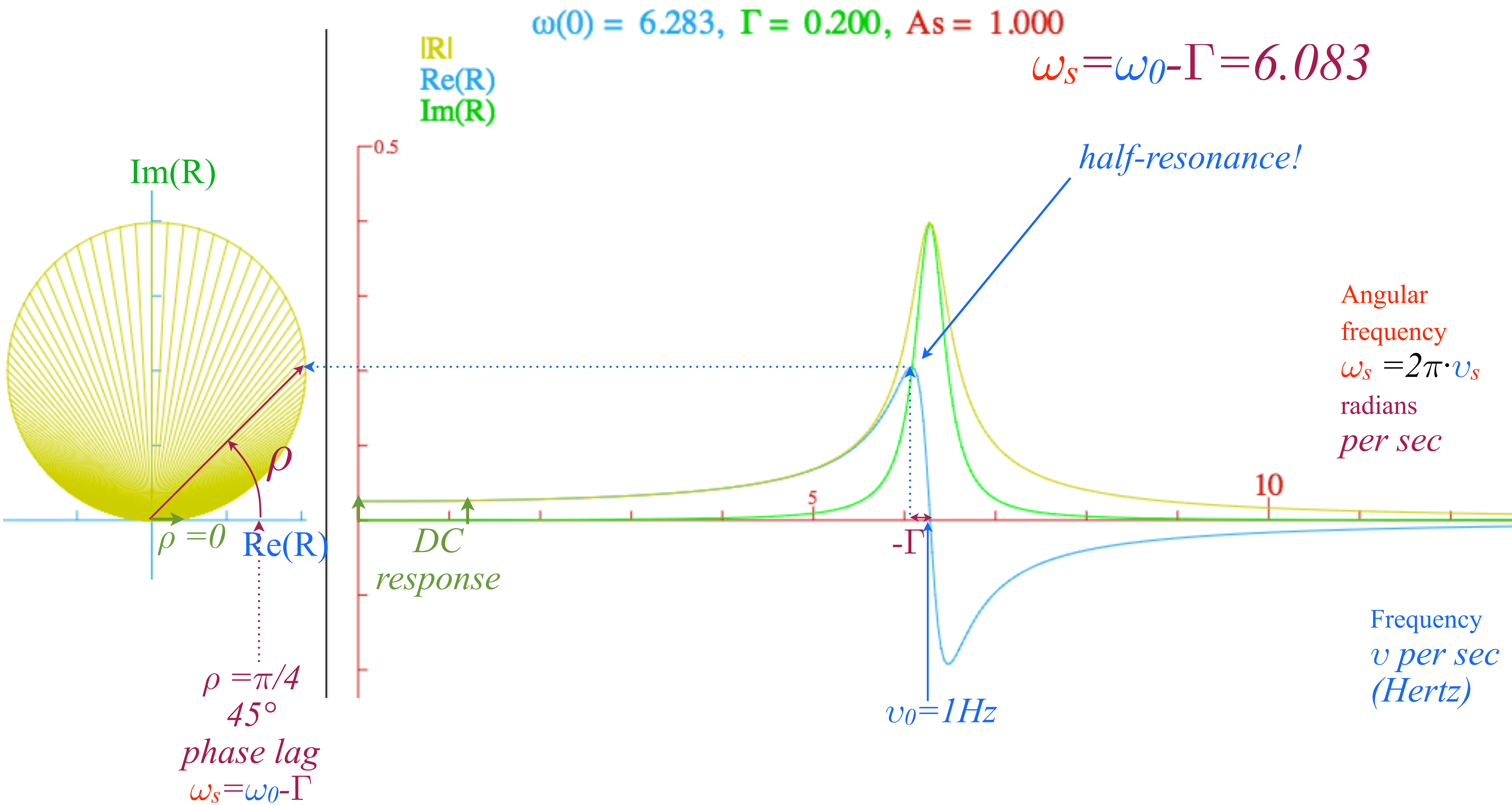
(Big "flat-tire")

DC response

[OscillIt Web Simulation](#)

[*Lorentz Response (Γ=1.6)*](#)

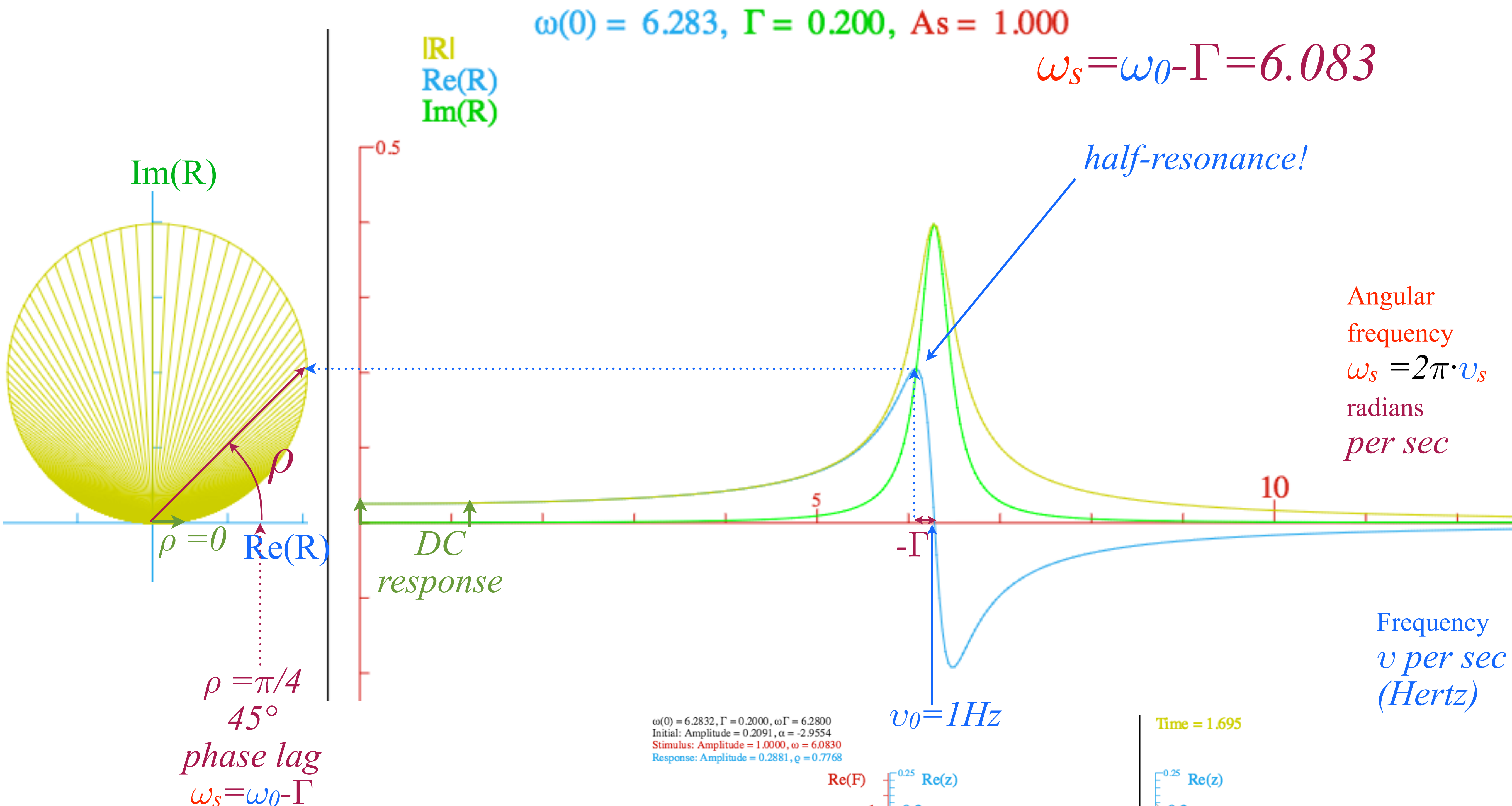
link to response

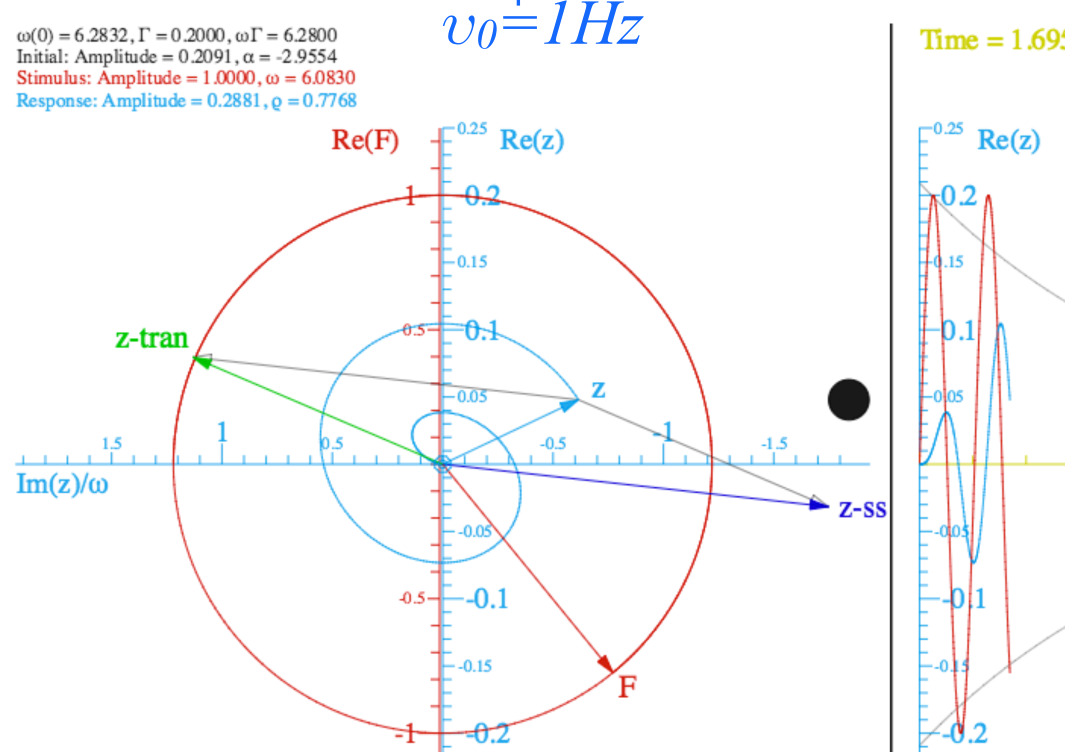


OscillIt Web Simulation

Lorentz Response (Γ=0.2)

link to time plot





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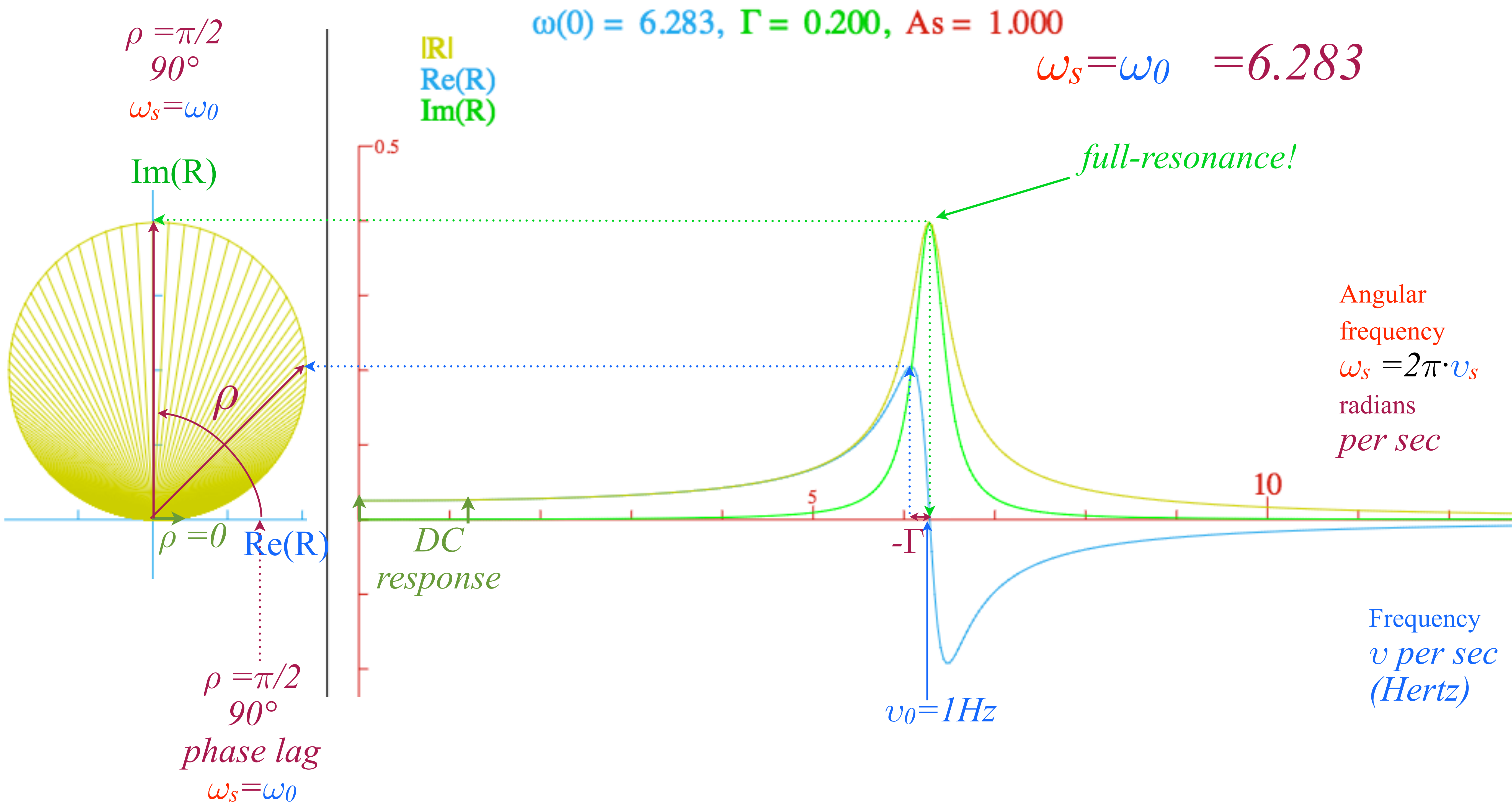

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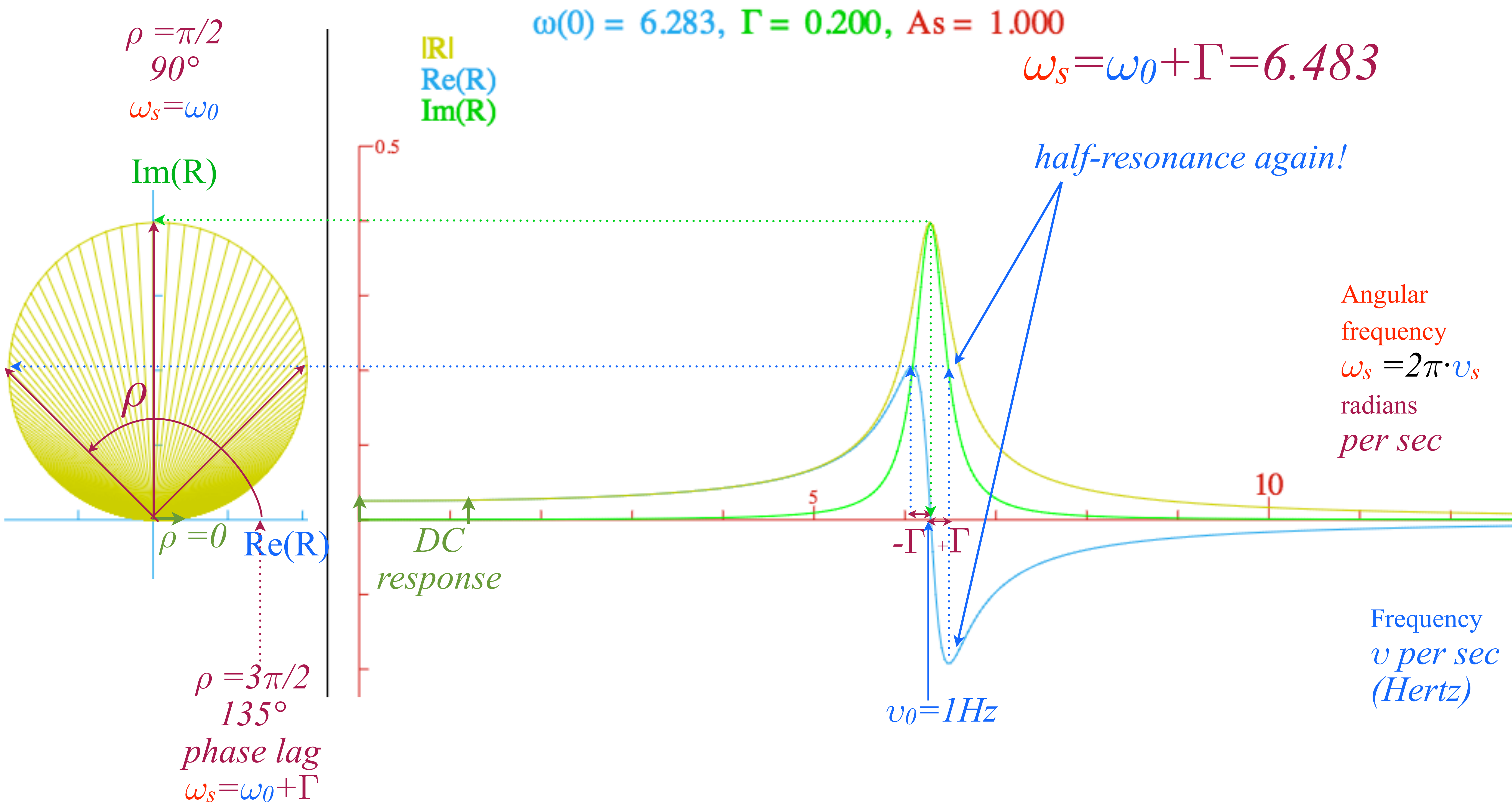
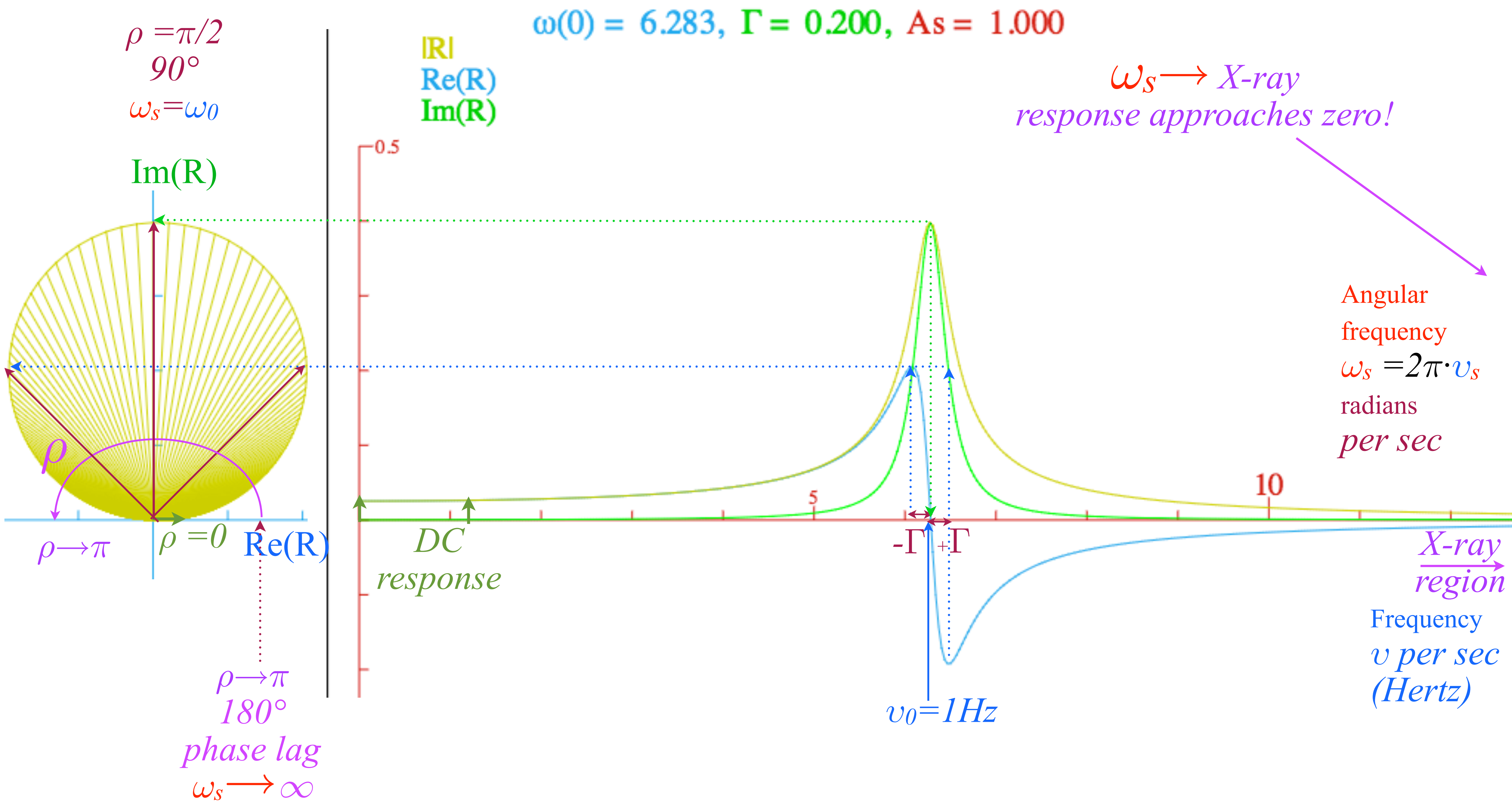


OscillIt Web Simulation

Lorentz Response (Γ=0.2)

link to time plot



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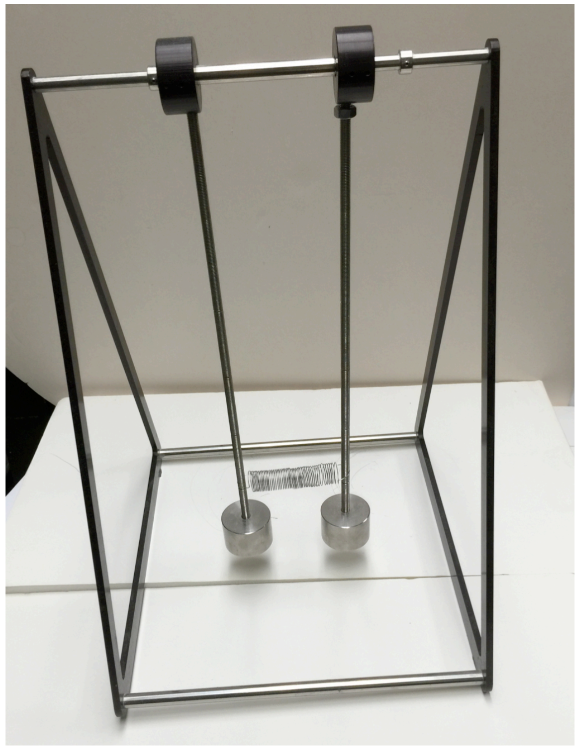

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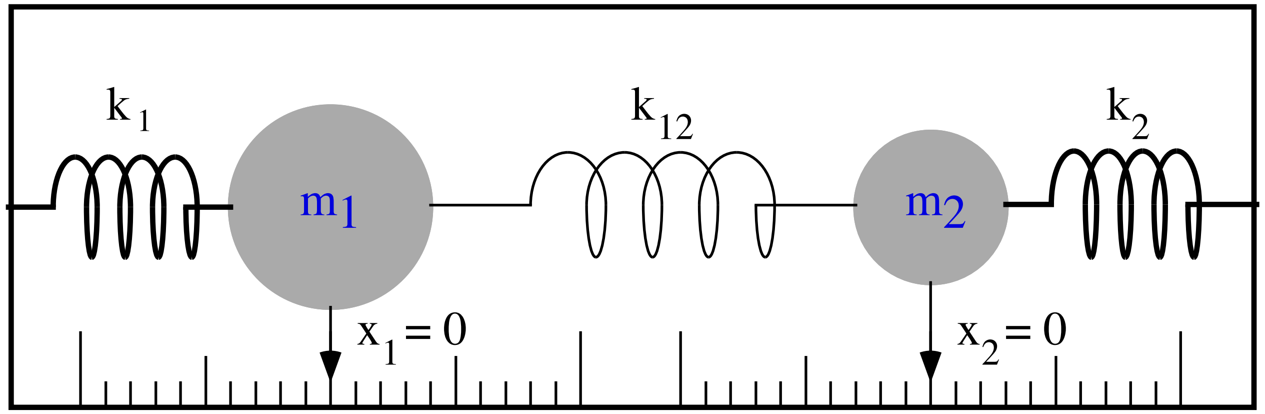


Fig. 3.3.1 Two 1-dimensional coupled oscillators

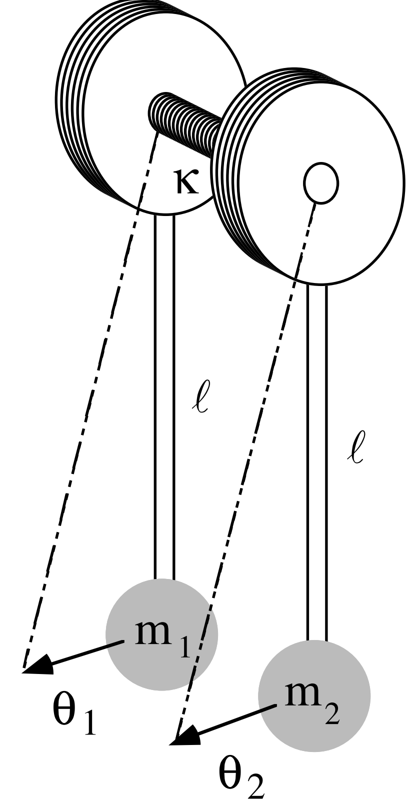


Fig. 3.3.2 Coupled pendulums

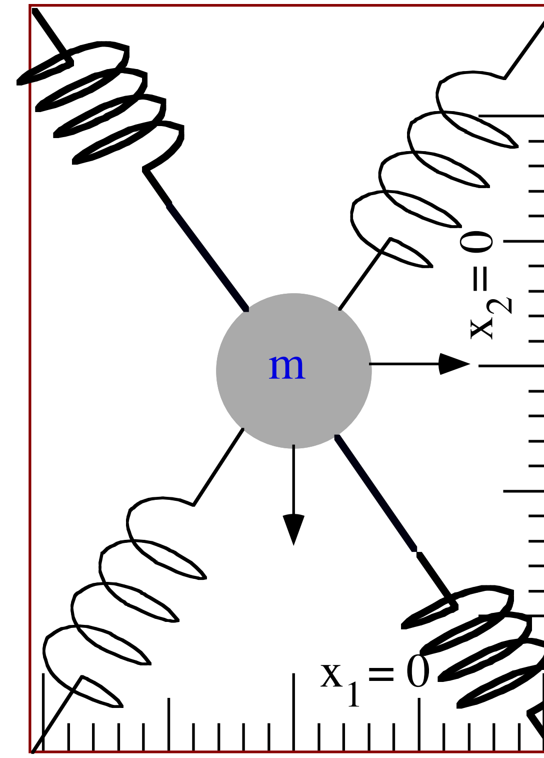
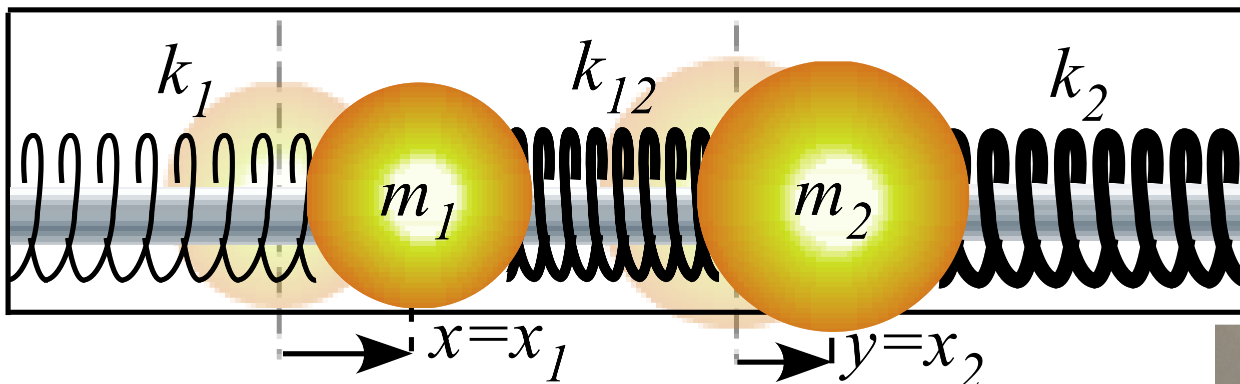
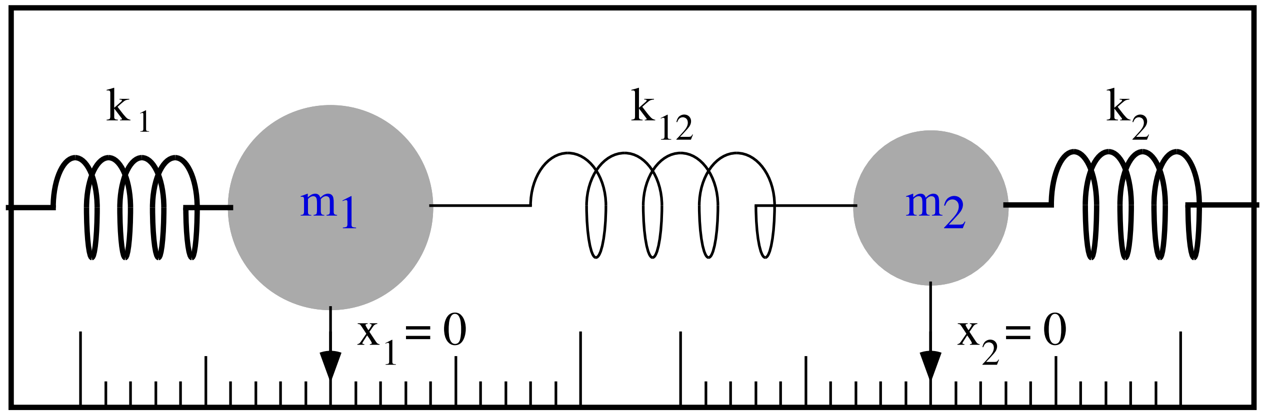


Fig. 3.3.3 One 2-dimensional coupled oscillator



2D harmonic oscillator energy



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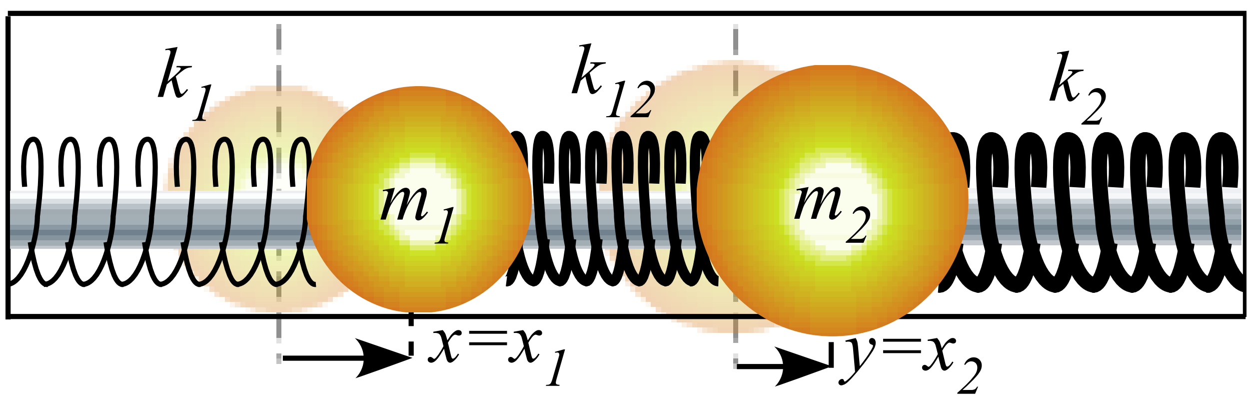

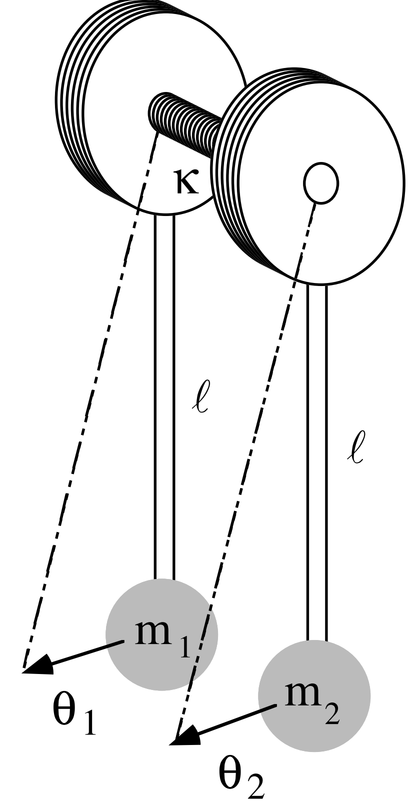

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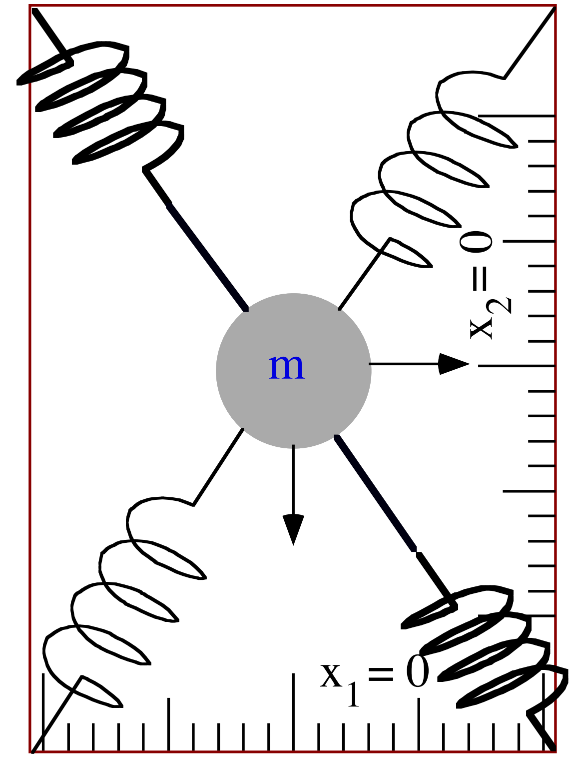

Fig. 3.3.3 One 2-dimensional coupled oscillator

2D HO kinetic energy $T(v_1, v_2)$

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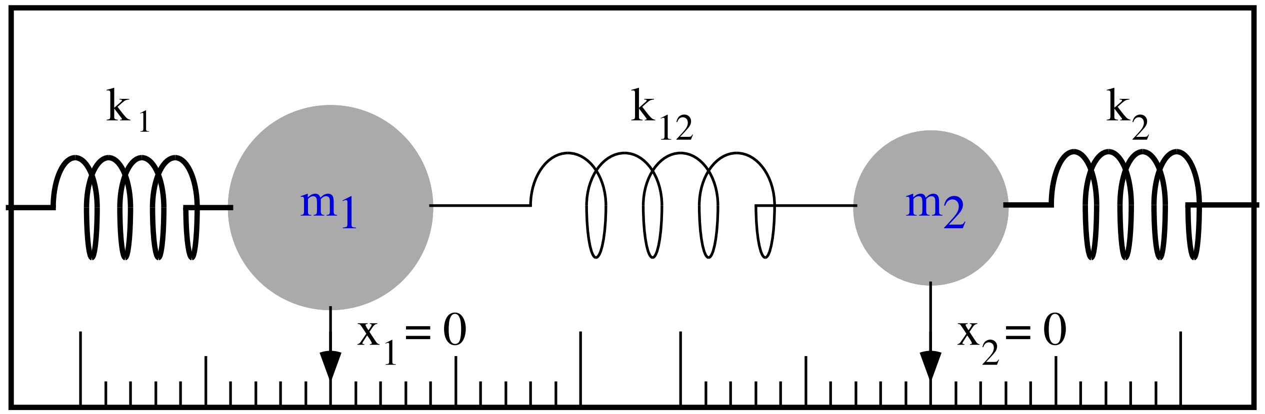
2D harmonic oscillator energy



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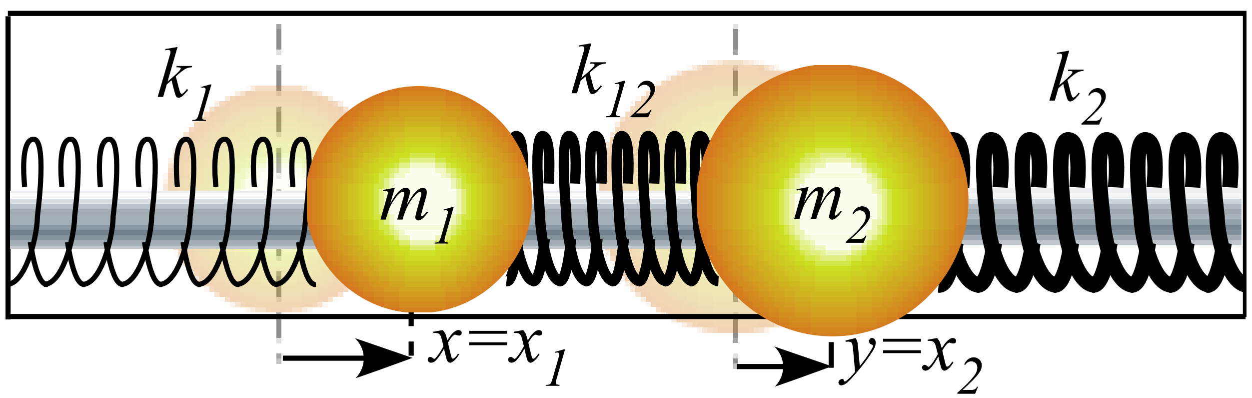

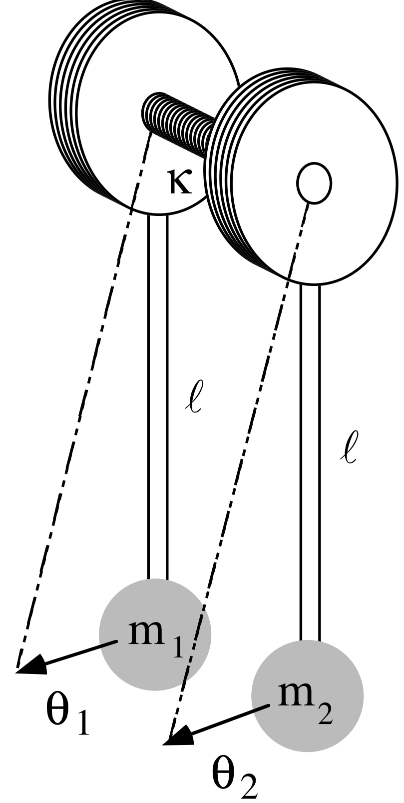

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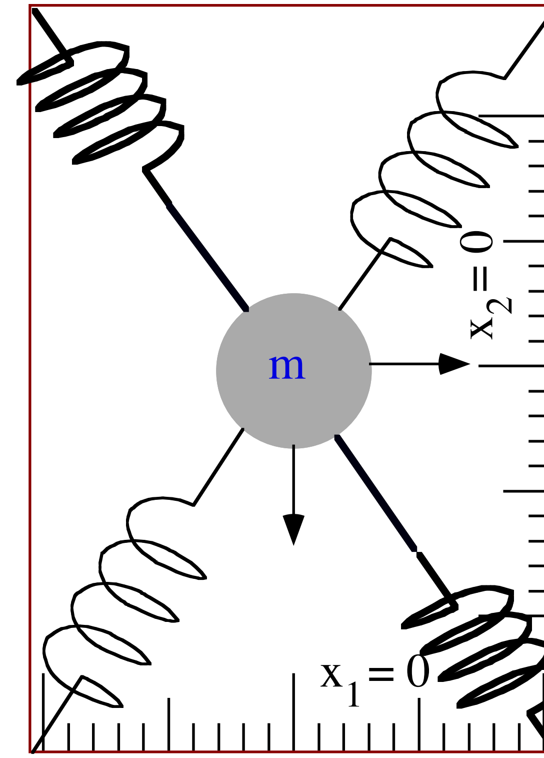

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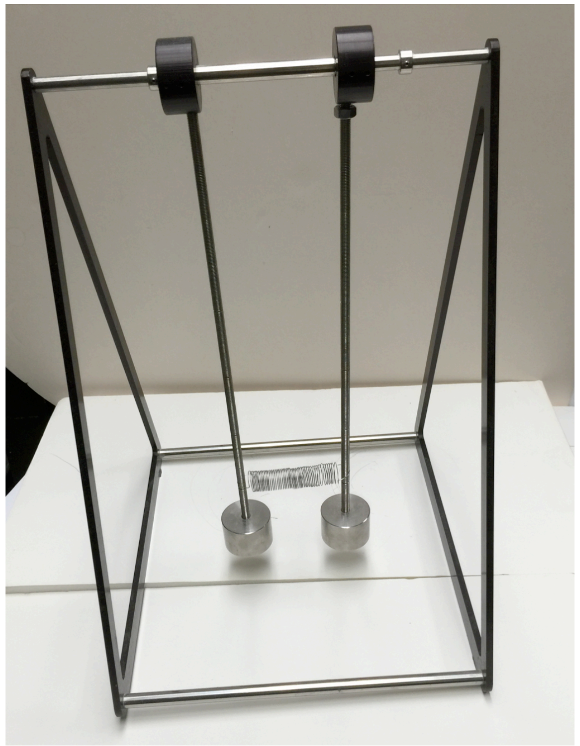

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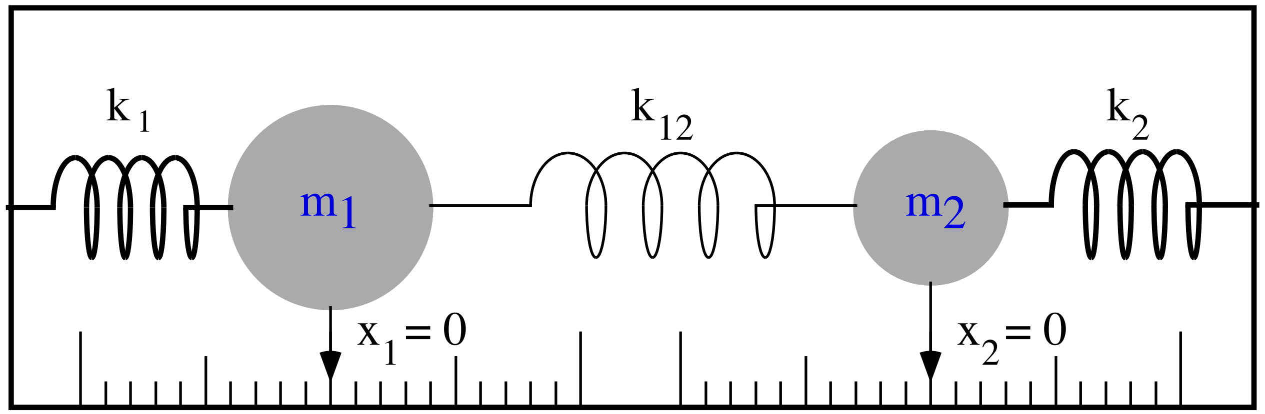

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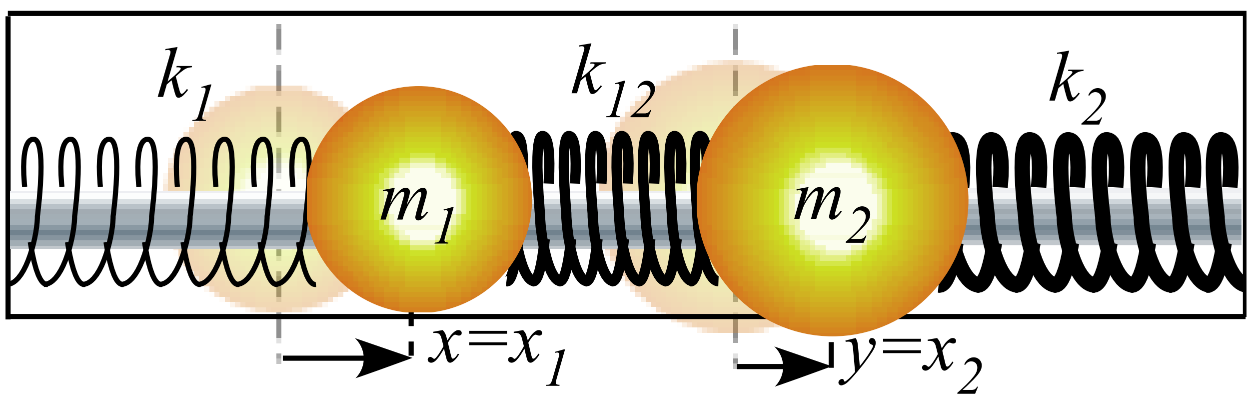

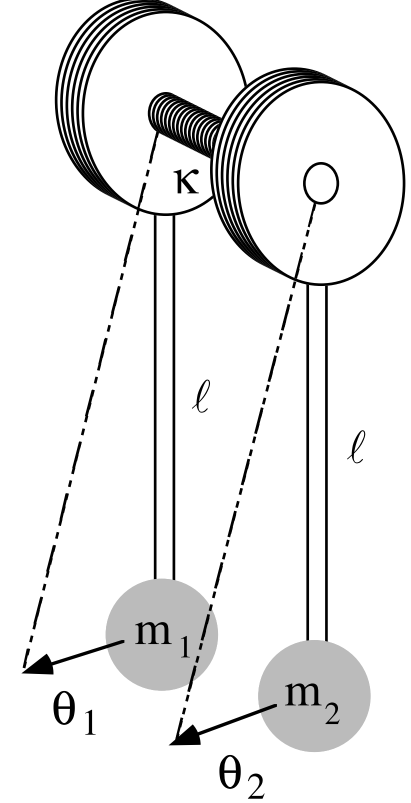

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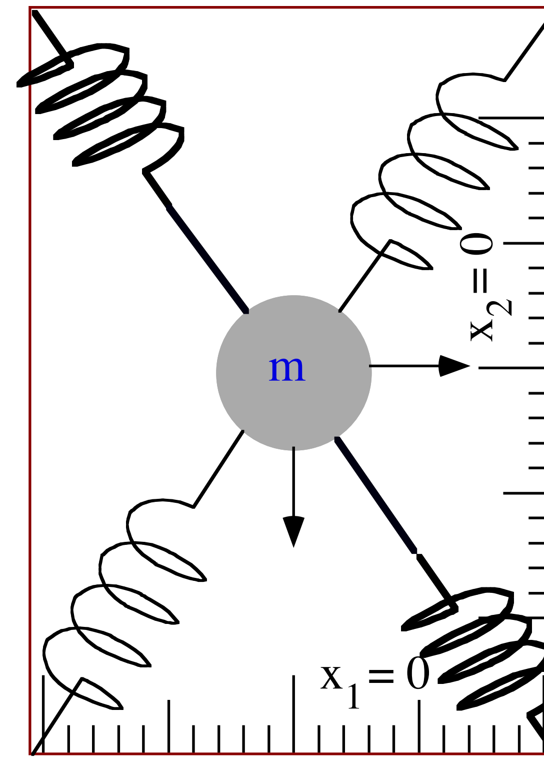

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$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

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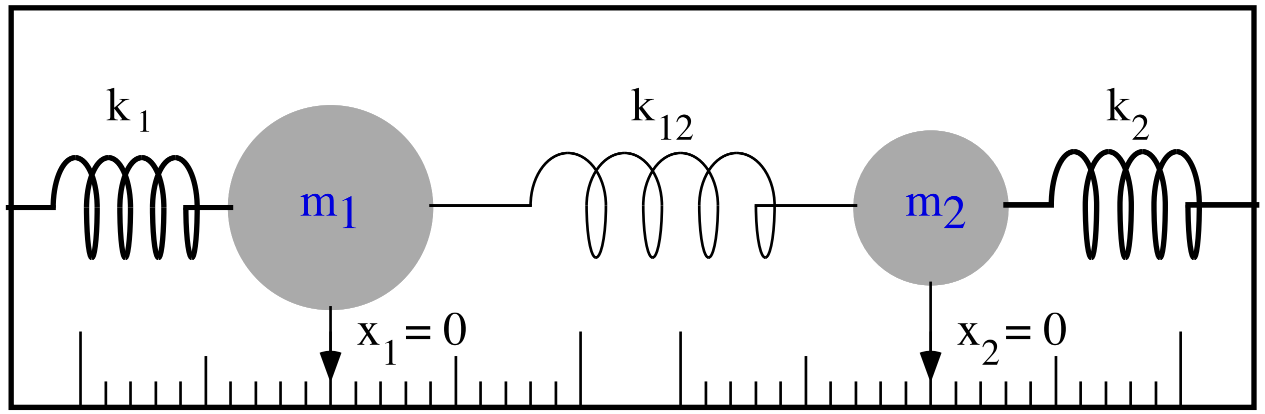
Lagrange-Newton equations for 2D HO

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_1} \right) = m_1 \ddot{x}_1 = F_1 = -\frac{\partial V}{\partial x_1} = -\left(k_1 + k_{12} \right) x_1 + k_{12} x_2$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_2} \right) = m_2 \ddot{x}_2 = F_2 = -\frac{\partial V}{\partial x_2} = k_{12} x_1 - \left(k_2 + k_{12} \right) x_2$$

2D harmonic oscillator equations



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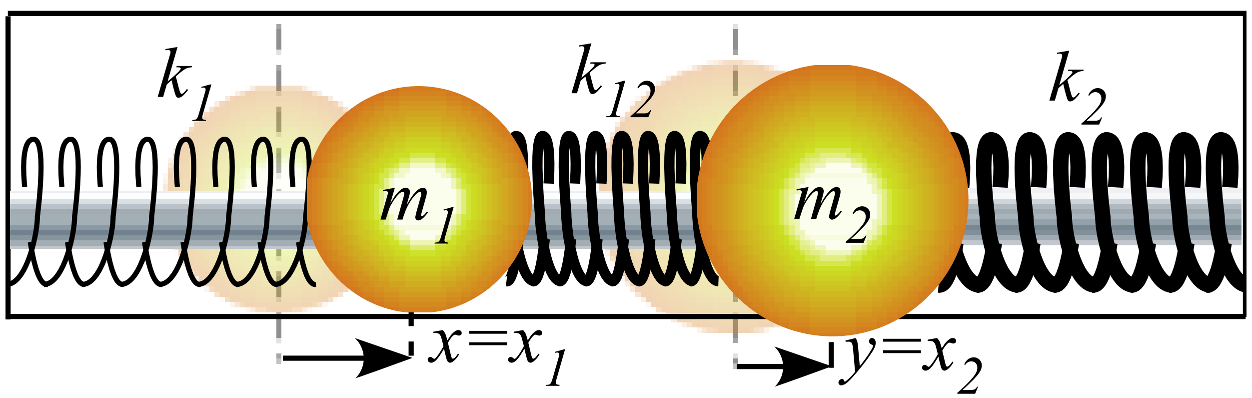

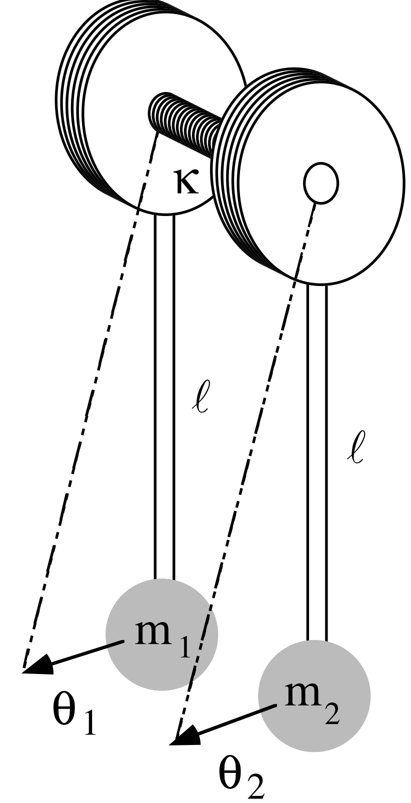

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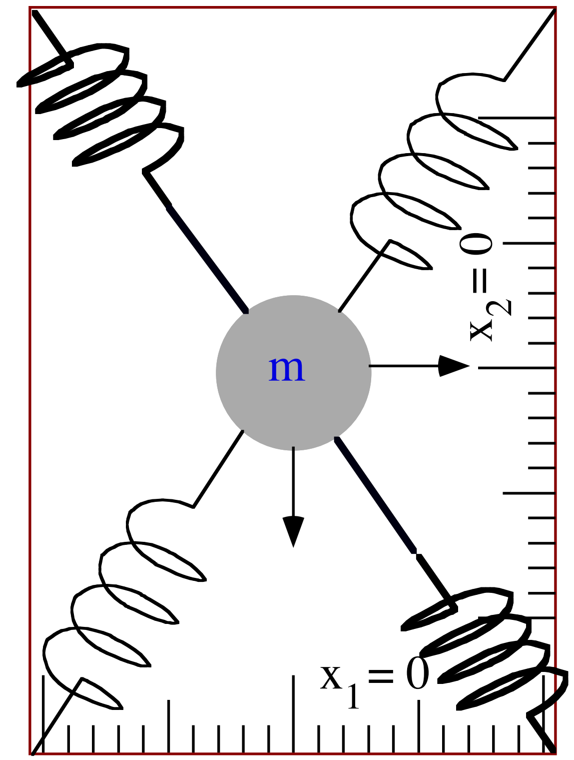

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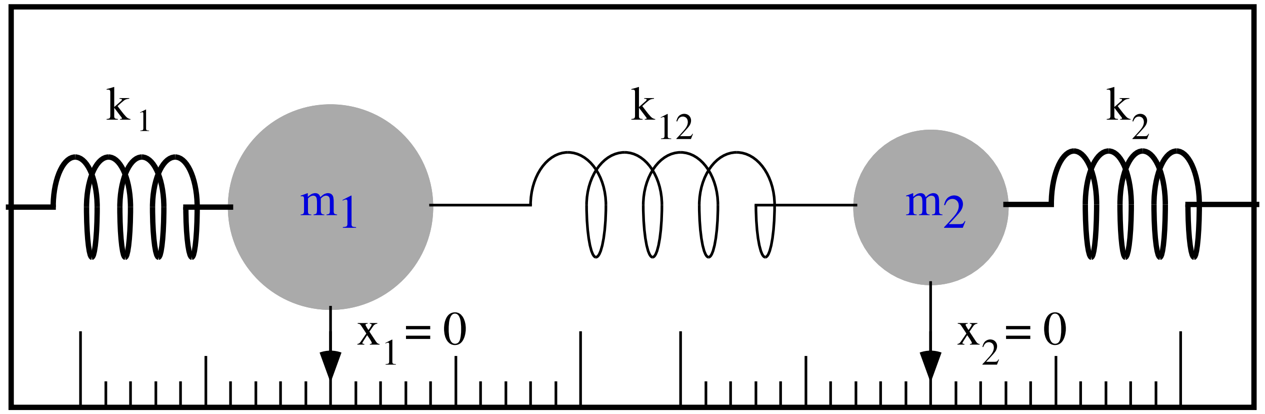
2D HO Matrix operator equations

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2D harmonic oscillator equations



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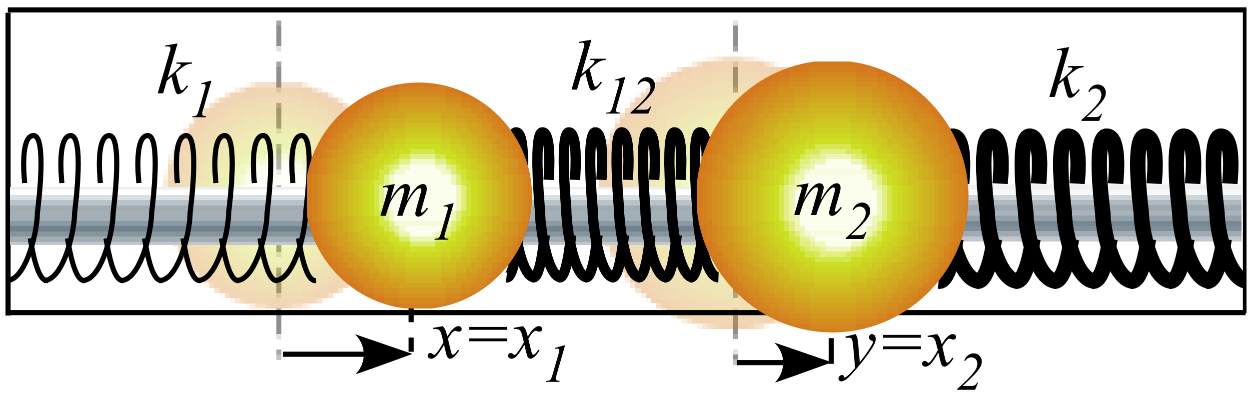

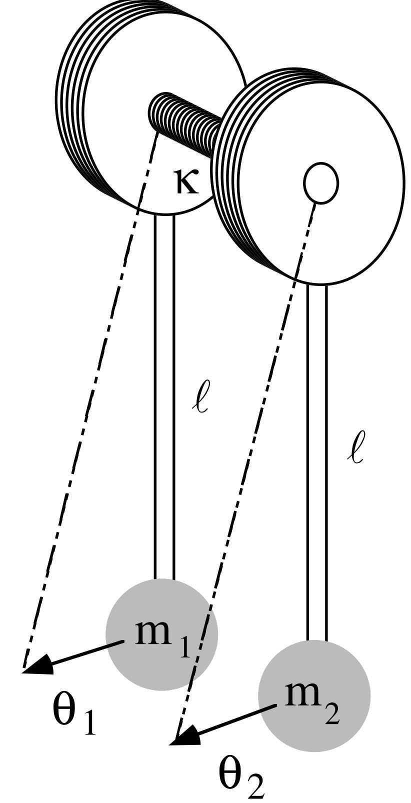

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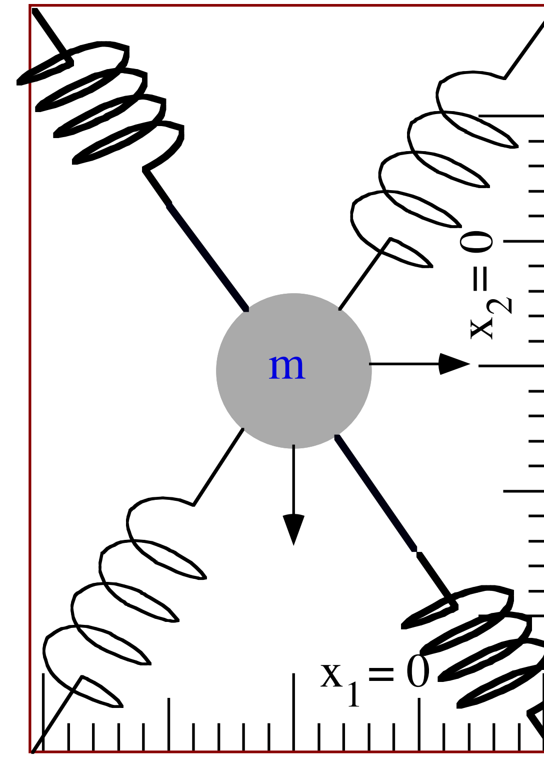

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$$= \frac{1}{2}(k_1 + k_{12}) x_1^2 - k_{12} x_1 x_2 + \frac{1}{2}(k_2 + k_{12}) x_2^2$$

Lagrange-Newton equations for 2D HO

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1 = F_1 = -\frac{\partial V}{\partial x_1} = -(k_1 + k_{12}) x_1 + k_{12} x_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = m_2 \ddot{x}_2 = F_2 = -\frac{\partial V}{\partial x_2} = k_{12} x_1 - (k_2 + k_{12}) x_2$$

2D HO Matrix operator equations

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Matrix operator notation:

$$\mathbf{M} \bullet \left| \ddot{\mathbf{x}} \right\rangle = - \mathbf{K} \bullet \left| \mathbf{x} \right\rangle$$

2D harmonic oscillator equations

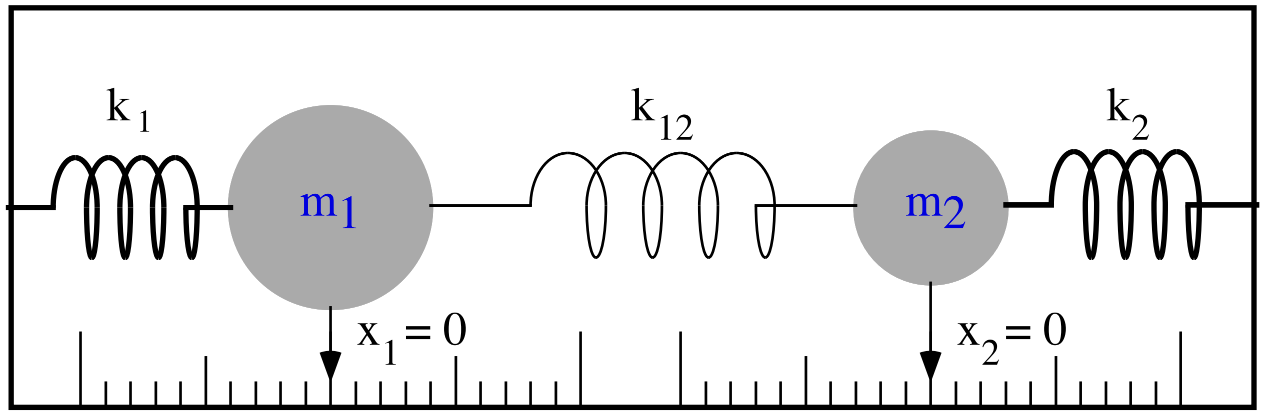
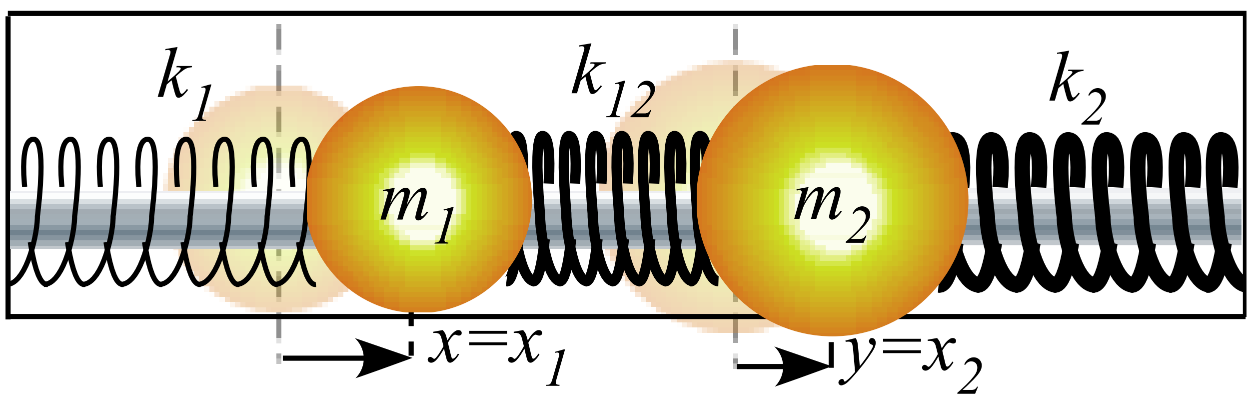
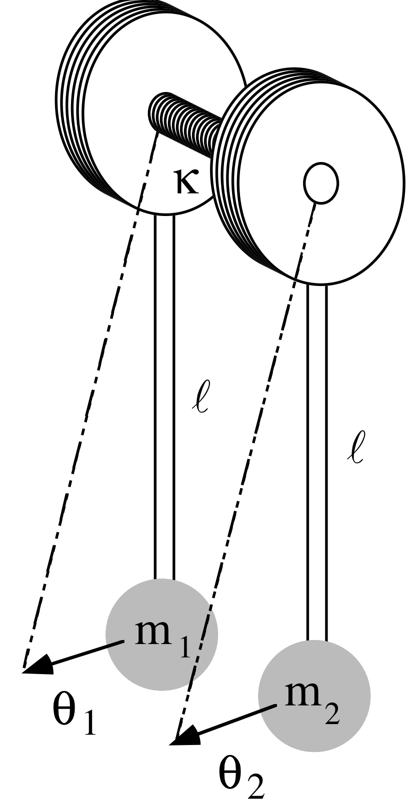


Fig. 3.3.1 Two 1-dimensional coupled oscillators





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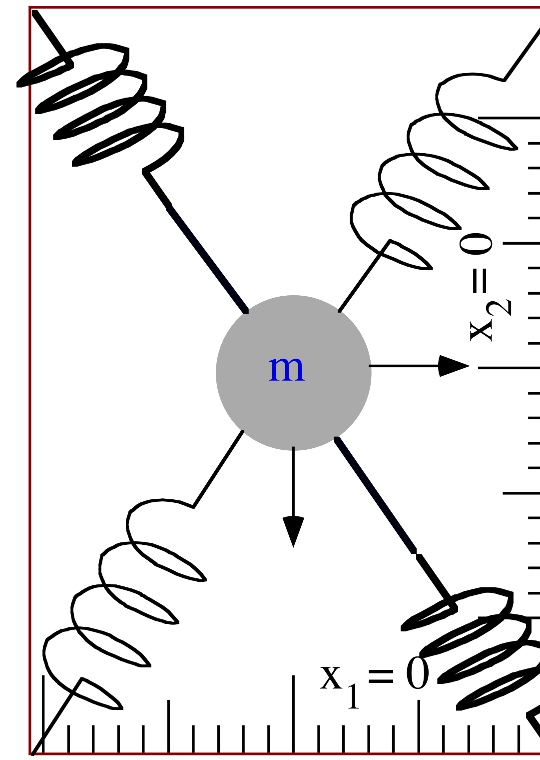

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$$= \frac{1}{2}\langle\dot{\mathbf{x}}|\mathbf{M}|\dot{\mathbf{x}}\rangle$$

2D HO potential energy $V(x_1, x_2)$

$$V = \frac{1}{2}(k_1 + k_{12})x_1^2 - k_{12}x_1x_2 + \frac{1}{2}(k_2 + k_{12})x_2^2$$

$$= \frac{1}{2}\langle\mathbf{x}|\mathbf{K}|\mathbf{x}\rangle \quad \text{where: } \mathbf{K} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix}$$

Lagrange-Newton equations for 2D HO

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m_1\ddot{x}_1 = F_1 = -\frac{\partial V}{\partial x_1} = -(k_1 + k_{12})x_1 + k_{12}x_2$$

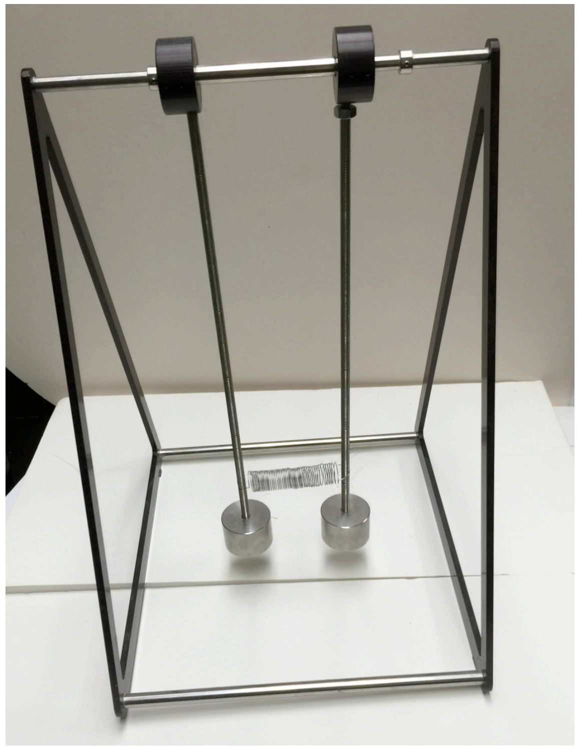
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = m_2\ddot{x}_2 = F_2 = -\frac{\partial V}{\partial x_2} = k_{12}x_1 - (k_2 + k_{12})x_2$$

2D HO Matrix operator equations

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Matrix operator notation:

$$\mathbf{M}\bullet|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$$



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So equation of motion is simply: $|\ddot{\mathbf{x}}\rangle=-\mathbf{K}|\mathbf{x}\rangle$

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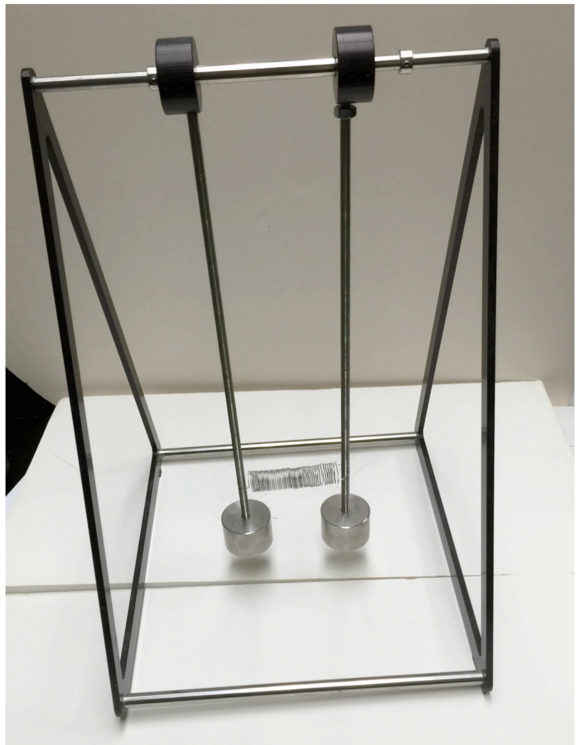

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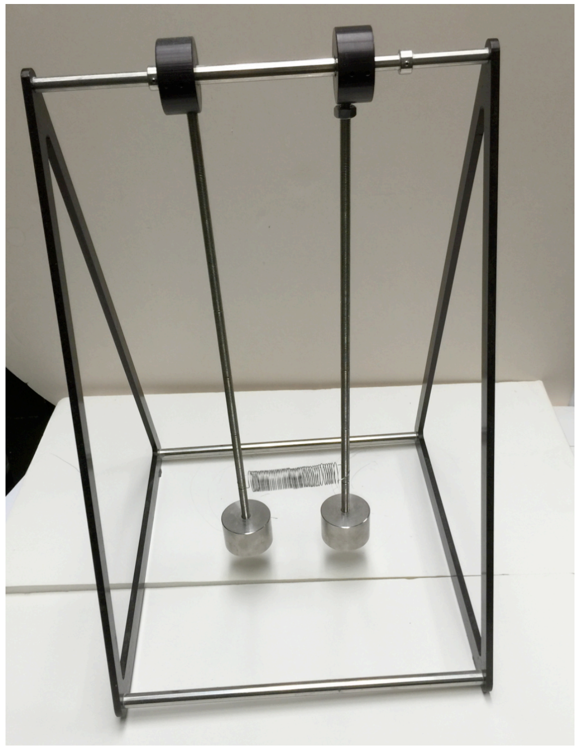
2D classical HO same as the U(2) quantum 2-state system

...if we set **K**-spring matrix to squared quantum operator $\mathbf{H}^2$

$$\mathbf{K} = \begin{pmatrix} k_{11} & k_{12}-i\cdot j_{12} \\ k_{12}-i\cdot j_{12} & k_{22} \end{pmatrix} = \begin{pmatrix} G & H-i\cdot J \\ H+i\cdot J & K \end{pmatrix} = \mathbf{H}^2 = \begin{pmatrix} A^2+B^2+C^2 & (B\text{-}iC)(A+D) \\ (B+iC)(A+D) & D^2+B^2+C^2 \end{pmatrix}$$



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Decompose the Hamiltonian operator **H** into four *ABCD* *symmetry operators*

(Labeled to provide dynamic mnemonics as well as colorful analogies)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + D\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = A\mathbf{e}_{11} + B\boldsymbol{\sigma}_B + C\boldsymbol{\sigma}_C + D\mathbf{e}_{22}$$

$$= \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Introducing ABCD Hamilton Pauli spinor symmetry expansion

Decompose the Hamiltonian operator **H** into four *ABCD symmetry operators*
(Labeled to provide dynamic mnemonics as well as colorful analogies)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + D\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = A\mathbf{e}_{11} + B\boldsymbol{\sigma}_B + C\boldsymbol{\sigma}_C + D\mathbf{e}_{22}$$

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$$\mathbf{H} = \frac{A-D}{2}\boldsymbol{\sigma}_A + B\,\boldsymbol{\sigma}_B + C\,\boldsymbol{\sigma}_C + \frac{A+D}{2}\boldsymbol{\sigma}_0$$

...current-carrier...

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex-Coriolis-cyclotron-curly...)*

Color scheme based on traffic signals

STOP (standing waves)

CAUTION (stretched waves)

GO (moving waves)

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Decompose the Hamiltonian operator **H** into four *ABCD* *symmetry operators*
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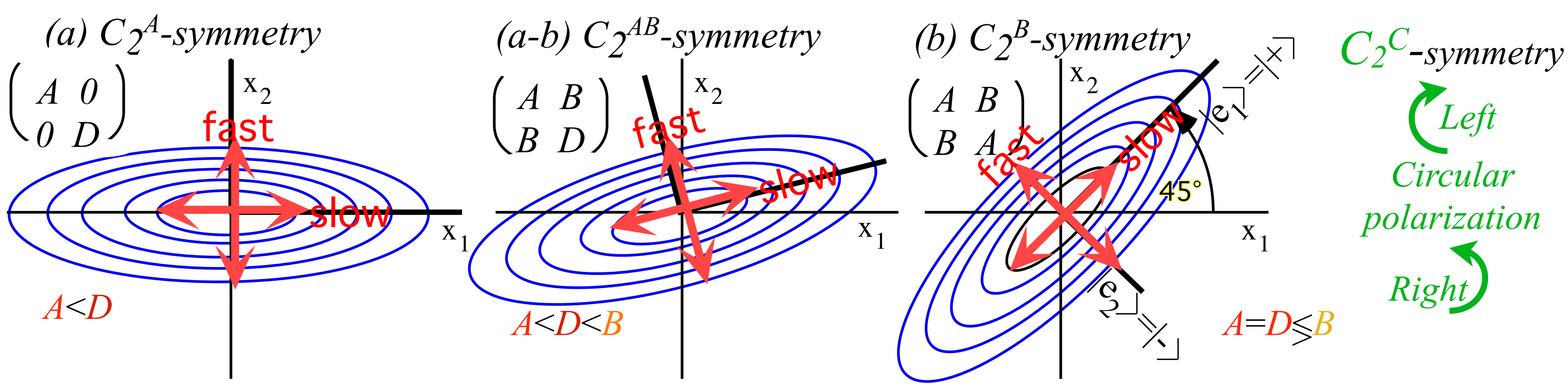
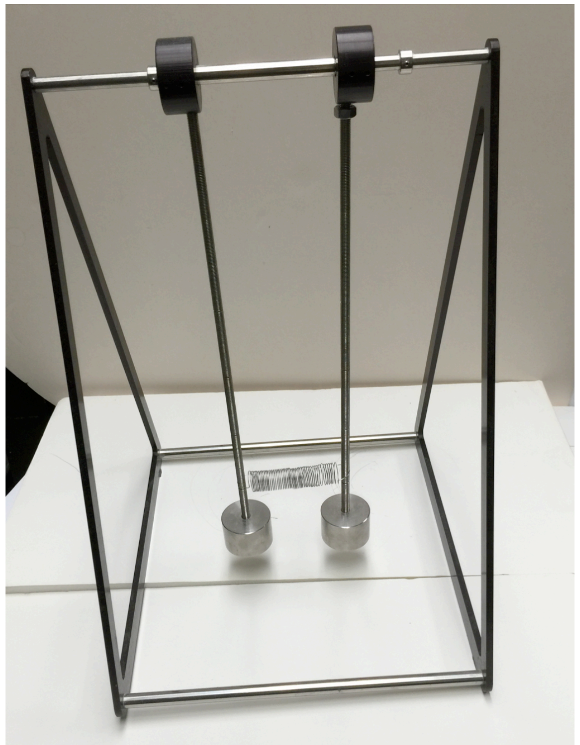


Fig. 3.4.1 Potentials for (a) C_2^A-asymmetric-diagonal, (ab) C_2^{AB}-mixed , (b) C_2^B-bilateral U(2)system.



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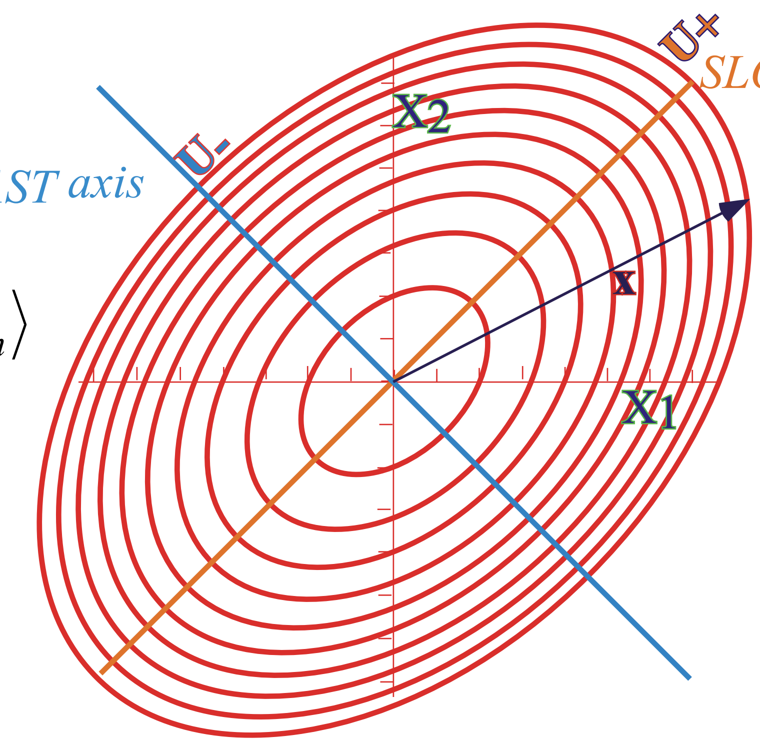
- *Functional spectral decomposition*
- *Orthonormality vs. Completeness vis-a`-vis Operator vs. State*
- *Lagrange functional interpolation formula*
  - *Diagonalizing Transformations (D-Ttran) from projectors*

2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical V-contours

$$V = \frac{1}{2}(k_1+k_{12})x_1^2 - k_{12}x_1x_2 + \frac{1}{2}(k_2+k_{12})x_2^2 = \frac{1}{2}\langle \mathbf{x}|\mathbf{K}|\mathbf{x}\rangle = \frac{1}{2}\mathbf{x}\bullet\mathbf{K}\bullet\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(a) PE Contours



What direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ is the *same* as $\mathbf{K}|\mathbf{x}\rangle$??

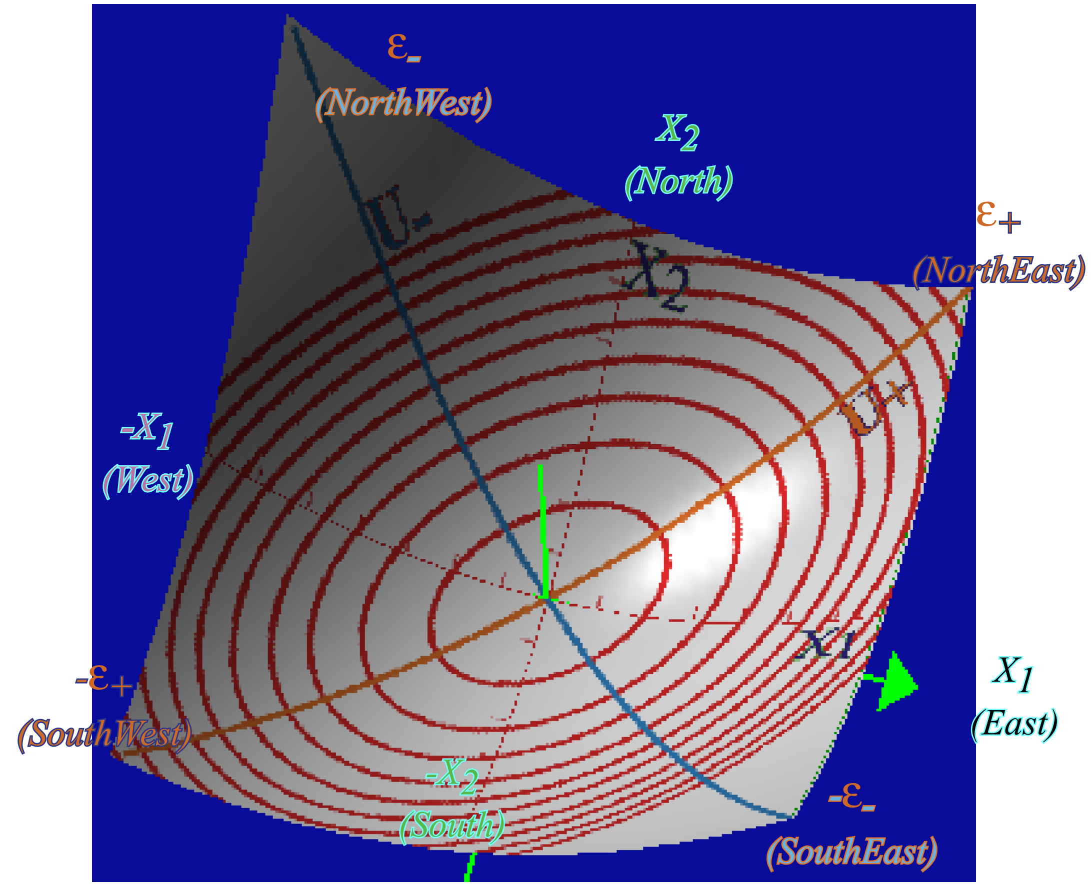


Fig. 3.3.4 Plot of potential function $V(x_1,x_2)$ showing elliptical $V(x_1,x_2)$=const. level curves.

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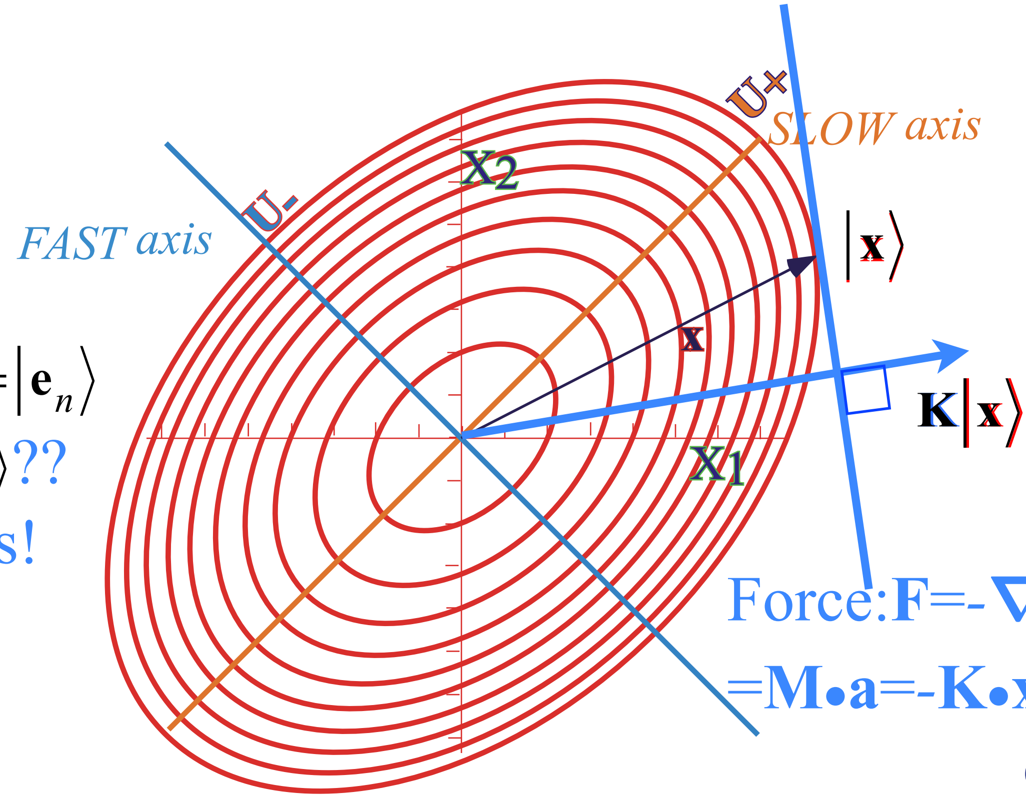
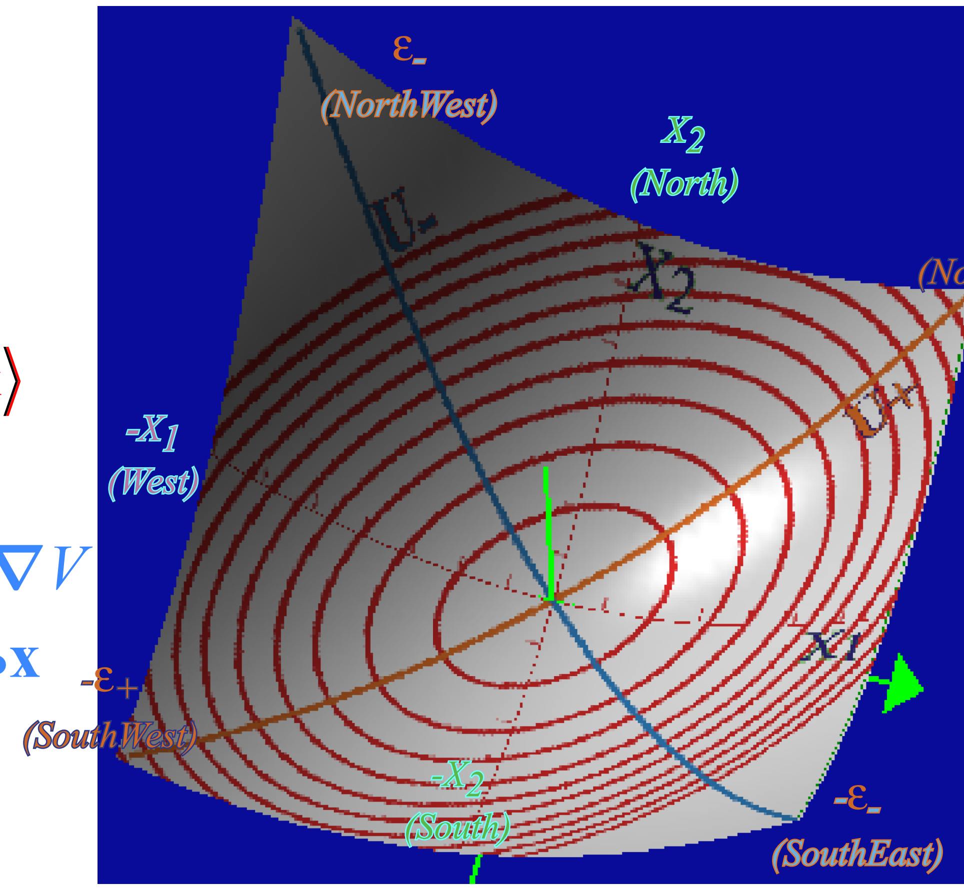


What direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ is the *same* as $\mathbf{K}|\mathbf{x}\rangle$?? Not <u>most</u> directions!

Force: $\mathbf{F}$=-∇V =$\mathbf{M}\bullet\mathbf{a}$=-$\mathbf{K}\bullet\mathbf{x}$



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$$\mathbf{F} = -\nabla V = -\frac{\partial V}{\partial \mathbf{x}}$$

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} -\frac{\partial V}{\partial x_1} \\ -\frac{\partial V}{\partial x_2} \end{pmatrix}$$

(details of gradient expression)

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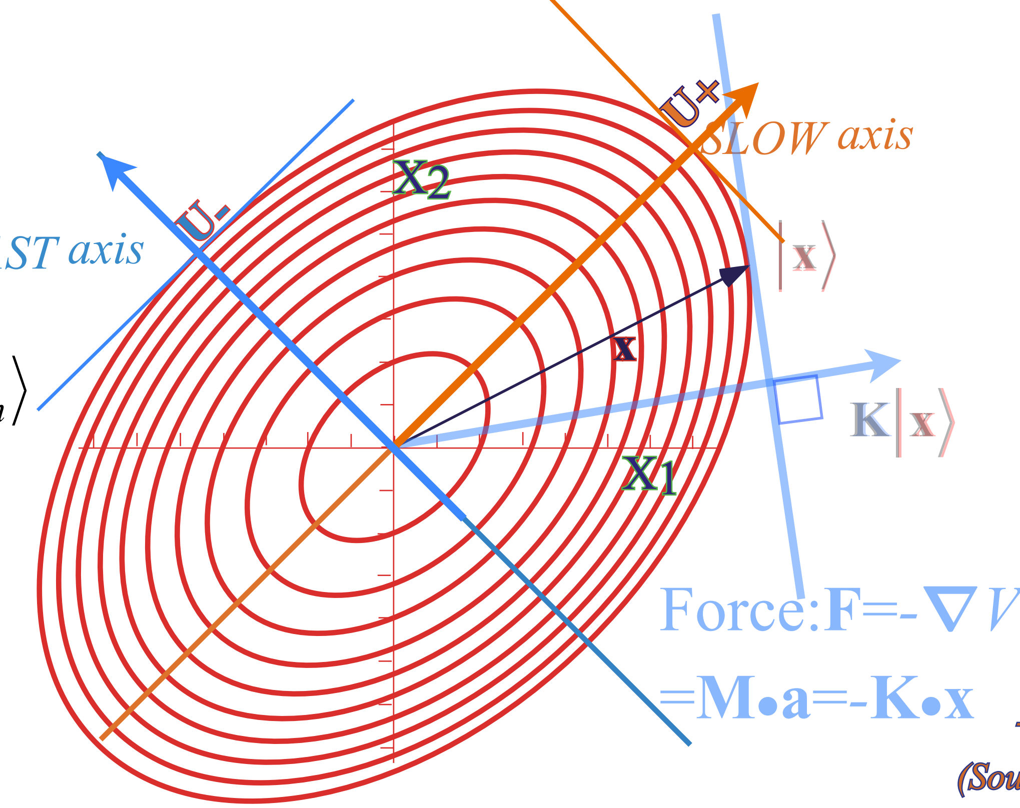

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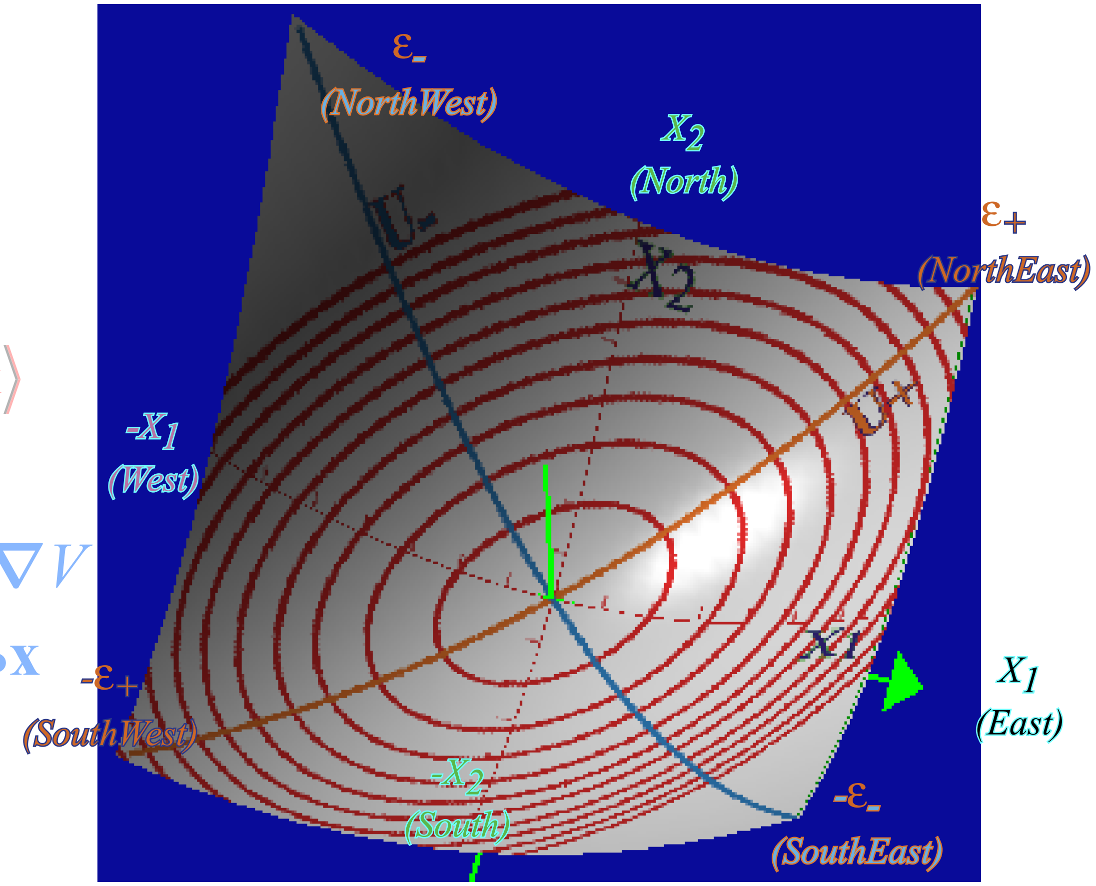
Not most directions!
Only extremal axes work. (major or minor axes)



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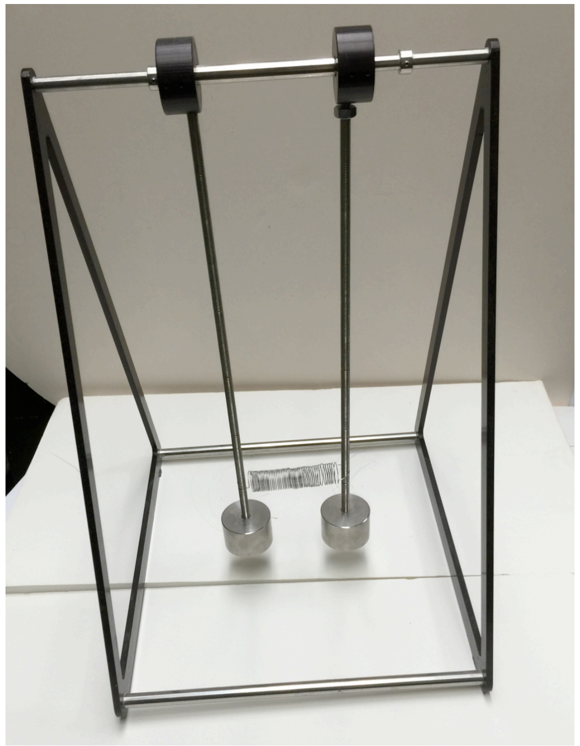

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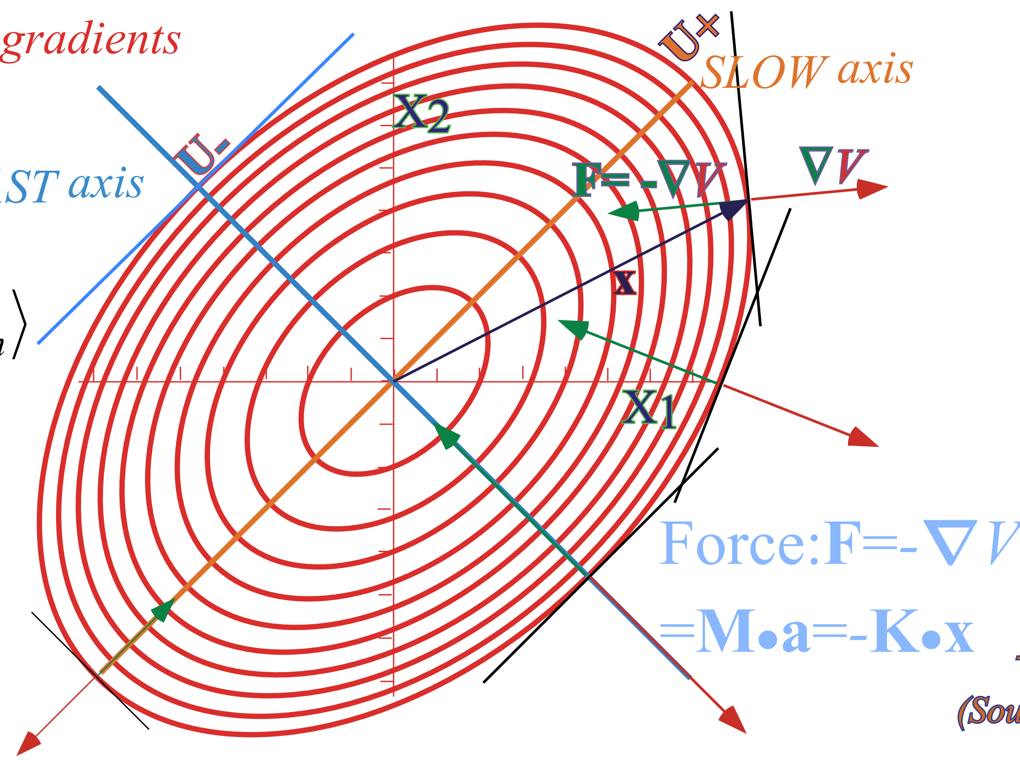
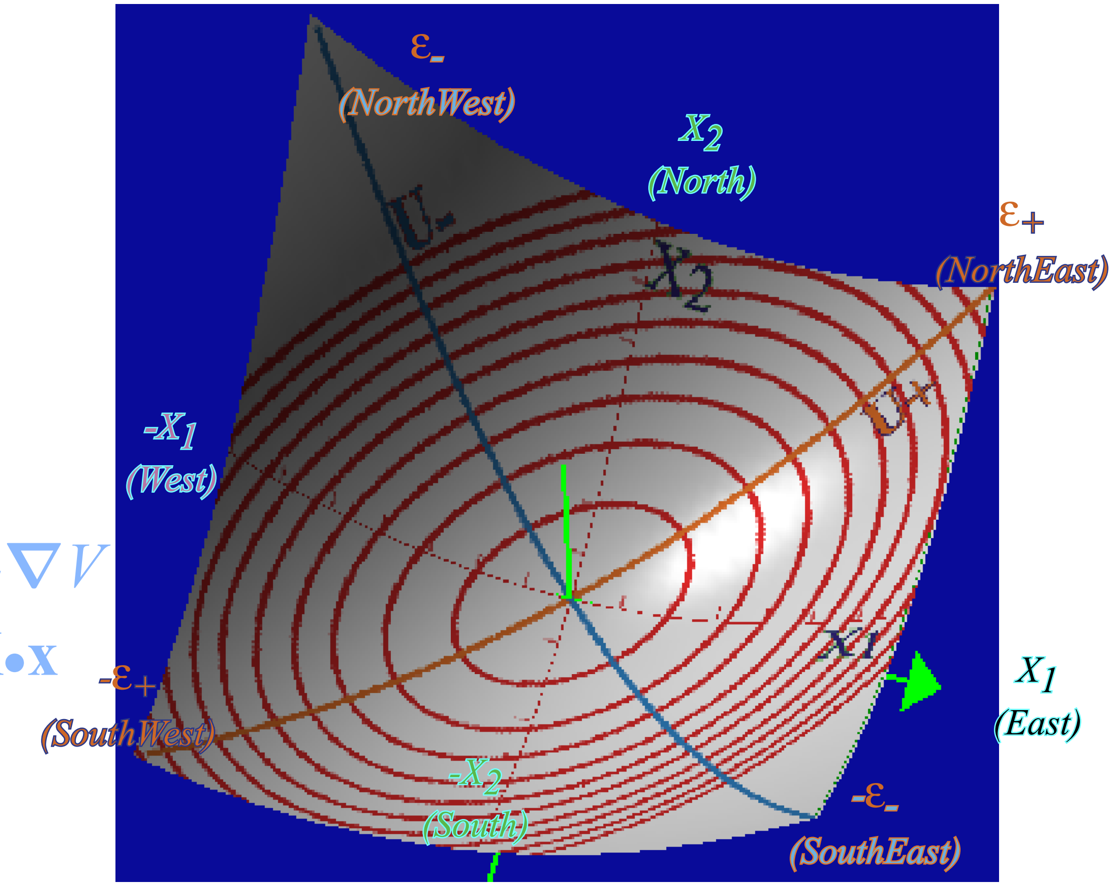
(a) PE Contours and gradients



What direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ is the *same* as $\mathbf{K}|\mathbf{x}\rangle$?? Not <u>most</u> directions! Only extremal axes work. (major or minor axes)



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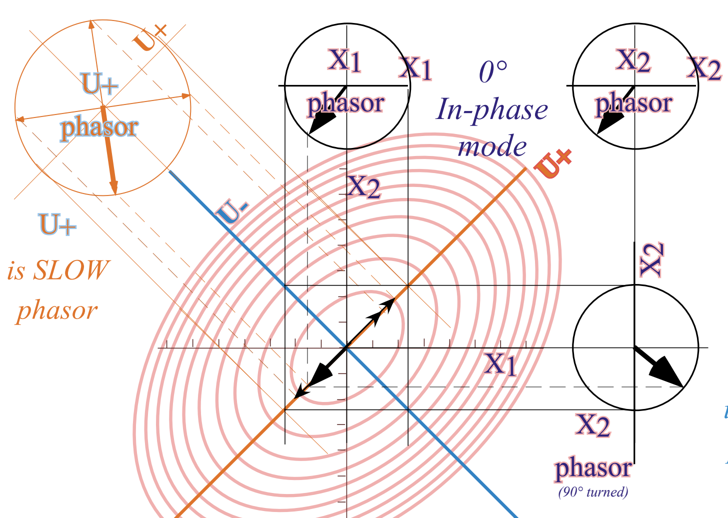
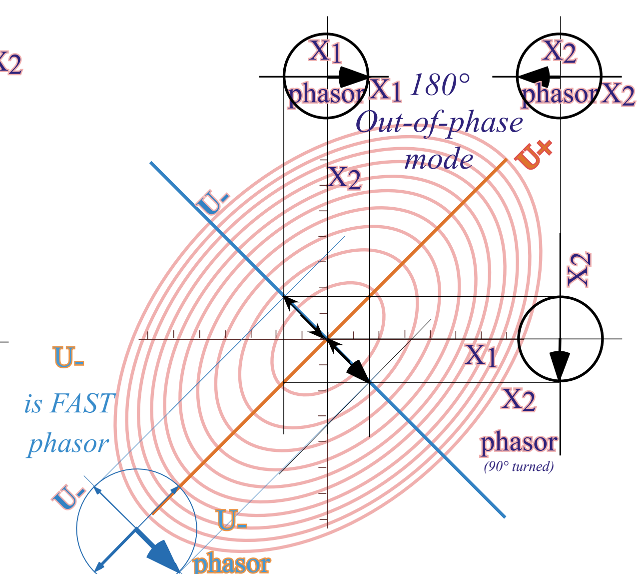
(b) Symmetric U+ Coordinate SLOW Mode



(c) Anti-symmetric U- Coordinate FAST Mode



2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical V-contours (Here: $k_1 = k = k_2$)

$$V = \frac{1}{2}(k+k_{12})x_1^2 - k_{12}x_1x_2 + \frac{1}{2}(k+k_{12})x_2^2 = \frac{1}{2}\langle\mathbf{x}|\mathbf{K}|\mathbf{x}\rangle = \frac{1}{2}\mathbf{x}\bullet\mathbf{K}\bullet\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} k+k_{12} & -k_{12} \\ -k_{12} & k+k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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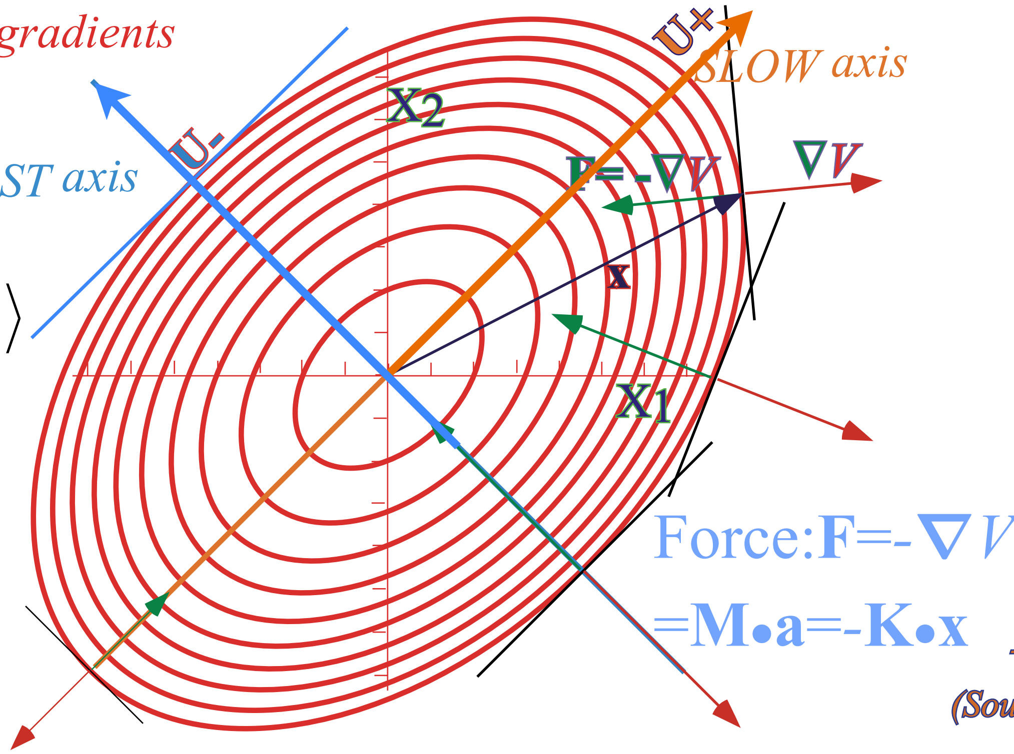

What direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ is the *same* as $\mathbf{K}|\mathbf{x}\rangle$??
Not most directions!
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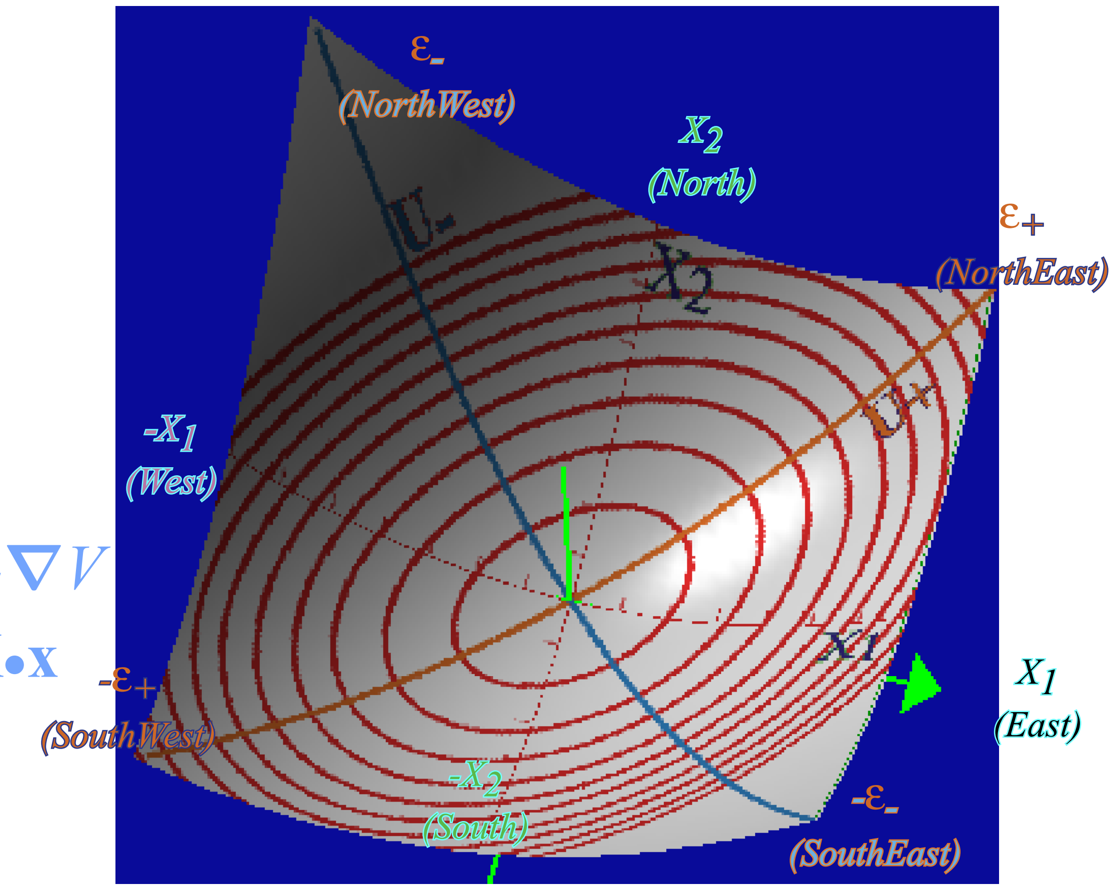

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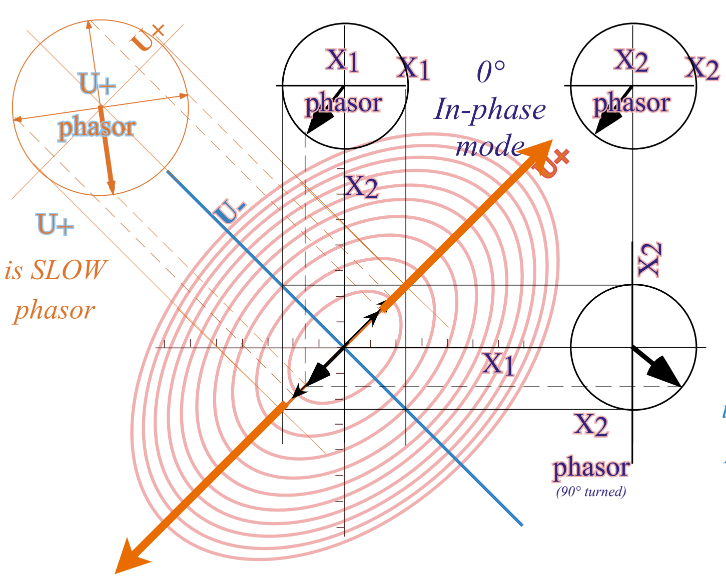
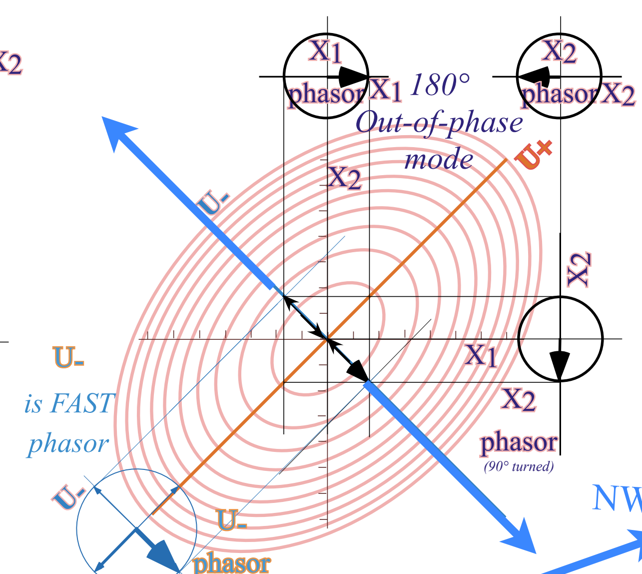
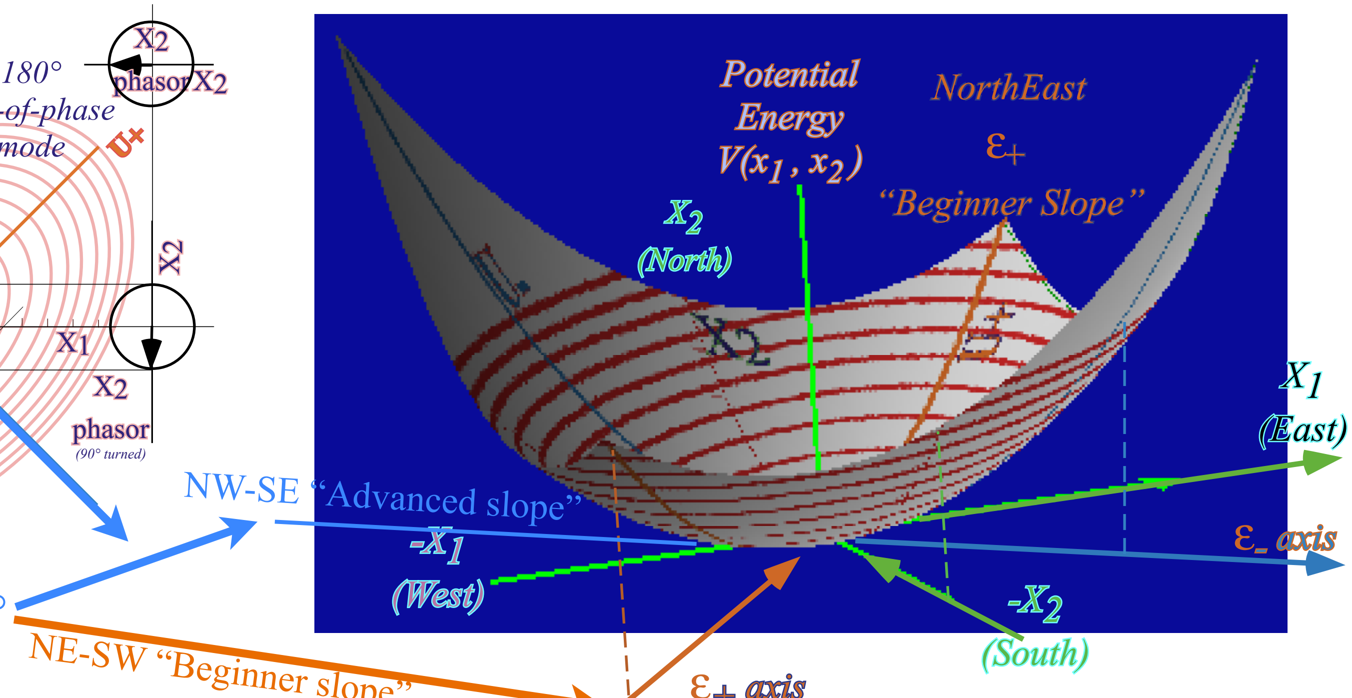


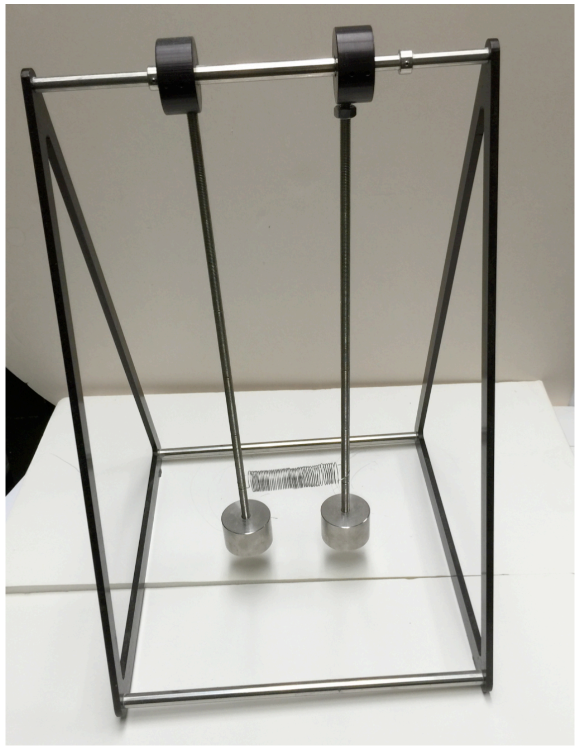
(c) Anti-symmetric U- Coordinate FAST Mode





With Bilateral symmetry ($k_1 = k = k_2$) the extremal axes lie at ±45°

Fig. 3.3.5 Topography lines of potential function $V(x_1,x_2)$ and orthogonal ε_+ and ε_- normal mode slopes



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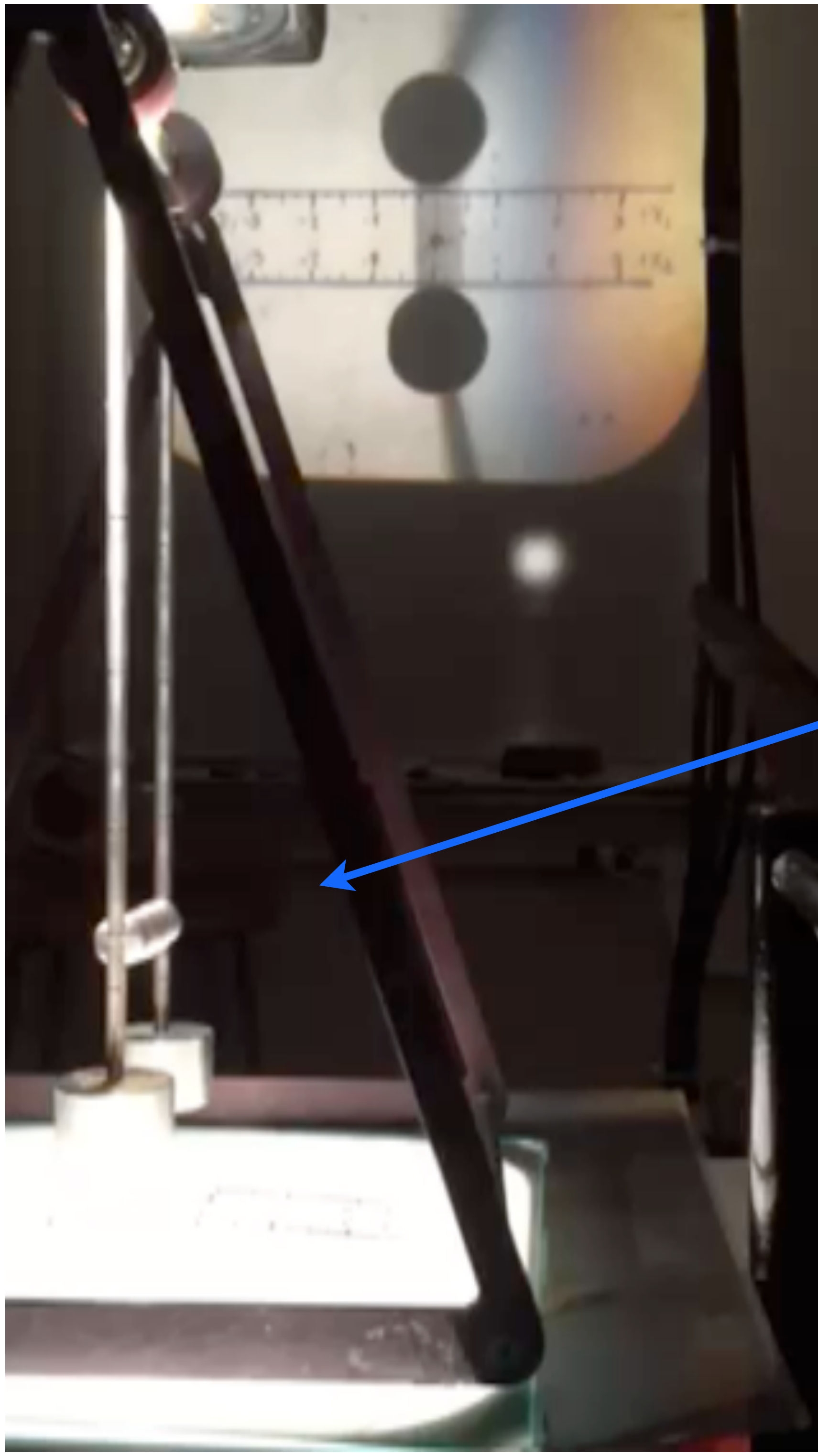
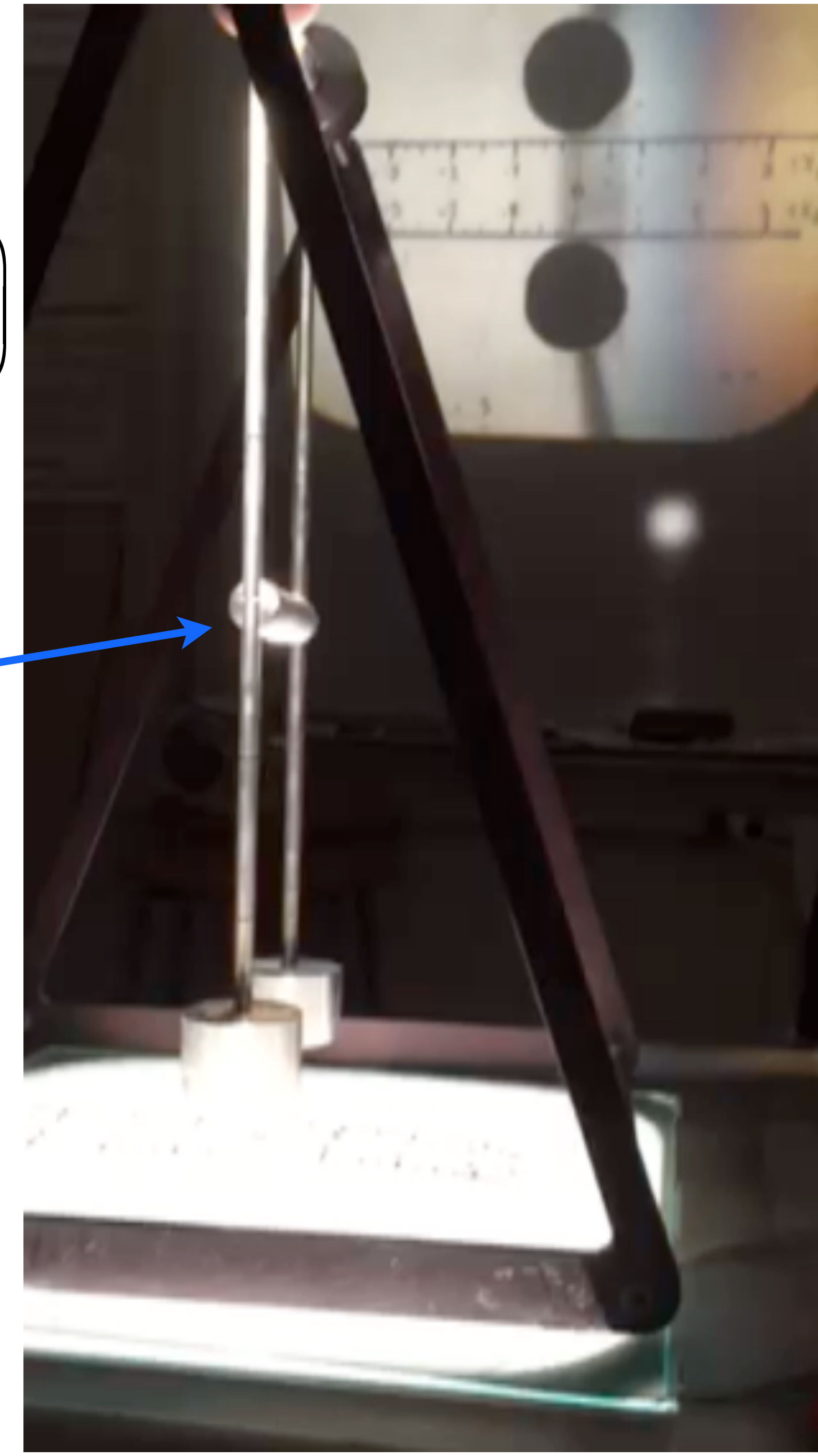


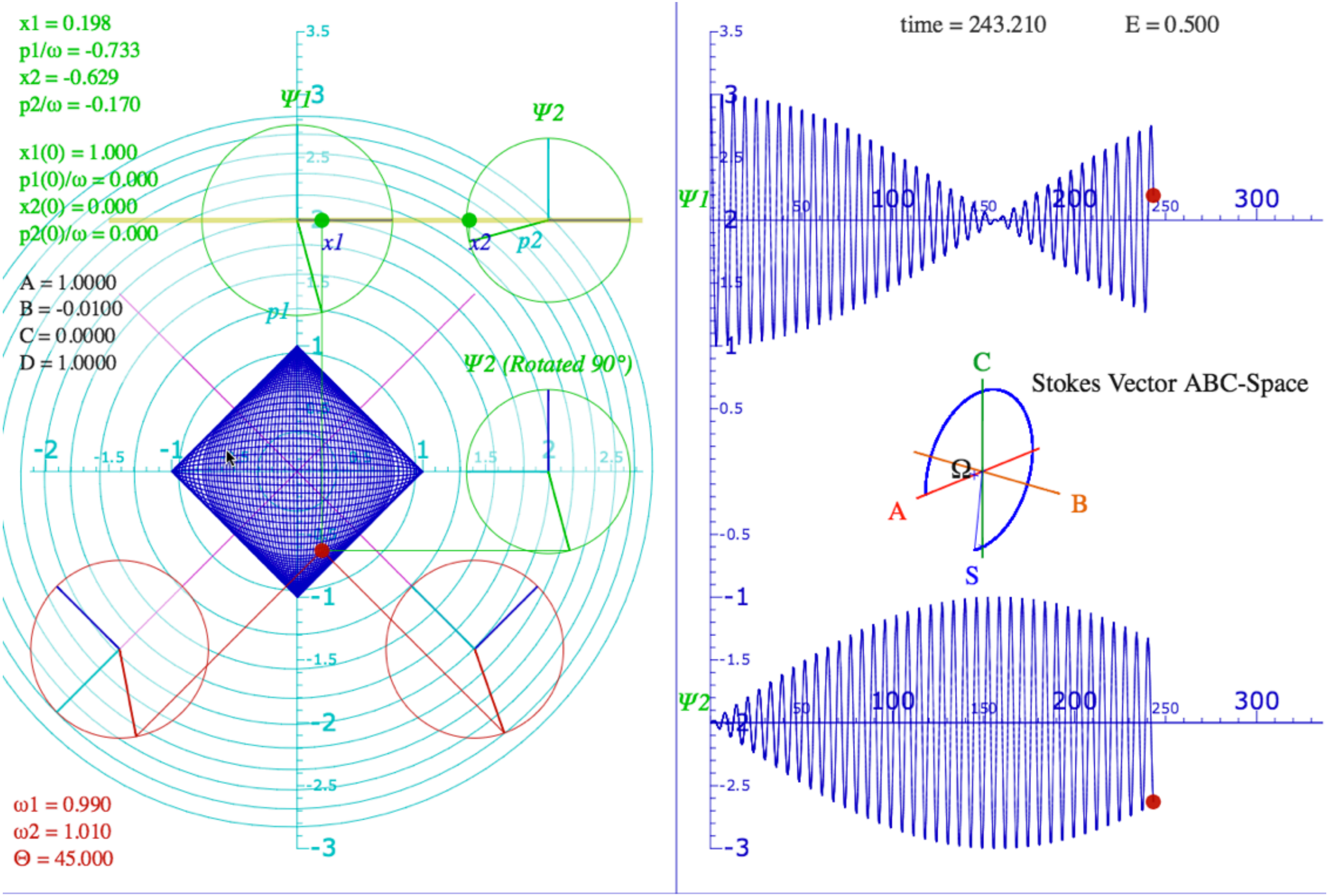
B-Type coupling

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$$

Spring set
down
for
STRONGER
coupling
(higher |B|)

Spring set
up
for
weaker
coupling
(lower |B|)





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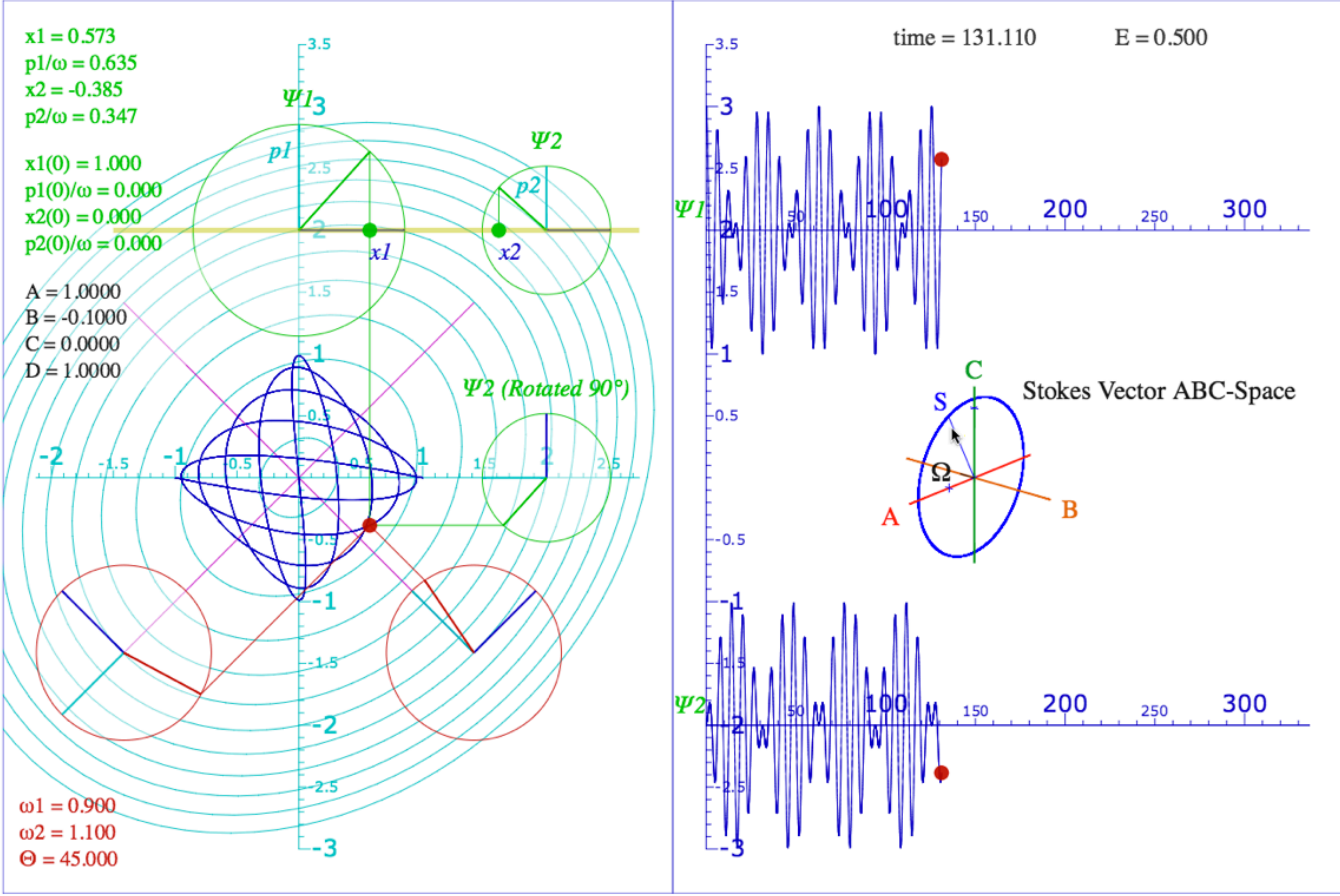

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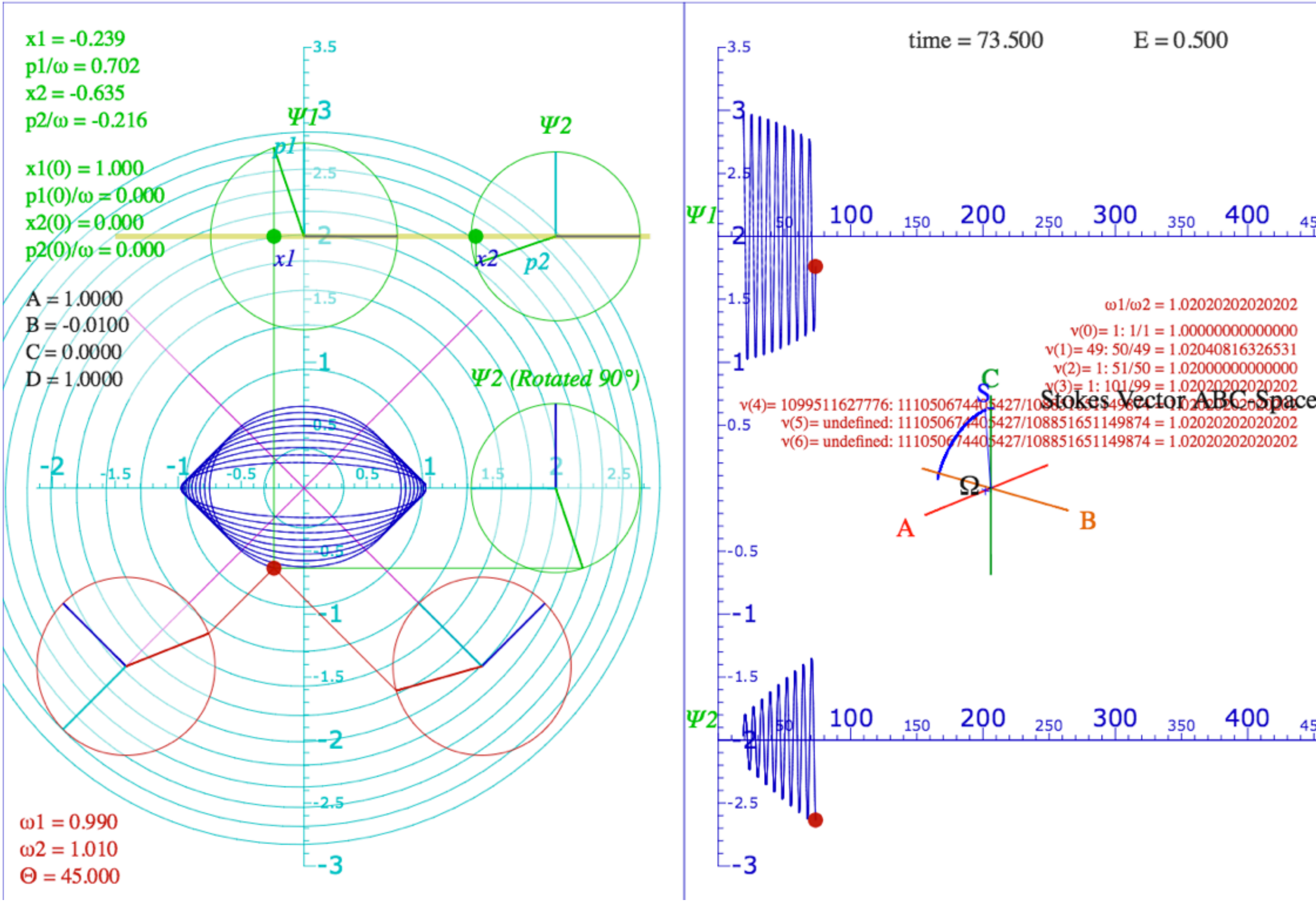

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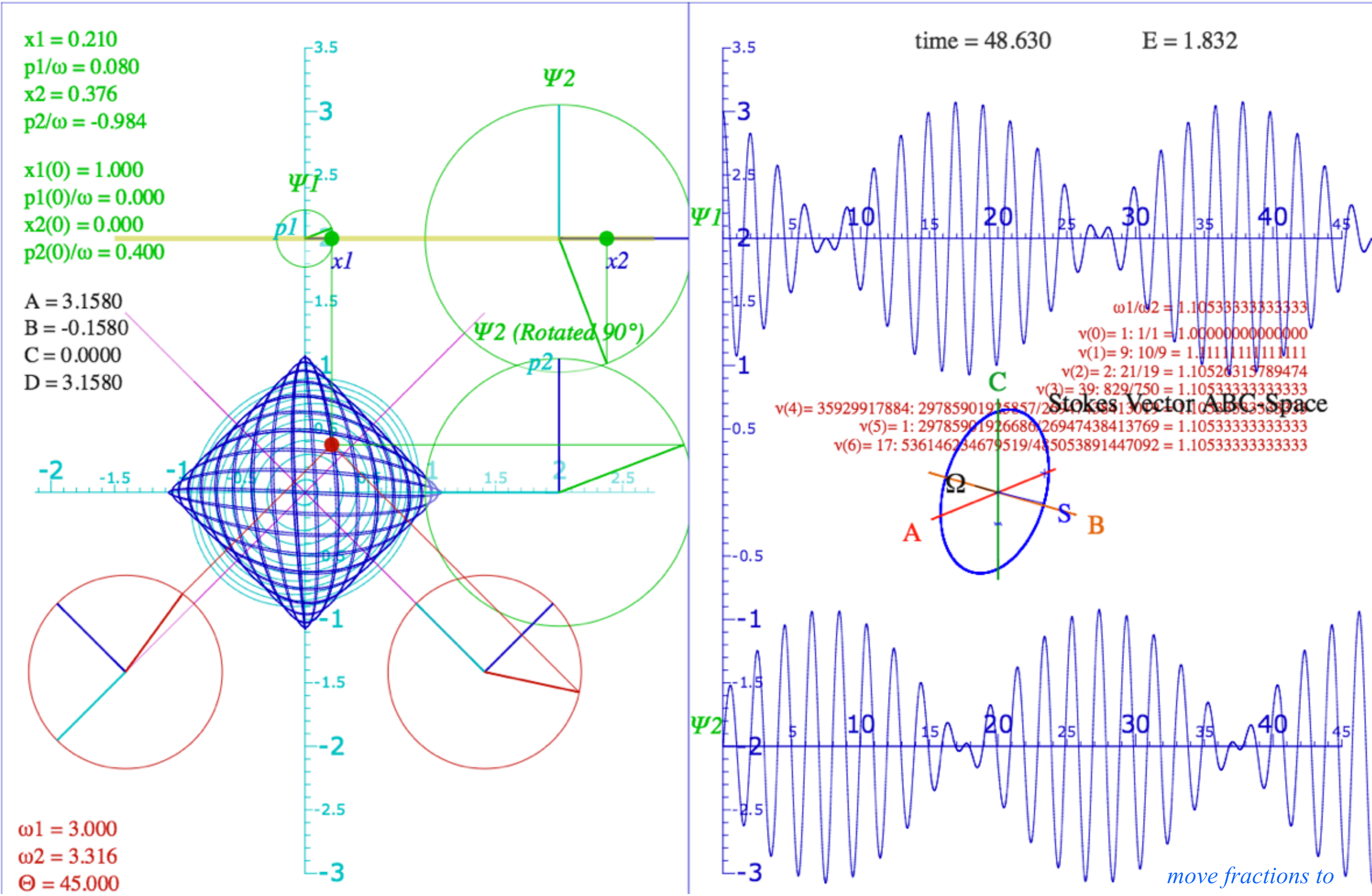
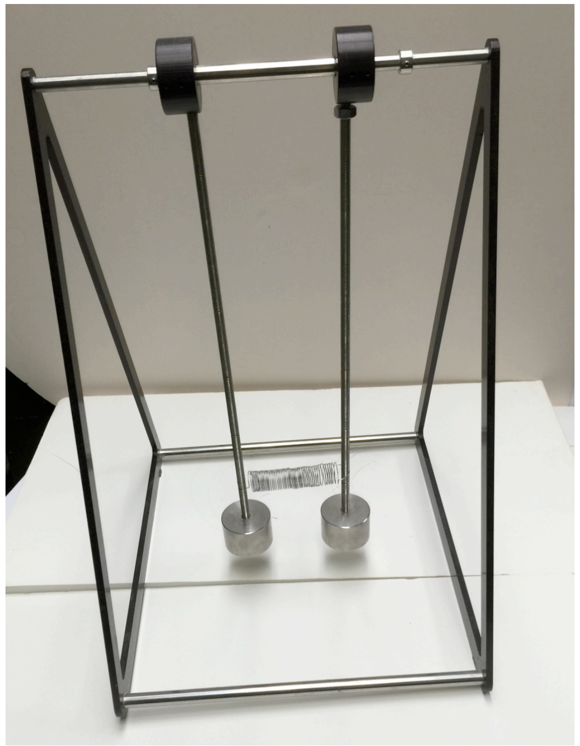
p1/ω = 0.702
x2 = -0.635
p2/ω = -0.216
x1(0) = 1.000
p1(0)/ω = 0.000
x2(0) = 0.000
p2(0)/ω = 0.000
A = 1.0000
B = -0.0100
C = 0.0000
D = 1.0000
ω1 = 0.990
ω2 = 1.010
Θ = 45.000
Ψ1
Ψ2
p1
x1
x2
p2
Ψ2 (Rotated 90°)
time = 73.500
E = 0.500
ω1/ω2 = 1.02020202020202
ν(0)= 1: 1/1 = 1.00000000000000
ν(1)= 49: 50/49 = 1.02040816326531
ν(2)= 1: 51/50 = 1.02000000000000
ν(3)= 1: 101/99 = 1.02020202020202
ν(4)= 1099511627776: 111050674405427/108851651149874 = 1.02020202020202
ν(5)= undefined: 111050674405427/108851651149874 = 1.02020202020202
ν(6)= undefined: 111050674405427/108851651149874 = 1.02020202020202
Stokes Vector ABC-Space
C
S
Ω
A
B



move fractions to bottom of slide

[BoxIt (Beating) Simulation]



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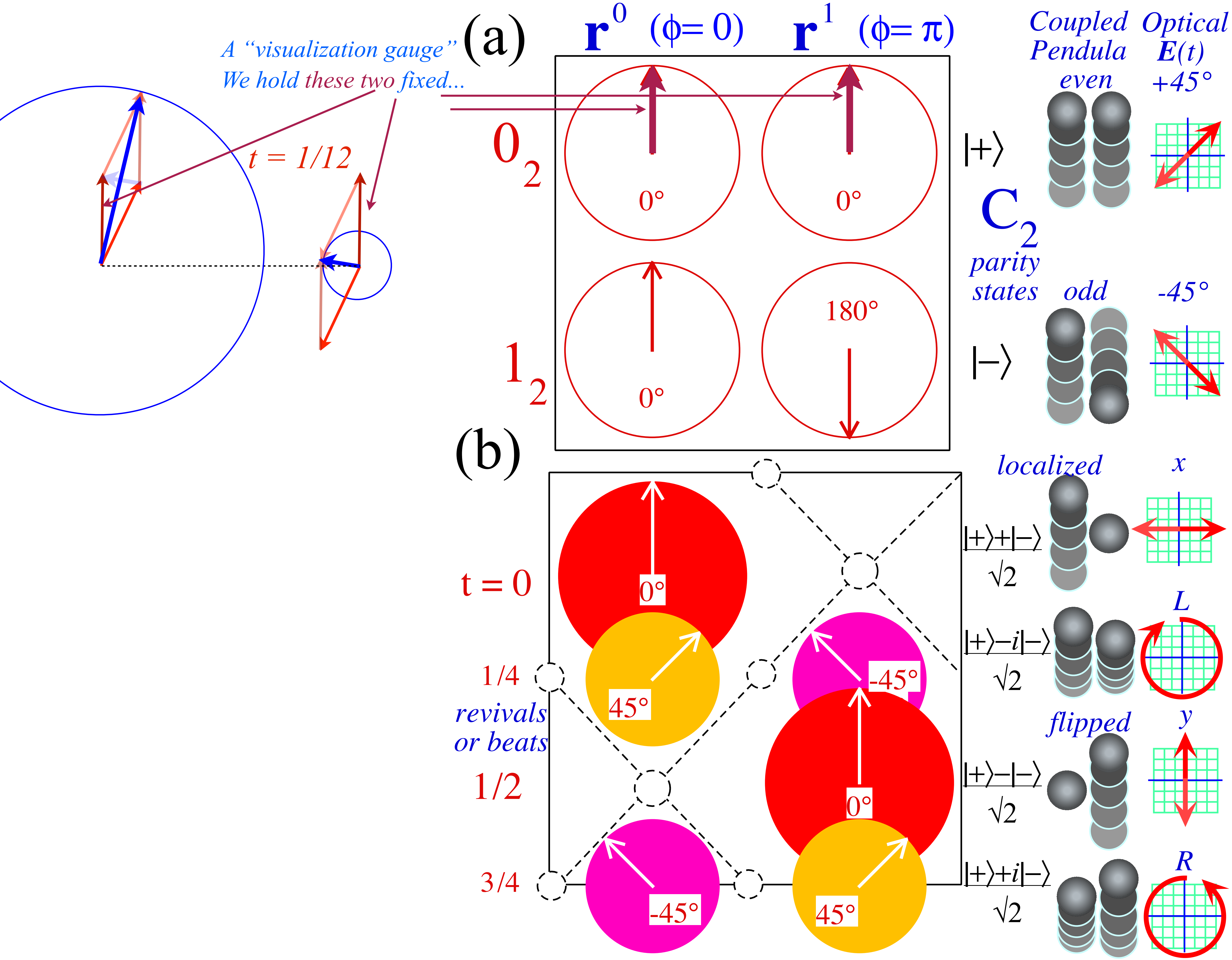

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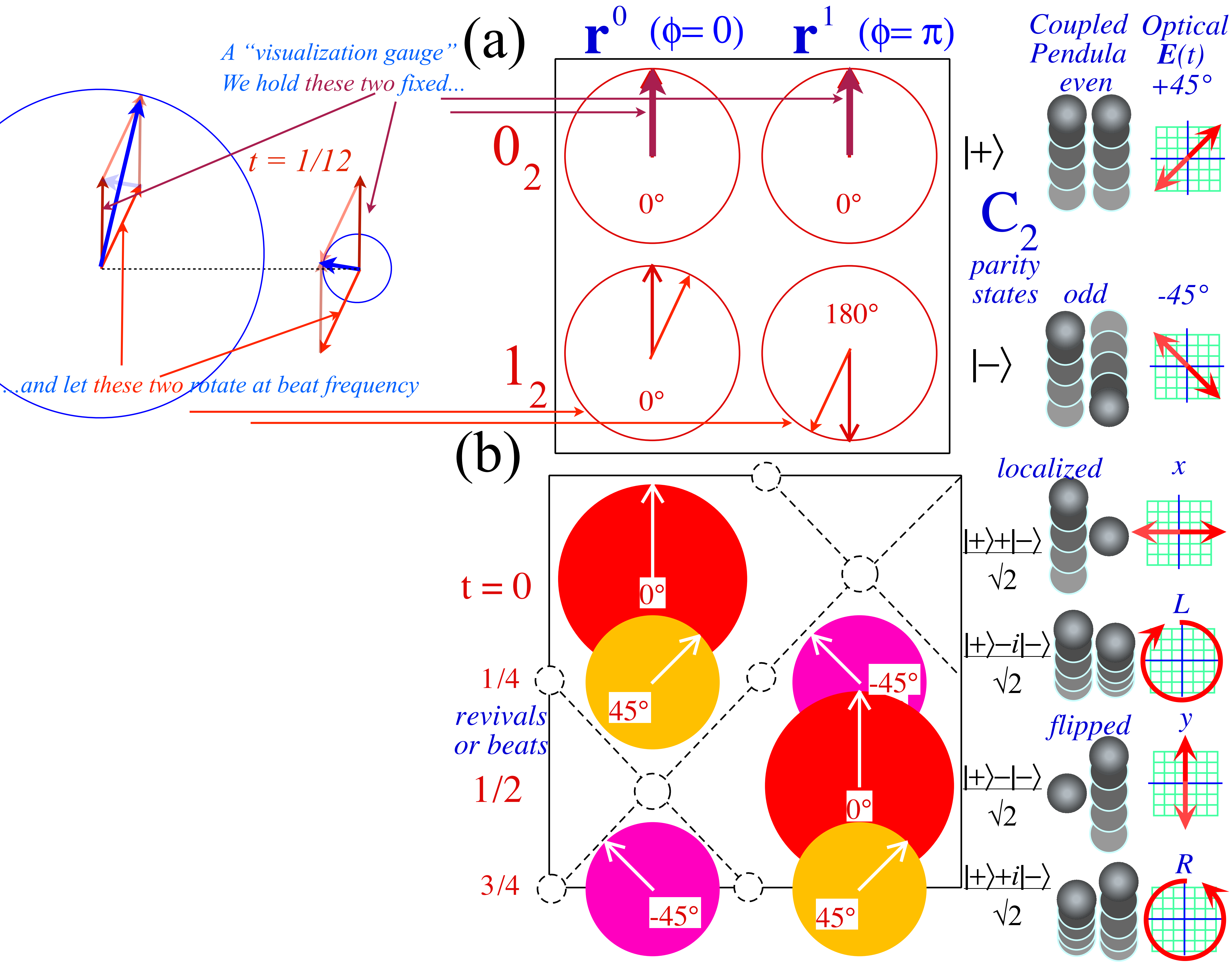

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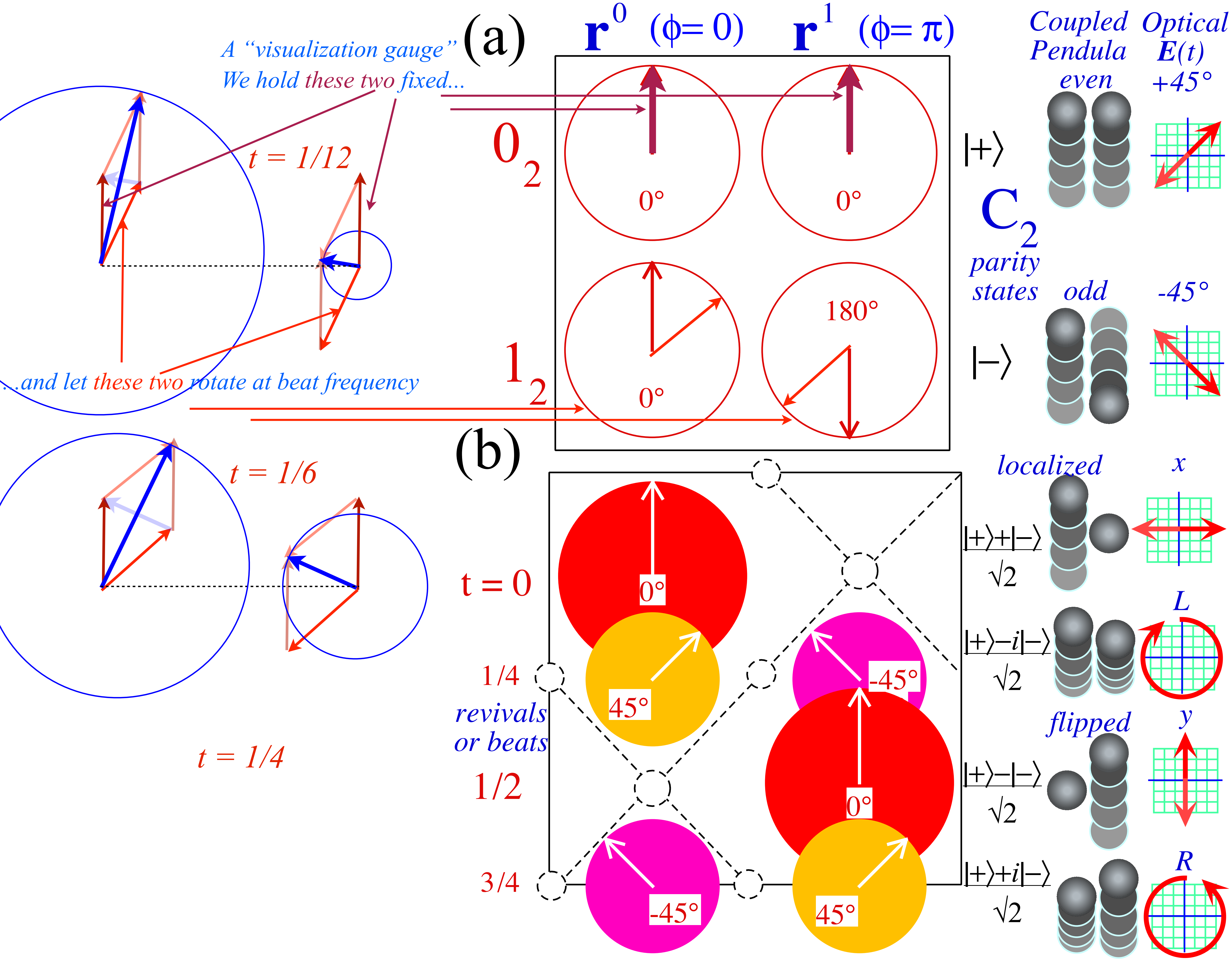

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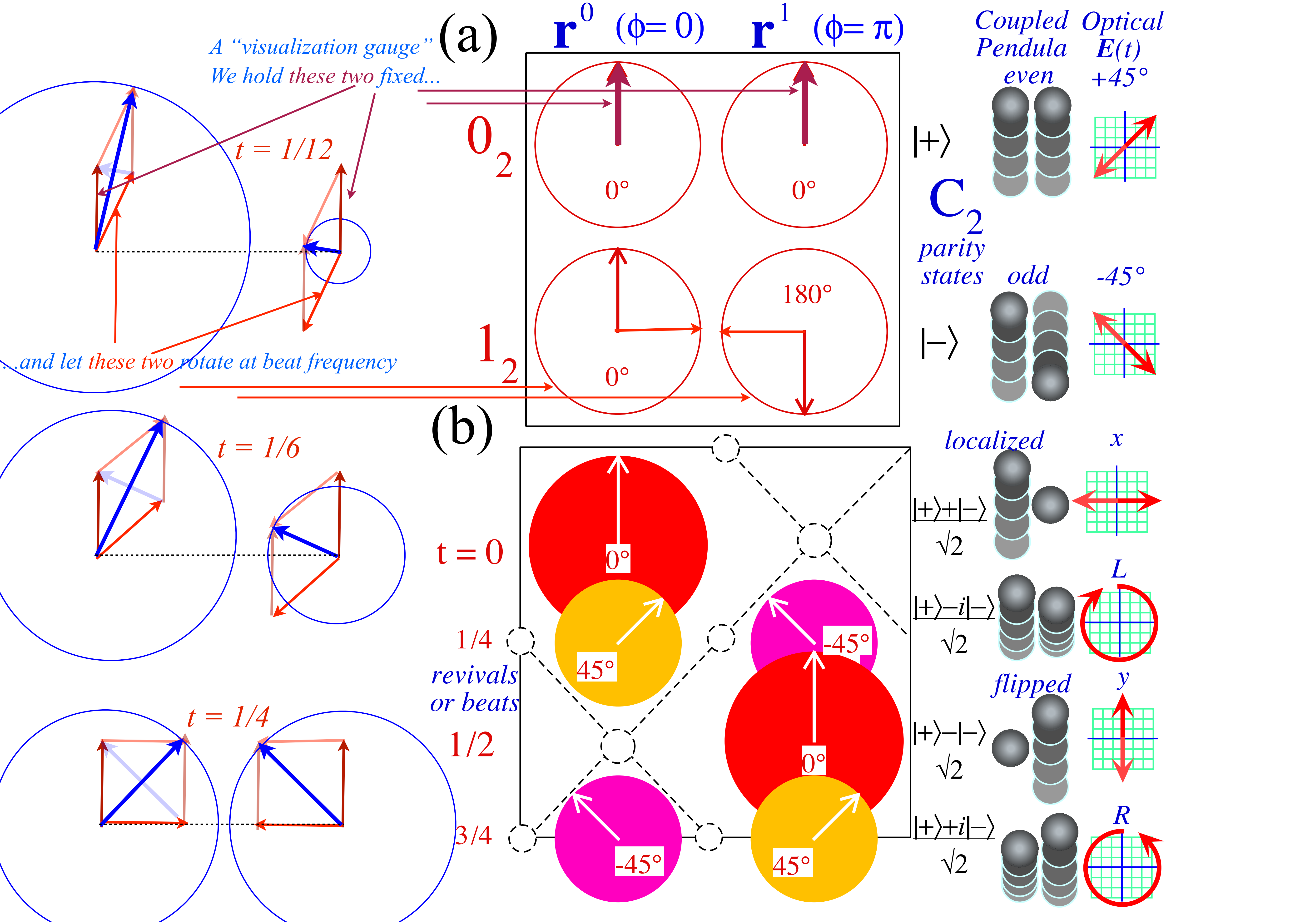

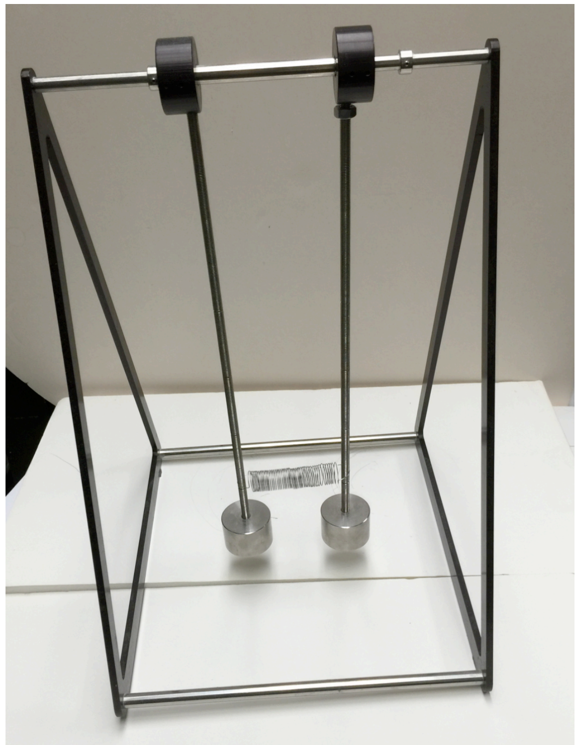

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➜ *Eigensolutions by matrix-algebra with example* $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$
Secular equation
Hamilton-Cayley equation and projectors
Idempotent projectors (how eigenvalues⇒eigenvectors)
Operator orthonormality and Completeness (Idempotent means: **P·P=P***)*
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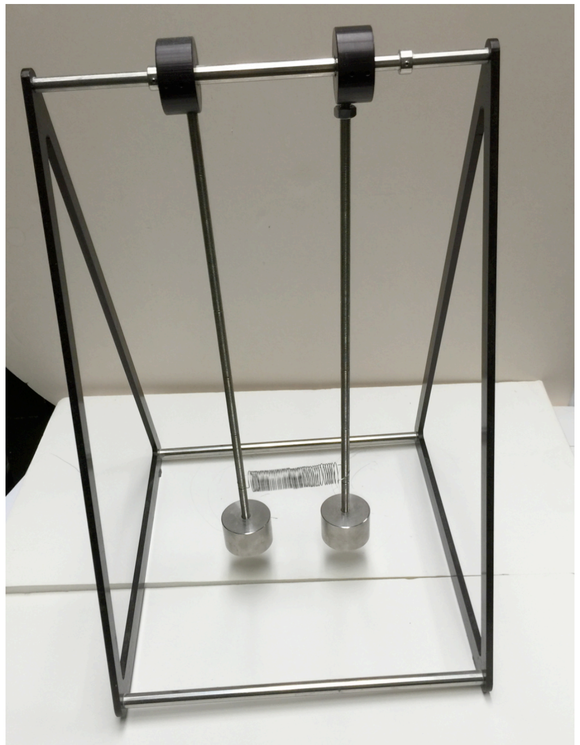

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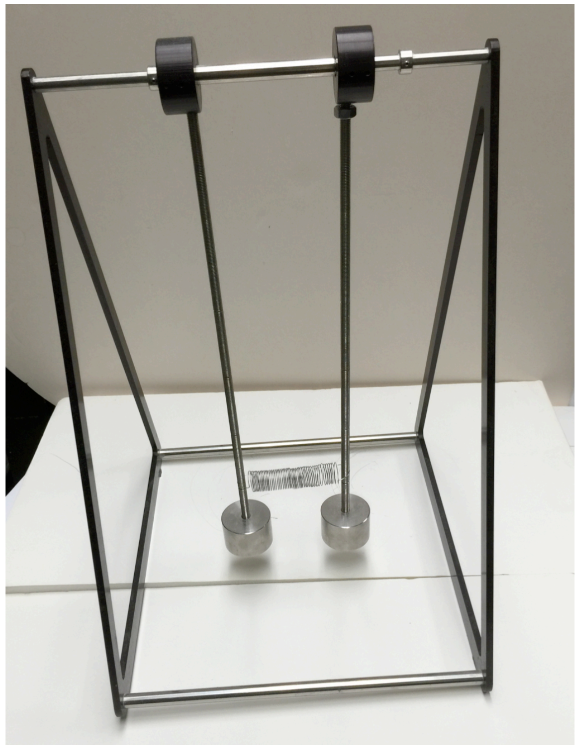

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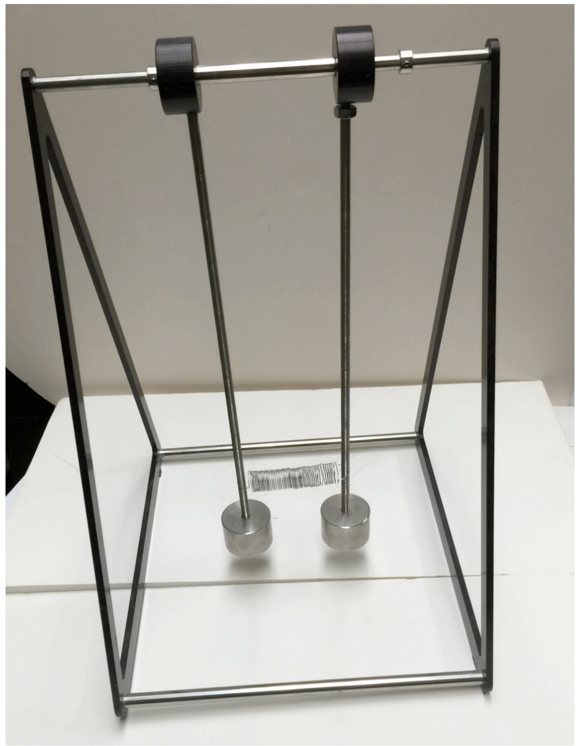

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$$\mathbf{p}_2=(\mathbf{M}-\varepsilon_1\mathbf{1})(\quad\mathbf{1}\quad)\cdots(\mathbf{M}-\varepsilon_n\mathbf{1})$$
$$\vdots$$
$$\mathbf{p}_n=(\mathbf{M}-\varepsilon_1\mathbf{1})(\mathbf{M}-\varepsilon_2\mathbf{1})\cdots(\quad\mathbf{1}\quad)$$

(Assume distinct e-values here: *Non-degeneracy clause*)
$\varepsilon_j\neq\varepsilon_k\neq\ldots$

Each $\mathbf{p}_k$ contains *eigen-bra-kets* since: $(\mathbf{M}-\varepsilon_k\mathbf{1})\mathbf{p}_k=0$ or: $\mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$.

$$\mathbf{M}|\varepsilon\rangle=\begin{pmatrix}4&1\\3&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\varepsilon\begin{pmatrix}x\\y\end{pmatrix}\ \text{or: }\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

Trying to solve by Kramer's inversion:

$$x=\frac{\det\begin{vmatrix}\begin{pmatrix}0&1\\0&2-\varepsilon\end{pmatrix}\end{vmatrix}}{\det\begin{vmatrix}\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\end{vmatrix}}\quad\text{and}\quad y=\frac{\det\begin{vmatrix}\begin{pmatrix}4-\varepsilon&0\\3&0\end{pmatrix}\end{vmatrix}}{\det\begin{vmatrix}\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\end{vmatrix}}$$

Only possible non-zero $\{x,y\}$ if denominator is zero, too!

$$0=\det|\mathbf{M}-\varepsilon\cdot\mathbf{1}|=\det\begin{vmatrix}\begin{pmatrix}4&1\\3&2\end{pmatrix}-\varepsilon\begin{pmatrix}1&0\\0&1\end{pmatrix}\end{vmatrix}=\det\begin{vmatrix}\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\end{vmatrix}$$

$$0=(4-\varepsilon)(2-\varepsilon)-1\cdot3=8-6\varepsilon+\varepsilon^2-1\cdot3=\varepsilon^2-6\varepsilon+5$$
$$0=\varepsilon^2-Trace(\mathbf{M})\varepsilon+\det(\mathbf{M})=\varepsilon^2-6\varepsilon+5$$

$$0=(\varepsilon-1)(\varepsilon-5)\ \text{ so let: }\ \varepsilon_1=1\ \text{ and: }\ \varepsilon_2=5$$

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$$\mathbf{p}_1=(\mathbf{1})(\mathbf{M}-5\cdot\mathbf{1})=\begin{pmatrix}4-5&1\\3&2-5\end{pmatrix}=\begin{pmatrix}-1&1\\3&-3\end{pmatrix}$$

$$\mathbf{p}_2=(\mathbf{M}-1\cdot\mathbf{1})(\mathbf{1})=\begin{pmatrix}4-1&1\\3&2-1\end{pmatrix}=\begin{pmatrix}3&1\\3&1\end{pmatrix}$$

$$\mathbf{M}\mathbf{p}_1=\begin{pmatrix}4&1\\3&2\end{pmatrix}\cdot\begin{pmatrix}-1&1\\3&-3\end{pmatrix}=1\cdot\begin{pmatrix}-1&1\\3&-3\end{pmatrix}=1\cdot\mathbf{p}_1$$

$$\mathbf{M}\mathbf{p}_2=\begin{pmatrix}4&1\\3&2\end{pmatrix}\cdot\begin{pmatrix}3&1\\3&1\end{pmatrix}=5\cdot\begin{pmatrix}3&1\\3&1\end{pmatrix}=5\cdot\mathbf{p}_2$$

Matrix-algebraic method for finding eigenvector and eigenvalues *With example matrix* $\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$

An *eigenvector* $|\varepsilon_k\rangle$ of **M** is in a direction that is left unchanged by **M**.

$$\mathbf{M}|\varepsilon_k\rangle=\varepsilon_k|\varepsilon_k\rangle,\ \text{or: } (\mathbf{M}-\varepsilon_k\mathbf{1})|\varepsilon_k\rangle=\mathbf{0}$$

ε_k is *eigenvalue* associated with eigenvector $|\varepsilon_k\rangle$ direction.

A change of basis to $\{|\varepsilon_1\rangle,|\varepsilon_2\rangle,\cdots|\varepsilon_n\rangle\}$ called *diagonalization* gives

$$\begin{pmatrix}\langle\varepsilon_1|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_1|\mathbf{M}|\varepsilon_n\rangle\\ \langle\varepsilon_2|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_2|\mathbf{M}|\varepsilon_n\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle\varepsilon_n|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_n|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_n|\mathbf{M}|\varepsilon_n\rangle\end{pmatrix}=\begin{pmatrix}\varepsilon_1&0&\cdots&0\\0&\varepsilon_2&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&\varepsilon_n\end{pmatrix}$$

First step in finding eigenvalues: Solve *secular equation*

$$\det|\mathbf{M}-\varepsilon\mathbf{1}|=0=(-1)^n\left(\varepsilon^n+a_1\varepsilon^{n-1}+a_2\varepsilon^{n-2}+\ldots+a_{n-1}\varepsilon\ +a_n\right)$$

where:

$$\boxed{a_1=-Trace\mathbf{M},\cdots,\ a_k=(-1)^k\sum\text{diagonal k-by-k minors of }\mathbf{M},\cdots,\ \ a_n=(-1)^n\det|\mathbf{M}|}$$

Secular equation has n-factors, one for each eigenvalue.

$$\det|\mathbf{M}-\varepsilon\mathbf{1}|=0=(-1)^n(\varepsilon-\varepsilon_1)(\varepsilon-\varepsilon_2)\cdots(\varepsilon-\varepsilon_n)$$

Each ε replaced by **M** and each ε_k by ε_k **1** gives *Hamilton-Cayley* matrix equation.

$$\mathbf{0}=(\mathbf{M}-\varepsilon_1\mathbf{1})(\mathbf{M}-\varepsilon_2\mathbf{1})\cdots(\mathbf{M}-\varepsilon_n\mathbf{1})$$

Obviously true if **M** has diagonal form. (But, that's circular logic. Faith needed!)

Replace j^{th} HC-factor by (**1**) to make *projection operators* $\mathbf{p}_k=\prod_{j\neq k}(\mathbf{M}-\varepsilon_j\mathbf{1})$

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Notice $\mathbf{p}_k$ *commutes with* **M**,..
since $\mathbf{M}^1$, $\mathbf{M}^2$,..*commute with* **M**.

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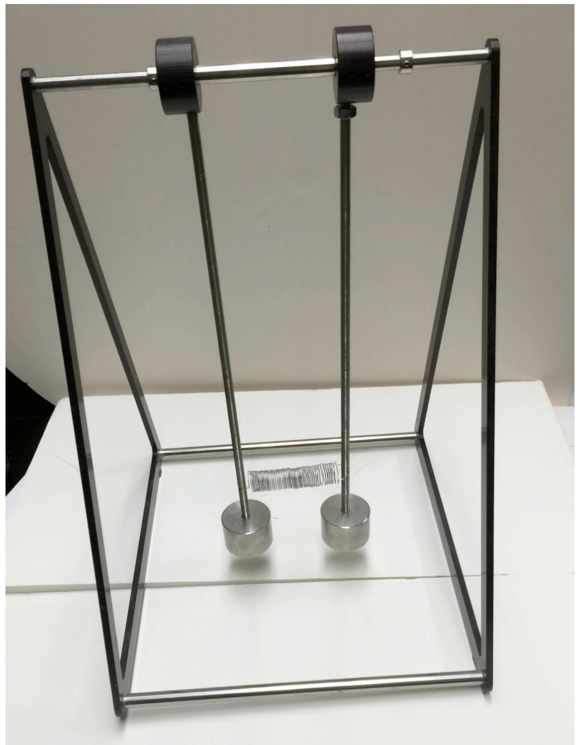

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$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}) \qquad \mathbf{M}\mathbf{p}_k = \varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$$

Multiplication properties of $\mathbf{p}_j$:

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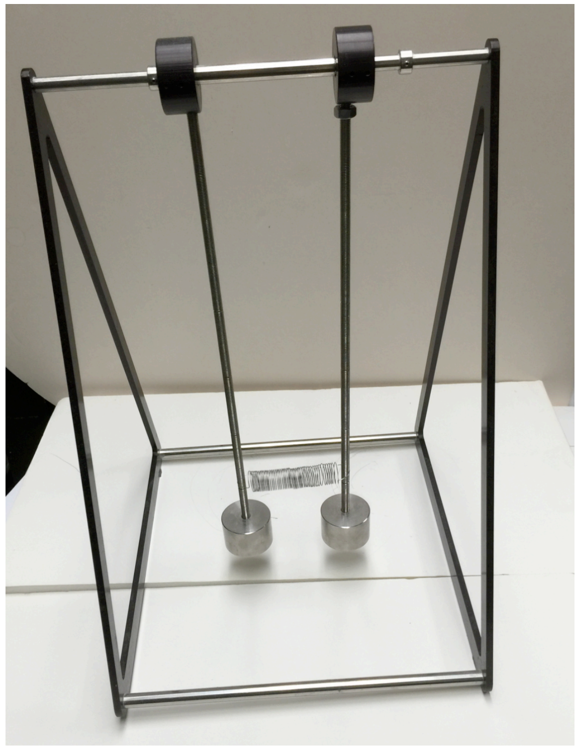

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"Gauge" scale factors that only affect plots

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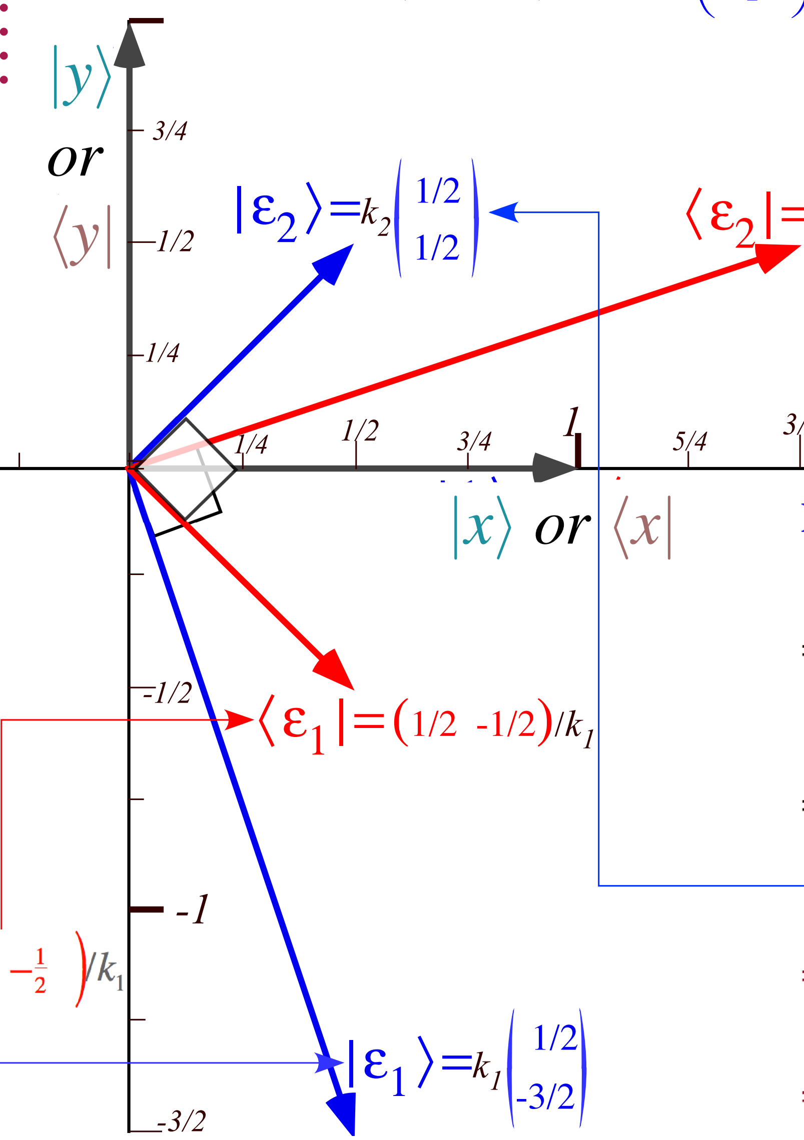
Eigen-bra-ket projectors of matrix: $\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$



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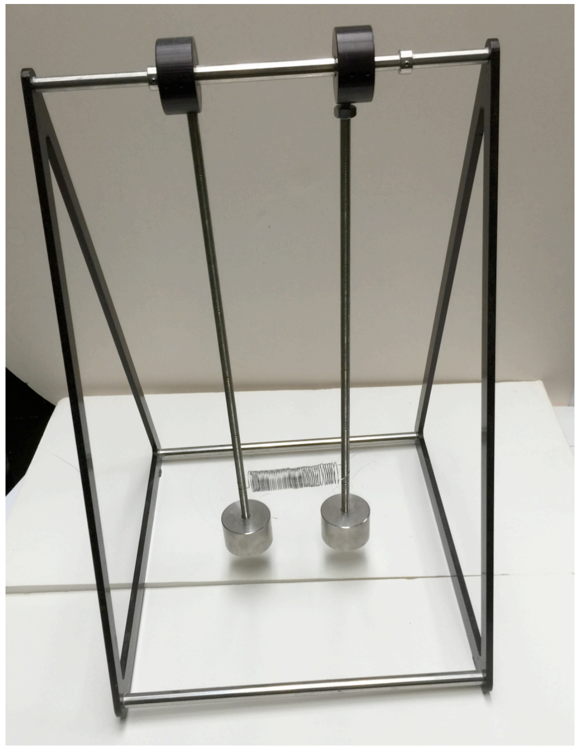

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$$\mathbf{p}_j\mathbf{p}_k=\mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})=\prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}) \qquad \mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$$

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Last step:

make *Idempotent Projectors:* $\mathbf{P}_k=\dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}=\dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$

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With example matrix $\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$

$$\mathbf{p}_1=(\mathbf{M}-5\cdot\mathbf{1})=\begin{pmatrix}-1&1\\3&-3\end{pmatrix} \qquad \mathbf{p}_2=(\mathbf{M}-1\cdot\mathbf{1})=\begin{pmatrix}3&1\\3&1\end{pmatrix} \qquad \mathbf{p}_1\mathbf{p}_2=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

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$$\mathbf{P}_1=\frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)}=\frac{1}{4}\begin{pmatrix}1&-1\\-3&3\end{pmatrix}=k_1\begin{pmatrix}\frac12\\-\frac32\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac12&-\frac12\end{pmatrix}}{k_1}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

"Gauge" scale factors that only affect plots

$$\mathbf{P}_2=\frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)}=\frac{1}{4}\begin{pmatrix}3&1\\3&1\end{pmatrix}=k_2\begin{pmatrix}\frac12\\\frac12\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac32&\frac12\end{pmatrix}}{k_2}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

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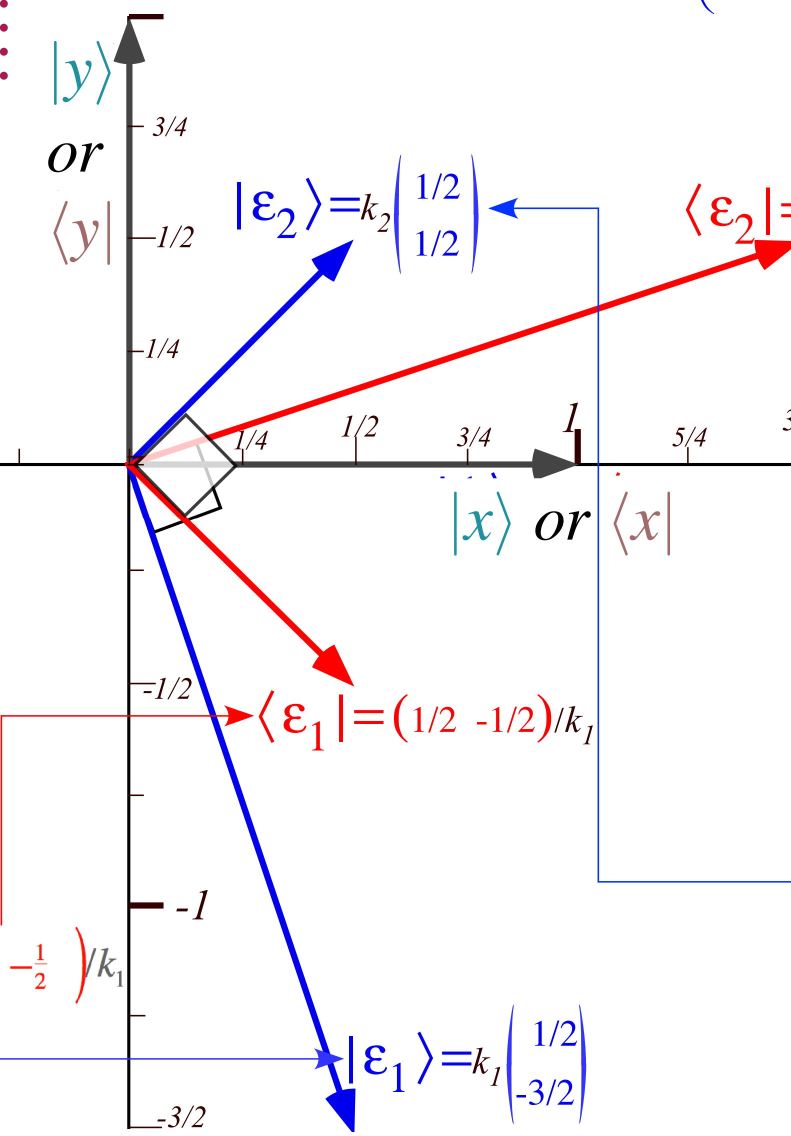

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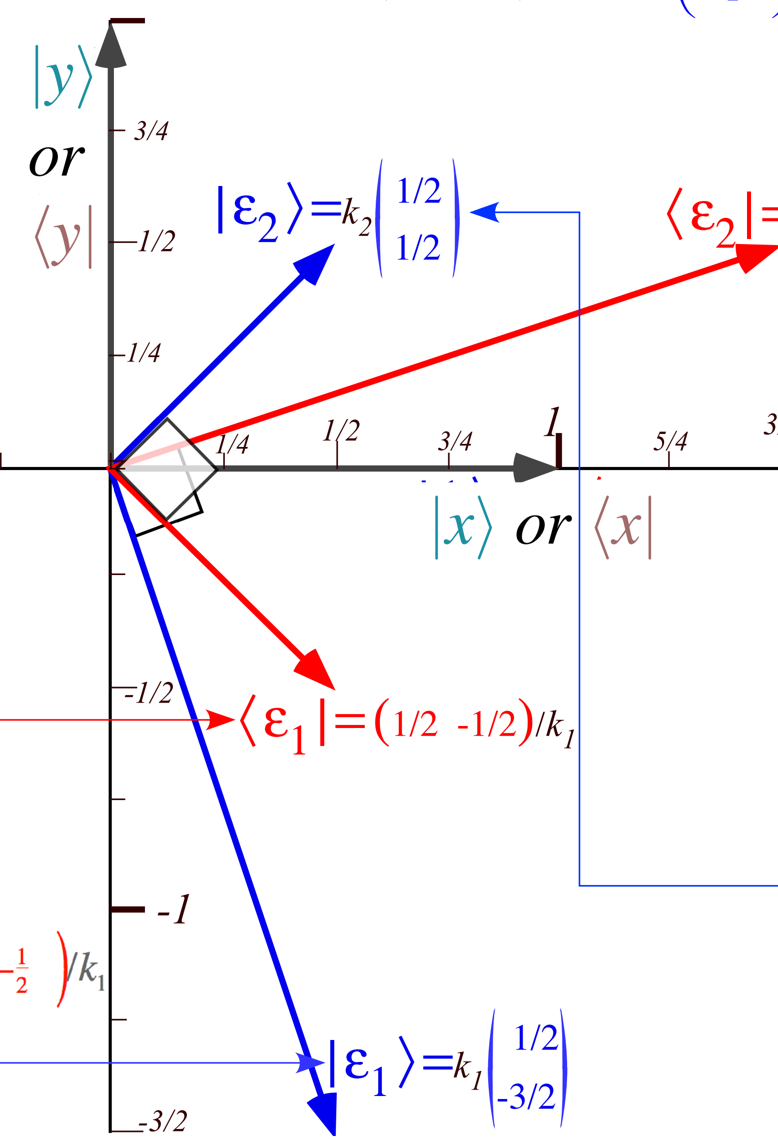

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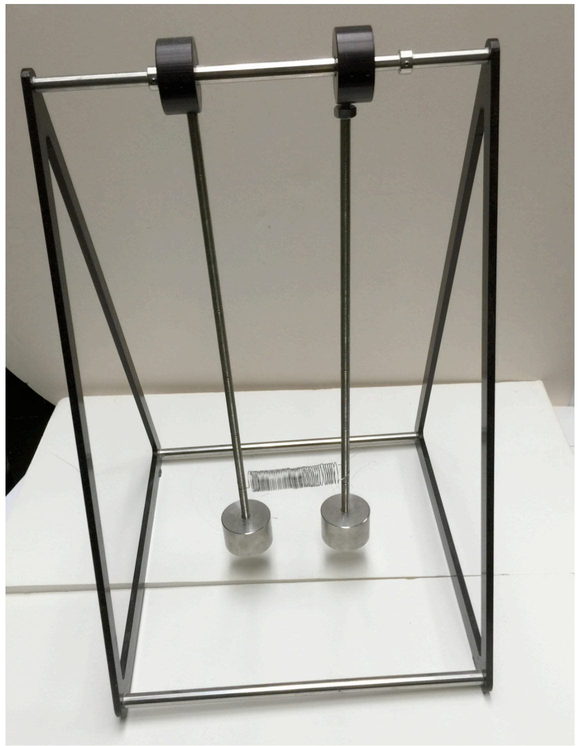

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$$\mathbf{M} = \mathbf{M}\mathbf{P}_1 + \mathbf{M}\mathbf{P}_2 + ... + \mathbf{M}\mathbf{P}_n = \varepsilon_1\mathbf{P}_1 + \varepsilon_2\mathbf{P}_2 + ... + \varepsilon_n\mathbf{P}_n$$

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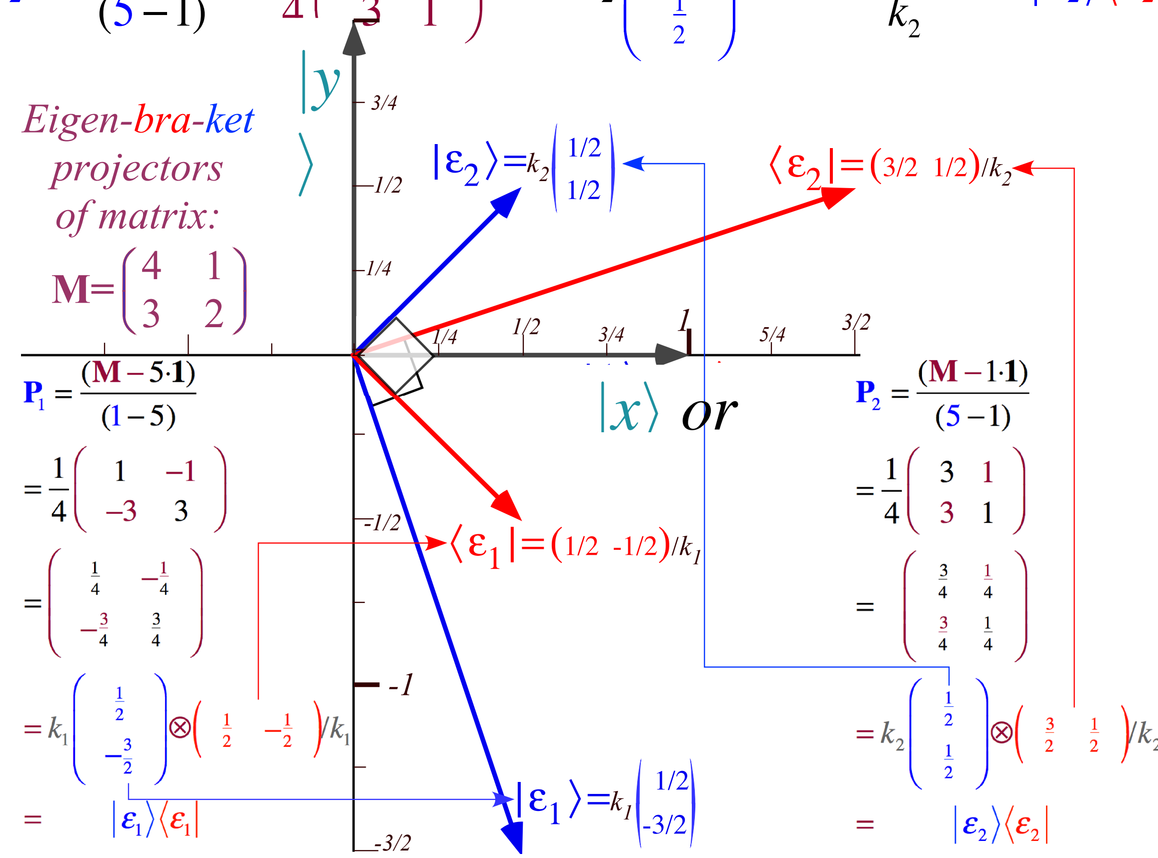

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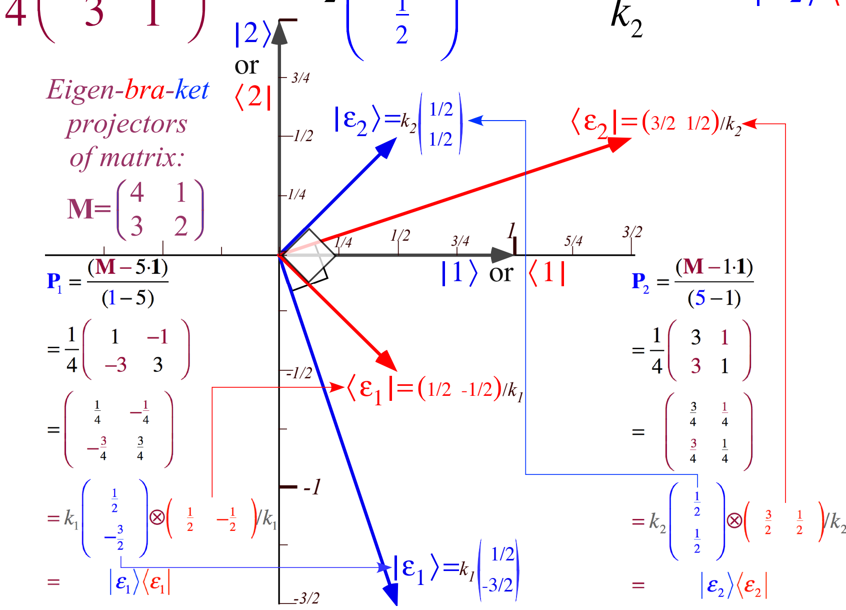

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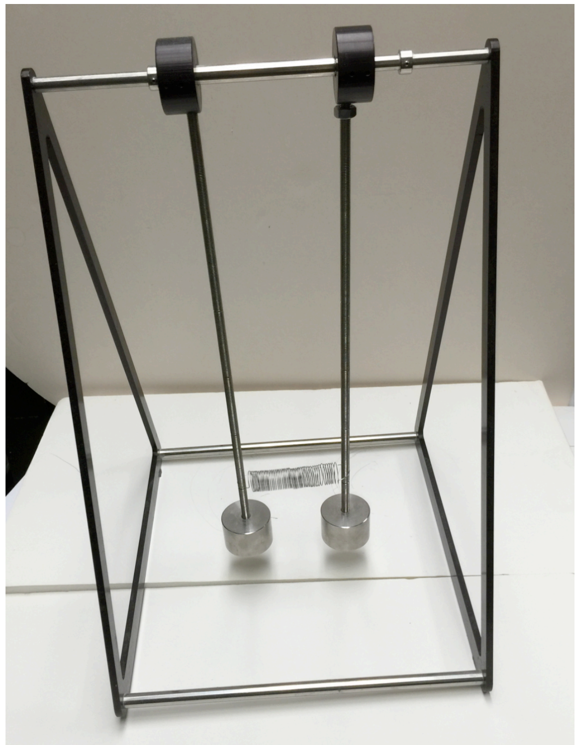

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...and *Functional Spectral Decomposition* of any function *f(***M***)* of **M**

$$f(\mathbf{M})==f(\varepsilon_1)\mathbf{P}_1+f(\varepsilon_2)\mathbf{P}_2+...+f(\varepsilon_n)\mathbf{P}_n$$

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$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}) \qquad \mathbf{M}\mathbf{p}_k = \varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$$

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$$\mathbf{p}_j\mathbf{p}_k = \prod_{m\neq k}(\varepsilon_j\mathbf{p}_j-\varepsilon_m\mathbf{p}_j) = \mathbf{p}_j\prod_{m\neq k}(\varepsilon_j-\varepsilon_m) = \begin{cases}\mathbf{0} & if: j\neq k\\ \mathbf{p}_k\prod_{m\neq k}(\varepsilon_k-\varepsilon_m) & if: j=k\end{cases}$$

Last step:

make *Idempotent Projectors:* $\mathbf{P}_k = \dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} = \dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$

(*Idempotent means:* **P·P=P**)

$$\mathbf{P}_j\mathbf{P}_k = \begin{cases}\mathbf{0} & if: j\neq k\\ \mathbf{P}_k & if: j=k\end{cases}$$

$\mathbf{M}\mathbf{p}_k = \varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$

implies:

$\mathbf{M}\mathbf{P}_k = \varepsilon_k\mathbf{P}_k = \mathbf{P}_k\mathbf{M}$

$$\mathbf{M} = \begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix}$$

$$\mathbf{p}_1 = (\mathbf{M}-5\cdot\mathbf{1}) = \begin{pmatrix}-1 & 1\\ 3 & -3\end{pmatrix} \qquad \mathbf{p}_1\mathbf{p}_2 = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

$$\mathbf{p}_2 = (\mathbf{M}-1\cdot\mathbf{1}) = \begin{pmatrix}3 & 1\\ 3 & 1\end{pmatrix}$$

Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix}1 & -1\\ -3 & 3\end{pmatrix} = k_1\begin{pmatrix}\frac{1}{2}\\ -\frac{3}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix}3 & 1\\ 3 & 1\end{pmatrix} = k_2\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{3}{2} & \frac{1}{2}\end{pmatrix}}{k_2} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

The $\mathbf{P}_j$ are *Mutually Ortho-Normal* as are bra-ket $\langle\varepsilon_j|$and$|\varepsilon_j\rangle$ inside $\mathbf{P}_j$'s

$$\begin{pmatrix}\langle\varepsilon_1|\varepsilon_1\rangle & \langle\varepsilon_1|\varepsilon_2\rangle\\ \langle\varepsilon_2|\varepsilon_1\rangle & \langle\varepsilon_2|\varepsilon_2\rangle\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

...and the $\mathbf{P}_j$ satisfy a *Completeness Relation:*

$$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$$
$$= |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2| + \ldots + |\varepsilon_n\rangle\langle\varepsilon_n|$$

$$\mathbf{P}_1 + \mathbf{P}_2 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2|$$

$$\mathbf{M} = \begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix} = 1\mathbf{P}_1 + 5\mathbf{P}_2 = 1|1\rangle\langle 1| + 5|2\rangle\langle 2| = 1\begin{pmatrix}\frac{1}{4} & -\frac{1}{4}\\ -\frac{3}{4} & \frac{3}{4}\end{pmatrix} + 5\begin{pmatrix}\frac{3}{4} & \frac{1}{4}\\ \frac{3}{4} & \frac{1}{4}\end{pmatrix}$$

Eigen-operators $\mathbf{M}\mathbf{P}_k = \varepsilon_k\mathbf{P}_k$ then give *Spectral Decomposition* of operator **M**

$$\mathbf{M} = \mathbf{M}\mathbf{P}_1 + \mathbf{M}\mathbf{P}_2 + \ldots + \mathbf{M}\mathbf{P}_n = \varepsilon_1\mathbf{P}_1 + \varepsilon_2\mathbf{P}_2 + \ldots + \varepsilon_n\mathbf{P}_n$$

...and *Functional Spectral Decomposition* of any function *f*(**M**) of **M**

$$f(\mathbf{M}) = f(\varepsilon_1)\mathbf{P}_1 + f(\varepsilon_2)\mathbf{P}_2 + \ldots + f(\varepsilon_n)\mathbf{P}_n$$

Example:

$$\mathbf{M}^{50} = \begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix} = 1^{50}\begin{pmatrix}\frac{1}{4} & -\frac{1}{4}\\ -\frac{3}{4} & \frac{3}{4}\end{pmatrix} + 5^{50}\begin{pmatrix}\frac{3}{4} & \frac{1}{4}\\ \frac{3}{4} & \frac{1}{4}\end{pmatrix} = \frac{1}{4}\begin{pmatrix}1+3\cdot 5^{50} & 5^{50}-1\\ 3\cdot 5^{50}-3 & 5^{50}+3\end{pmatrix}$$

Matrix and operator Spectral Decompositons

$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}) \qquad \mathbf{M}\mathbf{p}_k = \varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k = \prod_{m\neq k}(\varepsilon_j\mathbf{p}_j - \varepsilon_m\mathbf{p}_j) = \mathbf{p}_j\prod_{m\neq k}(\varepsilon_j-\varepsilon_m) = \begin{cases} \mathbf{0} & if: j\neq k \\ \mathbf{p}_k\prod_{m\neq k}(\varepsilon_k-\varepsilon_m) & if: j=k \end{cases}$$

Last step:

make *Idempotent Projectors:* $\mathbf{P}_k = \dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} = \dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$

(*Idempotent means:* **P·P=P**)

$$\mathbf{P}_j\mathbf{P}_k = \begin{cases} \mathbf{0} & if: j\neq k \\ \mathbf{P}_k & if: j=k \end{cases}$$

$\mathbf{M}\mathbf{p}_k = \varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$

implies:

$\mathbf{M}\mathbf{P}_k = \varepsilon_k\mathbf{P}_k = \mathbf{P}_k\mathbf{M}$

$$\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$$

$$\mathbf{p}_1 = (\mathbf{M}-5\cdot\mathbf{1}) = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \qquad \mathbf{p}_1\mathbf{p}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{p}_2 = (\mathbf{M}-1\cdot\mathbf{1}) = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1\begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}}{k_2} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

The $\mathbf{P}_j$ are *Mutually Ortho-Normal*
as are bra-ket $\langle\varepsilon_j|$and$|\varepsilon_j\rangle$ inside $\mathbf{P}_j$'s

$$\begin{pmatrix} \langle\varepsilon_1|\varepsilon_1\rangle & \langle\varepsilon_1|\varepsilon_2\rangle \\ \langle\varepsilon_2|\varepsilon_1\rangle & \langle\varepsilon_2|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

...and the $\mathbf{P}_j$ satisfy a
Completeness Relation:

$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + ... + \mathbf{P}_n$

$= |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2| + ... + |\varepsilon_n\rangle\langle\varepsilon_n|$

$$\mathbf{P}_1 + \mathbf{P}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2|$$

$$\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = 1\mathbf{P}_1 + 5\mathbf{P}_2 = 1|1\rangle\langle 1| + 5|2\rangle\langle 2| = 1\begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} + 5\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

Eigen-operators $\mathbf{M}\mathbf{P}_k = \varepsilon_k\mathbf{P}_k$ then give *Spectral Decomposition* of operator **M**

$$\mathbf{M} = \mathbf{M}\mathbf{P}_1 + \mathbf{M}\mathbf{P}_2 + ... + \mathbf{M}\mathbf{P}_n = \varepsilon_1\mathbf{P}_1 + \varepsilon_2\mathbf{P}_2 + ... + \varepsilon_n\mathbf{P}_n$$

...and *Functional Spectral Decomposition* of any function *f*(**M**) of **M**

$$f(\mathbf{M}) == f(\varepsilon_1)\mathbf{P}_1 + f(\varepsilon_2)\mathbf{P}_2 + ... + f(\varepsilon_n)\mathbf{P}_n$$

Examples:

$$\mathbf{M}^{50} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = 1^{50}\begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} + 5^{50}\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1+3\cdot 5^{50} & 5^{50}-1 \\ 3\cdot 5^{50}-3 & 5^{50}+3 \end{pmatrix}$$

$$\sqrt{\mathbf{M}} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = \pm\sqrt{1}\begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} \pm \sqrt{5}\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{4}+\frac{3\sqrt{5}}{4} & -\frac{1}{4}+\frac{\sqrt{}}{4} \\ -\frac{3}{4}+\frac{3\sqrt{5}}{4} & \frac{3}{4}+\frac{\sqrt{}}{4} \end{pmatrix}$$

Orthonormality vs. Completeness

$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}) \qquad \mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k = \prod_{m\neq k}(\varepsilon_j\mathbf{p}_j-\varepsilon_m\mathbf{p}_j) = \mathbf{p}_j\prod_{m\neq k}(\varepsilon_j-\varepsilon_m) = \begin{cases}\mathbf{0} & if: j\neq k\\ \mathbf{p}_k\prod_{m\neq k}(\varepsilon_k-\varepsilon_m) & if: j=k\end{cases}$$

Last step:

make *Idempotent Projectors:* $\mathbf{P}_k = \dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} = \dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$

(Idempotent means: **P·P=P**)

$$\mathbf{P}_j\mathbf{P}_k = \begin{cases}\mathbf{0} & if: j\neq k\\ \mathbf{P}_k & if: j=k\end{cases}$$

$\mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$

implies:

$\mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k=\mathbf{P}_k\mathbf{M}$

$$\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$$

$$\mathbf{p}_1=(\mathbf{M}-5\cdot\mathbf{1})=\begin{pmatrix}-1&1\\3&-3\end{pmatrix} \qquad \mathbf{p}_1\mathbf{p}_2=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$$\mathbf{p}_2=(\mathbf{M}-1\cdot\mathbf{1})=\begin{pmatrix}3&1\\3&1\end{pmatrix}$$

Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1=\frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)}=\frac{1}{4}\begin{pmatrix}1&-1\\-3&3\end{pmatrix}=k_1\begin{pmatrix}\frac{1}{2}\\-\frac{3}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\end{pmatrix}}{k_1}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

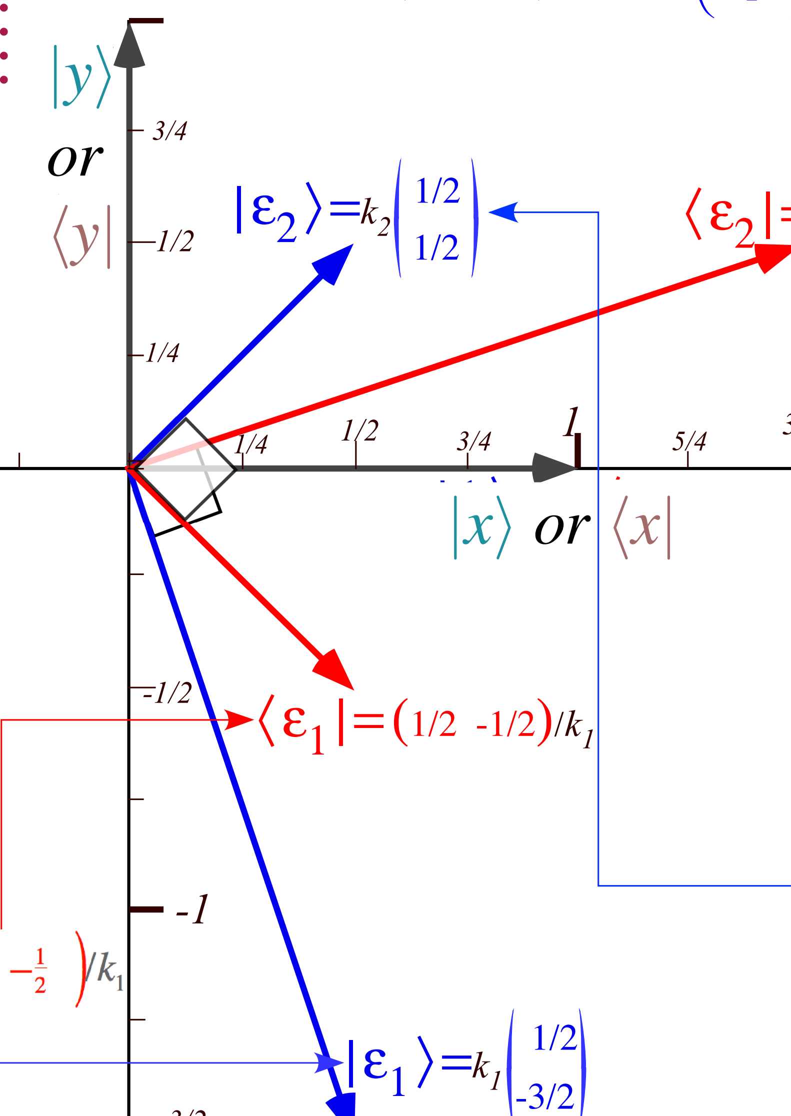
"Gauge" scale factors that only affect plots

$$\mathbf{P}_2=\frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)}=\frac{1}{4}\begin{pmatrix}3&1\\3&1\end{pmatrix}=k_2\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{3}{2}&\frac{1}{2}\end{pmatrix}}{k_2}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

The $\mathbf{P}_j$ are *Mutually Ortho-Normal* as are bra-ket $\langle\varepsilon_j|$ and $|\varepsilon_j\rangle$ inside $\mathbf{P}_j$'s

$$\begin{pmatrix}\langle\varepsilon_1|\varepsilon_1\rangle&\langle\varepsilon_1|\varepsilon_2\rangle\\\langle\varepsilon_2|\varepsilon_1\rangle&\langle\varepsilon_2|\varepsilon_2\rangle\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

Eigen-bra-ket projectors of matrix: $\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$



...and the $\mathbf{P}_j$ satisfy a *Completeness Relation:*

$\mathbf{1}= \mathbf{P}_1 + \mathbf{P}_2 +\ldots+ \mathbf{P}_n$

$=|\varepsilon_1\rangle\langle\varepsilon_1|+|\varepsilon_2\rangle\langle\varepsilon_2|+\ldots+|\varepsilon_n\rangle\langle\varepsilon_n|$

$$\mathbf{P}_1+\mathbf{P}_2=\begin{pmatrix}1&0\\0&1\end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|+|\varepsilon_2\rangle\langle\varepsilon_2|$$

$$\mathbf{P}_1=\frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)}=\frac{1}{4}\begin{pmatrix}1&-1\\-3&3\end{pmatrix}=\begin{pmatrix}\frac{1}{4}&-\frac{1}{4}\\-\frac{3}{4}&\frac{3}{4}\end{pmatrix}=k_1\begin{pmatrix}\frac{1}{2}\\-\frac{3}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\end{pmatrix}/k_1=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)}=\frac{1}{4}\begin{pmatrix}3&1\\3&1\end{pmatrix}=\begin{pmatrix}\frac{3}{4}&\frac{1}{4}\\\frac{3}{4}&\frac{1}{4}\end{pmatrix}=k_2\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{3}{2}&\frac{1}{2}\end{pmatrix}/k_2=|\varepsilon_2\rangle\langle\varepsilon_2|$$

$\{|x\rangle,|y\rangle\}$-orthonormality with $\{|\varepsilon_1\rangle,|\varepsilon_2\rangle\}$-completeness

$$\langle x|y\rangle=\delta_{x,y}=\langle x|\mathbf{1}|y\rangle=\langle x|\varepsilon_1\rangle\langle\varepsilon_1|y\rangle+\langle x|\varepsilon_2\rangle\langle\varepsilon_2|y\rangle.$$

$\{|\varepsilon_1\rangle,|\varepsilon_2\rangle\}$-orthonormality with $\{|x\rangle,|y\rangle\}$-completeness

$$\langle\varepsilon_i|\varepsilon_j\rangle=\delta_{i,j}=\langle\varepsilon_i|\mathbf{1}|\varepsilon_j\rangle=\langle\varepsilon_i|x\rangle\langle x|\varepsilon_j\rangle+\langle\varepsilon_i|y\rangle\langle y|\varepsilon_j\rangle$$