Introduction to coupled oscillation and eigenmodes
(Ch. 3-4 of Unit 2)

Review of 1D FDHO (Forced-Damped-Harmonic Oscillator) response
Amplitude and phase variation due to resonance

2D harmonic oscillator (2D-HO) equations of motion
Lagrangian and matrix forms

2D harmonic oscillator equation eigensolutions (normal modes)
2D classical HO compared to U(2) quantum 2-state system
Introducing ABCD Hamilton Pauli spinor symmetry expansion

Eigensolutions by geometry for 2D-HO with bilateral (B-Type) symmetry
Symmetric (low frequency) mode versus antisymmetric (high frequency) mode
Mixed mode beat dynamics (with constant $\pi/2$ phase-lag)
Geometry of phase and polarization

Eigensolutions by matrix-algebra with example $M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$
Secular equation
Hamilton-Cayley equation and projectors
Idempotent projectors (how eigenvalues $\Rightarrow$ eigenvectors)
Operator orthonormality and Completeness (Idempotent means: $P \cdot P = P$)

Spectral Decompositions
Functional spectral decomposition
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Spectral Decompositions
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Resonance Region
Resonance Response
DC Response
High \( \omega \) Fall-Off

\[ \text{Resonance Region (FWHM)} \]

\[ \omega (\text{radian/sec}) \]
\[ \nu (\text{Hertz}) \]

\[ \text{Stimulus Frequency} \]

\[ \text{Resonant Response} \]

\[ \text{DC Response} \]

\[ \text{Re} \ G_{\omega_0} (\omega) = \frac{\omega_0^2 - \omega_s^2}{\left( \omega_0^2 - \omega_s^2 \right)^2 + (2\Gamma \omega_s)^2} \]

\[ \text{Im} \ G_{\omega_0} (\omega) = \frac{2\Gamma \omega_s}{\left( \omega_0^2 - \omega_s^2 \right)^2 + (2\Gamma \omega_s)^2} \]

\[ \rho = \tan^{-1} \left( \frac{2\Gamma \omega_s}{\omega_0^2 - \omega_s^2} \right) \]

\[ AAF = \frac{\text{Resonant response}}{\text{DC response}} = \frac{|G_{\omega_0} (\omega_s = \omega_0)|}{|G_{\omega_0} (0)|} = \frac{1 / (2\Gamma \omega_0)}{1 / \omega_0^2} = \frac{\omega_0}{2\Gamma} \equiv q \quad \text{(angular quality factor)} \]

\[ \text{OscillIt Web Simulation} \]
\[ \text{Lorentz Response (}\Gamma = 0.2) \]

\[ \text{Fig. 2.2.6 Anatomy of oscillator Green-Lorentz response function plots} \]
\[ \omega(0) = 6.283, \quad \Gamma = 0.200, \quad \text{As} = 1.000 \]

\[ \omega_s = \text{near zero} \]

Angular frequency \( \omega_s = 2\pi \cdot \nu \) radians per sec

Frequency \( \nu \) per sec (Hertz)

DC response

\( \nu_0 = 1 \text{Hz} \)

\( \rho = 0 \)

DC response

OscillIt Web Simulation
Lorentz Response (\( \Gamma = 0.2 \))

link to time plot
Response functions for very low q oscillator

\[ \omega(0) = 1.919, \quad \Gamma = 1.591, \quad A_s = 1.000 \]

(Big "flat-tire")

OscillIt Web Simulation
Lorentz Response (\(\Gamma = 1.6\))
\[ \omega(0) = 6.283, \quad \Gamma = 0.200, \quad A_s = 1.000 \]
\[ \omega_s = \omega_0 - \Gamma = 6.083 \]

Angular frequency \( \omega_s = 2\pi \cdot \nu_s \) radians per sec

Frequency \( \nu \) per sec (Hertz)

\[ \rho = \pi/4 \]

45° phase lag

\[ \omega_s = \omega_0 - \Gamma \]

OscillIt Web Simulation
Lorentz Response \((\Gamma = 0.2)\)

link to time plot
\[ \omega(0) = 6.283, \quad \Gamma = 0.200, \quad A_s = 1.000 \]

\[ \omega_s = \omega_0 - \Gamma = 6.083 \]

Angular frequency \( \omega_s = 2\pi \cdot \nu_s \) radians per sec.

Frequency \( \nu \) per sec (Hertz)

OscillIt Web Simulation
Lorentz Response (\( \Gamma = 0.2 \))
\[ \rho = \pi/2 \quad 90^\circ \]

Angular frequency \( \omega_s = 2\pi \cdot \nu_s \) radians per sec

\[ \omega_s = \omega_0 = 6.283 \]

Full-resonance!

OscillIt Web Simulation
Lorentz Response \((\Gamma = 0.2)\)

link to time plot

Wednesday, March 23, 2016
\[ \rho = \pi/2 \quad 90^\circ \]

Angular frequency \( \omega_s = 2\pi \cdot \nu_s \) radians per sec

Frequency \( \nu_0 = 1 \text{Hz} \)

Lorentz Response (\( \Gamma = 0.2 \))

OscillIt Web Simulation

\[ \omega(0) = 6.283, \quad \Gamma = 0.200, \quad A_s = 1.000 \]

\[ \omega_s = \omega_0 + \Gamma = 6.483 \]

Phase lag \( \omega_s = \omega_0 + \Gamma \)

Half-resonance again!

DC response

Frequency \( \nu \) per sec (Hertz)

Link to time plot
$\rho = \frac{\pi}{2}$

$90^\circ$

$\omega_s = \omega_0$

$\omega(0) = 6.283$, $\Gamma = 0.200$, $A_s = 1.000$

Angular frequency $\omega_s = 2\pi \cdot \nu_s$ radians per sec

Frequency $\nu$ per sec (Hertz)

$\nu_0 = 1$ Hz

$\omega_s \rightarrow X$-ray region response approaches zero!

OscillIt Web Simulation

Lorentz Response ($\Gamma = 0.2$)

link to time plot

Wednesday, March 23, 2016
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2D harmonic oscillators

Fig. 3.3.1 Two 1-dimensional coupled oscillators

Fig. 3.3.2 Coupled pendulums

Fig. 3.3.3 One 2-dimensional coupled oscillator
2D harmonic oscillator energy

\[
\begin{align*}
\kappa & = 0 \\
\theta & = \theta_1 = \theta_2
\end{align*}
\]

Fig. 3.3.1 Two 1-dimensional coupled oscillators

\[
\begin{align*}
x &= x_1 \\
y &= x_2
\end{align*}
\]

Fig. 3.3.2 Coupled pendulums

\[
\begin{align*}
m_1 & = m_1 \\
m_2 & = m_2
\end{align*}
\]

Fig. 3.3.3 One 2-dimensional coupled oscillator

2D HO kinetic energy \( T(v_1, v_2) \)

\[
T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2
\]
2D harmonic oscillator energy

\[
V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 + \frac{1}{2} k_{12} (x_1 - x_2)^2
\]

\[
V = \frac{1}{2} (k_1 + k_{12}) x_1^2 - k_{12} x_1 x_2 + \frac{1}{2} (k_2 + k_{12}) x_2^2
\]

2D HO kinetic energy \( T(v_1, v_2) \)

\[
T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2
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Fig. 3.3.1 Two 1-dimensional coupled oscillators

Fig. 3.3.2 Coupled pendulums

Fig. 3.3.3 One 2-dimensional coupled oscillator

2D HO kinetic energy $T(v_1, v_2)$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

2D HO potential energy $V(x_1, x_2)$

$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 + \frac{1}{2} k_{12} (x_1 - x_2)^2$$

$$= \frac{1}{2} (k_1 + k_{12}) x_1^2 - k_{12} x_1 x_2 + \frac{1}{2} (k_2 + k_{12}) x_2^2$$

Lagrange-Newton equations for 2D HO

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}_1} \right) = m_1 \ddot{x}_1 = F_1 = - \frac{\partial V}{\partial x_1} = - (k_1 + k_{12}) x_1 + k_{12} x_2$$

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}_2} \right) = m_2 \ddot{x}_2 = F_2 = - \frac{\partial V}{\partial x_2} = k_{12} x_1 - (k_2 + k_{12}) x_2$$
2D harmonic oscillator equations

\[ T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 \]

2D HO kinetic energy \( T(v_1, v_2) \)

\[ V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 + \frac{1}{2} k_{12} (x_1 - x_2)^2 \]

2D HO potential energy \( V(x_1, x_2) \)

\[
\begin{pmatrix}
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}_1} \right)
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}_2} \right)
\end{pmatrix} =
\begin{pmatrix}
m_1 & 0 \\
0 & m_2
\end{pmatrix}
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix}
= -\begin{pmatrix}
\frac{\partial V}{\partial x_1}
\frac{\partial V}{\partial x_2}
\end{pmatrix}
= -\begin{pmatrix}
(k_1 + k_{12}) x_1 + k_{12} x_2 \\
k_{12} x_1 - (k_2 + k_{12}) x_2
\end{pmatrix}
\]

Lagrange-Newton equations for 2D HO

\[
\begin{pmatrix}
m_1 & 0 \\
0 & m_2
\end{pmatrix}
\begin{pmatrix}
\ddot{x}_1 \\
\ddot{x}_2
\end{pmatrix}
= -\begin{pmatrix}
\frac{\partial V}{\partial x_1}
\frac{\partial V}{\partial x_2}
\end{pmatrix}
= k_1 x_1 - (k_2 + k_{12}) x_2
\]

2D HO Matrix operator equations
2D harmonic oscillator equations

2D HO kinetic energy $T(v_1, v_2)$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{y}_2^2$$

2D HO potential energy $V(x_1, x_2)$

$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 + \frac{1}{2} k_{12} (x_1 - x_2)^2$$

$$= \frac{1}{2} (k_1 + k_{12}) x_1^2 - k_{12} x_1 x_2 + \frac{1}{2} (k_2 + k_{12}) x_2^2$$

Lagrange-Newton equations for 2D HO

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}_1} \right) = m_1 \ddot{x}_1 = F_1 = -\frac{\partial V}{\partial x_1} = -(k_1 + k_{12}) x_1 + k_{12} x_2$$

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{y}_2} \right) = m_2 \ddot{y}_2 = F_2 = -\frac{\partial V}{\partial y_2} = k_{12} x_1 - (k_2 + k_{12}) x_2$$

2D HO Matrix operator equations

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Matrix operator notation:

$$\mathbf{M} \cdot \ddot{\mathbf{x}} = -\mathbf{K} \cdot \mathbf{x}$$
**2D harmonic oscillator equations**

![Diagram of 2D harmonic oscillators](image)

### 2D HO kinetic energy \( T(v_1, v_2) \)

\[
T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2
\]

\[
= \frac{1}{2} \langle \dot{x} | M | \dot{x} \rangle
\]

### 2D HO potential energy \( V(x_1, x_2) \)

\[
V = \frac{1}{2} (k_1 + k_{12}) x_1^2 - k_{12} x_1 x_2 + \frac{1}{2} (k_2 + k_{12}) x_2^2
\]

where: \( K = \begin{pmatrix}
  k_1 + k_{12} & -k_{12} \\
  -k_{12} & k_2 + k_{12}
\end{pmatrix} \)

### Lagrange-Newton equations for 2D HO

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}_1} \right) = m_1 \ddot{x}_1 = F_1 = -\frac{\partial V}{\partial x_1} = -(k_1 + k_{12}) x_1 + k_{12} x_2
\]

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}_2} \right) = m_2 \ddot{x}_2 = F_2 = -\frac{\partial V}{\partial x_2} = k_{12} x_1 - (k_2 + k_{12}) x_2
\]

### 2D HO Matrix operator equations

\[
\begin{pmatrix}
  m_1 & 0 \\
  0 & m_2
\end{pmatrix}
\begin{pmatrix}
  \ddot{x}_1 \\
  \ddot{x}_2
\end{pmatrix}
= -\begin{pmatrix}
  k_1 + k_{12} & -k_{12} \\
  -k_{12} & k_2 + k_{12}
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}
\]

Matrix operator notation:

\[
M \cdot \dot{x} = -K \cdot x
\]
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2D harmonic oscillator equation solutions

1. May rewrite equation \( \mathbf{M} \cdot \ddot{\mathbf{x}} = -\mathbf{K} \cdot \mathbf{x} \) in acceleration matrix form: \( \ddot{x} = -A \cdot x \) where: \( A = \mathbf{M}^{-1} \cdot \mathbf{K} \)

\[
\begin{pmatrix}
\dddot{x}_1 \\
\dddot{x}_2
\end{pmatrix} = -
\begin{pmatrix}
m_1 & 0 \\
0 & m_2
\end{pmatrix}^{-1}
\begin{pmatrix}
k_1 + k_{12} & -k_{12} \\
-k_{12} & k_2 + k_{12}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} = -
\begin{pmatrix}
\frac{k_1 + k_{12}}{m_1} & -\frac{k_{12}}{m_1} \\
-\frac{k_{12}}{m_2} & \frac{k_2 + k_{12}}{m_2}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]
2D harmonic oscillator equation solutions

1. May rewrite equation $M \cdot \ddot{x} = -K \cdot x$ in acceleration matrix form: $\ddot{x} = -Ax$ where $A = M^{-1} \cdot K$

\[
\begin{pmatrix}
\ddot{x}_1 \\
\ddot{x}_2
\end{pmatrix} = -\begin{pmatrix}
m_1 & 0 \\
0 & m_2
\end{pmatrix}^{-1} \begin{pmatrix}
k_1 + k_{12} & -k_{12} \\
-k_{12} & k_2 + k_{12}
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} = -\begin{pmatrix}
\frac{k_1 + k_{12}}{m_1} & \frac{-k_{12}}{m_1} \\
\frac{-k_{12}}{m_2} & \frac{k_2 + k_{12}}{m_2}
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

2. Need to find eigenvectors $|e_1\rangle, |e_2\rangle, \ldots$ of acceleration matrix such that: $A|e_n\rangle = \epsilon_n|e_n\rangle = \omega_n^2|e_n\rangle$
2D harmonic oscillator equation solutions

1. May rewrite equation \( \mathbf{M} \cdot \ddot{\mathbf{x}} = -\mathbf{K} \cdot \mathbf{x} \) in acceleration matrix form: \( \ddot{\mathbf{x}} = -\mathbf{A} \cdot \mathbf{x} \) where: \( \mathbf{A} = \mathbf{M}^{-1} \cdot \mathbf{K} \)

\[
\begin{pmatrix}
\ddot{x}_1 \\
\ddot{x}_2
\end{pmatrix}
= -
\begin{pmatrix}
m_1 & 0 \\
0 & m_2
\end{pmatrix}^{-1}
\begin{pmatrix}
k_1 + k_{12} & -k_{12} \\
-k_{12} & k_2 + k_{12}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
= -
\begin{pmatrix}
\frac{k_1 + k_{12}}{m_1} & \frac{-k_{12}}{m_1} \\
\frac{-k_{12}}{m_2} & \frac{k_2 + k_{12}}{m_2}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

2. Need to find eigenvectors \( |e_1\rangle, |e_2\rangle, \ldots \) of acceleration matrix such that: \( \mathbf{A} |e_n\rangle = \varepsilon_n |e_n\rangle = \omega_n^2 |e_n\rangle \)

Then equations decouple to: \( \frac{d^2}{dt^2} |e_1\rangle \equiv \ddot{e}_1 = -\mathbf{A} |e_1\rangle = -\varepsilon_1 |e_1\rangle = -\omega_1^2 |e_1\rangle \) so: \( |e_1(t)\rangle = e^{-i\omega_1 t} |e_1(0)\rangle \)

where \( \varepsilon_1 \) is 1st eigenvalue and \( \omega_1 \) is 1st eigenfrequency
2D harmonic oscillator equation solutions

1. May rewrite equation \( \mathbf{M} \cdot \ddot{\mathbf{x}} = -\mathbf{K} \cdot \mathbf{x} \) in acceleration matrix form: \( \ddot{\mathbf{x}} = -\mathbf{A} \cdot \mathbf{x} \) where: \( \mathbf{A} = \mathbf{M}^{-1} \cdot \mathbf{K} \)

\[
\begin{pmatrix}
\ddot{x}_1 \\
\ddot{x}_2
\end{pmatrix} = -
\begin{pmatrix}
m_1 & 0 \\
0 & m_2
\end{pmatrix}^{-1}
\begin{pmatrix}
k_1 + k_{12} & -k_{12} \\
-k_{12} & k_2 + k_{12}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} = -
\begin{pmatrix}
k_1 + k_{12} & -k_{12} \\
-k_{12} & k_2 + k_{12}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

2. Need to find eigenvectors \( |e_1\rangle, |e_2\rangle, \ldots \) of acceleration matrix such that: \( \mathbf{A} |e_n\rangle = \epsilon_n |e_n\rangle = \omega_n^2 |e_n\rangle \)

Then equations decouple to: \( \frac{d^2}{dt^2} |e_1\rangle = \mathbf{-A} |e_1\rangle = -\epsilon_1 |e_1\rangle = -\omega_1^2 |e_1\rangle \) so: \( |e_1(t)\rangle = e^{-i\omega_1 t} |e_1(0)\rangle \)

where \( \epsilon_1 \) is 1\textsuperscript{st} eigenvalue and \( \omega_1 \) is 1\textsuperscript{st} eigenfrequency

and: \( \frac{d^2}{dt^2} |e_2\rangle = \mathbf{-A} |e_2\rangle = -\epsilon_2 |e_2\rangle = -\omega_2^2 |e_2\rangle \) so: \( |e_2(t)\rangle = e^{-i\omega_2 t} |e_2(0)\rangle \)

where \( \epsilon_2 \) is 2\textsuperscript{nd} eigenvalue and \( \omega_2 \) is 2\textsuperscript{nd} eigenfrequency
2D harmonic oscillator equation solutions

1. May rewrite equation \( \mathbf{M} \cdot \ddot{\mathbf{x}} = -\mathbf{K} \cdot \mathbf{x} \) in acceleration matrix form: 

\[
\begin{pmatrix}
\ddot{x}_1 \\
\ddot{x}_2
\end{pmatrix} = -\begin{pmatrix}
m_1 & 0 \\
0 & m_2
\end{pmatrix}^{-1} \begin{pmatrix}
k_1 + k_{12} & -k_{12} \\
-k_{12} & k_2 + k_{12}
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} = -\begin{pmatrix}
k_1 + k_{12} & -k_{12} \\
-k_{12} & k_2 + k_{12}
\end{pmatrix} \begin{pmatrix}
m_1 \\
m_2
\end{pmatrix}^{-1} \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

2. Need to find eigenvectors \( |e_1\rangle, |e_2\rangle, \ldots \) of acceleration matrix such that: 

\[
\mathbf{A} |e_n\rangle = \epsilon_n |e_n\rangle = \omega_n^2 |e_n\rangle
\]

Then equations decouple to: 

\[
\frac{d^2}{dt^2} |e_1\rangle \equiv |\ddot{e}_1\rangle = -\mathbf{A} |e_1\rangle = -\epsilon_1 |e_1\rangle = -\omega_1^2 |e_1\rangle \quad \text{so:} \quad |e_1(t)\rangle = e^{-i\omega_1 t} |e_1(0)\rangle
\]

where \( \epsilon_1 \) is 1\textsuperscript{st} eigenvalue and \( \omega_1 \) is 1\textsuperscript{st} eigenfrequency

and: 

\[
\frac{d^2}{dt^2} |e_2\rangle \equiv |\ddot{e}_2\rangle = -\mathbf{A} |e_2\rangle = -\epsilon_2 |e_2\rangle = -\omega_2^2 |e_2\rangle \quad \text{so:} \quad |e_2(t)\rangle = e^{-i\omega_2 t} |e_2(0)\rangle
\]

where \( \epsilon_2 \) is 2\textsuperscript{nd} eigenvalue and \( \omega_2 \) is 2\textsuperscript{nd} eigenfrequency

To introduce eigensolutions we take a simple case of unit masses \((m_1 = 1 = m_2)\)

So equation of motion is simply: 

\[
|\ddot{x}\rangle = -\mathbf{K} |x\rangle
\]
2D harmonic oscillator equation solutions

1. May rewrite equation \( \mathbf{M} \cdot \ddot{\mathbf{x}} = -\mathbf{K} \cdot \mathbf{x} \) in acceleration matrix form: \( \ddot{\mathbf{x}} = -\mathbf{A} \mathbf{x} \) where: \( \mathbf{A} = \mathbf{M}^{-1} \cdot \mathbf{K} \)

\[
\begin{pmatrix}
\ddot{x}_1 \\
\ddot{x}_2 
\end{pmatrix} = -\begin{pmatrix} m_1 & 0 \\
0 & m_2 \end{pmatrix}^{-1} \begin{pmatrix} k_1 + k_{12} & -k_{12} \\
-k_{12} & k_2 + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\
x_2 \end{pmatrix} = -\begin{pmatrix} \frac{k_1 + k_{12}}{m_1} & \frac{-k_{12}}{m_1} \\
\frac{-k_{12}}{m_2} & \frac{k_2 + k_{12}}{m_2} \end{pmatrix} \begin{pmatrix} x_1 \\
x_2 \end{pmatrix}
\]

2. Need to find eigenvectors \( |e_1\rangle, |e_2\rangle \),... of acceleration matrix such that: \( \mathbf{A} |e_n\rangle = \varepsilon_n |e_n\rangle = \omega_n^2 |e_n\rangle \)

Then equations decouple to: \( \frac{d^2}{dt^2} |e_1\rangle = -\mathbf{A} |e_1\rangle = -\varepsilon_1 |e_1\rangle = -\omega_1^2 |e_1\rangle \) so: \( |e_1(t)\rangle = e^{-i\omega_1 t} |e_1(0)\rangle \)

where \( \varepsilon_1 \) is 1st eigenvalue and \( \omega_1 \) is 1st eigenfrequency

and: \( \frac{d^2}{dt^2} |e_2\rangle = -\mathbf{A} |e_2\rangle = -\varepsilon_2 |e_2\rangle = -\omega_2^2 |e_2\rangle \) so: \( |e_2(t)\rangle = e^{-i\omega_2 t} |e_2(0)\rangle \)

where \( \varepsilon_2 \) is 2nd eigenvalue and \( \omega_2 \) is 2nd eigenfrequency

To introduce eigensolutions we take a simple case of unit masses \( (m_1=1=m_2) \)

So equation of motion is simply: \( \ddot{\mathbf{x}} = -\mathbf{K} \mathbf{x} \)

Eigenvectors \( |\mathbf{x}\rangle = |e_n\rangle \) are in special directions where \( \ddot{\mathbf{x}} = -\mathbf{K} \mathbf{x} \) is in same direction as \( |\mathbf{x}\rangle \)
Review of 1D FDHO (Forced-Damped-Harmonic Oscillator) response
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2D harmonic oscillator (2D-HO) equations of motion
Lagrangian and matrix forms

2D harmonic oscillator equation eigensolutions (normal modes)
2D classical HO compared to U(2) quantum 2-state system
Introducing $ABCD$ Hamilton Pauli spinor symmetry expansion

Eigensolutions by geometry for 2D-HO with bilateral (B-Type) symmetry
Symmetric (low frequency) mode versus antisymmetric (high frequency) mode
Mixed mode beat dynamics (with constant $\pi/2$ phase-lag)

Eigensolutions by matrix-algebra with example $M=\text{Secular equation}$
$\text{Hamilton-Cayley equation and projectors}$
$\text{Idempotent projectors (how eigenvalues} \Rightarrow \text{eigenvectors)}$
$\text{Operator orthonormality and Completeness (Idempotent means: } P \cdot P = P)$
$\text{Spectral Decompositions}$
$\text{Functional spectral decomposition}$
2D classical HO compared to $U(2)$ quantum 2-state system

Classical Newton-Hooke-Stokes equation $\ddot{z} = -K \cdot z$

Quantum Schrödinger-Pauli equation $i\hbar \frac{d}{dt} |\Psi\rangle = H |\Psi\rangle$

$$\frac{d^2}{dt^2} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = - \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

versus

$$i\hbar \frac{d}{dt} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$
2D classical HO compared to U(2) quantum 2-state system

Classical Newton-Hooke-Stokes equation
\[
\frac{d^2}{dt^2}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = -\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}
\]

Quantum Schrodinger-Pauli equation
\[
\hbar \frac{d}{dt}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}
\]

Modern theorists use natural units so \( \hbar = 1.05 \cdot 10^{-34} \) equals \( \hbar = 1 \)
2D classical HO compared to U(2) quantum 2-state system

Classical Newton-Hooke-Stokes equation \( \dddot{z} = -K \cdot z \)

Quantum Schrodinger-Pauli equation \( i \hbar \frac{d}{dt} |\Psi\rangle = H |\Psi\rangle \)

\[
\begin{align*}
\frac{d^2}{dt^2} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &= - \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad \text{versus} \quad i \hbar \frac{d}{dt} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \\
\frac{d^2}{dt^2} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &= - \begin{pmatrix} G & H - iJ \\ H + iJ & K \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad \text{versus} \quad i \hbar \frac{d}{dt} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}
\end{align*}
\]

Modern theorists use natural units so \( \hbar = 1.05 \cdot 10^{-34} \) equals \( \hbar = 1 \)

Let us square the quantum operator \( i \frac{d}{dt} = \begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix} \)
2D classical HO compared to \( U(2) \) quantum 2-state system

Classical Newton-Hooke-Stokes equation \( \ddot{z} = -K \dot{z} \)

\[
\frac{d^2}{dt^2} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = -\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}
\]

Quantum Schrödinger-Pauli equation

\( i \hbar \frac{d}{dt} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \)

\[
\frac{d^2}{dt^2} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = -\begin{pmatrix} G & H - iJ \\ H + iJ & K \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}
\]

versus

\[
\frac{i \hbar}{\hbar} \frac{d}{dt} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}
\]

Modern theorists use natural units so \( \hbar = 1.05 \cdot 10^{-34} \) equals \( \hbar = 1 \)

Let us square the quantum operator

\[
\frac{i d}{d \tau} \frac{i d}{d \tau} = \begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix} \begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix} = \begin{pmatrix} A^2 + B^2 + C^2 & (B-iC)(A+D) \\ (B+iC)(A+D) & D^2 + B^2 + C^2 \end{pmatrix}
\]
2D classical HO compared to $U(2)$ quantum 2-state system

Classical Newton-Hooke-Stokes equation $\ddot{z} = -K \dot{z}$

Quantum Schrodinger-Pauli equation $i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$

Let us square the quantum operator

$\frac{d^2}{dt^2} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = -\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

versus

$\hbar \frac{d}{dt} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$

$\frac{d^2}{dt^2} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = -\begin{pmatrix} G & H - iJ \\ H + iJ & K \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

versus

$\hbar \frac{d}{dt} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$

Modern theorists use natural units so $\hbar = 1.05 \times 10^{-34}$ equals $\hbar = 1$.

Let us square the quantum operator

$i \frac{d}{dt} = \begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix}$

$i \frac{d}{dt} \frac{d}{dt} = \begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix} \begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix} = \begin{pmatrix} A^2 + B^2 + C^2 & (B-iC)(A+D) \\ (B+iC)(A+D) & D^2 + B^2 + C^2 \end{pmatrix}$

$-\frac{d^2}{dt^2} = H^2 = \begin{pmatrix} A^2 + B^2 + C^2 & (B-iC)(A+D) \\ (B+iC)(A+D) & D^2 + B^2 + C^2 \end{pmatrix}$
**2D classical HO compared to U(2) quantum 2-state system**

Classical Newton-Hooke-Stokes equation \( \dot{\mathbf{z}} = -\mathbf{K} \cdot \mathbf{z} \) versus Quantum Schrodinger-Pauli equation \( i\hbar \frac{d}{dt} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \)

\[
\begin{align*}
\frac{d^2}{dt^2} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &= - \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\
\frac{d^2}{dt^2} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &= - \begin{pmatrix} G & H - iJ \\ H + iJ & K \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}
\end{align*}
\]

Modern theorists use natural units so \( \hbar = 1.05 \cdot 10^{-34} \) equals \( \hbar = 1 \)

**Let us square the quantum operator**

\[
\begin{align*}
i \frac{d}{dt} i \frac{d}{dt} &= \begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix} \begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix} = \begin{pmatrix} A^2 + B^2 + C^2 & (B - iC)(A + D) \\ (B + iC)(A + D) & D^2 + B^2 + C^2 \end{pmatrix} \\
- \frac{d^2}{dt^2} &= \mathbf{H}^2 = \begin{pmatrix} A^2 + B^2 + C^2 & (B - iC)(A + D) \\ (B + iC)(A + D) & D^2 + B^2 + C^2 \end{pmatrix}
\end{align*}
\]

**2D classical HO same as the U(2) quantum 2-state system**

...if we set \( \mathbf{K} \)-spring matrix to squared quantum operator \( \mathbf{H}^2 \)

\[
\mathbf{K} = \begin{pmatrix} k_{11} & k_{12} - i \cdot j_{12} \\ k_{12} - i \cdot j_{12} & k_{22} \end{pmatrix} = \begin{pmatrix} G & H - i \cdot J \\ H + i \cdot J & K \end{pmatrix} = \mathbf{H}^2 = \begin{pmatrix} A^2 + B^2 + C^2 & (B - iC)(A + D) \\ (B + iC)(A + D) & D^2 + B^2 + C^2 \end{pmatrix}
\]
Review of 1D **FDHO** *(Forced-Damped-Harmonic Oscillator)* response
Amplitude and phase variation due to resonance

2D harmonic oscillator (2D-HO) equations of motion
Lagrangian and matrix forms

2D harmonic oscillator equation eigensolutions (normal modes)
2D classical HO compared to U(2) quantum 2-state system

Introducing *ABCD* Hamilton Pauli spinor symmetry expansion

Eigensolutions by geometry for 2D-HO with bilateral (B-Type) symmetry
Symmetric *(low frequency)* mode versus antisymmetric *(high frequency)* mode
Mixed mode beat dynamics *(with constant π/2 phase-lag)*

Eigensolutions by matrix-algebra *with example M=
Secular equation
Hamilton-Cayley equation and projectors
*Idempotent* projectors *(how eigenvalues⇒eigenvectors)*
Operator orthonormality and Completeness *(Idempotent means: P•P=P)*
Spectral Decompositions
Functional spectral decomposition
Introducing $ABCD$ Hamilton Pauli spinor symmetry expansion
Decompose the Hamiltonian operator $\mathbf{H}$ into four $ABCD$ symmetry operators
(Labeled to provide dynamic mnemonics as well as colorful analogies)

$$
\begin{pmatrix}
A & B - iC \\
B + iC & D
\end{pmatrix} = A\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix} + B\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} + C\begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix} + D\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix} = A\sigma_{11} + B\sigma_B + C\sigma_C + D\sigma_{22}
$$

$$
= \frac{A - D}{2}\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix} + B\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} + C\begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix} + \frac{A + D}{2}\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
$$
Introducing $ABCD$ Hamilton Pauli spinor symmetry expansion

Decompose the Hamiltonian operator $\mathbf{H}$ into four $ABCD$ symmetry operators (Labeled to provide dynamic mnemonics as well as colorful analogies)

$$\begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix} = A \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + B \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + D \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = A\mathbf{e}_{11} + B\sigma_B + C\sigma_C + D\mathbf{e}_{22}$$

$$\mathbf{H} = \frac{A - D}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + B \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A + D}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Symmetry archetypes: $A$ (Asymmetric-diagonal) | $B$ (Bilateral-balanced) | $C$ (Chiral-circular-complex-Coriolis-cyclotron-curly...)

Color scheme based on traffic signals

- **STOP** (standing waves)
- **CAUTION** (stretched waves)
- **GO** (moving waves)
Introducing $ABCD$ Hamilton Pauli spinor symmetry expansion

Decompose the Hamiltonian operator $H$ into four $ABCD$ symmetry operators

(Labeled to provide dynamic mnemonics as well as colorful analogies)

$$
\begin{pmatrix}
A & B - iC \\
B + iC & D
\end{pmatrix} =
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix} +
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} +
\begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix} +
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}
= Ae_{11} + B \sigma_B + C \sigma_C + De_{22}
$$

$$
= \frac{A - D}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + B \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A + D}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
$$

$$
H = \frac{A - D}{2} \sigma_A + B \sigma_B + C \sigma_C + \frac{A + D}{2} \sigma_0
$$

Symmetry archetypes: $A$ (Asymmetric-diagonal) | $B$ (Bilateral-balanced) | $C$ (Chiral-circular-complex-Coriolis-cyclotron-curly...)

Color scheme based on traffic signals

- STOP (standing waves)
- CAUTION (stretched waves)
- GO (moving waves)

---

Fig. 3.4.1 Potentials for (a) $C_2^A$-asymmetric-diagonal, (ab) $C_2^{AB}$-mixed, (b) $C_2^B$-bilateral $U(2)$ system.
Review of 1D **FDHO** (*Forced-Damped-Harmonic Oscillator*) response

Amplitude and phase variation due to resonance

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Lagrangian and matrix forms

2D harmonic oscillator equation eigensolutions (normal modes)

2D classical HO compared to U(2) quantum 2-state system

Introducing **ABCD** Hamilton Pauli spinor symmetry expansion

#### Eigensolutions by geometry for 2D-HO with bilateral (B-Type) symmetry

Symmetric (**low frequency**) mode versus antisymmetric (**high frequency**) mode

Mixed mode beat dynamics (with constant $\pi/2$ phase-lag)

Eigensolutions by matrix-algebra with example $M=$

- Secular equation
- Hamilton-Cayley equation and projectors
- Idempotent projectors (how eigenvalues $\Rightarrow$ eigenvectors)
- Operator orthonormality and Completeness (*Idempotent means: $P \cdot P = P$*)

Spectral Decompositions

- Functional spectral decomposition
- Orthonormality vs. Completeness vis-a`-vis Operator vs. State
- Lagrange functional interpolation formula
- Diagonalizing Transformations (D-Ttran) from projectors
2D HO potential energy \( V(x_1, x_2) \) quadratic form defines layers of elliptical \( V \)-contours

\[
V = \frac{1}{2} (k_1 + k_{12}) x_1^2 - k_{12} x_1 x_2 + \frac{1}{2} (k_2 + k_{12}) x_2^2 = \frac{1}{2} \langle x | K | x \rangle = \frac{1}{2} x \cdot K \cdot x = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

What direction \(|x\rangle = |e_n\rangle\) is the same as \(K|x\rangle\)??

Fig. 3.3.4 Plot of potential function \( V(x_1, x_2) \) showing elliptical \( V(x_1,x_2)=\text{const.} \) level curves.
2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical $V$-contours

$$
V = \frac{1}{2} (k_1 + k_{12}) x_1^2 - k_{12} x_1 x_2 + \frac{1}{2} (k_2 + k_{12}) x_2^2 = \frac{1}{2} \langle x | K | x \rangle = \frac{1}{2} x \cdot K \cdot x
$$

$$
= \begin{pmatrix}
  k_1 + k_{12} & -k_{12} \\
  -k_{12} & k_2 + k_{12}
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}
$$

$\mathbf{F} = -\nabla V = -\frac{\partial V}{\partial \mathbf{x}}$

$$
\begin{pmatrix}
  F_1 \\
  F_2
\end{pmatrix}
= \begin{pmatrix}
  -\frac{\partial V}{\partial x_1} \\
  -\frac{\partial V}{\partial x_2}
\end{pmatrix}
$$

Fig. 3.3.4 Plot of potential function $V(x_1, x_2)$ showing elliptical $V(x_1, x_2)=\text{const.}$ level curves.

What direction $|\mathbf{x}\rangle = |e_n\rangle$ is the same as $K|\mathbf{x}\rangle$??
Not most directions!

Force:$\mathbf{F} = -\nabla V$
$= M \cdot \mathbf{a} = -K \cdot \mathbf{x}$

(details of gradient expression)
2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical $V$-contours

$$V = \frac{1}{2}(k_1 + k_{12})x_1^2 - k_{12}x_1x_2 + \frac{1}{2}(k_2 + k_{12})x_2^2 = \frac{1}{2} \langle x | K | x \rangle = \frac{1}{2} x \cdot K \cdot x = \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) \left( \begin{array}{cc} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{array} \right) \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right)$$

What direction $|x\rangle = |e_n\rangle$ is the same as $K|x\rangle$??
Not most directions!
Only extremal axes work. (major or minor axes)

Force: $F = -\nabla V = -\frac{\partial V}{\partial x}$

$$\left( \begin{array}{c} F_1 \\ F_2 \end{array} \right) = \left( \begin{array}{c} -\frac{\partial V}{\partial x_1} \\ -\frac{\partial V}{\partial x_2} \end{array} \right)$$

(details of gradient expression)

Fig. 3.3.4 Plot of potential function $V(x_1, x_2)$ showing elliptical $V(x_1, x_2) = \text{const.}$ level curves.
Review of 1D \textit{FDHO} (\textit{Forced-Damped-Harmonic Oscillator}) response
Amplitude and phase variation due to resonance

2D harmonic oscillator (2D-HO) equations of motion
Lagrangian and matrix forms

2D harmonic oscillator equation eigensolutions (normal modes)
2D classical HO compared to \textit{U(2)} quantum 2-state system
Introducing \textit{ABCD} Hamilton Pauli spinor symmetry expansion

\textbf{Eigensolutions by geometry for 2D-HO with bilateral (B-Type) symmetry}
Symmetric (low frequency) mode versus antisymmetric (high frequency) mode
Mixed mode beat dynamics (with constant $\pi/2$ phase-lag)

\textbf{Eigensolutions by matrix-algebra with example $M=$}
Secular equation
Hamilton-Cayley equation and projectors
Idempotent projectors (how eigenvalues $\Rightarrow$ eigenvectors)
Operator orthonormality and Completeness (Idempotent means: $P \cdot P = P$)
Spectral Decompositions
Functional spectral decomposition
2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical $V$-contours

$$V = \frac{1}{2} \left( k + k_{12} \right) x_1^2 - k_{12} x_1 x_2 + \frac{1}{2} \left( k + k_{12} \right) x_2^2 = \frac{1}{2} \langle x | K | x \rangle = \frac{1}{2} x \cdot K \cdot x = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} k + k_{12} & -k_{12} \\ -k_{12} & k + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(a) PE Contours and gradients

What direction $\langle x \rangle = | e_n \rangle$ is the same as $K | x \rangle$?? Not most directions! Only extremal axes work. (major or minor axes)

(b) Symmetric $U^+$ Coordinate SLOW Mode

(c) Anti-symmetric $U^-$ Coordinate FAST Mode

![Figure 3.3.4 Plot of potential function $V(x_1, x_2)$ showing elliptical $V(x_1, x_2)$=const. level curves.](image)
2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical $V$-contours (Here: $k_1 = k_2$)

$$V = \frac{1}{2}(k + k_{12})x_1^2 - k_{12}x_1x_2 + \frac{1}{2}(k + k_{12})x_2^2 = \frac{1}{2}\langle x | K | x \rangle = \frac{1}{2} x \cdot K \cdot x = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} k + k_{12} & -k_{12} \\ -k_{12} & k + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(a) PE Contours and gradients

What direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ is the same as $|K|\mathbf{x}\rangle$? Not most directions! Only extremal axes work. (major or minor axes)

(b) Symmetric $\mathbf{u}^+$ Coordinate

SLOW Mode

(c) Anti-symmetric $\mathbf{u}^-$ Coordinate

FAST Mode

With Bilateral symmetry ($k_1 = k_2$) the extremal axes lie at $\pm 45^\circ$

Fig. 3.3.5 Topography lines of potential function $V(x_1, x_2)$ and orthogonal $\varepsilon_+$ and $\varepsilon_-$ normal mode slopes
Review of 1D FDHO (Forced-Damped-Harmonic Oscillator) response
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Mixed mode beat dynamics (with constant $\pi/2$ phase-lag)
Geometry of phase and polarization

Eigensolutions by matrix-algebra with example $M=$
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Operator orthonormality and Completeness (Idempotent means: $P \cdot P = P$)
Spectral Decompositions
Functional spectral decomposition

Wednesday, March 23, 2016
B-Type coupling

\[
\begin{pmatrix}
A & B - iC \\
B + iC & D
\end{pmatrix} = 
\begin{pmatrix}
A & B \\
B & A
\end{pmatrix}
\]

Spring set \textit{down} for STRONGER coupling (higher |B|)

Spring set \textit{up} for weaker coupling (lower |B|)
x1 = 0.198
p1/\omega = -0.733
x2 = -0.629
p2/\omega = -0.170

x1(0) = 1.000
p1(0)/\omega = 0.000
x2(0) = 0.000
p2(0)/\omega = 0.000

A = 1.0000
B = -0.0100
C = 0.0000
D = 1.0000

\omega_1 = 0.990
\omega_2 = 1.010
\Theta = 45.000

Stokes Vector ABC-Space

time = 243.210
E = 0.500

http://www.uark.edu/ua/modphys/markup/BoxItWeb.html?
AU2=1.0&BU2=-0.01&CU2=0.0&DU2=1.0&xInitial=1.0&yInitial=0.0&pxInitial=0.0&pyInitial=0.0&wantBoxLines=0&wantPELevels=1&timeMax=330.0
$x_1 = 0.210$
$p_1/\omega = 0.080$
$x_2 = 0.376$
$p_2/\omega = -0.984$

$x_1(0) = 1.000$
$p_1(0)/\omega = 0.000$
$x_2(0) = 0.000$
$p_2(0)/\omega = 0.400$

A = 3.1580
B = -0.1580
C = 0.0000
D = 3.1580

$\omega_1 = 3.000$
$\omega_2 = 3.316$
$\Theta = 45.000$

BoxIt (Beating) Simulation

Stokes Vector ABC Space

move fractions to bottom of slide
Review of 1D FDHO (Forced-Damped-Harmonic Oscillator) response
Amplitude and phase variation due to resonance

2D harmonic oscillator (2D-HO) equations of motion
Lagrangian and matrix forms

2D harmonic oscillator equation eigensolutions (normal modes)
2D classical HO compared to U(2) quantum 2-state system
Introducing ABCD Hamilton Pauli spinor symmetry expansion

Eigensolutions by geometry for 2D-HO with bilateral (B-Type) symmetry
Symmetric (low frequency) mode versus antisymmetric (high frequency) mode
Mixed mode beat dynamics (with constant $\pi/2$ phase-lag)
Geometry of phase and polarization

Eigensolutions by matrix-algebra with example $M=$
Secular equation
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Functional spectral decomposition
2D-HO beats and mixed mode geometry

A “visualization gauge”
We hold these two fixed...

(a) \[ r^0 (\phi = 0) \quad r^1 (\phi = \pi) \]

0\[2\]
\[0^\circ\]
\[0^\circ\]

1\[2\]
\[0^\circ\]
\[180^\circ\]

(b) \[ t = 0 \]
\[ t = 1/12 \]

1/4 revivals or beats

1/2

3/4

\[ r^0 = \sqrt{2} \]
\[ r^1 = \sqrt{2} \]

\( |+\rangle + i|\rangle \)
\( \sqrt{2} \)

\( |+\rangle - i|\rangle \)
\( \sqrt{2} \)

\( |+\rangle + i|\rangle \)
\( \sqrt{2} \)

Coupled Pendula even

Optical even

parity states odd

even

odd

localized

flipped

Wednesday, March 23, 2016
2D-HO beats and mixed mode geometry

A “visualization gauge”
We hold these two fixed...

and let these two rotate at beat frequency

(a) \[
\begin{align*}
&\mathbf{r}^0 (\phi = 0) \\
&\mathbf{r}^1 (\phi = \pi)
\end{align*}
\]

(b)

We hold these two fixed...

and let these two rotate at beat frequency

\[t = 1/12\]

|+\rangle C_2 parity states odd -45°

|−\rangle

|+\rangle +|−\rangle \sqrt{2}

|+\rangle -i|−\rangle \sqrt{2}

|+\rangle +i|−\rangle \sqrt{2}

|+\rangle +|−\rangle \sqrt{2}

Localized

x

L

y

R

Revivals

or beats

1/4

1/2

3/4
2D-HO beats and mixed mode geometry

A “visualization gauge”

We hold these two fixed...

and let these two rotate at beat frequency

(a) \( r^0 \ (\phi = 0) \) \( r^1 \ (\phi = \pi) \)

\[ t = 1/12 \]

\[ 0 \]

\[ 1 \]

(b) \[ t = 0 \]

\[ 1/4 \]

\[ 1/2 \]

\[ 3/4 \]

\[ \phi = 0 \]

\[ \phi = \pi \]

Coupled Pendula

Optical \( E(t) \)

| + ⟩

| − ⟩

\( C_2 \)

parity states

odd

-45°

localized

| + ⟩ + − ⟩ \( \sqrt{2} \)

| + ⟩ − i| − ⟩ \( \sqrt{2} \)

| + ⟩ i| − ⟩ \( \sqrt{2} \)

flipped

x

L

y

R

\( \theta = 45° \)

\( \theta = -45° \)

revivals or beats

Wednesday, March 23, 2016
2D-HO beats and mixed mode geometry

A “visualization gauge”
We hold these two fixed...

and let these two rotate at beat frequency

(a) \( r^0 (\phi = 0) \) \( r^1 (\phi = \pi) \)

\[ t = 1/12 \]

\[ t = 1/6 \]

\[ t = 0 \]

1/4 revivals or beats

\[ 1/2 \]

\[ 3/4 \]

(b) localized

\[ \frac{1}{\sqrt{2}} \]

\[ \frac{1}{\sqrt{2}} \]

\[ \frac{1}{\sqrt{2}} \]

\[ \frac{1}{\sqrt{2}} \]

\[ \sqrt{2} \]

\[ \sqrt{2} \]

\[ \sqrt{2} \]

\[ \sqrt{2} \]

Coupled Pendula

Optical \( E(t) \)

parity states

even

odd

\( \uparrow \rangle \)

\( \downarrow \rangle \)

\( \uparrow \rangle \)

\( \downarrow \rangle \)

\( \uparrow \rangle + |\downarrow \rangle \frac{1}{\sqrt{2}} \)

\( \uparrow \rangle - i|\downarrow \rangle \frac{1}{\sqrt{2}} \)

\( \uparrow \rangle + i|\downarrow \rangle \frac{1}{\sqrt{2}} \)

\( \uparrow \rangle - |\downarrow \rangle \frac{1}{\sqrt{2}} \)

\( \downarrow \rangle + |\uparrow \rangle \frac{1}{\sqrt{2}} \)

\( \downarrow \rangle - i|\uparrow \rangle \frac{1}{\sqrt{2}} \)

\( \downarrow \rangle + i|\uparrow \rangle \frac{1}{\sqrt{2}} \)

\( \uparrow \rangle - |\downarrow \rangle \frac{1}{\sqrt{2}} \)

\( \uparrow \rangle + i|\downarrow \rangle \frac{1}{\sqrt{2}} \)

\( \downarrow \rangle + |\uparrow \rangle \frac{1}{\sqrt{2}} \)

\( \downarrow \rangle - i|\uparrow \rangle \frac{1}{\sqrt{2}} \)

\( \uparrow \rangle + |\downarrow \rangle \frac{1}{\sqrt{2}} \)

\( \downarrow \rangle + i|\uparrow \rangle \frac{1}{\sqrt{2}} \)

\( \uparrow \rangle - |\downarrow \rangle \frac{1}{\sqrt{2}} \)

\( \uparrow \rangle + i|\downarrow \rangle \frac{1}{\sqrt{2}} \)

\( \downarrow \rangle + |\uparrow \rangle \frac{1}{\sqrt{2}} \)

\( \downarrow \rangle - i|\uparrow \rangle \frac{1}{\sqrt{2}} \)

\( \uparrow \rangle + |\downarrow \rangle \frac{1}{\sqrt{2}} \)

\( \downarrow \rangle + i|\uparrow \rangle \frac{1}{\sqrt{2}} \)

\( \uparrow \rangle - |\downarrow \rangle \frac{1}{\sqrt{2}} \)

\( \uparrow \rangle + i|\downarrow \rangle \frac{1}{\sqrt{2}} \)

\( \downarrow \rangle + |\uparrow \rangle \frac{1}{\sqrt{2}} \)

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An eigenvector $|\varepsilon_k\rangle$ of $\mathbf{M}$ is in a direction that is left unchanged by $\mathbf{M}$.

$$\mathbf{M}|\varepsilon_k\rangle = \varepsilon_k|\varepsilon_k\rangle,$$ or: $$(\mathbf{M} - \varepsilon_k \mathbf{1})|\varepsilon_k\rangle = 0$$

$\varepsilon_k$ is eigenvalue associated with eigenvector $|\varepsilon_k\rangle$ direction.

A change of basis to \{\{\varepsilon_1\},\{\varepsilon_2\},\ldots,\{\varepsilon_n\}\} called diagonalization gives

$$\left(\begin{array}{cccc}
\langle \varepsilon_1 | \mathbf{M} | \varepsilon_1 \rangle & \langle \varepsilon_1 | \mathbf{M} | \varepsilon_2 \rangle & \cdots & \langle \varepsilon_1 | \mathbf{M} | \varepsilon_n \rangle \\
\langle \varepsilon_2 | \mathbf{M} | \varepsilon_1 \rangle & \langle \varepsilon_2 | \mathbf{M} | \varepsilon_2 \rangle & \cdots & \langle \varepsilon_2 | \mathbf{M} | \varepsilon_n \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle \varepsilon_n | \mathbf{M} | \varepsilon_1 \rangle & \langle \varepsilon_n | \mathbf{M} | \varepsilon_2 \rangle & \cdots & \langle \varepsilon_n | \mathbf{M} | \varepsilon_n \rangle
\end{array}\right) \left(\begin{array}{c}
\varepsilon_1 \\
\varepsilon_2 \\
\vdots \\
\varepsilon_n
\end{array}\right) = \left(\begin{array}{c}
\varepsilon_1 \\
0 \\
\vdots \\
0
\end{array}\right)$$

**Matrix-algebraic method for finding eigenvector and eigenvalues**

*With example matrix $\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$*

$$\mathbf{M}|\varepsilon\rangle = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \varepsilon \begin{pmatrix} x \\ y \end{pmatrix} \text{ or: } \begin{pmatrix} 4 - \varepsilon & 1 \\ 3 & 2 - \varepsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
An eigenvector $|\epsilon_k\rangle$ of $M$ is in a direction that is left unchanged by $M$.

$$M|\epsilon_k\rangle = \epsilon_k|\epsilon_k\rangle, \quad \text{or:} \quad (M - \epsilon_k \mathbf{1})|\epsilon_k\rangle = 0$$

$\epsilon_k$ is eigenvalue associated with eigenvector $|\epsilon_k\rangle$ direction.

A change of basis to $\{|\epsilon_1\rangle, |\epsilon_2\rangle, \ldots, |\epsilon_n\rangle\}$ called diagonalization gives

$$
\begin{pmatrix}
  \langle \epsilon_1 | M | \epsilon_1 \rangle & \langle \epsilon_1 | M | \epsilon_2 \rangle & \cdots & \langle \epsilon_1 | M | \epsilon_n \rangle \\
  \langle \epsilon_2 | M | \epsilon_1 \rangle & \langle \epsilon_2 | M | \epsilon_2 \rangle & \cdots & \langle \epsilon_2 | M | \epsilon_n \rangle \\
  \vdots & \vdots & \ddots & \vdots \\
  \langle \epsilon_n | M | \epsilon_1 \rangle & \langle \epsilon_n | M | \epsilon_2 \rangle & \cdots & \langle \epsilon_n | M | \epsilon_n \rangle
\end{pmatrix}
\begin{pmatrix}
  \epsilon_1 \\
  \epsilon_2 \\
  \vdots \\
  \epsilon_n
\end{pmatrix}
= \begin{pmatrix}
  \epsilon_1 & 0 & \cdots & 0 \\
  0 & \epsilon_2 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & \epsilon_n
\end{pmatrix}
$$

Matrix-algebraic method for finding eigenvector and eigenvalues

With example matrix

$$M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$$

$$M|\epsilon\rangle = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \epsilon \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{or:} \quad \begin{pmatrix} 4 - \epsilon & 1 \\ 3 & 2 - \epsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Trying to solve by Kramer's inversion:

$$x = \frac{\det \begin{pmatrix} 0 & 1 \\ 0 & 2 - \epsilon \end{pmatrix}}{\det \begin{pmatrix} 4 - \epsilon & 1 \\ 3 & 2 - \epsilon \end{pmatrix}} \quad \text{and} \quad y = \frac{\det \begin{pmatrix} 4 - \epsilon & 0 \\ 3 & 0 \end{pmatrix}}{\det \begin{pmatrix} 4 - \epsilon & 1 \\ 3 & 2 - \epsilon \end{pmatrix}}$$
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Eigensolutions by matrix-algebra with example M=
\[
\begin{pmatrix}
4 & 1 \\
3 & 2
\end{pmatrix}
\]

- Secular equation
- Hamilton-Cayley equation and projectors
  - Idempotent projectors (how eigenvalues ⇒ eigenvectors)
  - Operator orthonormality and Completeness (Idempotent means: P·P=P)
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Matrix-algebraic method for finding eigenvector and eigenvalues

An **eigenvector** \(|\epsilon_k\rangle\) of \(M\) is in a direction that is left unchanged by \(M\).

\[ M|\epsilon_k\rangle = \epsilon_k|\epsilon_k\rangle, \quad \text{or:} \quad (M - \epsilon_k \mathbf{1})|\epsilon_k\rangle = 0 \]

\(\epsilon_k\) is **eigenvalue** associated with eigenvector \(|\epsilon_k\rangle\) direction.

A change of basis to \(|\epsilon_1\rangle, |\epsilon_2\rangle, \ldots, |\epsilon_n\rangle\} \) called **diagonalization** gives

\[
\begin{bmatrix}
  \langle \epsilon_1|M|\epsilon_1 \rangle & \langle \epsilon_1|M|\epsilon_2 \rangle & \cdots & \langle \epsilon_1|M|\epsilon_n \rangle \\
  \langle \epsilon_2|M|\epsilon_1 \rangle & \langle \epsilon_2|M|\epsilon_2 \rangle & \cdots & \langle \epsilon_2|M|\epsilon_n \rangle \\
  \vdots & \vdots & \ddots & \vdots \\
  \langle \epsilon_n|M|\epsilon_1 \rangle & \langle \epsilon_n|M|\epsilon_2 \rangle & \cdots & \langle \epsilon_n|M|\epsilon_n \rangle \\
\end{bmatrix}
\begin{bmatrix}
  \epsilon_1 \\
  \epsilon_2 \\
  \vdots \\
  \epsilon_n \\
\end{bmatrix}
= \begin{bmatrix}
  \epsilon_1 & 0 & \cdots & 0 \\
  0 & \epsilon_2 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & \epsilon_n \\
\end{bmatrix}
\]

First step in finding eigenvalues: Solve **secular equation**

\[
\det[M - \epsilon \mathbf{1}] = 0 = (-1)^n \left( \epsilon^n + a_1 \epsilon^{n-1} + a_2 \epsilon^{n-2} + \ldots + a_{n-1} \epsilon + a_n \right)
\]

where:

\[
a_i = -\text{Trace}(M) \cdots, \quad a_k = (-1)^k \sum \text{diagonal k-by-k minors of } M, \cdots, \quad a_n = (-1)^n \det[M]
\]

**With example matrix** \(M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}\)

\[
M|\epsilon\rangle = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \epsilon \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{or:} \quad \begin{pmatrix} 4-\epsilon & 1 \\ 3 & 2-\epsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

Trying to solve by Kramer's inversion:

\[
x = \frac{\det\left[ \begin{array}{cc} 0 & 1 \\ 0 & 2-\epsilon \end{array} \right]}{\det\left[ \begin{array}{cc} 4-\epsilon & 1 \\ 3 & 2-\epsilon \end{array} \right]} \quad \text{and} \quad y = \frac{\det\left[ \begin{array}{cc} 4-\epsilon & 0 \\ 3 & 0 \end{array} \right]}{\det\left[ \begin{array}{cc} 4-\epsilon & 1 \\ 3 & 2-\epsilon \end{array} \right]}
\]

Only possible non-zero \(\{x,y\}\) if denominator is zero, too!

\[
0 = \det[M - \epsilon \cdot \mathbf{1}] = \det\left[ \begin{array}{cc} 4 & 1 \\ 3 & 2 \end{array} \right] - \epsilon \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \det\left[ \begin{array}{cc} 4-\epsilon & 1 \\ 3 & 2-\epsilon \end{array} \right] = 0 = (4-\epsilon)(2-\epsilon) - 1 \cdot 3 = 8 - 6\epsilon + \epsilon^2 - 1 \cdot 3 = \epsilon^2 - 6\epsilon + 5
\]

\[
0 = \epsilon^2 - \text{Trace}(M)\epsilon + \det(M)
\]
Matrix-algebraic method for finding eigenvector and eigenvalues

With example matrix $M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

An **eigenvector** $|\varepsilon_k\rangle$ of $M$ is in a direction that is left unchanged by $M$.

$$M|\varepsilon_k\rangle = \varepsilon_k|\varepsilon_k\rangle, \text{ or } (M - \varepsilon_k \cdot 1)|\varepsilon_k\rangle = 0$$

$\varepsilon_k$ is **eigenvalue** associated with eigenvector $|\varepsilon_k\rangle$ direction.

A change of basis to $\{|\varepsilon_1\rangle, |\varepsilon_2\rangle, \ldots, |\varepsilon_n\rangle\}$ called **diagonalization** gives

$$\begin{pmatrix}
\langle \varepsilon_1 | M | \varepsilon_1 \rangle & \langle \varepsilon_1 | M | \varepsilon_2 \rangle & \cdots & \langle \varepsilon_1 | M | \varepsilon_n \rangle \\
\langle \varepsilon_2 | M | \varepsilon_1 \rangle & \langle \varepsilon_2 | M | \varepsilon_2 \rangle & \cdots & \langle \varepsilon_2 | M | \varepsilon_n \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle \varepsilon_n | M | \varepsilon_1 \rangle & \langle \varepsilon_n | M | \varepsilon_2 \rangle & \cdots & \langle \varepsilon_n | M | \varepsilon_n \rangle 
\end{pmatrix}
= \begin{pmatrix}
\varepsilon_1 & 0 & \cdots & 0 \\
0 & \varepsilon_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \varepsilon_n 
\end{pmatrix}$$

First step in finding eigenvalues: Solve **secular equation**

$$\det[M - \varepsilon \cdot 1] = 0 = (-1)^n \left( \varepsilon^n + a_1 \varepsilon^{n-1} + a_2 \varepsilon^{n-2} + \ldots + a_{n-1} \varepsilon + a_n \right)$$

where:

$$a_i = -\text{Trace} M, \ldots, a_k = (-1)^k \sum \text{diagonal k-by-k minors of } M, \ldots, a_n = (-1)^n \det[M]$$

Secular equation has $n$-factors, one for each eigenvalue.

$$\det[M - \varepsilon \cdot 1] = 0 = (-1)^n (\varepsilon - \varepsilon_1)(\varepsilon - \varepsilon_2) \cdots (\varepsilon - \varepsilon_n)$$

$$\text{Trying to solve by Kramer's inversion:}$$

$$\begin{pmatrix} 0 & 1 \\ 4 - \varepsilon & 2 - \varepsilon \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \text{ or: } \begin{pmatrix} 4 - \varepsilon & 1 \\ 3 & 2 - \varepsilon \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Only possible non-zero $\{x,y\}$ if denominator is zero, too!

$$0 = \det[M - \varepsilon \cdot 1] = \det\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} - \varepsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \det\begin{pmatrix} 4 - \varepsilon & 1 \\ 3 & 2 - \varepsilon \end{pmatrix}$$

$$0 = (4 - \varepsilon)(2 - \varepsilon) - 1 \cdot 3 = 8 - 6\varepsilon + \varepsilon^2 - 1 \cdot 3 = \varepsilon^2 - 6\varepsilon + 5$$

$$0 = \varepsilon^2 - \text{Trace}(M)\varepsilon + \det(M) = \varepsilon^2 - 6\varepsilon + 5$$

$$0 = (\varepsilon - 1)(\varepsilon - 5) \text{ so let: } \varepsilon_1 = 1 \text{ and: } \varepsilon_2 = 5$$
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\[ M|\varepsilon_k\rangle = \varepsilon_k|\varepsilon_k\rangle, \quad \text{or: } (M - \varepsilon_k I)|\varepsilon_k\rangle = 0 \]

\(\varepsilon_k\) is eigenvalue associated with eigenvector \(|\varepsilon_k\rangle\) direction.

A change of basis to \(\{|e_1\rangle, |e_2\rangle, \ldots, |e_n\rangle\}\) called diagonalization gives

\[
\begin{pmatrix}
    \langle e_1|M|e_1\rangle & \langle e_1|M|e_2\rangle & \cdots & \langle e_1|M|e_n\rangle \\
    \langle e_2|M|e_1\rangle & \langle e_2|M|e_2\rangle & \cdots & \langle e_2|M|e_n\rangle \\
    \vdots & \vdots & \ddots & \vdots \\
    \langle e_n|M|e_1\rangle & \langle e_n|M|e_2\rangle & \cdots & \langle e_n|M|e_n\rangle
\end{pmatrix}
\begin{pmatrix}
    \varepsilon_1 \\
    \varepsilon_2 \\
    \vdots \\
    \varepsilon_n
\end{pmatrix}
= \begin{pmatrix}
    \varepsilon_1 \\
    0 \\
    \vdots \\
    0
\end{pmatrix}
\]

First step in finding eigenvalues: Solve secular equation

\[ \det|M - \varepsilon I| = 0 = (-1)^n \left( \varepsilon^n + a_1 \varepsilon^{n-1} + a_2 \varepsilon^{n-2} + \ldots + a_{n-1} \varepsilon + a_n \right) \]

where:

\[ a_1 = -\text{Trace}M, \ldots, a_k = (-1)^k \sum \text{diagonal k-by-k minors of } M, \ldots, a_n = (-1)^n \det|M| \]

Secular equation has \(n\)-factors, one for each eigenvalue.

\[ \det|M - \varepsilon I| = 0 = (-1)^n (\varepsilon - \varepsilon_1)(\varepsilon - \varepsilon_2)\cdots(\varepsilon - \varepsilon_n) \]

Each \(\varepsilon\) replaced by \(M\) and each \(\varepsilon_k\) by \(\varepsilon_k I\) gives Hamilton-Cayley matrix equation.

\[ 0 = (M - \varepsilon_1 I)(M - \varepsilon_2 I)\cdots(M - \varepsilon_n I) \]

Obviously true if \(M\) has diagonal form. (But, that’s circular logic. Faith needed!)
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\[ \text{Eigensolutions by matrix-algebra with example } M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \]

- Secular equation
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An **eigenvector** \( |\epsilon_k\rangle \) of \( M \) is in a direction that is left unchanged by \( M \).

\[
M|\epsilon_k\rangle = \epsilon_k|\epsilon_k\rangle, \quad \text{or:} \quad (M - \epsilon_k I)|\epsilon_k\rangle = 0
\]

\( \epsilon_k \) is **eigenvalue** associated with eigenvector \( |\epsilon_k\rangle \) direction.

A change of basis to \( \{ |\epsilon_1\rangle, |\epsilon_2\rangle, \ldots |\epsilon_n\rangle \} \) called **diagonalization** gives

\[
\begin{pmatrix}
\langle \epsilon_1 | M | \epsilon_1 \rangle & \langle \epsilon_1 | M | \epsilon_2 \rangle & \cdots & \langle \epsilon_1 | M | \epsilon_n \rangle \\
\langle \epsilon_2 | M | \epsilon_1 \rangle & \langle \epsilon_2 | M | \epsilon_2 \rangle & \cdots & \langle \epsilon_2 | M | \epsilon_n \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle \epsilon_n | M | \epsilon_1 \rangle & \langle \epsilon_n | M | \epsilon_2 \rangle & \cdots & \langle \epsilon_n | M | \epsilon_n \rangle
\end{pmatrix} =
\begin{pmatrix}
\epsilon_1 & 0 & \cdots & 0 \\
0 & \epsilon_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \epsilon_n
\end{pmatrix}
\]

First step in finding eigenvalues: Solve **secular equation**

\[
\det[M - \epsilon I] = 0 = (-1)^n \left( \epsilon^n + a_1 \epsilon^{n-1} + a_2 \epsilon^{n-2} + \ldots + a_{n-1} \epsilon + a_n \right)
\]

where:

\[
a_1 = -\text{Trace} M, \ldots, a_k = (-1)^k \sum \text{diagonal k-by-k minors of } M, \ldots, a_n = (-1)^n \det[M]
\]

Secular equation has \( n \)-factors, one for each eigenvalue.

\[
\det[M - \epsilon I] = 0 = (-1)^n (\epsilon - \epsilon_1)(\epsilon - \epsilon_2)\cdots(\epsilon - \epsilon_n)
\]

Each \( \epsilon \) replaced by \( M \) and each \( \epsilon_k \) by \( \epsilon_k I \) gives **Hamilton-Cayley** matrix equation.

\[
0 = (M - \epsilon_1 I)(M - \epsilon_2 I)\cdots(M - \epsilon_n I)
\]

Obviously true if \( M \) has diagonal form. (But, that’s circular logic. Faith needed!)

Replace \( j \)th HC-factor by (1) to make **projection operators**

\[
p_1 = (1) (M - \epsilon_1 I) \cdots (M - \epsilon_j I)
\]

\[
p_2 = (M - \epsilon_1 I) (1) \cdots (M - \epsilon_j I)
\]

\[
p_n = (M - \epsilon_1 I) (M - \epsilon_2 I) \cdots (1)
\]

With example matrix \( M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \)

\[
M|\epsilon\rangle = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \epsilon \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{or:} \quad \begin{pmatrix} 4 - \epsilon & 1 \\ 3 & 2 - \epsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

Trying to solve by Kramer’s inversion:

\[
\begin{pmatrix} 0 & 1 \\ 0 & 2 - \epsilon \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 4 - \epsilon & 1 \\ 3 & 2 - \epsilon \end{pmatrix}
\]

Only possible non-zero \( \{x, y\} \) if denominator is zero, too!

\[
0 = \det[M - \epsilon I] = \det \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} - \epsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \det \begin{pmatrix} 4 - \epsilon & 1 \\ 3 & 2 - \epsilon \end{pmatrix}
\]

\[
0 = (4 - \epsilon)(2 - \epsilon) - 1 \cdot 3 = 8 - 6\epsilon + \epsilon^2 - 1 \cdot 3 = \epsilon^2 - 6\epsilon + 5
\]

\[
0 = \epsilon^2 - \text{Trace}(M)\epsilon + \det(M) = \epsilon^2 - 6\epsilon + 5
\]

\[
0 = (\epsilon - 1)(\epsilon - 5) \quad \text{so let:} \quad \epsilon_1 = 1 \quad \text{and:} \quad \epsilon_2 = 5
\]

\[
0 = M^2 - 6M + 51 = (M - 1)(M - 5)I
\]

\[
\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}^2 - 6 \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} + 5 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
p_1 = (1)(M - 5 I) = \begin{pmatrix} 4 - 5 & 1 \\ 3 & 2 - 5 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}
\]

\[
p_2 = (M - 1 I)(1) = \begin{pmatrix} 4 - 1 & 1 \\ 3 & 2 - 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}
\]
Matrix-algebraic method for finding eigenvector and eigenvalues

An eigenvector \( |\epsilon_k\rangle \) of \( M \) is in a direction that is left unchanged by \( M \).

\[ M|\epsilon_k\rangle = \epsilon_k |\epsilon_k\rangle, \quad \text{or: } (M - \epsilon_k I)|\epsilon_k\rangle = 0 \]

\( \epsilon_k \) is eigenvalue associated with eigenvector \( |\epsilon_k\rangle \) direction.

A change of basis to \( \{|\epsilon_1\rangle, |\epsilon_2\rangle, \ldots, |\epsilon_n\rangle\} \) called diagonalization gives

\[
\begin{pmatrix}
|\epsilon_1\rangle M |\epsilon_1\rangle & |\epsilon_1\rangle M |\epsilon_2\rangle & \cdots & |\epsilon_1\rangle M |\epsilon_n\rangle \\
|\epsilon_2\rangle M |\epsilon_1\rangle & |\epsilon_2\rangle M |\epsilon_2\rangle & \cdots & |\epsilon_2\rangle M |\epsilon_n\rangle \\
\vdots & \vdots & \ddots & \vdots \\
|\epsilon_n\rangle M |\epsilon_1\rangle & |\epsilon_n\rangle M |\epsilon_2\rangle & \cdots & |\epsilon_n\rangle M |\epsilon_n\rangle
\end{pmatrix}
= 
\begin{pmatrix}
\epsilon_1 & 0 & \cdots & 0 \\
0 & \epsilon_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \epsilon_n
\end{pmatrix}
\]

First step in finding eigenvalues: Solve secular equation

\[ \det[M - \epsilon I] = 0 = (-1)^n \left( \epsilon^n + a_1 \epsilon^{n-1} + a_2 \epsilon^{n-2} + \ldots + a_{n-1} \epsilon + a_n \right) \]

where:

\[ a_i = -\text{Trace}M, \ldots, a_k = (-1)^k \sum \text{diagonal k-by-k minors of } M, \ldots, a_n = (-1)^n \det[M] \]

Secular equation has \( n \)-factors, one for each eigenvalue.

\[ \det[M - \epsilon I] = 0 = (-1)^n (\epsilon - \epsilon_1)(\epsilon - \epsilon_2)\cdots(\epsilon - \epsilon_n) \]

Each \( \epsilon \) replaced by \( M \) and each \( \epsilon_k \) by \( \epsilon_k I \) gives Hamilton-Cayley matrix equation.

\[ 0 = (M - \epsilon_1 I)(M - \epsilon_2 I)\cdots(M - \epsilon_n I) \]

Obviously true if \( M \) has diagonal form. (But, that’s circular logic. Faith needed!)

Replace \( j \)-th HC-factor by (1) to make projection operators

\[ p_k = \prod_{j \neq k} (M - \epsilon_j I) \]

\[ p_1 = (1)(M - \epsilon_2 I)\cdots(M - \epsilon_n I) \\
p_2 = (M - \epsilon_1 I)(1)\cdots(M - \epsilon_n I) \] (Assume distinct \( \epsilon \)-values here: Non-degeneracy clause

\[ \epsilon_j \neq \epsilon_k \neq \ldots \]

Each \( p_k \) contains eigen-bra-kets since: \( (M - \epsilon_k I)p_k = 0 \) or: \( Mp_k = \epsilon_k p_k = p_k M \).
Matrix-algebraic method for finding eigenvector and eigenvalues

An eigenvector $|\epsilon_k\rangle$ of $M$ is in a direction that is left unchanged by $M$.

$$M|\epsilon_k\rangle = \epsilon_k |\epsilon_k\rangle,$$ or: $(M - \epsilon_k 1)|\epsilon_k\rangle = 0$

$\epsilon_k$ is eigenvalue associated with eigenvector $|\epsilon_k\rangle$ direction.

A change of basis to $\{|\epsilon_1\rangle, |\epsilon_2\rangle, \ldots, |\epsilon_n\rangle\}$ called diagonalization gives

$$
\begin{bmatrix}
\langle \epsilon_1 | M | \epsilon_1 \rangle & \langle \epsilon_1 | M | \epsilon_2 \rangle & \cdots & \langle \epsilon_1 | M | \epsilon_n \rangle \\
\langle \epsilon_2 | M | \epsilon_1 \rangle & \langle \epsilon_2 | M | \epsilon_2 \rangle & \cdots & \langle \epsilon_2 | M | \epsilon_n \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle \epsilon_n | M | \epsilon_1 \rangle & \langle \epsilon_n | M | \epsilon_2 \rangle & \cdots & \langle \epsilon_n | M | \epsilon_n \rangle
\end{bmatrix}
= 
\begin{bmatrix}
\epsilon_1 & 0 & \cdots & 0 \\
0 & \epsilon_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \epsilon_n
\end{bmatrix}
$$

First step in finding eigenvalues: Solve secular equation

$$\det[M - \epsilon 1] = 0 = (-1)^n \left( \epsilon^n + a_1 \epsilon^{n-1} + a_2 \epsilon^{n-2} + \ldots + a_{n-1} \epsilon + a_n \right)$$

where:

$$a_i = -\text{Trace} M, \ldots, a_k = (-1)^k \sum \text{diagonal } k\text{-by-}k \text{ minors of } M, \ldots, a_n = (-1)^n \det[M]$$

Secular equation has $n$ factors, one for each eigenvalue.

$$\det[M - \epsilon 1] = 0 = (-1)^n (\epsilon - \epsilon_1)(\epsilon - \epsilon_2)\ldots(\epsilon - \epsilon_n)$$

Each $\epsilon$ replaced by $M$ and each $\epsilon_k$ by $\epsilon_k 1$ gives Hamilton-Cayley matrix equation.

$$0 = (M - \epsilon_1 1)(M - \epsilon_2 1)\ldots(M - \epsilon_n 1)$$

Obviously true if $M$ has diagonal form. (But, that’s circular logic. Faith needed!)

Replace $j^{th}$ HC-factor by (1) to make projection operators

$$p_k = \prod_{j \neq k} (M - \epsilon_j 1)$$

$$p_1 = (1\quad 0) (M - \epsilon_2 1)\ldots(M - \epsilon_n 1)$$

$$p_2 = (M - \epsilon_1 1) (1\quad 0) \ldots(M - \epsilon_n 1)$$

Assume distinct $\epsilon$-values here: Non-degeneracy clause

$$\epsilon_j \neq \epsilon_k \neq \ldots$$

Accordingly, $p_k$ commutes with $M$,

$$M p_k = \epsilon_k p_k = p_k M$$

since $M^1, M^2, \ldots$ commute with $M$.

With example matrix

$$M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$$

$$M|\epsilon\rangle = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \epsilon \begin{pmatrix} x \\ y \end{pmatrix}$$

or:

$$M = \begin{pmatrix} 4 - \epsilon & 1 \\ 3 & 2 - \epsilon \end{pmatrix}$$

$$M|\epsilon\rangle = \begin{pmatrix} 4 - \epsilon & 1 \\ 3 & 2 - \epsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Trying to solve by Kramer’s inversion:

$$x = \frac{\det \begin{pmatrix} 0 & 1 \\ 3 & 2 - \epsilon \end{pmatrix}}{\det \begin{pmatrix} 4 - \epsilon & 1 \\ 3 & 2 - \epsilon \end{pmatrix}}$$

and

$$y = \frac{\det \begin{pmatrix} 4 - \epsilon & 1 \\ 3 & 2 - \epsilon \end{pmatrix}}{\det \begin{pmatrix} 4 - \epsilon & 0 \\ 3 & 2 - \epsilon \end{pmatrix}}$$

Only possible non-zero $\{x,y\}$ if denominator is zero, too!

$$0 = \det[M - \epsilon 1] = \det \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} - \epsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \det \begin{pmatrix} 4 - \epsilon & 1 \\ 3 & 2 - \epsilon \end{pmatrix}$$

$$0 = (4 - \epsilon)(2 - \epsilon) - 1\cdot 3 = 8 - 6\epsilon + \epsilon^2 - 1\cdot 3 = \epsilon^2 - 6\epsilon + 5$$

$$0 = \epsilon^2 - \text{Trace}(M) \epsilon + \det(M) = \epsilon^2 - 6\epsilon + 5$$

$$0 = (\epsilon - 1)(\epsilon - 5)$$

so let: $\epsilon_1 = 1$ and: $\epsilon_2 = 5$

$$0 = M^2 - 6M + 51 = (M - 1\cdot 1)(M - 5\cdot 1)$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}^2 - 6 \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} + 5 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$p_1 = (1\quad 0)(M - 5\cdot 1) = \begin{pmatrix} 4 & 5 \\ 3 & 2 - 5 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}$$

$$p_2 = (M - 1\cdot 1) = \begin{pmatrix} 4 & 1 \\ 3 & 2 - 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

$$M p_1 = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & -3 \end{pmatrix} = 1\cdot p_1$$

$$M p_2 = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = 5 \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = 5\cdot p_2$$
Review of 1D FDHO (Forced-Damped-Harmonic Oscillator) response
Amplitude and phase variation due to resonance

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Lagrangian and matrix forms

2D harmonic oscillator equation eigensolutions (normal modes)
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Mixed mode beat dynamics (with constant $\pi/2$ phase-lag)
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\textbf{Idempotent} projectors (low eigenvalues $\Rightarrow$ eigenvectors)
Operator orthonormality and Completeness (Idempotent means: $P \cdot P = P$)
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Functional spectral decomposition
Matrix-algebraic method for finding eigenvector and eigenvalues

\[ p_j p_k = p_j \prod_{m \neq k} (M - \varepsilon_m 1) = \prod_{m \neq k} (p_j M - \varepsilon_m p_j 1) \]

Multiplication properties of \( p_j \):

\[ p_j p_k = \prod_{m \neq k} (\varepsilon_j p_j - \varepsilon_m p_j) = \prod_{m \neq k} (\varepsilon_j - \varepsilon_m) = \begin{cases} 
0 & \text{if } j \neq k \\
\prod_{m \neq k} (\varepsilon_k - \varepsilon_m) & \text{if } j = k 
\end{cases} \]

With example matrix

\[ M = \begin{pmatrix} 
4 & 1 \\
3 & 2 
\end{pmatrix} \]

\[ p_1 = (M - 5 \cdot 1) = \begin{pmatrix} 
-1 & 1 \\
3 & -3 
\end{pmatrix} \]

\[ p_2 = (M - 1 \cdot 1) = \begin{pmatrix} 
3 & 1 \\
3 & 1 
\end{pmatrix} \]

\[ p_1 p_2 = \begin{pmatrix} 
0 & 0 \\
0 & 0 
\end{pmatrix} \]
Matrix-algebraic method for finding eigenvector and eigenvalues

Multiplication properties of $p_j$:

$$p_j p_k = \prod_{m \neq k} (\varepsilon_j p_j - \varepsilon_m p_j) = p_j \prod_{m \neq k} (\varepsilon_j - \varepsilon_m)$$

Last step:

make **Idempotent Projectors**: $P_k = \prod_{m \neq k} (\varepsilon_k - \varepsilon_m)$

(Idempotent means: $P \cdot P = P$)

With example matrix $M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

$$p_1 = (M - 5 \cdot 1) = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}$$

$$p_2 = (M - 1 \cdot 1) = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

$$P_1 = \frac{(M - 5 \cdot 1)}{(1 - 5)} = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix}$$

$$P_2 = \frac{(M - 1 \cdot 1)}{(5 - 1)} = \frac{1}{4} \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$
Matrix-algebraic method for finding eigenvector and eigenvalues

\[ \mathbf{p}_j \mathbf{p}_k = \mathbf{p}_j \prod_{m \neq k} (\mathbf{M} - \varepsilon_m \mathbf{1}) = \prod_{m \neq k} (\mathbf{p}_j \mathbf{M} - \varepsilon_m \mathbf{p}_j \mathbf{1}) \quad \text{M} \mathbf{p}_k = \varepsilon_k \mathbf{p}_k = \mathbf{p}_k \mathbf{M} \]

Multiplication properties of \( \mathbf{p}_j \):

\[ \mathbf{p}_j \mathbf{p}_k = \prod_{m \neq k} (\varepsilon_j \mathbf{p}_j - \varepsilon_m \mathbf{p}_j) = \mathbf{p}_j \prod_{m \neq k} (\varepsilon_j - \varepsilon_m) = \begin{cases} 0 & \text{if } j \neq k \\ \mathbf{p}_k \prod_{m \neq k} (\varepsilon_k - \varepsilon_m) & \text{if } j = k \end{cases} \]

Last step:
make **Idempotent Projectors**: \( \mathbf{P}_k = \prod_{m \neq k} (\varepsilon_k - \varepsilon_m) \)

(Idempotent means: \( \mathbf{P} \cdot \mathbf{P} = \mathbf{P} \))

\[ \mathbf{P}_j \mathbf{P}_k = \begin{cases} 0 & \text{if } j \neq k \\ \mathbf{P}_k & \text{if } j = k \end{cases} \quad \mathbf{M} \mathbf{P}_k = \varepsilon_k \mathbf{p}_k = \mathbf{p}_k \mathbf{M} \]

With example matrix

\[ \mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \]

\[ \mathbf{p}_1 = (\mathbf{M} - 5 \cdot \mathbf{1}) = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \quad \mathbf{p}_1 \mathbf{p}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]

\[ \mathbf{p}_2 = (\mathbf{M} - 1 \cdot \mathbf{1}) = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \]

\[ \mathbf{P}_1 = \frac{(\mathbf{M} - 5 \cdot \mathbf{1})}{(1 - 5)} = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} \]

\[ \mathbf{P}_2 = \frac{(\mathbf{M} - 1 \cdot \mathbf{1})}{(5 - 1)} = \frac{1}{4} \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \]

Wednesday, March 23, 2016
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$\textbf{Idempotent}$ projectors (how eigenvalues $\Rightarrow$ eigenvectors)
Operator orthonormality and Completeness ($\textbf{Idempotent}$ means: $\textbf{P} \cdot \textbf{P} = \textbf{P}$)
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\[ p_j p_k = p_j \prod_{m \neq k} (M - \varepsilon_m 1) = \prod_{m \neq k} (p_j M - \varepsilon_m p_j 1) \]

Multiplication properties of \( p_j \):

\[ p_j p_k = \prod_{m \neq k} (\varepsilon_j p_j - \varepsilon_m p_j) = p_j \prod_{m \neq k} (\varepsilon_j - \varepsilon_m) = \begin{cases} 0 & \text{if } j \neq k \\ p_k \prod_{m \neq k} (\varepsilon_k - \varepsilon_m) & \text{if } j = k \end{cases} \]

Last step:

make **Idempotent Projectors**: \( P_k = \prod_{m \neq k} (M - \varepsilon_m 1) / \prod_{m \neq k} (\varepsilon_k - \varepsilon_m) \)

(Idempotent means: \( P \cdot P = P \))

\[ P_j P_k = \begin{cases} 0 & \text{if } j \neq k \\ P_k & \text{if } j = k \end{cases} \]

implies:

\[ M P_k = \varepsilon_k P_k = p_k M \]

**With example matrix**

\[ M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \]

\[ p_1 = (M - 5 \cdot 1) = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \]

\[ p_2 = (M - 1 \cdot 1) = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \]

Factoring bra-kets into “Ket-Bras”:

\[ P_1 = (M - 5 \cdot 1) / (1 - 5) = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1 \begin{pmatrix} 1/2 \\ -3/2 \end{pmatrix} \otimes k_1 \]

\[ P_2 = (M - 1 \cdot 1) / (5 - 1) = \frac{1}{4} \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2 \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \otimes k_2 \]

“Gauge” scale factors that only affect plots.
Matrix-algebraic method for finding eigenvector and eigenvalues

\[ p_j p_k = p_j \prod_{m \neq k} (M - \varepsilon_m 1) = \prod_{m \neq k} (p_j M - \varepsilon_m p_j) \]

Multiplication properties of \( p_j \):

\[ p_j p_k = \prod_{m \neq k} (\varepsilon_j p_j - \varepsilon_m p_j) = p_j \prod_{m \neq k} (\varepsilon_j - \varepsilon_m) = \begin{cases} 0 & \text{if } j \neq k \\ p_k & \text{if } j = k \end{cases} \]

Last step:

make **Idempotent Projectors**: \( P_k = \prod_{m \neq k} (\varepsilon_k - \varepsilon_m) \)

(Idempotent means: \( P \cdot P = P \))

\[ P_j P_k = \begin{cases} 0 & \text{if } j \neq k \\ P_k & \text{if } j = k \end{cases} \]

\[ M \mathbf{p}_k = \varepsilon_k \mathbf{p}_k = \mathbf{p}_k M \]

With example matrix

\[ M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \]

\[ p_1 = (M - 5 \cdot 1) = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \]

\[ p_2 = (M - 1 \cdot 1) = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \]

Factoring bra-kets into “Ket-Bras:

\[ |\varepsilon_1\rangle \langle \varepsilon_1| = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ -3 & 3 \end{pmatrix} = k_1 \begin{pmatrix} 1/2 \\ -3/2 \end{pmatrix} \]

\[ |\varepsilon_2\rangle \langle \varepsilon_2| = \frac{1}{4} \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2 \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix} \]
Review of 1D FDHO (Forced-Damped-Harmonic Oscillator) response
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\[ p_j p_k = p_j \prod_{m \neq k} (M - \varepsilon_m \textbf{1}) = \prod_{m \neq k} (p_j M - \varepsilon_m p_j \textbf{1}) \]

Multiplication properties of \( p_j \):

\[ p_j p_k = \prod_{m \neq k} (\varepsilon_j p_j - \varepsilon_m p_j) = p_j \prod_{m \neq k} (\varepsilon_j - \varepsilon_m) = \begin{cases} 0 & \text{if } j \neq k \\ p_k & \text{if } j = k \end{cases} \]

Last step:

make **Idempotent Projectors**: \( p_k = \prod_{m \neq k} (\varepsilon_k - \varepsilon_m) \)

(Idempotent means: \( P \cdot P = P \))

\[ \text{implies:} \]

\[ p_k = \prod_{m \neq k} (M - \varepsilon_m \textbf{1}) \]

\[ M p_k = \varepsilon_k p_k = p_k M \]

\[ MP_k = \varepsilon_k p_k = p_k M \]

The \( p_j \) are **Mutually Ortho-Normal** as are bra-ket \( \langle \varepsilon_j | \text{and} | \varepsilon_i \rangle \) inside \( p_j \)'s

\[ \begin{bmatrix} \langle \varepsilon_1 | \varepsilon_1 \rangle & \langle \varepsilon_1 | \varepsilon_2 \rangle \\ \langle \varepsilon_2 | \varepsilon_1 \rangle & \langle \varepsilon_2 | \varepsilon_2 \rangle \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

**Eigen-bra-ket projectors of matrix:**

\[ M = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \]

With example matrix

\[ M = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \]

\[ p_1 = (M - 5 \cdot \textbf{1}) = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \]

\[ p_2 = (M - 1 \cdot \textbf{1}) = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \]

Factoring bra-kets into "Ket-Bras:"

\[ p_1 = (M - 5 \cdot \textbf{1}) = \frac{1}{4} \begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix} = k_1 \begin{bmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{bmatrix} \]

"Gauge" scale factors that only affect plots

\[ p_2 = (M - 1 \cdot \textbf{1}) = \frac{1}{4} \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} = k_2 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \]

**Example matrix**

\[ M = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \]

\[ p_1 = (M - 5 \cdot \textbf{1}) = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \]

\[ p_2 = (M - 1 \cdot \textbf{1}) = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \]
Matrix-algebraic method for finding eigenvector and eigenvalues:

\[ p_j p_k = \prod_{m \neq j, k} (p_j - \varepsilon_m p_j) \]

Multiplication properties of \( p_j \):

\[ p_j p_k = \prod_{m \neq k} (p_j - \varepsilon_m p_j) = p_j \prod_{m \neq k} (p_j - \varepsilon_m) \]

Last step:

make **Idempotent Projectors**: \( p_j p_k = \prod_{m \neq j, k} (p_j - \varepsilon_m) \)

(Idempotent means: \( P \cdot P = P \))

\[ p_j p_k = \begin{cases} 0 & \text{if } j \neq k \\ p_k & \text{if } j = k \end{cases} \]

With example matrix:

\[ M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \]

\[ p_1 = (M - \varepsilon_1) = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \]

\[ p_2 = (M - \varepsilon_2) = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \]

Factoring bra-kets into “Ket-Bras:

\[ \langle \varepsilon_1 | \varepsilon_1 \rangle \]

“Gauge” scale factors that only affect plots

\[ \langle \varepsilon_2 | \varepsilon_2 \rangle \]

Eigen-bra-ket projectors of matrix:

\[ M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \]

\[ p_1 = \frac{(M - \varepsilon_1)}{(1 - 5)} = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} \]

\[ p_2 = \frac{(M - \varepsilon_2)}{(5 - 1)} = \frac{1}{4} \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \]

The \( p_j \) are **Mutually Ortho-Normal**
as are bra-ket \( \langle \varepsilon_j | \varepsilon_j \rangle \) inside \( p_j \)’s

\[ \langle \varepsilon_j | \varepsilon_j \rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

...and the \( p_j \) satisfy a **Completeness Relation**:

\[ 1 = \sum_{j=1}^{n} \langle \varepsilon_j | \varepsilon_j \rangle \]

\[ = |\varepsilon_j \rangle \langle \varepsilon_j | + \cdots + |\varepsilon_n \rangle \langle \varepsilon_n | \]
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Mixed mode beat dynamics (with constant \(\pi/2\) phase-lag)
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Matrix-algebraic method for finding eigenvector and eigenvalues

\[ p_j p_k = p_j \prod_{m \neq k} (M - \varepsilon_m 1) \]

Multiplication properties of \( p_j \):

\[ p_j p_k = \prod_{m \neq k} (\varepsilon_j - \varepsilon_m p_j) = p_j \prod_{m \neq k} (\varepsilon_j - \varepsilon_m) \]

Last step: make **Idempotent Projectors**: \( P_k = \prod_{m \neq k} (\varepsilon_k - \varepsilon_m) \)

(Idempotent means: \( P \cdot P = P \))

\[ M \varepsilon_k p_k = p_k M \]

Factoring bra-kets into “Ket-Bra’s”:

\[ \begin{align*}
    p_1 &= \begin{pmatrix} 1 & 1 \\ -3 & 3 \end{pmatrix}, \quad p_2 = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}, \\
    P_1 &= \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix}, \quad P_2 = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}
\end{align*} \]

Eigen-operators \( M \varepsilon_k p_k = p_k M \) then give **Spectral Decomposition** of operator \( M \)

\[ M = \sum_{k=1}^{n} \varepsilon_k p_k \]

With example matrix

\[ M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \]

Eigen-operators \( M \varepsilon_k p_k = p_k M \)

\[ p_1 = \begin{pmatrix} 1 & 1 \\ -3 & 3 \end{pmatrix}, \quad p_2 = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \]

\[ P_1 = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix}, \quad P_2 = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix} \]

“Gauge” scale factors that only affect plots

\[ \begin{align*}
    \varepsilon_1 &= \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \quad \varepsilon_2 &= \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \\
    |\varepsilon_1\rangle &= \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \quad |\varepsilon_2\rangle &= \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}
\end{align*} \]

Eigenvectors

\[ \begin{align*}
    \varepsilon_1 &= \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \quad \varepsilon_2 &= \begin{pmatrix} 3 \\ 1 \end{pmatrix}
\end{align*} \]

\[ \begin{align*}
    |\varepsilon_1\rangle &= \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \quad |\varepsilon_2\rangle &= \begin{pmatrix} 3 \\ 1 \end{pmatrix}
\end{align*} \]
Matrix-algebraic method for finding eigenvector and eigenvalues

\[ p_j p_k = p_j \prod_{m \neq k} (M - \varepsilon_m 1) = \prod_{m \neq k} (p_j M - \varepsilon_m p_j 1) \]

**Multiplication properties of** \( p_j \):

\[ p_j p_k = \prod_{m \neq k} (\varepsilon_j - \varepsilon_m p_j) = p_j \prod_{m \neq k} (\varepsilon_j - \varepsilon_m) = \begin{cases} 0 & \text{if } j \neq k \\ p_k \prod_{m \neq k} (\varepsilon_k - \varepsilon_m) & \text{if } j = k \end{cases} \]

Last step:

**make Idempotent Projectors:** \( P_k = \prod_{m \neq k} (M - \varepsilon_m 1) \)

(Idempotent means: \( P \cdot P = P \))

\[ P_j P_k = \begin{cases} 0 & \text{if } j \neq k \\ P_k & \text{if } j = k \end{cases} \]

The \( P_j \) are **Mutually Ortho-Normal** as are bra-ket \( \langle \varepsilon_j \rangle \) and \( |\varepsilon_j\rangle \) inside \( P_j \)’s

\[ \begin{bmatrix} \langle \varepsilon_1 | \varepsilon_1 \rangle & \langle \varepsilon_1 | \varepsilon_2 \rangle \\ \langle \varepsilon_2 | \varepsilon_1 \rangle & \langle \varepsilon_2 | \varepsilon_2 \rangle \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

...and the \( P_j \) satisfy a **Completeness Relation:**

\[ 1 = P_1 + P_2 + \ldots + P_n \]

\[ = |\varepsilon_1\rangle \langle \varepsilon_1| + |\varepsilon_2\rangle \langle \varepsilon_2| + \ldots + |\varepsilon_n\rangle \langle \varepsilon_n| \]

**Eigen-operators** \( MP_k = \varepsilon_k p_k \) then give **Spectral Decomposition** of operator \( M \)

\[ M = MP_1 + MP_2 + \ldots + MP_n = \varepsilon_1 P_1 + \varepsilon_2 P_2 + \ldots + \varepsilon_n P_n \]

**With example matrix**

\[ M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \]

\[ p_1 = (M - 5 \cdot 1) = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \]

\[ p_2 = (M - 1 \cdot 1) = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \]

**Factoring bra-kets into “Ket-Bra”**

\[ p_1 p_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]

**Eigen-operators**

\[ |\varepsilon_1\rangle \langle \varepsilon_1|, |\varepsilon_2\rangle \langle \varepsilon_2|, \ldots, |\varepsilon_n\rangle \langle \varepsilon_n| \]

\[ |\varepsilon_1\rangle \langle \varepsilon_1| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ |\varepsilon_2\rangle \langle \varepsilon_2| = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \]

\[ |\varepsilon_3\rangle \langle \varepsilon_3| = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix} \]

**Eigen-bra-ket**

\[ \langle \varepsilon_1| = |\varepsilon_1\rangle \]

\[ \langle \varepsilon_2| = |\varepsilon_2\rangle \]

\[ \langle \varepsilon_3| = |\varepsilon_3\rangle \]

**Spectral Decomposition**: 

\[ M = |\varepsilon_1\rangle \langle \varepsilon_1| + |\varepsilon_2\rangle \langle \varepsilon_2| + 5 |\varepsilon_3\rangle \langle \varepsilon_3| \]

\[ = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \]
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\[ p_j p_k = p_j \prod_{m \neq k} (M - \varepsilon_m 1) = \prod_{m \neq k} (p_j M - \varepsilon_m p_j 1) \quad \text{MP}_k = \varepsilon_k p_k = p_k M \]

Multiplication properties of \( p_j \):

\[ p_j p_k = \prod_{m \neq k} (\varepsilon_j - \varepsilon_m p_j) = p_j \prod_{m \neq k} (\varepsilon_j - \varepsilon_m) = \begin{cases} 0 & \text{if } j \neq k \\ p_k \prod_{m \neq k} (\varepsilon_k - \varepsilon_m) & \text{if } j = k \end{cases} \]

Last step:
make **Idempotent Projectors**: \( \textbf{P}_k = \frac{p_k}{\prod_{m \neq k} (\varepsilon_k - \varepsilon_m)} \)

(Idempotent means: \( \textbf{P} \cdot \textbf{P} = \textbf{P} \))

\[ \textbf{P}_j \cdot \textbf{P}_k = \begin{cases} 0 & \text{if } j \neq k \\ \textbf{P}_k & \text{if } j = k \end{cases} \]

\[ \text{MP}_k = \varepsilon_k \textbf{P}_k = p_k M \]

implies:

\[ \text{MP}_k = \varepsilon_k \textbf{P}_k = p_k M \]

The \( \textbf{P}_j \) are **Mutually Ortho-Normal**
as are bra-ket \( \langle \varepsilon_j | \text{and} | \varepsilon_i \rangle \) inside \( \textbf{P}_j \)’s

\[
\begin{pmatrix}
\langle \varepsilon_i | \varepsilon_i \rangle & \langle \varepsilon_i | \varepsilon_2 \rangle \\
\langle \varepsilon_2 | \varepsilon_1 \rangle & \langle \varepsilon_2 | \varepsilon_2 \rangle
\end{pmatrix} =
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

...and the \( \textbf{P}_j \) satisfy a **Completeness Relation**:

\[ 1 = \textbf{P}_1 + \textbf{P}_2 + \ldots + \textbf{P}_n \]

\[ = | \varepsilon_1 \rangle \langle \varepsilon_1 | + | \varepsilon_2 \rangle \langle \varepsilon_2 | + \ldots + | \varepsilon_n \rangle \langle \varepsilon_n | \]

\[ = | \varepsilon_1 \rangle \langle \varepsilon_1 | + | \varepsilon_2 \rangle \langle \varepsilon_2 | \]

\[ = | \varepsilon_1 \rangle \langle \varepsilon_1 | + | \varepsilon_2 \rangle \langle \varepsilon_2 | \]

\[ \quad \quad \quad \quad = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \]

\[ \text{M} = \begin{pmatrix}
4 & 1 \\
3 & 2
\end{pmatrix} \]

Eigen-operators \( \text{MP}_k = \varepsilon_k \textbf{P}_k \) then give **Spectral Decomposition** of operator \( \textbf{M} \)

\[ \textbf{M} = \text{MP}_1 + \text{MP}_2 + \ldots + \text{MP}_n = \varepsilon_1 \textbf{P}_1 + \varepsilon_2 \textbf{P}_2 + \ldots + \varepsilon_n \textbf{P}_n \]

...and **Functional Spectral Decomposition** of any function \( f(\textbf{M}) \) of \( \textbf{M} \)

\[ f(\textbf{M}) = f(\varepsilon_1) \textbf{P}_1 + f(\varepsilon_2) \textbf{P}_2 + \ldots + f(\varepsilon_n) \textbf{P}_n \]
Matrix and operator Spectral Decompositions

\[ p_j p_k = p_j \prod_{m \neq k} (M - \varepsilon_m 1) = \prod_{m \neq k} (p_j M - \varepsilon_m p_j 1) \]

Multiplication properties of \( p_j \):

\[ p_j p_k = \prod_{m \neq k} (\varepsilon_j - \varepsilon_m p_j) = p_j \prod_{m \neq k} (\varepsilon_j - \varepsilon_m) = \begin{cases} 0 & \text{if } j \neq k \\ p_k & \text{if } j = k \end{cases} \prod_{m \neq k} (\varepsilon_k - \varepsilon_m) \]

Last step: make **Idempotent Projectors**: \( P_k = \prod_{m \neq k} (M - \varepsilon_m 1) \)

(Idempotent means: \( P \cdot P = P \))

\[ M p_k = \varepsilon_k p_k = p_k M \]

The \( P_j \) are **Mutually Ortho-Normal**

as are bra-ket \( \langle \varepsilon_j | \) and \( | \varepsilon_j \rangle \) inside \( P_j \)’s

\[ \begin{pmatrix} \langle \varepsilon_1 | \varepsilon_1 \rangle & \langle \varepsilon_1 | \varepsilon_2 \rangle \\ \langle \varepsilon_2 | \varepsilon_1 \rangle & \langle \varepsilon_2 | \varepsilon_2 \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

...and the \( p_j \) satisfy a **Completeness Relation**:

\[ I = P_1 + P_2 + \ldots + P_n \]

\[ = | \varepsilon_1 \rangle \langle \varepsilon_1 | + | \varepsilon_2 \rangle \langle \varepsilon_2 | + \ldots + | \varepsilon_n \rangle \langle \varepsilon_n | \]

\[ = | \varepsilon_1 \rangle \langle \varepsilon_1 | + | \varepsilon_2 \rangle \langle \varepsilon_2 | + \ldots + | \varepsilon_n \rangle \langle \varepsilon_n | \]

Eigen-operators \( M p_k = \varepsilon_k p_k \) then give **Spectral Decomposition** of operator \( M \)

\[ M = M P_1 + M P_2 + \ldots + M P_n = \varepsilon_1 P_1 + \varepsilon_2 P_2 + \ldots + \varepsilon_n P_n \]

...and **Functional Spectral Decomposition** of any function \( f(M) \) of \( M \)

\[ f(M) = f(\varepsilon_1) P_1 + f(\varepsilon_2) P_2 + \ldots + f(\varepsilon_n) P_n \]
Matrix and operator Spectral Decompositions

\[ p_j p_k = p_j \prod_{m \neq k} (M - \varepsilon_m 1) = \prod_{m \neq k} (p_j M - \varepsilon_m p_j 1) \]

Multiplication properties of \( p_j \):

\[ p_j p_k = \prod_{m \neq k} (\varepsilon_j - \varepsilon_m p_j) = p_j \prod_{m \neq k} (\varepsilon_j - \varepsilon_m) = \begin{cases} 0 & \text{if } j \neq k \\ p_k \prod_{m \neq k} (\varepsilon_k - \varepsilon_m) & \text{if } j = k \end{cases} \]

Last step:

make Idempotent Projectors:

\[ P_k = \prod_{m \neq k} (M - \varepsilon_m 1) \]

(Idempotent means: \( P \cdot P = P \))

implies:

\[ M P_k = \varepsilon_k P_k = P_k M \]

\[ P_j P_k = \begin{cases} 0 & \text{if } j \neq k \\ P_k & \text{if } j = k \end{cases} \]

The \( P_j \) are Mutually Ortho-Normal as are bra-ket \( \langle \varepsilon_j | \) and \( | \varepsilon_j \rangle \) inside \( P_j \)’s

\[ \begin{pmatrix} \langle \varepsilon_1 | \varepsilon_1 \rangle & \langle \varepsilon_1 | \varepsilon_2 \rangle \\ \langle \varepsilon_2 | \varepsilon_1 \rangle & \langle \varepsilon_2 | \varepsilon_2 \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

...and the \( P_j \) satisfy a Completeness Relation:

\[ 1 = P_1 + P_2 + ... + P_n \]

\[ = | \varepsilon_1 \rangle \langle \varepsilon_1 | + | \varepsilon_2 \rangle \langle \varepsilon_2 | + ... + | \varepsilon_n \rangle \langle \varepsilon_n | = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \]

Eigen-operators \( M P_k = \varepsilon_k P_k \) then give Spectral Decomposition of operator \( M \)

\[ M = MP_1 + MP_2 + ... + MP_n = \varepsilon_1 P_1 + \varepsilon_2 P_2 + ... + \varepsilon_n P_n \]

...and Functional Spectral Decomposition of any function \( f(M) \) of \( M \)

\[ f(M) = f(\varepsilon_1) P_1 + f(\varepsilon_2) P_2 + ... + f(\varepsilon_n) P_n \]

Examples:

\[ M^{50} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = l^{50} \begin{pmatrix} \frac{4}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{4}{3} \end{pmatrix} + 5^{50} \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} = \begin{pmatrix} 1 + 3 \cdot 5^{50} & 5^{50} - 1 \\ 5^{50} - 3 & 5^{50} + 3 \end{pmatrix} \]

\[ \sqrt{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = \pm \sqrt{5} \begin{pmatrix} \frac{4}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{4}{3} \end{pmatrix} + \sqrt{5} \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \pm \frac{1}{3} \sqrt{5} & \frac{1}{3} \pm \frac{1}{3} \sqrt{5} \\ \frac{1}{3} \pm \frac{1}{3} \sqrt{5} & \frac{3}{4} \pm \frac{1}{4} \sqrt{5} \end{pmatrix} \]
Orthonormality vs. Completeness

\[ p_j p_k = P_j \prod_{m \neq k} (M - \varepsilon_m 1) = P_j \prod_{m \neq k} (p_j M - \varepsilon_m p_j 1) \]

Multiplication properties of \( p_j \):

\[ p_j p_k = \prod_{m \neq k} (\varepsilon_j - \varepsilon_m p_j) = p_j \prod_{m \neq k} (\varepsilon_j - \varepsilon_m) = \left\{ \begin{array}{ll} 0 & \text{if } j \neq k \\ p_k \prod_{m \neq k} (\varepsilon_k - \varepsilon_m) & \text{if } j = k \end{array} \right. \]

Last step:

make *Idempotent Projectors*: \( P_k = P_j \prod_{m \neq k} (M - \varepsilon_m 1) = \prod_{m \neq k} (M - \varepsilon_m 1) \)

(Idempotent means: \( P \cdot P = P \))

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\[ \begin{bmatrix} \langle \varepsilon_1 | \varepsilon_1 \rangle & \langle \varepsilon_1 | \varepsilon_2 \rangle \\ \langle \varepsilon_2 | \varepsilon_1 \rangle & \langle \varepsilon_2 | \varepsilon_2 \rangle \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

...and the \( p_j \) satisfy a *Completeness Relation*:

\[ 1 = P_1 + P_2 + \ldots + P_n \]

\[ = | \varepsilon_1 \rangle \langle \varepsilon_1 | + | \varepsilon_2 \rangle \langle \varepsilon_2 | + \ldots + | \varepsilon_n \rangle \langle \varepsilon_n | \]

*Eigen-bra-ket projectors of matrix:*

\[ \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \]

Factoring bra-kets into “Ket-Bra’s”

\[ p_1 = (M -5 \cdot 1) = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \]

\[ p_2 = (M - 1 \cdot 1) = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \]

<table>
<thead>
<tr>
<th>( p_1 ) or ( p_2 )</th>
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