Lecture 13 to 14
Tue.-Thu. 3.01-3.03.2016

Complex Variables, Series, and Field Coordinates

(Ch. 10 of Unit 1)

1. The Story of e (A Tale of Great $Interest$)
  How good are those power series?
  Taylor-Maclaurin series, imaginary interest, and complex exponentials

Lecture 14 Tue. 10.15
starts here

2. What good are complex exponentials?
  Easy trig
  Easy 2D vector analysis
  Easy oscillator phase analysis
  Easy rotation and "dot" or "cross" products
3. Easy 2D vector calculus
  Easy 2D vector derivatives
  Easy 2D source-free field theory
  Easy 2D vector field-potential theory
4. Riemann-Cauchy relations (What's analytic? What's not?)
  Easy 2D curvilinear coordinate discovery
  Easy 2D circulation and flux integrals
  Easy 2D monopole, dipole, and 2^n-pole analysis
  Easy 2^n-multipole field and potential expansion
  Easy stereo-projection visualization
  Cauchy integrals, Laurent-Maclaurin series
5. Mapping and Non-analytic 2D source field analysis

1. *Complex numbers provide "automatic trigonometry"*
2. *Complex numbers add like vectors.*
3. *Complex exponentials $Ae^{-i\omega t}$ track position and velocity using Phasor Clock.*
4. *Complex products provide 2D rotation operations.*
5. *Complex products provide 2D "dot"(•) and "cross"(x) products.*

2D Applications: E&B-fields, heat flow, hydro-dynamics, surface-shape,...

6. *Complex derivative contains "divergence"($\nabla\cdot\mathbf{F}$) and "curl"($\nabla\times\mathbf{F}$) of 2D vector field*
7. *Invent source-free 2D vector fields [$\nabla\cdot\mathbf{F}=0$ and $\nabla\times\mathbf{F}=0$]*
8. *Complex potential ϕ contains "scalar"($\mathbf{F}=\nabla\Phi$) and "vector"($\mathbf{F}=\nabla\times\mathbf{A}$) potentials*

   The *half*-n'-*half* results: (Riemann-Cauchy Derivative Relations)
9. *Complex potentials define 2D Orthogonal Curvilinear Coordinates (OCC) of field*
10. *Complex integrals $\int f(z)dz$ count 2D "circulation"($\int\mathbf{F}\cdot d\mathbf{r}$) and "flux"($\int\mathbf{F}\times d\mathbf{r}$)*
11. *Complex integrals define 2D monopole fields and potentials*
12. *Complex derivatives give 2D dipole fields*
13. *More derivatives give 2D 2^N-pole fields…*
14. *...and 2^N-pole multipole expansions of fields and potentials...*
15. *...and Laurent Series...*
16. *...and non-analytic source analysis.*

Lecture 15 Thur. 10.17
starts here

,...quantum probability current flow, 2^n-pole fields,...

The Story of e (A Tale of Great $Interest$)

Simple *interest* at some rate r based on a 1 year period.

You gave a principal $p(0)$ to the bank and some time t later they would pay you $p(t)=(1+r\cdot t)p(0)$.

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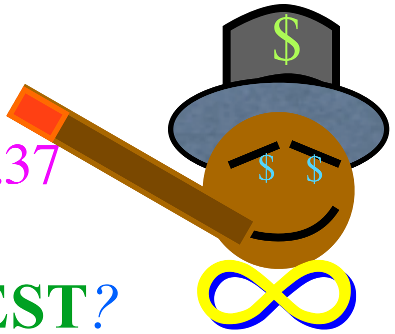

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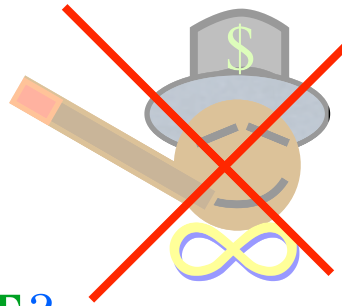

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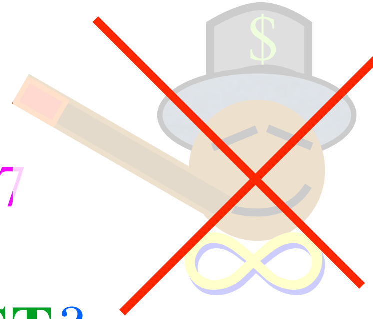

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+25¢

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$$p^{\frac{1}{4}}(t)=(1+r\cdot\tfrac{t}{4})^{4}p(0)=\left(\tfrac{5}{4}\right)^{4}\cdot 1=\tfrac{625}{256}=2.44$$

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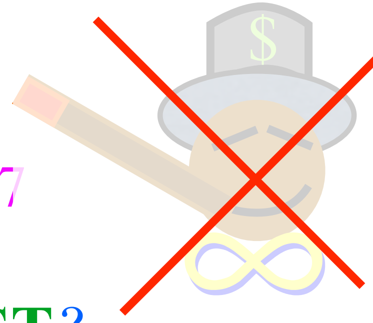

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+17¢

Monthly: $p^{\frac{1}{12}}(t)=(1+r\cdot\frac{t}{12})^{12}p(0)=\left(\frac{13}{12}\right)^{12}\cdot 1=2.613$

+8¢

Weekly: $p^{\frac{1}{52}}(t)=(1+r\cdot\frac{t}{52})^{52}p(0)=\left(\frac{53}{52}\right)^{52}\cdot 1=2.693$

+2¢

Daily: $p^{\frac{1}{365}}(t)=(1+r\cdot\frac{t}{365})^{365}p(0)=\left(\frac{366}{365}\right)^{365}\cdot 1=2.7145$

+1¢

Hrly: $p^{\frac{1}{8760}}(t)=(1+r\cdot\frac{t}{8760})^{8760}p(0)=\left(\frac{8761}{8760}\right)^{8760}\cdot 1=2.7181$

Interest product formula is really inefficient: 10^6 products for 6-figures! .. .10^9 products for 9 ...

$$p^{1/m}(1) = (1+\tfrac{1}{m})^{m} \xrightarrow[m\to\infty]{} \mathbf{2.718281828459..}$$

$=e$

Let: $m\cdot r\cdot t=n$

or: $1/m= r\cdot t/n$

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$p^{1/m}(1)$ = **2.718**1459268 for m = 10,000

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Can improve computational efficiency using binomial theorem:

$$(x+y)^n = x^n + n\cdot x^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \ldots + n\cdot xy^{n-1} + y^n$$

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Define: *Factorials*(!):

$0! = 1 = 1!$, $2! = 1\cdot 2$, $3! = 1\cdot 2\cdot 3, \ldots$

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Precision order:

(o=1)-e-series = **2.**00000 =1+1

(o=2)-e-series = **2.**50000 =1+1+1/2

(o=3)-e-series = **2.**66667 =1+1+1/2+1/6

(o=4)-e-series = **2.7**0833 =1+1+1/2+1/6+1/24

(o=5)-e-series = **2.71**667 =1+1+1/2+1/6+1/24+1/120

(o=6)-e-series = **2.718**05 =1+1+1/2+1/6+1/24+1/120+1/720

(o=7)-e-series = **2.7182**5

(o=8)-e-series = **2.71828**

About 12 summed quotients
for 6-figure precision (A lot better!)

Power Series Good! *Need general power series development*

Start with a general power series with constant coefficients c_0, c_1, *etc.*

Set $t=0$ to get $c_0 = x(0)$.

$$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + \ldots + c_n t^n +$$

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Rate of change of position *x(t)* is *velocity v(t)*.

Set $t=0$ to get $c_1 = v(0)$.

$$v(t) = \frac{d}{dt} x(t) = 0 + c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + 5c_5 t^4 + ... + nc_n t^{n-1} +$$

Power Series Good! Need general power series development

Start with a general power series with constant coefficients c_0, c_1, *etc.*

Set $t=0$ to get $c_0 = x(0)$.

$$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + \ldots + c_n t^n +$$

Rate of change of position $x(t)$ is *velocity $v(t)$*.

Set $t=0$ to get $c_1 = v(0)$.

$$v(t) = \frac{d}{dt} x(t) = 0 + c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + 5c_5 t^4 + \ldots + nc_n t^{n-1} +$$

Change of velocity $v(t)$ is *acceleration $a(t)$*.

Set $t=0$ to get $c_2 = \frac{1}{2} a(0)$.

$$a(t) = \frac{d}{dt} v(t) = 0 + 2c_2 + 2\cdot 3c_3 t + 3\cdot 4c_4 t^2 + 4\cdot 5c_5 t^3 + \ldots + n(n-1)c_n t^{n-2} +$$

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Change of acceleration $a(t)$ is *jerk* $j(t)$. (*Jerk* is NASA term.)

Set $t=0$ to get $c_3 = \frac{1}{3!} j(0)$.

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Change of jerk $j(t)$ is *inauguration $i(t)$*. (Be silly like NASA!)

Set $t=0$ to get $c_4 = \frac{1}{4!} i(0)$.

$$i(t) = \frac{d}{dt} j(t) = 0 + 2\cdot 3\cdot 4c_4 + 2\cdot 3\cdot 4\cdot 5c_5 t + \ldots + n(n-1)(n-2)(n-3)c_n t^{n-4} +$$

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Gives Maclaurin (or Taylor) power series

$$\boxed{x(t) = x(0) + v(0)t + \tfrac{1}{2!} a(0)t^2 + \tfrac{1}{3!} j(0)t^3 + \tfrac{1}{4!} i(0)t^4 + \tfrac{1}{5!} r(0)t^5 + \ldots + \tfrac{1}{n!} x^{(n)} t^n +}$$

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$\uparrow$

Good old UP I formula!

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$$\boxed{\boxed{x(t) = x(0) + v(0)t + \tfrac{1}{2!}a(0)t^2} + \tfrac{1}{3!}j(0)t^3 + \tfrac{1}{4!}i(0)t^4 + \tfrac{1}{5!}r(0)t^5 + \ldots + \tfrac{1}{n!}x^{(n)}t^n +}$$

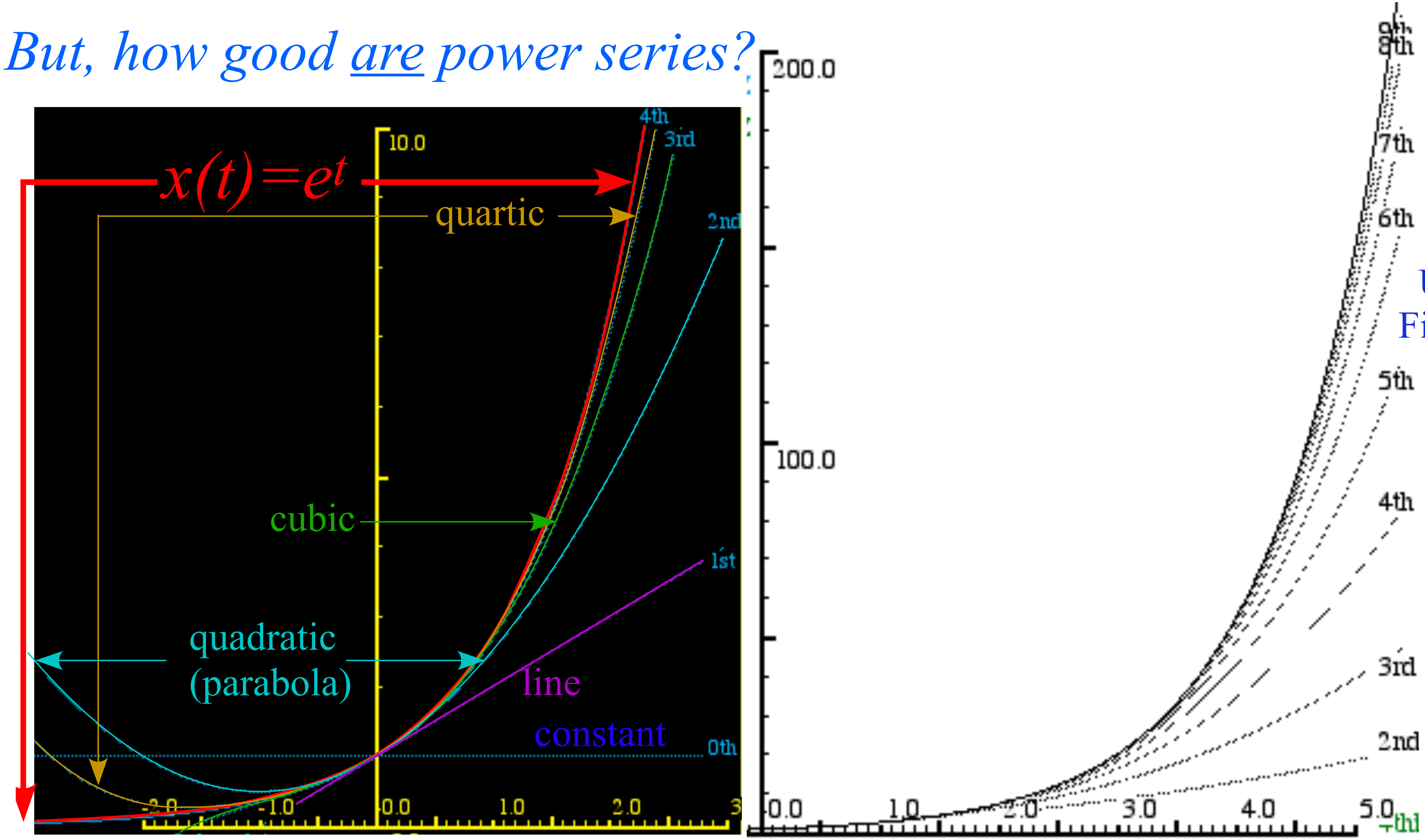
Setting all initial values to $1 = x(0) = v(0) = a(0) = j(0) = i(0) = \ldots$

Good old UP I formula!

gives *exponential:*

$$e^t = 1 + t + \tfrac{1}{2!}t^2 + \tfrac{1}{3!}t^3 + \tfrac{1}{4!}t^4 + \tfrac{1}{5!}t^5 + \ldots + \tfrac{1}{n!}t^n +$$

But, how good are power series?



Unit 1
Fig. 10.2

Gives Maclaurin (or Taylor) power series

$$x(t) = x(0) + v(0)t + \frac{1}{2!}a(0)t^2 + \frac{1}{3!}j(0)t^3 + \frac{1}{4!}i(0)t^4 + \frac{1}{5!}r(0)t^5 + \ldots + \frac{1}{n!}x^{(n)}t^n +$$

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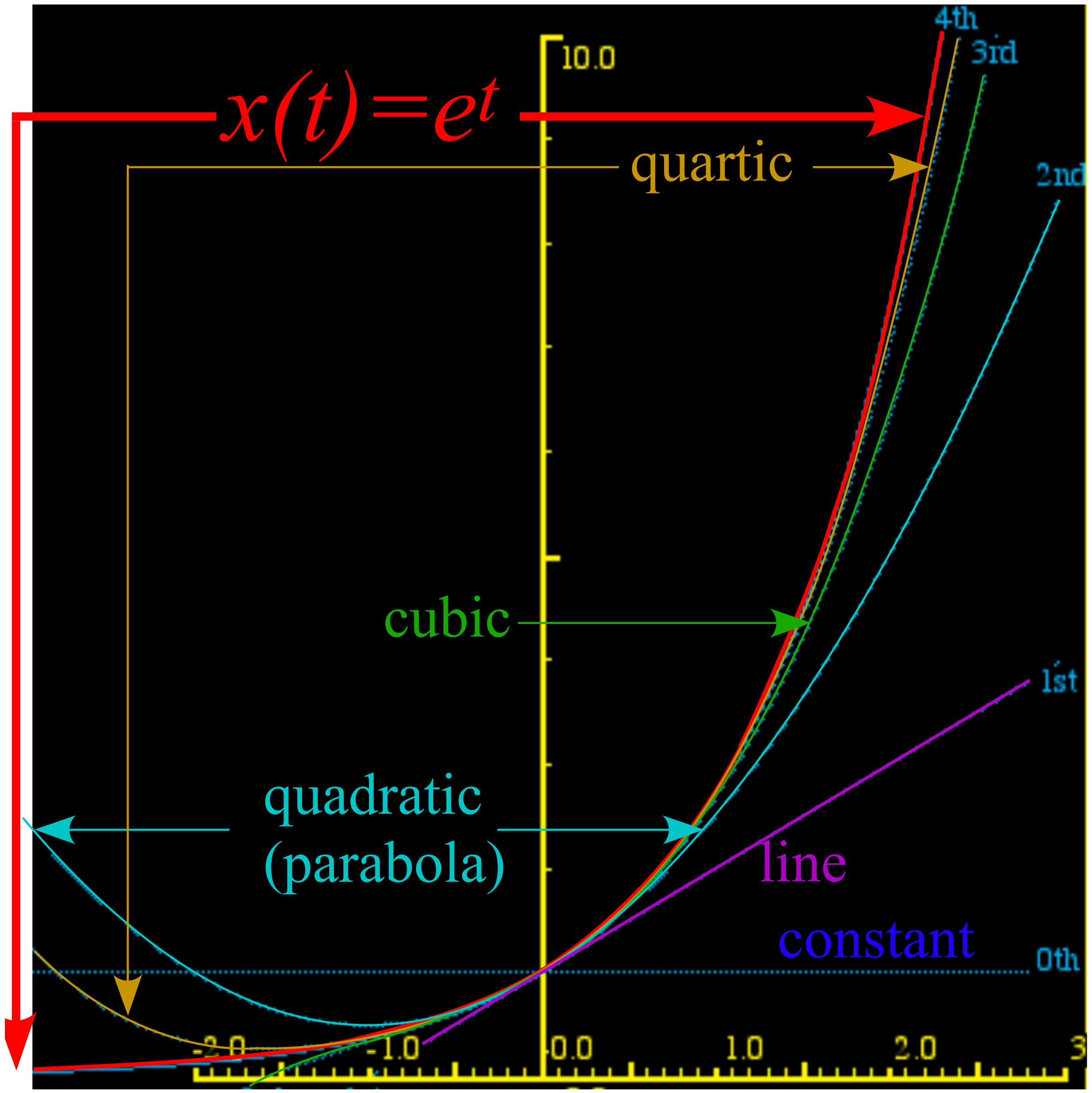

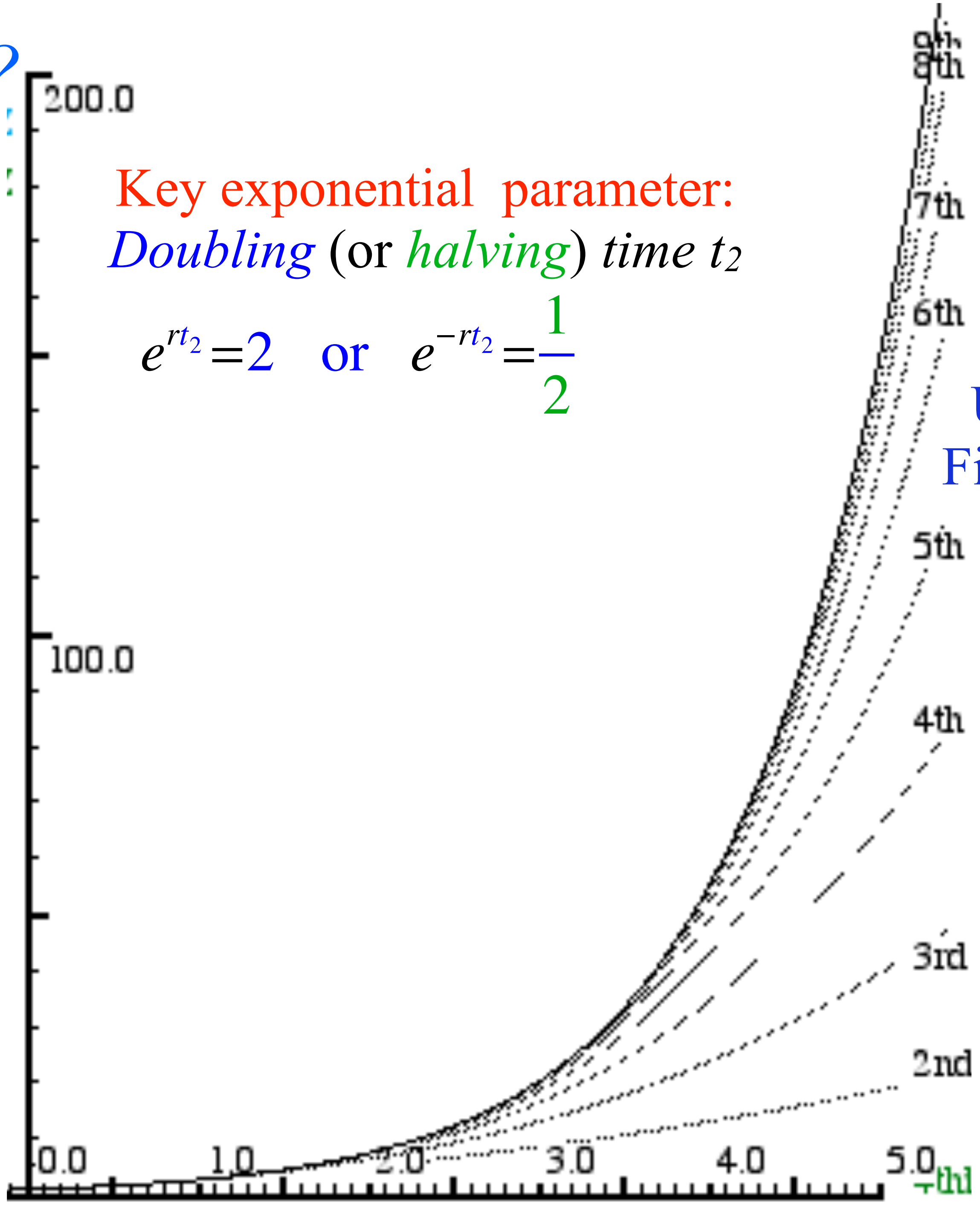

Key exponential parameter:
Doubling (or *halving*) *time* t_2

$$e^{rt_2}=2 \quad \text{or} \quad e^{-rt_2}=\frac{1}{2}$$

Unit 1
Fig. 10.2

Gives Maclaurin (or Taylor) power series

$$x(t)=x(0)+v(0)t+\frac{1}{2!}a(0)t^2+\frac{1}{3!}j(0)t^3+\frac{1}{4!}i(0)t^4+\frac{1}{5!}r(0)t^5+\ldots+\frac{1}{n!}x^{(n)}t^n+$$

Setting all initial values to *1= x(0) = v(0) = a(0) = j(0) = i(0) =* ....

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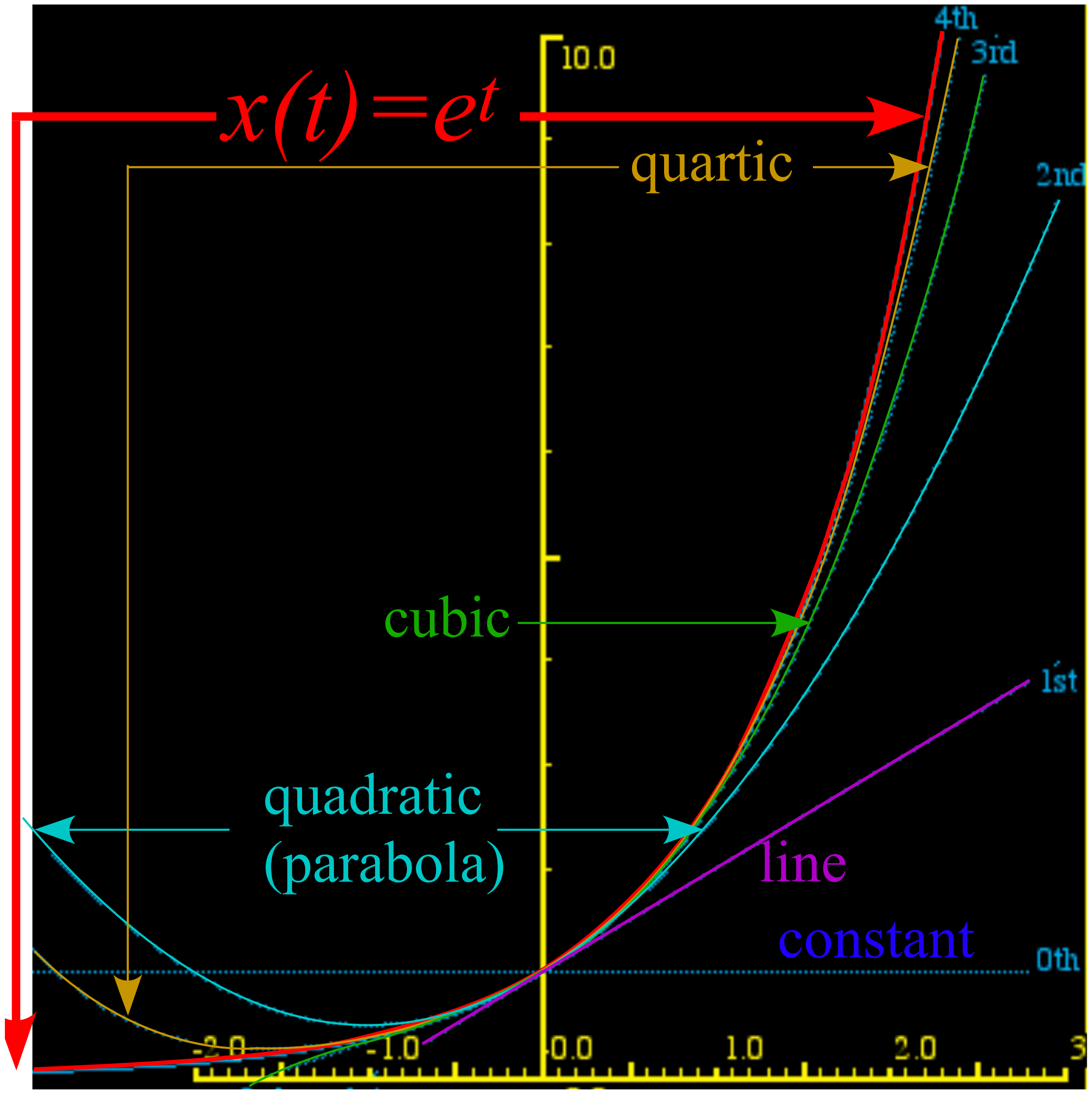
But, how good are power series?



$x(t)=e^t$

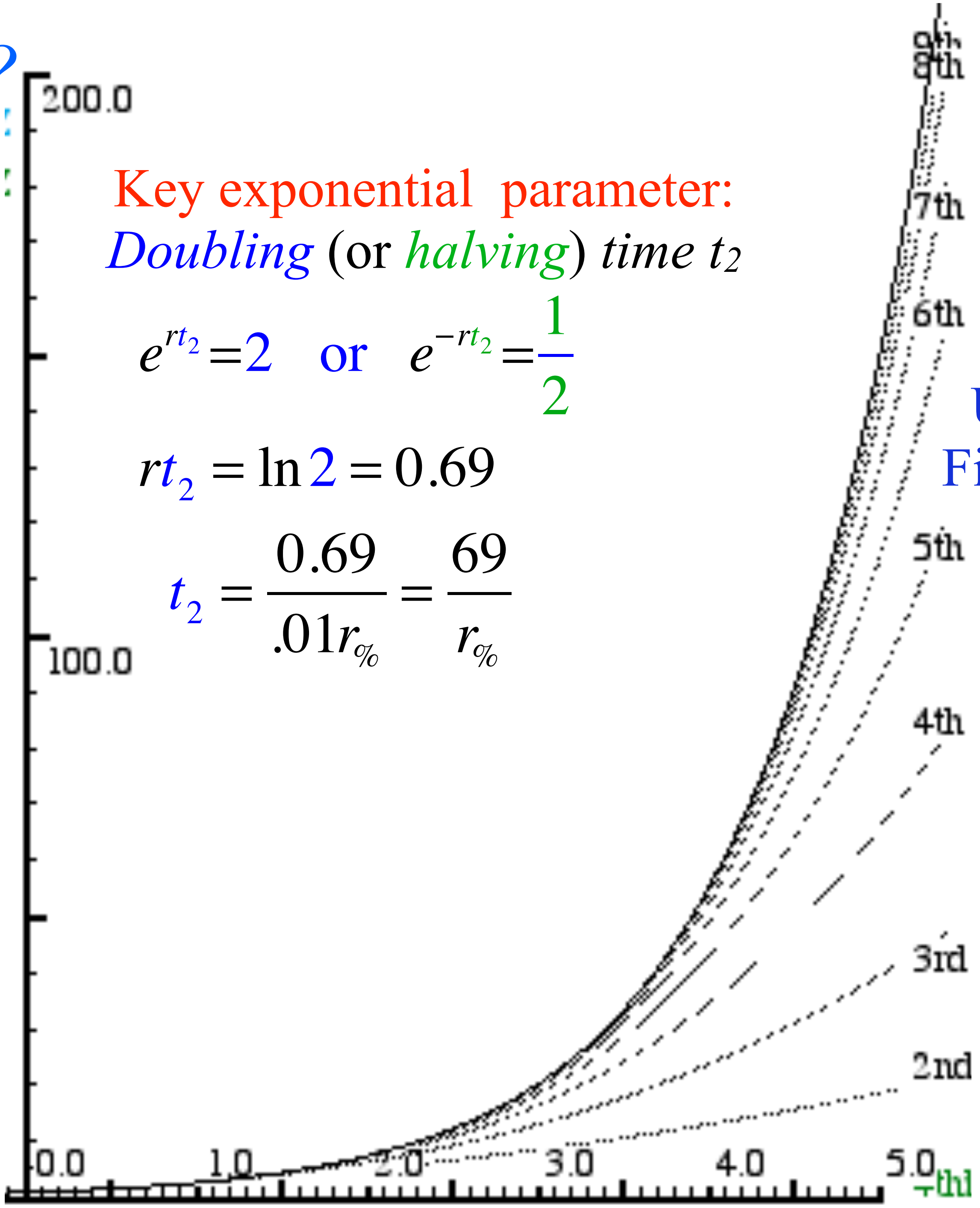
quartic

cubic

quadratic (parabola)

line

constant



Key exponential parameter:

Doubling (or *halving*) *time* t_2

$e^{rt_2}=2$ or $e^{-rt_2}=\frac{1}{2}$

$rt_2=\ln 2=0.69$

$t_2=\frac{0.69}{.01r_\%}=\frac{69}{r_\%}$

Unit 1
Fig. 10.2

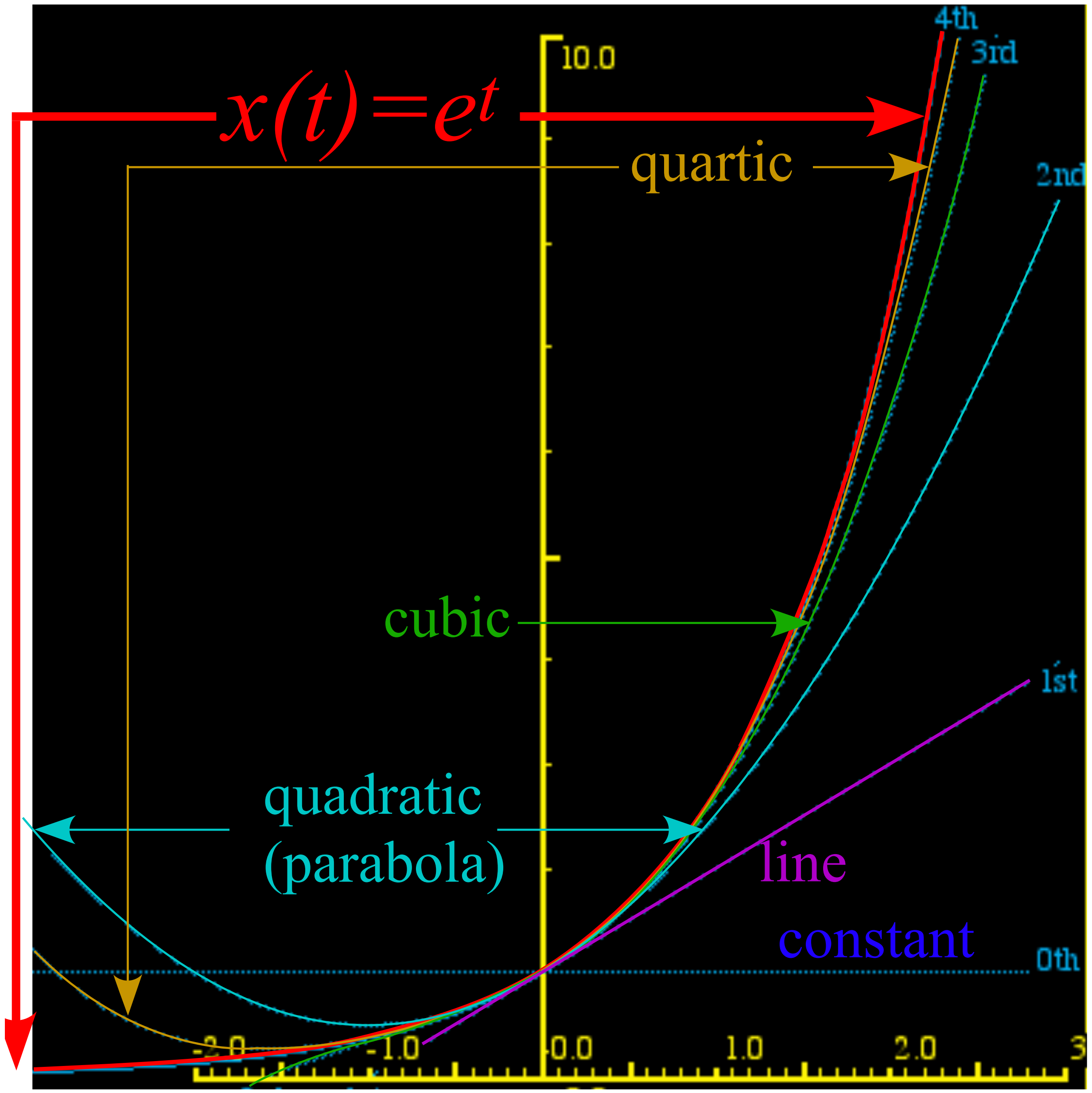
Gives Maclaurin (or Taylor) power series

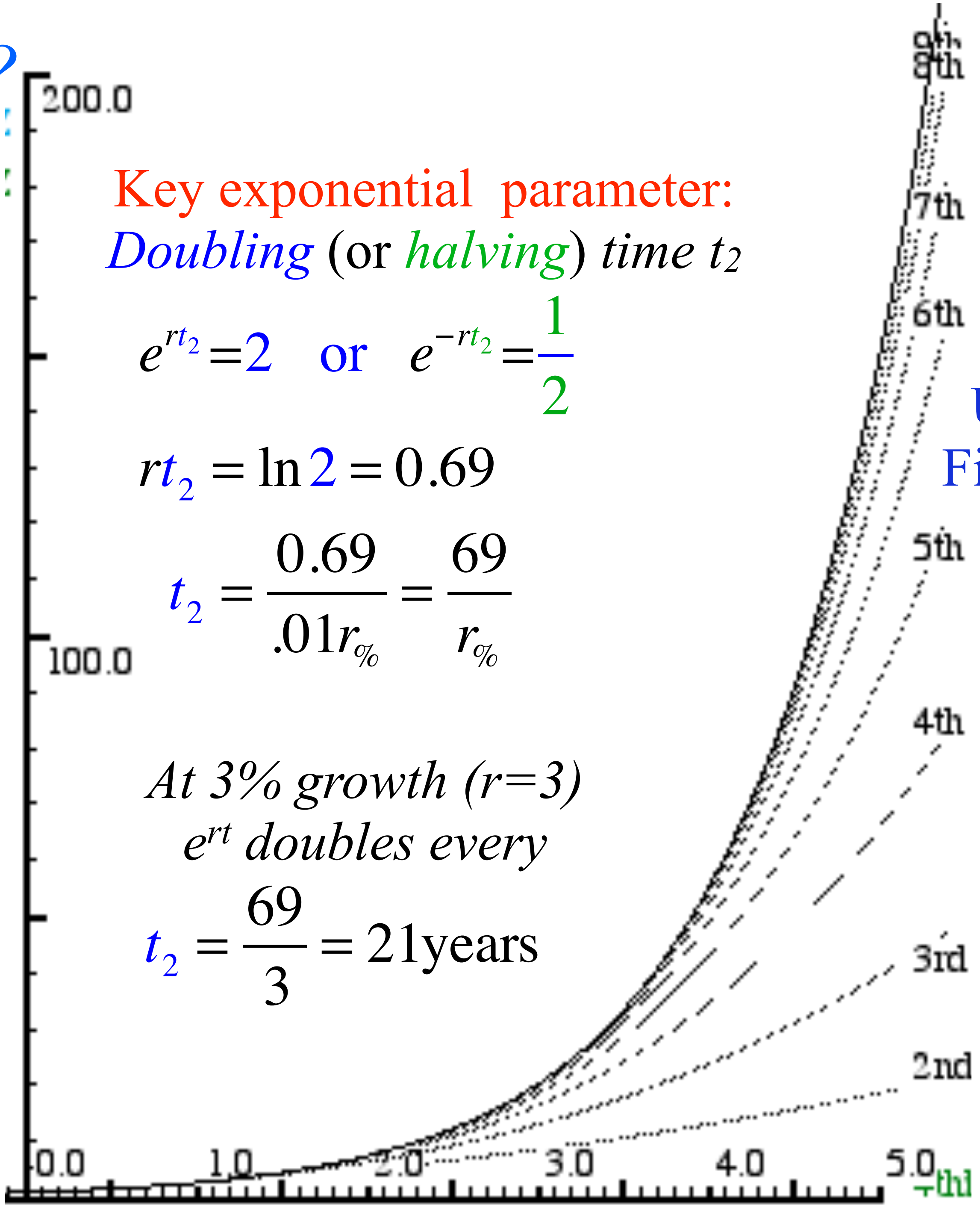
$$x(t)=x(0)+v(0)t+\frac{1}{2!}a(0)t^2+\frac{1}{3!}j(0)t^3+\frac{1}{4!}i(0)t^4+\frac{1}{5!}r(0)t^5+\ldots+\frac{1}{n!}x^{(n)}t^n+$$

Setting all initial values to $1=x(0)=v(0)=a(0)=j(0)=i(0)=\ldots$.

gives *exponential:* $e^t=1+t+\frac{1}{2!}t^2+\frac{1}{3!}t^3+\frac{1}{4!}t^4+\frac{1}{5!}t^5+\ldots+\frac{1}{n!}t^n+$

But, how good are power series?





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Doubling (or *halving*) *time* t_2

$$e^{rt_2}=2 \quad \text{or} \quad e^{-rt_2}=\frac{1}{2}$$

$$rt_2=\ln 2=0.69$$

$$t_2=\frac{0.69}{.01r_{\%}}=\frac{69}{r_{\%}}$$

At 3% growth (r=3)
e^{rt} *doubles every*

$$t_2=\frac{69}{3}=21\text{years}$$

Unit 1
Fig. 10.2

Gives Maclaurin (or Taylor) power series

$$\boxed{x(t)=x(0)+v(0)t+\tfrac{1}{2!}a(0)t^2+\tfrac{1}{3!}j(0)t^3+\tfrac{1}{4!}i(0)t^4+\tfrac{1}{5!}r(0)t^5+\ldots+\tfrac{1}{n!}x^{(n)}t^n+}$$

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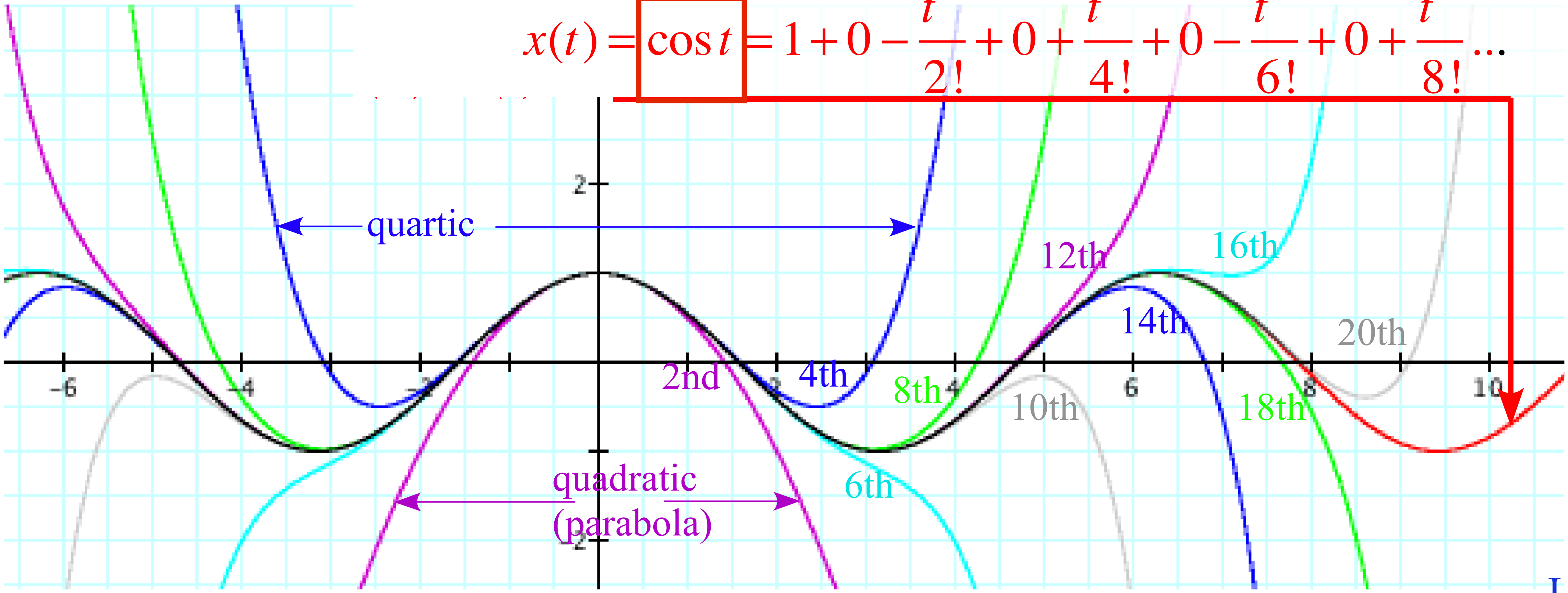
$$e^t=1+t+\tfrac{1}{2!}t^2+\tfrac{1}{3!}t^3+\tfrac{1}{4!}t^4+\tfrac{1}{5!}t^5+\ldots+\tfrac{1}{n!}t^n+$$

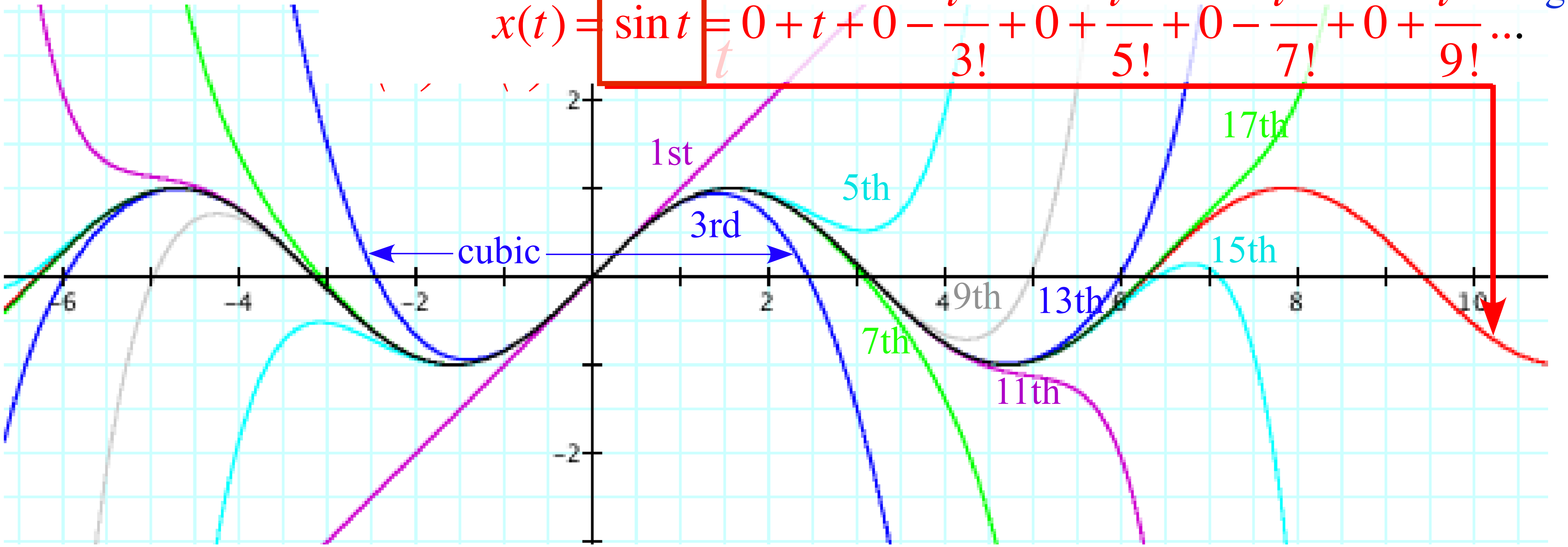
How good are power series? Depends...

$$x(t)=\boxed{\cos t}=1+0-\frac{t^2}{2!}+0+\frac{t^4}{4!}+0-\frac{t^6}{6!}+0+\frac{t^8}{8!}\ldots$$



Unit 1
Fig. 10.3

$$x(t)=\boxed{\sin t}=0+t+0-\frac{t^3}{3!}+0+\frac{t^5}{5!}+0-\frac{t^7}{7!}+0+\frac{t^9}{9!}\ldots$$



1. The Story of e (A Tale of Great $Interest$)

How good are those power series?

Taylor-Maclaurin series,

→ *imaginary interest, and complex exponentials*

Suppose the fancy bankers really went bonkers and made interest rate *r* an *imaginary number* *r*=*i*θ.

Imaginary number $i=\sqrt{-1}$ powers have *repeat-after-4-pattern:* $i^0=1$, $i^1=i$, $i^2=-1$, $i^3=-i$, $i^4=1$,*etc...*

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \ldots \qquad \text{(From exponential series)}$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \ldots \qquad (i=\sqrt{-1} \text{ imples: } i^1=i,\ i^2=-1,\ i^3=-i,\ i^4=+1,\ i^5=i,\ldots)$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right) + \left(i\theta - i\frac{\theta^3}{3!} + i\frac{\theta^5}{5!} - \ldots\right)$$

Suppose the fancy bankers really went bonkers and made interest rate *r* an *imaginary number* $r=i\theta$.

Imaginary number $i=\sqrt{-1}$ powers have *repeat-after-4-pattern:* $i^0=1,\ i^1=i,\ i^2=-1,\ i^3=-i,\ i^4=1,etc...$

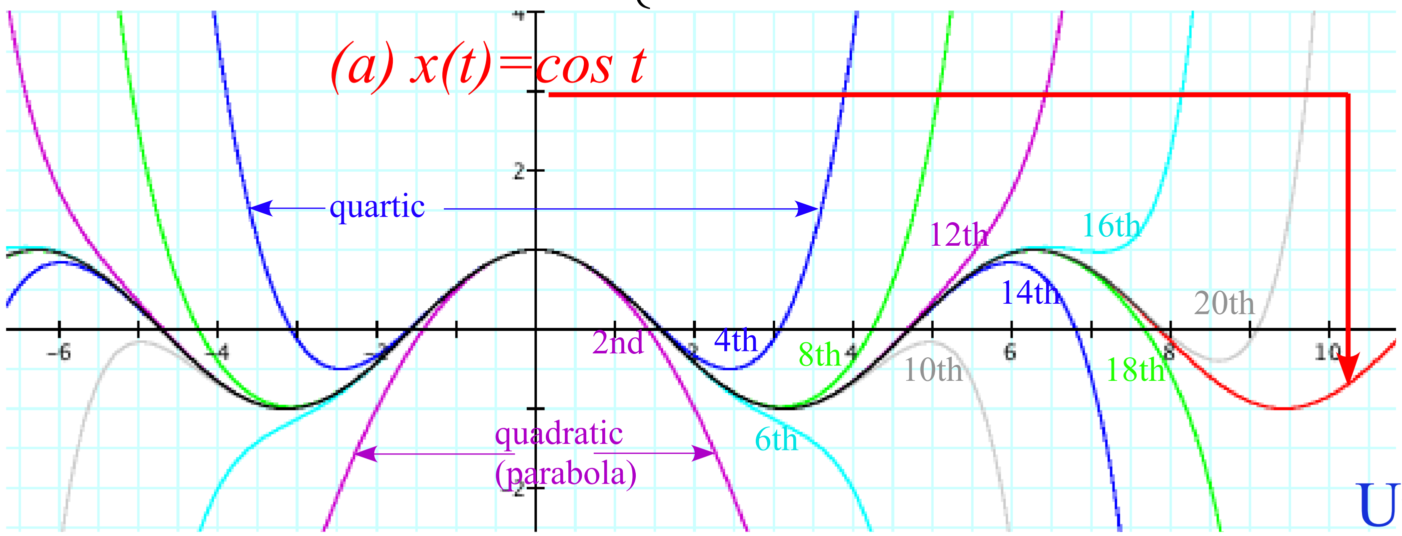
$$e^{i\theta}=1+i\theta+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}+\frac{(i\theta)^4}{4!}+\frac{(i\theta)^5}{5!}+\ldots \qquad \text{(From exponential series)}$$

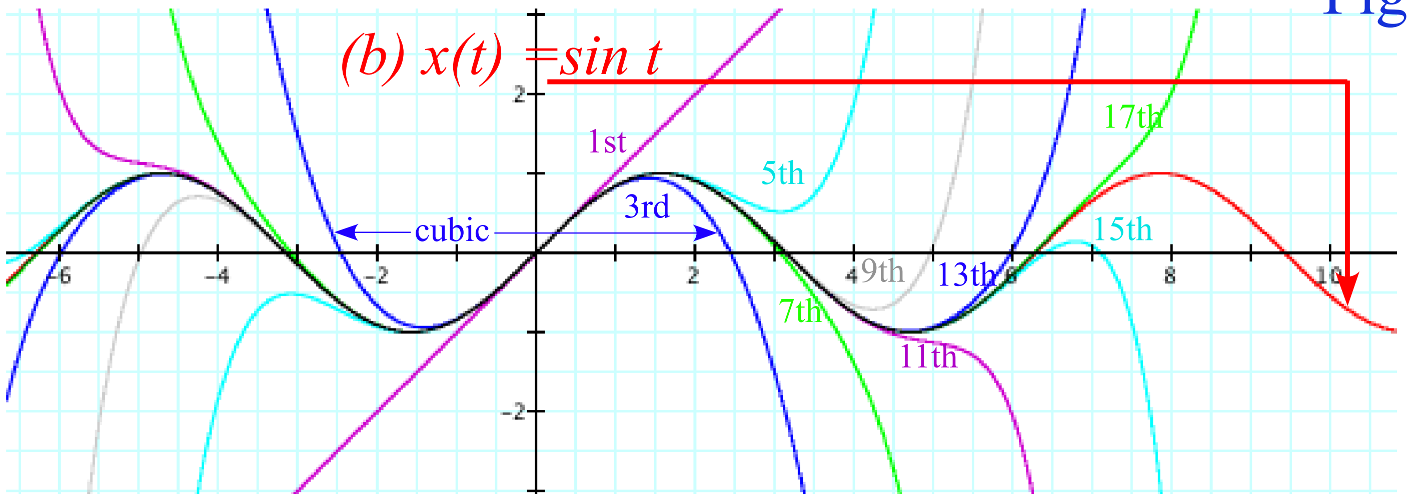
$$=1+i\theta-\frac{\theta^2}{2!}-i\frac{\theta^3}{3!}+\frac{\theta^4}{4!}+i\frac{\theta^5}{5!}-\ldots \qquad (i=\sqrt{-1}\text{ imples: }i^1=i,\ i^2=-1,\ i^3=-i,\ i^4=+1,\ i^5=i,\ldots)$$

$$=\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\ldots\right)+\left(i\theta-i\frac{\theta^3}{3!}+i\frac{\theta^5}{5!}-\ldots\right) \qquad \text{To match series for } \begin{cases} cosine:\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots \\ sine:\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots \end{cases}$$

$$\boxed{e^{i\theta}=\cos\theta+i\sin\theta}$$

Euler-DeMoivre Theorem





Unit 1
Fig. 10.3

Suppose the fancy bankers really went bonkers and made interest rate *r* an *imaginary number* $r=i\theta$.

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$$e^{i\theta}=1+i\theta+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}+\frac{(i\theta)^4}{4!}+\frac{(i\theta)^5}{5!}+\ldots$$

(From exponential series)

$$=1+i\theta-\frac{\theta^2}{2!}\quad-i\frac{\theta^3}{3!}\quad+\frac{\theta^4}{4!}\quad+i\frac{\theta^5}{5!}\quad-\ldots$$

($i=\sqrt{-1}$ imples: $i^1=i$, $i^2=-1$, $i^3=-i$, $i^4=+1$, $i^5=i$,...)

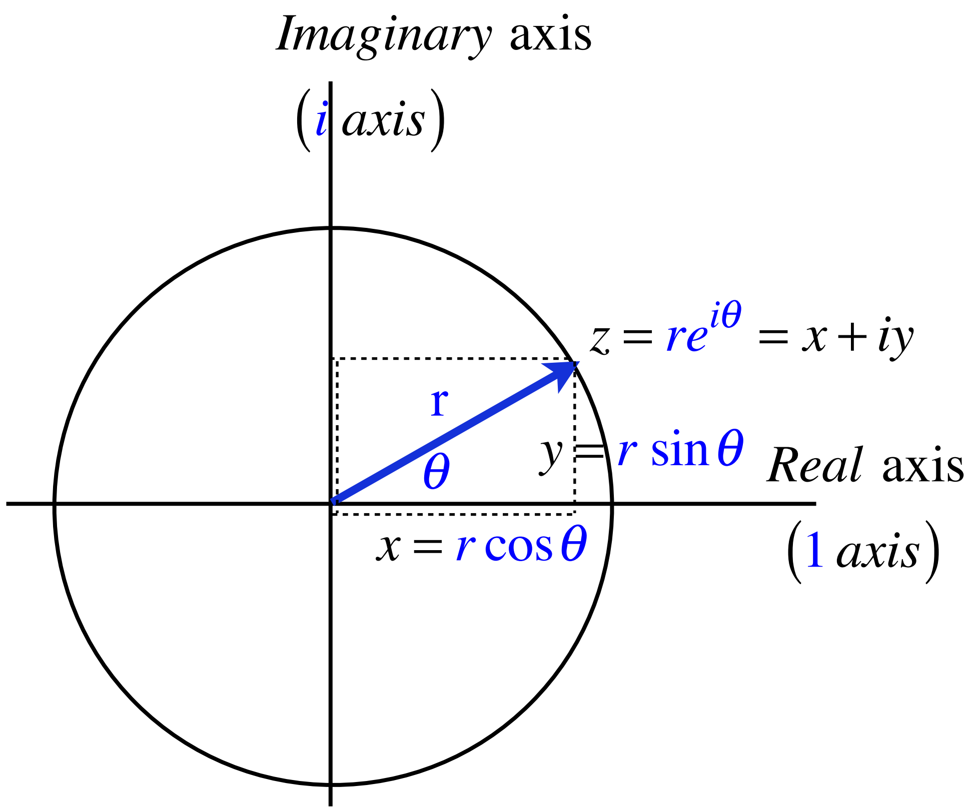
$$=\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\ldots\right)+\left(i\theta-i\frac{\theta^3}{3!}+i\frac{\theta^5}{5!}-\ldots\right)$$

To match series for

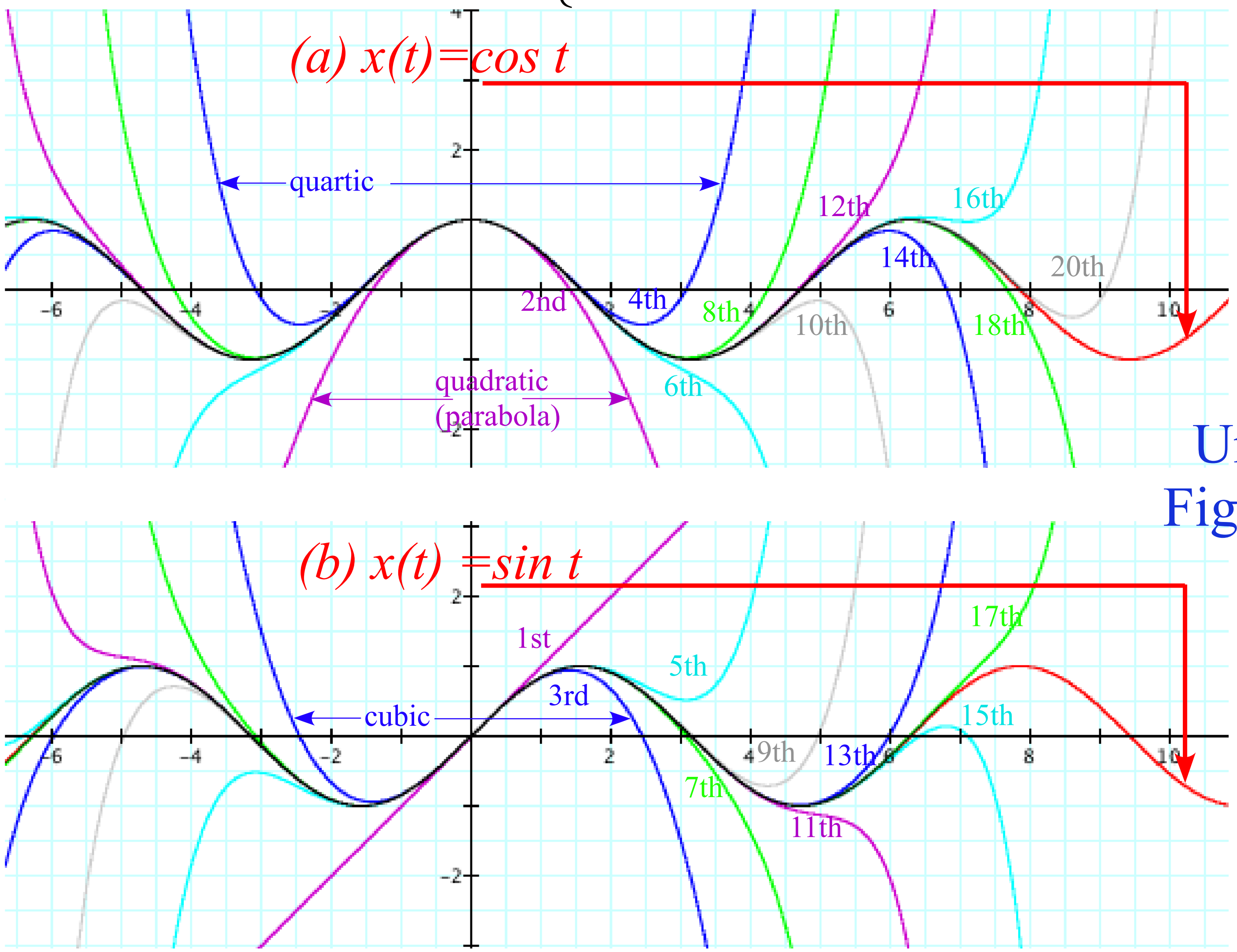
$$\begin{cases} cosine: \cos x=1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+\cdots \\ sine: \sin x=x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+\cdots \end{cases}$$

$$\boxed{e^{i\theta}\ =\ \cos\theta\ +\ i\sin\theta}$$

Euler-DeMoivre Theorem



$$re^{i\theta}\ =\ r\cos\theta+\ i\sin\theta$$



Unit 1
Fig. 10.3

Suppose the fancy bankers really went bonkers and made interest rate r an *imaginary number* $r=i\theta$.

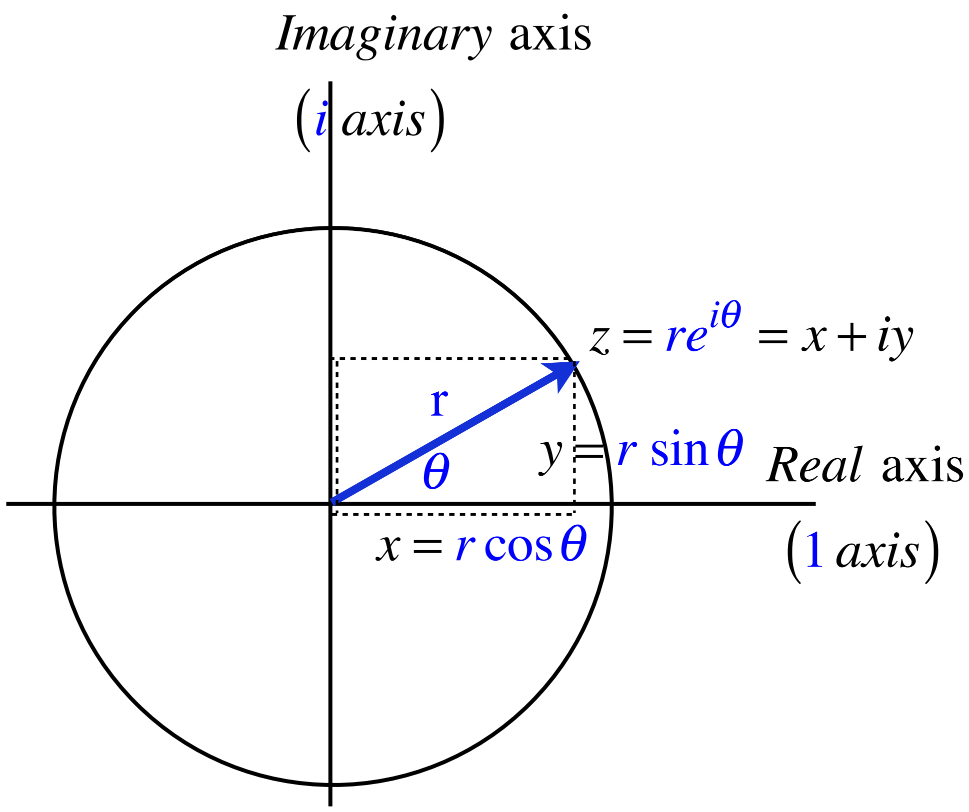
Imaginary number $i=\sqrt{-1}$ powers have *repeat-after-4-pattern:* $i^0=1$, $i^1=i$, $i^2=-1$, $i^3=-i$, $i^4=1$,*etc...*

Polar form (r,θ) *Cartesian form (x,y)*

$$z = re^{i\theta} = r\cos\theta + i\,r\sin\theta = x+iy$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Euler-DeMoivre Theorem



$$re^{i\theta} = r\cos\theta + i\sin\theta$$

Suppose the fancy bankers really went bonkers and made interest rate *r* an *imaginary number* $r=i\theta$.

Imaginary number $i=\sqrt{-1}$ powers have *repeat-after-4-pattern:* $i^0=1$, $i^1=i$, $i^2=-1$, $i^3=-i$, $i^4=1$, *etc...*

Polar form (r,θ) *Cartesian form (x,y)*

$$\boxed{\begin{array}{c} e^{i\theta} = \cos\theta + i\sin\theta \\ \textit{Euler-DeMoivre Theorem} \end{array}}$$

$$z = re^{i\theta} = r\cos\theta + i\,r\sin\theta = x+iy$$

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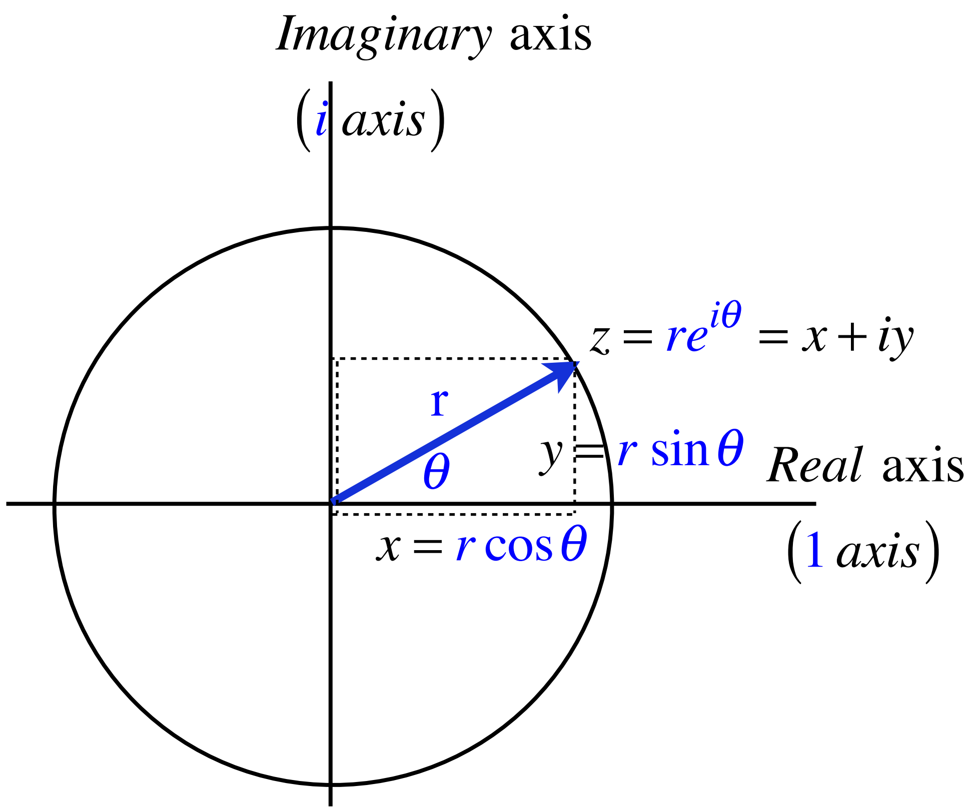

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$$z = re^{i\theta} = r\cos\theta + i\,r\sin\theta = x+iy$$

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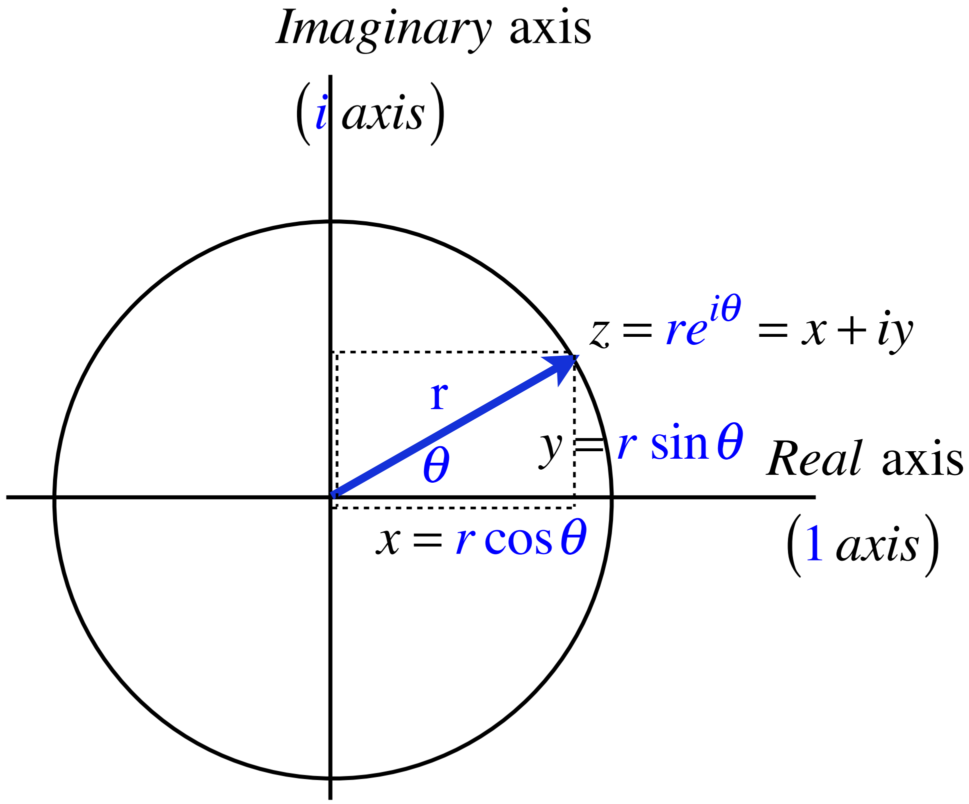
Polar form to Cartesian form

$$\frac{z+z^*}{2} = r\frac{e^{+i\theta}+e^{-i\theta}}{2} = r\cos\theta \qquad = x$$

$$\frac{z-z^*}{2i} = r\frac{e^{+i\theta}+e^{-i\theta}}{2i} = \qquad r\sin\theta = \qquad y$$



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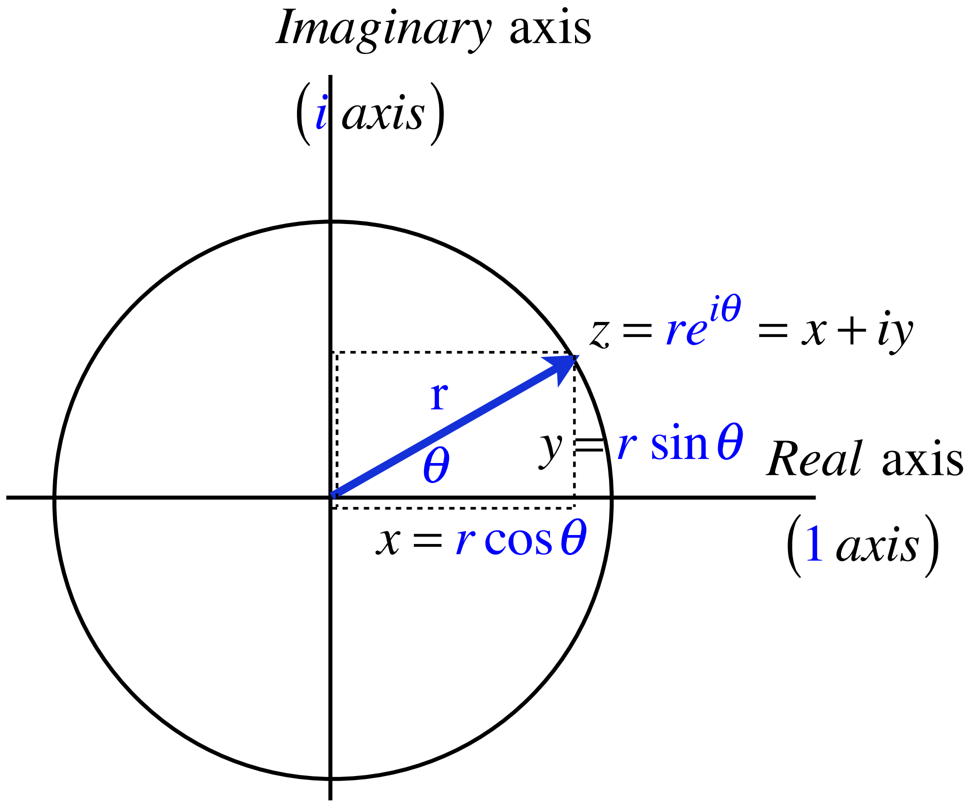
Polar form from Cartesian form

$$z^*z = r^2 = x^2 + y^2 \qquad \sqrt{z^*z} = r = \sqrt{x^2+y^2}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x} \qquad \theta = ATAN\frac{y}{x} = atan2(y,x)$$



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Suppose the fancy bankers really went bonkers and made interest rate r an *imaginary number* $r=i\theta$.

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Polar form (r,θ) *Cartesian form* (x,y)

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Euler-DeMoivre Theorem

$$z = re^{i\theta} = r\cos\theta + i\,r\sin\theta = x+iy$$

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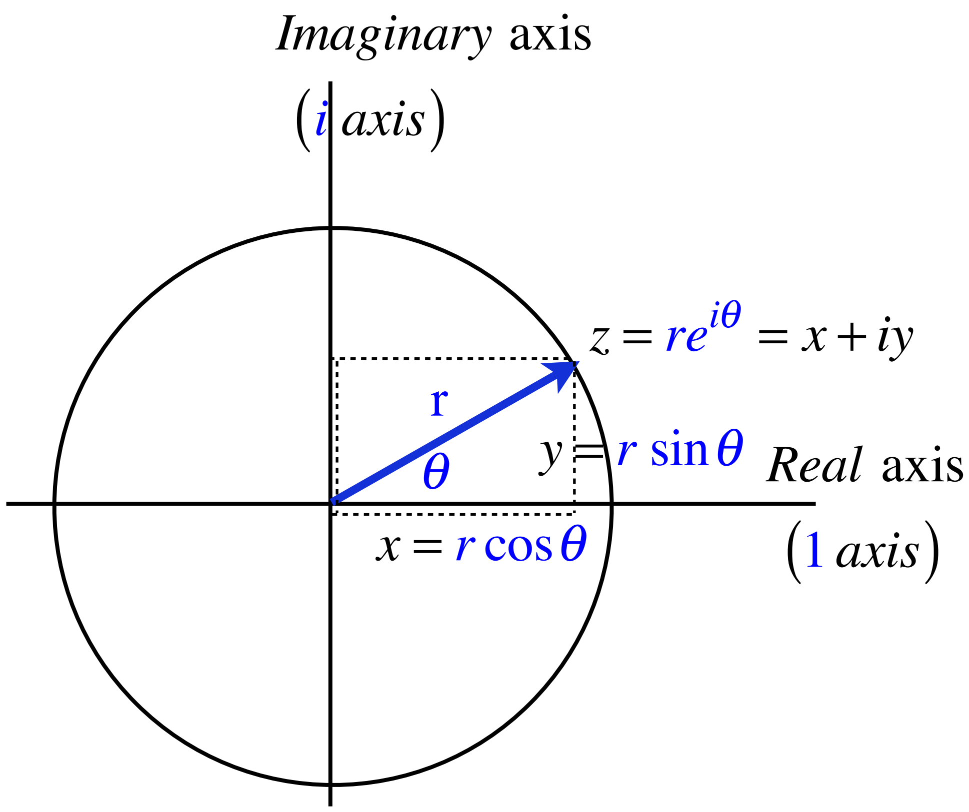
$$\boxed{r = \sqrt{z^*z} = |z| \text{ is } \textit{Modulus}} \qquad = atan2(y,x)$$



$$re^{i\theta} = r\cos\theta + i\sin\theta$$

2. What Good Are Complex Exponentials?

Easy trig
→ *Easy 2D vector analysis*
→ *Easy oscillator phase analysis*
Easy rotation and "dot" or "cross" products

What Good Are Complex Exponentials?

1. Complex numbers provide "automatic trigonometry"

Can't remember is cos$(a+b)$ or sin$(a+b)$? Just factor $e^{i(a+b)} = e^{ia}e^{ib}$...

$$
\begin{aligned}
e^{i(a+b)} &= e^{ia} \qquad e^{ib} \\
\cos(a+b) + i\sin(a+b) &= (\cos a + i\sin a)\,(\cos b + i\sin b) \\
\boxed{\cos(a+b)} + i\,\boxed{\sin(a+b)} &= \boxed{[\cos a\cos b - \sin a\sin b]} + i\,\boxed{[\sin a\cos b + \cos a\sin b]}
\end{aligned}
$$

What Good Are Complex Exponentials?

1. Complex numbers provide "automatic trigonometry"

Can't remember is $\cos(a+b)$ or $\sin(a+b)$? Just factor $e^{i(a+b)} = e^{ia}e^{ib}$...

$$e^{i(a+b)} \quad = \quad e^{ia} \quad e^{ib}$$

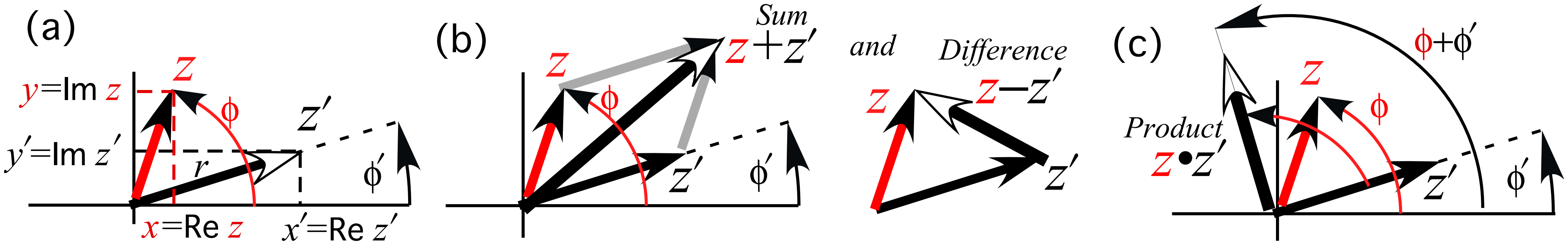
$$\cos(a+b) + i\sin(a+b) = (\cos a + i\sin a)(\cos b + i\sin b)$$

$$\boxed{\cos(a+b)} + i\,\boxed{\sin(a+b)} = \boxed{[\cos a\cos b - \sin a\sin b]} + i\,\boxed{[\sin a\cos b + \cos a\sin b]}$$

2. Complex numbers add like vectors.

$$z_{sum} = z + z' = (x + iy) + (x' + iy') = (x + x') + i(y + y')$$

$$z_{diff} = z - z' = (x + iy) - (x' + iy') = (x - x') + i(y - y')$$



Unit 1
Fig. 10.6

$$|z_{SUM}| = \sqrt{(z+z')^*(z+z')} = \sqrt{\left(re^{i\phi} + r'e^{i\phi'}\right)^*\left(re^{i\phi} + r'e^{i\phi'}\right)} = \sqrt{\left(re^{-i\phi} + r'e^{-i\phi'}\right)\left(re^{i\phi} + r'e^{i\phi'}\right)}$$

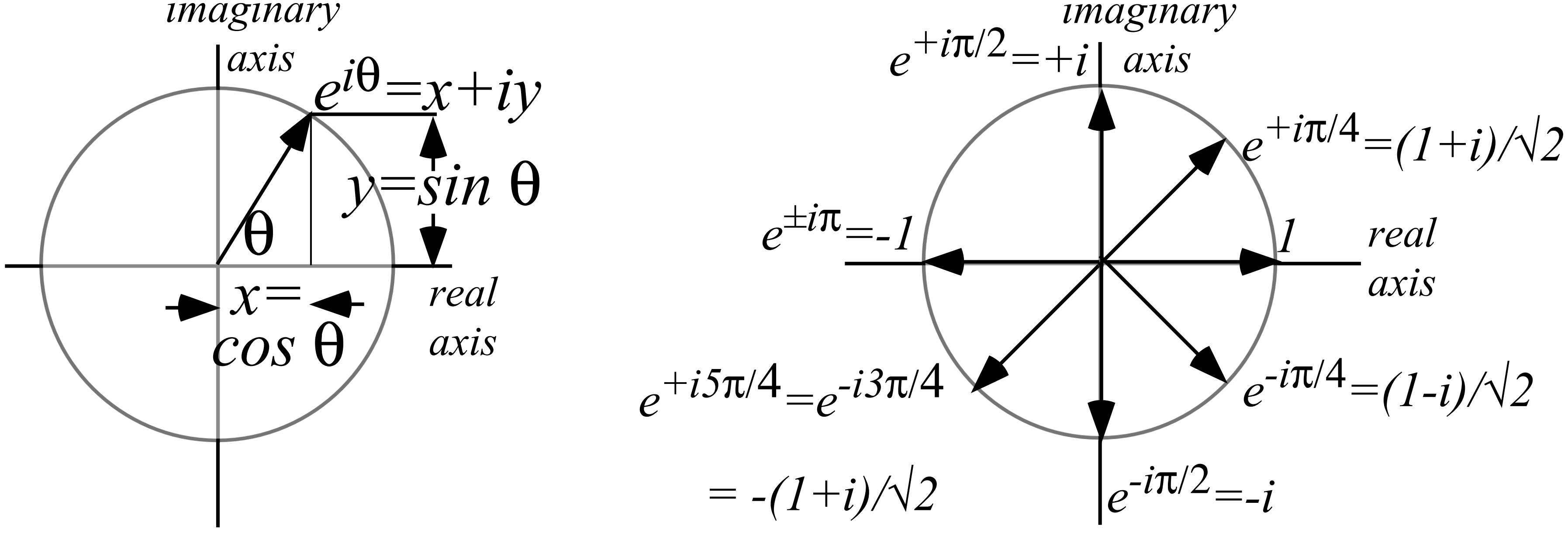
$$= \sqrt{r^2 + r'^2 + rr'\left(e^{i(\phi-\phi')} + e^{-i(\phi-\phi')}\right)} = \sqrt{r^2 + r'^2 + 2rr'\cos(\phi-\phi')}$$

(quick derivation of Cosine Law)

What Good Are Complex Exponentials? (contd.)

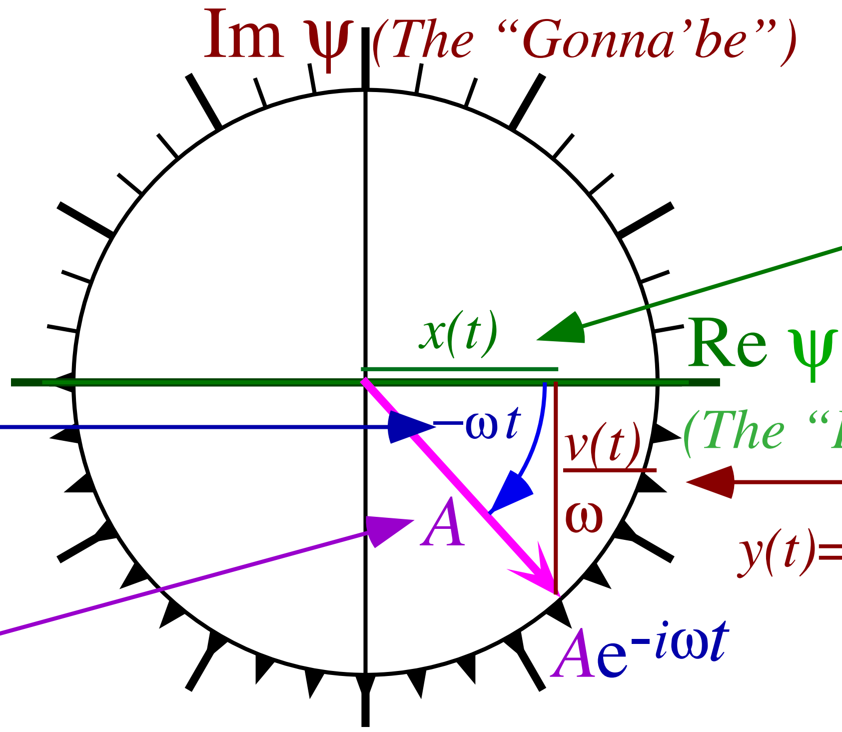
3.Complex exponentials $Ae^{-i\omega t}$ track position <u>and</u> velocity using Phasor Clock.

(a) Complex plane and unit vectors



(b) Quantum Phasor Clock $\psi = Ae^{-i\omega t} = A\cos\omega t - i\,A\sin\omega t = x+iy$

Unit 1
Fig. 10.5



Re ψ
$x(t) = A\cos\omega t$
<u>CARTESIAN COMPONENTS</u>

Im ψ
$y(t)=v(t)/\omega= -A\sin\omega t$

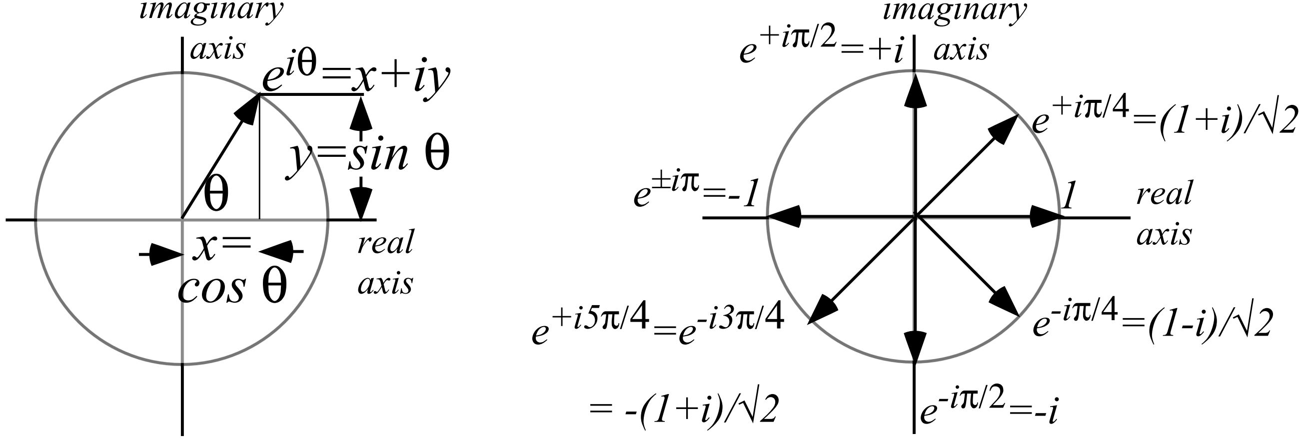
Phase angle or *Argument*
$\theta=-\omega t$ = ATAN$[v(t)/\omega x(t)]$
<u>POLAR COMPONENTS</u>

Magnitude or *Modulus*
$A = |\psi| = \sqrt{\psi^*\psi}$

What Good Are Complex Exponentials? (contd.)

3.Complex exponentials $Ae^{-i\omega t}$ track position and velocity using Phasor Clock.

(a) Complex plane and unit vectors



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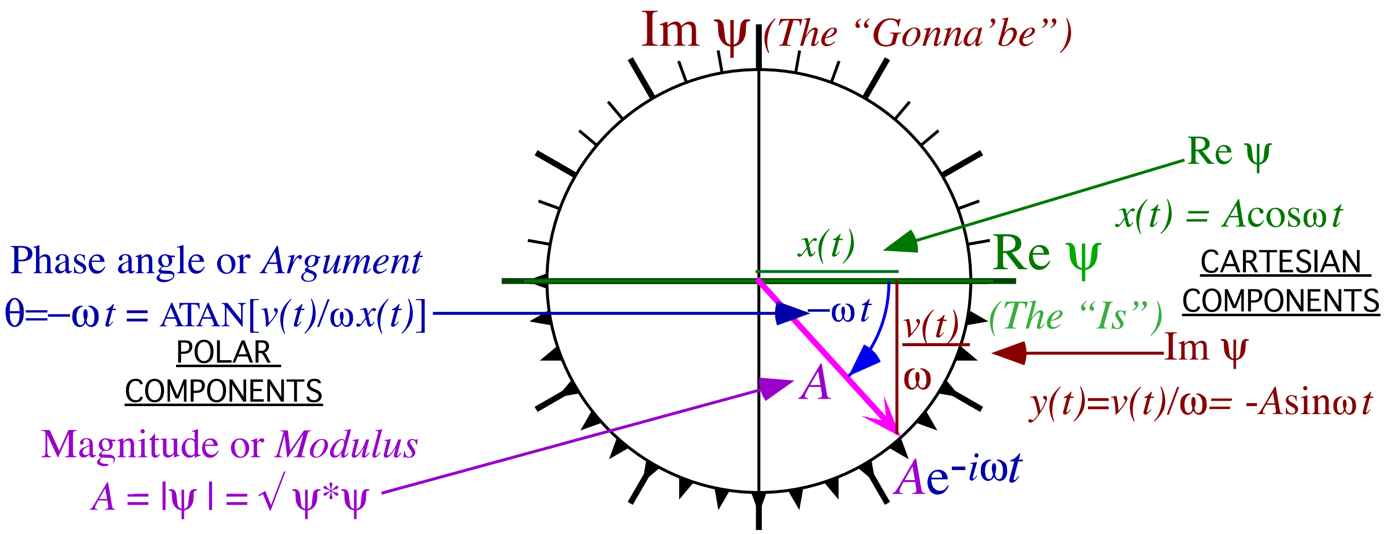

Unit 1
Fig. 10.5

Some Rect-vs-Polar *relations worth remembering*

Cartesian (x,y) form

$$\begin{cases} \psi_x = \mathrm{Re}\,\psi(t) = x(t) = A\cos\omega t = \dfrac{\psi+\psi^*}{2} \\ \psi_y = \mathrm{Im}\,\psi(t) = \dfrac{v(t)}{\omega} = -A\sin\omega t = \dfrac{\psi-\psi^*}{2i} \end{cases}$$

$$\psi = re^{+i\theta} = re^{-i\omega t} = r(\cos\omega t - i\sin\omega t)$$

$$\psi^* = re^{-i\theta} = re^{+i\omega t} = r(\cos\omega t + i\sin\omega t)$$

Polar (r,θ) form

$$\begin{cases} r = A = |\psi| = \sqrt{\psi_x{}^2+\psi_y{}^2} = \sqrt{\psi^*\psi} \\ \theta = -\omega t = \arctan(\psi_y/\psi_x) \end{cases}$$

$$\cos\theta = \tfrac{1}{2}(e^{+i\theta}+e^{-i\theta}) \qquad \mathrm{Re}\,\psi = \frac{\psi+\psi^*}{2}$$

$$\sin\theta = \tfrac{1}{2i}(e^{+i\theta}-e^{-i\theta}) \qquad \mathrm{Im}\,\psi = \frac{\psi-\psi^*}{2i}$$

2. What Good Are Complex Exponentials?

Easy trig

Easy 2D vector analysis

Easy oscillator phase analysis

→ *Easy rotation and “dot” or “cross” products*

What Good Are Complex Exponentials? (contd.)

4. Complex products provide 2D rotation operations.

$$e^{i\phi}\cdot z = (\cos\phi + i\sin\phi)\cdot(x + iy) = x\cos\phi - y\sin\phi \quad + i \quad (x\sin\phi + y\cos\phi)$$

$$\mathbf{R}_{+\phi}\bullet\mathbf{r} = (x\cos\phi - y\sin\phi)\hat{\mathbf{e}}_x + (x\sin\phi + y\cos\phi)\hat{\mathbf{e}}_y$$

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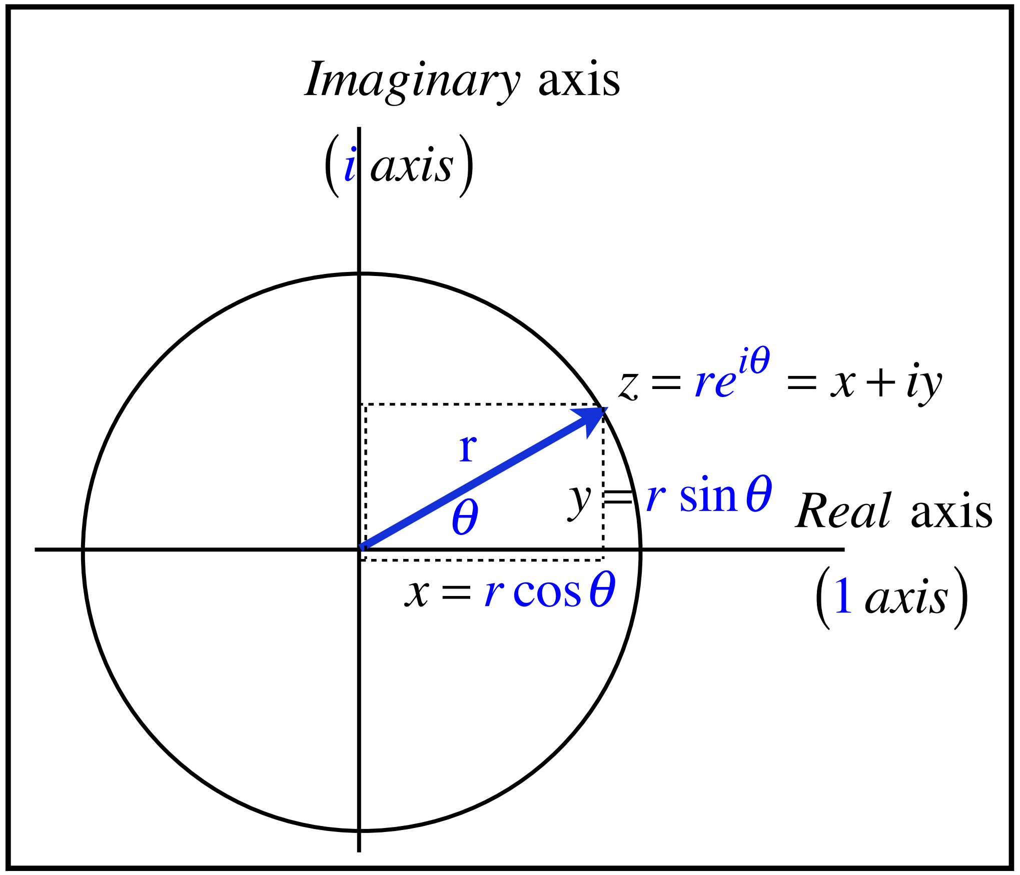
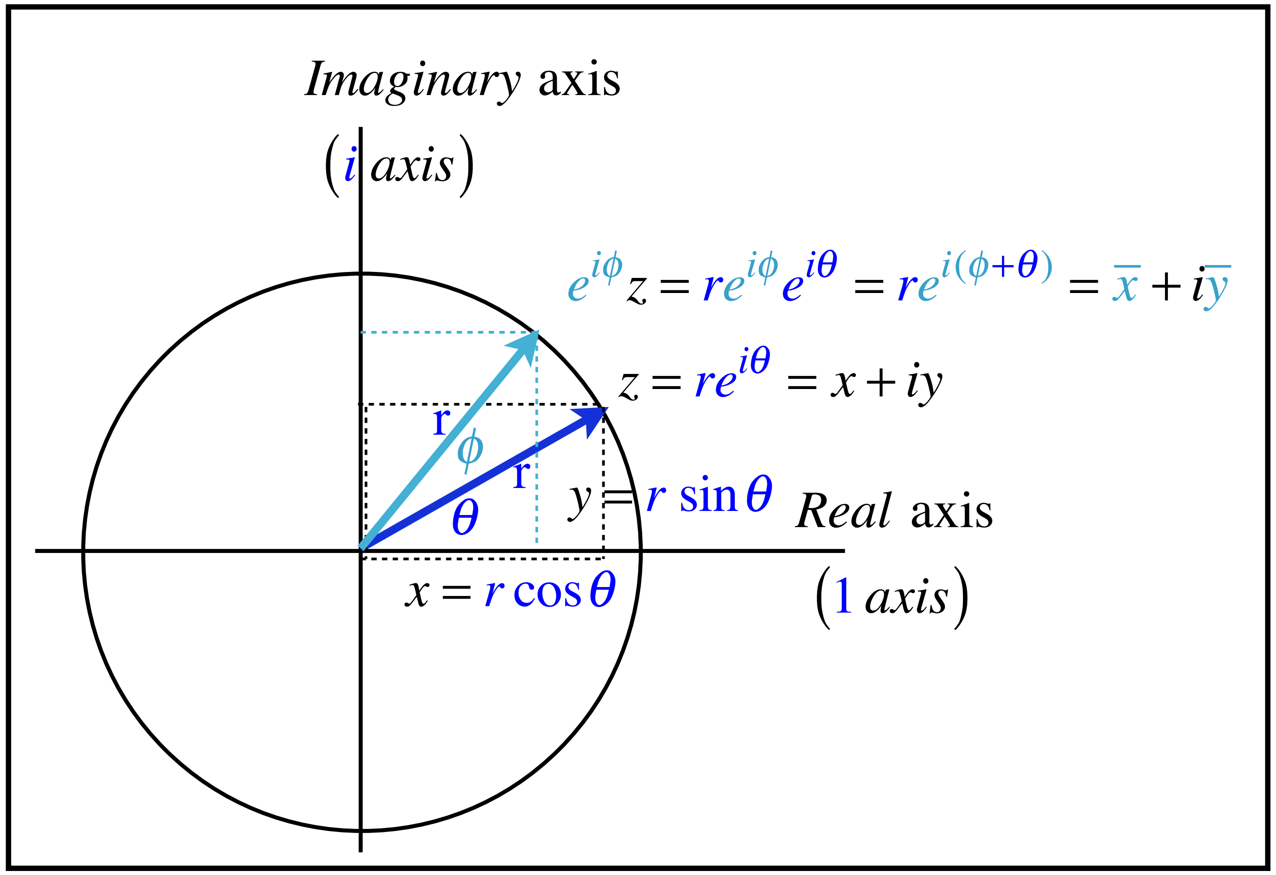
$e^{i\phi}$ acts on this: $z = re^{i\theta}$

to give this: $e^{i\phi}z = re^{i\phi}e^{i\theta} = re^{i(\phi+\theta)}$





What Good Are Complex Exponentials? (contd.)

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5. Complex products provide 2D “dot”(•) and “cross”(x) products.

Two complex numbers $A=A_x+iA_y$ and $B=B_x+iB_y$ and their “star” (*)-product A^*B.

$$A^*B = (A_x+iA_y)^*(B_x+iB_y) = (A_x-iA_y)(B_x+iB_y)$$
$$= (A_xB_x + A_yB_y) + i(A_xB_y - A_yB_x)$$

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$$A*B = (A_x+iA_y)^*(B_x+iB_y) = (A_x-iA_y)(B_x+iB_y)$$

$$= (A_xB_x + A_yB_y) + i(A_xB_y - A_yB_x) = \mathbf{A}\bullet\mathbf{B} + i\,|\mathbf{A}\times\mathbf{B}|_{Z\perp(x,y)}$$

Real part is scalar or "dot"(•) product **A•B**.

Imaginary part is vector or "cross"(×) product, but just the *Z*-component *normal* to *xy*-plane.

What Good Are Complex Exponentials? (contd.)

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Rewrite A^*B in polar form.

$$A^*B = (|A|e^{i\theta_A})^*(|B|e^{i\theta_B}) = |A|e^{-i\theta_A}|B|e^{i\theta_B} = |A||B|e^{i(\theta_B-\theta_A)}$$

$$= |A||B|\cos(\theta_B - \theta_A) + i|A||B|\sin(\theta_B - \theta_A) = \mathbf{A}\bullet\mathbf{B} + i\,|\,\mathbf{A}\times\mathbf{B}\,|_{Z\perp(x,y)}$$

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$$\mathbf{A}\bullet\mathbf{B} = |A||B|\cos(\theta_B-\theta_A)$$
$$= |A|\cos\theta_A|B|\cos\theta_B + |A|\sin\theta_A|B|\sin\theta_B$$
$$= \quad A_xB_x \quad + \quad A_yB_y$$

$$|\,\mathbf{A}\times\mathbf{B}\,| = |A||B|\sin(\theta_B-\theta_A)$$
$$= |A|\cos\theta_A|B|\sin\theta_B - |A|\sin\theta_A|B|\cos\theta_B$$
$$= \quad A_xB_y \quad - \quad A_yB_x$$

What Good are complex variables?

- *Easy 2D vector calculus*
  - → *Easy 2D vector derivatives*
    - *Easy 2D source-free field theory*
      - *Easy 2D vector field-potential theory*

What Good Are Complex Exponentials? (contd.)

6. Complex derivative contains "divergence"(∇•F) and "curl"(∇xF) of 2D vector field

Relation of (z,z^*) to $(x=\text{Re}z, y=\text{Im}z)$ defines a z-derivative $\frac{df}{dz}$ and "star" z^*-derivative. $\frac{df}{dz^*}$

$$z = x + iy \qquad z^* = x - iy$$

$$x = \tfrac{1}{2}(z + z^*) \qquad y = \tfrac{1}{2i}(z - z^*) = \tfrac{-i}{2}(z - z^*)$$

Applying chain-rule

$$\frac{df}{dz} = \frac{\partial x}{\partial z}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{i}{2}\frac{\partial f}{\partial y} \qquad \frac{d}{dz} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y}$$

$$\frac{df}{dz^*} = \frac{\partial x}{\partial z^*}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z^*}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{i}{2}\frac{\partial f}{\partial y} \qquad \frac{d}{dz^*} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{i}{2}\frac{\partial}{\partial y}$$

Discussion of partial derivatives ∂f/∂x and chain-rule df = ∂f/∂x dx + ∂f/∂y dy

http://www.uark.edu/ua/modphys/pdfs/CMwBang_Pdfs/CMwBang_Lectures_2015/CMwithBang_Lect.9_9.22.15.pdf

What Good Are Complex Exponentials? (contd.)

6. Complex derivative contains “divergence”(∇•F) and “curl”(∇xF) of 2D vector field

Relation of *(z,z*)* to *(x*=Re*z,y*=Im*z)* defines a *z*-derivative $\frac{df}{dz}$ and “star” *z**-derivative. $\frac{df}{dz^*}$

$$z = x + iy \qquad x = \tfrac{1}{2}(z + z^*)$$

$$z^* = x - iy \qquad y = \tfrac{1}{2i}(z - z^*) = \tfrac{-i}{2}(z - z^*)$$

$$\frac{df}{dz} = \frac{\partial x}{\partial z}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{i}{2}\frac{\partial f}{\partial y} \qquad \frac{d}{dz} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y}$$

$$\frac{df}{dz^*} = \frac{\partial x}{\partial z^*}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z^*}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{i}{2}\frac{\partial f}{\partial y} \qquad \frac{d}{dz^*} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{i}{2}\frac{\partial}{\partial y}$$

$$\frac{d}{dz} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y}$$

Derivative chain-rule shows real part of $\frac{df}{dz}$ has 2D divergence $\nabla\bullet\mathbf{f}$ and imaginary part has curl $\nabla\times\,\mathbf{f}$.

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6. Complex derivative contains "divergence"($\nabla\cdot\mathbf{F}$) and "curl"($\nabla\times\mathbf{F}$) of 2D vector field

Relation of *(z,z*)* to *(x=*Re*z,y=*Im*z)* defines a *z*-derivative $\frac{df}{dz}$ and "star" *z**-derivative. $\frac{df}{dz*}$

$$z = x + iy \qquad x = \tfrac{1}{2}(z + z*)$$

$$z^* = x - iy \qquad y = \tfrac{1}{2i}(z - z*) = \tfrac{-i}{2}(z - z*)$$

$$\boxed{\frac{df}{dz} = \frac{\partial x}{\partial z}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{i}{2}\frac{\partial f}{\partial y}} \qquad \frac{d}{dz} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y}$$

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7. Invent source-free 2D vector fields [$\nabla\cdot\mathbf{F}=0$ *and* $\nabla\times\mathbf{F}=0$]

We can invent *source-free 2D vector fields* that are both *zero-divergence* and *zero-curl*.
Take any function *f(z)*, conjugate it (change all *i*'s to –*i*) to give *f*(z*)* for which $\boxed{\frac{df^*}{dz} = 0}$

What Good Are Complex Exponentials? (contd.)

6. Complex derivative contains "divergence"($\nabla\cdot\mathbf{F}$) and "curl"($\nabla\times\mathbf{F}$) of 2D vector field

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For example: if *f(z)=a·z* then $f^*(z^*)=a\cdot z^*=a(x-iy)$ is not function of *z* so it has *zero z-derivative*.

$\mathbf{F}=(F_x,F_y)=(f^*_x,f^*_y)=(a\cdot x,-a\cdot y)$ *has zero divergence:* $\nabla\bullet\mathbf{F}=0$ *and has zero curl:* $|\nabla\times\mathbf{F}|=0$.

$$\nabla\bullet\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} = \frac{\partial(ax)}{\partial x} + \frac{\partial F(-ay)}{\partial y} = 0 \qquad |\nabla\times\mathbf{F}|_{Z\perp(x,y)} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \frac{\partial(-ay)}{\partial x} - \frac{\partial F(ax)}{\partial y} = 0$$

A *DFL* field **F** (*Divergence-Free-Laminar*)

What Good Are Complex Exponentials? (contd.)

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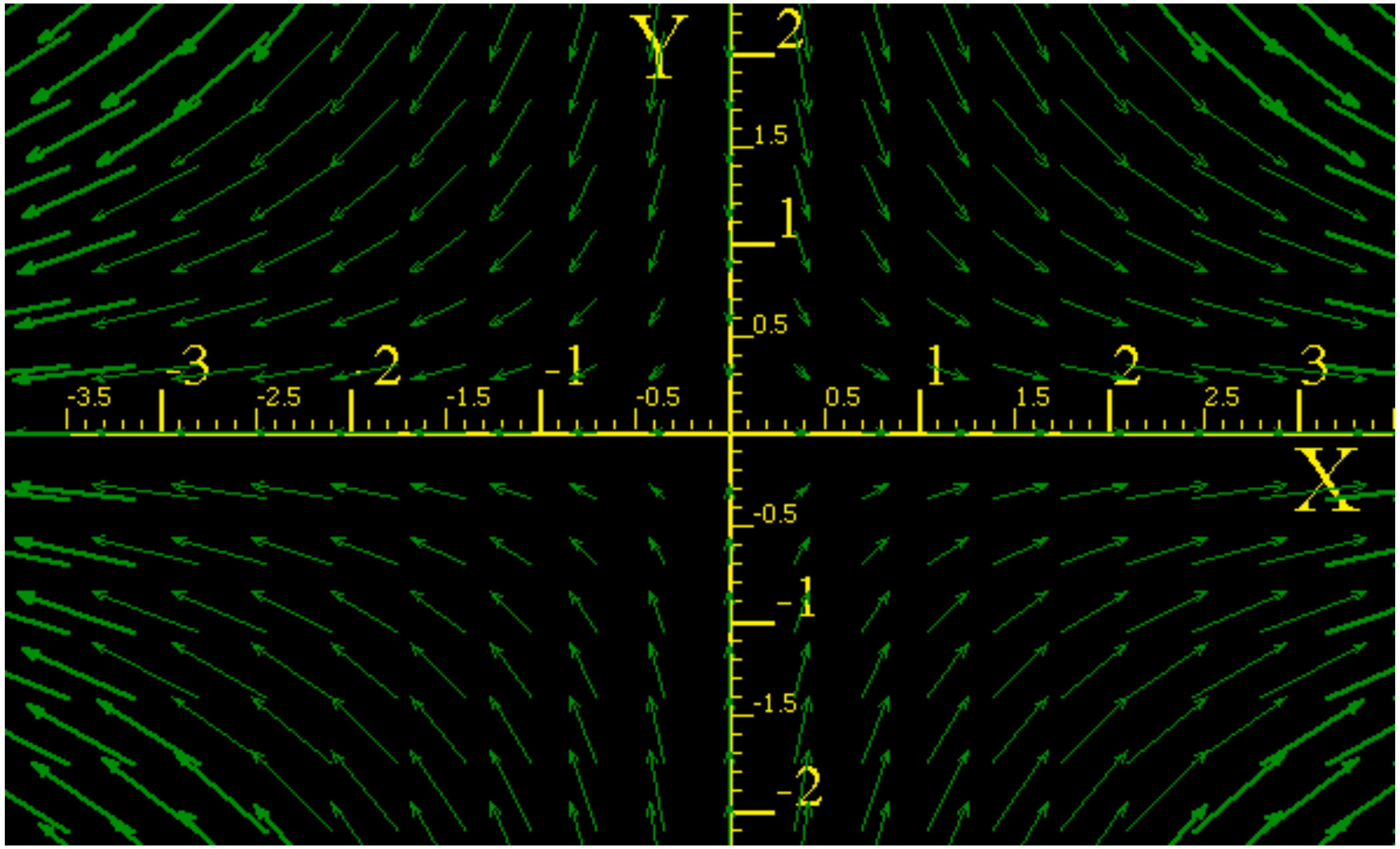

precursor to
Unit 1
Fig. 10.7

$\mathbf{F}=(f^*_x,f^*_y)=(a\cdot x,-a\cdot y)$ is a *divergence-free laminar (DFL)* field.

What Good are complex variables?

Easy 2D vector calculus
Easy 2D vector derivatives
Easy 2D source-free field theory
→ *Easy 2D vector field-potential theory*

What Good Are Complex Exponentials? (contd.)

8. Complex potential ϕ contains "scalar"($\mathbf{F}=\nabla\mathbf{\Phi}$) and "vector"($\mathbf{F}=\nabla x\mathbf{A}$) potentials

Any *DFL* field **F** is a gradient of a *scalar potential field* **Φ** or a curl of a *vector potential field* **A**.

$$\mathbf{F}=\nabla\mathbf{\Phi} \qquad\qquad \mathbf{F}=\nabla\times\mathbf{A}$$

A *complex potential* $\phi(z)=\mathbf{\Phi}(x,y)+i\mathbf{A}(x,y)$ exists whose z-derivative is $f(z)=d\phi/dz$.

Its complex conjugate $\phi^*(z^*)=\mathbf{\Phi}(x,y)-i\mathbf{A}(x,y)$ has z^*-derivative $f^*(z^*)=d\phi^*/dz^*$ giving *DFL* field **F**.

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To find $\phi=\mathbf{\Phi}+i\mathbf{A}$ integrate $f(z)=a\cdot z$ to get ϕ and isolate real ($\mathrm{Re}\,\phi=\mathbf{\Phi}$) and imaginary ($\mathrm{Im}\,\phi=\mathbf{A}$) parts.

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$$f(z)=\frac{d\phi}{dz} \quad \Rightarrow \quad \phi = \quad \Phi \quad +i\ \mathbf{A}=\int f\cdot dz=\int az\cdot dz=\frac{1}{2}\ az^2$$

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$$= \overbrace{\tfrac{1}{2}a(x^2-y^2)}^{\mathbf{\Phi}} + i\,\overbrace{axy}^{\mathbf{A}}$$

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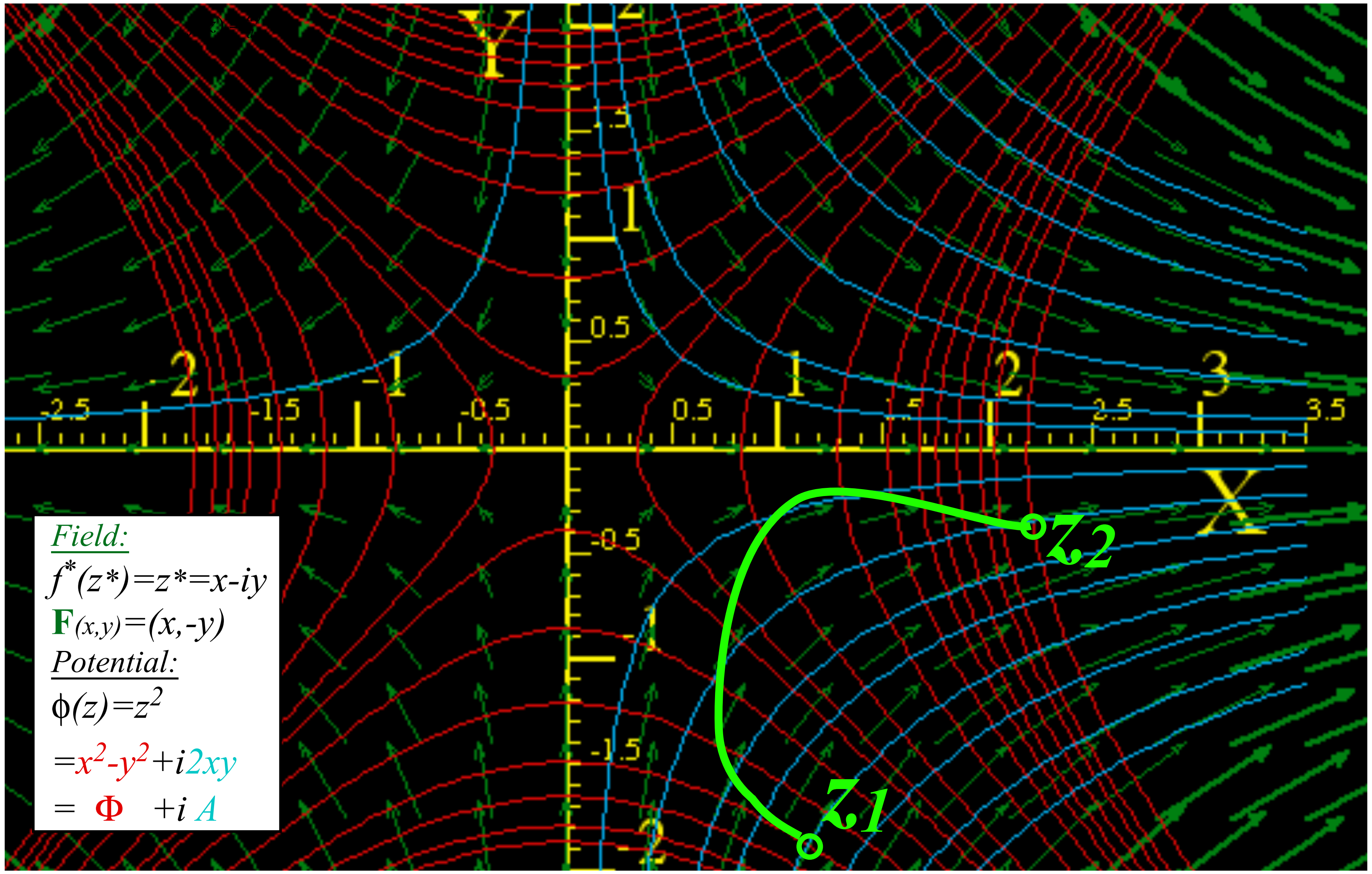
Unit 1
Fig. 10.7



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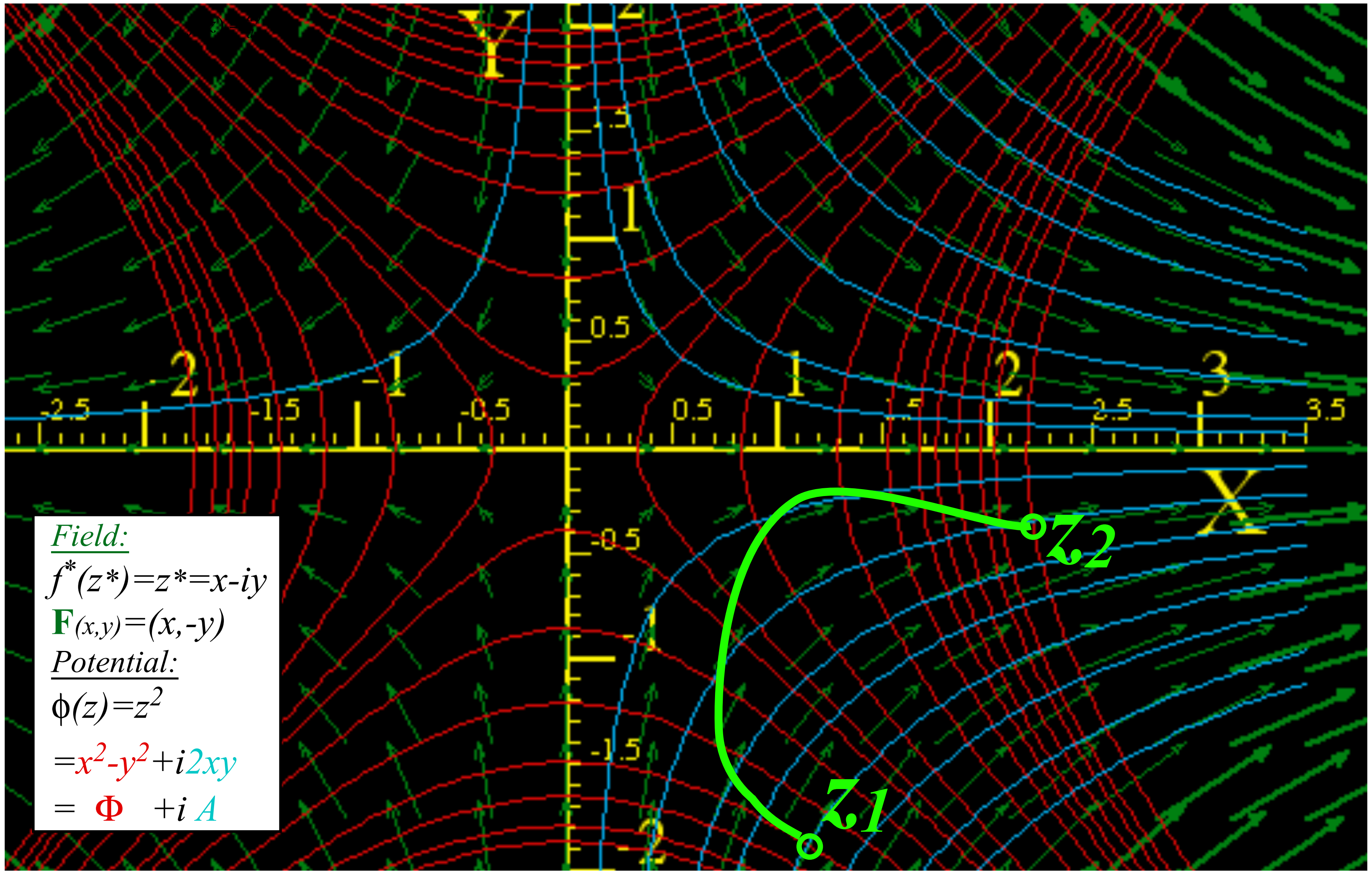
BONUS!
Get a free
coordinate
system!

Unit 1
Fig. 10.7



Field:
$f^*(z^*)=z^*=x-iy$
$\mathbf{F}_{(x,y)}=(x,-y)$
Potential:
$\phi(z)=z^2$
$=x^2-y^2+i2xy$
$=\Phi\ +i\,A$

The (Φ,A) grid is a GCC coordinate system*:

$q^1=\Phi=(x^2-y^2)/2=$ *const.*

$q^2=\mathbf{A}=(xy)=$ *const.*

*Actually it's OCC.

What Good are complex variables?

Easy 2D vector calculus

Easy 2D vector derivatives

Easy 2D source-free field theory

→ *Easy 2D vector field-potential theory*

→ *The half-n'-half results: (Riemann-Cauchy Derivative Relations)*

What Good Are Complex Exponentials? (contd.)

*8. (contd.) Complex potential ϕ contains “scalar”(**F=**$\nabla\Phi$) and “vector”(**F=**∇x**A**) potentials*

...and either one (or half-n’-half!) works just as well.

Derivative $\frac{d\phi^*}{dz^*}$ has 2D gradient $\nabla\Phi=\begin{pmatrix}\frac{\partial\Phi}{\partial x}\\ \frac{\partial\Phi}{\partial y}\end{pmatrix}$ of scalar Φ and curl $\nabla\times\mathbf{A}=\begin{pmatrix}\frac{\partial\mathbf{A}}{\partial y}\\ -\frac{\partial\mathbf{A}}{\partial x}\end{pmatrix}$ of vector **A** (*and they’re equal!*)

$f(z)=\frac{d\phi}{dz} \quad \Rightarrow$

$$\frac{d}{dz^*}\phi^*=\frac{d}{dz^*}(\Phi-i\mathbf{A})=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(\Phi-i\mathbf{A})=\frac{1}{2}\overbrace{\left(\frac{\partial\Phi}{\partial x}+i\frac{\partial\Phi}{\partial y}\right)}+\frac{1}{2}\overbrace{\left(\frac{\partial\mathbf{A}}{\partial y}-i\frac{\partial\mathbf{A}}{\partial x}\right)}=\frac{1}{2}\nabla\Phi+\frac{1}{2}\nabla\times\mathbf{A}$$

$$\frac{d}{dz}=\frac{1}{2}\frac{\partial}{\partial x}-\frac{i}{2}\frac{\partial}{\partial y}$$

$$\frac{d}{dz^*}=\frac{1}{2}\frac{\partial}{\partial x}+\frac{i}{2}\frac{\partial}{\partial y}$$

What Good Are Complex Exponentials? (contd.)

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$f(z)=\frac{d\phi}{dz} \quad\Rightarrow$

$$\frac{d}{dz^*}\phi^* = \frac{d}{dz^*}(\mathbf{\Phi}-i\,\mathbf{A}) = \frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(\mathbf{\Phi}-i\,\mathbf{A}) = \frac{1}{2}\overbrace{\left(\frac{\partial\mathbf{\Phi}}{\partial x}+i\frac{\partial\mathbf{\Phi}}{\partial y}\right)}+\frac{1}{2}\overbrace{\left(\frac{\partial\mathbf{A}}{\partial y}-i\frac{\partial\mathbf{A}}{\partial x}\right)} = \frac{1}{2}\nabla\mathbf{\Phi}+\frac{1}{2}\nabla\times\mathbf{A}$$

Note, *mathematician definition* of force field **F**=+∇**Φ** replaces usual physicist’s definition **F**=-∇**Φ**

What Good Are Complex Exponentials? (contd.)

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$f(z)=\frac{d\phi}{dz} \quad \Rightarrow$

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Given ϕ:

$$\boxed{\begin{aligned}\phi &= \quad \Phi \quad + i\ \mathbf{A}\\ &= \tfrac{1}{2}a(x^2-y^2) + i\ axy\end{aligned}}$$

The half-n'-half result

find:

$$\nabla\Phi = \begin{pmatrix}\frac{\partial\Phi}{\partial x}\\ \frac{\partial\Phi}{\partial y}\end{pmatrix} = \begin{pmatrix}\frac{\partial}{\partial x}\frac{a}{2}(x^2-y^2)\\ \frac{\partial}{\partial y}\frac{a}{2}(x^2-y^2)\end{pmatrix} = \begin{pmatrix}ax\\ -ay\end{pmatrix} = \mathbf{F}$$

or find:

$$\nabla\times\mathbf{A} = \begin{pmatrix}\frac{\partial\mathbf{A}}{\partial y}\\ -\frac{\partial\mathbf{A}}{\partial x}\end{pmatrix} = \begin{pmatrix}\frac{\partial}{\partial y}axy\\ -\frac{\partial}{\partial x}axy\end{pmatrix} = \begin{pmatrix}ax\\ -ay\end{pmatrix} = \mathbf{F}$$

What Good Are Complex Exponentials? (contd.)

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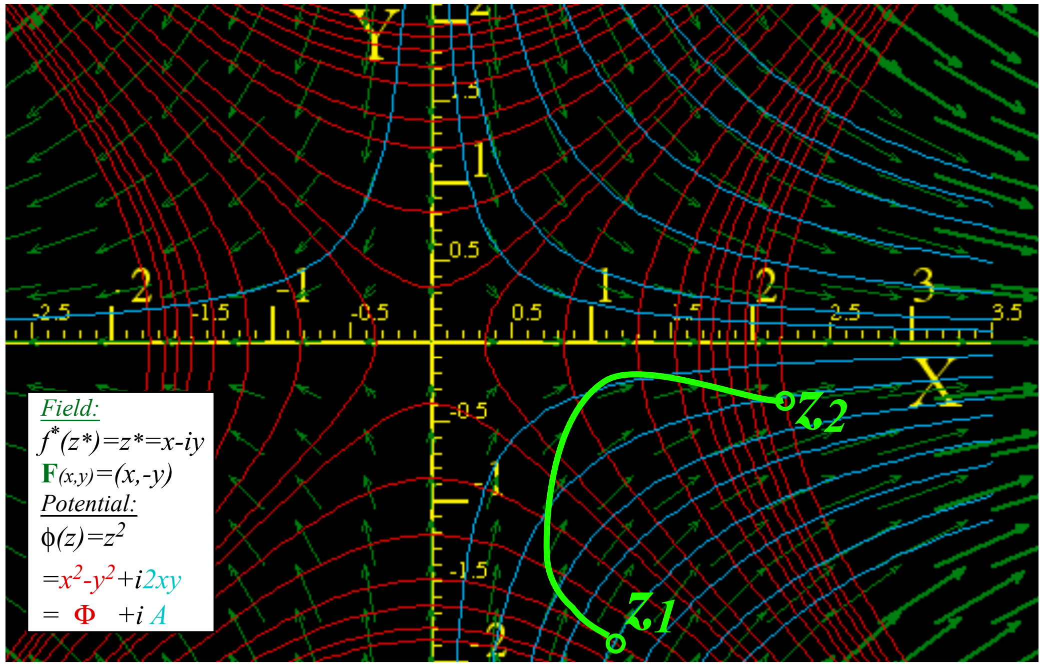
Scalar *static potential lines* Φ=*const.* and vector *flux potential lines* $\mathbf{A}$=*const.* define *DFL field-net*.



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The half-n'-half result

$$\frac{d}{dz^*}\phi^*=\frac{d}{dz^*}(\Phi-i\mathbf{A})=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(\Phi-i\mathbf{A})=\frac{1}{2}\left(\frac{\partial\Phi}{\partial x}+i\frac{\partial\Phi}{\partial y}\right)+\frac{1}{2}\left(\frac{\partial\mathbf{A}}{\partial y}-i\frac{\partial\mathbf{A}}{\partial x}\right)=\frac{1}{2}\nabla\Phi+\frac{1}{2}\nabla\times\mathbf{A}$$

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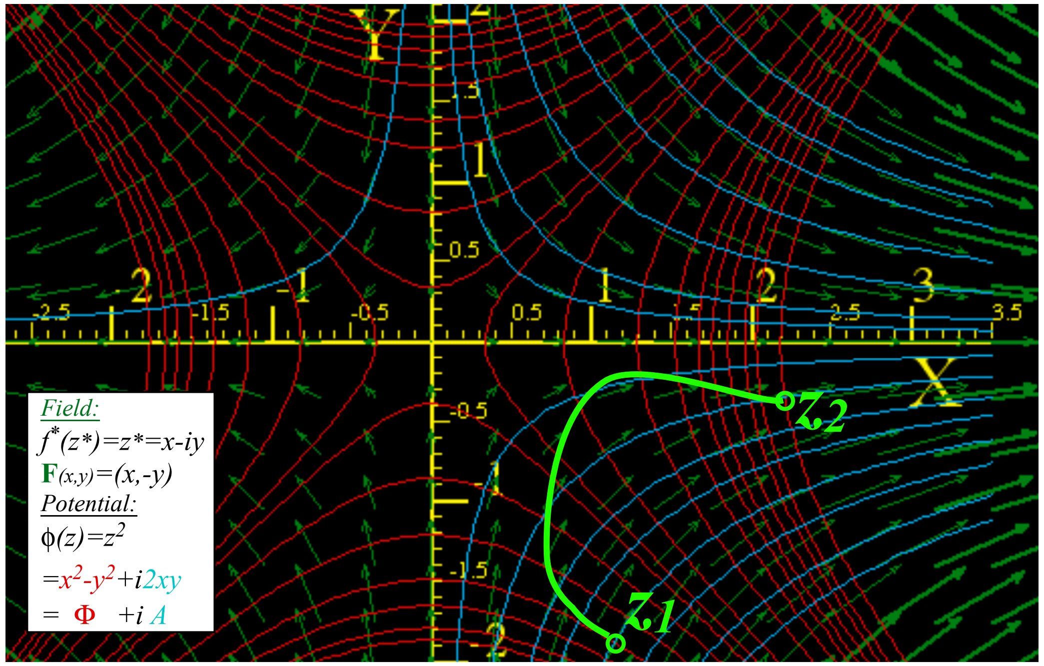
$$\nabla\Phi=\begin{pmatrix}\frac{\partial\Phi}{\partial x}\\ \frac{\partial\Phi}{\partial y}\end{pmatrix}=\begin{pmatrix}\frac{\partial}{\partial x}\frac{a}{2}(x^2-y^2)\\ \frac{\partial}{\partial y}\frac{a}{2}(x^2-y^2)\end{pmatrix}=\begin{pmatrix}ax\\ -ay\end{pmatrix}=\mathbf{F}$$

or find:

$$\nabla\times\mathbf{A}=\begin{pmatrix}\frac{\partial\mathbf{A}}{\partial y}\\ -\frac{\partial\mathbf{A}}{\partial x}\end{pmatrix}=\begin{pmatrix}\frac{\partial}{\partial y}axy\\ -\frac{\partial}{\partial x}axy\end{pmatrix}=\begin{pmatrix}ax\\ -ay\end{pmatrix}=\mathbf{F}$$

Scalar *static potential lines* Φ=*const.* and vector *flux potential lines* $\mathbf{A}$=*const.* define *DFL field-net*.



The half-n'-half results

are called

Riemann-Cauchy

Derivative Relations

$$\frac{\partial\Phi}{\partial x}=\frac{\partial\mathbf{A}}{\partial y}\quad\text{is:}\quad\frac{\partial\mathrm{Re}f(z)}{\partial x}=\frac{\partial\mathrm{Im}f(z)}{\partial y}$$

$$\frac{\partial\Phi}{\partial y}=-\frac{\partial\mathbf{A}}{\partial x}\quad\text{is:}\quad\frac{\partial\mathrm{Re}f(z)}{\partial y}=-\frac{\partial\mathrm{Im}f(z)}{\partial x}$$

4. Riemann-Cauchy conditions *What’s analytic? (...and what’s not?)*

Review (z,z) to (x,y) transformation relations*

$$z = x + iy \qquad x = \tfrac{1}{2}(z + z^*)$$

$$z^* = x - iy \qquad y = \tfrac{1}{2i}(z - z^*)$$

$$\frac{df}{dz} = \frac{\partial x}{\partial z}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2i}\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f$$

$$\frac{df}{dz^*} = \frac{\partial x}{\partial z^*}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z^*}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2i}\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f$$

Criteria for a field function $f = f_x(x,y) + i\, f_y(x,y)$ to be an ***analytic function f(z)*** *of $z=x+iy$:*

First, f(z) must <u>not</u> be a function of $z^*=x-iy$, that is: $\frac{df}{dz^*} = 0$

This implies f(z) satisfies differential equations known as the ***Riemann-Cauchy conditions***

$$\frac{df}{dz^*} = 0 = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} - \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} + \frac{\partial f_x}{\partial y}\right) \textit{ implies}: \quad \frac{\partial f_x}{\partial x} = \frac{\partial f_y}{\partial y} \quad \textit{and}: \quad \frac{\partial f_y}{\partial x} = -\frac{\partial f_x}{\partial y}$$

$$\frac{df}{dz} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) = \frac{\partial f_x}{\partial x} + i\frac{\partial f_y}{\partial x} = \frac{\partial f_y}{\partial y} - i\frac{\partial f_x}{\partial y} = \frac{\partial}{\partial x}(f_x + i f_y) = \frac{\partial}{\partial iy}(f_x + i f_y)$$

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Criteria for a field function $f = f_x(x,y) + i\,f_y(x,y)$ to be an ***analytic function f(z*)*** *of $z^*=x-iy$:*

First, f(z) must <u>not</u> be a function of $z=x+iy$, that is:* $\frac{df}{dz} = 0$

This implies f(z) satisfies differential equations we call* ***Anti-Riemann-Cauchy conditions***

$$\frac{df}{dz} = 0 = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(f_x + i\,f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) = \text{implies : } \frac{\partial f_x}{\partial x} = -\frac{\partial f_y}{\partial y} \quad \text{and :} \quad \frac{\partial f_y}{\partial x} = \frac{\partial f_x}{\partial y}$$

$$\frac{df}{dz^*} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} - \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} + \frac{\partial f_x}{\partial y}\right) = \frac{\partial f_x}{\partial x} + i\frac{\partial f_y}{\partial x} = -\frac{\partial f_y}{\partial y} + i\frac{\partial f_x}{\partial y} = \frac{\partial}{\partial x}(f_x + i f_y) = -\frac{\partial}{\partial iy}(f_x + i f_y)$$

What's analytic? (...and what's not?)

Example: Is $f(x,y) = 2x + iy$ an analytic function of $z=x+iy$?

What's analytic? (...and what's <u>not</u>?)

Example: Q: Is $f(x,y) = 2x + i4y$ an analytic function of $z=x+iy$?

Well, test it using definitions: $z = x + iy$ and: $z^* = x - iy$

or: $x = (z+z^*)/2$ and: $y = -i(z-z^*)/2$

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$$= \quad z+z^* \quad + \quad (2z-2z^*)$$

What's analytic? (...and what's not?)

Example: Q: Is $f(x,y) = 2x + i4y$ an analytic function of $z=x+iy$?

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$$\begin{aligned} f(x,y) = 2x + i4y &= 2\,(z+z^*)/2 + i4(-i(z-z^*)/2) \\ &= z+z^* + (2z-2z^*) \\ &= 3z-z^* \end{aligned}$$

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A: **NO!** *It's a function of z and z* so not analytic for either.*

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$$
\begin{aligned}
f(x,y) = 2x + i4y &= 2\,(z+z^*)/2 + i4(-i(z-z^*)/2) \\
&= z+z^* + (2z-2z^*) \\
&= 3z-z^*
\end{aligned}
$$

A: NO! *It's a function of z and z* so not analytic for either.*

Example 2: Q: Is $r(x,y) = x^2 + y^2$ an analytic function of $z=x+iy$?

A: NO! *$r(xy)=z^*z$ is a function of z and z* so not analytic for either.*

What's analytic? (...and what's not?)

Example: Q: Is $f(x,y) = 2x + i4y$ an analytic function of $z=z+iy$?

Well, test it using definitions: $z = x + iy$ and: $z^* = x - iy$

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A: NO! *It's a function of z and z^* so not analytic for either.*

Example 2: Q: Is $r(x,y) = x^2 + y^2$ an analytic function of $z=x+iy$?

A: NO! $r(xy)=z^*z$ *is a function of z and z^* so not analytic for either.*

Example 3: Q: Is $s(x,y) = x^2-y^2 + 2ixy$ an analytic function of $z=x+iy$?

A: YES! $s(xy)=(x+iy)^2 =z^2$ *is analytic function of z.* (Yay!)

4. Riemann-Cauchy conditions *What's analytic? (...and what's not?)*

Easy 2D circulation and flux integrals

Easy 2D curvilinear coordinate discovery

Easy 2D monopole, dipole, and 2^n-pole analysis

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Easy stereo-projection visualization

What Good Are Complex Exponentials? (contd.)

*9. Complex integrals ∫ f(z)dz count 2D "circulation"(∫**F•dr**) and "flux"(∫**F**x**dr**)*

Integral of *f(z)* between point z_1 and point z_2 is potential difference $\Delta\phi = \phi(z_2) - \phi(z_1)$

$$\Delta\phi = \phi(z_2) - \phi(z_1) = \int_{z_1}^{z_2} f(z)dz = \underbrace{\mathbf{\Phi}(x_2, y_2) - \mathbf{\Phi}(x_1, y_1)}_{\Delta\mathbf{\Phi}} + i\,[\underbrace{\mathbf{A}(x_2, y_2) - \mathbf{A}(x_1, y_1)}_{\Delta\mathbf{A}}]$$

$$\Delta\phi = \Delta\mathbf{\Phi} + i\,\Delta\mathbf{A}$$

In *DFL*-field **F**, $\Delta\phi$ is independent of the integration path *z(t)* connecting z_1 and z_2.

What Good Are Complex Exponentials? (contd.)

*9. Complex integrals ∫ f(z)dz count 2D "circulation"(∫***F•dr***) and "flux"(∫***Fxdr***)*

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$$\Delta\phi = \qquad \Delta\mathbf{\Phi} \qquad + i \qquad \Delta\mathbf{A}$$

In *DFL*-field **F**, $\Delta\phi$ is independent of the integration path *z(t)* connecting z_1 and z_2.

$$\int f(z)dz = \int \left(f^*(z^*)\right)^* dz = \int \left(f^*(z^*)\right)^* \left(dx + i\,dy\right) = \int \left(f_x^* + i\,f_y^*\right)^* \left(dx + i\,dy\right) = \int \left(f_x^* - i\,f_y^*\right)\left(dx + i\,dy\right)$$

$$= \int (f_x^* dx + f_y^* dy) + i \int (f_x^* dy - f_y^* dx)$$

$$= \int \mathbf{F} \bullet d\mathbf{r} \qquad + i \int \mathbf{F} \times d\mathbf{r} \bullet \hat{\mathbf{e}}_Z$$

$$= \int \mathbf{F} \bullet d\mathbf{r} \qquad + i \int \mathbf{F} \bullet d\mathbf{r} \times \hat{\mathbf{e}}_Z$$

$$= \int \mathbf{F} \bullet d\mathbf{r} \qquad + i \int \mathbf{F} \bullet d\mathbf{S} \qquad \text{where:} \quad d\mathbf{S} = d\mathbf{r} \times \hat{\mathbf{e}}_Z$$

What Good Are Complex Exponentials? (contd.)

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$$= \int (f_x^* dx + f_y^* dy) + i \int (f_x^* dy - f_y^* dx)$$

$$= \int \mathbf{F} \bullet d\mathbf{r} \quad + i \int \mathbf{F} \times d\mathbf{r} \bullet \hat{\mathbf{e}}_Z$$

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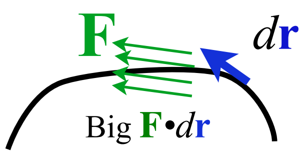
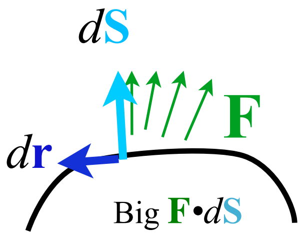


Real part $\int_1^2 \mathbf{F} \bullet d\mathbf{r} = \Delta\mathbf{\Phi}$
sums **F** projections *along* path d**r** that is, *circulation* on path to get $\Delta\mathbf{\Phi}$.

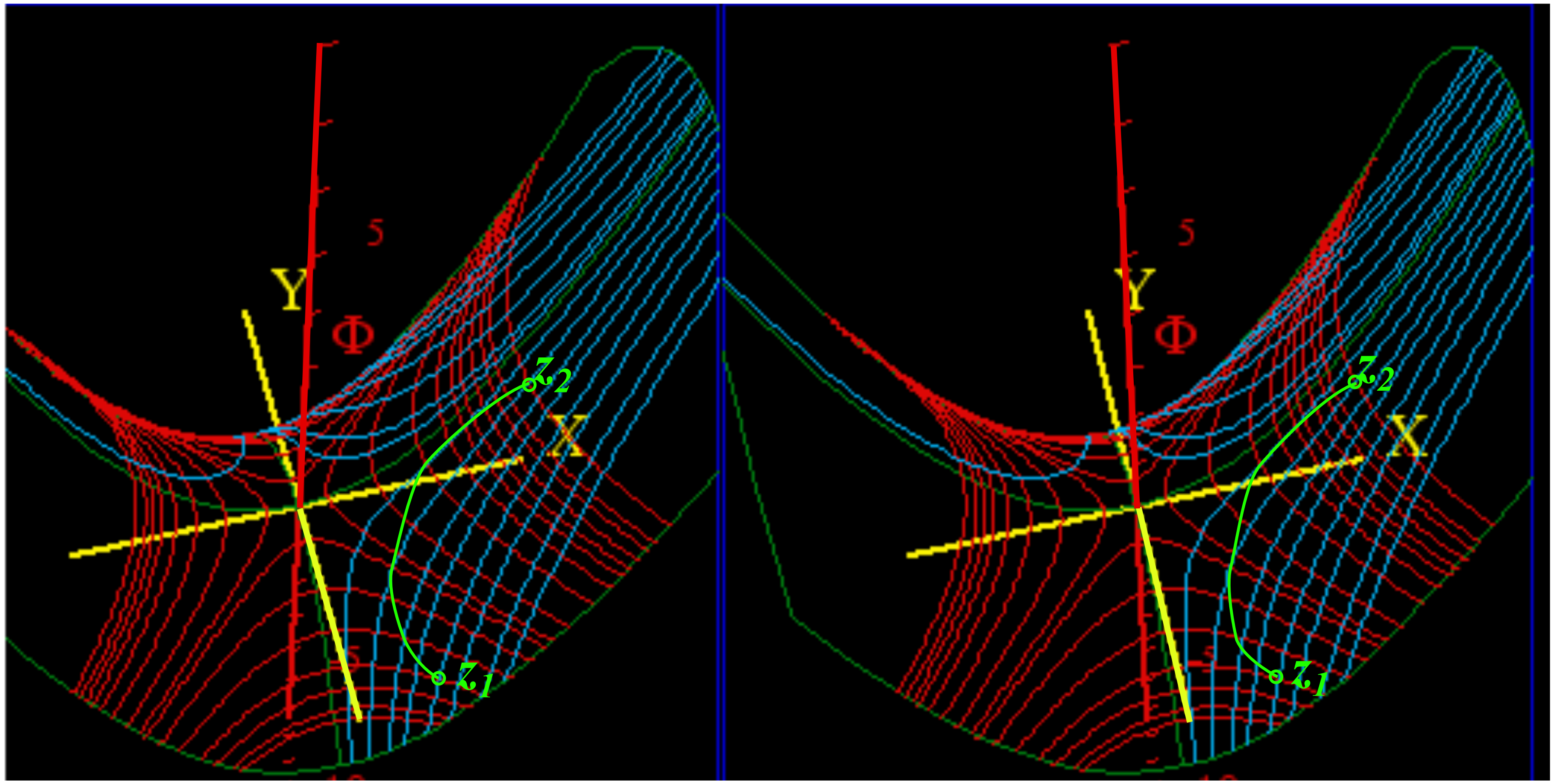


Imaginary part $\int_1^2 \mathbf{F} \bullet d\mathbf{S} = \Delta\mathbf{A}$
sums **F** projection *across* path d**r** that is, *flux* thru surface elements d**S**=d**r**×**e**$_Z$ normal to d**r** to get $\Delta\mathbf{A}$.

Here the scalar potential $\Phi=(x^2-y^2)/2$ is stereo-plotted vs. (x,y)

The $\Phi=(x^2-y^2)/2=const.$ curves are topography lines

The $A=(xy)=const.$ curves are streamlines normal to topography lines



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What Good Are Complex Exponentials? (contd.)

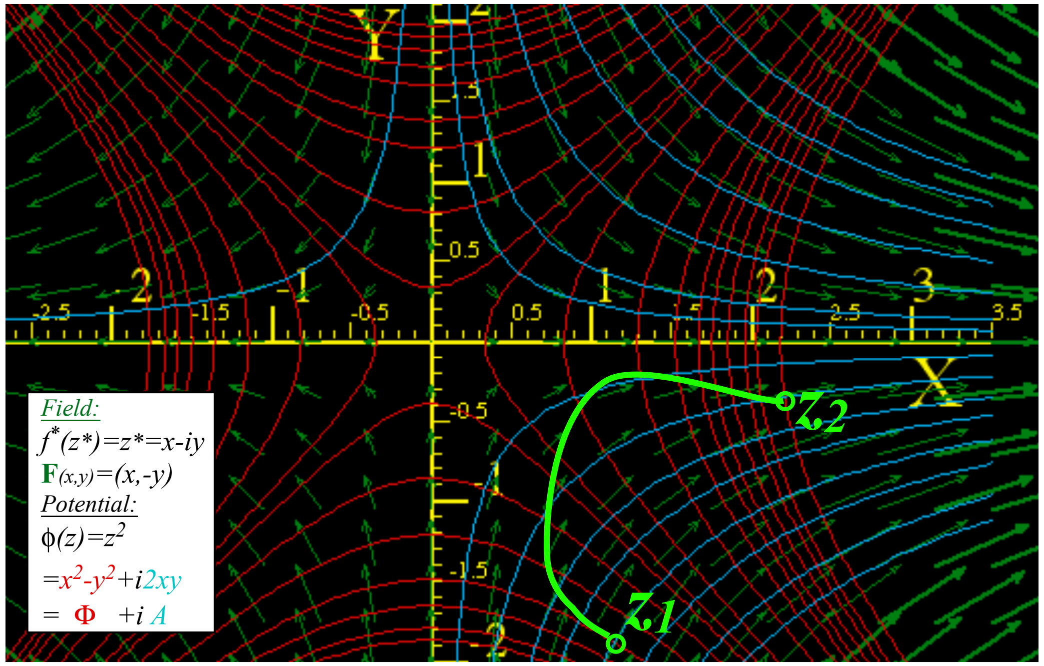
10. Complex potentials define 2D Orthogonal Curvilinear Coordinates (OCC) of field

The (Φ, A) grid is a GCC coordinate system*:

$q^1 = \Phi = (x^2 - y^2)/2 = const.$

$q^2 = A = (xy) = const.$

*Actually it's OCC.



$$Kajobian = \begin{pmatrix} \frac{\partial q^1}{\partial x} & \frac{\partial q^1}{\partial y} \\ \frac{\partial q^2}{\partial x} & \frac{\partial q^2}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial \Phi}{\partial x} & \frac{\partial \Phi}{\partial y} \\ \frac{\partial A}{\partial x} & \frac{\partial A}{\partial y} \end{pmatrix} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{matrix} \leftarrow \mathbf{E}^{\Phi} \\ \leftarrow \mathbf{E}^{A} \end{matrix}$$

$$Jacobian = \begin{pmatrix} \frac{\partial x}{\partial q^1} & \frac{\partial x}{\partial q^2} \\ \frac{\partial y}{\partial q^1} & \frac{\partial y}{\partial q^2} \end{pmatrix} = \begin{array}{c} \begin{pmatrix} \frac{\partial x}{\partial \Phi} & \frac{\partial x}{\partial A} \\ \frac{\partial y}{\partial \Phi} & \frac{\partial y}{\partial A} \end{pmatrix} \\ \begin{matrix} \uparrow & \uparrow \\ \mathbf{E}_{\Phi} & \mathbf{E}_{A} \end{matrix} \end{array} = \frac{1}{r^2} \begin{array}{c} \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \\ \begin{matrix} \uparrow & \uparrow \\ \mathbf{E}_{\Phi} & \mathbf{E}_{A} \end{matrix} \end{array}$$

$$Metric\,tensor = \begin{pmatrix} g_{\Phi\Phi} & g_{\Phi A} \\ g_{A\Phi} & g_{AA} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_{\Phi} \bullet \mathbf{E}_{\Phi} & \mathbf{E}_{\Phi} \bullet \mathbf{E}_{A} \\ \mathbf{E}_{A} \bullet \mathbf{E}_{\Phi} & \mathbf{E}_{A} \bullet \mathbf{E}_{A} \end{pmatrix} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \end{pmatrix} \text{ where: } r^2 = x^2 + y^2$$

What Good Are Complex Exponentials? (contd.)

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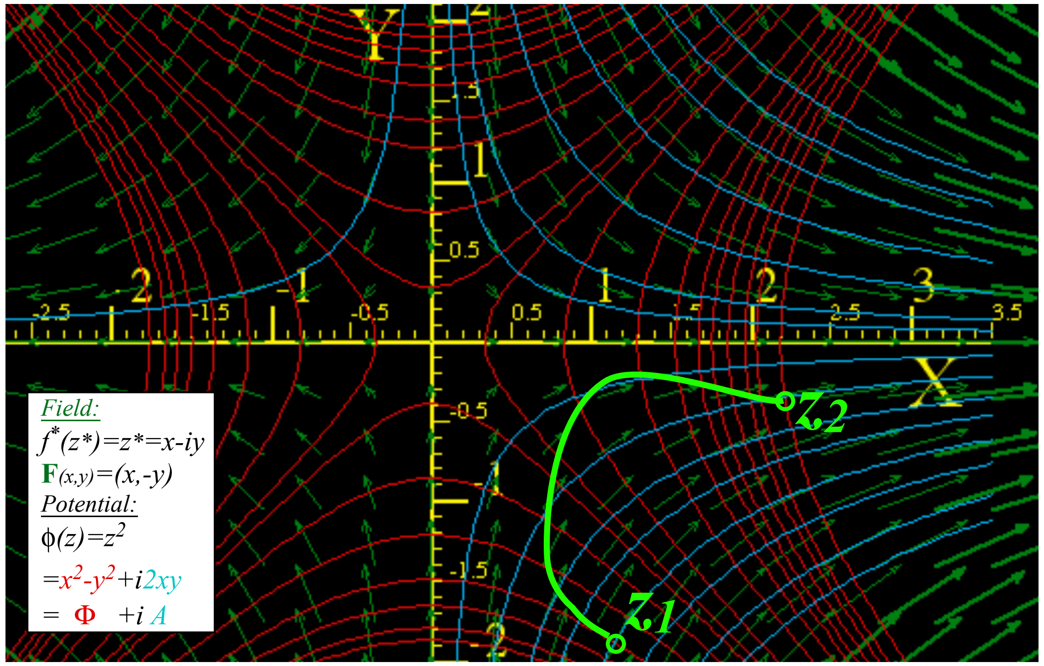

$$Kajobian=\begin{pmatrix}\frac{\partial q^1}{\partial x} & \frac{\partial q^1}{\partial y}\\ \frac{\partial q^2}{\partial x} & \frac{\partial q^2}{\partial y}\end{pmatrix}=\begin{pmatrix}\frac{\partial \Phi}{\partial x} & \frac{\partial \Phi}{\partial y}\\ \frac{\partial A}{\partial x} & \frac{\partial A}{\partial y}\end{pmatrix}=\begin{pmatrix}x & -y\\ y & x\end{pmatrix}\begin{array}{l}\leftarrow \mathbf{E}^{\Phi}\\ \leftarrow \mathbf{E}^{A}\end{array}$$

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Riemann-Cauchy Derivative Relations make coordinates orthogonal

$$\nabla\Phi=\begin{pmatrix}\frac{\partial\Phi}{\partial x}\\ \frac{\partial\Phi}{\partial y}\end{pmatrix}=\begin{pmatrix}\frac{\partial}{\partial x}\frac{a}{2}(x^2-y^2)\\ \frac{\partial}{\partial y}\frac{a}{2}(x^2-y^2)\end{pmatrix}=\begin{pmatrix}ax\\ -ay\end{pmatrix}=\mathbf{F}$$

The half-n'-half results assure

$$\mathbf{E}_{\Phi}\bullet\mathbf{E}_{A}=\frac{\partial\Phi}{\partial x}\frac{\partial A}{\partial x}+\frac{\partial\Phi}{\partial y}\frac{\partial A}{\partial y}$$
$$=-\frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial y}+\frac{\partial\Phi}{\partial y}\frac{\partial\Phi}{\partial x}=0$$

$$\nabla\times\mathbf{A}=\begin{pmatrix}\frac{\partial\mathbf{A}}{\partial y}\\ -\frac{\partial\mathbf{A}}{\partial x}\end{pmatrix}=\begin{pmatrix}\frac{\partial}{\partial y}axy\\ -\frac{\partial}{\partial x}axy\end{pmatrix}=\begin{pmatrix}ax\\ -ay\end{pmatrix}=\mathbf{F}$$

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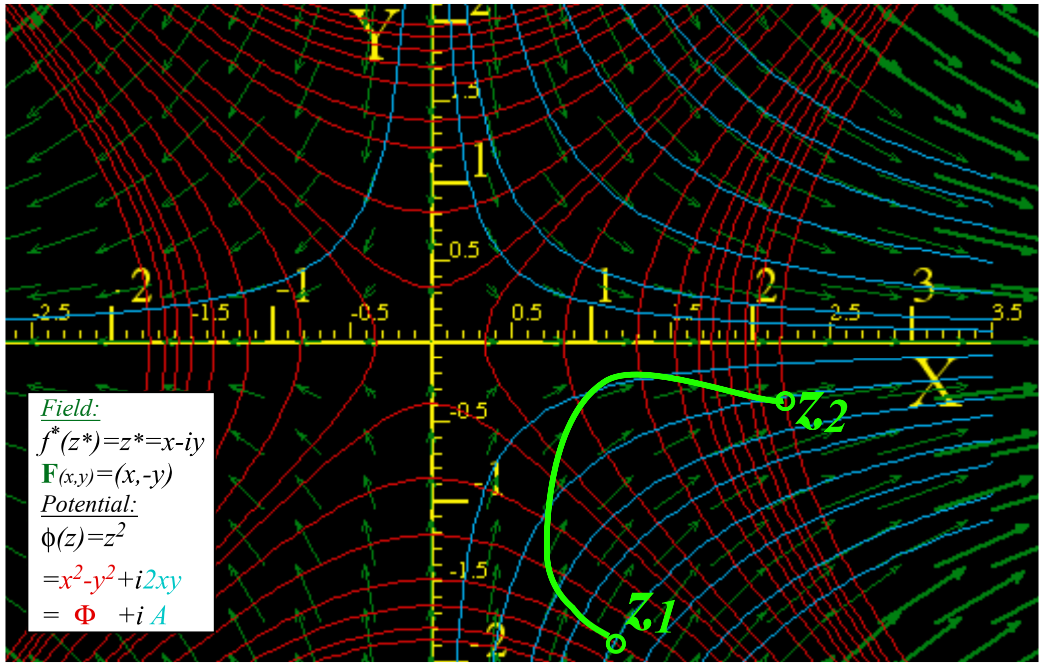
$q^2 = A = (xy) = const.$

*Actually it's OCC.



$$\mathit{Kajobian} = \begin{pmatrix} \frac{\partial q^1}{\partial x} & \frac{\partial q^1}{\partial y} \\ \frac{\partial q^2}{\partial x} & \frac{\partial q^2}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial \Phi}{\partial x} & \frac{\partial \Phi}{\partial y} \\ \frac{\partial A}{\partial x} & \frac{\partial A}{\partial y} \end{pmatrix} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{array}{l} \leftarrow \mathbf{E}^{\Phi} \\ \leftarrow \mathbf{E}^{A} \end{array}$$

$$\mathit{Jacobian} = \begin{pmatrix} \frac{\partial x}{\partial q^1} & \frac{\partial x}{\partial q^2} \\ \frac{\partial y}{\partial q^1} & \frac{\partial y}{\partial q^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \Phi} & \frac{\partial x}{\partial A} \\ \frac{\partial y}{\partial \Phi} & \frac{\partial y}{\partial A} \\ \uparrow & \uparrow \\ \mathbf{E}_{\Phi} & \mathbf{E}_{A} \end{pmatrix} = \frac{1}{r^2} \begin{pmatrix} x & y \\ -y & x \\ \uparrow & \uparrow \\ \mathbf{E}_{\Phi} & \mathbf{E}_{A} \end{pmatrix}$$

$$\mathit{Metric\,tensor} = \begin{pmatrix} g_{\Phi\Phi} & g_{\Phi A} \\ g_{A\Phi} & g_{AA} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_{\Phi} \bullet \mathbf{E}_{\Phi} & \mathbf{E}_{\Phi} \bullet \mathbf{E}_{A} \\ \mathbf{E}_{A} \bullet \mathbf{E}_{\Phi} & \mathbf{E}_{A} \bullet \mathbf{E}_{A} \end{pmatrix} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \end{pmatrix} \text{ where: } r^2 = x^2 + y^2$$

Riemann-Cauchy Derivative Relations make coordinates orthogonal

$$\nabla \Phi = \begin{pmatrix} \frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} \frac{a}{2}(x^2 - y^2) \\ \frac{\partial}{\partial y} \frac{a}{2}(x^2 - y^2) \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

The half-n'-half results assure

$$\mathbf{E}_{\Phi} \bullet \mathbf{E}_{A} = \frac{\partial \Phi}{\partial x}\frac{\partial A}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{\partial A}{\partial y} = -\frac{\partial \Phi}{\partial x}\frac{\partial \Phi}{\partial y} + \frac{\partial \Phi}{\partial y}\frac{\partial \Phi}{\partial x} = 0$$

$$\nabla \times \mathbf{A} = \begin{pmatrix} \frac{\partial A}{\partial y} \\ -\frac{\partial A}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y} axy \\ -\frac{\partial}{\partial x} axy \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

Zero divergence requirement: $0 = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} = \frac{\partial}{\partial x}\frac{\partial \Phi}{\partial x} + \frac{\partial}{\partial y}\frac{\partial \Phi}{\partial y} = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$ *potential Φ obeys Laplace equation*

What Good Are Complex Exponentials? (contd.)

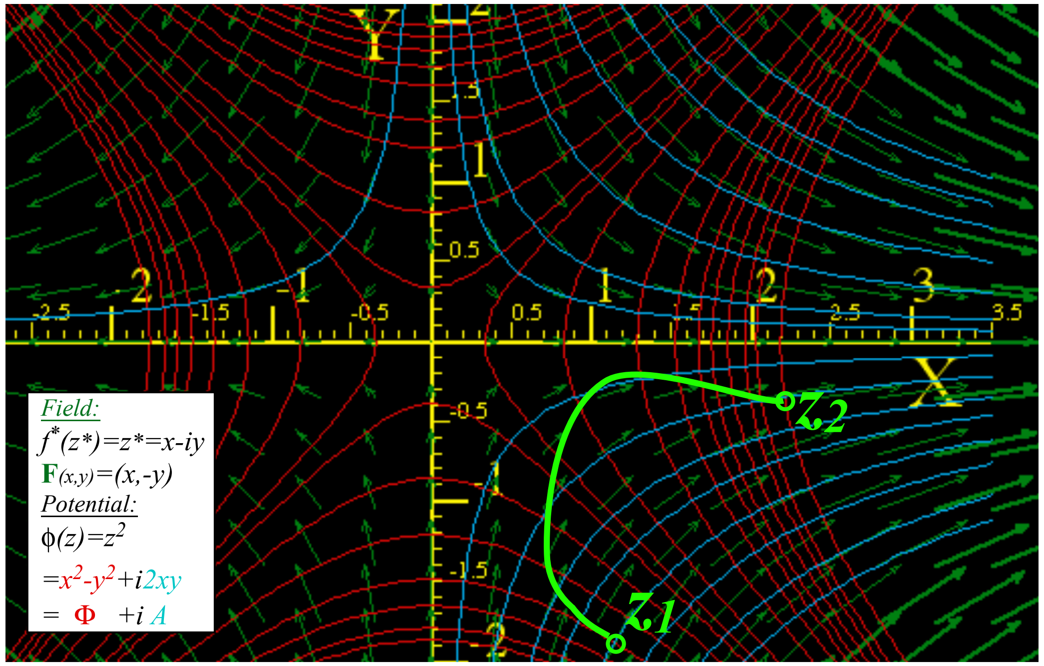
10. Complex potentials define 2D Orthogonal Curvilinear Coordinates (OCC) of field

The (Φ,**A**) grid is a GCC coordinate system*:

$q^1 = \Phi = (x^2-y^2)/2 = \text{const.}$

$q^2 = \mathbf{A} = (xy) = \text{const.}$

*Actually it's OCC.



$$Kajobian = \begin{pmatrix} \frac{\partial q^1}{\partial x} & \frac{\partial q^1}{\partial y} \\ \frac{\partial q^2}{\partial x} & \frac{\partial q^2}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial \Phi}{\partial x} & \frac{\partial \Phi}{\partial y} \\ \frac{\partial A}{\partial x} & \frac{\partial A}{\partial y} \end{pmatrix} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{matrix} \leftarrow \mathbf{E}^{\Phi} \\ \leftarrow \mathbf{E}^{A} \end{matrix}$$

$$Jacobian = \begin{pmatrix} \frac{\partial x}{\partial q^1} & \frac{\partial x}{\partial q^2} \\ \frac{\partial y}{\partial q^1} & \frac{\partial y}{\partial q^2} \end{pmatrix} = \begin{array}{c} \begin{pmatrix} \frac{\partial x}{\partial \Phi} & \frac{\partial x}{\partial A} \\ \frac{\partial y}{\partial \Phi} & \frac{\partial y}{\partial A} \end{pmatrix} \\ \begin{matrix} \uparrow & \uparrow \\ \mathbf{E}_{\Phi} & \mathbf{E}_{A} \end{matrix} \end{array} = \frac{1}{r^2} \begin{array}{c} \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \\ \begin{matrix} \uparrow & \uparrow \\ \mathbf{E}_{\Phi} & \mathbf{E}_{A} \end{matrix} \end{array}$$

$$Metric\,tensor = \begin{pmatrix} g_{\Phi\Phi} & g_{\Phi A} \\ g_{A\Phi} & g_{AA} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_{\Phi}\bullet\mathbf{E}_{\Phi} & \mathbf{E}_{\Phi}\bullet\mathbf{E}_{A} \\ \mathbf{E}_{A}\bullet\mathbf{E}_{\Phi} & \mathbf{E}_{A}\bullet\mathbf{E}_{A} \end{pmatrix} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \end{pmatrix} \text{ where: } r^2 = x^2+y^2$$

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$$\nabla\Phi = \begin{pmatrix} \frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x}\frac{a}{2}(x^2-y^2) \\ \frac{\partial}{\partial y}\frac{a}{2}(x^2-y^2) \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

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$$\mathbf{E}_{\Phi}\bullet\mathbf{E}_{A} = \frac{\partial \Phi}{\partial x}\frac{\partial A}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{\partial A}{\partial y} = -\frac{\partial \Phi}{\partial x}\frac{\partial \Phi}{\partial y} + \frac{\partial \Phi}{\partial y}\frac{\partial \Phi}{\partial x} = 0$$

$$\nabla\times\mathbf{A} = \begin{pmatrix} \frac{\partial \mathbf{A}}{\partial y} \\ -\frac{\partial \mathbf{A}}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y}axy \\ -\frac{\partial}{\partial x}axy \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

or Riemann-Cauchy

Zero divergence requirement: $0 = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} = \frac{\partial}{\partial x}\frac{\partial \Phi}{\partial x} + \frac{\partial}{\partial y}\frac{\partial \Phi}{\partial y} = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$

and so does **A**

potential Φ obeys Laplace equation

4. *Riemann-Cauchy conditions* *What's analytic? (...and what's <u>not</u>?)*

Easy 2D circulation and flux integrals

Easy 2D curvilinear coordinate discovery

→ *Easy 2D monopole, dipole, and* 2^n*-pole analysis*

Easy 2^n*-multipole field and potential expansion*

Easy stereo-projection visualization

What Good Are Complex Exponentials? (contd.)

11. Complex integrals define 2D monopole fields and potentials

Of all power-law fields $f(z)=az^n$ one lacks a power-law potential $\phi(z)=\frac{a}{n+1}z^{n+1}$. It is the $n = -1$ case.

Unit monopole field: $f(z)=\frac{1}{z}=z^{-1}$ $f(z)=\frac{a}{z}=az^{-1}$ Source-a monopole

It has a *logarithmic potential* $\phi(z)=a\cdot\ln(z)=a\cdot\ln(x+iy)$.

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$$\phi(z)=\quad \Phi \quad + \quad i\mathbf{A}=\int f(z)dz=\int \frac{a}{z}dz=a\ln(z)$$

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$$\phi(z)= \quad \Phi \quad + \quad i\mathbf{A} = \int f(z)dz = \int \frac{a}{z}dz = a\ln(z) = a\ln(re^{i\theta})$$

$$= a\ln(r) \; + \; i\,a\theta$$

What Good Are Complex Exponentials? (contd.)

11. Complex integrals define 2D monopole fields and potentials

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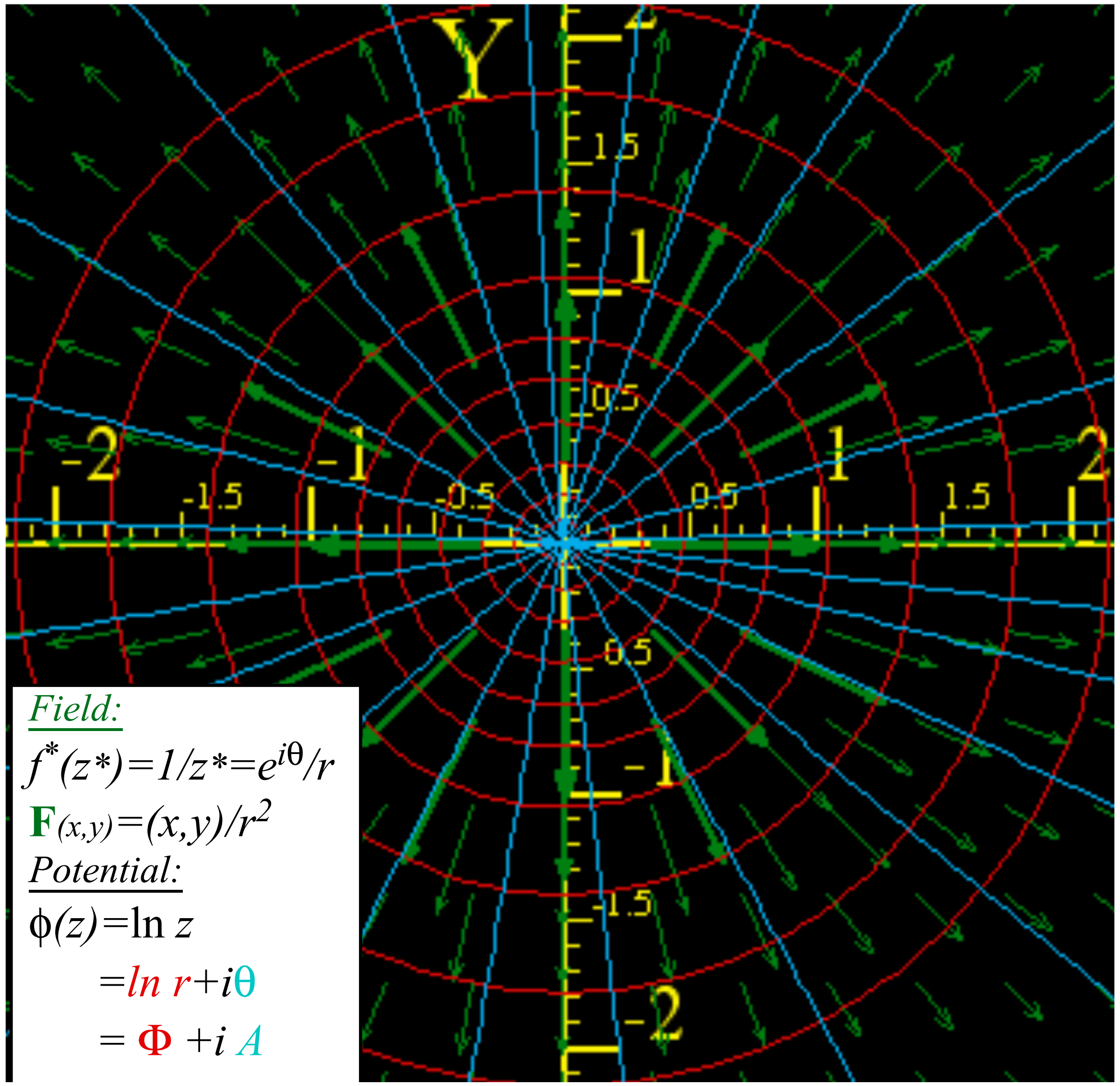
$$\phi(z)=\Phi+i\mathbf{A}=\int f(z)dz=\int\frac{a}{z}dz=a\ln(z)=a\ln(re^{i\theta})$$

$$=\overbrace{a\ln(r)}+i\,\overbrace{a\theta}$$

(a) Unit Z-line-flux field f(z)=1/z



Field:

$f^*(z^*)=1/z^*=e^{i\theta}/r$

$\mathbf{F}_{(x,y)}=(x,y)/r^2$

Potential:

$\phi(z)=\ln z$

$=\ln r+i\theta$

$=\Phi+i\,A$

What Good Are Complex Exponentials? (contd.)

11. Complex integrals define 2D monopole fields and potentials

Of all power-law fields $f(z)=az^n$ one lacks a power-law potential $\phi(z)=\frac{a}{n+1}z^{n+1}$. It is the $n=-1$ case.

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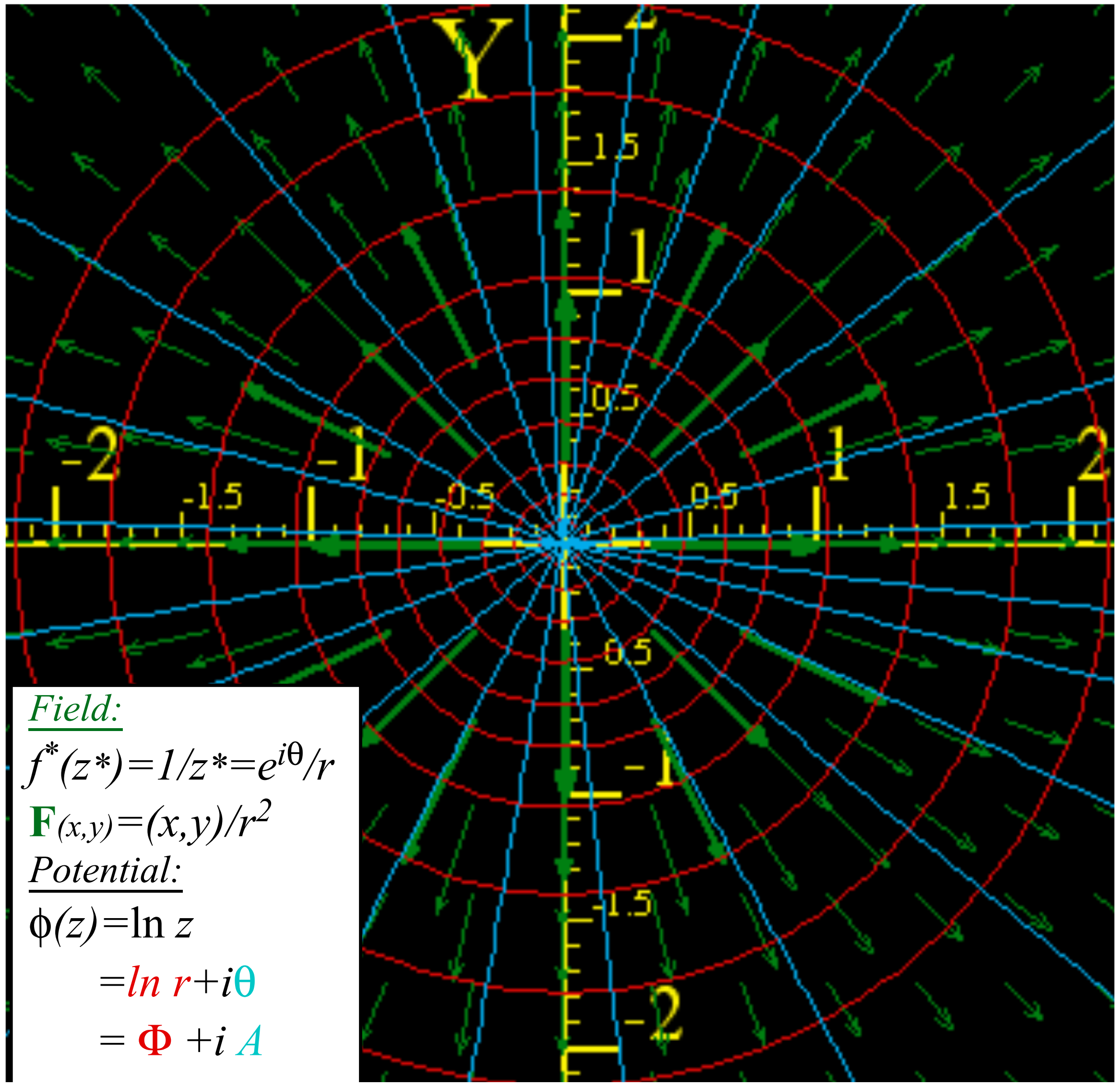
$$\phi(z) = \Phi + i\mathbf{A} = \int f(z)dz = \int \frac{a}{z}dz = a\ln(z) = a\ln(re^{i\theta})$$

$$= \overbrace{a\ln(r)} + i\,\overbrace{a\theta}$$

(a) *Unit Z-line-flux field f(z)=1/z*



Field:

$f^*(z^*)=1/z^*=e^{i\theta}/r$

$\mathbf{F}(x,y)=(x,y)/r^2$

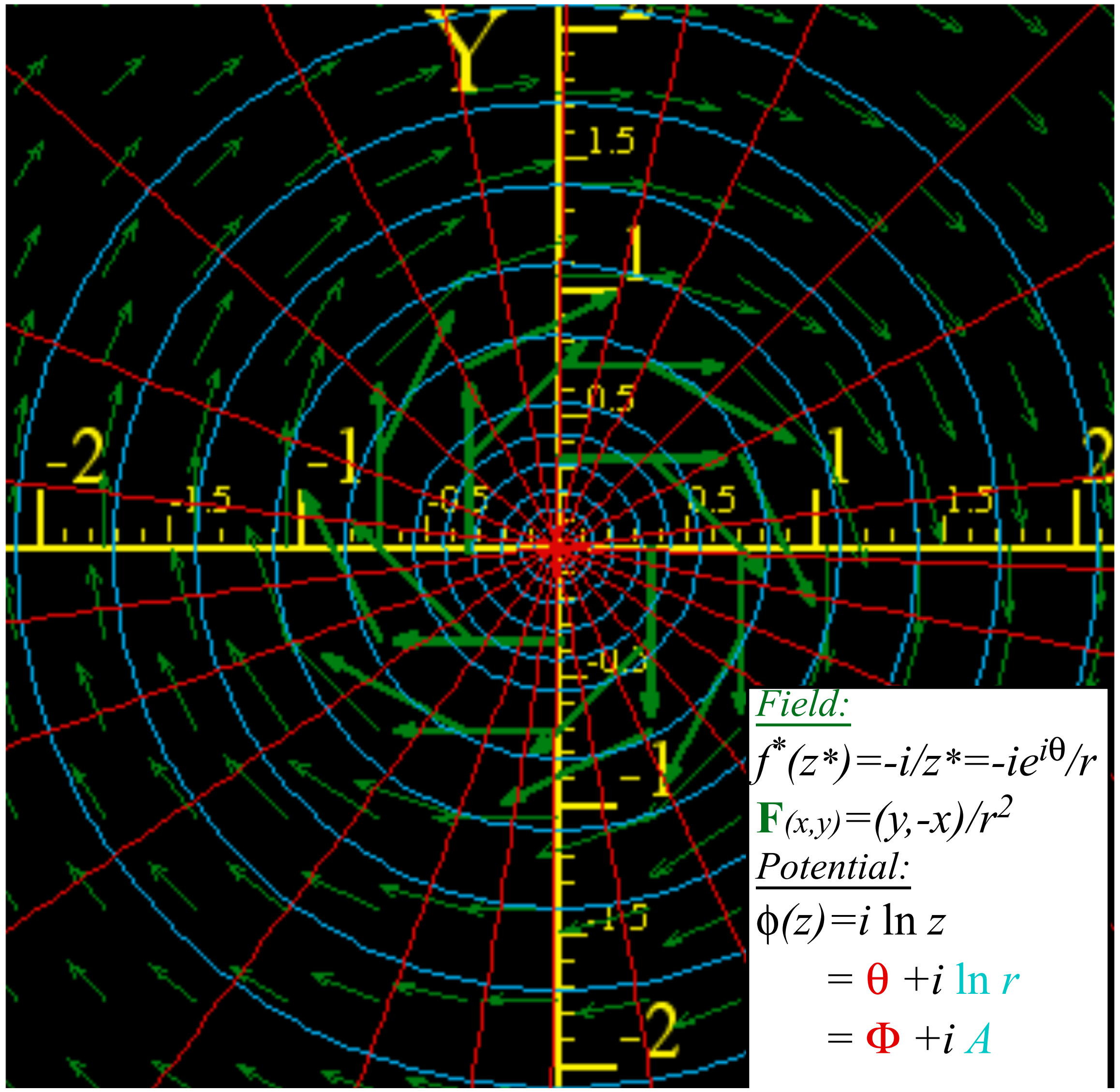
Potential:

$\phi(z)=\ln z$

$=\ln r+i\theta$

$=\Phi+iA$

(b) *Unit Z-line-vortex field f(z)=i/z*



Field:

$f^*(z^*)=-i/z^*=-ie^{i\theta}/r$

$\mathbf{F}(x,y)=(y,-x)/r^2$

Potential:

$\phi(z)=i\ln z$

$=\theta+i\ln r$

$=\Phi+iA$

What Good Are Complex Exponentials? (contd.)

11. Complex integrals define 2D monopole fields and potentials

Of all power-law fields $f(z)=az^n$ one lacks a power-law potential $\phi(z)=\frac{a}{n+1}z^{n+1}$. It is the $n = -1$ case.

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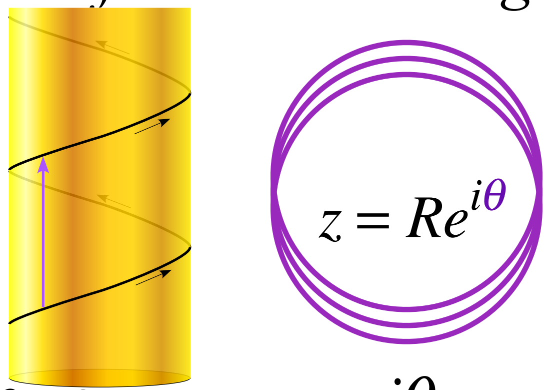
$$\phi(z) = \underbrace{\Phi}_{} + i\underbrace{A}_{} = \int f(z)dz = \int \tfrac{a}{z}dz = a\ln(z) = a\ln(re^{i\theta})$$

$$= a\ln(r) + i\,a\theta$$

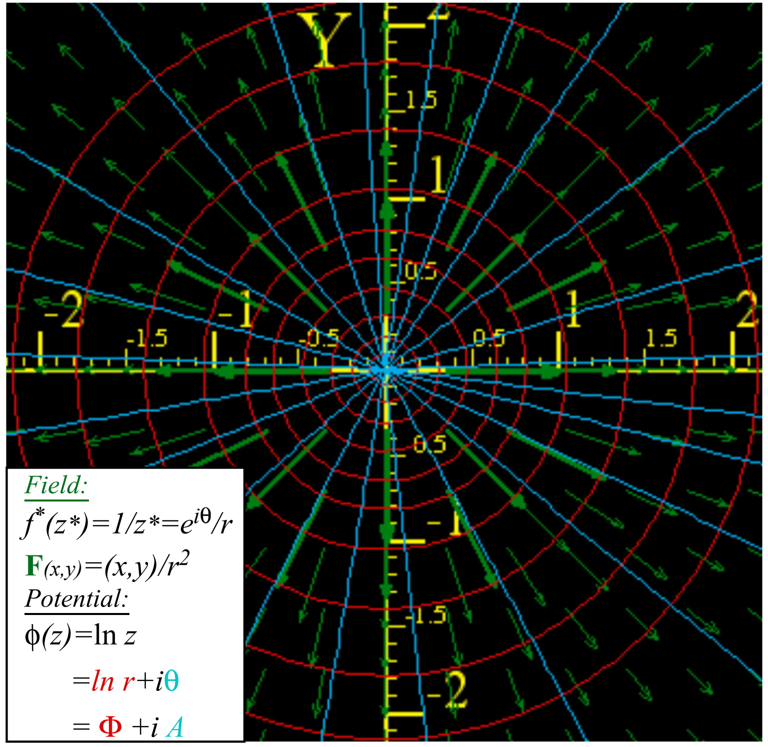
A monopole field is the only power-law field whose integral (potential) depends on path of integration.



path that goes N times around origin (r=0) at constant r = R.

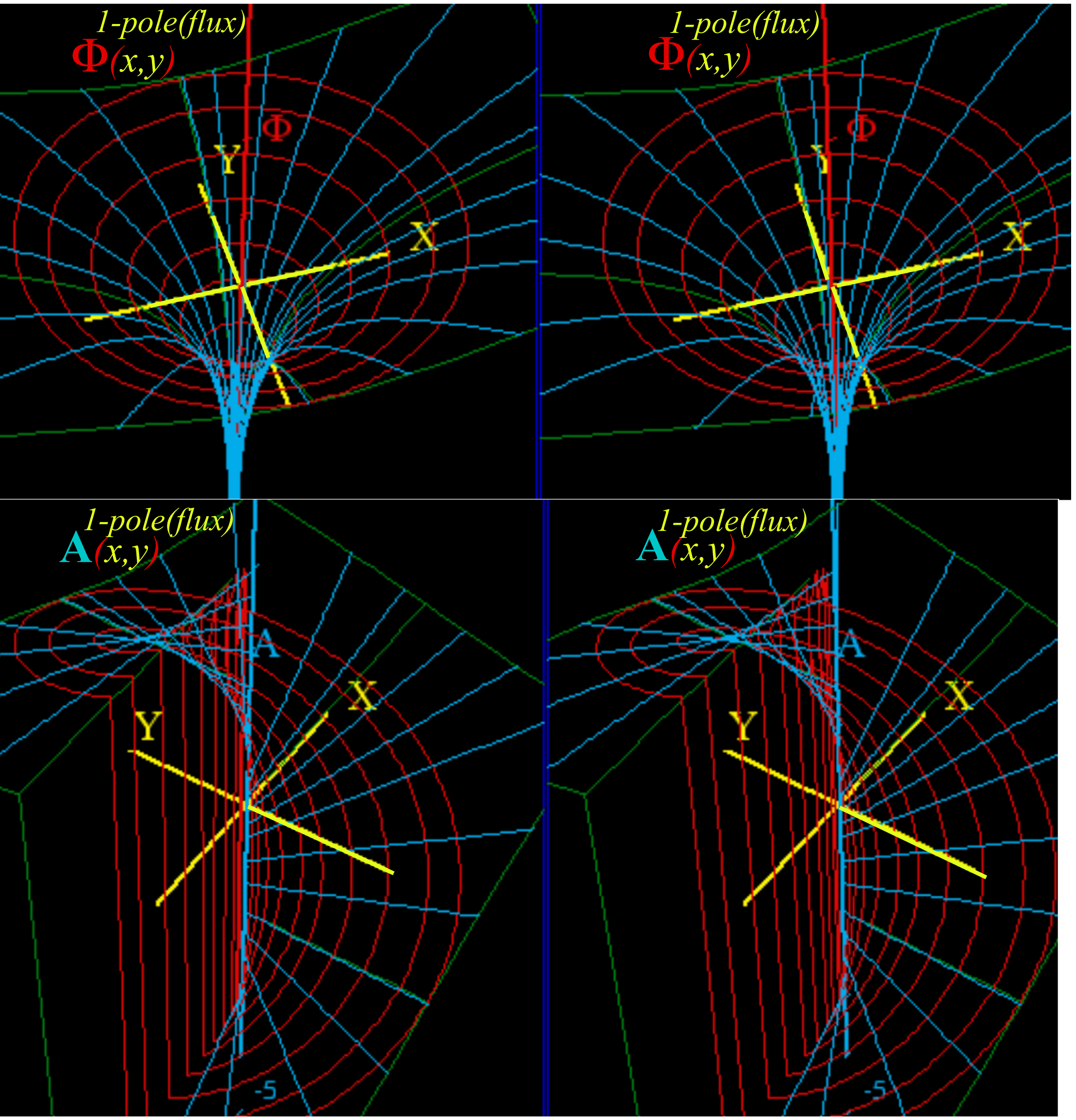
$$\Delta\phi = \oint f(z)dz = a\oint\frac{dz}{z} = a\int_{\theta=0}^{\theta=2\pi N}\frac{d(Re^{i\theta})}{Re^{i\theta}} = a\int_{\theta=0}^{\theta=2\pi N} i\,d\theta = ai\,\theta\Big|_0^{2\pi N} = 2a\pi iN$$

(a) Unit Z-line-flux field f(z)=1/z



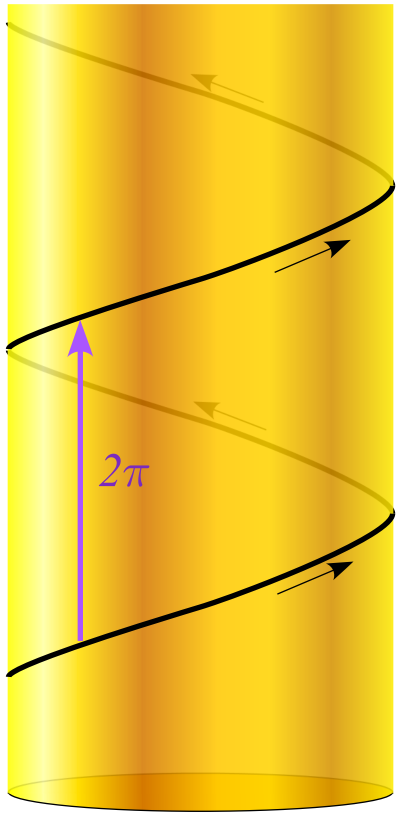
$$\phi(z) = \underbrace{\Phi} + i\underbrace{\mathbf{A}} = \int f(z)dz = \int \frac{a}{z}dz = a\ln(re^{i\theta})$$

$$= \ln(r) + i\theta$$

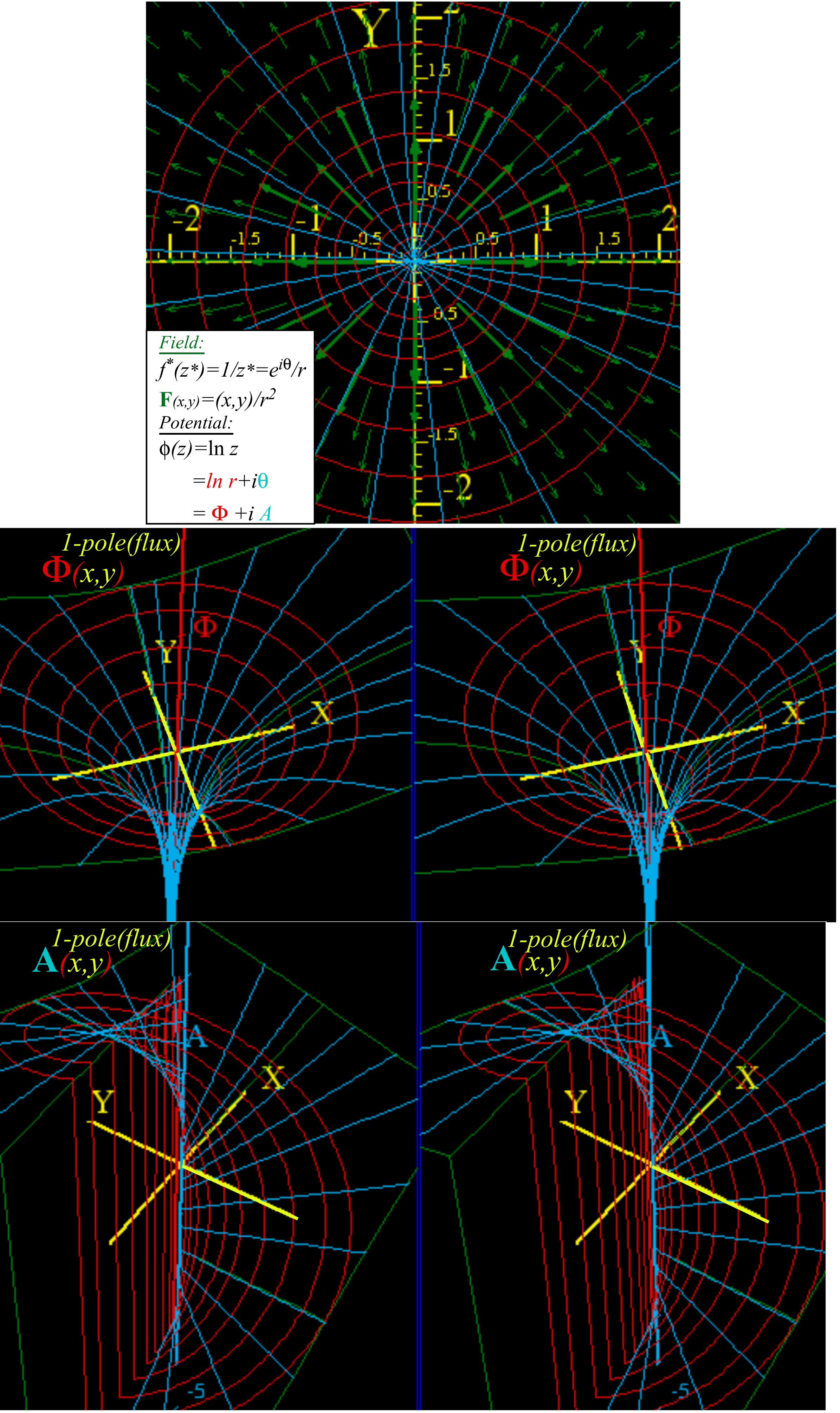


Each turn around origin

*adds 2πi to vector potential i***A**



(a) Unit Z-line-flux field f(z)=1/z



Field:

$f^*(z^*)=1/z^*=e^{i\theta}/r$

$\mathbf{F}_{(x,y)}=(x,y)/r^2$

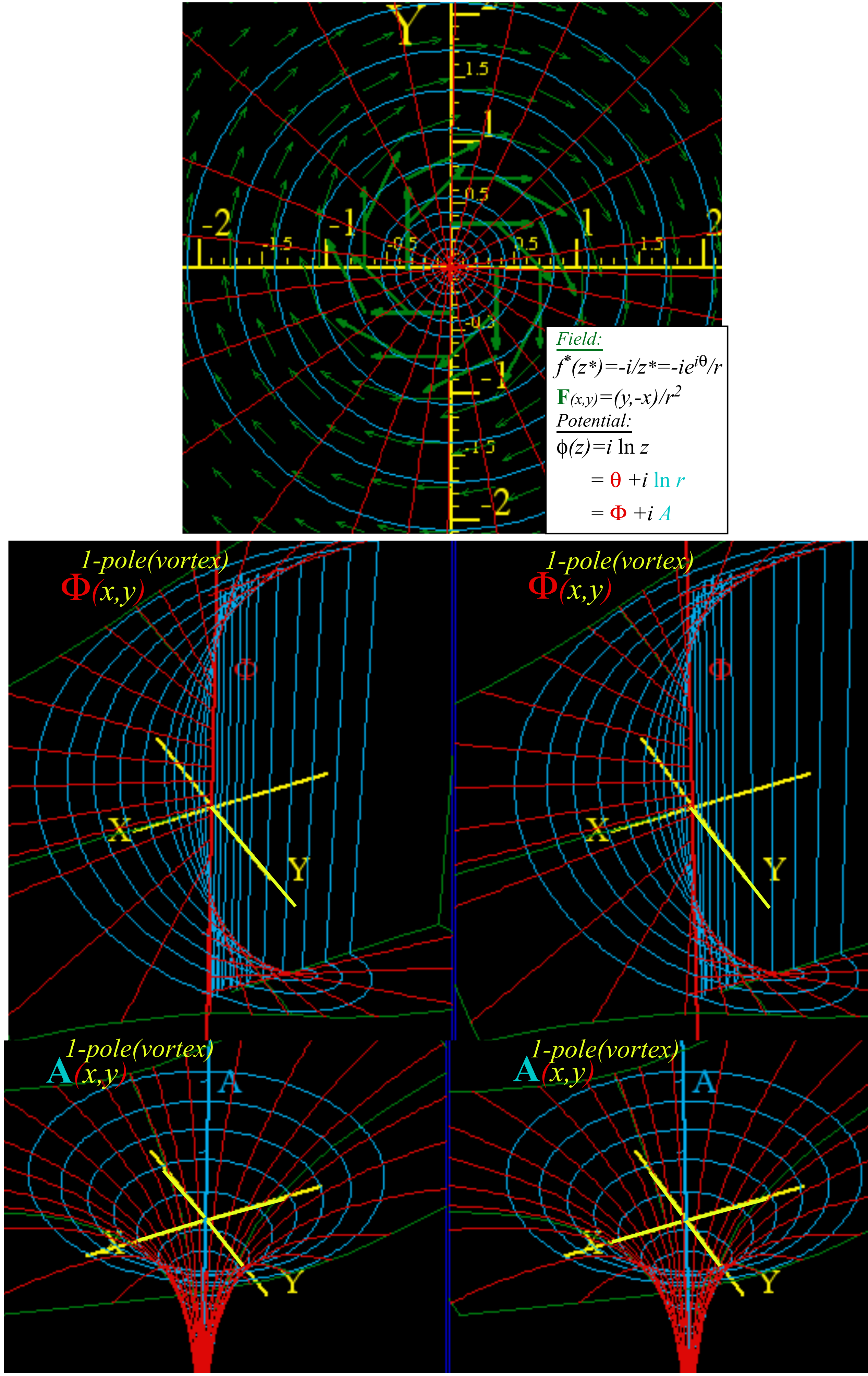
Potential:

$\phi(z)=\ln z$

$=\ln r+i\theta$

$=\Phi+i\,A$

(b) Unit Z-line-vortex field f(z)=i/z



Field:

$f^*(z^*)=-i/z^*=-ie^{i\theta}/r$

$\mathbf{F}_{(x,y)}=(y,-x)/r^2$

Potential:

$\phi(z)=i\ln z$

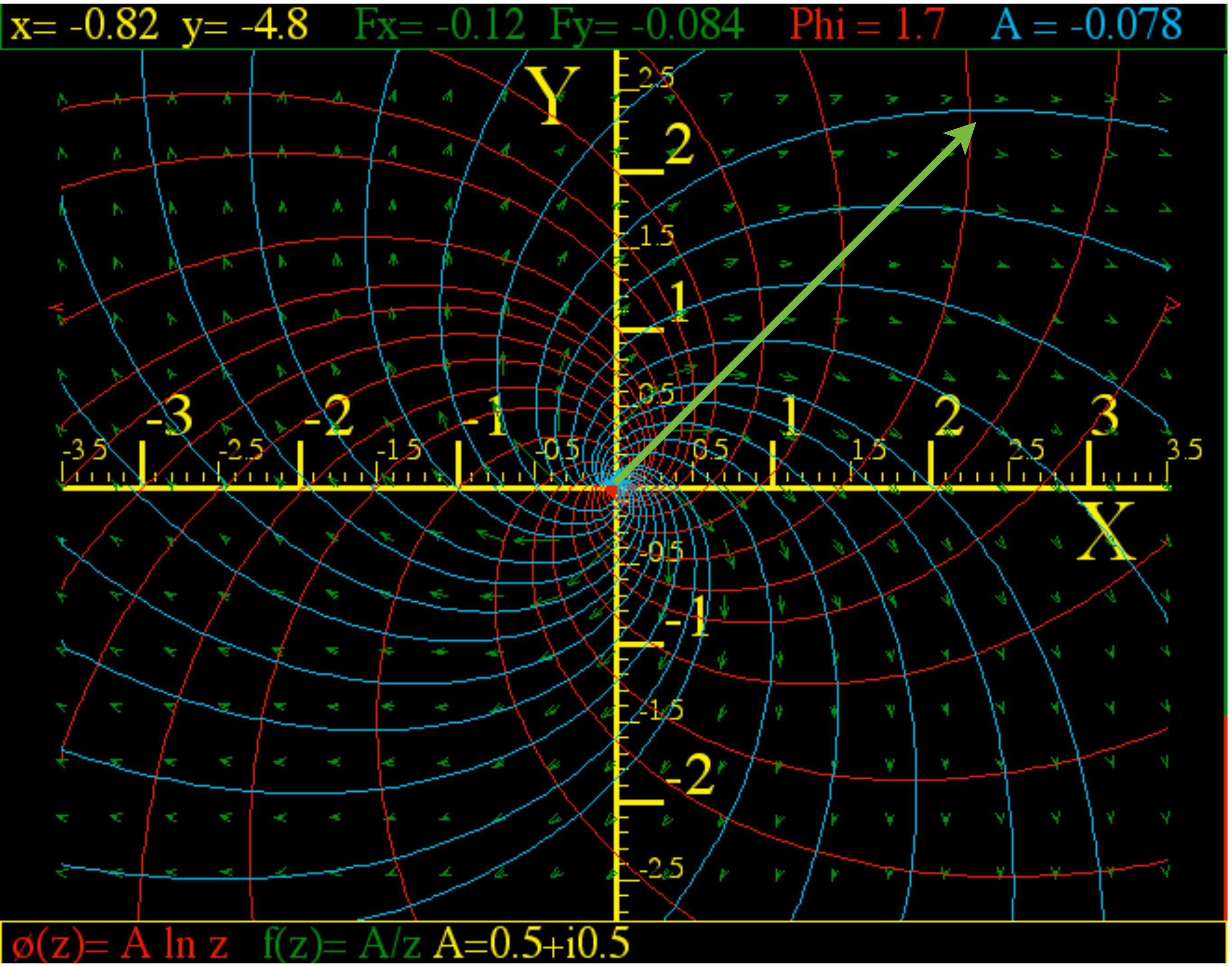
$=\theta+i\ln r$

$=\Phi+i\,A$

What Good Are Complex Exponentials? (contd.)

$f(z) = (0.5 + i0.5)/z = e^{i\pi/4}/z\sqrt{2}$

"Vortex"

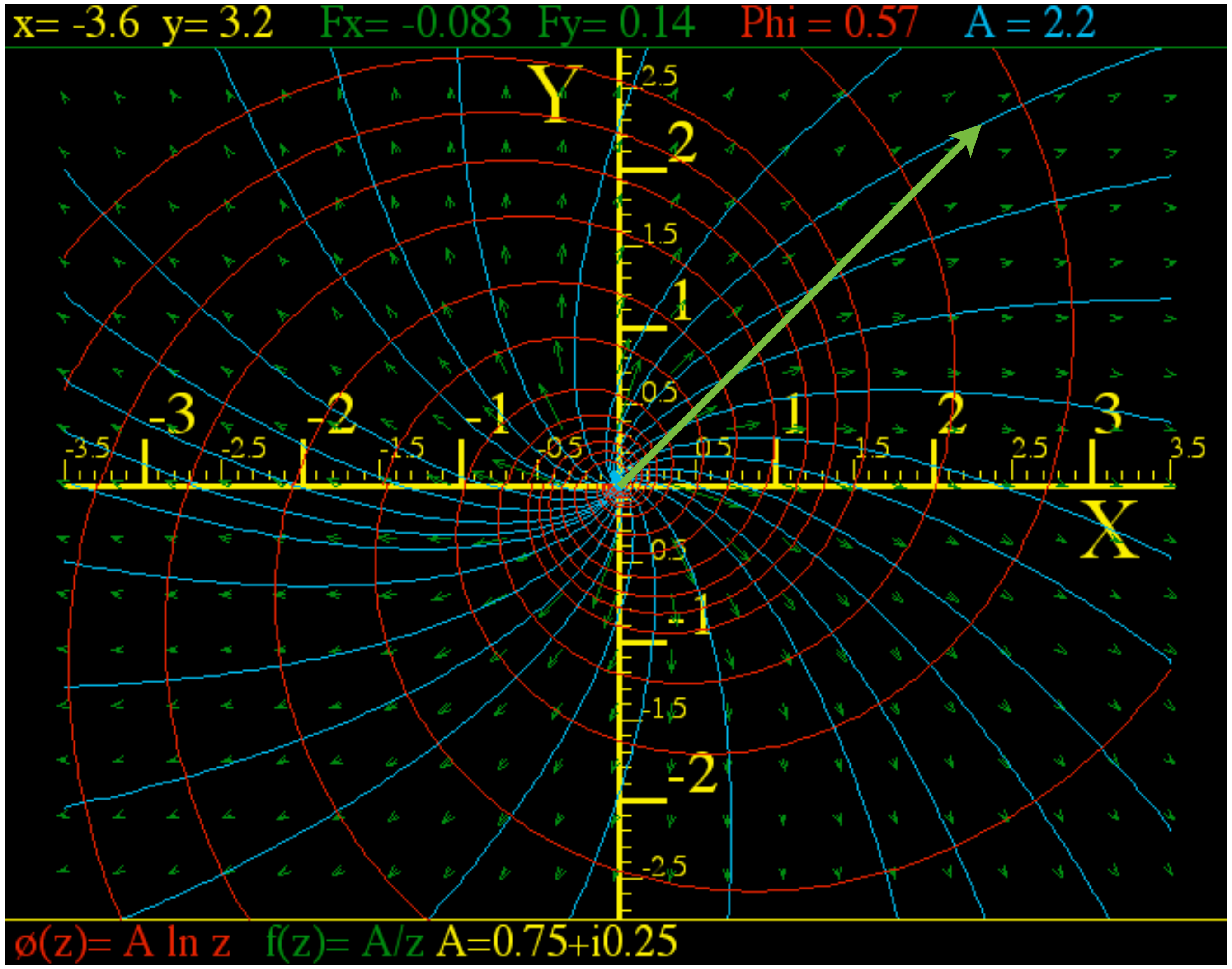


x= -0.82 y= -4.8 Fx= -0.12 Fy= -0.084 Phi = 1.7 A = -0.078

ø(z)= A ln z f(z)= A/z A=0.5+i0.5

$f(z) = (0.75 + i0.25)/z = e^{i18^\circ}/z\sqrt{n}$

"Hurricane"



x= -3.6 y= 3.2 Fx= -0.083 Fy= 0.14 Phi = 0.57 A = 2.2

ø(z)= A ln z f(z)= A/z A=0.75+i0.25

4. Riemann-Cauchy conditions *What's analytic? (...and what's not?)*

Easy 2D circulation and flux integrals

Easy 2D curvilinear coordinate discovery

→ *Easy 2D monopole, dipole, and 2^n-pole analysis*

Easy 2^n-multipole field and potential expansion

Easy stereo-projection visualization

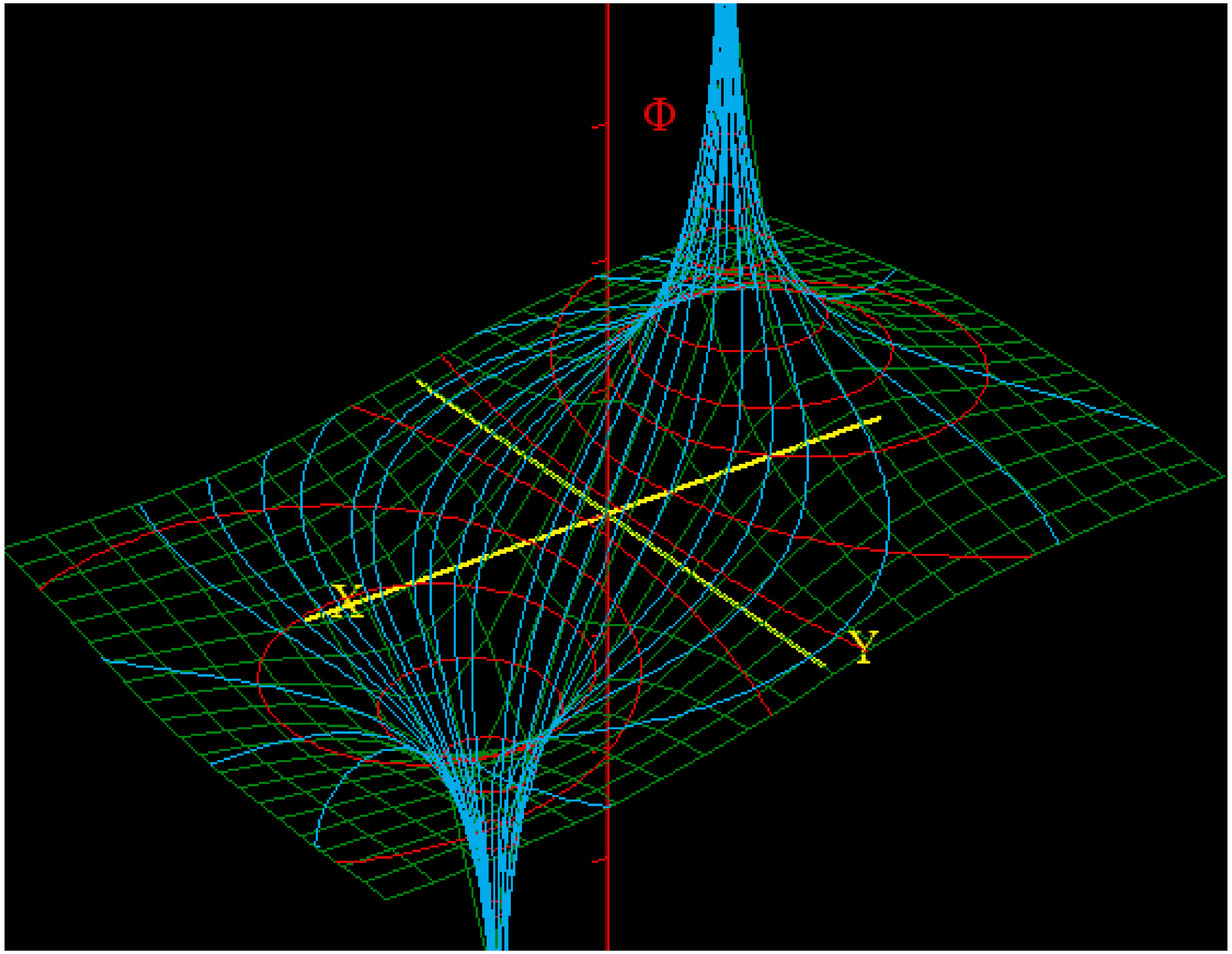
12. Complex derivatives give 2D dipole fields

Start with $f(z)=az^{-1}$: 2D line *monopole field* and is its *monopole potential* $\phi(z)=a\ln z$ of source strength a.

$$f^{1\text{-}pole}(z)=\frac{a}{z}=\frac{d\phi^{1\text{-}pole}}{dz} \qquad \phi^{1\text{-}pole}(z)=a\ln z$$

Now let these two line-sources of equal but opposite source constants $+a$ and $-a$ be located at $z=\pm\Delta/2$ separated by a small interval Δ. This sum (actually difference) of $f^{1\text{-}pole}$-fields is called a *dipole field*.

$$f^{dipole}(z)=\frac{a}{z+\frac{\Delta}{2}}-\frac{a}{z-\frac{\Delta}{2}}=\frac{-a\cdot\Delta}{z^2-\frac{\Delta^2}{4}} \qquad \phi^{dipole}(z)=a\ln(z-\tfrac{\Delta}{2})-a\ln(z+\tfrac{\Delta}{2})=a\ln\frac{z-\frac{\Delta}{2}}{z+\frac{\Delta}{2}}$$



So-called "physical dipole" has finite Δ

(+)(−) separation

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If interval Δ is *tiny* and is divided out we get a *point-dipole field* $f^{2\text{-}pole}$ that is the *z*-derivative of $f^{1\text{-}pole}$.

$$f^{2\text{-}pole}=\frac{-a}{z^2}=\frac{df^{1\text{-}pole}}{dz}=\frac{d\phi^{2\text{-}pole}}{dz} \qquad \phi^{2\text{-}pole}=\frac{a}{z}=\frac{d\phi^{1\text{-}pole}}{dz}$$

12. Complex derivatives give 2D dipole fields

Start with $f(z)=az^{-1}$: 2D line *monopole field* and is its *monopole potential* $\phi(z)=a\ln z$ of source strength a.

$$f^{1\text{-}pole}(z)=\frac{a}{z}=\frac{d\phi^{1\text{-}pole}}{dz} \qquad \phi^{1\text{-}pole}(z)=a\ln z$$

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$$f^{dipole}(z)=\frac{a}{z+\frac{\Delta}{2}}-\frac{a}{z-\frac{\Delta}{2}}=\frac{-a\cdot\Delta}{z^2-\frac{\Delta^2}{4}} \qquad \phi^{dipole}(z)=a\ln(z-\tfrac{\Delta}{2})-a\ln(z+\tfrac{\Delta}{2})=a\ln\frac{z-\frac{\Delta}{2}}{z+\frac{\Delta}{2}}$$

If interval Δ is *tiny* and is divided out we get a *point-dipole field* $f^{2\text{-}pole}$ that is the z-derivative of $f^{1\text{-}pole}$.

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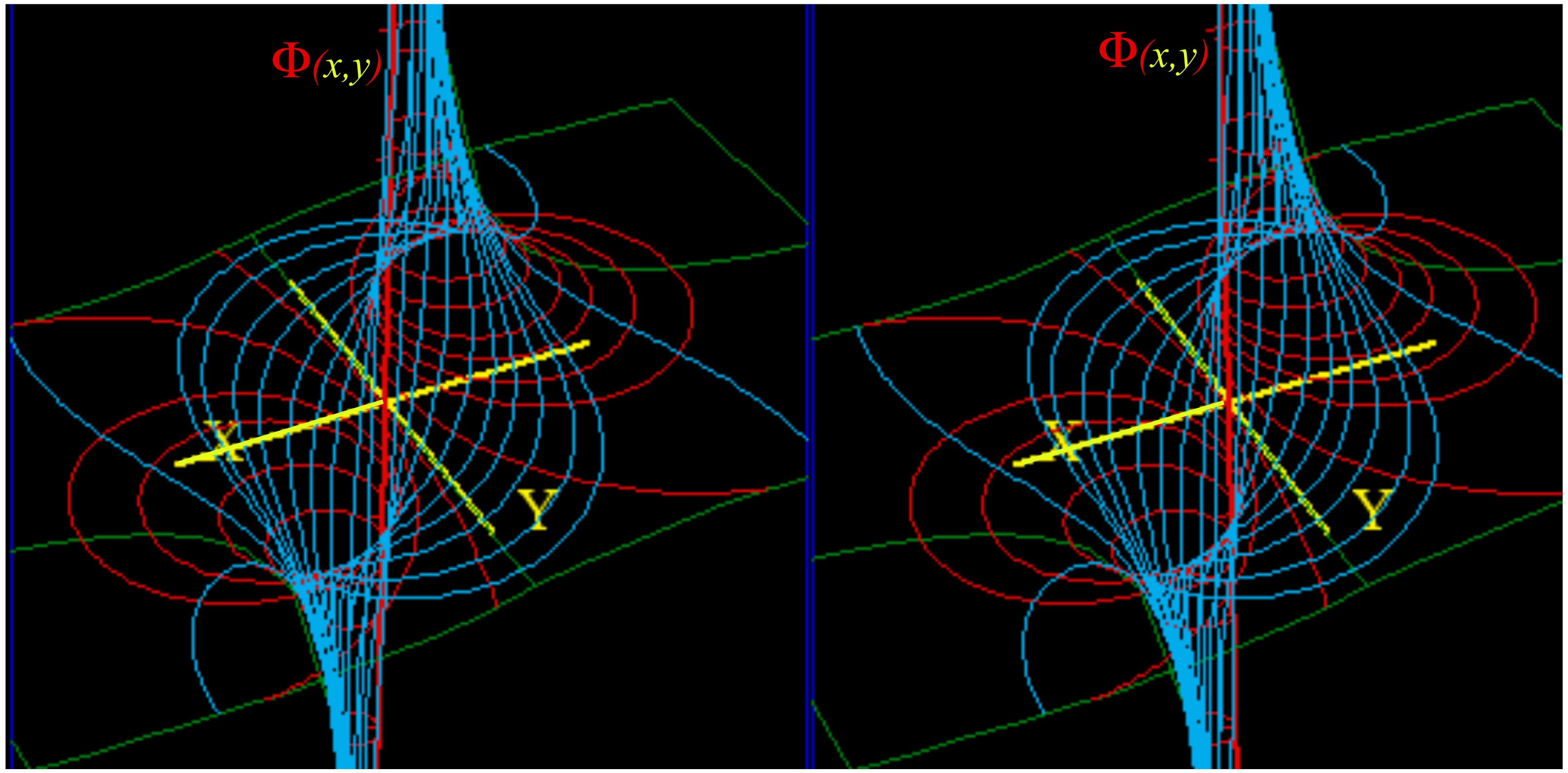


12. Complex derivatives give 2D dipole fields

Start with *f(z)=az*$^{-1}$: 2D line *monopole field* and is its *monopole potential* $\phi(z)=a\ln z$ of source strength *a*.

$$f^{1\text{-}pole}(z)=\frac{a}{z}=\frac{d\phi^{1\text{-}pole}}{dz} \qquad\qquad \phi^{1\text{-}pole}(z)=a\ln z$$

Now let these two line-sources of equal but opposite source constants +*a* and –*a* be located at *z*=±Δ/2 separated by a small interval Δ. This sum (actually difference) of $f^{1\text{-}pole}$-fields is called a *dipole field*.

$$f^{dipole}(z)=\frac{a}{z+\frac{\Delta}{2}}-\frac{a}{z-\frac{\Delta}{2}}=\frac{-a\cdot\Delta}{z^2-\frac{\Delta^2}{4}} \qquad \phi^{dipole}(z)=a\ln(z-\tfrac{\Delta}{2})-a\ln(z+\tfrac{\Delta}{2})=a\ln\frac{z-\frac{\Delta}{2}}{z+\frac{\Delta}{2}}$$

If interval Δ is *tiny* and is divided out we get a *point-dipole field* $f^{\,2\text{-}pole}$ that is the *z*-derivative of $f^{\,1\text{-}pole}$.

$$f^{2\text{-}pole}=\frac{-a}{z^2}=\frac{df^{1\text{-}pole}}{dz}=\frac{d\phi^{2\text{-}pole}}{dz} \qquad\qquad \phi^{2\text{-}pole}=\frac{a}{z}=\frac{d\phi^{1\text{-}pole}}{dz}$$

A *point-dipole potential* $\phi^{2\text{-}pole}$ (whose *z*-derivative is $f^{\,2\text{-}pole}$) is a *z*-derivative of $\phi^{1\text{-}pole}$.

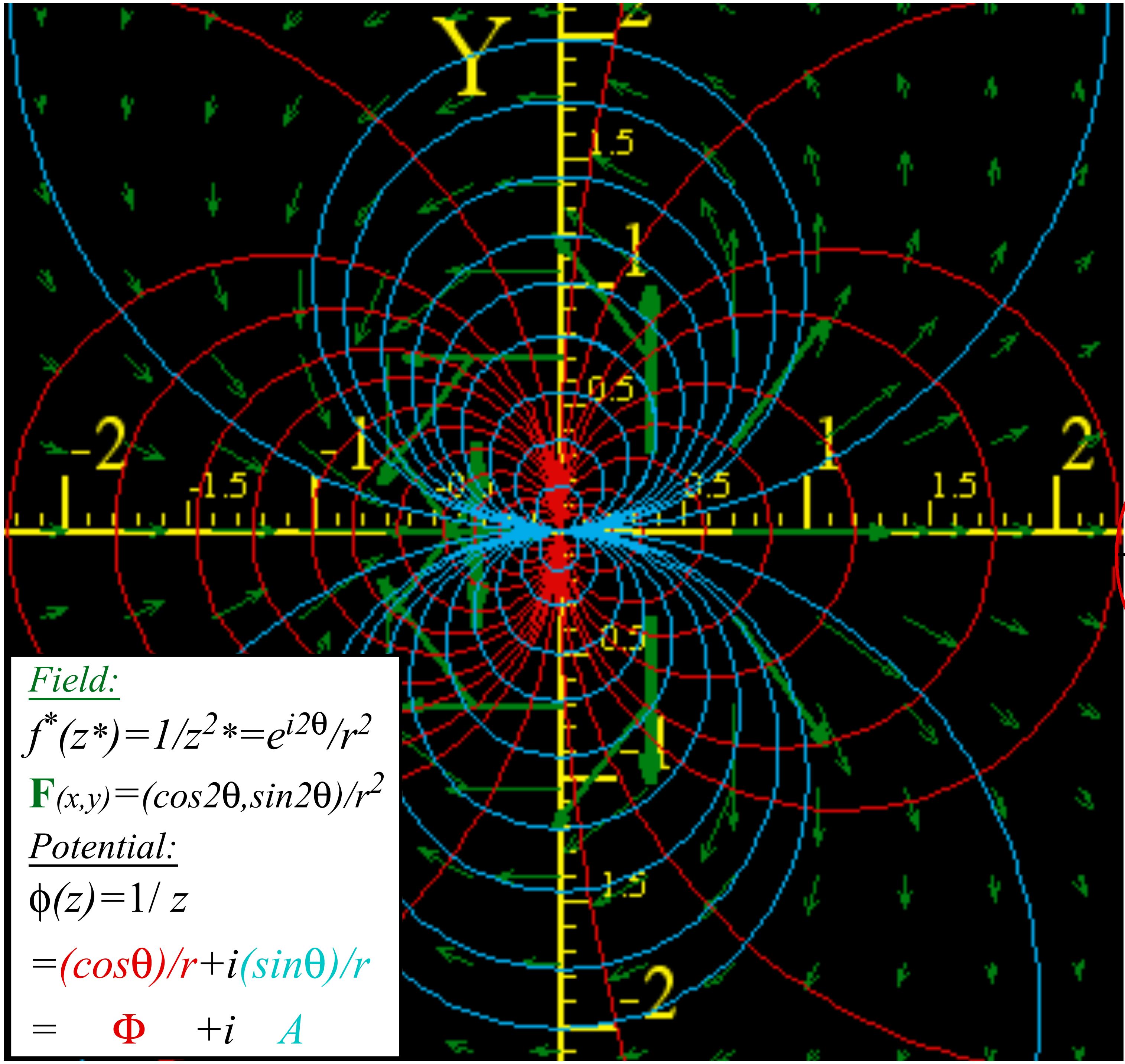
$$\phi^{2\text{-}pole}=\frac{a}{z}=\frac{a}{x+iy}=\frac{a}{x+iy}\frac{x-iy}{x-iy}=\frac{ax}{x^2+y^2}+i\frac{-ay}{x^2+y^2}=\frac{a}{r}\cos\theta-i\frac{a}{r}\sin\theta$$

$$=\Phi^{2\text{-}pole}+i\,\mathbf{A}^{2\text{-}pole}$$

A *point-dipole potential* $\phi^{2\text{-}pole}$ (whose z-derivative is $f^{2\text{-}pole}$) is a z-derivative of $\phi^{1\text{-}pole}$.

$$\phi^{2\text{-}pole}=\frac{a}{z}=\frac{a}{x+iy}=\frac{a}{x+iy}\frac{x-iy}{x-iy}=\frac{ax}{x^2+y^2}+i\frac{-ay}{x^2+y^2}=\frac{a}{r}\cos\theta-i\frac{a}{r}\sin\theta$$

$$=\Phi^{2\text{-}pole}+i\,A^{2\text{-}pole}$$



Field:

$f^*(z^*)=1/z^{2*}=e^{i2\theta}/r^2$

$\mathbf{F}(x,y)=(\cos 2\theta,\sin 2\theta)/r^2$

Potential:

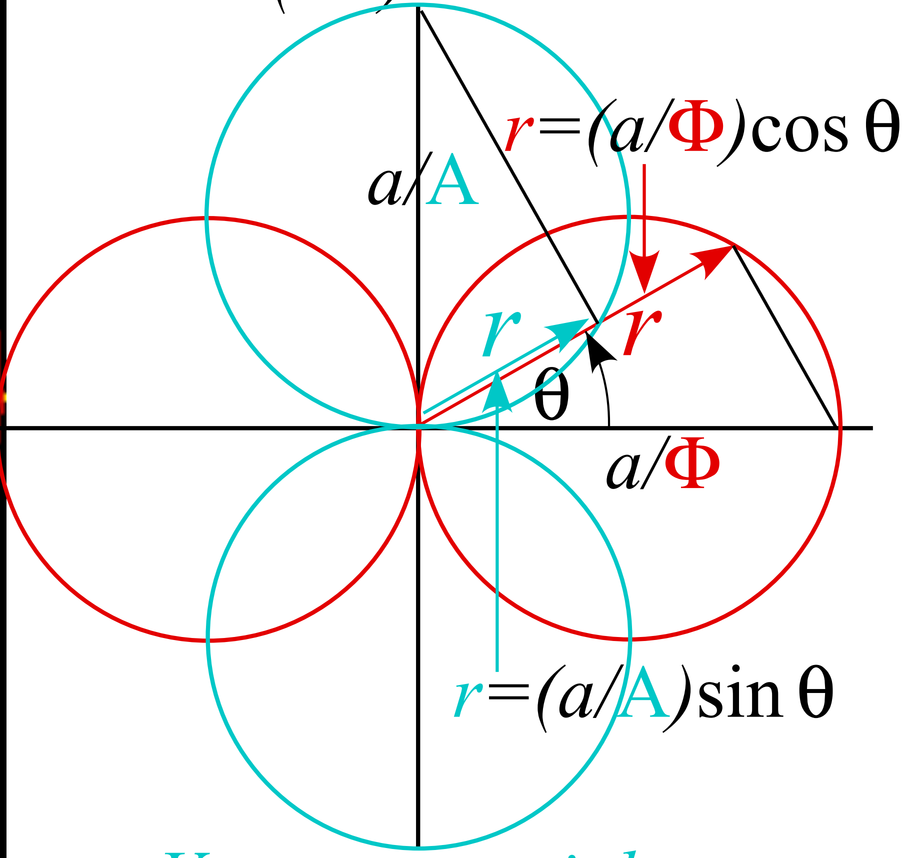
$\phi(z)=1/z$

$=(\cos\theta)/r+i(\sin\theta)/r$

$=\quad\Phi\quad+i\quad A$

Scalar potentials

$\Phi=(a/r)\cos\theta=const.$



Vector potentials

$A=(a/r)\sin\theta=const.$

2^n-*pole analysis* *(quadrupole:2^2=4-pole, octapole:2^3=8-pole,…, pole dancer,*

What if we put a *(-)*copy of a *2-pole* near its original?

Well, the result is *4-pole* or *quadrupole* field $f^{\text{4-pole}}$ and potential $\phi^{\text{4-pole}}$.

Each a z-derivative of $f^{\text{2-pole}}$ and $\phi^{\text{2-pole}}$.

$$f^{4\text{-}pole} = \frac{a}{z^3} = \frac{1}{2}\frac{df^{2\text{-}pole}}{dz} = \frac{d\phi^{4\text{-}pole}}{dz} \qquad \phi^{4\text{-}pole} = -\frac{a}{2z^2} = \frac{1}{2}\frac{d\phi^{2\text{-}pole}}{dz}$$

2^n-pole analysis *(quadrupole:2^2=4-pole, octapole:2^3=8-pole,…, pole dancer,*

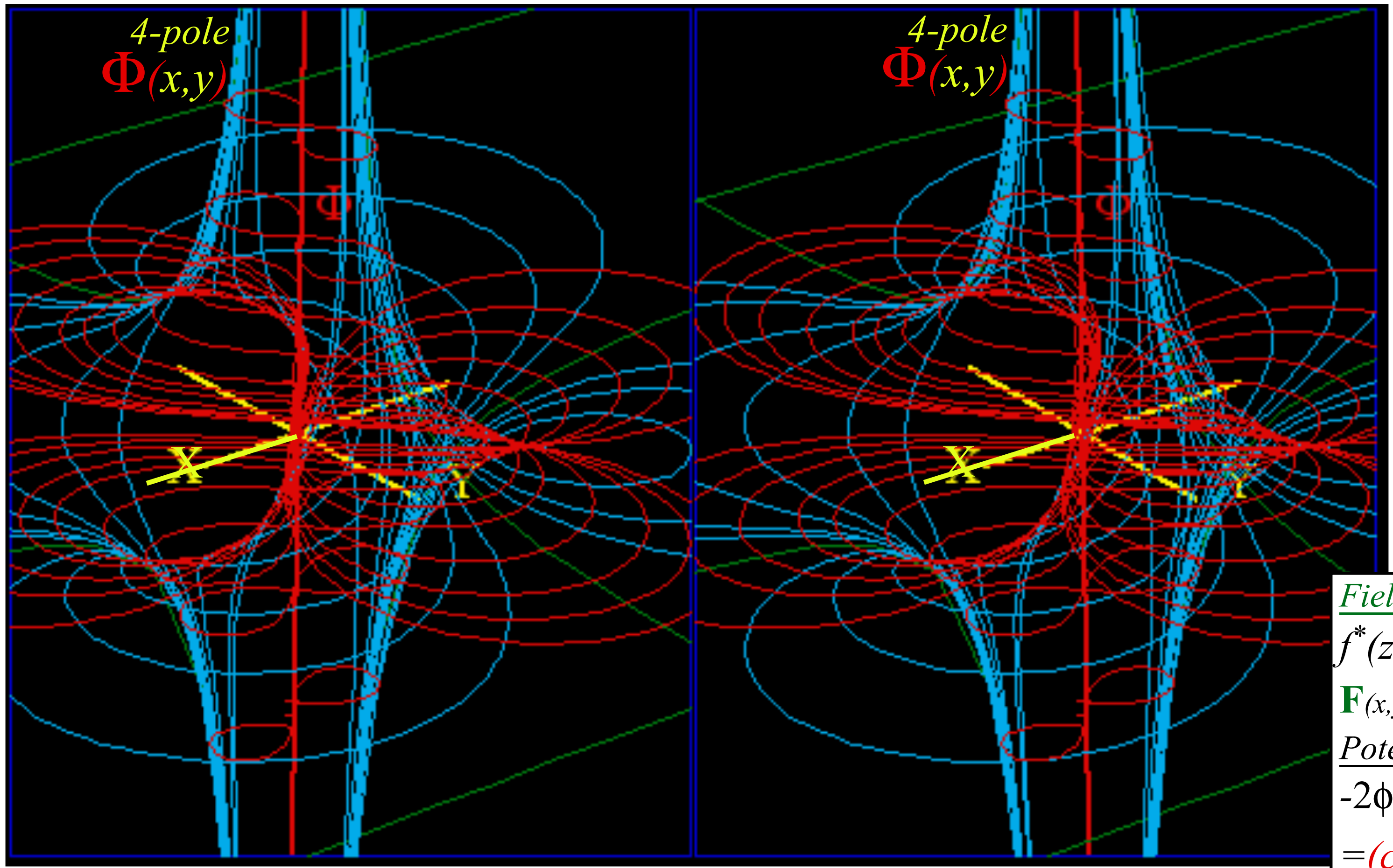
What if we put a *(-)*copy of a *2-pole* near its original?

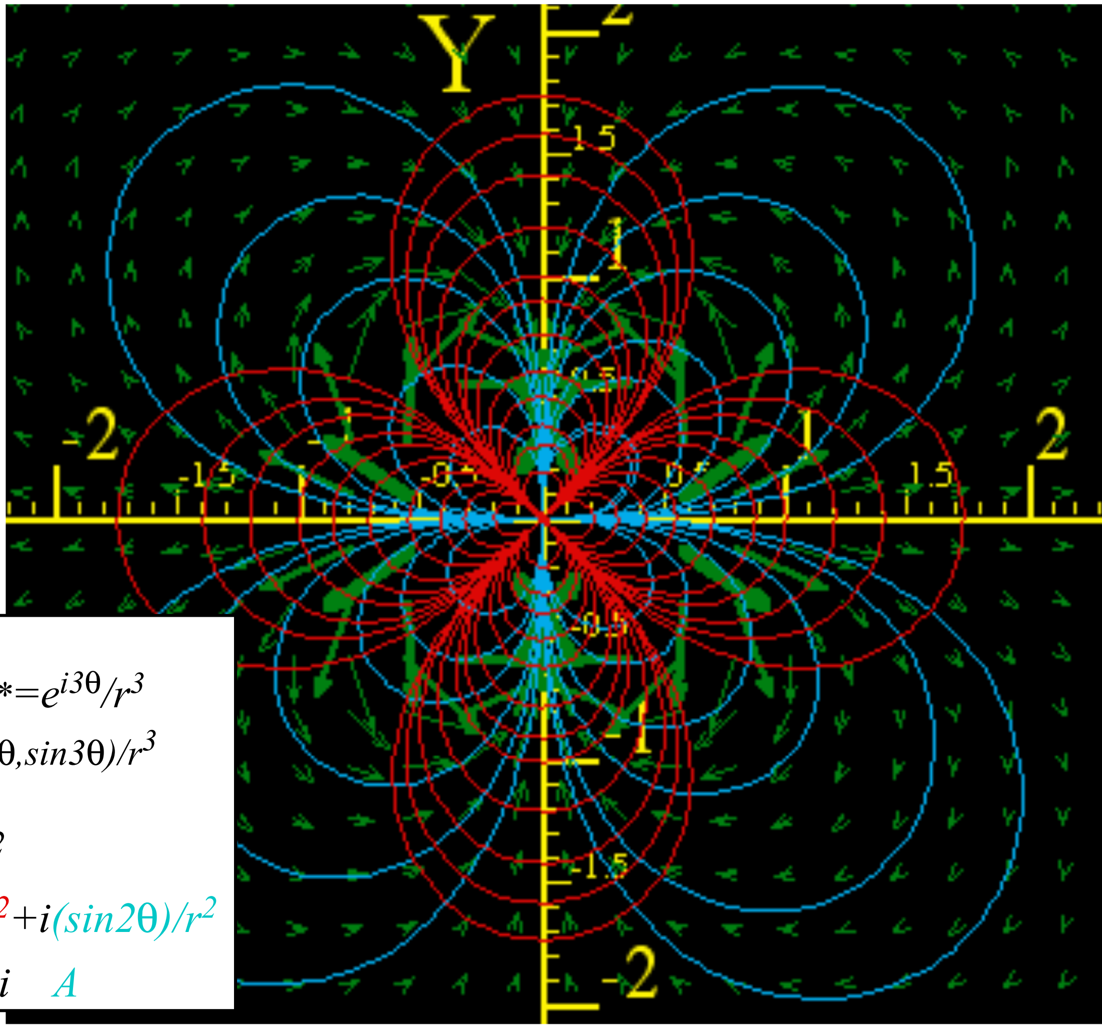
Well, the result is *4-pole* or *quadrupole* field $f^{4\text{-}pole}$ and potential $\phi^{4\text{-}pole}$.

Each a *z*-derivative of $f^{2\text{-}pole}$ and $\phi^{2\text{-}pole}$.

$$f^{4\text{-}pole} = \frac{a}{z^3} = \frac{1}{2}\frac{df^{2\text{-}pole}}{dz} = \frac{d\phi^{4\text{-}pole}}{dz}$$

$$\phi^{4\text{-}pole} = -\frac{a}{2z^2} = \frac{1}{2}\frac{d\phi^{2\text{-}pole}}{dz}$$





Field:
$f^*(z^*)=1/z^{3*}=e^{i3\theta}/r^3$
$\mathbf{F}_{(x,y)}=(\cos 3\theta,\sin 3\theta)/r^3$
Potential:
$-2\phi(z)=1/z^2$
$=(\cos 2\theta)/r^2+i(\sin 2\theta)/r^2$
$= \quad \Phi \quad +i \quad A$

4. Riemann-Cauchy conditions *What's analytic? (...and what's <u>not</u>?)*

- *Easy 2D circulation and flux integrals*
  - *Easy 2D curvilinear coordinate discovery*
    - *Easy 2D monopole, dipole, and 2^n-pole analysis*
      - *Easy 2^n-multipole field and potential expansion*
        - *Easy stereo-projection visualization*

2^n-pole analysis: Laurent series *(Generalization of Maclaurin-Taylor series)*

Laurent series or *multipole expansion* of a given complex field function *f(z)* around *z=0*.

$$\frac{d\phi}{dz}=f(z)=\ldots a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \ldots$$

⋯	2^2-pole	2^1-pole	2^0-pole	2^1-pole	2^2-pole	2^3-pole	2^4-pole	2^5-pole	2^6-pole	⋯
	(quadrupole)	*(dipole)*	*(monopole)*	*(dipole)*	*(quadrupole)*	*(octapole)*	*(hexadecapole)*			
	at $z=0$	at $z=0$	at $z=0$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$	

$\int f\,dz =$

$$\phi(z)=\ldots\frac{a_{-3}}{-2}z^{-2} + \frac{a_{-2}}{-1}z^{-1} + a_{-1}\ln z + a_0 z + \frac{a_1}{2}z^2 + \frac{a_2}{3}z^3 + \frac{a_3}{4}z^4 + \frac{a_4}{5}z^5 + \frac{a_5}{6}z^6 + \ldots$$

All field terms $a_{m-1}z^{m-1}$ except *1-pole* $\frac{a_{-1}}{z}$ have potential term $a_{m-1}z^m/m$ of a 2^m-*pole.*

These are located at $z=0$ for $m<0$ and at $z=\infty$ for $m>0$.

$$\phi(z)=\ldots\overset{(\text{octapole})_0}{\frac{a_{-4}}{-3}z^{-3}} + \overset{(\text{quadrupole})_0}{\frac{a_{-3}}{-2}z^{-2}} + \overset{(\text{dipole})_0}{\frac{a_{-2}}{-1}z^{-1}} + \overset{(\text{monopole})}{a_{-1}\ln z} + \overset{(\text{dipole})_\infty}{a_0 z} + \overset{(\text{quadrupole})_\infty}{\frac{a_1}{2}z^2} + \overset{(\text{octapole})_\infty}{\frac{a_2}{3}z^3} + \ldots$$

2^n-pole analysis: Laurent series *(Generalization of Maclaurin-Taylor series)*

Laurent series or *multipole expansion* of a given complex field function $f(z)$ around $z=0$.

$$\frac{d\phi}{dz}=f(z)=\ldots a_{-3}z^{-3}\;+\;a_{-2}z^{-2}\;+\;a_{-1}z^{-1}\;+\;a_0\;+\;a_1 z\;+\;a_2 z^2\;+\;a_3 z^3\;+\;a_4 z^4\;+\;a_5 z^5\;+\ldots$$

$\cdots$										$\cdots$
	2^2-pole	2^1-pole	2^0-pole	2^1-pole	2^2-pole	2^3-pole	2^4-pole	2^5-pole	2^6-pole	
	(quadrupole)	*(dipole)*	*(monopole)*	*(dipole)*	*(quadrupole)*	*(octapole)*	*(hexadecapole)*			
	at $z=0$	at $z=0$	at $z=0$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$	

$$\int f\,dz=$$

$$\phi(z)=\ldots\frac{a_{-3}}{-2}z^{-2}\;+\;\frac{a_{-2}}{-1}z^{-1}\;+\;a_{-1}\ln z\;+\;a_0 z\;+\;\frac{a_1}{2}z^2\;+\;\frac{a_2}{3}z^3\;+\;\frac{a_3}{4}z^4\;+\;\frac{a_4}{5}z^5+\;\frac{a_5}{6}z^6\;+\ldots$$

All field terms $a_{m-1}z^{m-1}$ except *1-pole* $\frac{a_{-1}}{z}$ have potential term $a_{m-1}z^m/m$ of a *2^m-pole.*

These are located at $z=0$ for $m<0$ and at $z=\infty$ for $m>0$.

(octapole)$_0$ *(quadrupole)$_0$* *(dipole)$_0$* *(monopole)* *(dipole)$_\infty$* *(quadrupole)$_\infty$* *(octapole)$_\infty$*

$$\phi(z)=\ldots\frac{a_{-3}}{-2}z^{-2}\;+\frac{a_{-3}}{-2}z^{-2}\;+\;\frac{a_{-2}}{-1}z^{-1}\;+\;a_{-1}\ln z\;+\;a_0 z\;+\;\frac{a_1}{2}z^2\;+\;\frac{a_2}{3}z^3\;+\ldots$$

$$\phi(w)=\ldots\frac{a_{-4}}{-3}w^{-3}+\frac{a_{-3}}{-2}w^{-2}+\;\frac{a_{-2}}{-1}w^{-1}\;+\;a_{-1}\ln w\;+\;a_0 w\;+\;\frac{a_1}{2}w^2\;+\;\frac{a_2}{3}w^3\;+\ldots$$

(*with* $z=w^{-1}$)

2^n-pole analysis: Laurent series *(Generalization of Maclaurin-Taylor series)*

Laurent series or *multipole expansion* of a given complex field function *f(z)* around *z=0*.

$$\frac{d\phi}{dz}=f(z)=\ldots a_{-3}z^{-3}+a_{-2}z^{-2}+a_{-1}z^{-1}+a_0+a_1z+a_2z^2+a_3z^3+a_4z^4+a_5z^5+\ldots$$

$a_{-3}z^{-3}$	$a_{-2}z^{-2}$	$a_{-1}z^{-1}$	a_0	a_1z	a_2z^2	a_3z^3	a_4z^4	a_5z^5
$\cdots$ 2^2-pole	2^1-pole	2^0-pole	2^1-pole	2^2-pole	2^3-pole	2^4-pole	2^5-pole	2^6-pole $\cdots$
(quadrupole)	*(dipole)*	*(monopole)*	*(dipole)*	*(quadrupole)*	*(octapole)*	*(hexadecapole)*		
at *z=0*	at *z=0*	at *z=0*	at *z=∞*	at *z=∞*	at *z=∞*	at *z=∞*	at *z=∞*	at *z=∞*

$$\int f\,dz=$$

$$\phi(z)=\ldots\frac{a_{-3}}{-2}z^{-2}+\frac{a_{-2}}{-1}z^{-1}+a_{-1}\ln z+a_0z+\frac{a_1}{2}z^2+\frac{a_2}{3}z^3+\frac{a_3}{4}z^4+\frac{a_4}{5}z^5+\frac{a_5}{6}z^6+\ldots$$

All field terms $a_{m-1}z^{m-1}$ except *1-pole* $\frac{a_{-1}}{z}$ have potential term $a_{m-1}z^m/m$ of a 2^m*-pole.*

These are located at $z=0$ for $m<0$ and at $z=\infty$ for $m>0$.

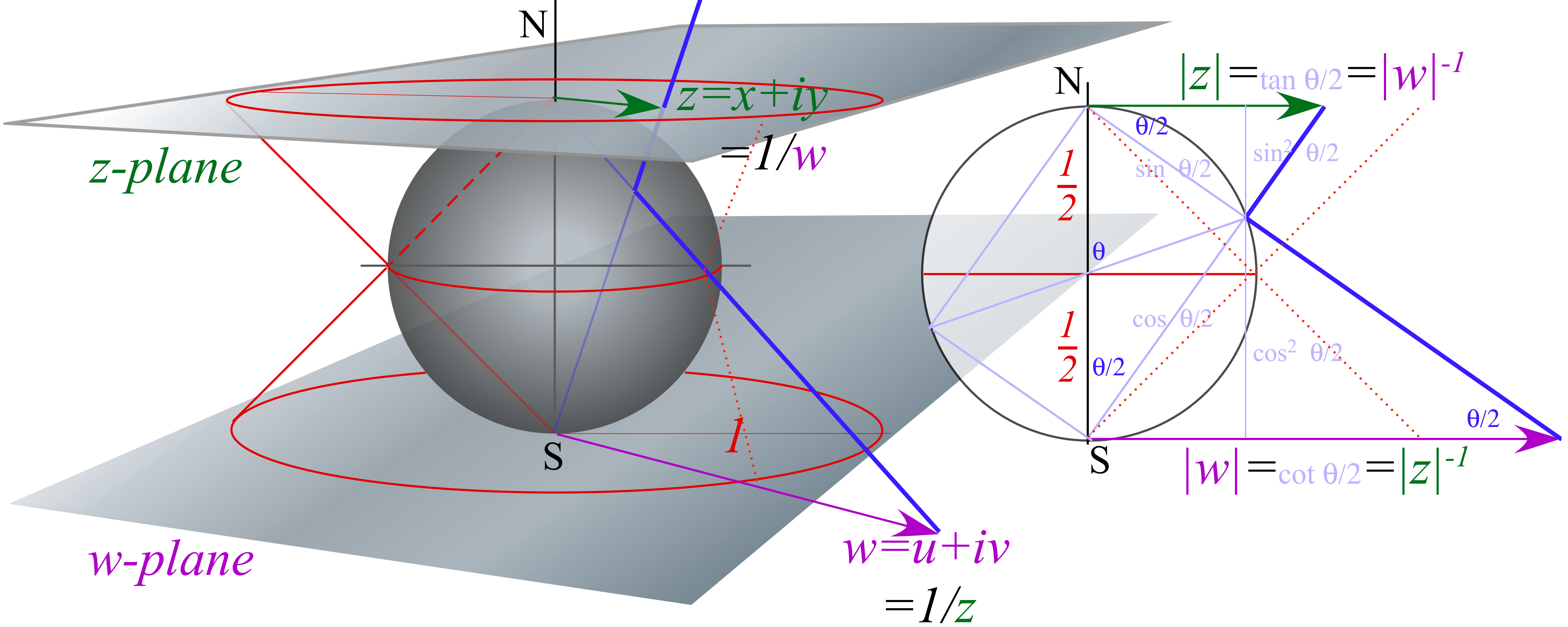
$$\phi(z)=\ldots\overset{(\text{octapole})_0}{\frac{a_{-4}}{-3}z^{-3}}+\overset{(\text{quadrupole})_0}{\frac{a_{-3}}{-2}z^{-2}}+\overset{(\text{dipole})_0}{\frac{a_{-2}}{-1}z^{-1}}+\overset{(\text{monopole})}{a_{-1}\ln z}+\overset{(\text{dipole})_\infty}{a_0z}+\overset{(\text{quadrupole})_\infty}{\frac{a_1}{2}z^2}+\overset{(\text{octapole})_\infty}{\frac{a_2}{3}z^3}+\ldots$$

$$\phi(w)=\ldots\frac{a_{-4}}{-3}w^{-3}+\frac{a_{-3}}{-2}w^{-2}+\frac{a_{-2}}{-1}w^{-1}+a_{-1}\ln w+a_0w+\frac{a_1}{2}w^2+\frac{a_2}{3}w^3+\ldots$$

(*with* $z\rightarrow w$)

$$=\ldots\frac{a_2}{3}z^{-3}+\frac{a_1}{2}z^{-2}+a_0z^{-1}-a_{-1}\ln z+\frac{a_{-2}}{-1}z+\frac{a_{-3}}{-2}z^2+\frac{a_{-4}}{-3}z^3+\ldots$$

(*with* $w=z^{-1}$)



$$\phi(z) = \ldots\frac{a_{-4}}{-3}z^{-3} + \frac{a_{-3}}{-2}z^{-2} + \frac{a_{-2}}{-1}z^{-1} + a_{-1}\ln z + a_0 z + \frac{a_1}{2}z^2 + \frac{a_2}{3}z^3 + \ldots$$

(octapole)₀ *(quadrupole)₀* *(dipole)₀* *(monopole)* *(dipole)∞* *(quadrupole)∞* *(octapole)∞*

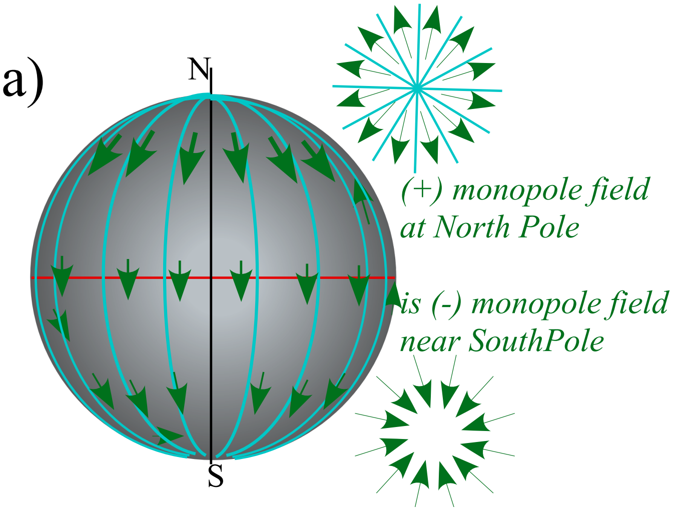
$$\phi(w) = \ldots\frac{a_{-4}}{-3}w^{-3} + \frac{a_{-3}}{-2}w^{-2} + \frac{a_{-2}}{-1}w^{-1} + a_{-1}\ln w + a_0 w + \frac{a_1}{2}w^2 + \frac{a_2}{3}w^3 + \ldots$$

(*with* $z \rightarrow w$)

$$= \ldots\frac{a_2}{3}z^{-2} + \frac{a_1}{2}z^{-2} + a_0 z^{-1} - a_{-1}\ln z + \frac{a_{-2}}{-1}z + \frac{a_{-3}}{-2}z^2 + \frac{a_{-4}}{-3}z^3 + \ldots$$

(*with* $w=z^{-1}$)

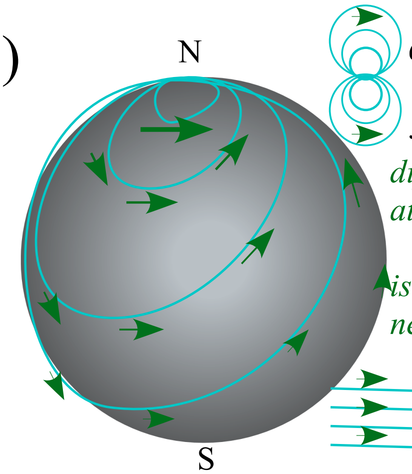
(a)



(+) monopole field at North Pole

is (-) monopole field near SouthPole

(b)



$\phi(z) = \frac{a_{-2}}{-1}z^{-1}$

$f(z) = a_{-2}z^{-2}$

dipole field centered at North Pole

is constant field near SouthPole

$\phi(w) = a_0 w$

$f(w) = a_0$

$\phi(z) = \frac{a_{-3}}{-2}z^{-2}$

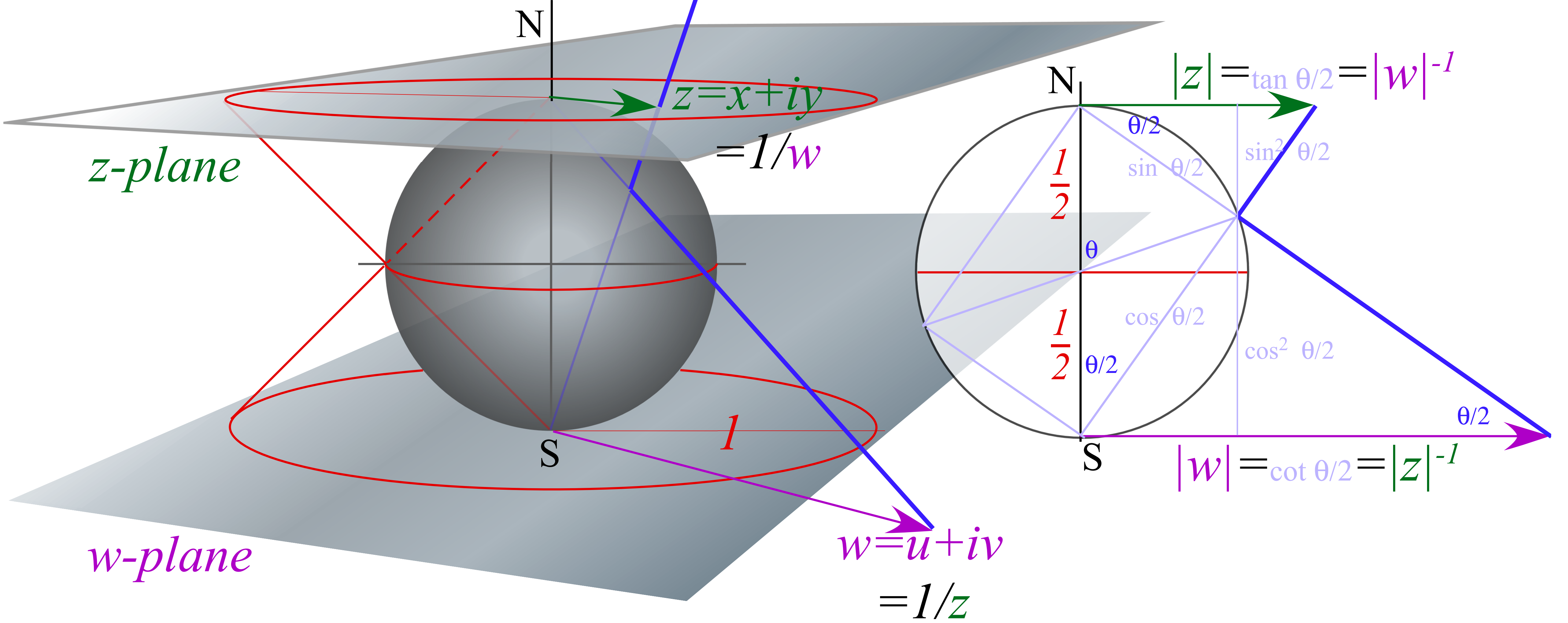
$f(z) = a_{-3}z^{-3}$

quadrupole field centered at North Pole

is quadratic field near South Pole

$\phi(w) = a_0 w^2$

$f(w) = a_1 w$



$$\phi(z)=\ldots\frac{a_{-4}}{-3}z^{-3}+\frac{a_{-3}}{-2}z^{-2}+\frac{a_{-2}}{-1}z^{-1}+a_{-1}\ln z+a_0 z+\frac{a_1}{2}z^{2}+\frac{a_2}{3}z^{3}+\ldots$$

(octapole)$_0$ *(quadrupole)$_0$* *(dipole)$_0$* *(monopole)* *(dipole)$_\infty$* *(quadrupole)$_\infty$* *(octapole)$_\infty$*

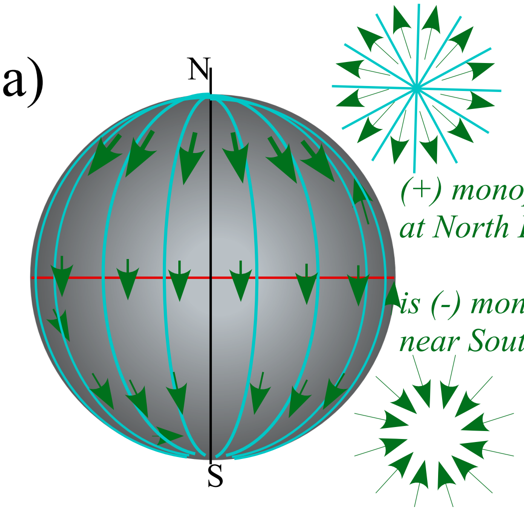
$$\phi(w)=\ldots\frac{a_{-4}}{-3}w^{-3}+\frac{a_{-3}}{-2}w^{-2}+\frac{a_{-2}}{-1}w^{-1}+a_{-1}\ln w+a_0 w+\frac{a_1}{2}w^{2}+\frac{a_2}{3}w^{3}+\ldots$$

(with $z\rightarrow w$)

$$=\ldots\frac{a_2}{3}z^{-2}+\frac{a_1}{2}z^{-2}+a_0 z^{-1}-a_{-1}\ln z+\frac{a_{-2}}{-1}z+\frac{a_{-3}}{-2}z^{2}+\frac{a_{-4}}{-3}z^{3}+\ldots$$

(with $w=z^{-1}$)

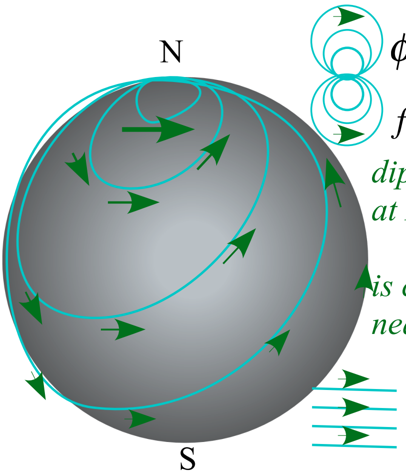
(a)



(+) monopole field at North Pole

is (-) monopole field near SouthPole

(b)



$$\phi(z)=\frac{a_{-2}}{-1}z^{-1}$$

$$f(z)=a_{-2}z^{-2}$$

dipole field centered at North Pole

is constant field near SouthPole

$$\phi(w)=a_0 w$$

$$f(w)=a_0$$

$$\phi(z)=\frac{a_{-3}}{-2}z^{-2}$$

$$f(z)=a_{-3}z^{-3}$$

quadrupole field centered at North Pole

is quadratic field near South Pole

$$\phi(w)=a_0 w^{2}$$

$$f(w)=a_1 w$$

$$f(z) = \ldots a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \ldots$$

Of all 2^m-*pole* field terms $a_{m-1}z^{m-1}$, only the $m=0$ monopole $a_{-1}z^{-1}$ has a non-zero loop integral (10.39).

$$\oint f(z)dz = \oint a_{-1}z^{-1}dz = 2\pi i a_{-1}$$

$$a_{-1} = \frac{1}{2\pi i}\oint f(z)dz$$

$$f(z) = \dots a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \dots$$

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$$\oint f(z)dz = \oint a_{-1}z^{-1}dz = 2\pi i a_{-1} \qquad a_{-1} = \tfrac{1}{2\pi i}\oint f(z)dz$$

This $m=1$-*pole* constant-a_{-1} formula is just the first in a series of *Laurent coefficient expressions*.

$$\cdots a_{-3} = \tfrac{1}{2\pi i}\oint z^2 f(z)dz \;,\; a_{-2} = \tfrac{1}{2\pi i}\oint z^1 f(z)dz \;,\; a_{-1} = \tfrac{1}{2\pi i}\oint f(z)dz \;,\; a_0 = \tfrac{1}{2\pi i}\oint \frac{f(z)}{z}dz \;,\; a_1 = \tfrac{1}{2\pi i}\oint \frac{f(z)}{z^2}dz \;,\cdots$$

$$f(z) = \ldots a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \ldots$$

Of all 2^m*-pole* field terms $a_{m-1}z^{m-1}$, only the $m=0$ monopole $a_{-1}z^{-1}$ has a non-zero loop integral (10.39).

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Source analysis starts with 1-pole loop integrals $\oint z^{-1}dz = 2\pi i$ or, with origin shifted $\oint (z-a)^{-1}dz = 2\pi i$.

$$f(z) = \ldots a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \ldots$$

Of all 2^m-*pole* field terms $a_{m-1}z^{m-1}$, only the $m=0$ monopole $a_{-1}z^{-1}$ has a non-zero loop integral (10.39).

$$\oint f(z)dz = \oint a_{-1}z^{-1}dz = 2\pi i a_{-1} \qquad a_{-1} = \tfrac{1}{2\pi i}\oint f(z)dz$$

This $m=1$-*pole* constant-a_{-1} formula is just the first in a series of *Laurent coefficient expressions*.

$$\cdots a_{-3} = \tfrac{1}{2\pi i}\oint z^2 f(z)dz \text{ , } a_{-2} = \tfrac{1}{2\pi i}\oint z^1 f(z)dz \text{ , } a_{-1} = \tfrac{1}{2\pi i}\oint f(z)dz \text{ , } a_0 = \tfrac{1}{2\pi i}\oint \frac{f(z)}{z}dz \text{ , } a_1 = \tfrac{1}{2\pi i}\oint \frac{f(z)}{z^2}dz \text{ ,} \cdots$$

Source analysis starts with 1-pole loop integrals $\oint z^{-1}dz = 2\pi i$ or, with origin shifted $\oint (z-a)^{-1}dz = 2\pi i$.

They hold for any loop about point-a. Function *f(z)* is just *f(a)* on a *tiny* circle around point-a.

(assume *tiny* circle around $z=a$)

$$\oint \frac{f(z)}{z-a}dz = \oint \frac{f(a)}{z-a}dz = f(a)\oint \frac{1}{z-a}dz = 2\pi i f(a)$$

(but any contour that doesn't “touch a gives same answer)

$$f(z)=\ldots a_{-3}z^{-3} \;+\; a_{-2}z^{-2} \;+\; a_{-1}z^{-1} \;+\; a_0 \;+\; a_1 z \;+\; a_2 z^2 \;+\; a_3 z^3 \;+\; a_4 z^4 \;+\; a_5 z^5 \;+\ldots$$

Of all 2^m-*pole* field terms $a_{m-1}z^{m-1}$, only the $m=0$ monopole $a_{-1}z^{-1}$ has a non-zero loop integral (10.39).

$$\oint f(z)dz=\oint a_{-1}z^{-1}dz=2\pi i a_{-1} \qquad\qquad a_{-1}=\tfrac{1}{2\pi i}\oint f(z)dz$$

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$$\cdots a_{-3}=\tfrac{1}{2\pi i}\oint z^2 f(z)dz\;,\; a_{-2}=\tfrac{1}{2\pi i}\oint z^1 f(z)dz\;,\; a_{-1}=\tfrac{1}{2\pi i}\oint f(z)dz\;,\; a_0=\tfrac{1}{2\pi i}\oint \frac{f(z)}{z}dz\;,\; a_1=\tfrac{1}{2\pi i}\oint \frac{f(z)}{z^2}dz\;,\cdots$$

Source analysis starts with 1-pole loop integrals $\oint z^{-1}dz=2\pi i$ or, with origin shifted $\oint (z-a)^{-1}dz=2\pi i$.

They hold for any loop about point-*a*. Function *f(z)* is just *f(a)* on a tiny circle around point-*a*.

(assume tiny circle around $z=a$)

$$\oint \frac{f(z)}{z-a}dz=\oint \frac{f(a)}{z-a}dz=f(a)\oint \frac{1}{z-a}dz=2\pi i f(a)$$

(but any contour that doesn’t “touch *a* gives same answer)

$$\boxed{f(a)=\frac{1}{2\pi i}\oint \frac{f(z)}{z-a}dz}$$

The *f(a)* result is called a *Cauchy integral*.

$$f(z) = \ldots a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \ldots$$

Of all 2^m-*pole* field terms $a_{m-1}z^{m-1}$, only the $m=0$ monopole $a_{-1}z^{-1}$ has a non-zero loop integral (10.39).

$$\oint f(z)dz = \oint a_{-1}z^{-1}dz = 2\pi i a_{-1} \qquad a_{-1} = \tfrac{1}{2\pi i}\oint f(z)dz$$

This $m=1$-*pole* constant-a_{-1} formula is just the first in a series of *Laurent coefficient expressions*.

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Source analysis starts with 1-pole loop integrals $\oint z^{-1}dz = 2\pi i$ or, with origin shifted $\oint (z-a)^{-1}dz = 2\pi i$.

They hold for any loop about point-a. Function $f(z)$ is just $f(a)$ on a tiny circle around point-a.

(assume tiny circle around $z=a$)

$$\oint \frac{f(z)}{z-a}dz = \oint \frac{f(a)}{z-a}dz = f(a)\oint \frac{1}{z-a}dz = 2\pi i f(a) \qquad \boxed{f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a}dz}$$

(but any contour that doesn't "touch a gives same answer)

The $f(a)$ result is called a *Cauchy integral*. Then repeated a-derivatives gives a sequence of them.

$$\frac{df(a)}{da} = \frac{1}{2\pi i}\oint \frac{f(z)}{(z-a)^2}dz \ ,$$

$$f(z) = \dots a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \dots$$

Of all 2^m-*pole* field terms $a_{m-1}z^{m-1}$, only the $m=0$ monopole $a_{-1}z^{-1}$ has a non-zero loop integral (10.39).

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Source analysis starts with 1-pole loop integrals $\oint z^{-1}dz = 2\pi i$ or, with origin shifted $\oint (z-a)^{-1}dz = 2\pi i$.

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$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n \qquad where:\ a_n = \frac{1}{2\pi i}\oint \frac{f(z)}{(z-a)^{n+1}}dz \left(= \frac{1}{n!}\frac{d^n f(a)}{da^n} \quad for:\ n \geq 0 \right)$$

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5. Mapping and Non-analytic 2D source field analysis

The half-n'-half results

are called

Riemann-Cauchy

Derivative Relations

$$\boxed{\begin{array}{l}\frac{\partial \Phi}{\partial x}=\frac{\partial \mathbf{A}}{\partial y}\\ \frac{\partial \Phi}{\partial y}=-\frac{\partial \mathbf{A}}{\partial x}\end{array}}\begin{array}{l}\text{is:}\\ \text{is:}\end{array}\boxed{\begin{array}{l}\frac{\partial \mathrm{Re}\,\phi(z)}{\partial x}=\frac{\partial \mathrm{Im}\,\phi(z)}{\partial y}\\ \frac{\partial \mathrm{Re}\,\phi(z)}{\partial y}=-\frac{\partial \mathrm{Im}\,\phi(z)}{\partial x}\end{array}}\begin{array}{l}\text{or:}\\ \text{or:}\end{array}\boxed{\begin{array}{l}\frac{\partial \mathrm{Re}\,f(z)}{\partial x}=\frac{\partial \mathrm{Im}\,f(z)}{\partial y}\\ \frac{\partial \mathrm{Re}\,f(z)}{\partial y}=-\frac{\partial \mathrm{Im}\,f(z)}{\partial x}\end{array}}\begin{array}{l}\text{is:}\\ \text{is:}\end{array}\boxed{\begin{array}{l}\frac{\partial f_x(z)}{\partial x}=\frac{\partial f_y(z)}{\partial y}\\ \frac{\partial f_x(z)}{\partial y}=-\frac{\partial f_y(z)}{\partial x}\end{array}}$$

RC applies to analytic potential $\phi(z)=\Phi+i\mathbf{A}$ *and analytic field* $f(z)=f_x+if_y$ *and any analytic function*

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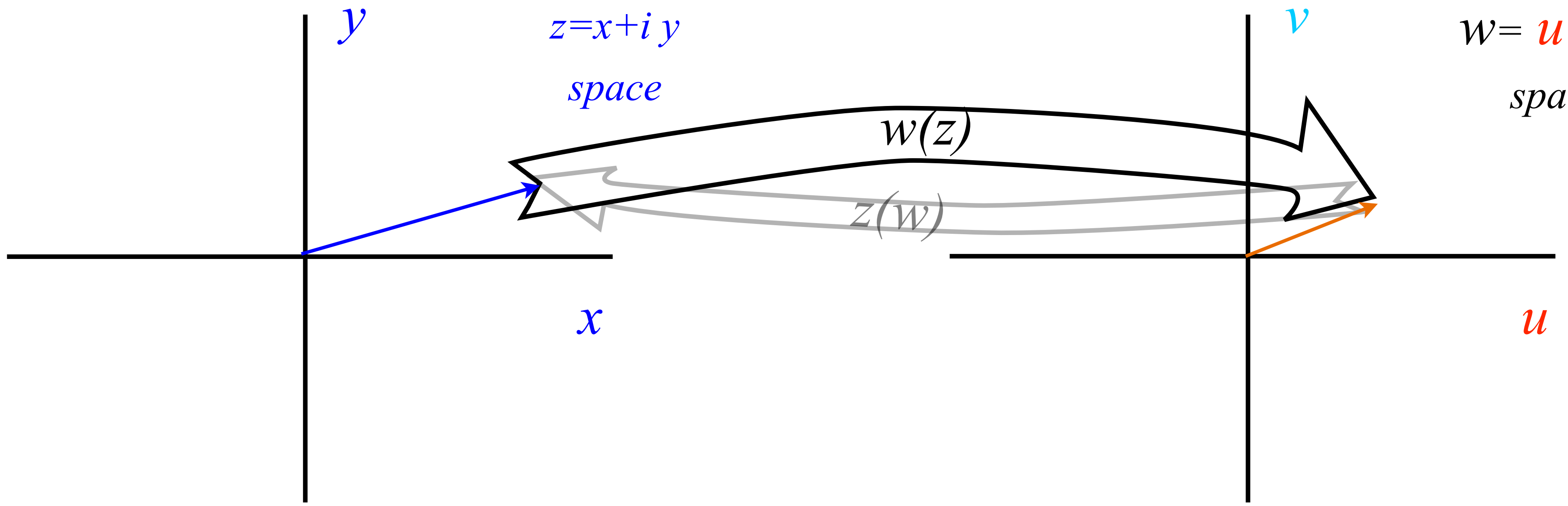
Common notation for mapping: $w(z)=u+iv$



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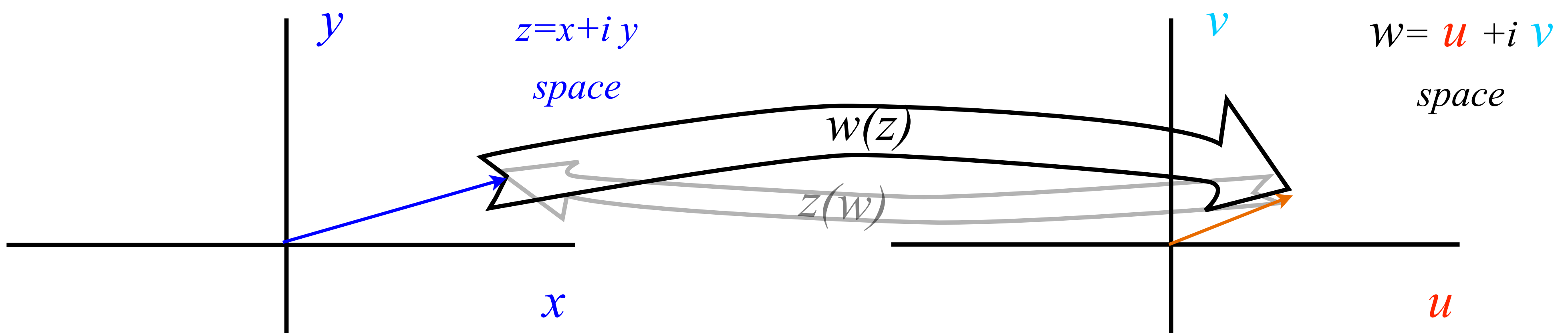

Jacobian for mapping:

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$$

$$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy$$

$$\begin{pmatrix} du\\ dv\end{pmatrix}=\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}\begin{pmatrix} dx\\ dy\end{pmatrix}$$

Complex derivative for mapping:

$$\frac{dw}{dz}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)(u+iv)=\frac{1}{2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{i}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

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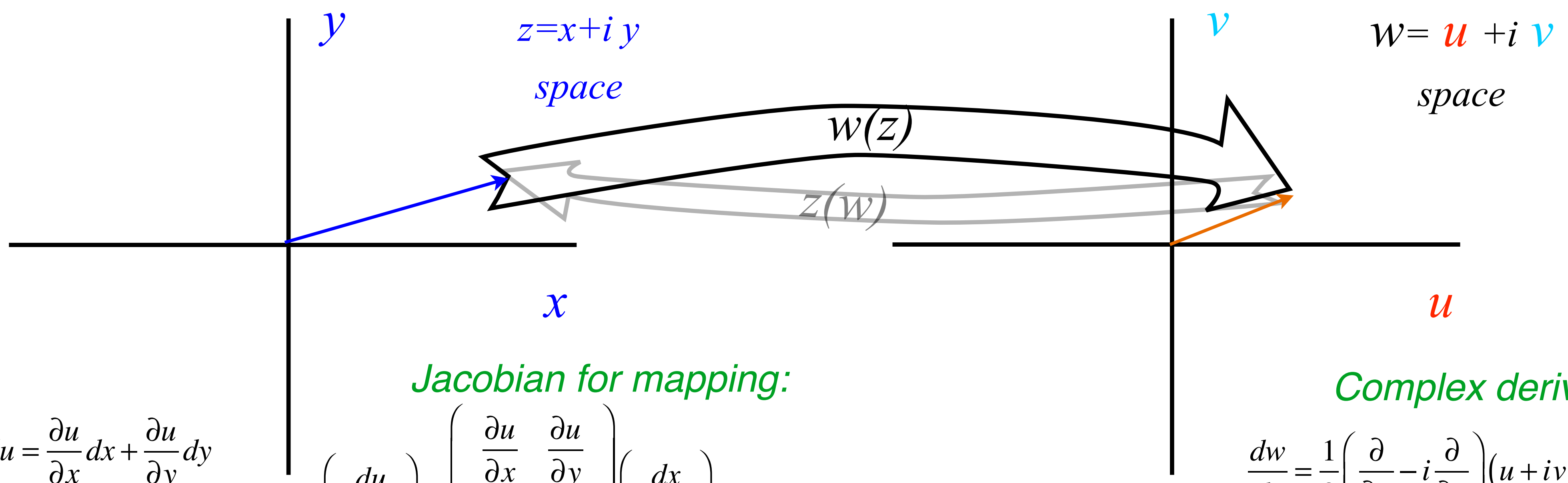

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$$=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}\qquad=\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x}$$

Complex derivative abs-square:

$$\left|\frac{dw}{dz}\right|^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2$$

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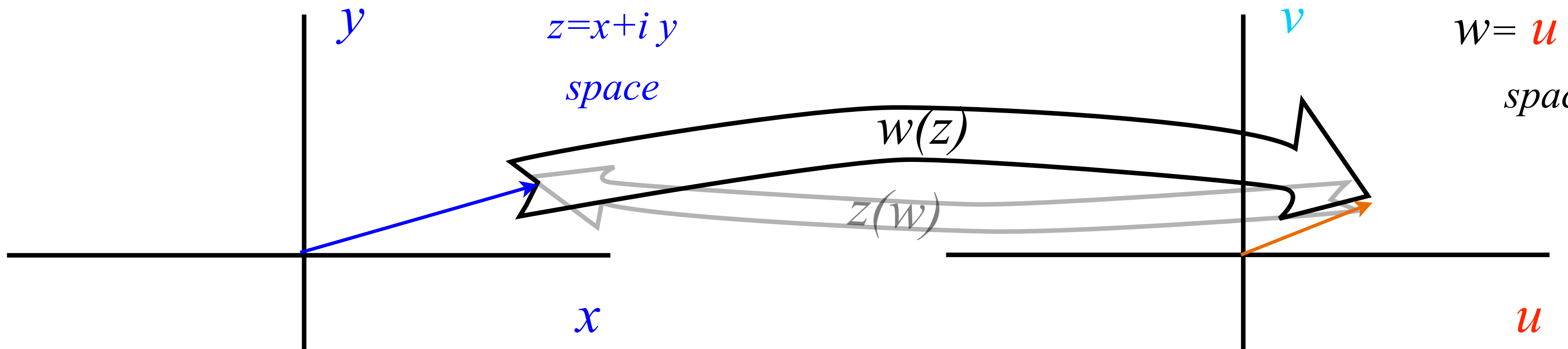

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$$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy$$

Jacobian for mapping:

$$\begin{pmatrix} du \\ dv \end{pmatrix}=\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix}$$

$$=\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix}=\begin{pmatrix} \frac{\partial v}{\partial y} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix}$$

Complex derivative for mapping:

$$\frac{dw}{dz}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)(u+iv)=\frac{1}{2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{i}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

$$=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}\qquad=\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x}$$

Complex derivative abs-square:

$$\left|\frac{dw}{dz}\right|^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2=\det|J|$$

...equals Jacobian Determinant

The half-n'-half results
are called
Riemann-Cauchy
Derivative Relations

$$\boxed{\begin{aligned}\frac{\partial\Phi}{\partial x}&=\frac{\partial A}{\partial y}\\ \frac{\partial\Phi}{\partial y}&=-\frac{\partial A}{\partial x}\end{aligned}}\ \begin{aligned}&\text{is:}\\&\text{is:}\end{aligned}\ \boxed{\begin{aligned}\frac{\partial \mathrm{Re}\,\phi(z)}{\partial x}&=\frac{\partial \mathrm{Im}\,\phi(z)}{\partial y}\\ \frac{\partial \mathrm{Re}\,\phi(z)}{\partial y}&=-\frac{\partial \mathrm{Im}\,\phi(z)}{\partial x}\end{aligned}}\ \begin{aligned}&\text{or:}\\&\text{or:}\end{aligned}\ \boxed{\begin{aligned}\frac{\partial \mathrm{Re}\,f(z)}{\partial x}&=\frac{\partial \mathrm{Im}\,f(z)}{\partial y}\\ \frac{\partial \mathrm{Re}\,f(z)}{\partial y}&=-\frac{\partial \mathrm{Im}\,f(z)}{\partial x}\end{aligned}}\ \begin{aligned}&\text{is:}\\&\text{is:}\end{aligned}\ \boxed{\begin{aligned}\frac{\partial f_x(z)}{\partial x}&=\frac{\partial f_y(z)}{\partial y}\\ \frac{\partial f_x(z)}{\partial y}&=-\frac{\partial f_y(z)}{\partial x}\end{aligned}}$$

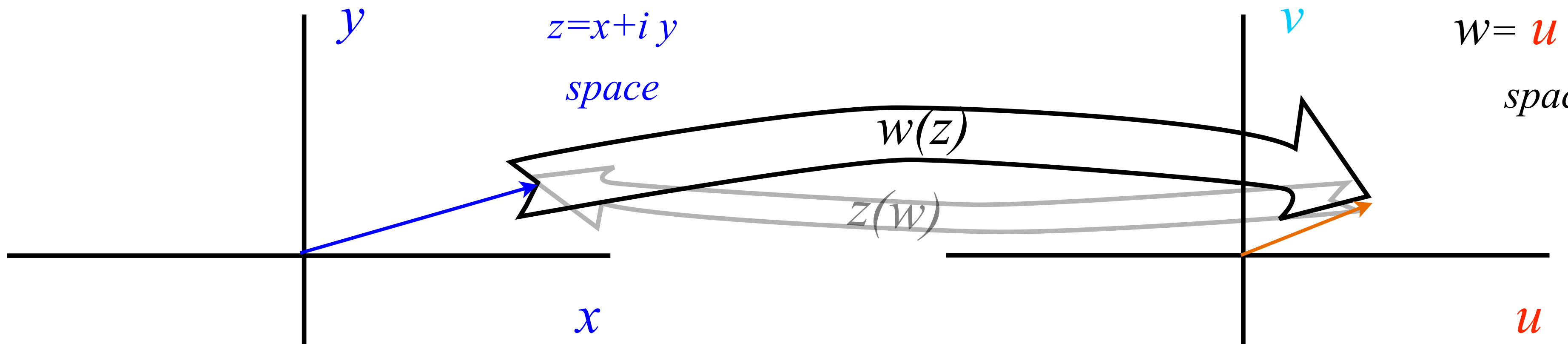
RC applies to analytic potential $\phi(z)=\Phi+iA$ and analytic field $f(z)=f_x+if_y$ and any analytic function

Common notation for mapping: $w(z)=u+iv$

Important result:

$$dw=\sqrt{J}\cdot e^{i\theta}\cdot dz$$

is scaled rotation of dz.



$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$$

$$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy$$

Jacobian for mapping is scaled rotation:

$$\begin{pmatrix}du\\dv\end{pmatrix}=\begin{pmatrix}\frac{\partial u}{\partial x}&\frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x}&\frac{\partial v}{\partial y}\end{pmatrix}\begin{pmatrix}dx\\dy\end{pmatrix}=\sqrt{\det J}\begin{pmatrix}\cos\theta&-\sin\theta\\ \sin\theta&\cos\theta\end{pmatrix}\begin{pmatrix}dx\\dy\end{pmatrix}$$

$$=\begin{pmatrix}\frac{\partial u}{\partial x}&\frac{\partial u}{\partial y}\\ -\frac{\partial u}{\partial y}&\frac{\partial u}{\partial x}\end{pmatrix}\begin{pmatrix}dx\\dy\end{pmatrix}=\begin{pmatrix}\frac{\partial v}{\partial y}&-\frac{\partial v}{\partial x}\\ \frac{\partial v}{\partial x}&\frac{\partial v}{\partial y}\end{pmatrix}\begin{pmatrix}dx\\dy\end{pmatrix}$$

Complex derivative for mapping:

$$\frac{dw}{dz}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)(u+iv)=\frac{1}{2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{i}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

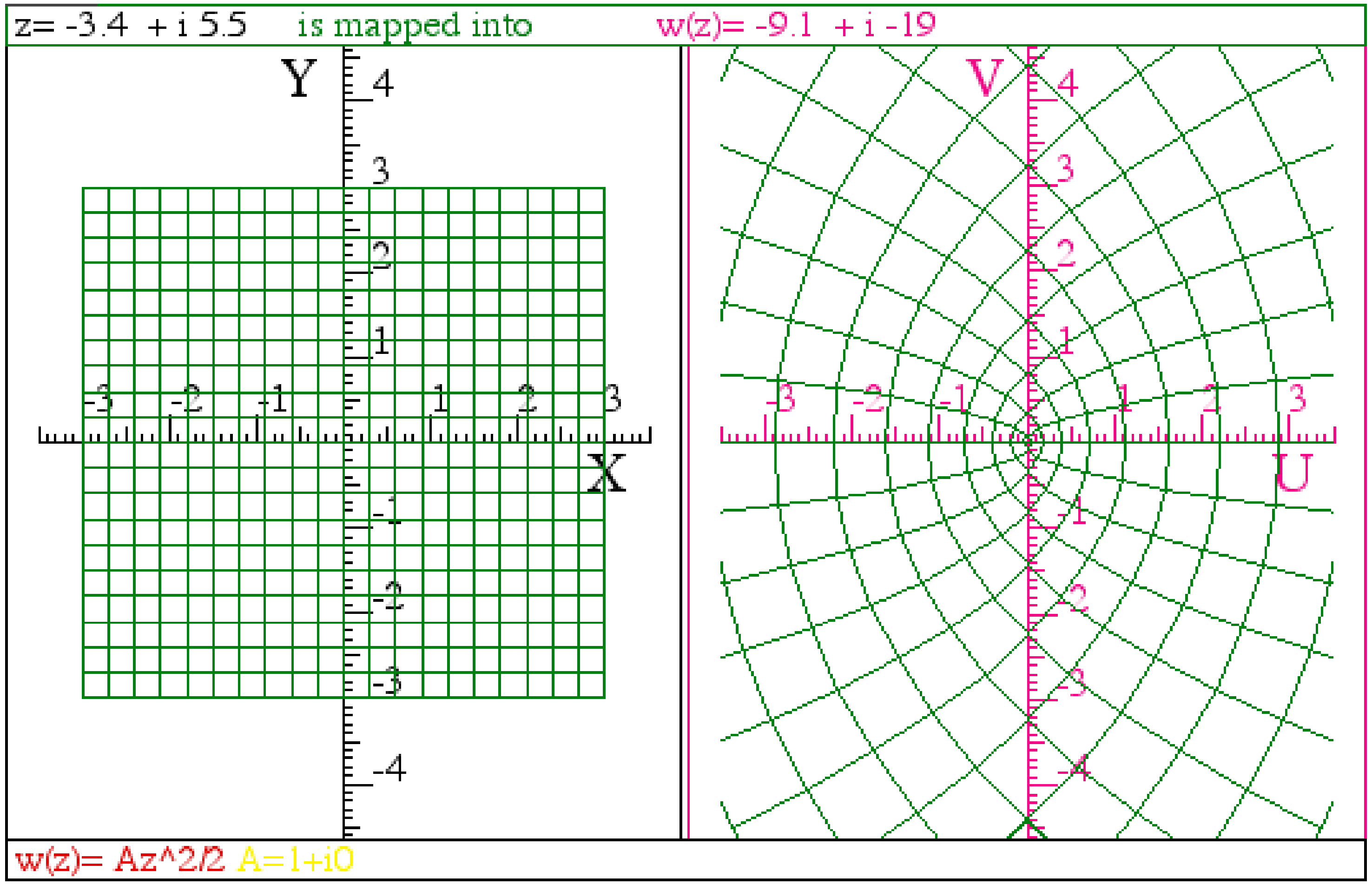
$$=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}\qquad=\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x}$$

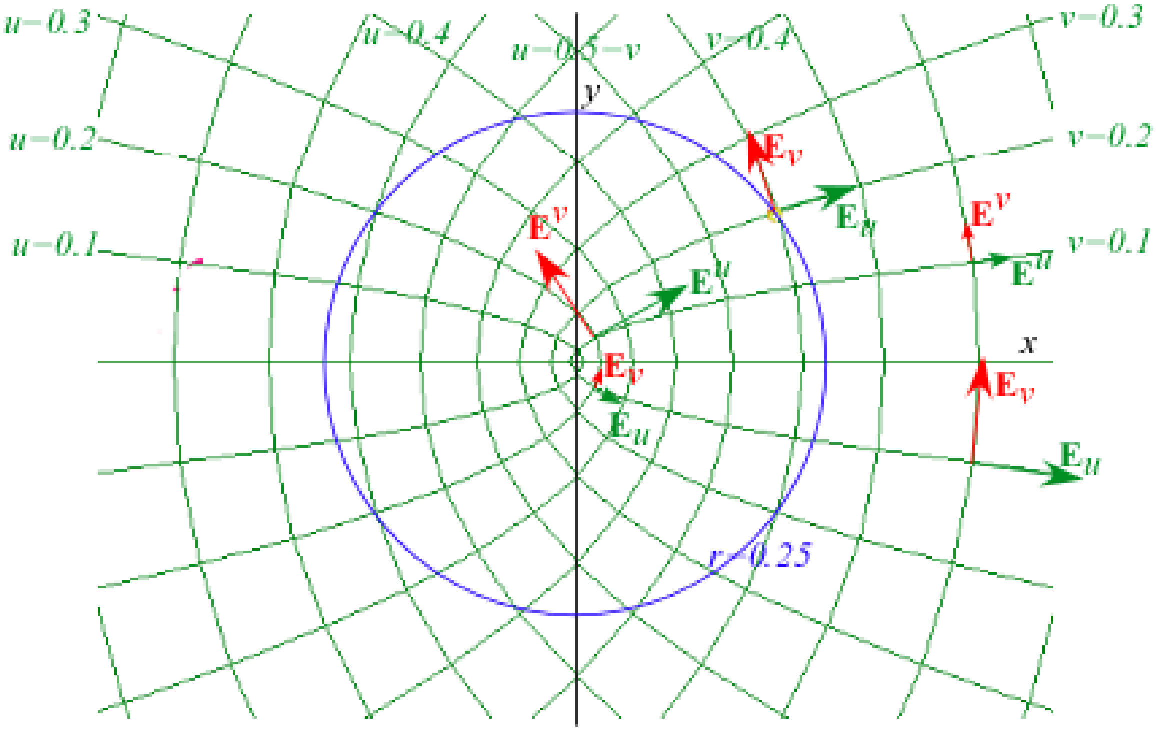
Complex derivative abs-square:

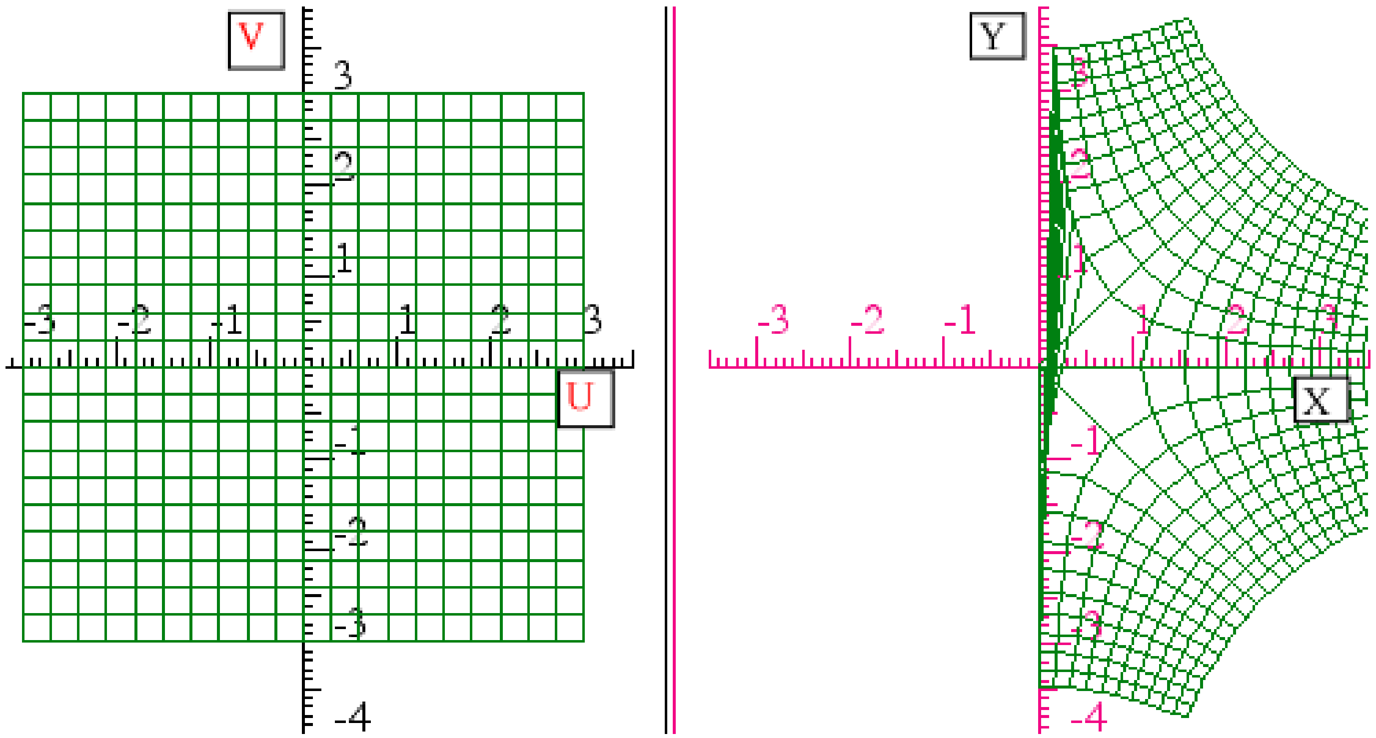
$$\left|\frac{dw}{dz}\right|^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2=\det|J|$$

...equals Jacobian Determinant

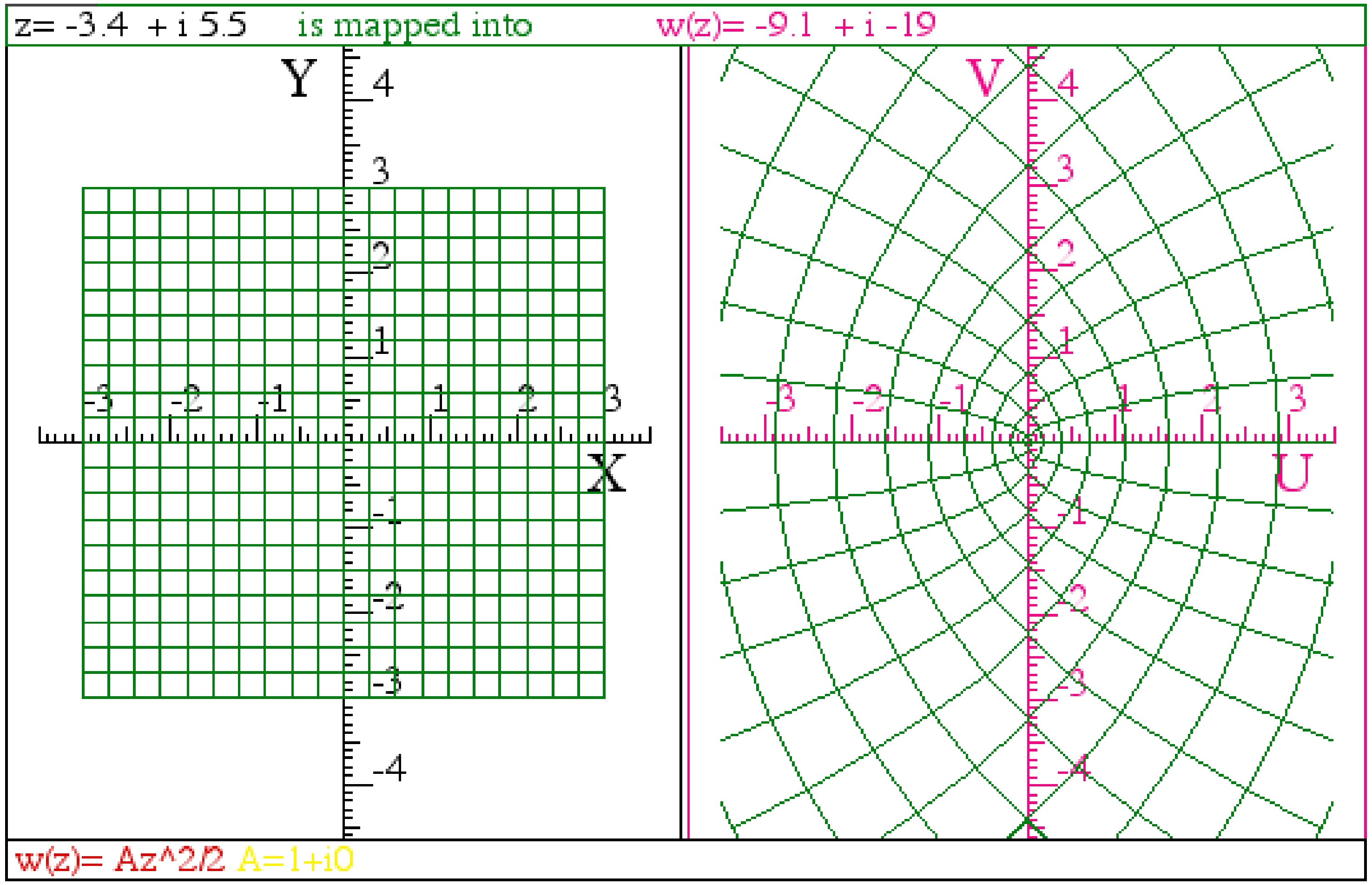
$w(z)=z^2$ gives parabolic OCC



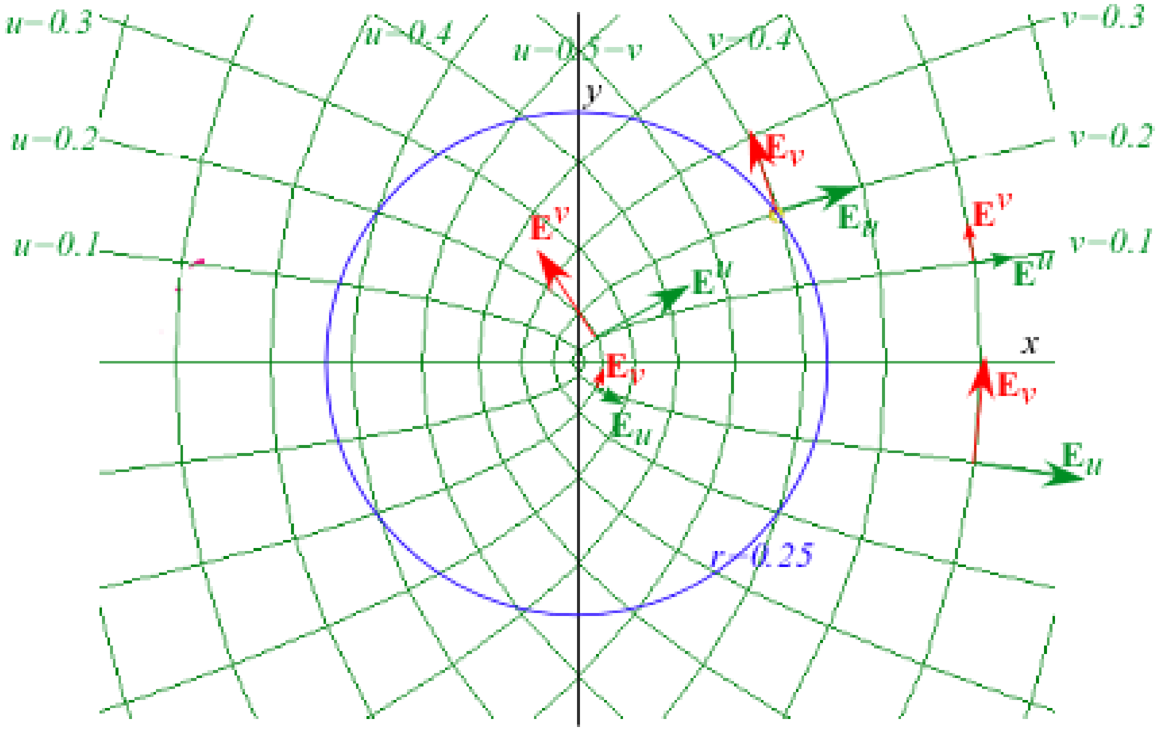




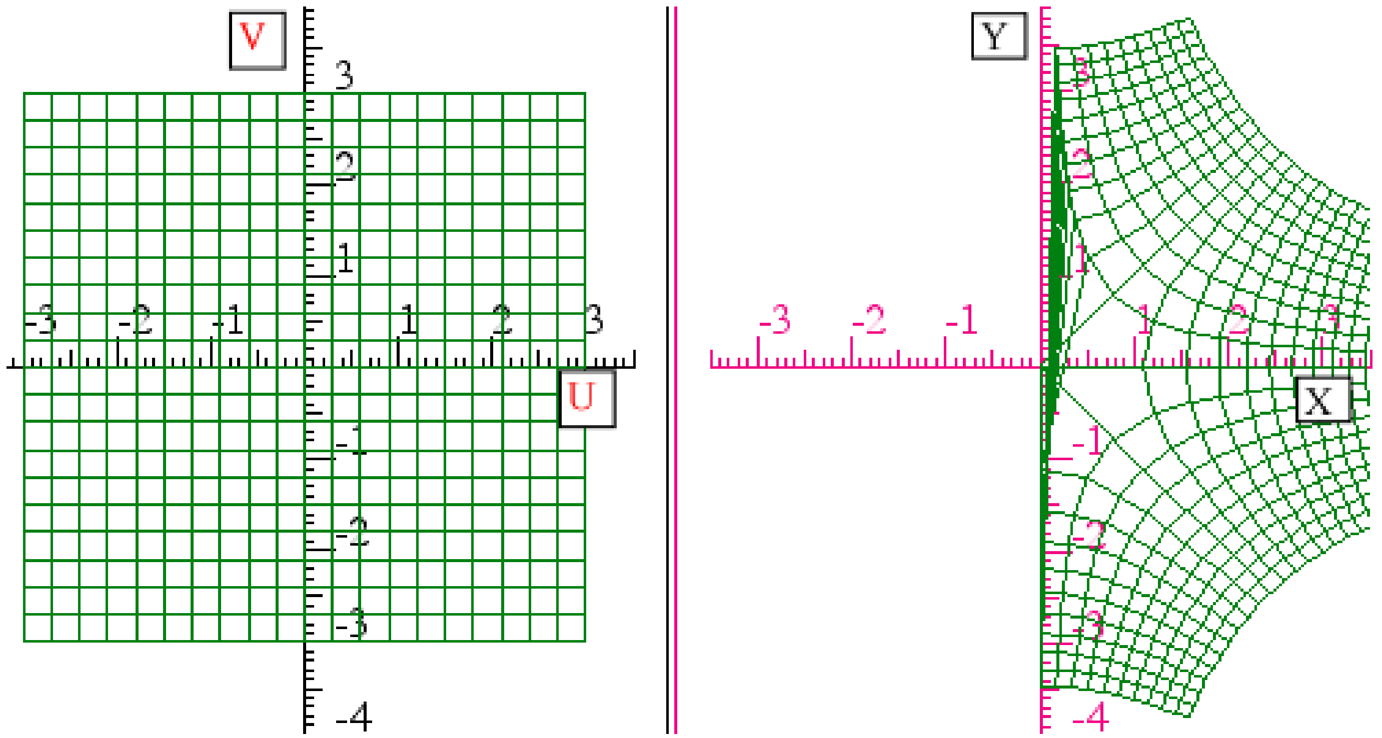
$w(z)=z^2$ gives parabolic OCC


z= -3.4 + i 5.5 is mapped into w(z)= -9.1 + i -19

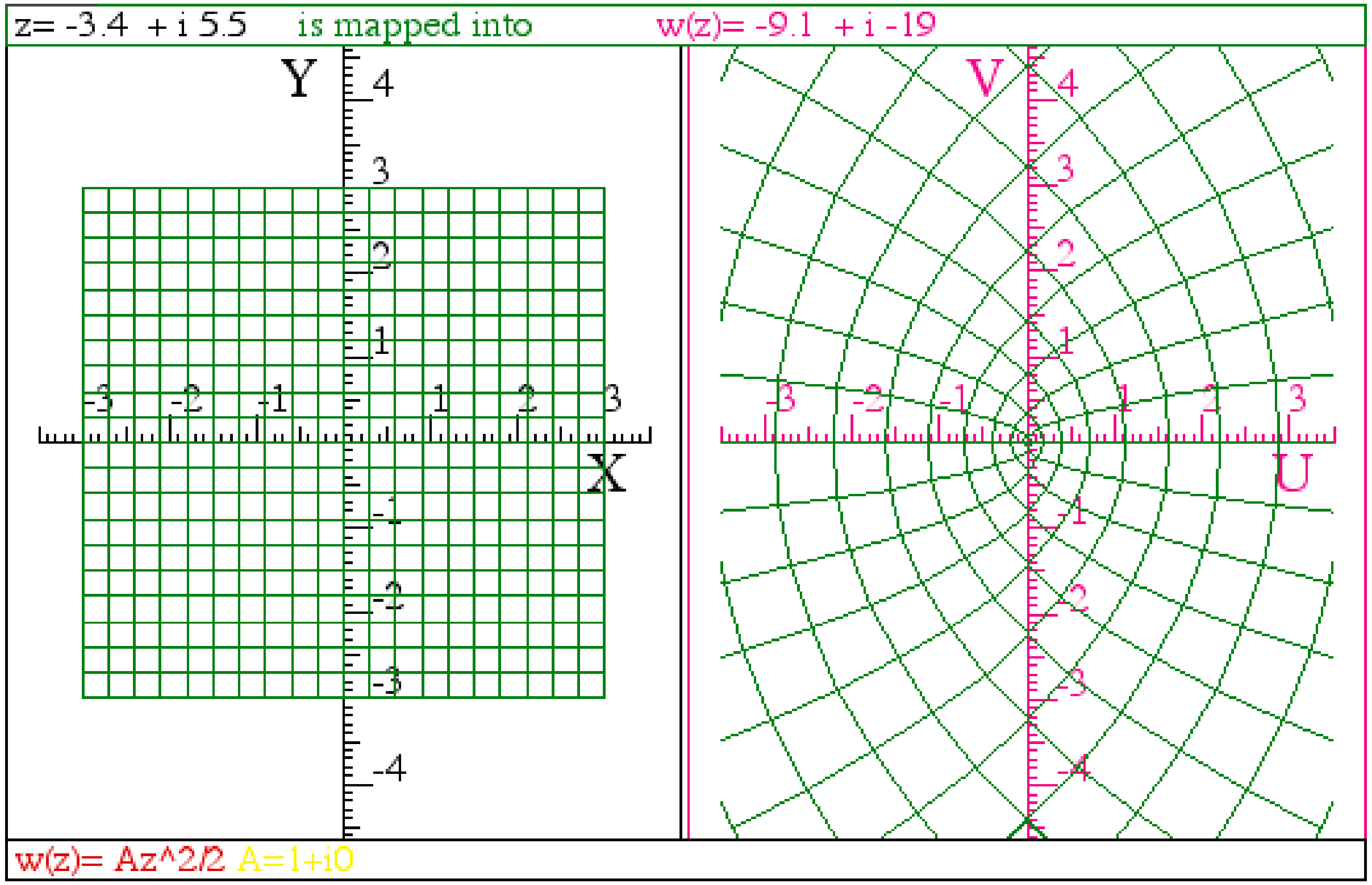
w(z)= Az^2/2 A=1+i0

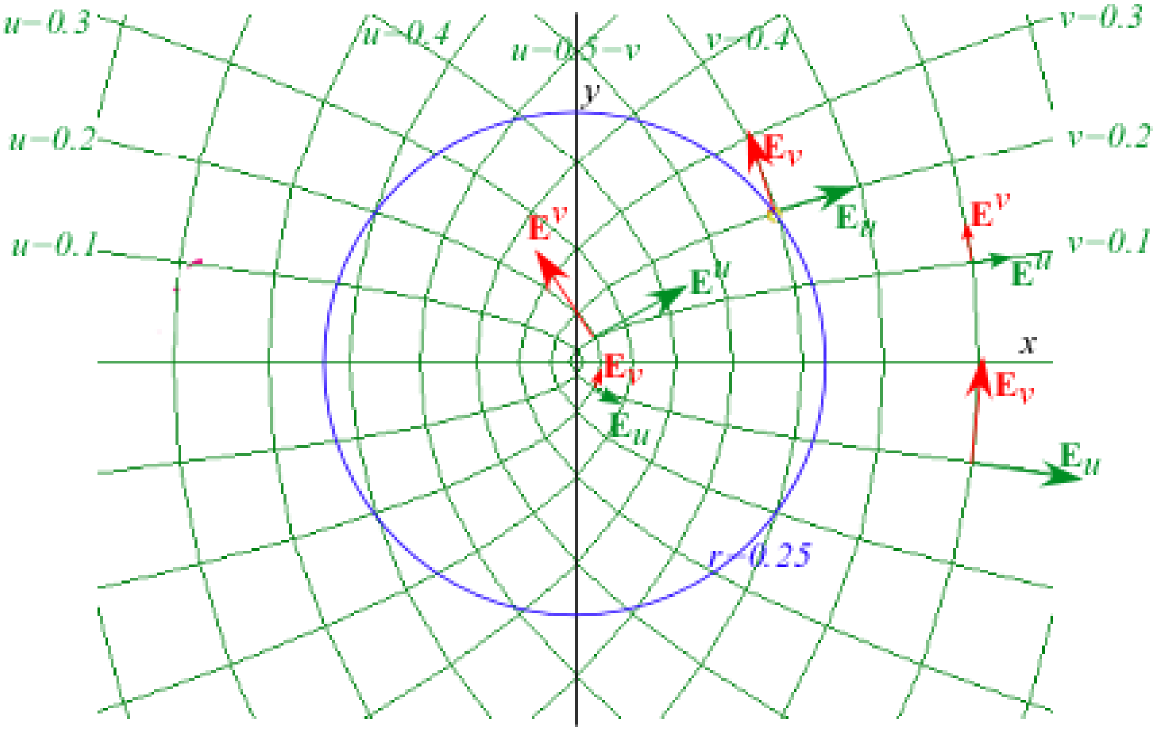


Inverse: $z(w)=w^{1/2}$ gives hyperbolic OCC

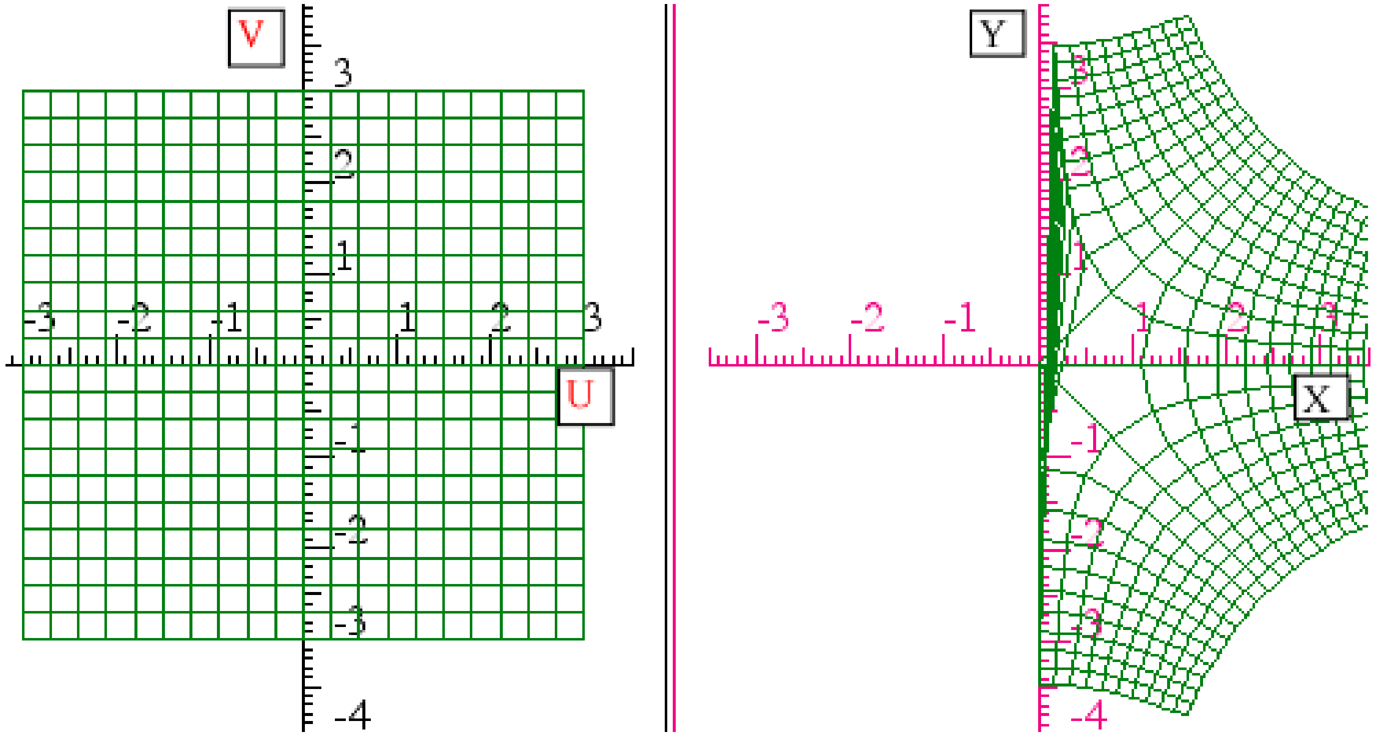


$w(z) = z^2$ gives parabolic OCC





Inverse: $z(w) = w^{1/2}$ gives hyperbolic OCC



$w = (u + iv) = z^2 = (x + iy)^2$ is analytic function of z and w

Expansion: $u = x^2 - y^2$ and $v = 2xy$ may be solved using $|w| = |z^2| = |z|^2$

Expansion: $|w| = \sqrt{u^2 + v^2} = x^2 + y^2 = |z|^2$

Solution: $x^2 = \dfrac{u + \sqrt{u^2 + v^2}}{2}$ $\quad y^2 = \dfrac{-u + \sqrt{u^2 + v^2}}{2}$

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{E}}^u \\ \bar{\mathbf{E}}^v \end{pmatrix} = \begin{pmatrix} 2x & -2y \\ +2y & 2x \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{E}}_u & \bar{\mathbf{E}}_v \end{pmatrix} = \frac{\begin{pmatrix} 2x & +2y \\ -2y & 2x \end{pmatrix}}{4\left(x^2 + y^2\right)}$$

Non-analytic potential, force, and source field functions

A general 2D complex field may have:

1. non-analytic *potential field function* $\phi(z,z^*)=\Phi(x,y)+iA(x,y)$,
2. non-analytic *force field function* $f(z,z^*) = fx(x,y) + ify(x,y)$,
3. non-analytic *source distribution function* $s(z,z^*) = \rho(x,y) + i\,I(x,y)$.

Source definitions are made to generalize the **f*** field equations (10.33) based on relations (10.31) and (10.32).

$$2\frac{df^*}{dz}=s^*(z,z^*) \qquad\qquad 2\frac{df}{dz^*}=s(z,z^*)$$

Field equations for the potentials are like (10.33) with an extra factor of *2*.

$$2\frac{d\phi}{dz}=f(z,z^*) \qquad\qquad 2\frac{d\phi^*}{dz^*}=f^*(z,z^*)$$

Source equations (10.46) expand like (10.32) into a real and imaginary parts of divergence and curl terms.

$$s^*(z,z^*)=2\frac{df^*}{dz}=\left[\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right]\left[f_x^*(x,y)+if_y^*(x,y)\right] \quad =\rho-i\,I\ , \quad \text{where: } f_x^*=f_x\ , \ \text{ and: } f_y^*=-f_y$$

$$=\left[\frac{\partial f_x^*}{\partial x}+\frac{\partial f_y^*}{\partial y}\right]+i\left[\frac{\partial f_y^*}{\partial x}-\frac{\partial f_x^*}{\partial y}\right] \quad =\left[\nabla\bullet\mathbf{f}^*\right]+i\left[\nabla\times\mathbf{f}^*\right]_Z$$

Real part: *Poisson scalar source equation* (*charge density* ρ). Imaginary part: *Biot-Savart vector source equation*(*current density I*)

$$\nabla\bullet\mathbf{f}^*=\rho \qquad\qquad \nabla\times\mathbf{f}^*=-I$$

Field equations (10.47) expand into Re and Im parts; *x* and *y* components of grad Φ and curlA_Z from potential $\phi=\Phi+iA$ or $\phi^*=\Phi-iA$.

$$f^*(z,z^*)=2\frac{d\phi^*}{dz^*}=\left[\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right](\Phi-iA)=f_x^*+if_y^*$$

$$=\left[\frac{\partial\Phi}{\partial x}+i\frac{\partial\Phi}{\partial y}\right]+\left[\frac{\partial A}{\partial y}-i\frac{\partial A}{\partial x}\right]=\left[\nabla\Phi\right]+\left[\nabla\times\mathbf{A}_Z\right]$$

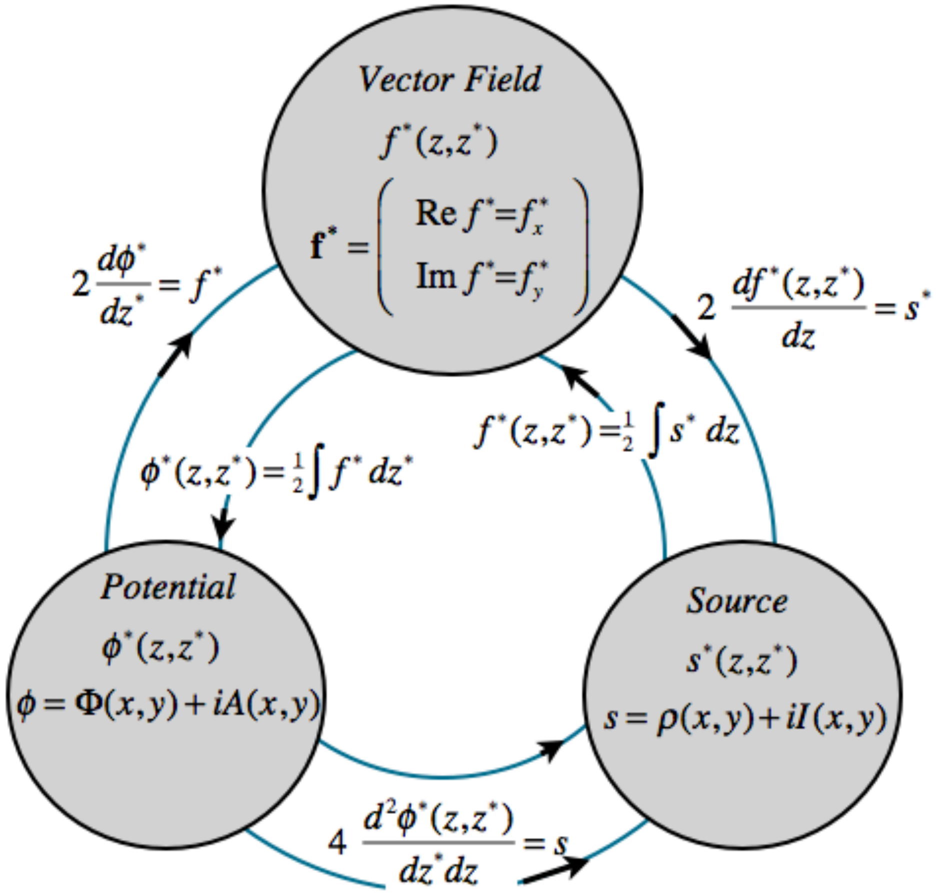
Two parts: gradient of scalar potential called the *longitudinal field* $\mathbf{f}_\mathbf{L}^*$ and curl of a vector potential called the *transverse field* $\mathbf{f}_\mathbf{T}^*$.

$$\mathbf{f}^*=\mathbf{f}_\mathbf{L}^*+\mathbf{f}_\mathbf{T}^* \qquad\qquad \mathbf{f}_\mathbf{L}^*=\nabla\Phi \qquad\qquad \mathbf{f}_\mathbf{T}^*=\nabla\times\mathbf{A}$$

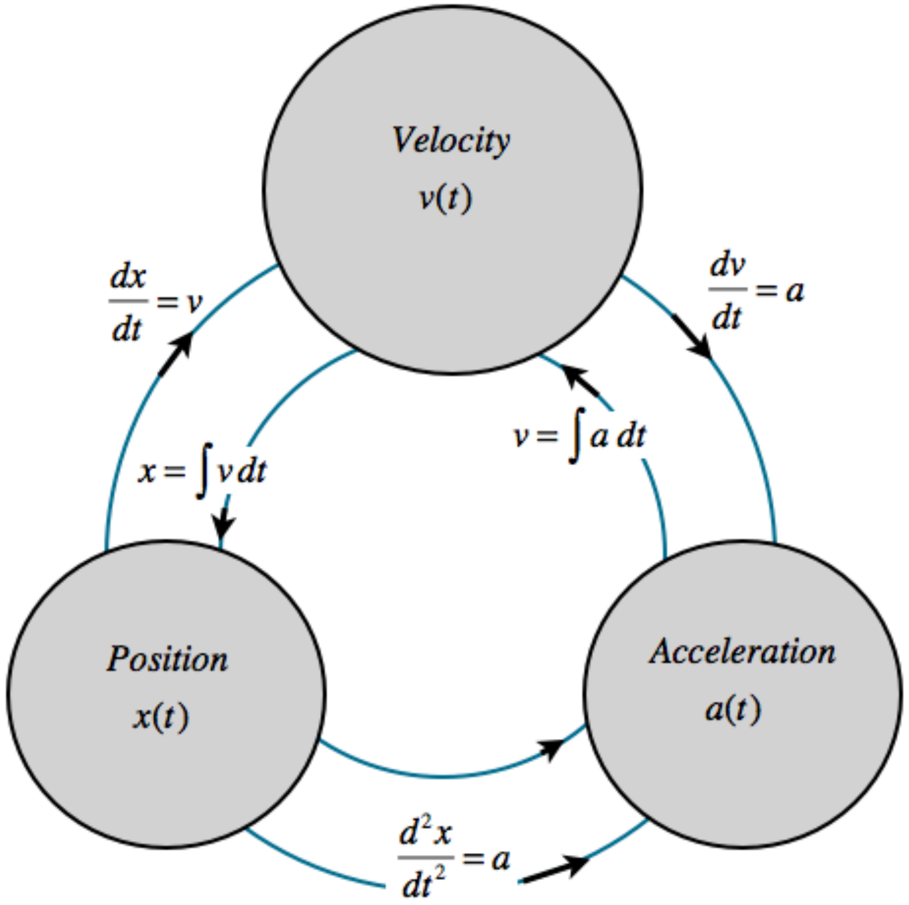
(For source-free analytic functions these two fields are identical.)

Potential, force, and source field equations vs. position, velocity, and acceleration equations

Field equations



Newton equations



Example 1

Consider a non-analytic field $f(z) = (z^*)^2$ or $f^*(z) = z^2$.

The non-analytic potential function follows by integrating

$$s^*(z,z^*) = 2\frac{df^*}{dz} = 4z = 4x + i4y,$$

$$or: \quad \rho = 4x, \quad and: \quad I = -4y.$$

$$\phi(z,z^*) = \frac{1}{2}\int f(z)\,dz = \frac{1}{2}\int (z^*)^2\,dz = \frac{z(z^*)^2}{2} = \frac{(x+iy)(x^2-y^2-i2xy)}{2},$$

$$or: \quad \Phi = \frac{x^3+xy^2}{2}, \quad and: \; A = \frac{-y^3-yx^2}{2}.$$

The longitudinal field $\mathbf{f}_\mathbf{T}^*$ is quite different from the transverse field $\mathbf{f}_\mathbf{L}^*$.

$\mathbf{f}_\mathbf{L}^* + \mathbf{f}_\mathbf{T}^*$

$$\mathbf{f}_\mathbf{L}^* = \nabla\Phi = \nabla\left(\frac{x^3+xy^2}{2}\right) = \begin{pmatrix} \frac{3x^2+y^2}{2} \\ xy \end{pmatrix}, \quad \mathbf{f}_\mathbf{T}^* = \nabla\times\mathbf{A} = \nabla\times\left(\frac{-y^3-yx^2}{2}\mathbf{e_z}\right) = \begin{pmatrix} \frac{\partial A}{\partial y} \\ -\frac{\partial A}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{-3y^2-x^2}{2} \\ xy \end{pmatrix}.$$

The longitudinal field $\mathbf{f}_\mathbf{L}^*$ has no curl and the transverse field $\mathbf{f}_\mathbf{T}^*$ has no divergence. The sum field has both making a violent storm, indeed, as shown by a plot of in Fig. 10.17.

$$\mathbf{f}^* = \mathbf{f}_\mathbf{L}^* + \mathbf{f}_\mathbf{T}^* = \begin{pmatrix} \frac{3x^2+y^2}{2} \\ xy \end{pmatrix} + \begin{pmatrix} \frac{-3y^2-x^2}{2} \\ xy \end{pmatrix} = \begin{pmatrix} x^2-y^2 \\ 2xy \end{pmatrix}, \quad \nabla\bullet\mathbf{f}^* = \nabla\bullet\mathbf{f}_\mathbf{L}^* = 4x = \rho, \quad \nabla\times\mathbf{f}^* = \nabla\times\mathbf{f}_\mathbf{T}^* = 4y = -I.$$