Group Theory in Quantum Mechanics

Lectures 9-10 (2.14-16.17)

Applications of U(2) and R(3) representations

(Quantum Theory for Computer Age - Ch. 10A-B of Unit 3)
(Principles of Symmetry, Dynamics, and Spectroscopy - Sec. 1-3 of Ch. 5 and Ch. 7)

Review: Fundamental Euler $\mathbf{R}(\alpha\beta\gamma)$ and Darboux $\mathbf{R}[\varphi\vartheta\Theta]$ representations of U(2) and R(3)

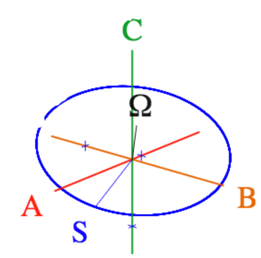
- Euler $\mathbf{R}(\alpha\beta\gamma)$ derived from Darboux $\mathbf{R}[\varphi\vartheta\Theta]$ and vice versa
- Euler $\mathbf{R}(\alpha\beta\gamma)$ rotation $\Theta=0-4\pi$-sequence $[\varphi\vartheta]$ fixed (and "real-world" applications)

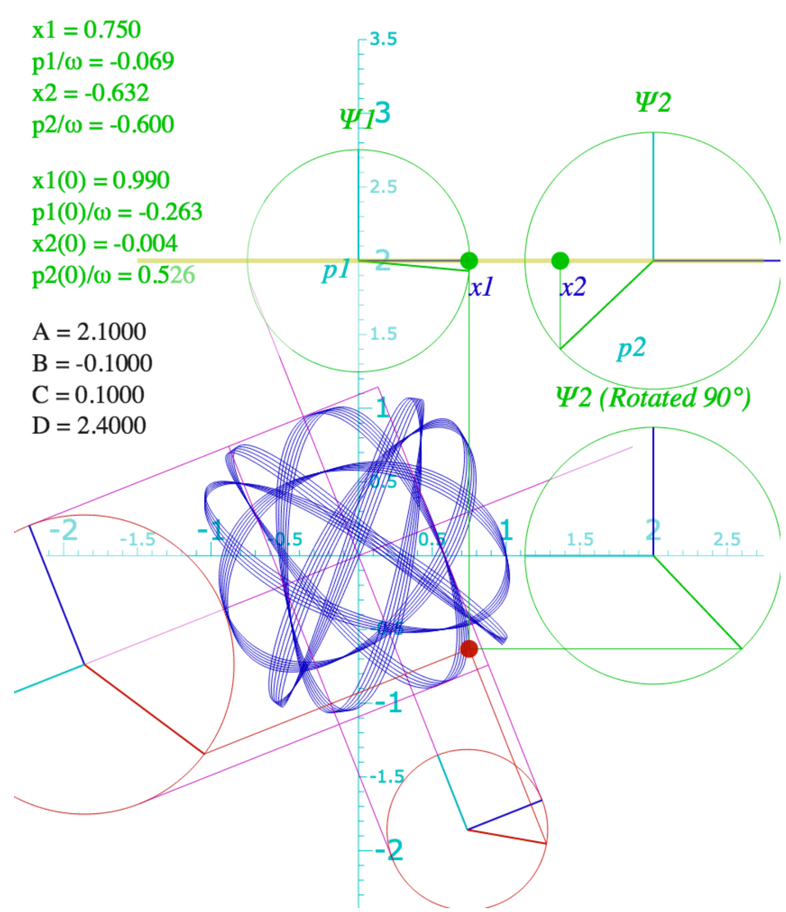
U(2) density operator approach to symmetry dynamics

- Bloch equation for density operator
- Quick U(2) way to find eigen-solutions for 2-by-2 $\mathbf{H} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes

- Asymmetric-Diagonal A-Type motion
- Bilateral-Balanced B-Type motion
- Circular-Coriolis... C-Type motion





The ABC's of U(2) dynamics-Mixed modes

- AB-Type motion and Wigner's Avoided-Symmetry-Crossings
- ABC-Type elliptical polarized motion

Ellipsometry using U(2) symmetry coordinates

- Conventional amp-phase ellipse coordinates
- Euler Angle $(\alpha\beta\gamma)$ ellipse coordinates

Review: Fundamental Euler **R***(αβγ) and Darboux* **R**[φϑΘ] *representations of U(2) and R(3)*

Euler **R***(αβγ) derived from Darboux* **R**[φϑΘ] *and vice versa*

Euler **R***(αβγ) rotation* Θ=0−4π*-sequence* [φϑ] *fixed (and "real-world" applications)*

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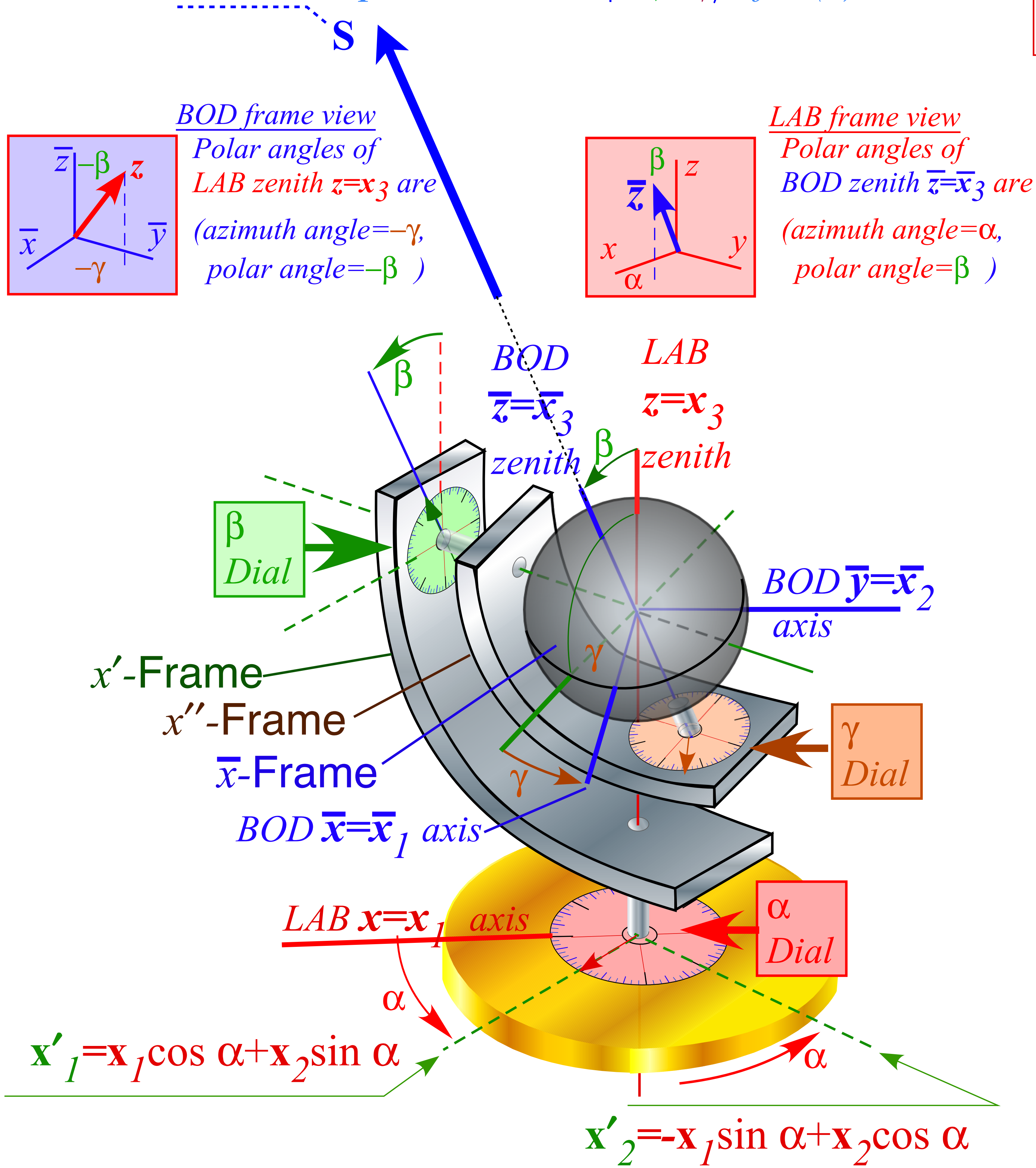
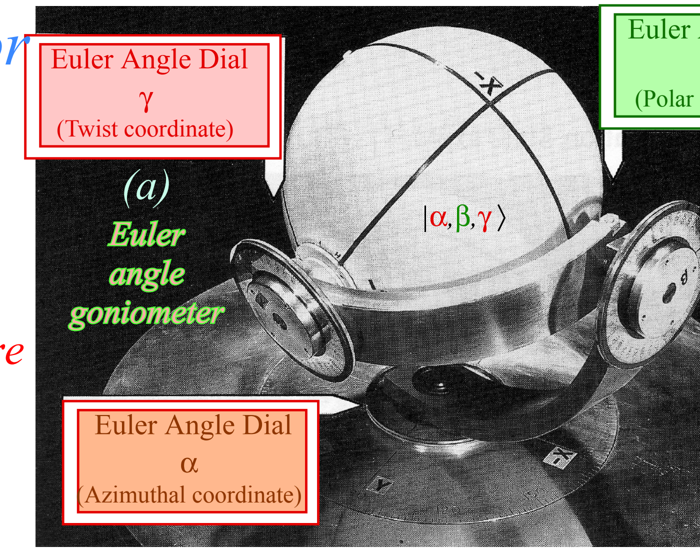
Euler Angle (αβγ) ellipse coordinates

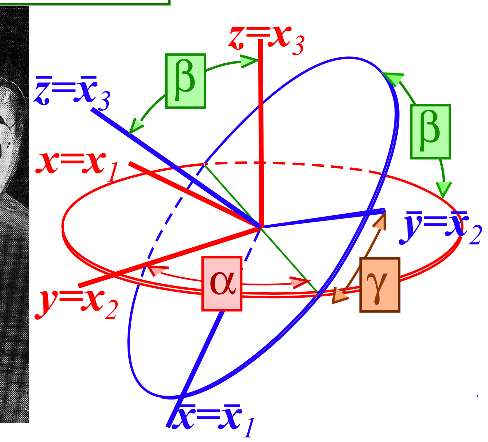
Euler's rotation state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, and $\mathbf{R}(0,0,\gamma)$

3D-real **S**-vector represents state $|\alpha,\beta,\gamma\rangle$ of U(2) oscillator





An astronomer's diagram



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Euler angles

Under Construction!

Web based U(2) Calculator - Euler State

Fig. 10.A.3-4 Mechanical device demonstrating Euler angles (α,β,γ)

Euler's rotation state definition using rotations **R**$(\alpha,0,0)$, **R**$(0,\beta,0)$, *and* **R**$(0,0,\gamma)$

Spin-1/2 (2D-complex spinor) case

$$\left|a\right\rangle = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow\right\rangle$$

$$= \mathbf{R}[\alpha \text{ about } Z]\cdot\mathbf{R}[\beta \text{ about } Y]\cdot\mathbf{R}[\gamma \text{ about } Z]\left|\uparrow\right\rangle$$

$$= \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}\begin{pmatrix} A \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} A \\ 0 \end{pmatrix}$$

$$= A\begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

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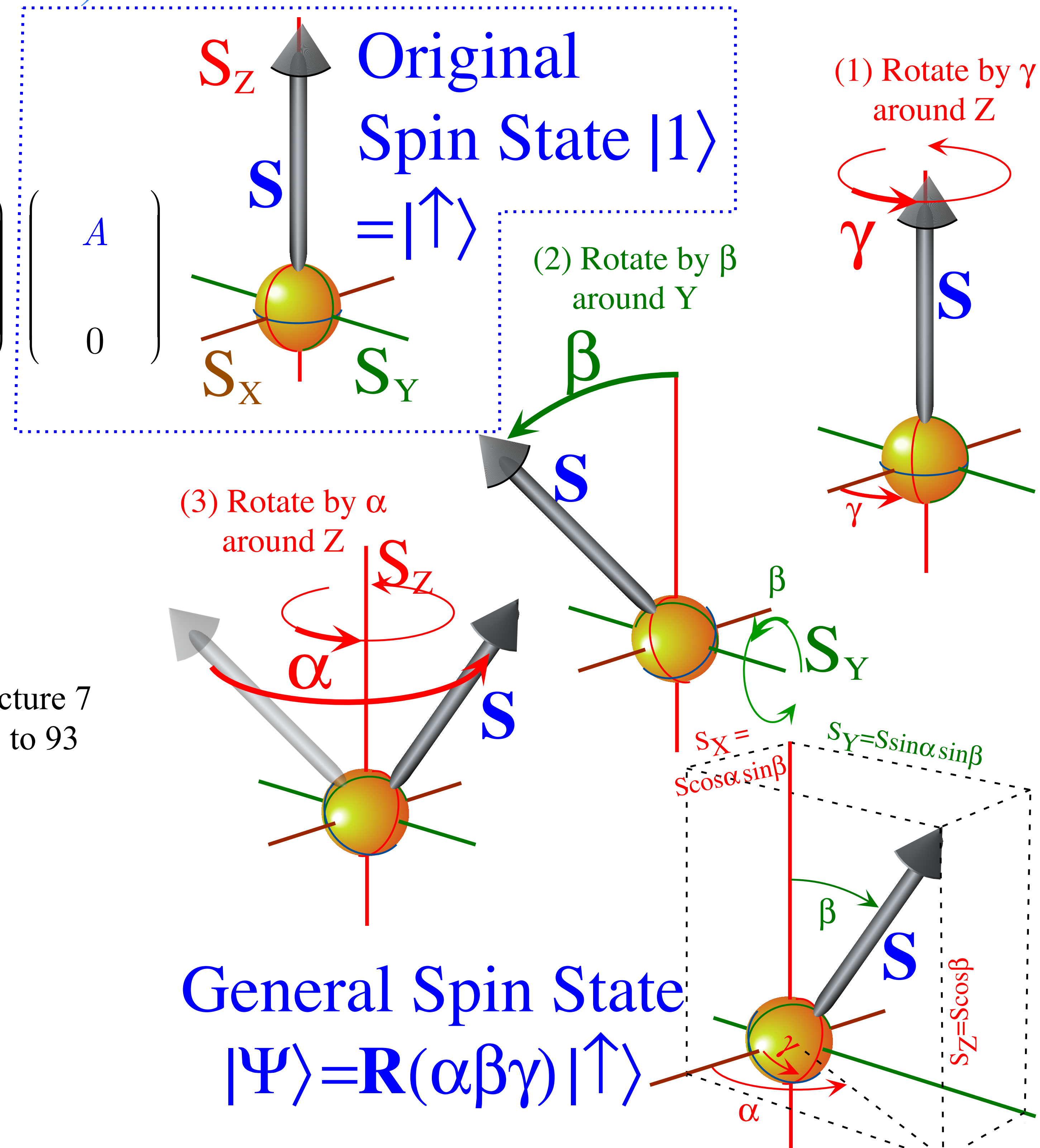

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A = S_Z$, Balance $S_B = S_X$, and Chirality $S_C = S_Y$

Each point $\{E_1, E_2\}$ defines *2D-HO phase space* or analogous Ψ-space given by 2D amplitude array: $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = A \begin{pmatrix} e^{-i\frac{\alpha}{2}} \cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}} \sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}}$

This defines real *3D spin vector* (S_A, S_B, S_C) "pointing" to a polarization ellipse or state.

Asymmetry $S_A = \frac{1}{2}(a|\sigma_A|a) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_1 - a_2^* a_2\right] = \frac{1}{2}\left[x_1^2 + p_1^2 - x_2^2 - p_2^2\right] = \frac{I}{2}\left[\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right] = \frac{I}{2}\cos\beta$

Balance $S_B = \frac{1}{2}(a|\sigma_B|a) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_2 + a_2^* a_1\right] = \left[p_1 p_2 + x_1 x_2\right] = I\left[-\sin\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} + \cos\frac{\alpha+\gamma}{2}\cos\frac{\alpha-\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{I}{2}\cos\alpha\sin\beta$

Chirality $S_C = \frac{1}{2}(a|\sigma_C|a) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{-i}{2}\left[a_1^* a_2 - a_2^* a_1\right] = \left[x_1 p_2 - x_2 p_1\right] = I\left[\cos\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} - \cos\frac{\alpha-\gamma}{2}\cdot -\sin\frac{\alpha+\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{I}{2}\sin\alpha\sin\beta$

Three ways to picture U(2) spin or pseudo-spin states

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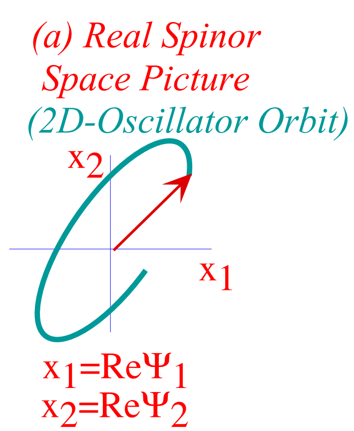

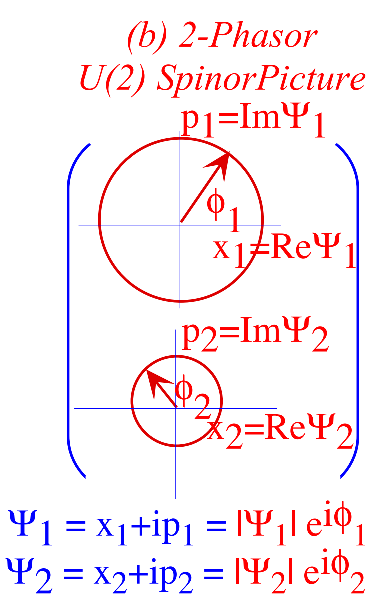

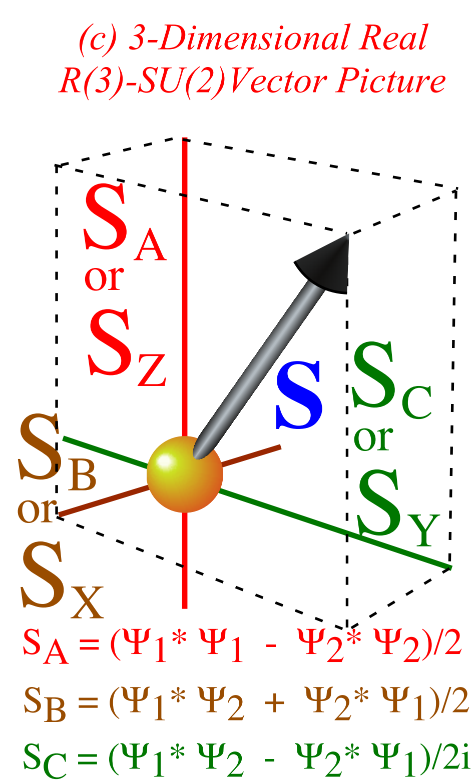

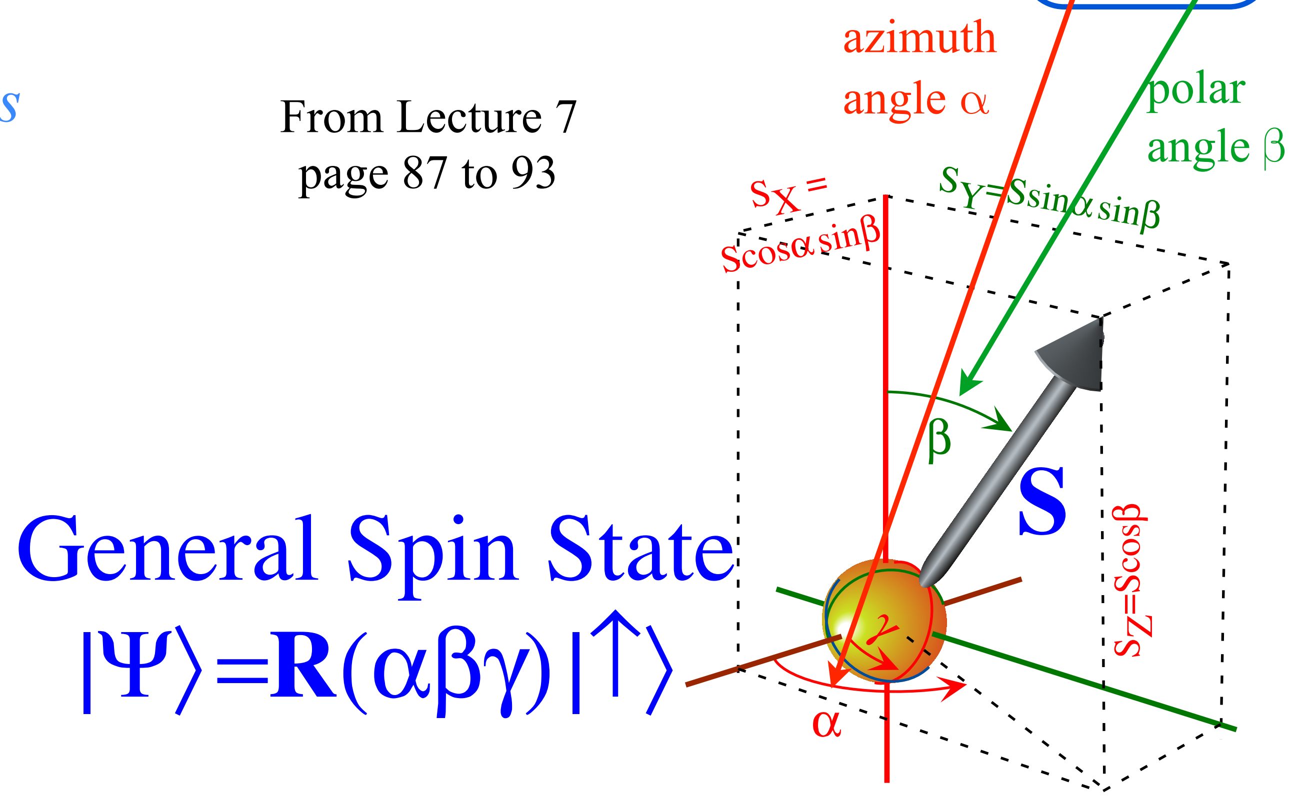

(a) Ellipsometry

(b) U(2) phasors

(c) 3D real R(3) vectors

Fig. 10.5.2 Spinor, phasor, and vector descriptions of 2-state systems .

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

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This defines real *3D spin vector* (S_A, S_B, S_C) "pointing" to a polarization ellipse or state.

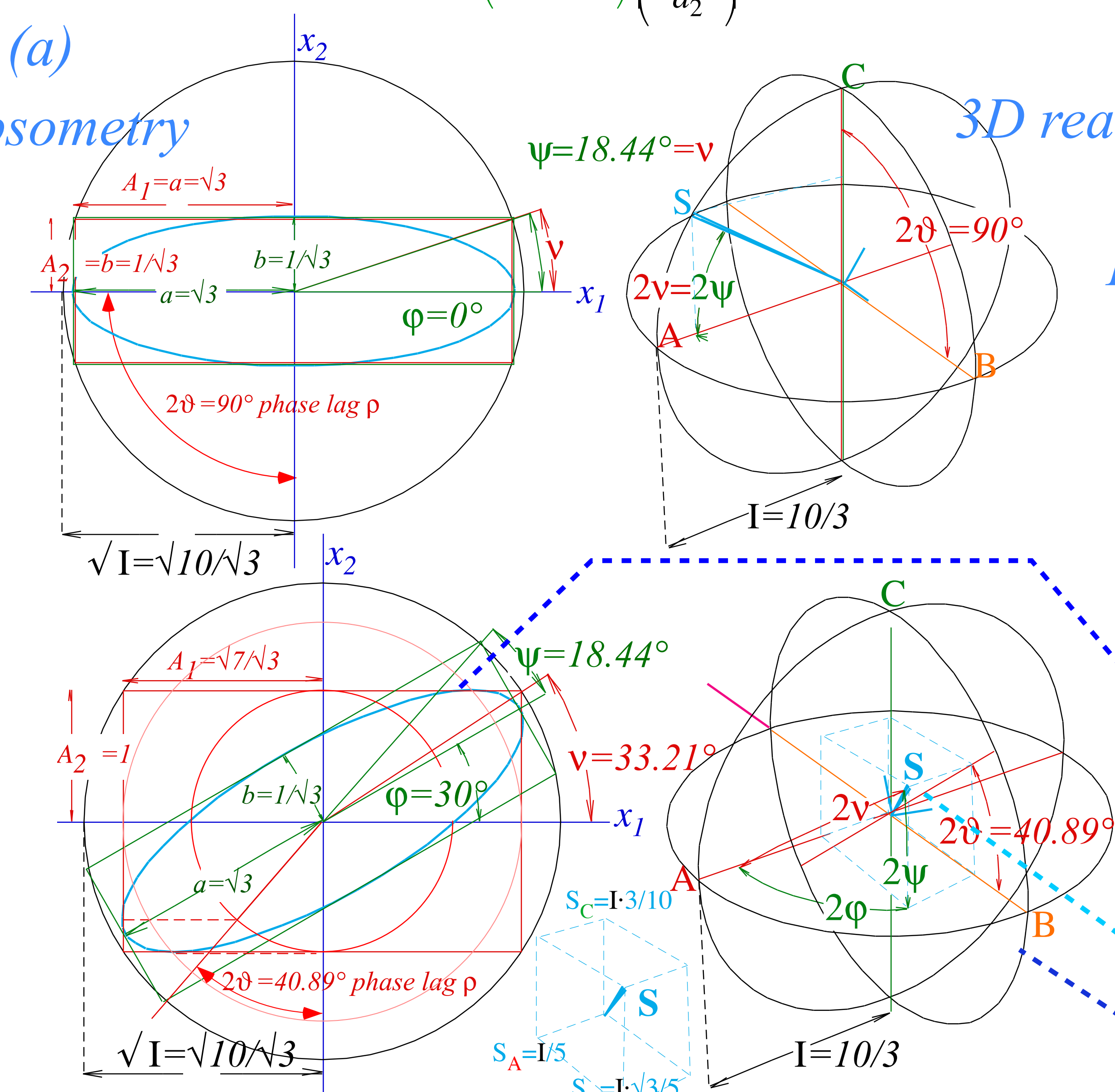
Asymmetry $S_A = \frac{1}{2}\langle a|\sigma_A|a\rangle = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_1 - a_2^* a_2\right] = \frac{1}{2}\left[x_1^2 + p_1^2 - x_2^2 - p_2^2\right] = \frac{I}{2}\left[\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right] = \frac{I}{2}\cos\beta$

Balance $S_B = \frac{1}{2}\langle a|\sigma_B|a\rangle = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_2 + a_2^* a_1\right] = \left[p_1 p_2 + x_1 x_2\right] = I\left[-\sin\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} + \cos\frac{\alpha+\gamma}{2}\cos\frac{\alpha-\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{I}{2}\cos\alpha\sin\beta$

Chirality $S_C = \frac{1}{2}\langle a|\sigma_C|a\rangle = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{-i}{2}\left[a_1^* a_2 - a_2^* a_1\right] = \left[x_1 p_2 - x_2 p_1\right] = I\left[\cos\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} - \cos\frac{\alpha-\gamma}{2}\cdot -\sin\frac{\alpha+\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{I}{2}\sin\alpha\sin\beta$

(a) Ellipsometry

(c) 3D real R(3) **S**-vectors



Ellipsometry of U(2) states detailed at end of this Lecture

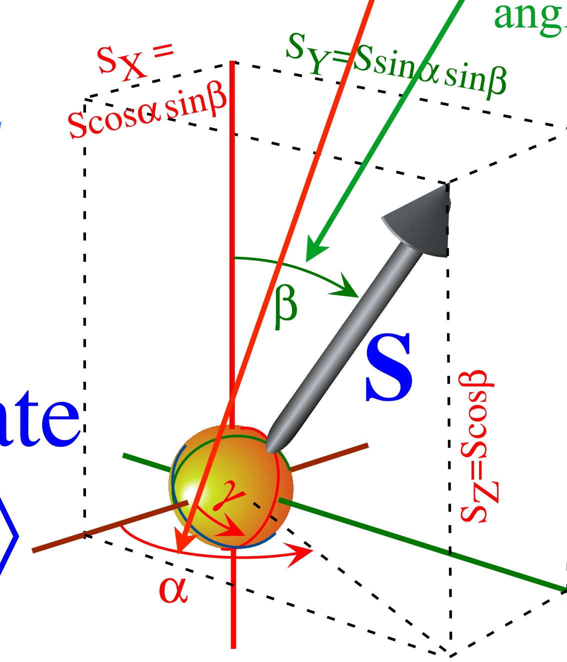
General Spin State

$|\Psi\rangle = \mathbf{R}(\alpha\beta\gamma)|\uparrow\rangle$

Complex U(2) ellipse of any state corresponds to a single point **S** *in R(3) on the Stoke's sphere*

azimuth angle α

polar angle β



Note phase or "gauge" angle γ is killed in R(3) a^*a-squares but lives on in U(2).

*Here spin-rotor **S**-polar coordinates are Euler angles*

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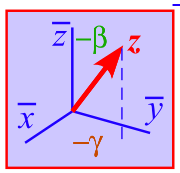
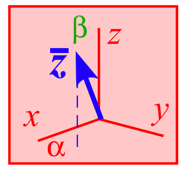
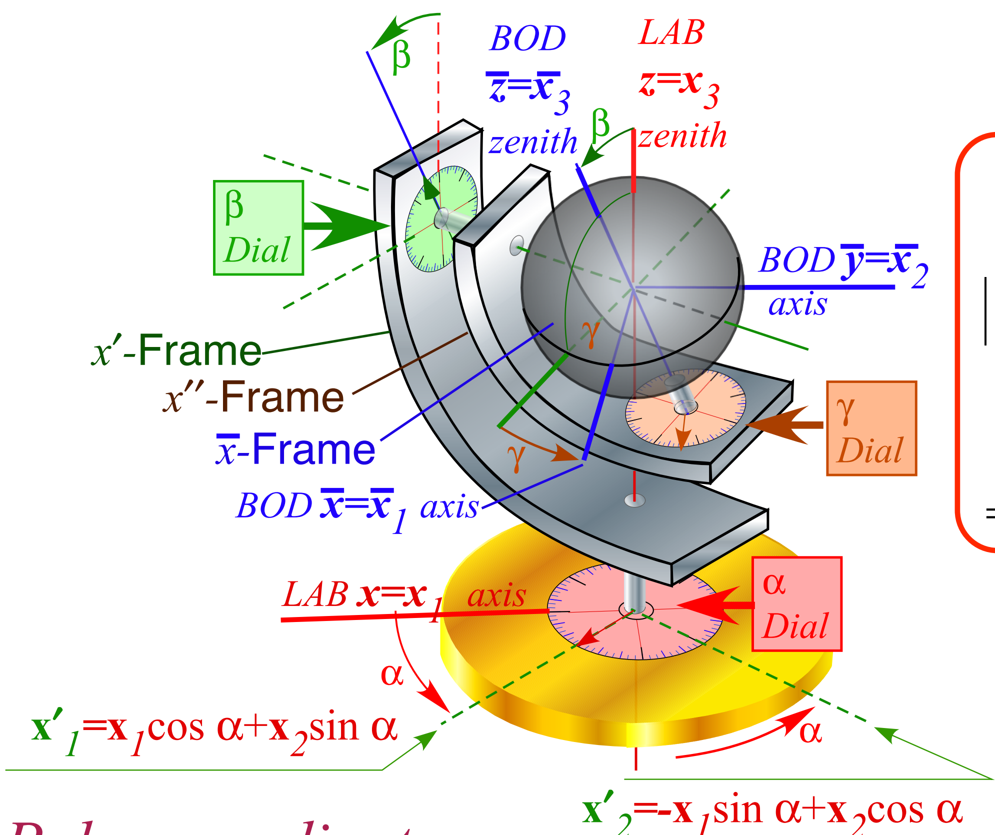
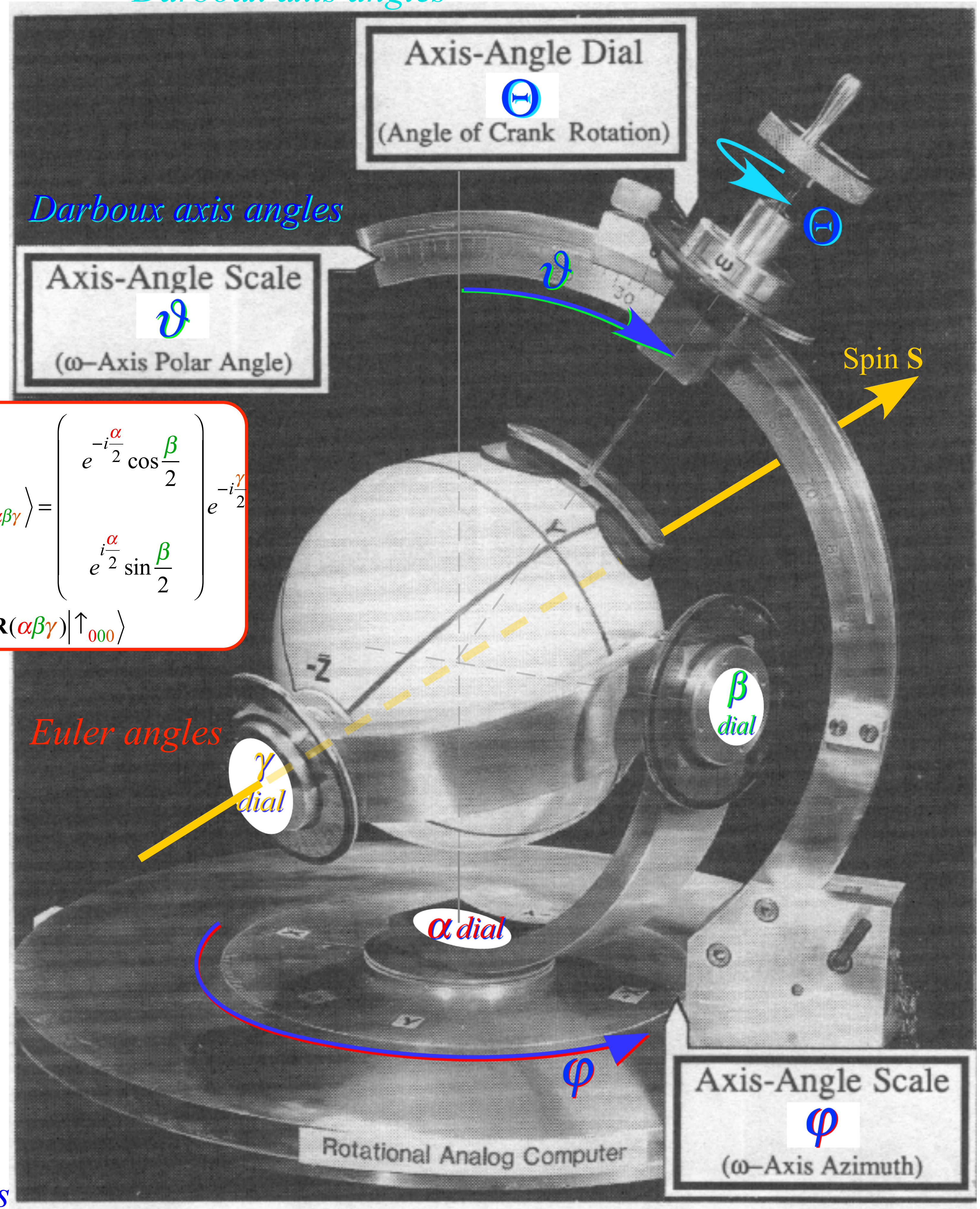






$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}}$$

$$= \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$



Polar coordinates for unit Spin vector $\hat{\mathbf{S}}$

$$\hat{S}_X = \cos\alpha \ \sin\beta$$
$$\hat{S}_Y = \sin\alpha \ \sin\beta$$
$$\hat{S}_Z = \cos\beta$$

Spin State

$$\left|\alpha\beta\gamma\right\rangle = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow\right\rangle$$

Euler angles

Polar coordinates for unit axis vector $\hat{\mathbf{\Theta}}$

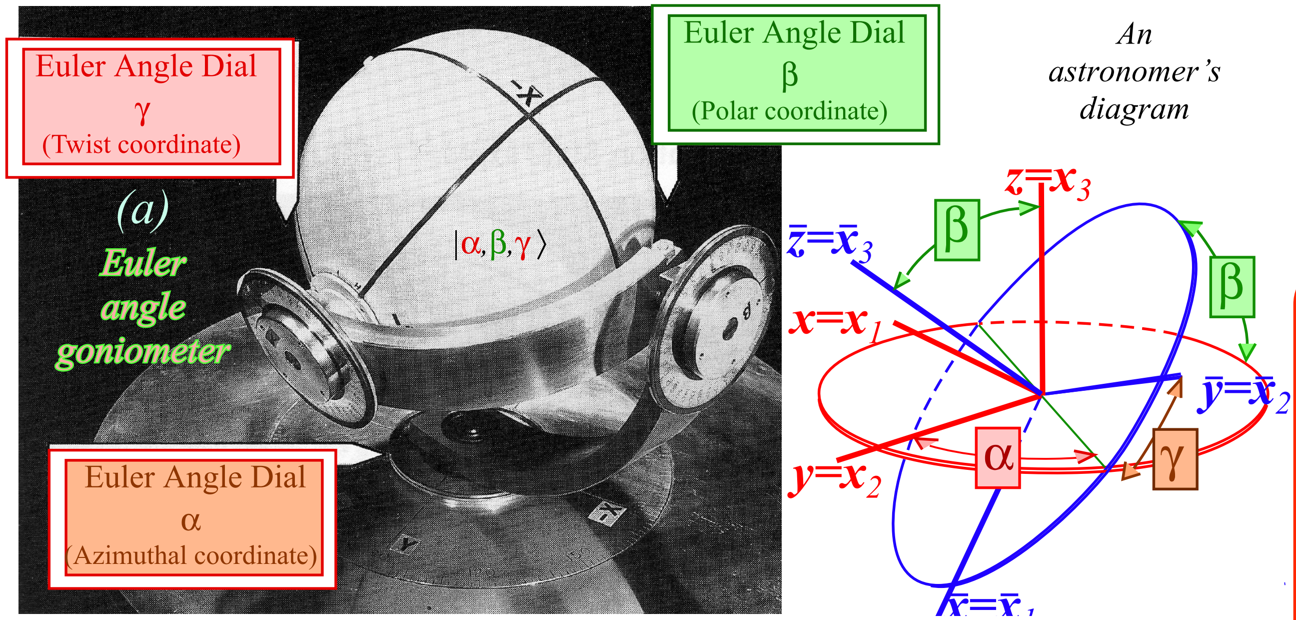
$$\hat{\Theta}_X = \cos\varphi \ \sin\vartheta$$
$$\hat{\Theta}_Y = \sin\varphi \ \sin\vartheta$$
$$\hat{\Theta}_Z = \cos\vartheta$$

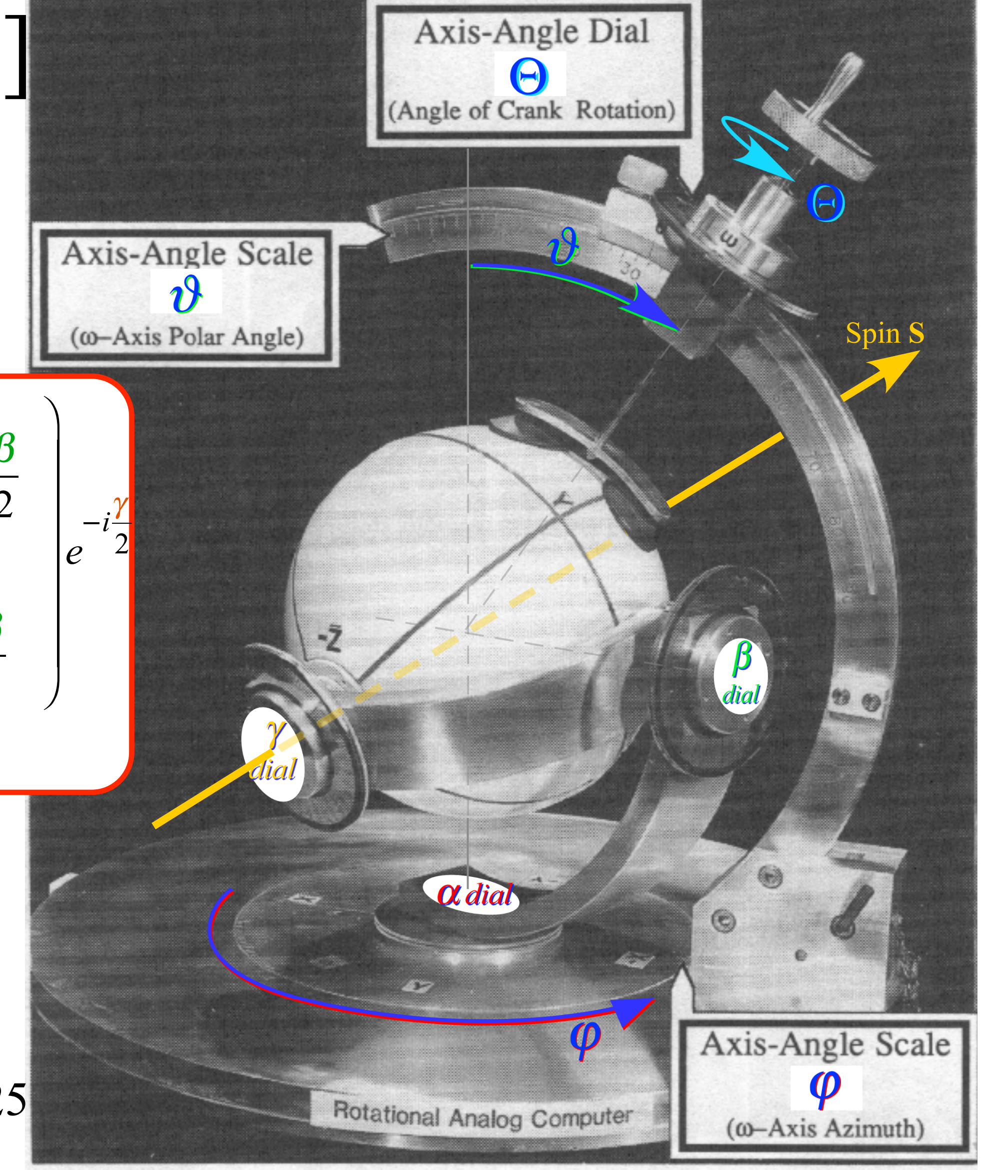
Operator

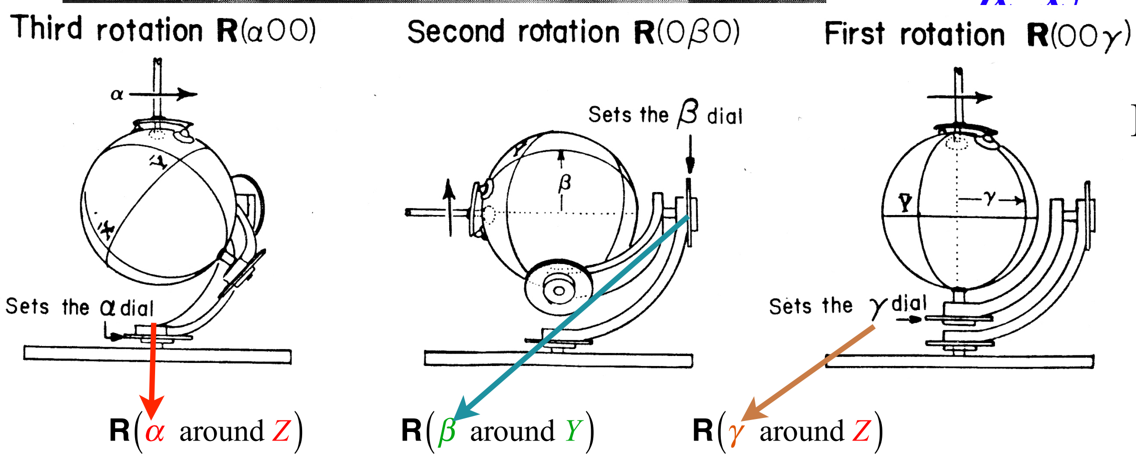
$$\left|[\varphi\vartheta\Theta]\right\rangle = \mathbf{R}[\varphi\vartheta\Theta]\left|\uparrow\right\rangle$$

Darboux axis angles

Euler $\mathbf{R}(\alpha\beta\gamma)$ versus Darboux $\mathbf{R}[\varphi\vartheta\Theta]$







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$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$

$$\mathbf{R}(\alpha\beta\gamma) = \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix} =$$

$$= \cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2} - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2} - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

$$\mathbf{R}[\vec{\Theta}] = \begin{pmatrix} \cos\frac{\Theta}{2} - i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}(\hat{\Theta}_X - i\hat{\Theta}_Y) \\ -i\sin\frac{\Theta}{2}(\hat{\Theta}_X + i\hat{\Theta}_Y) & \cos\frac{\Theta}{2} + i\hat{\Theta}_Z\sin\frac{\Theta}{2} \end{pmatrix} = \mathbf{R}[\varphi\vartheta\Theta] = e^{-i\mathbf{H}t}$$

$$= \cos\frac{\Theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\hat{\Theta}_X\sin\frac{\Theta}{2} - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\hat{\Theta}_Y\sin\frac{\Theta}{2} - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\hat{\Theta}_Z\sin\frac{\Theta}{2}$$

$\hat{\Theta}_X$: $\cos\varphi\ \sin\vartheta$ $\hat{\Theta}_Y$: $\sin\varphi\ \sin\vartheta$ $\hat{\Theta}_Z$: $\cos\vartheta$

$$= \begin{pmatrix} \cos\frac{\Theta}{2} - i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta + i\cos\varphi\sin\vartheta) \\ \sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta - i\cos\varphi\sin\vartheta) & \cos\frac{\Theta}{2} + i\cos\vartheta\sin\frac{\Theta}{2} \end{pmatrix}$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ is simpler to form than Θ-axis *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$.

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ *($\alpha\beta\gamma$ make better coordinates)*

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$

$-p_2 = \sin[(\gamma-\alpha)/2]\ \sin\beta/2 = \hat{\Theta}_X\sin\Theta/2 = \cos\varphi\ \sin\vartheta\ \ \sin\Theta/2$

$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y\sin\Theta/2 = \sin\varphi\ \sin\vartheta\ \ \sin\Theta/2$

$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z\sin\Theta/2 = \cos\vartheta\ \sin\Theta/2$

Review: Fundamental Euler **R***(αβγ) and Darboux* **R**[φϑΘ] *representations of U(2) and R(3)*

Euler **R***(αβγ) derived from Darboux* **R**[φϑΘ] *and vice versa*

Euler **R***(αβγ) rotation* Θ*=0−4π-sequence* [φϑ] *fixed (and "real-world" applications)*

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Euler $\mathbf{R}(\alpha\beta\gamma)$ related to Darboux $\mathbf{R}[\varphi\vartheta\Theta]$

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ $\alpha\beta\gamma$ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2\end{pmatrix}$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$$

$$-p_2 = \sin[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_X\sin\Theta/2 = \cos\varphi\sin\vartheta\ \sin\Theta/2$$

$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y\sin\Theta/2 = \sin\varphi\sin\vartheta\ \sin\Theta/2$$

$$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z\sin\Theta/2 = \cos\vartheta\ \sin\Theta/2$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2 \qquad \tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\tfrac{\pi}{2}-\varphi]$$

$$(\gamma+\alpha)/2 = \tan^{-1}[\cos\vartheta\tan\Theta/2] \qquad (\gamma-\alpha)/2 = \frac{\pi}{2}-\varphi$$

$$\sin[(\gamma-\alpha)/2] = \sin[\tfrac{\pi}{2}-\varphi] = \cos\varphi$$

$$\sin\beta/2 = \sin\vartheta\ \sin\Theta/2$$

This gives *Euler angles* $(\alpha\beta\gamma)$ in terms of *Darboux angles* $[\varphi\vartheta\Theta]$

$$\alpha = \varphi - \pi/2 + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

$$\beta = 2\sin^{-1}(\sin\Theta/2\ \sin\vartheta)$$

$$\gamma = \pi/2 - \varphi + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

Inverse relations have *Darboux axis angles* $[\varphi\vartheta\Theta]$ in terms of *Euler angles* $(\alpha\beta\gamma)$

$$\varphi = (\alpha-\gamma+\pi)/2$$

$$\vartheta = \tan^{-1}[\tan\beta/2/\sin(\alpha+\gamma)/2]$$

$$\Theta = 2\cos^{-1}[\cos\beta/2\cos(\alpha+\gamma)/2]$$

$$\cos[(\gamma-\alpha)/2] = \cos[\tfrac{\pi}{2}-\varphi] = \sin\varphi$$

$$\frac{\cos[(\gamma-\alpha)/2]\sin\beta/2}{\sin[(\gamma+\alpha)/2]\cos\beta/2} = \sin\varphi\tan\vartheta \Rightarrow \frac{\tan\beta/2}{\sin[(\gamma+\alpha)/2]} = \tan\vartheta$$

Example: *Euler angles* $(\alpha=50°\ \beta=60°\ \gamma=70°)$

$$\varphi = (50°-70°+180°)/2 = 80°$$

$$\vartheta = \tan^{-1}[\tan 60°/2/\sin(50°+\gamma)/2] = 33.7°$$

$$\Theta = 2\cos^{-1}[\cos 60°/2\cos(50°+\gamma)/2] = 128.7°$$

Reverse check: $(\alpha\beta\gamma)$ in terms of $[\varphi\vartheta\Theta]$

$$\alpha = 80° - 90° + \tan^{-1}(\tan(128.7°/2)\cos 33.7°) = 50.007°$$

$$\beta = 2\sin^{-1}(\sin 128.7°/2\ \sin 33.7°) = 60.022°$$

$$\gamma = \pi/2 - 128.7° + \tan^{-1}(\tan(128.7°/2)) = 70.007°$$

Review: Fundamental Euler **R**$(\alpha\beta\gamma)$ *and Darboux* **R**$[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

Euler **R**$(\alpha\beta\gamma)$ *derived from Darboux* **R**$[\varphi\vartheta\Theta]$ *and vice versa*

Euler **R**$(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes

AB-Type motion and Wigner's Avoided-Symmetry-Crossings

ABC-Type elliptical polarized motion

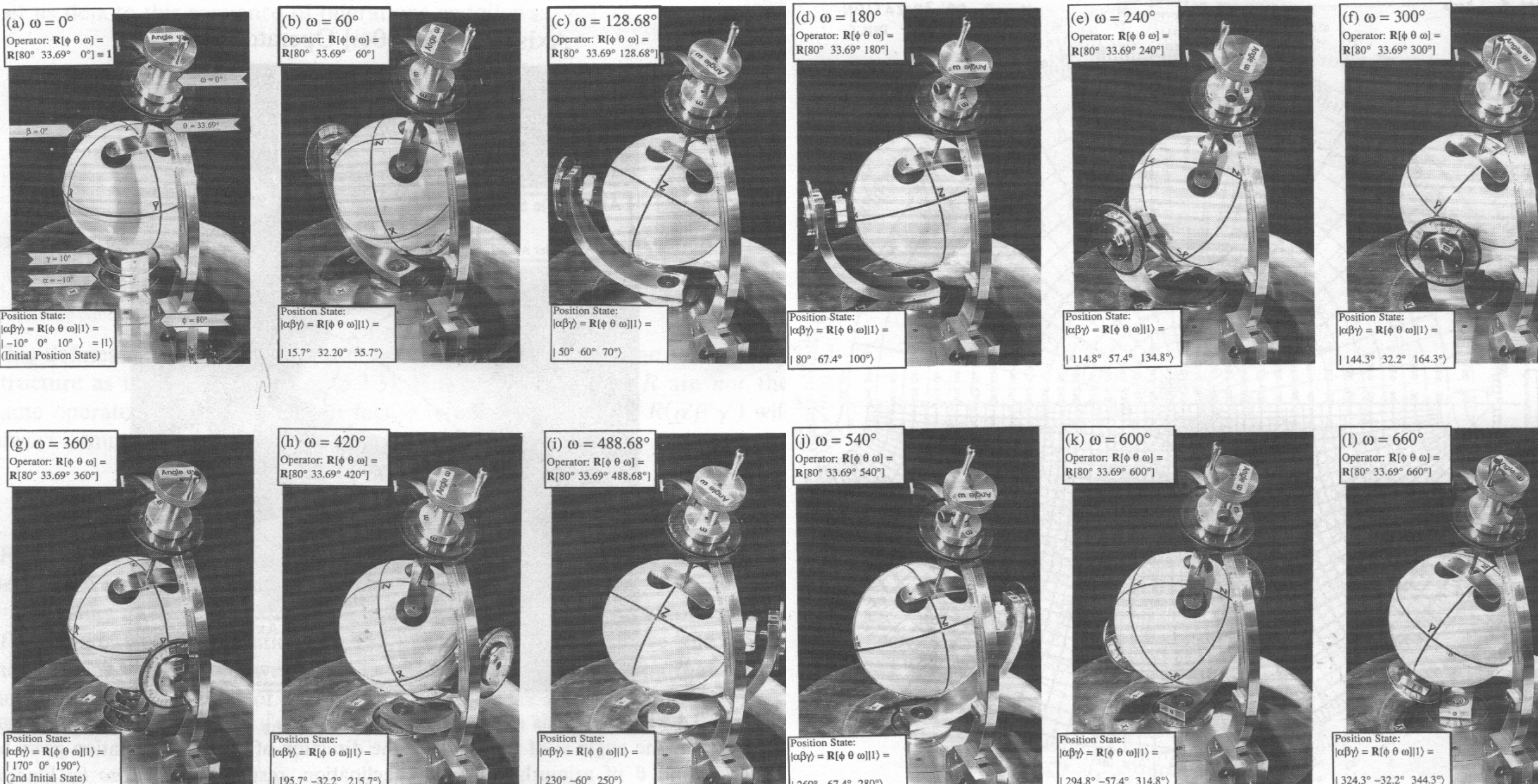
Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates

Euler Angle $(\alpha\beta\gamma)$ *ellipse coordinates*

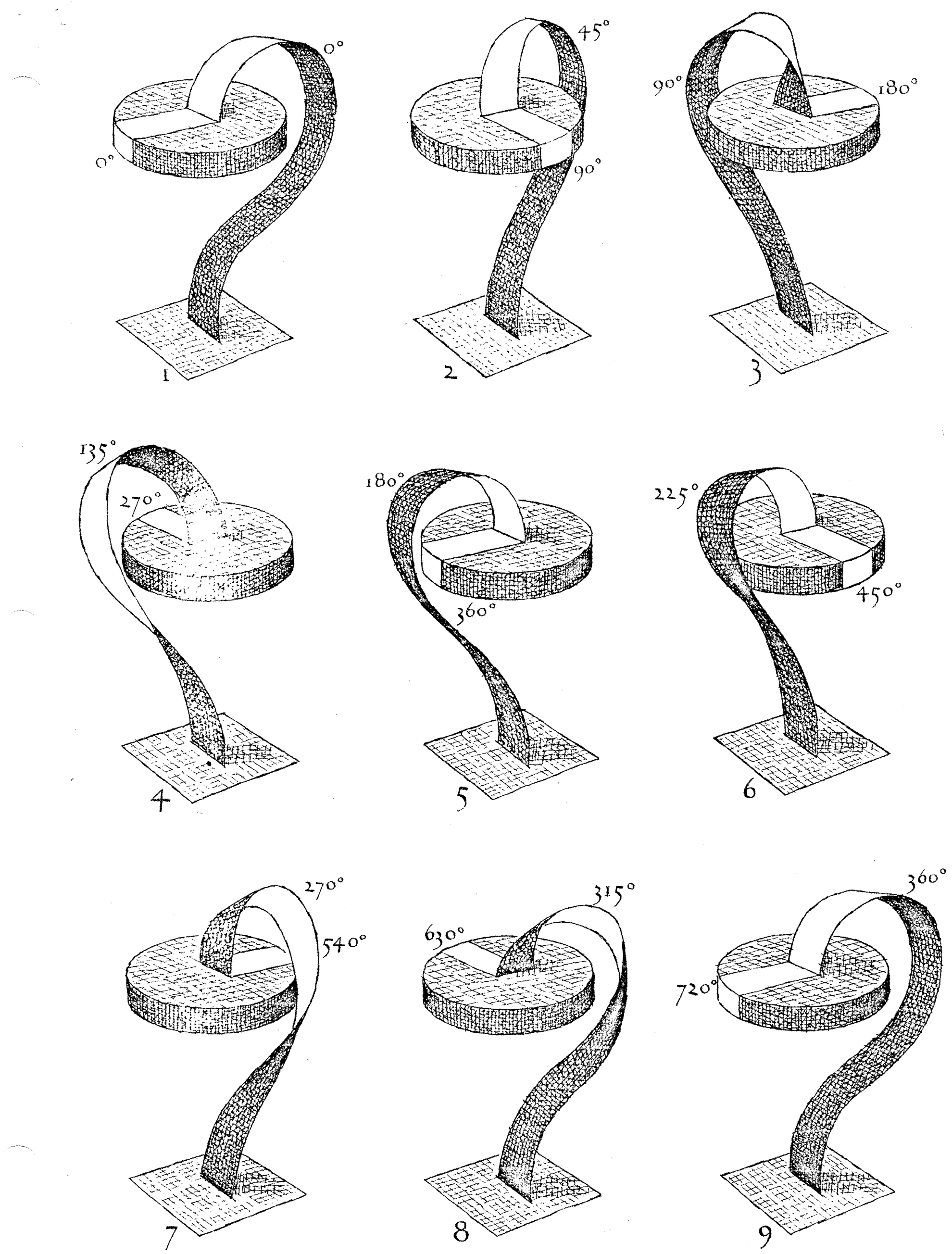
Euler **R**$(\alpha\beta\gamma)$ *rotation* Θ=0−4π*-sequence* [φϑ] *fixed*

Θ=0° Θ=60° Θ=128.7° Θ=180° Θ=240° Θ=300°



Θ=360° Θ=420° Θ=488.7° Θ=540° Θ=600° Θ=660°

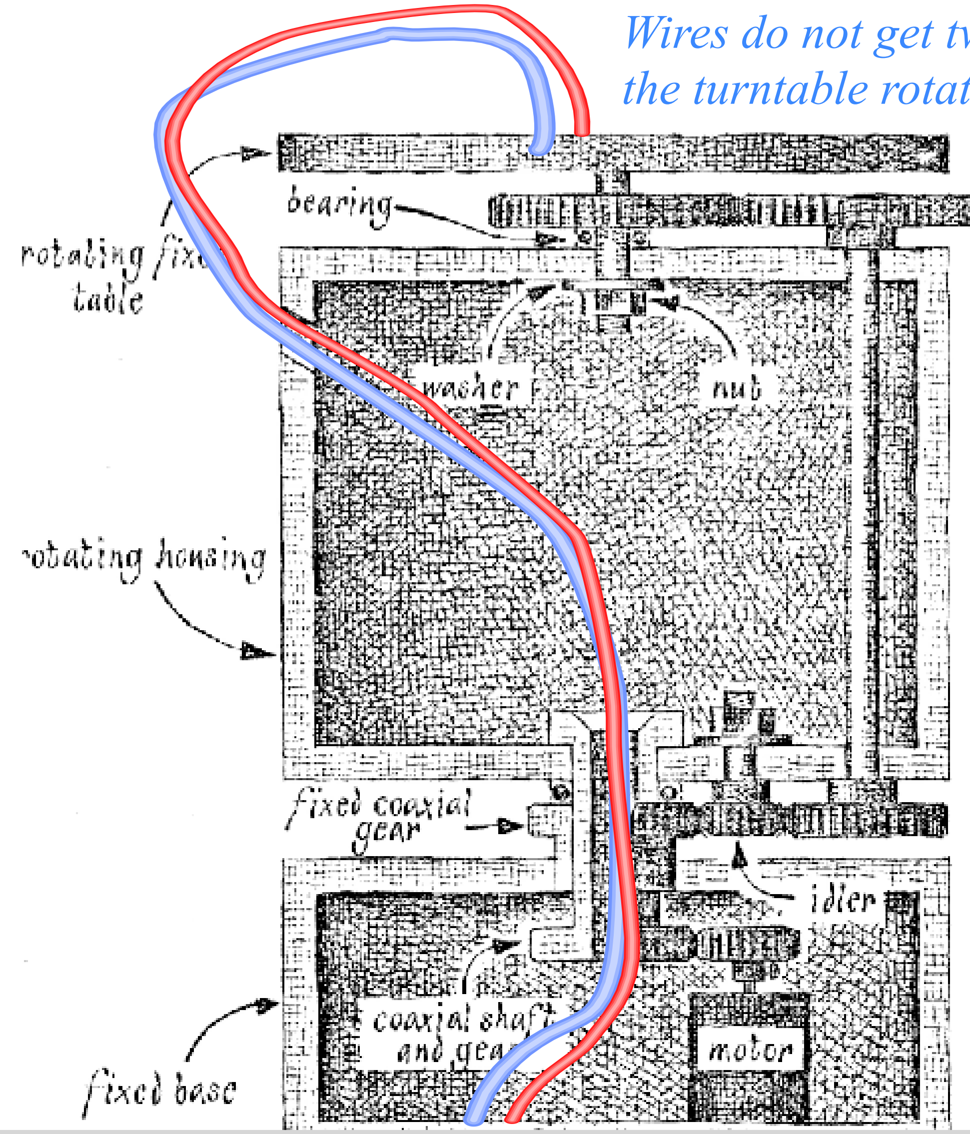
Some "real-world" applications of the U(2)-R(3) spinor-vector topology

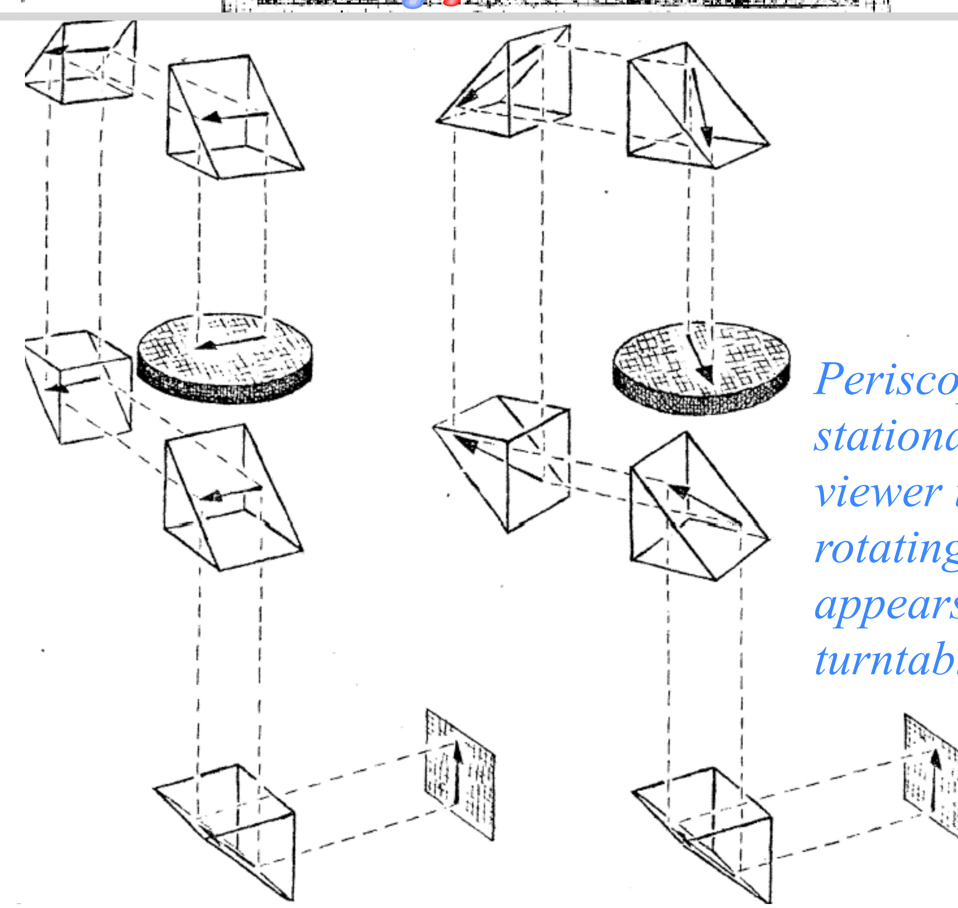


Sequential models of D. A. Adams' antitwister mechanism

From Scientific American December 1975-p.120-125

Wires do not get twisted up as the turntable rotates





Periscope allows stationary outside viewer to see into a rotating frame that appears fixed as the turntable rotates

Review: Fundamental Euler **R**$(\alpha\beta\gamma)$ *and Darboux* **R**$[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

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Euler **R**$(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*

R(3)-U(2) slide rule for converting **R**$(\alpha\beta\gamma)$ $\leftrightarrow$ **R**$[\varphi\vartheta\Theta]$ *and Sundial*

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

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Euler Angle $(\alpha\beta\gamma)$ *ellipse coordinates*

U(2) density operator approach to symmetry dynamics

Euler phase-angle coordinates (α, β, γ) *and norm N of quantum state* $|\Psi\rangle$

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \sqrt{N}\begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \sqrt{N}\begin{pmatrix} e^{-i\alpha/2}\cos\beta/2 \\ e^{i\alpha/2}\sin\beta/2 \end{pmatrix} e^{-i\gamma/2}$$

$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2$

$p_1 = -\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2$

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1/2 times $\boldsymbol{\sigma}$*-operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle = N = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N\underbrace{\left(p_1^2 + x_1^2 + p_2^2 + x_2^2\right)}_{4D\text{-}norm=1}$$

scaled by $\frac{1}{2}$:

$$\frac{1}{2}\left(|\Psi_1|^2 + |\Psi_2|^2\right) = \frac{N}{2}$$

$$\langle\Psi|\sigma_Z|\Psi\rangle = 2S_A = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N\left(p_1^2 + x_1^2 - p_2^2 - x_2^2\right)$$

scaled by $\frac{1}{2}$:

$$S_Z = S_A = \frac{1}{2}\left(|\Psi_1|^2 - |\Psi_2|^2\right) = \frac{N}{2}\left(\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right) = \frac{N}{2}\cos\beta$$

U(2) density operator approach to symmetry dynamics

$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1=-\sin[(\gamma+\alpha)/2]\cos\beta/2$

Euler phase-angle coordinates (α,β,γ) *and norm* N *of quantum state* $|\Psi\rangle$

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\beta/2\\ e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$

$x_2=\cos[(\gamma-\alpha)/2]\sin\beta/2$
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scaled by $\frac{1}{2}$:

$$\frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right)=\frac{N}{2}$$

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scaled by $\frac{1}{2}$:

$$S_Z=S_A=\frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right)=\frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)=\frac{N}{2}\cos\beta$$

$$\langle\Psi|\sigma_X|\Psi\rangle=2S_B=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1x_2+p_1p_2\right)$$

scaled by $\frac{1}{2}$:

$$S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2 \quad = N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}\quad=\frac{N}{2}\cos\alpha\,\sin\beta$$

U(2) density operator approach to symmetry dynamics

Euler phase-angle coordinates (α,β,γ) *and norm N of quantum state* $|\Psi\rangle$

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\beta/2\\ e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$

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scaled by $\frac{1}{2}$: $\frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right)=\frac{N}{2}$

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$$\langle\Psi|\sigma_X|\Psi\rangle=2S_B=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1x_2+p_1p_2\right)$$

scaled by $\frac{1}{2}$: $S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2 \quad =N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} \quad =\frac{N}{2}\cos\alpha\,\sin\beta$

$$\langle\Psi|\sigma_Y|\Psi\rangle=2S_C=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1p_2-x_2p_1\right)$$

scaled by $\frac{1}{2}$: $S_Y=S_C=\mathrm{Im}\,\Psi_1^*\Psi_2 \quad =N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} \quad =\frac{N}{2}\sin\alpha\,\sin\beta$

U(2) density operator approach to symmetry dynamics

Euler phase-angle coordinates (α,β,γ) *and norm N of quantum state* $|\Psi\rangle$

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\beta/2\\ e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$

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scaled by $\frac{1}{2}$: $\frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right)=\frac{N}{2}$

$$\langle\Psi|\sigma_Z|\Psi\rangle=2S_A=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\left(p_1^2+x_1^2-p_2^2-x_2^2\right)$$
scaled by $\frac{1}{2}$: $S_Z=S_A=\frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right)=\frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)=\frac{N}{2}\cos\beta$

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scaled by $\frac{1}{2}$: $S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2=N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\cos\alpha\sin\beta$

$$\langle\Psi|\sigma_Y|\Psi\rangle=2S_C=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1p_2-x_2p_1\right)$$
scaled by $\frac{1}{2}$: $S_Y=S_C=\mathrm{Im}\,\Psi_1^*\Psi_2=N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\sin\alpha\sin\beta$

The *density operator* $\rho=|\Psi\rangle\langle\Psi|=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}\otimes\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}=\begin{pmatrix}\Psi_1\Psi_1^* & \Psi_1\Psi_2^*\\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^*\end{pmatrix}=\begin{pmatrix}\rho_{11}&\rho_{12}\\ \rho_{21}&\rho_{22}\end{pmatrix}=\begin{pmatrix}\Psi_1^*\Psi_1 & \Psi_2^*\Psi_1\\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2\end{pmatrix}$

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scaled by $\frac{1}{2}$: $S_X = S_B = \mathrm{Re}\,\Psi_1^*\Psi_2 = N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{N}{2}\cos\alpha\sin\beta$

$$\langle\Psi|\sigma_Y|\Psi\rangle = 2S_C = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = 2N\left(x_1p_2 - x_2p_1\right)$$

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The *density operator* $\rho = |\Psi\rangle\langle\Psi| =$

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \otimes \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix} = \begin{pmatrix} \Psi_1\Psi_1^* & \Psi_1\Psi_2^* \\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^* \end{pmatrix} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \begin{pmatrix} \Psi_1^*\Psi_1 & \Psi_2^*\Psi_1 \\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2 \end{pmatrix}$$

$\rho_{11} = \Psi_1^*\Psi_1 = \frac{1}{2}N + S_Z$	$\rho_{12} = \Psi_2^*\Psi_1 = S_X - iS_Y,$
$\rho_{21} = \Psi_1^*\Psi_2 = S_X + iS_Y$	$\rho_{22} = \Psi_2^*\Psi_2 = \frac{1}{2}N - S_Z$

$$= \begin{pmatrix} \frac{1}{2}N + S_Z & S_X - iS_Y \\ S_X + iS_Y & \frac{1}{2}N - S_Z \end{pmatrix}$$

Norm: $N = \Psi_1^*\Psi_1 + \Psi_2^*\Psi_2$ ← ...2-by-2 *density operator* ρ

U(2) density operator approach to symmetry dynamics

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scaled by $\frac{1}{2}$: $S_X = S_B = \mathrm{Re}\,\Psi_1^*\Psi_2 = N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{N}{2}\cos\alpha\sin\beta$

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scaled by $\frac{1}{2}$: $S_Y = S_C = \mathrm{Im}\,\Psi_1^*\Psi_2 = N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{N}{2}\sin\alpha\sin\beta$

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Norm: $N = \Psi_1^*\Psi_1 + \Psi_2^*\Psi_2$

$$= \begin{pmatrix} \frac{1}{2}N + S_Z & S_X - iS_Y \\ S_X + iS_Y & \frac{1}{2}N - S_Z \end{pmatrix} = \frac{1}{2}N\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + S_X\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + S_Y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + S_Z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

↑ ρ

...so state *density operator* ρ has $\boldsymbol{\sigma}$-expansion

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$x_2=\cos[(\gamma-\alpha)/2]\sin\beta/2$
$p_2=-\sin[(\gamma-\alpha)/2]\sin\beta/2$

1/2 times $\boldsymbol{\sigma}$*-operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle = N = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N\underbrace{\left(p_1^2+x_1^2+p_2^2+x_2^2\right)}_{4D\text{-}norm=1}$$
scaled by $\frac{1}{2}$: $\quad \frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right) = \frac{N}{2}$

$$\langle\Psi|\sigma_Z|\Psi\rangle = 2S_A = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N\left(p_1^2+x_1^2-p_2^2-x_2^2\right)$$
scaled by $\frac{1}{2}$: $\quad S_Z = S_A = \frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right) = \frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right) = \frac{N}{2}\cos\beta$

$$\langle\Psi|\sigma_X|\Psi\rangle = 2S_B = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = 2N\left(x_1x_2+p_1p_2\right)$$
scaled by $\frac{1}{2}$: $\quad S_X = S_B = \mathrm{Re}\,\Psi_1^*\Psi_2 \quad = N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} \quad = \frac{N}{2}\cos\alpha\sin\beta$

$$\langle\Psi|\sigma_Y|\Psi\rangle = 2S_C = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = 2N\left(x_1p_2-x_2p_1\right)$$
scaled by $\frac{1}{2}$: $\quad S_Y = S_C = \mathrm{Im}\,\Psi_1^*\Psi_2 \quad = N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} \quad = \frac{N}{2}\sin\alpha\sin\beta$

The *density operator* $\rho = |\Psi\rangle\langle\Psi| =$

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \otimes \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix} = \begin{pmatrix} \Psi_1\Psi_1^* & \Psi_1\Psi_2^* \\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^* \end{pmatrix} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \begin{pmatrix} \Psi_1^*\Psi_1 & \Psi_2^*\Psi_1 \\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2 \end{pmatrix}$$

$\rho_{11} = \Psi_1^*\Psi_1 = \frac{1}{2}N + S_Z$	$\rho_{12} = \Psi_2^*\Psi_1 = S_X - iS_Y,$
$\rho_{21} = \Psi_1^*\Psi_2 = S_X + iS_Y$	$\rho_{22} = \Psi_2^*\Psi_2 = \frac{1}{2}N - S_Z$

Norm: $N = \Psi_1^*\Psi_1 + \Psi_2^*\Psi_2$

$$= \begin{pmatrix} \frac{1}{2}N + S_Z & S_X - iS_Y \\ S_X + iS_Y & \frac{1}{2}N - S_Z \end{pmatrix} = \frac{1}{2}N\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + S_X\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\sigma_X} + S_Y\underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\sigma_Y} + S_Z\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\sigma_Z}$$

$$\rho = \frac{1}{2}N\,\mathbf{1} + S_X\,\boldsymbol{\sigma}_X + S_Y\,\boldsymbol{\sigma}_Y + S_Z\,\boldsymbol{\sigma}_Z = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

...so state *density operator* ρ has $\boldsymbol{\sigma}$-expansion

U(2) density operator approach to symmetry dynamics

Euler phase-angle coordinates (α,β,γ) *and norm* N *of quantum state* $|\Psi\rangle$

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\beta/2\\ e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$

$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1=-\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2=\cos[(\gamma-\alpha)/2]\sin\beta/2$
$p_2=-\sin[(\gamma-\alpha)/2]\sin\beta/2$

1/2 times $\boldsymbol{\sigma}$*-operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle=N=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\underbrace{\left(p_1^2+x_1^2+p_2^2+x_2^2\right)}_{4D\text{-}norm=1}\quad \text{scaled by } \tfrac{1}{2}:\quad \frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right)=\frac{N}{2}$$

$$\langle\Psi|\sigma_Z|\Psi\rangle=2S_A=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\left(p_1^2+x_1^2-p_2^2-x_2^2\right)\quad \text{scaled by } \tfrac{1}{2}:\quad S_Z=S_A=\frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right)=\frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)=\frac{N}{2}\cos\beta$$

$$\langle\Psi|\sigma_X|\Psi\rangle=2S_B=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1x_2+p_1p_2\right)\quad \text{scaled by } \tfrac{1}{2}:\quad S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2=N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\cos\alpha\sin\beta$$

$$\langle\Psi|\sigma_Y|\Psi\rangle=2S_C=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1p_2-x_2p_1\right)\quad \text{scaled by } \tfrac{1}{2}:\quad S_Y=S_C=\mathrm{Im}\,\Psi_1^*\Psi_2=N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\sin\alpha\sin\beta$$

The *density operator* $\rho=|\Psi\rangle\langle\Psi|=$

$$\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}\otimes\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}=\begin{pmatrix}\Psi_1\Psi_1^* & \Psi_1\Psi_2^*\\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^*\end{pmatrix}=\begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix}=\begin{pmatrix}\Psi_1^*\Psi_1 & \Psi_2^*\Psi_1\\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2\end{pmatrix}$$

$\rho_{11}=\Psi_1^*\Psi_1=\frac{1}{2}N+S_Z$	$\rho_{12}=\Psi_2^*\Psi_1=S_X-iS_Y,$
$\rho_{21}=\Psi_1^*\Psi_2=S_X+iS_Y$	$\rho_{22}=\Psi_2^*\Psi_2=\frac{1}{2}N-S_Z$

Norm: $N=\Psi_1^*\Psi_1+\Psi_2^*\Psi_2$

$$=\begin{pmatrix}\frac{1}{2}N+S_Z & S_X-iS_Y\\ S_X+iS_Y & \frac{1}{2}N-S_Z\end{pmatrix}=\frac{1}{2}N\begin{pmatrix}1&0\\0&1\end{pmatrix}+S_X\begin{pmatrix}0&1\\1&0\end{pmatrix}+S_Y\begin{pmatrix}0&-i\\i&0\end{pmatrix}+S_Z\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$\rho=\frac{1}{2}N\,\mathbf{1}+S_X\,\sigma_X+S_Y\,\sigma_Y+S_Z\,\sigma_Z=\frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

...so state *density operator* ρ has $\boldsymbol{\sigma}$-expansion like *Hamiltonian operator* **H**

$$\begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix}=\mathbf{H}=\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}+B\begin{pmatrix}0&1\\1&0\end{pmatrix}+C\begin{pmatrix}0&-i\\i&0\end{pmatrix}$$

$$\mathbf{H}=\omega_0\,\sigma_0+\frac{\Omega_A}{2}\sigma_A+\frac{\Omega_B}{2}\sigma_B+\frac{\Omega_C}{2}\sigma_C=\omega_0\sigma_0+\frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

U(2) density operator approach to symmetry dynamics

Euler phase-angle coordinates (α, β, γ) *and norm N of quantum state* $|\Psi\rangle$

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \sqrt{N}\begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \sqrt{N}\begin{pmatrix} e^{-i\alpha/2}\cos\beta/2 \\ e^{i\alpha/2}\sin\beta/2 \end{pmatrix} e^{-i\gamma/2}$$

$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1 = -\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2$
$p_2 = -\sin[(\gamma-\alpha)/2]\sin\beta/2$

1/2 times $\boldsymbol{\sigma}$*-operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle = N = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N\left(p_1^2 + x_1^2 + p_2^2 + x_2^2\right) \quad \text{scaled by } \tfrac{1}{2}: \quad \frac{1}{2}\left(|\Psi_1|^2 + |\Psi_2|^2\right) = \frac{N}{2}$$

4D-norm=1

$$\langle\Psi|\sigma_Z|\Psi\rangle = 2S_A = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N\left(p_1^2 + x_1^2 - p_2^2 - x_2^2\right) \quad \text{scaled by } \tfrac{1}{2}: \quad S_Z = S_A = \frac{1}{2}\left(|\Psi_1|^2 - |\Psi_2|^2\right) = \frac{N}{2}\left(\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right) = \frac{N}{2}\cos\beta$$

$$\langle\Psi|\sigma_X|\Psi\rangle = 2S_B = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = 2N\left(x_1x_2 + p_1p_2\right) \quad \text{scaled by } \tfrac{1}{2}: \quad S_X = S_B = \mathrm{Re}\,\Psi_1^*\Psi_2 = N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{N}{2}\cos\alpha\sin\beta$$

$$\langle\Psi|\sigma_Y|\Psi\rangle = 2S_C = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = 2N\left(x_1p_2 - x_2p_1\right) \quad \text{scaled by } \tfrac{1}{2}: \quad S_Y = S_C = \mathrm{Im}\,\Psi_1^*\Psi_2 = N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{N}{2}\sin\alpha\sin\beta$$

The *density operator* $\rho = |\Psi\rangle\langle\Psi| =$

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \otimes \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix} = \begin{pmatrix} \Psi_1\Psi_1^* & \Psi_1\Psi_2^* \\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^* \end{pmatrix} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \begin{pmatrix} \Psi_1^*\Psi_1 & \Psi_2^*\Psi_1 \\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2 \end{pmatrix}$$

$\rho_{11} = \Psi_1^*\Psi_1 = \frac{1}{2}N + S_Z$	$\rho_{12} = \Psi_2^*\Psi_1 = S_X - iS_Y,$
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Norm: $N = \Psi_1^*\Psi_1 + \Psi_2^*\Psi_2$

$$= \begin{pmatrix} \frac{1}{2}N + S_Z & S_X - iS_Y \\ S_X + iS_Y & \frac{1}{2}N - S_Z \end{pmatrix} = \frac{1}{2}N\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + S_X\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + S_Y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + S_Z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\rho = \frac{1}{2}N\,\mathbf{1} + S_X\,\sigma_X + S_Y\,\sigma_Y + S_Z\,\sigma_Z = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

...so state *density operator* ρ has $\boldsymbol{\sigma}$-expansion like *Hamiltonian operator* **H**

$$\begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{H} = \omega_0\,\sigma_0 + \frac{\Omega_A}{2}\sigma_A + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C = \omega_0\sigma_0 + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\,\mathbf{1} + \Omega_A\,\mathbf{S}_A + \Omega_B\,\mathbf{S}_B + \Omega_C\mathbf{S}_C = \Omega_0\mathbf{1} + \vec{\Omega}\bullet\mathbf{S}$$

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

Review: Fundamental Euler **R**$(\alpha\beta\gamma)$ *and Darboux* **R**$[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

Euler **R**$(\alpha\beta\gamma)$ *derived from Darboux* **R**$[\varphi\vartheta\Theta]$ *and vice versa*
Euler **R**$(\alpha\beta\gamma)$ *rotation* Θ=0−4π*-sequence* $[\varphi\vartheta]$ *fixed*
R(3)-U(2) slide rule for converting **R**$(\alpha\beta\gamma)$ ↔ **R**$[\varphi\vartheta\Theta]$ *and Sundial*

U(2) density operator approach to symmetry dynamics
→ *Bloch equation for density operator*
Quick U(2) way to find eigen-solutions for 2-by-2 **H**

The ABC's of U(2) dynamics-Archetypes
Asymmetric-Diagonal A-Type motion
Bilateral-Balanced B-Type motion
Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes
AB-Type motion and Wigner's Avoided-Symmetry-Crossings
ABC-Type elliptical polarized motion

Ellipsometry using U(2) symmetry coordinates
Conventional amp-phase ellipse coordinates
Euler Angle $(\alpha\beta\gamma)$ *ellipse coordinates*

U(2) density operator approach to symmetry dynamics

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}} \bullet \boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0 \mathbf{1} + \frac{\vec{\Omega}}{2} \bullet \boldsymbol{\sigma}$$

Bloch equation for density operator

Ket equation (time *forward*) and "daggered" bra-equation (time *reversed*).

$$i\hbar|\dot{\Psi}\rangle = \mathbf{H}|\Psi\rangle, \quad \Leftarrow \textit{Daggar}^{\dagger} \Rightarrow \quad -i\hbar\langle\dot{\Psi}| = \langle\Psi|\mathbf{H}$$

Note: $\mathbf{H}^{\dagger} = \mathbf{H}$.

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\boldsymbol{\rho} = \frac{1}{2}\left(N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

Ket equation (time *forward*) and "daggered" bra-equation (time *reversed*).

$$i\hbar\left|\dot{\Psi}\right\rangle = \mathbf{H}\left|\Psi\right\rangle, \qquad \Leftarrow \textit{Daggar}^{\dagger} \Rightarrow \qquad -i\hbar\left\langle\dot{\Psi}\right| = \left\langle\Psi\right|\mathbf{H}$$

Note: $\mathbf{H}^{\dagger} = \mathbf{H}$.
$\boldsymbol{\rho}^{\dagger} = \boldsymbol{\rho}$

Combining these gives a time derivative of the density operator $\boldsymbol{\rho} = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho} = i\hbar\dot{\boldsymbol{\rho}} = i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right| + i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right| = \mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}} \bullet \boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0 \mathbf{1} + \frac{\vec{\Omega}}{2} \bullet \boldsymbol{\sigma}$$

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$$i\hbar|\dot{\Psi}\rangle = \mathbf{H}|\Psi\rangle, \quad \Leftarrow Daggar^{\dagger} \Rightarrow \quad -i\hbar\langle\dot{\Psi}| = \langle\Psi|\mathbf{H}$$

Note: $\mathbf{H}^{\dagger} = \mathbf{H}$.
$\rho^{\dagger} = \rho$

Combining these gives a time derivative of the density operator $\rho = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = i\hbar|\dot{\Psi}\rangle\langle\Psi| + i\hbar|\Psi\rangle\langle\dot{\Psi}| = \mathbf{H}|\Psi\rangle\langle\Psi| - |\Psi\rangle\langle\Psi|\mathbf{H}$$

The result is called a *Bloch equation*.

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = \mathbf{H}\rho - \rho\mathbf{H} = [\mathbf{H}, \rho]$$

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\boldsymbol{\Omega}}}{2}\bullet\boldsymbol{\sigma}$$

Ket equation (time *forward*) and "daggered" bra-equation (time *reversed*).

$$i\hbar\left|\dot{\Psi}\right\rangle = \mathbf{H}\left|\Psi\right\rangle, \quad \Leftarrow Daggar^{\dagger} \Rightarrow \quad -i\hbar\left\langle\dot{\Psi}\right| = \left\langle\Psi\right|\mathbf{H}$$

Note: $\mathbf{H}^{\dagger} = \mathbf{H}$.
$\rho^{\dagger} = \rho$

Combining these gives a time derivative of the density operator $\rho = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right| + i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right| = \mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

The result is called a *Bloch equation*.

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = \mathbf{H}\rho - \rho\mathbf{H} = \left[\mathbf{H},\rho\right]$$

Given ρ and $\mathbf{H}$ in terms *spin* $\mathbf{S}$-vector and *crank* $\boldsymbol{\Omega}$-vector:

$$\mathbf{H}\rho = \left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\frac{N}{2}\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma} + \hbar\Omega_0\vec{\mathbf{S}}\bullet\boldsymbol{\sigma} + \frac{\hbar}{2}\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)$$

$$-\rho\mathbf{H} = \left(\frac{N}{2}\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)\left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma} + \hbar\Omega_0\vec{\mathbf{S}}\bullet\boldsymbol{\sigma} + \frac{\hbar}{2}\left(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)$$

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\sigma$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\sigma$$

Ket equation (time *forward*) and "daggered" bra-equation (time *reversed*).

$$i\hbar|\dot{\Psi}\rangle = \mathbf{H}|\Psi\rangle, \quad \Leftarrow Daggar^{\dagger} \Rightarrow \quad -i\hbar\langle\dot{\Psi}| = \langle\Psi|\mathbf{H}$$

Note: $\mathbf{H}^{\dagger} = \mathbf{H}$.
$\rho^{\dagger} = \rho$

Combining these gives a time derivative of the density operator $\rho = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = i\hbar|\dot{\Psi}\rangle\langle\Psi| + i\hbar|\Psi\rangle\langle\dot{\Psi}| = \mathbf{H}|\Psi\rangle\langle\Psi| - |\Psi\rangle\langle\Psi|\mathbf{H}$$

The result is called a *Bloch equation*.

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = \mathbf{H}\rho - \rho\mathbf{H} = [\mathbf{H},\rho]$$

Given ρ and $\mathbf{H}$ in terms *spin* **S**-vector and *crank* **Ω**-vector:

$$\mathbf{H}\rho = \left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\Omega}\bullet\sigma\right)\left(\frac{N}{2}\mathbf{1} + \vec{\mathbf{S}}\bullet\sigma\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\Omega}\bullet\sigma + \hbar\Omega_0\vec{\mathbf{S}}\bullet\sigma + \frac{\hbar}{2}(\vec{\Omega}\bullet\sigma)(\vec{\mathbf{S}}\bullet\sigma)$$

$$-\rho\mathbf{H} = \left(\frac{N}{2}\mathbf{1} + \vec{\mathbf{S}}\bullet\sigma\right)\left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\Omega}\bullet\sigma\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\Omega}\bullet\sigma + \hbar\Omega_0\vec{\mathbf{S}}\bullet\sigma + \frac{\hbar}{2}(\vec{\mathbf{S}}\bullet\sigma)(\vec{\Omega}\bullet\sigma)$$

Last terms don't cancel if the *spin* **S** and *crank* **Ω** *point in different directions*.

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$
$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\boldsymbol{\Omega}}}{2}\bullet\boldsymbol{\sigma}$$

Ket equation (time *forward*) and "daggered" bra-equation (time *reversed*).

$$i\hbar|\dot{\Psi}\rangle = \mathbf{H}|\Psi\rangle, \qquad \Leftarrow Daggar^{\dagger} \Rightarrow \qquad -i\hbar\langle\dot{\Psi}| = \langle\Psi|\mathbf{H}$$

Note: $\mathbf{H}^{\dagger} = \mathbf{H}$. $\rho^{\dagger} = \rho$

Combining these gives a time derivative of the density operator $\rho = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = i\hbar|\dot{\Psi}\rangle\langle\Psi| + i\hbar|\Psi\rangle\langle\dot{\Psi}| = \mathbf{H}|\Psi\rangle\langle\Psi| - |\Psi\rangle\langle\Psi|\mathbf{H}$$

The result is called a *Bloch equation*.

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = \mathbf{H}\rho - \rho\mathbf{H} = [\mathbf{H},\rho]$$

$$(\mathbf{A}\bullet\boldsymbol{\sigma})(\mathbf{B}\bullet\boldsymbol{\sigma}) = A_\alpha B_\beta\sigma_\alpha\sigma_\beta = A_\alpha B_\beta(\delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma)$$
$$= A_\alpha B_\alpha + i\varepsilon_{\alpha\beta\gamma}A_\alpha B_\beta\sigma_\gamma$$
$$= \mathbf{A}\bullet\mathbf{B} + i(\mathbf{A}\times\mathbf{B})\cdot\boldsymbol{\sigma}$$

Given ρ and $\mathbf{H}$ in terms *spin* $\mathbf{S}$-vector and *crank* $\boldsymbol{\Omega}$-vector:

$$\mathbf{H}\rho = \left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\frac{N}{2}\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma} + \hbar\Omega_0\vec{\mathbf{S}}\bullet\boldsymbol{\sigma} + \frac{\hbar}{2}(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma})(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma})$$

$$-\rho\mathbf{H} = \left(\frac{N}{2}\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)\left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma} + \hbar\Omega_0\vec{\mathbf{S}}\bullet\boldsymbol{\sigma} + \frac{\hbar}{2}(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma})(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma})$$

This cancels   *This remains*

Last terms don't cancel if the *spin* $\mathbf{S}$ and *crank* $\boldsymbol{\Omega}$ *point in different directions*.

$$\mathbf{H}\rho - \rho\mathbf{H} = \frac{\hbar}{2}(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma})(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}) - \frac{\hbar}{2}(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma})(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma})$$

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\boldsymbol{\Omega}}}{2}\bullet\boldsymbol{\sigma}$$

Ket equation (time forward) and "daggered" bra-equation (time reversed).

$$i\hbar\left|\dot{\Psi}\right\rangle = \mathbf{H}\left|\Psi\right\rangle, \qquad \Leftarrow Daggar^{\dagger} \Rightarrow \qquad -i\hbar\left\langle\dot{\Psi}\right| = \left\langle\Psi\right|\mathbf{H}$$

Combining these gives a time derivative of the density operator $\rho = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right| + i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right| = \mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

The result is called a *Bloch equation*.

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = \mathbf{H}\rho - \rho\mathbf{H} = \left[\mathbf{H},\rho\right]$$

$$(\mathbf{A}\bullet\boldsymbol{\sigma})(\mathbf{B}\bullet\boldsymbol{\sigma}) = A_\alpha B_\beta \sigma_\alpha\sigma_\beta = A_\alpha B_\beta\left(\delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma\right)$$

$$= A_\alpha B_\alpha + i\varepsilon_{\alpha\beta\gamma}A_\alpha B_\beta\sigma_\gamma$$

$$= \mathbf{A}\bullet\mathbf{B} + i(\mathbf{A}\times\mathbf{B})\cdot\boldsymbol{\sigma}$$

This cancels | *This remains*

Given ρ and **H** in terms *spin* **S**-vector and *crank* **Ω**-vector:

$$\mathbf{H}\rho = \left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\frac{N}{2}\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma} + \hbar\Omega_0\vec{\mathbf{S}}\bullet\boldsymbol{\sigma} + \frac{\hbar}{2}(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma})(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma})$$

$$\rho\mathbf{H} = \left(\frac{N}{2}\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)\left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma} + \hbar\Omega_0\vec{\mathbf{S}}\bullet\boldsymbol{\sigma} + \frac{\hbar}{2}(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma})(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma})$$

Last terms don't cancel if the *spin* **S** and *crank* **Ω** *point in different directions*.

$$\mathbf{H}\rho - \rho\mathbf{H} = \frac{\hbar}{2}(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma})(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}) - \frac{\hbar}{2}(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma})(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma})$$

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = \frac{i\hbar}{2}(\vec{\boldsymbol{\Omega}}\times\vec{\mathbf{S}})\bullet\boldsymbol{\sigma} - \frac{i\hbar}{2}(\vec{\mathbf{S}}\times\vec{\boldsymbol{\Omega}})\bullet\boldsymbol{\sigma}$$

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\sigma$$
$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\sigma$$

Ket equation (time forward) and "daggered" bra-equation (time reversed).

$$i\hbar\left|\dot{\Psi}\right\rangle = \mathbf{H}\left|\Psi\right\rangle, \qquad \Leftarrow \textit{Daggar}^{\dagger} \Rightarrow \qquad -i\hbar\left\langle\dot{\Psi}\right| = \left\langle\Psi\right|\mathbf{H}$$

Note: $\mathbf{H}^{\dagger} = \mathbf{H}$.
$\rho^{\dagger} = \rho$

Combining these gives a time derivative of the density operator $\rho = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right| + i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right| = \mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

The result is called a *Bloch equation*.

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = \mathbf{H}\rho - \rho\mathbf{H} = [\mathbf{H},\rho]$$

$$(\mathbf{A}\bullet\sigma)(\mathbf{B}\bullet\sigma) = A_\alpha B_\beta\sigma_\alpha\sigma_\beta = A_\alpha B_\beta(\delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma)$$
$$= A_\alpha B_\alpha + i\varepsilon_{\alpha\beta\gamma}A_\alpha B_\beta\sigma_\gamma$$
$$= \mathbf{A}\bullet\mathbf{B} + i(\mathbf{A}\times\mathbf{B})\cdot\sigma$$

This cancels *This remains*

Given ρ and $\mathbf{H}$ in terms *spin* $\mathbf{S}$-vector and *crank* $\mathbf{\Omega}$-vector:

$$\mathbf{H}\rho = \left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\Omega}\bullet\sigma\right)\left(\frac{N}{2}\mathbf{1} + \vec{\mathbf{S}}\bullet\sigma\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\Omega}\bullet\sigma + \hbar\Omega_0\vec{\mathbf{S}}\bullet\sigma + \frac{\hbar}{2}(\vec{\Omega}\bullet\sigma)(\vec{\mathbf{S}}\bullet\sigma)$$

$$\rho\mathbf{H} = \left(\frac{N}{2}\mathbf{1} + \vec{\mathbf{S}}\bullet\sigma\right)\left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\Omega}\bullet\sigma\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\Omega}\bullet\sigma + \hbar\Omega_0\vec{\mathbf{S}}\bullet\sigma + \frac{\hbar}{2}(\vec{\mathbf{S}}\bullet\sigma)(\vec{\Omega}\bullet\sigma)$$

Last terms don't cancel if the *spin* $\mathbf{S}$ and *crank* $\mathbf{\Omega}$ *point in different directions*.

$$\mathbf{H}\rho - \rho\mathbf{H} = \frac{\hbar}{2}(\vec{\Omega}\bullet\sigma)(\vec{\mathbf{S}}\bullet\sigma) - \frac{\hbar}{2}(\vec{\mathbf{S}}\bullet\sigma)(\vec{\Omega}\bullet\sigma)$$

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = \frac{i\hbar}{2}(\vec{\Omega}\times\vec{\mathbf{S}})\bullet\sigma - \frac{i\hbar}{2}(\vec{\mathbf{S}}\times\vec{\Omega})\bullet\sigma$$

$$i\hbar\frac{\partial}{\partial t}\left(\frac{N}{2}\mathbf{1} + \vec{\mathbf{S}}\bullet\sigma\right) = i\hbar\dot{\vec{\mathbf{S}}}\bullet\sigma = i\hbar(\vec{\Omega}\times\mathbf{S})\bullet\sigma$$

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\rho = \frac{1}{2}\left(N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)$$
$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\boldsymbol{\Omega}}}{2}\bullet\boldsymbol{\sigma}$$

Ket equation (time forward) and "daggered" bra-equation (time reversed).

$$i\hbar\left|\dot{\Psi}\right\rangle = \mathbf{H}\left|\Psi\right\rangle, \qquad \Leftarrow \textit{Daggar}^{\dagger} \Rightarrow \qquad -i\hbar\left\langle\dot{\Psi}\right| = \left\langle\Psi\right|\mathbf{H}$$

Note: $\mathbf{H}^{\dagger} = \mathbf{H}$.
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Combining these gives a time derivative of the density operator $\rho = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right| + i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right| = \mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

The result is called a *Bloch equation*.

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = \mathbf{H}\rho - \rho\mathbf{H} = \left[\mathbf{H},\rho\right]$$

$$(\mathbf{A}\bullet\boldsymbol{\sigma})(\mathbf{B}\bullet\boldsymbol{\sigma}) = A_\alpha B_\beta \sigma_\alpha\sigma_\beta = A_\alpha B_\beta\left(\delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma\right)$$
$$= A_\alpha B_\alpha + i\varepsilon_{\alpha\beta\gamma}A_\alpha B_\beta\sigma_\gamma$$
$$= \mathbf{A}\bullet\mathbf{B} + i(\mathbf{A}\times\mathbf{B})\cdot\boldsymbol{\sigma}$$

Given ρ and $\mathbf{H}$ in terms *spin* $\mathbf{S}$-vector and *crank* $\boldsymbol{\Omega}$-vector:

$$\mathbf{H}\rho = \left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\frac{N}{2}\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma} + \hbar\Omega_0\vec{\mathbf{S}}\bullet\boldsymbol{\sigma} + \frac{\hbar}{2}(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma})(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma})$$

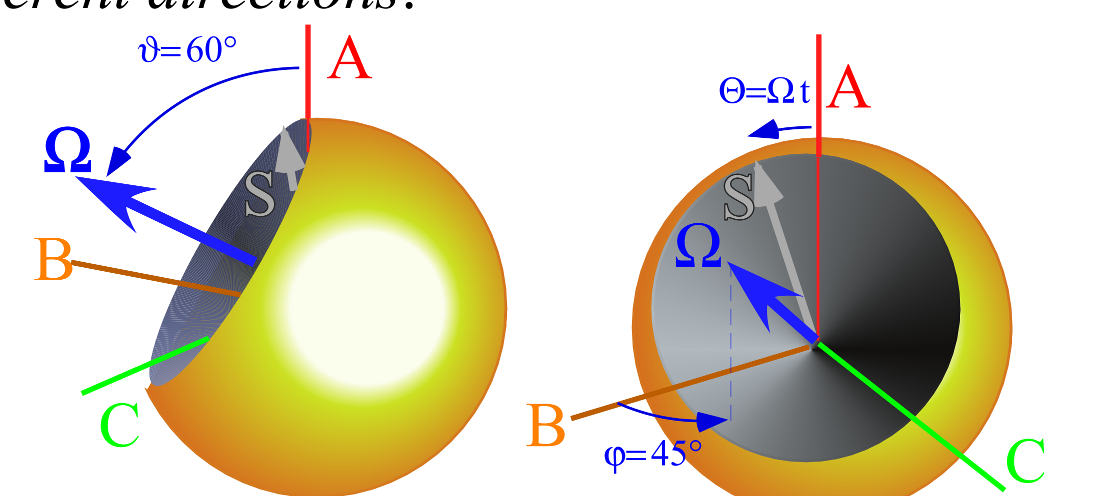
$$\rho\mathbf{H} = \left(\frac{N}{2}\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)\left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma} + \hbar\Omega_0\vec{\mathbf{S}}\bullet\boldsymbol{\sigma} + \frac{\hbar}{2}(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma})(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma})$$

Last terms don't cancel if the *spin* $\mathbf{S}$ and *crank* $\boldsymbol{\Omega}$ *point in different directions*.

$$\mathbf{H}\rho - \rho\mathbf{H} = \frac{\hbar}{2}(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma})(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}) - \frac{\hbar}{2}(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma})(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma})$$

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = \frac{i\hbar}{2}(\vec{\boldsymbol{\Omega}}\times\vec{\mathbf{S}})\bullet\boldsymbol{\sigma} - \frac{i\hbar}{2}(\vec{\mathbf{S}}\times\vec{\boldsymbol{\Omega}})\bullet\boldsymbol{\sigma}$$

$$i\hbar\frac{\partial}{\partial t}\left(\frac{N}{2}\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right) = i\hbar\dot{\vec{\mathbf{S}}}\bullet\boldsymbol{\sigma} = i\hbar(\vec{\boldsymbol{\Omega}}\times\mathbf{S})\bullet\boldsymbol{\sigma}$$



Factoring out $\bullet\sigma$ gives a classical/quantum *gyro-precession equation*. $$\frac{\partial\vec{\mathbf{S}}}{\partial t} = \dot{\vec{\mathbf{S}}} = \vec{\boldsymbol{\Omega}}\times\vec{\mathbf{S}}$$

*Review: Fundamental Euler **R**(αβγ) and Darboux **R**[φϑΘ] representations of U(2) and R(3)*

*Euler **R**(αβγ) derived from Darboux **R**[φϑΘ] and vice versa*

*Euler **R**(αβγ) rotation Θ=0−4π-sequence [φϑ] fixed*

*R(3)-U(2) slide rule for converting **R**(αβγ) ↔ **R**[φϑΘ] and Sundial*

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

→ *Quick U(2) way to find eigen-solutions for 2-by-2* **H**

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes

AB-Type motion and Wigner's Avoided-Symmetry-Crossings

ABC-Type elliptical polarized motion

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates

Euler Angle (αβγ) ellipse coordinates

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\mathbf{\Omega} = \mathbf{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix} = \mathbf{H}$$

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\mathbf{\Omega} = \mathbf{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\boldsymbol{\Omega} = \boldsymbol{\Theta}/t$

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$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{H} = \Omega_0 \quad \mathbf{1} \quad + \Omega_A \quad \mathbf{S}_A \quad + \Omega_B \quad \mathbf{S}_B \quad + \Omega_C \quad \mathbf{S}_C \quad = \Omega_0 \mathbf{1} + \vec{\Omega} \bullet \mathbf{S}$$

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\mathbf{\Omega} = \mathbf{\Theta}/t$

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$$\mathbf{H} = \Omega_0 \quad \mathbf{1} \quad + \Omega_A \quad \mathbf{S}_A \quad + \Omega_B \quad \mathbf{S}_B \quad + \Omega_C \quad \mathbf{S}_C \quad = \Omega_0\mathbf{1} + \vec{\Omega}\bullet\mathbf{S}$$

Step 2.Convert Cartesian to polar form: $(\Omega_A = \Omega\cos\vartheta,\quad \Omega_B = \Omega\cos\varphi\sin\vartheta,\quad \Omega_C = \Omega\sin\varphi\sin\vartheta)$

where: $\Omega_0 = \dfrac{A+D}{2}$ *and:* $\Omega = \sqrt{\Omega_A{}^2 + \Omega_B{}^2 + \Omega_C{}^2} = \sqrt{(A-D)^2 + 4B^2 + 4C^2}$

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\boldsymbol{\Omega} = \boldsymbol{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{H} = \Omega_0 \quad \mathbf{1} \quad + \Omega_A \quad \mathbf{S}_A \quad + \Omega_B \quad \mathbf{S}_B \quad + \Omega_C \quad \mathbf{S}_C \quad = \Omega_0 \mathbf{1} + \vec{\Omega} \bullet \mathbf{S}$$

Step 2.Convert Cartesian to polar form: $(\Omega_A = \Omega\cos\vartheta, \quad \Omega_B = \Omega\cos\varphi\sin\vartheta, \quad \Omega_C = \Omega\sin\varphi\sin\vartheta)$

where: $\Omega_0 = \frac{A+D}{2}$ *and:* $\Omega = \sqrt{\Omega_A{}^2 + \Omega_B{}^2 + \Omega_C{}^2} = \sqrt{(A-D)^2 + 4B^2 + 4C^2}$

Eigenvalues: $\Omega_\pm = \Omega_0 \pm \Omega/2$

$$= \frac{A + D \pm \sqrt{(A-D)^2 + 4B^2 + 4C^2}}{2}$$

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\mathbf{\Omega} = \mathbf{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

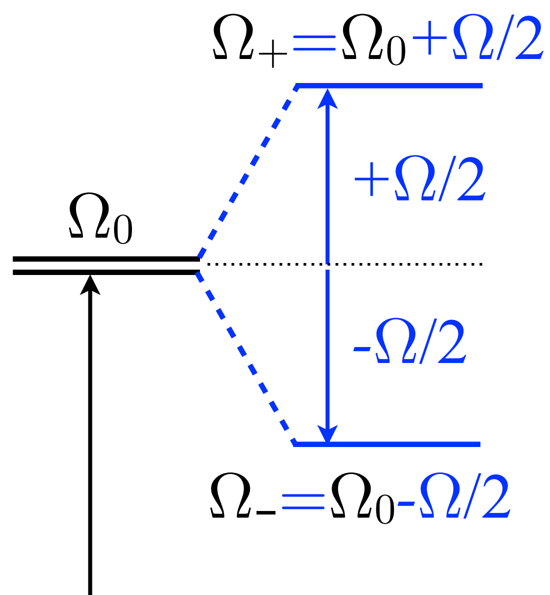
$$\mathbf{H} = \Omega_0 \mathbf{1} + \Omega_A \mathbf{S}_A + \Omega_B \mathbf{S}_B + \Omega_C \mathbf{S}_C = \Omega_0 \mathbf{1} + \vec{\Omega} \bullet \mathbf{S}$$

Step 2. Convert Cartesian to polar form: $(\Omega_A = \Omega\cos\vartheta,\quad \Omega_B = \Omega\cos\varphi\sin\vartheta,\quad \Omega_C = \Omega\sin\varphi\sin\vartheta)$

where: $\Omega_0 = \frac{A+D}{2}$ *and:* $\Omega = \sqrt{\Omega_A^2 + \Omega_B^2 + \Omega_C^2} = \sqrt{(A-D)^2 + 4B^2 + 4C^2}$

Eigenvalues: $\Omega_\pm = \Omega_0 \pm \Omega/2$

$$= \frac{A+D \pm \sqrt{(A-D)^2 + 4B^2 + 4C^2}}{2}$$



Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\boldsymbol{\Omega} = \boldsymbol{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{H} = \Omega_0 \quad \mathbf{1} \quad + \Omega_A \quad \mathbf{S}_A \quad + \Omega_B \quad \mathbf{S}_B \quad + \Omega_C \quad \mathbf{S}_C \quad = \Omega_0 \mathbf{1} + \vec{\Omega} \bullet \mathbf{S}$$

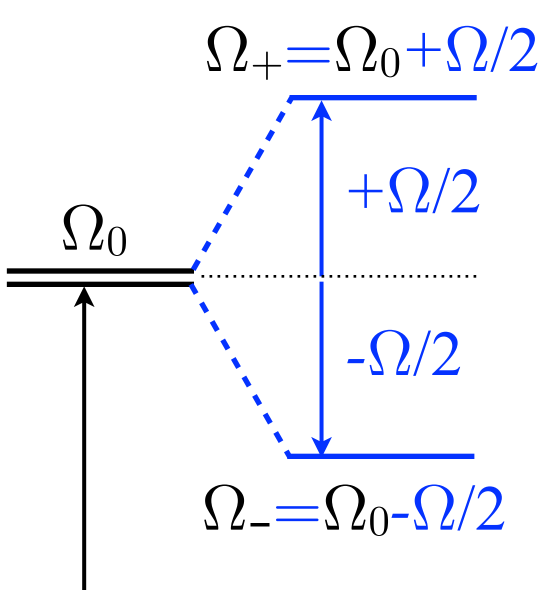
Step 2.Convert Cartesian to polar form: $(\Omega_A = \Omega\cos\vartheta,\quad \Omega_B = \Omega\cos\varphi\sin\vartheta,\quad \Omega_C = \Omega\sin\varphi\sin\vartheta)$

where: $\Omega_0 = \frac{A+D}{2}$ *and:* $\Omega = \sqrt{\Omega_A^{\,2} + \Omega_B^{\,2} + \Omega_C^{\,2}} = \sqrt{(A-D)^2 + 4B^2 + 4C^2}$

and: $\vartheta = \cos^{-1}(\Omega_A/\Omega)$, *and:* $\varphi = \cos^{-1}(\Omega_B/\Omega\sin\vartheta) = \cos^{-1}[\Omega_B/\sqrt{\Omega_B^{\,2} + \Omega_C^{\,2}}\,]$

Eigenvalues: $\Omega_\pm = \Omega_0 \pm \Omega/2$

$$= \frac{A+D \pm \sqrt{(A-D)^2 + 4B^2 + 4C^2}}{2}$$



Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\mathbf{\Omega} = \mathbf{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{H} = \Omega_0 \mathbf{1} + \Omega_A \mathbf{S}_A + \Omega_B \mathbf{S}_B + \Omega_C \mathbf{S}_C = \Omega_0\mathbf{1} + \vec{\Omega}\bullet\mathbf{S}$$

Step 2.Convert Cartesian to polar form: $(\Omega_A = \Omega\cos\vartheta,\quad \Omega_B = \Omega\cos\varphi\,\sin\vartheta,\quad \Omega_C = \Omega\sin\varphi\,\sin\vartheta)$

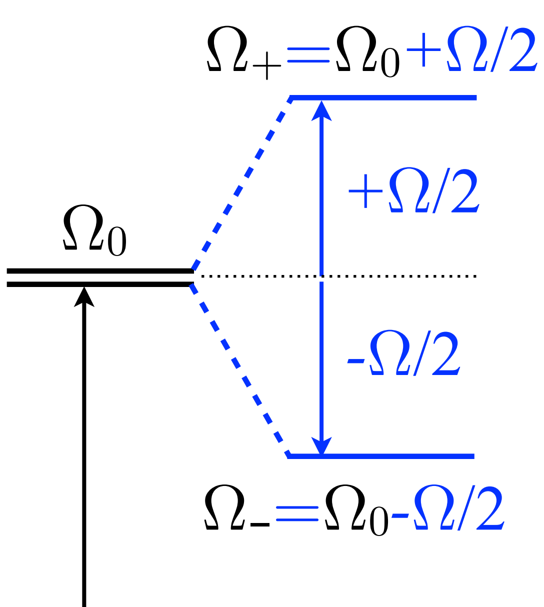
where: $\Omega_0 = \dfrac{A+D}{2}$ *and:* $\Omega = \sqrt{\Omega_A{}^2 + \Omega_B{}^2 + \Omega_C{}^2} = \sqrt{(A-D)^2 + 4B^2 + 4C^2}$

Eigenvalues: $\Omega_\pm = \Omega_0 \pm \Omega/2$

$$= \frac{A+D \pm \sqrt{(A-D)^2 + 4B^2 + 4C^2}}{2}$$

and: $\vartheta = \cos^{-1}(\Omega_A/\Omega)$, *and:* $\varphi = \cos^{-1}(\Omega_B/\Omega\sin\vartheta) = \cos^{-1}[\Omega_B/\sqrt{\Omega_B{}^2 + \Omega_C{}^2}\,]$

or: $\vartheta = \cos^{-1}[(A\text{-}D)/\sqrt{(A-D)^2 + 4B^2 + 4C^2}\,]$, $\quad\varphi = \cos^{-1}[B/\sqrt{B^2 + C^2}\,]$



Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\mathbf{\Omega} = \mathbf{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{H} = \Omega_0 \mathbf{1} + \Omega_A \mathbf{S}_A + \Omega_B \mathbf{S}_B + \Omega_C \mathbf{S}_C = \Omega_0 \mathbf{1} + \vec{\Omega} \bullet \mathbf{S}$$

Step 2. Convert Cartesian to polar form: $(\Omega_A = \Omega\cos\vartheta,\ \ \Omega_B = \Omega\cos\varphi\sin\vartheta,\ \ \Omega_C = \Omega\sin\varphi\sin\vartheta)$

where: $\Omega_0 = \dfrac{A+D}{2}$ *and:* $\Omega = \sqrt{\Omega_A{}^2 + \Omega_B{}^2 + \Omega_C{}^2} = \sqrt{(A-D)^2 + 4B^2 + 4C^2}$

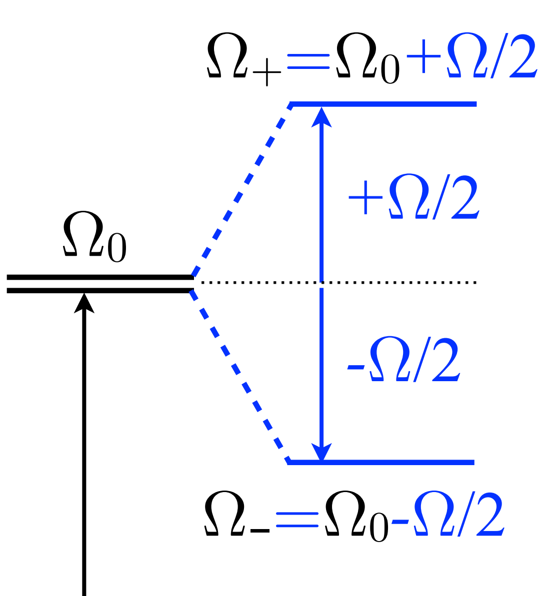
and: $\vartheta = \cos^{-1}(\Omega_A/\Omega)$, *and:* $\varphi = \cos^{-1}(\Omega_B/\Omega\sin\vartheta) = \cos^{-1}[\Omega_B/\sqrt{\Omega_B{}^2 + \Omega_C{}^2}]$

or: $\vartheta = \cos^{-1}[(A-D)/\sqrt{(A-D)^2 + 4B^2 + 4C^2}]$, $\varphi = \cos^{-1}[B/\sqrt{B^2 + C^2}]$

Eigenvalues: $\Omega_\pm = \Omega_0 \pm \Omega/2 = \dfrac{A+D \pm \sqrt{(A-D)^2 + 4B^2 + 4C^2}}{2}$

Step 3. To find eigenvectors replace Euler angles (azimuth α*, polar* β*) of Euler-state*

$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$



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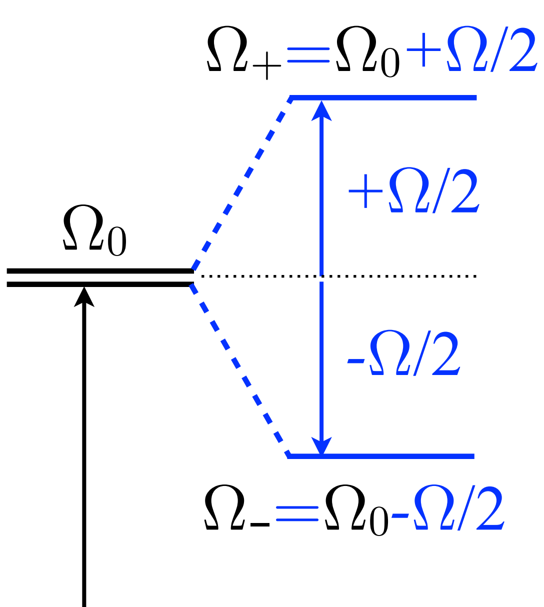

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\mathbf{\Omega} = \mathbf{\Theta}/t$

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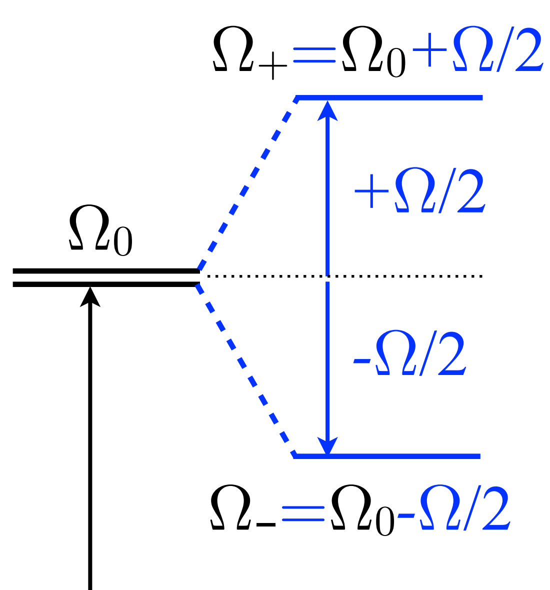
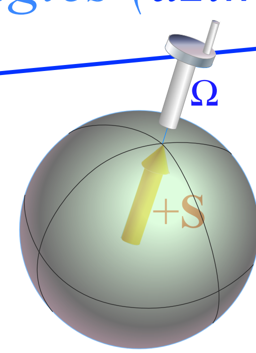
$$\left|\Omega_+\right\rangle = \begin{pmatrix} e^{-i\frac{\varphi}{2}}\cos\frac{\vartheta}{2} \\ e^{i\frac{\varphi}{2}}\sin\frac{\vartheta}{2} \end{pmatrix}$$





Spin +S
Up-Crank

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

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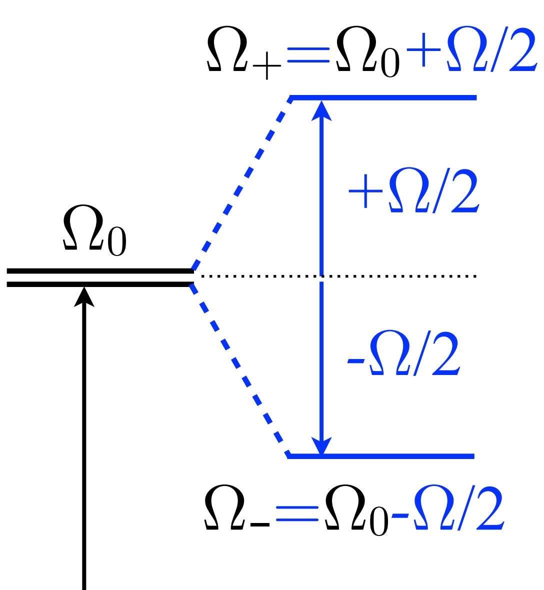

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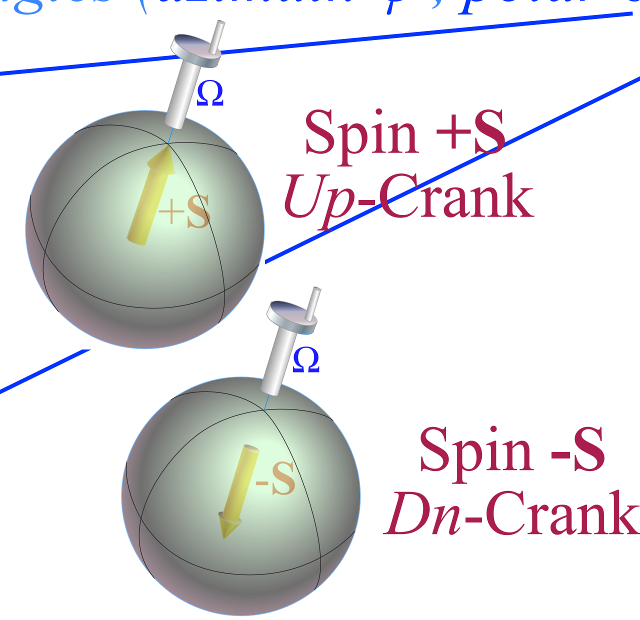

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

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$$\mathbf{H} = \Omega_0 \mathbf{1} + \Omega_A \mathbf{S}_A + \Omega_B \mathbf{S}_B + \Omega_C \mathbf{S}_C = \Omega_0 \mathbf{1} + \vec{\Omega} \bullet \mathbf{S}$$

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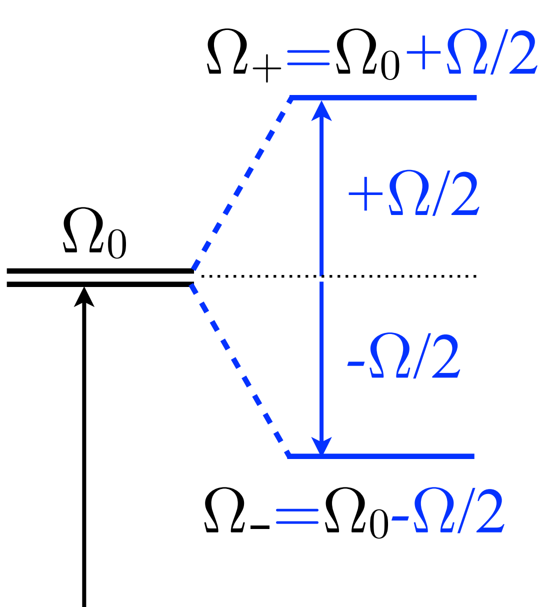

$$|\Omega_+\rangle = \begin{pmatrix} e^{-i\frac{\varphi}{2}}\cos\frac{\vartheta}{2} \\ e^{i\frac{\varphi}{2}}\sin\frac{\vartheta}{2} \end{pmatrix}$$

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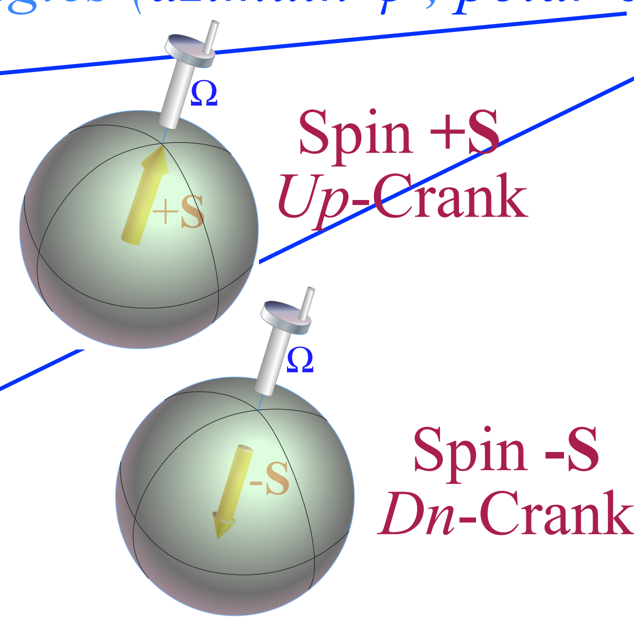

More reliable computation:

$\varphi = atan2(C, B)$

[$\tan^{-1}(C/B)$ *is unreliable*]

$\vartheta = atan2(2\sqrt{B^2 + C^2}, A-D)$

Quick U(2) way example for 2-by-2 **H**

Can you write down all eigensolutions to the following **H** -matrix in 60 seconds?

$$\mathbf{H} = \begin{pmatrix} 12 & \sqrt{6}(1-i) \\ \sqrt{6}(1+i) & 8 \end{pmatrix} = \begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix}$$

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$$\Omega_0 = \frac{A+D}{2} = 10$$

and: $\Omega = \sqrt{\Omega_A{}^2 + \Omega_B{}^2 + \Omega_C{}^2} = \sqrt{(A-D)^2 + 4B^2 + 4C^2} = \sqrt{(4)^2 + 4\sqrt{6}^2 + 4\sqrt{6}^2} = \sqrt{16 + 24 + 24} = \sqrt{64} = 8$

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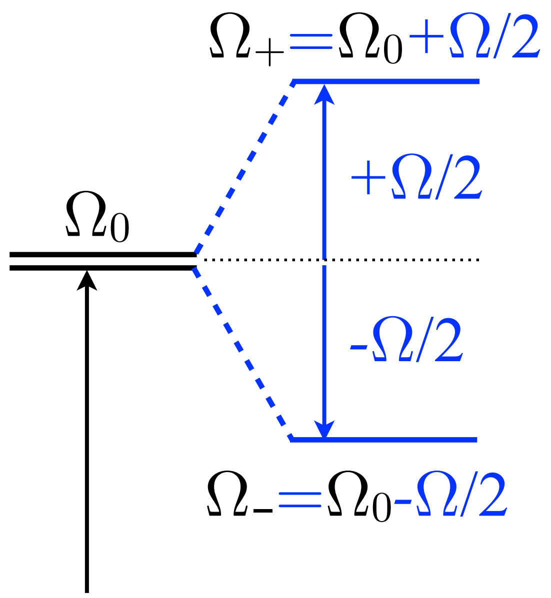

$\underline{|eigenvalue-1}$

$$\omega_\uparrow = 10 + \sqrt{\left(\frac{12-8}{2}\right)^2 + \left(\sqrt{6}\right)^2 + \left(\sqrt{6}\right)^2}$$
$$= 10 + 4 = 14$$

$\underline{|eigenvalue-2}$

$$\omega_\downarrow = 10 - \sqrt{\left(\frac{12-8}{2}\right)^2 + \left(\sqrt{6}\right)^2 + \left(\sqrt{6}\right)^2}$$
$$= 10 - 4 = 6$$

Quick U(2) way example for 2-by-2 **H**

Can you write down all eigensolutions to the following **H** -matrix in 60 seconds?

$$\mathbf{H}=\begin{pmatrix} 12 & \sqrt{6}(1-i) \\ \sqrt{6}(1+i) & 8 \end{pmatrix}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}=\begin{pmatrix} 10+4\cos\frac{\pi}{3} & 4\cos\frac{\pi}{4}\sin\frac{\pi}{3}-i4\sin\frac{\pi}{4}\sin\frac{\pi}{3} \\ 4\cos\frac{\pi}{4}\sin\frac{\pi}{3}+i4\sin\frac{\pi}{4}\sin\frac{\pi}{3} & 10-4\cos\frac{\pi}{3} \end{pmatrix}$$

$A=12,\quad B=\sqrt{6},\quad C=\sqrt{6},\quad D=8,$

Step 2.Convert Cartesian to polar form: ($\Omega_A=\Omega\cos\vartheta,\quad \Omega_B=\Omega\cos\varphi\sin\vartheta,\quad \Omega_C=\Omega\sin\varphi\sin\vartheta$)

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and: $\Omega=\sqrt{\Omega_A{}^2+\Omega_B{}^2+\Omega_C{}^2}=\sqrt{(A-D)^2+4B^2+4C^2}=\sqrt{(4)^2+4\sqrt{6}^2+4\sqrt{6}^2}=\sqrt{16+24+24}=\sqrt{64}=8$

or: $\vartheta=\cos^{-1}[(A\text{-}D)/\sqrt{(A-D)^2+4B^2+4C^2}\,]=\cos^{-1}[(4)/8]=\pi/3,$

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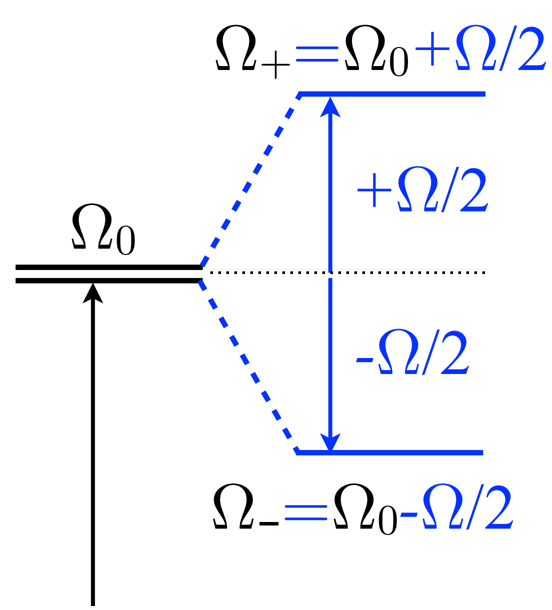

$\underline{|eigenvalue-1}$

$$\omega_\uparrow=10+\sqrt{\left(\frac{12-8}{2}\right)^2+\left(\sqrt{6}\right)^2+\left(\sqrt{6}\right)^2}$$
$$=10+4=14$$

$\underline{|eigenvalue-2}$

$$\omega_\downarrow=10-\sqrt{\left(\frac{12-8}{2}\right)^2+\left(\sqrt{6}\right)^2+\left(\sqrt{6}\right)^2}$$
$$=10-4=6$$

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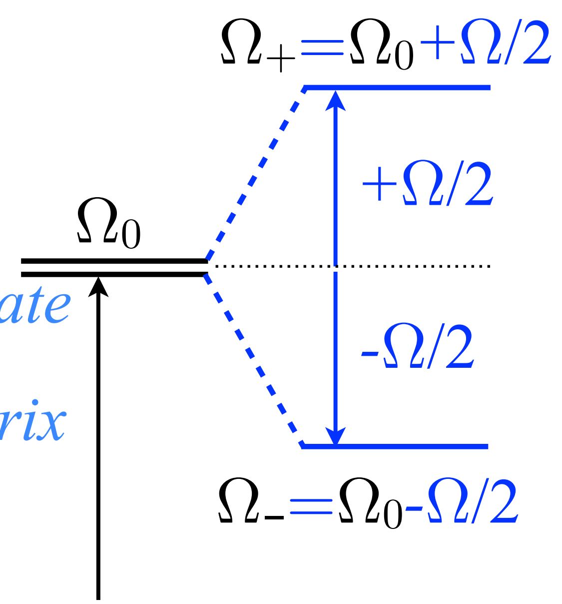

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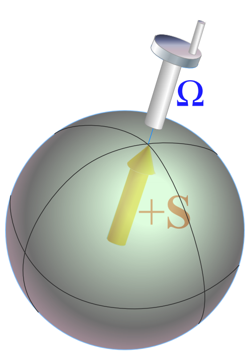
with the Darboux axis polar angles (azimuth φ , polar ϑ or ϑ±π) of **H***-matrix*

$\lfloor$ *eigenvalue* − 1

$$\omega_\uparrow=10+\sqrt{\left(\frac{12-8}{2}\right)^2+\left(\sqrt{6}\right)^2+\left(\sqrt{6}\right)^2}$$
$$=10+4=14$$

$\lfloor$ *eigenvector* − 1

$$|\uparrow\rangle=\begin{pmatrix}e^{-i\frac{\pi}{8}}\cos\frac{\pi}{6}\\ e^{+i\frac{\pi}{8}}\sin\frac{\pi}{6}\end{pmatrix}=\begin{pmatrix}1\\ e^{i\frac{\pi}{4}}\frac{\sqrt{3}}{3}\end{pmatrix}\frac{e^{-i\frac{\pi}{8}}\sqrt{3}}{2}$$

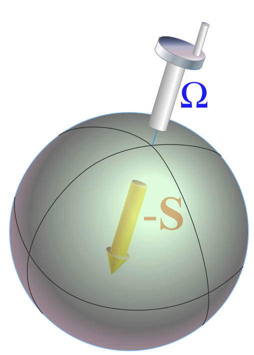


$\lfloor$ *eigenvalue* − 2

$$\omega_\downarrow=10-\sqrt{\left(\frac{12-8}{2}\right)^2+\left(\sqrt{6}\right)^2+\left(\sqrt{6}\right)^2}$$
$$=10-4=6$$

$\lfloor$ *eigenvector* − 2

$$|\downarrow\rangle=\begin{pmatrix}-e^{-i\frac{\pi}{8}}\sin\frac{\pi}{6}\\ e^{+i\frac{\pi}{8}}\cos\frac{\pi}{6}\end{pmatrix}=\begin{pmatrix}-e^{i\frac{\pi}{4}}\frac{\sqrt{3}}{3}\\ 1\end{pmatrix}\frac{e^{-i\frac{\pi}{8}}\sqrt{3}}{2}$$



Review: Fundamental Euler **R***(αβγ) and Darboux* **R**[φϑΘ] *representations of U(2) and R(3)*

Euler **R***(αβγ) derived from Darboux* **R**[φϑΘ] *and vice versa*

Euler **R***(αβγ) rotation* Θ=0−4π*-sequence* [φϑ] *fixed*

R(3)-U(2) slide rule for converting **R***(αβγ)* ↔ **R**[φϑΘ] *and Sundial*

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes

AB-Type motion and Wigner's Avoided-Symmetry-Crossings

ABC-Type elliptical polarized motion

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates

Euler Angle (αβγ) ellipse coordinates

The ABC's of U(2) dynamics

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Asymmetric Diagonal A-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^A|1\rangle & \langle 1|\mathbf{H}^A|2\rangle \\ \langle 2|\mathbf{H}^A|1\rangle & \langle 2|\mathbf{H}^A|2\rangle \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_A}{2}\sigma_A$$

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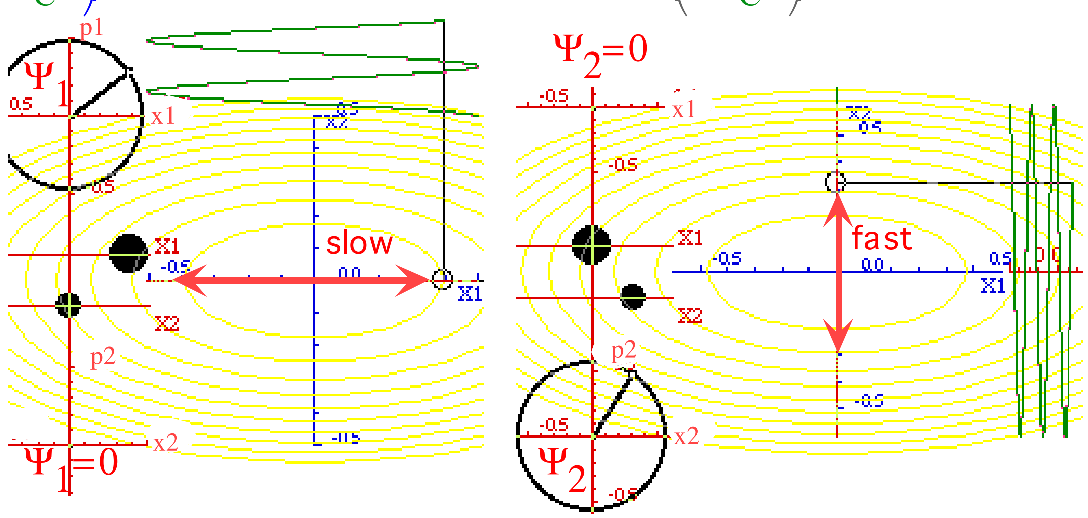

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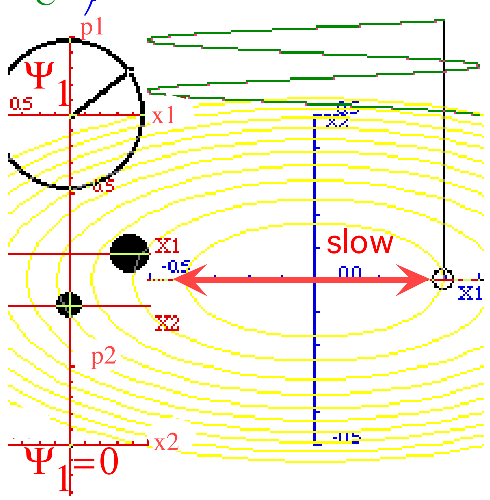

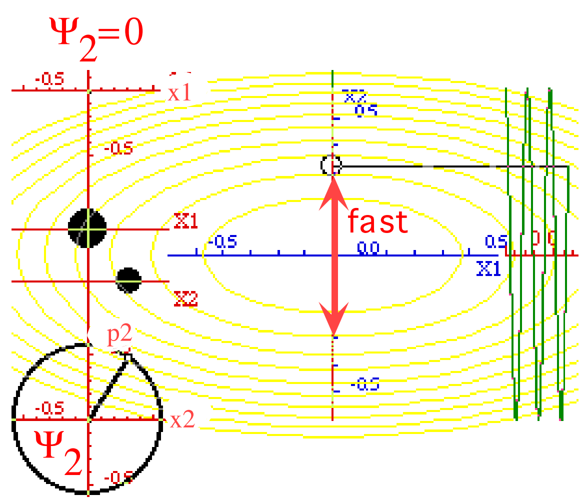

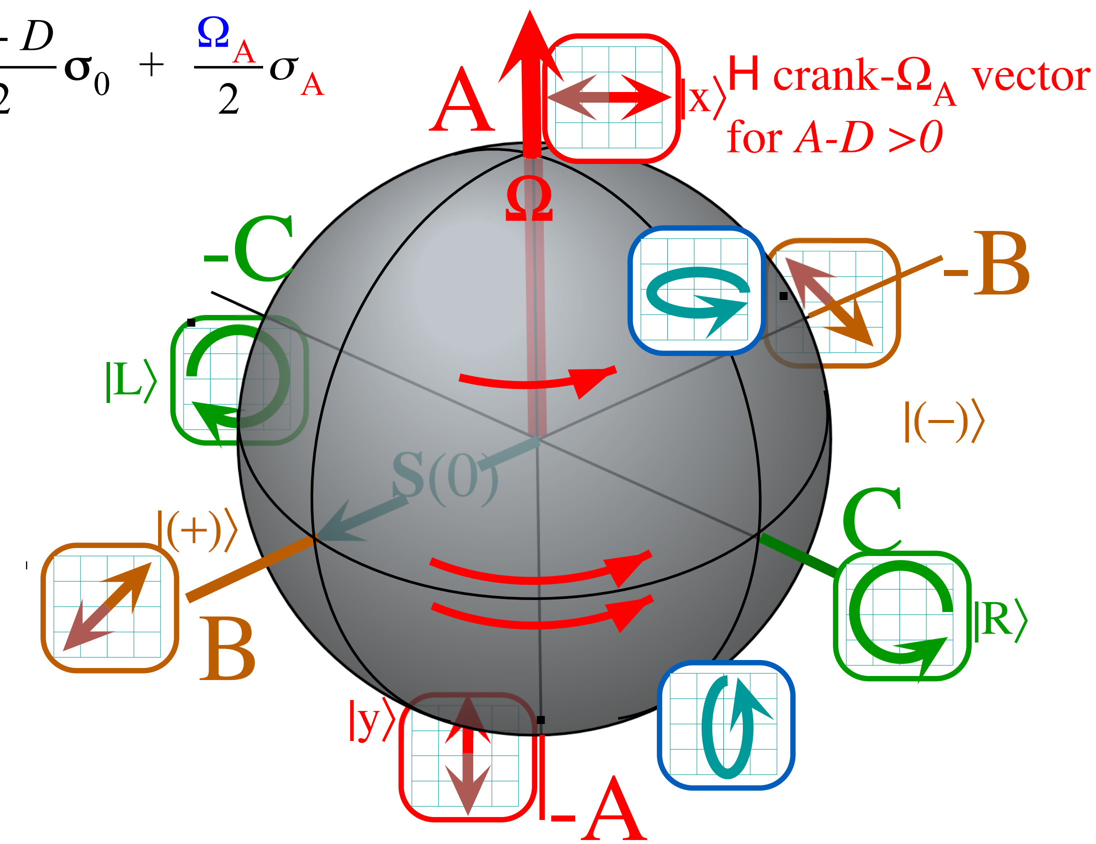

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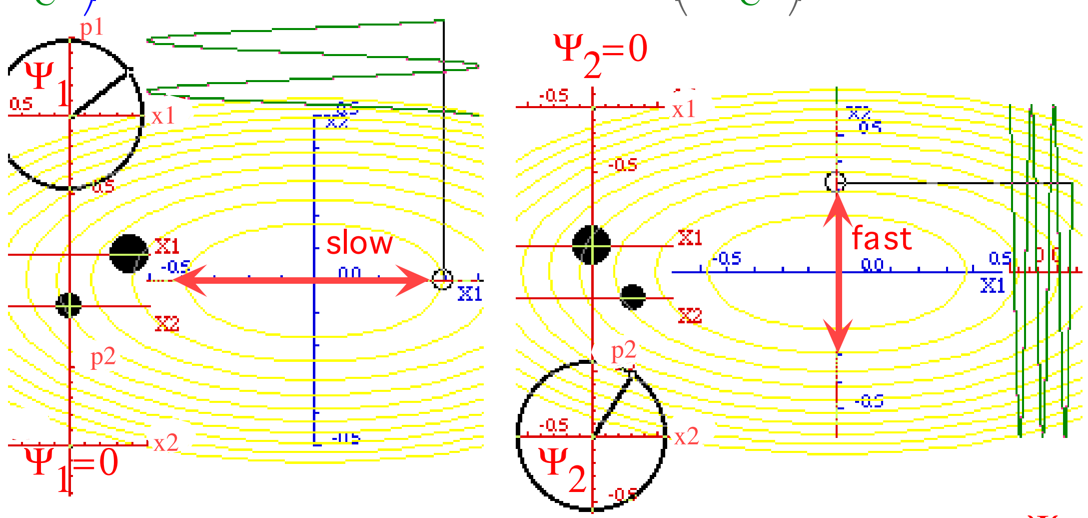
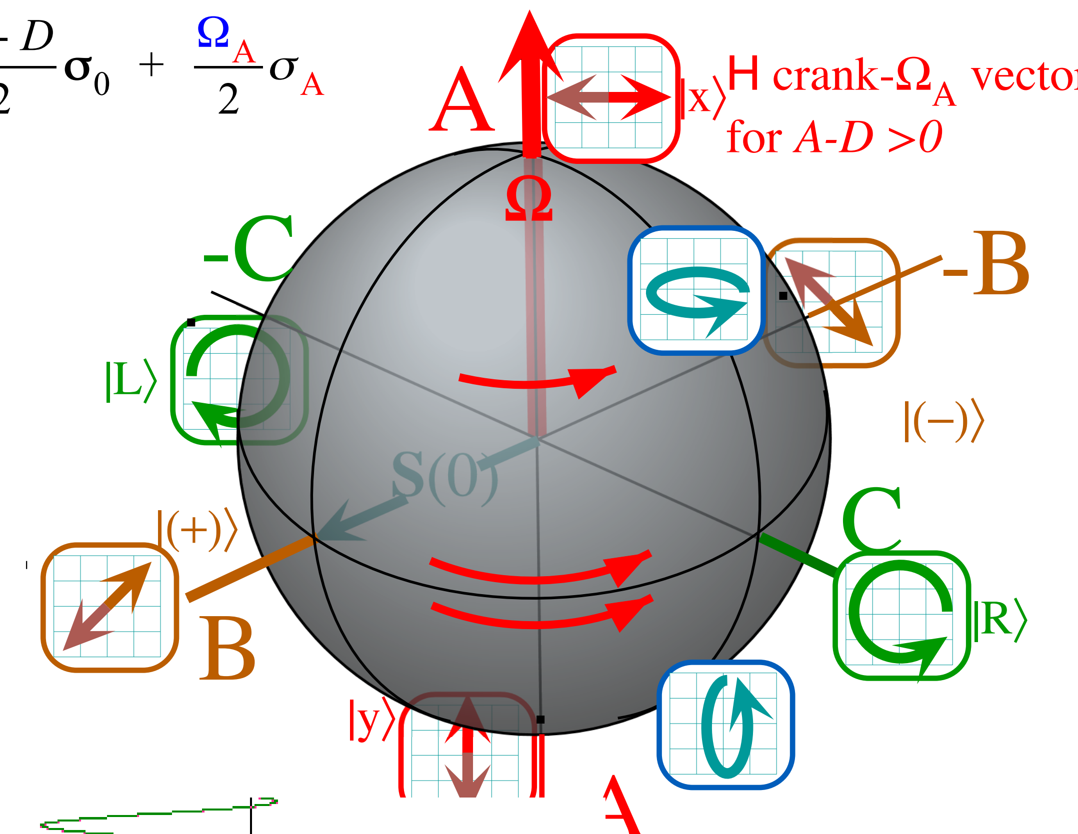
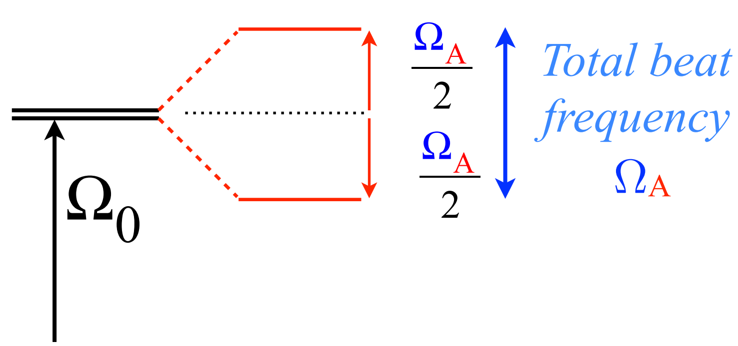
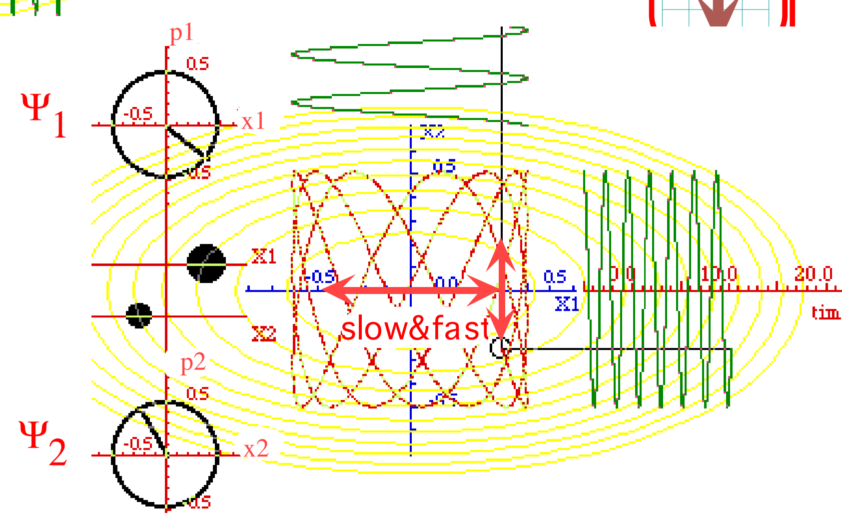




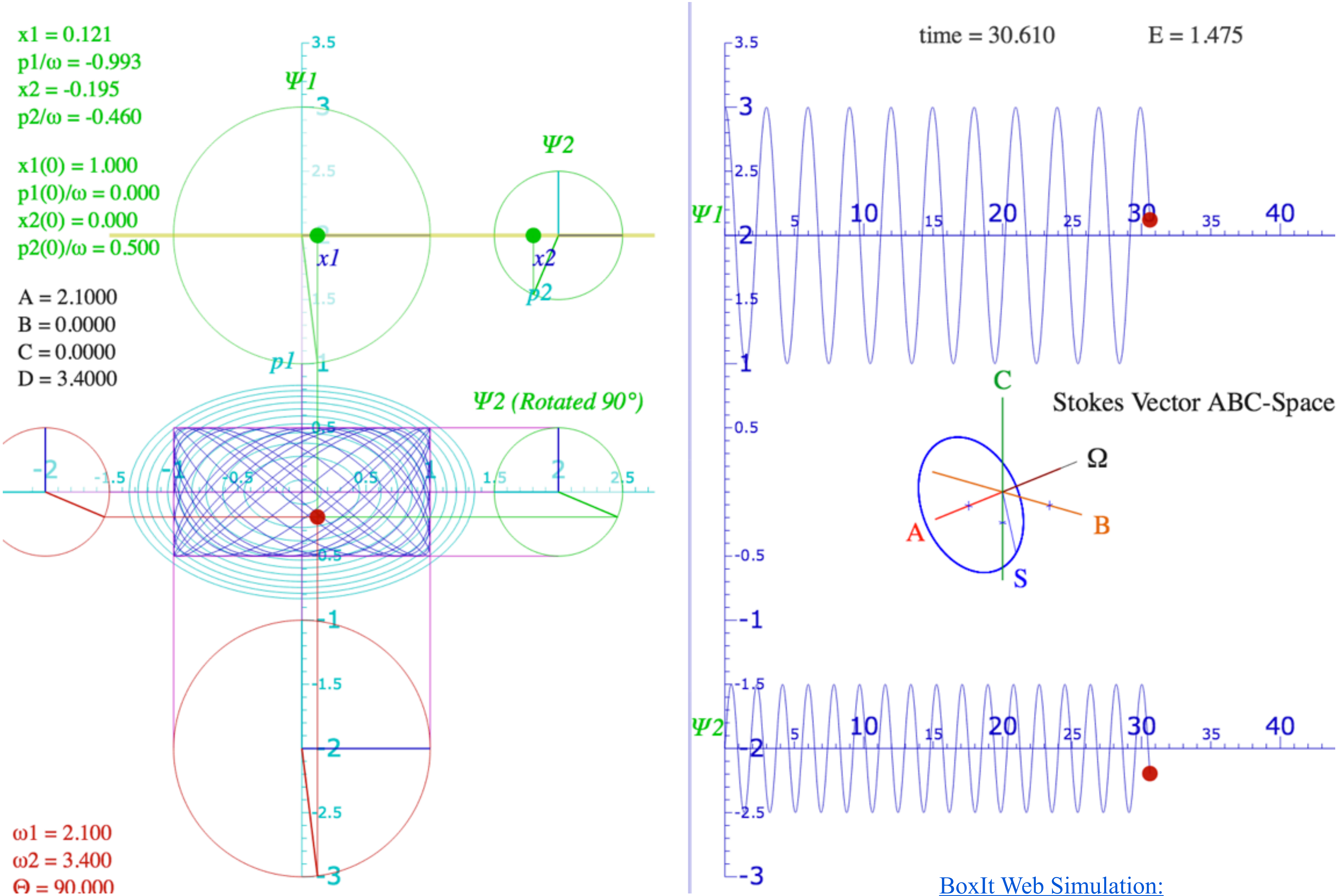
Beat dynamics:





BoxIt (A-Type)
Web Simulation

A-Type elliptical polarized motion



BoxIt Web Simulation:
A-Type with A=2.1, D=3.4

Review: Fundamental Euler **R***(αβγ) and Darboux* **R**[φϑΘ] *representations of U(2) and R(3)*

Euler **R***(αβγ) derived from Darboux* **R**[φϑΘ] *and vice versa*

Euler **R***(αβγ) rotation* Θ=0−4π*-sequence* [φϑ] *fixed*

R(3)-U(2) slide rule for converting **R***(αβγ)* ↔ **R**[φϑΘ] *and Sundial*

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

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➡ *Bilateral-Balanced B-Type motion*

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Bilateral-Balanced B-Type motion

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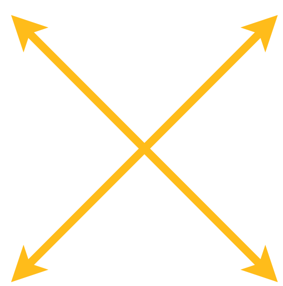

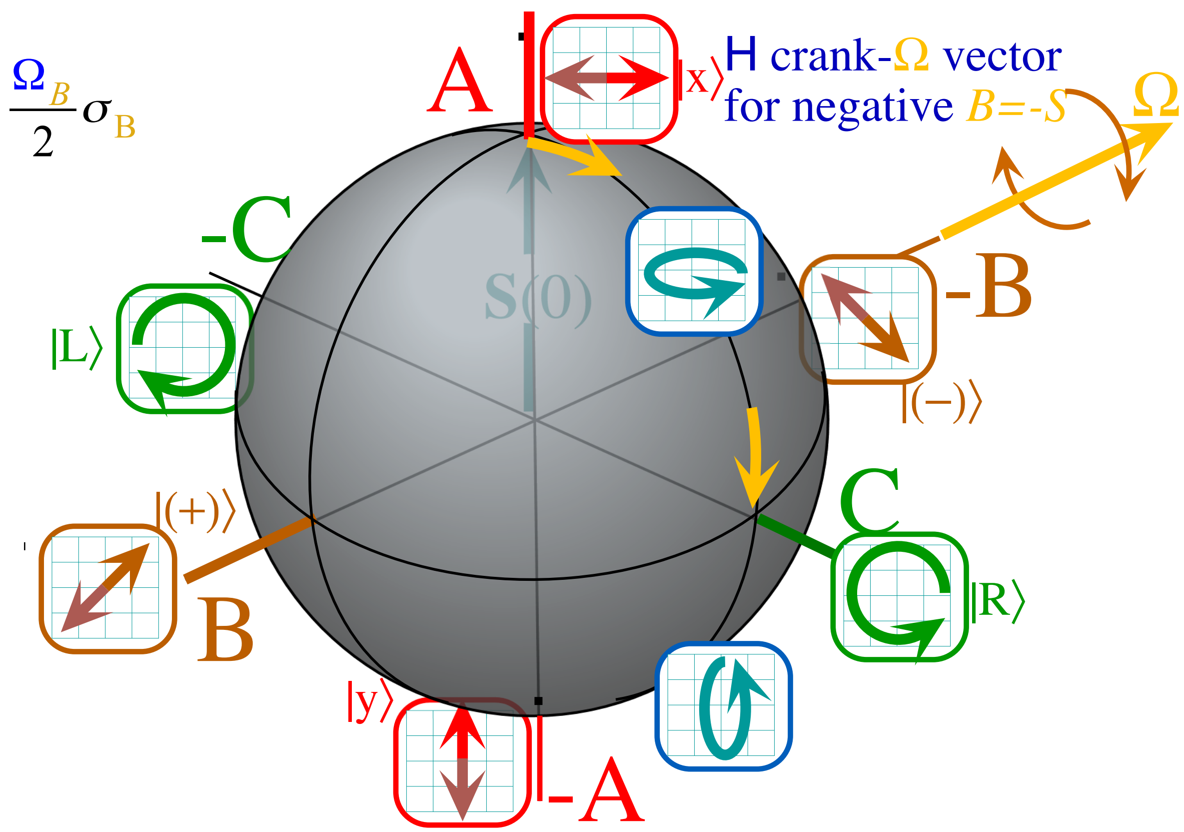

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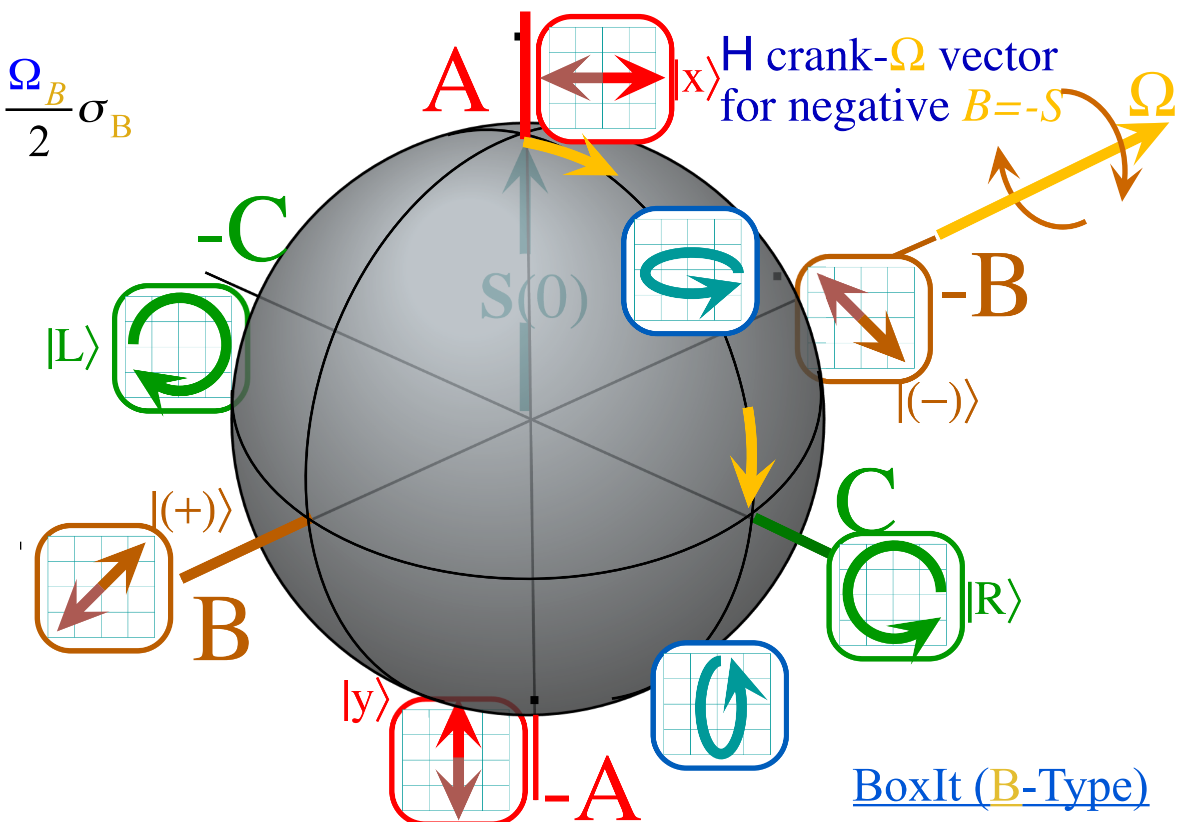
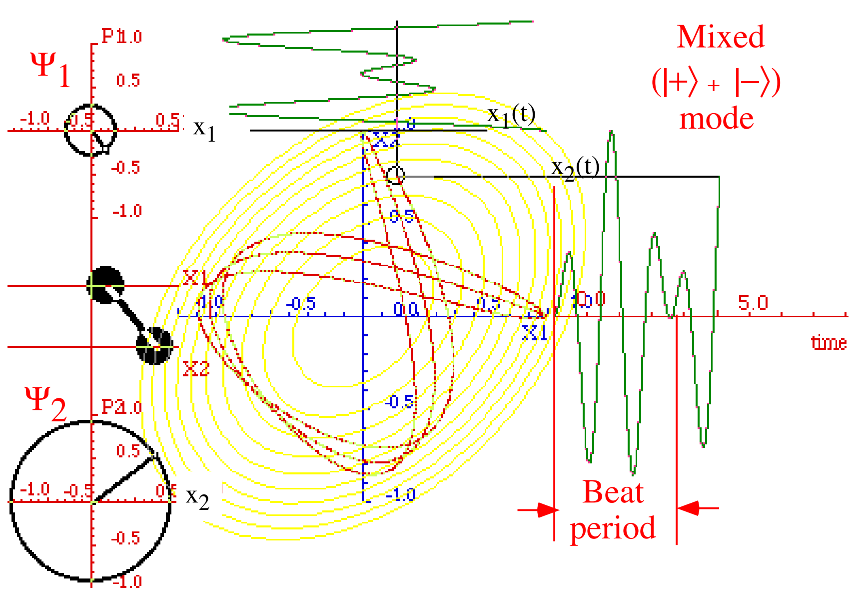


BoxIt (B-Type) Web Simulation

Beat dynamics:



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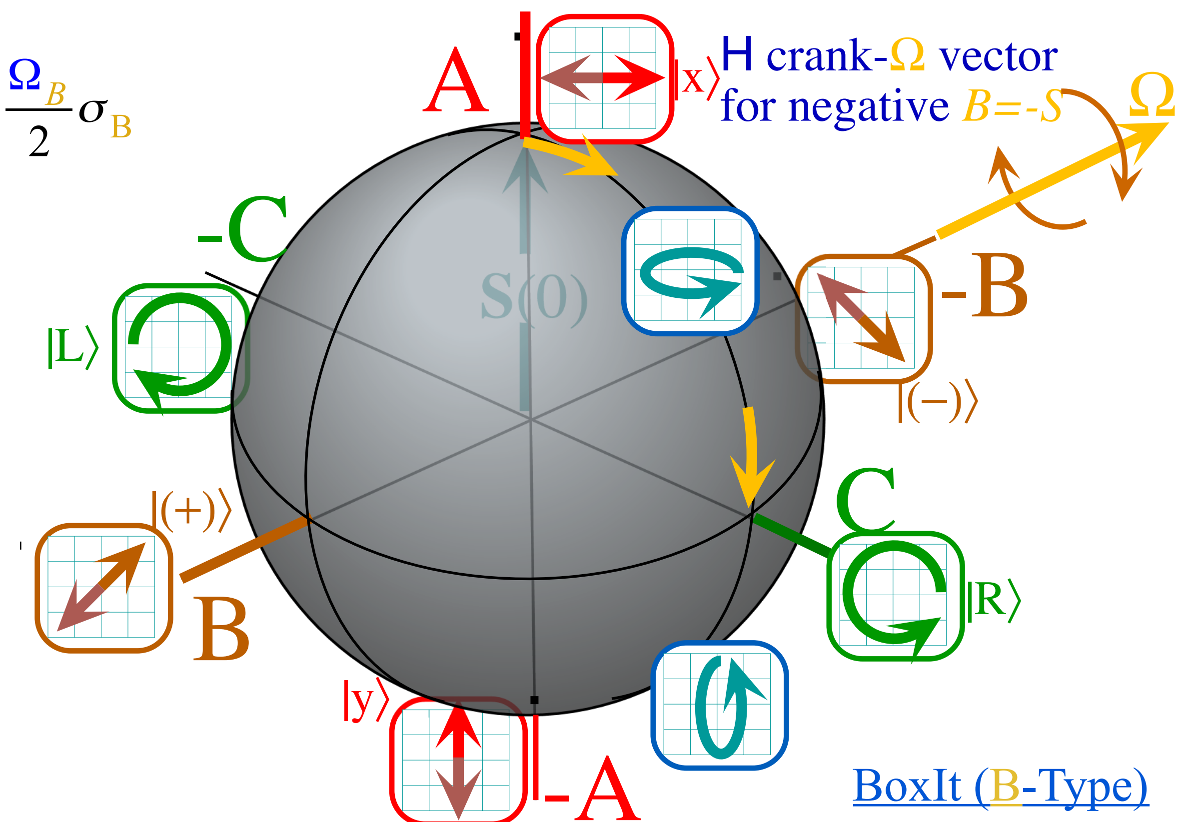

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Web Simulation

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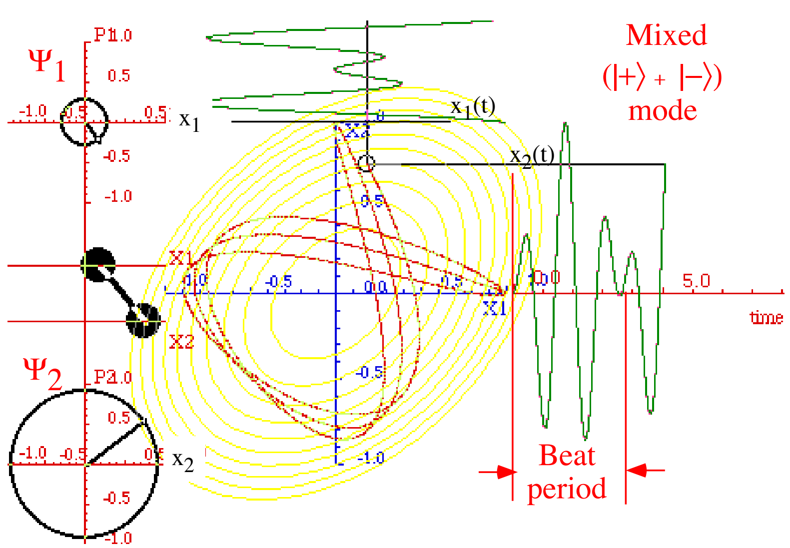
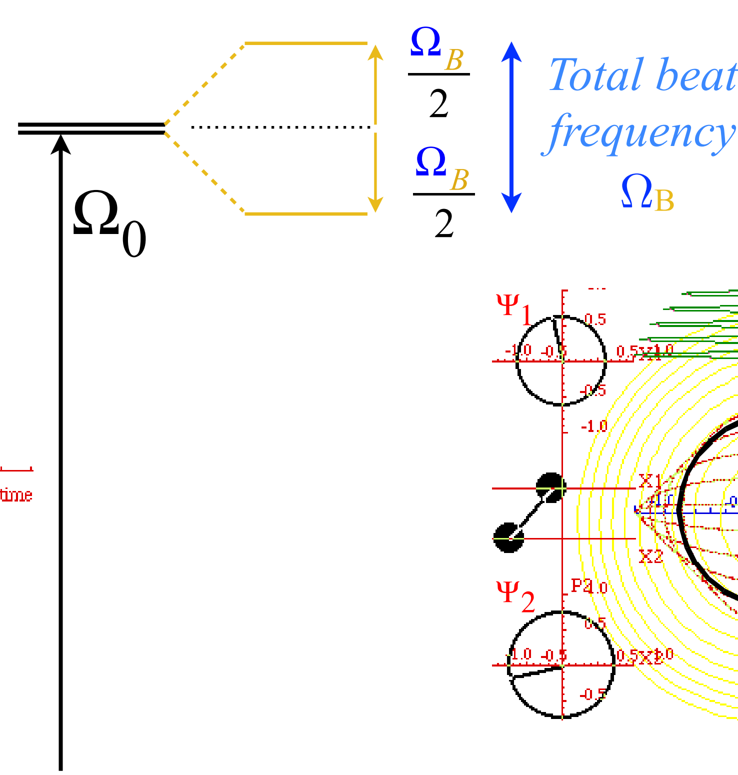
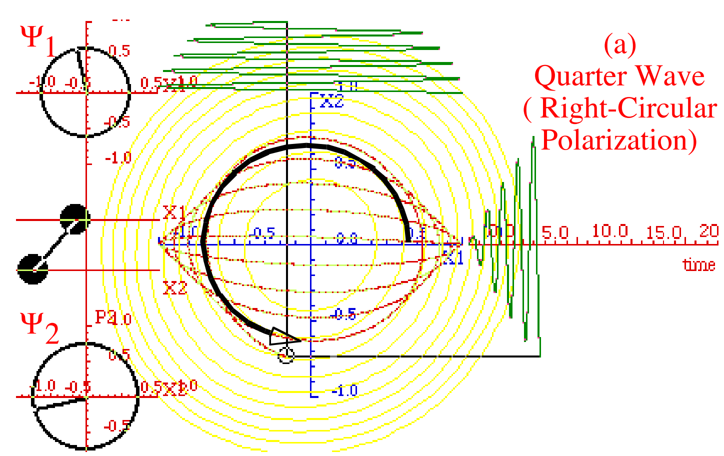
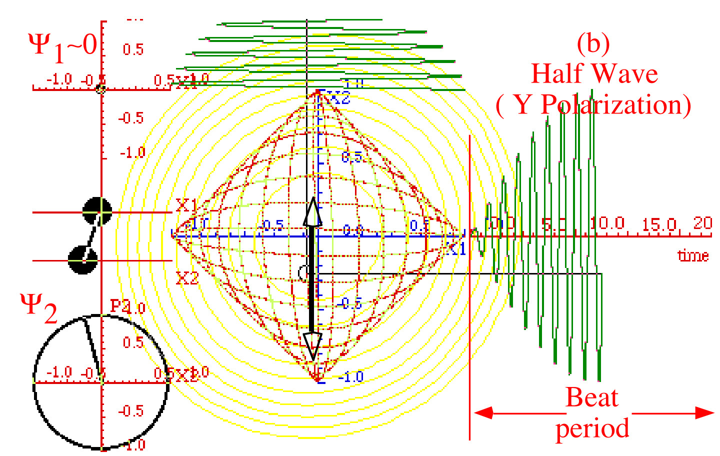
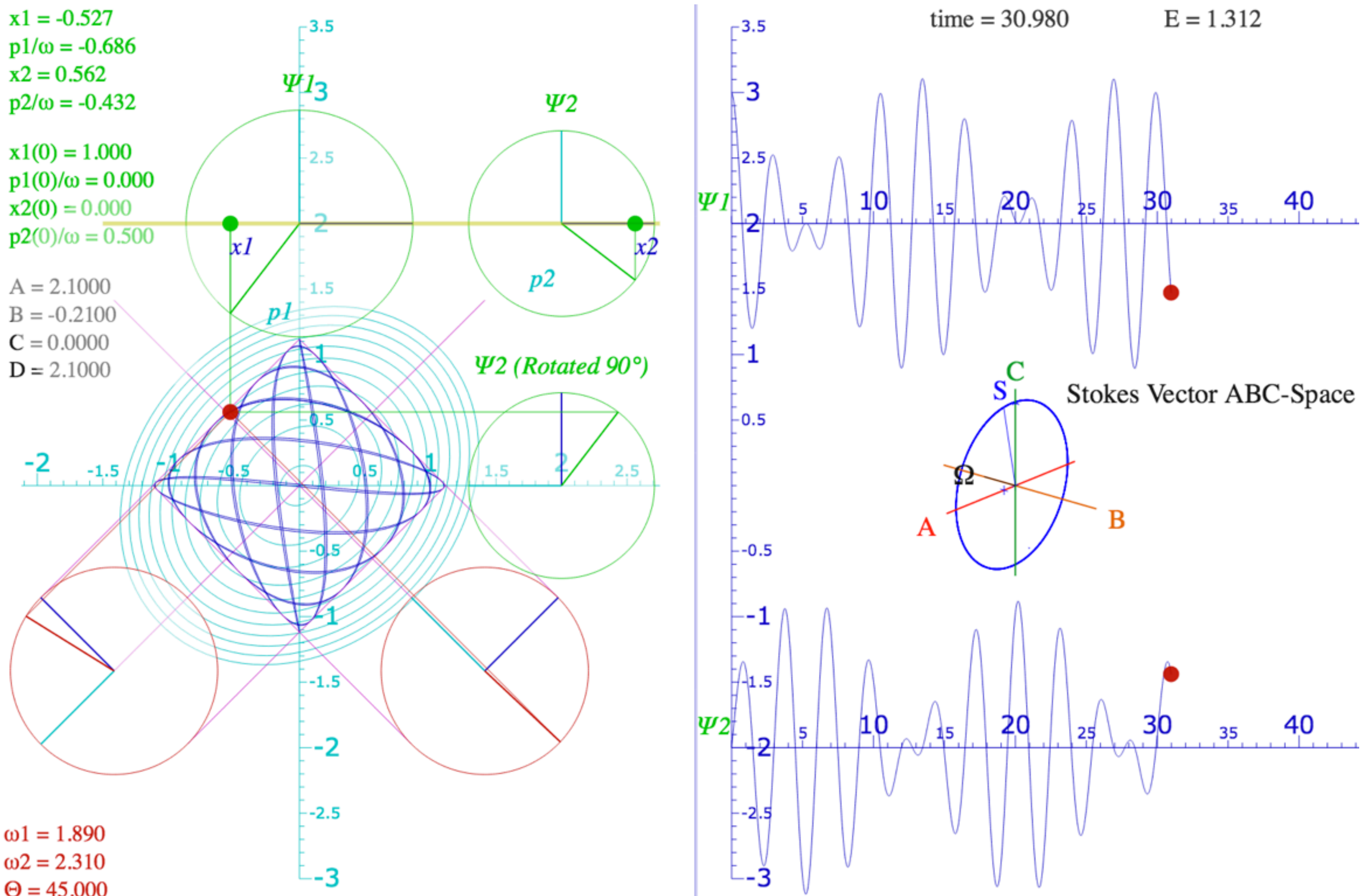








B-Type elliptical polarized motion



BoxIt Web Simulation:
B-Type with A, D=2.1; B=-0.21

Review: Fundamental Euler **R**$(\alpha\beta\gamma)$ *and Darboux* **R**$[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

Euler **R**$(\alpha\beta\gamma)$ *derived from Darboux* **R**$[\varphi\vartheta\Theta]$ *and vice versa*

Euler **R**$(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*

R(3)-U(2) slide rule for converting **R**$(\alpha\beta\gamma)$ ↔ **R**$[\varphi\vartheta\Theta]$ *and Sundial*

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

Bilateral-Balanced B-Type motion

→ *Circular-Coriolis… C-Type motion*

The ABC's of U(2) dynamics-Mixed modes

AB-Type motion and Wigner's Avoided-Symmetry-Crossings

ABC-Type elliptical polarized motion

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates

Euler Angle $(\alpha\beta\gamma)$ *ellipse coordinates*

The ABC's of U(2) dynamics

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$
$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

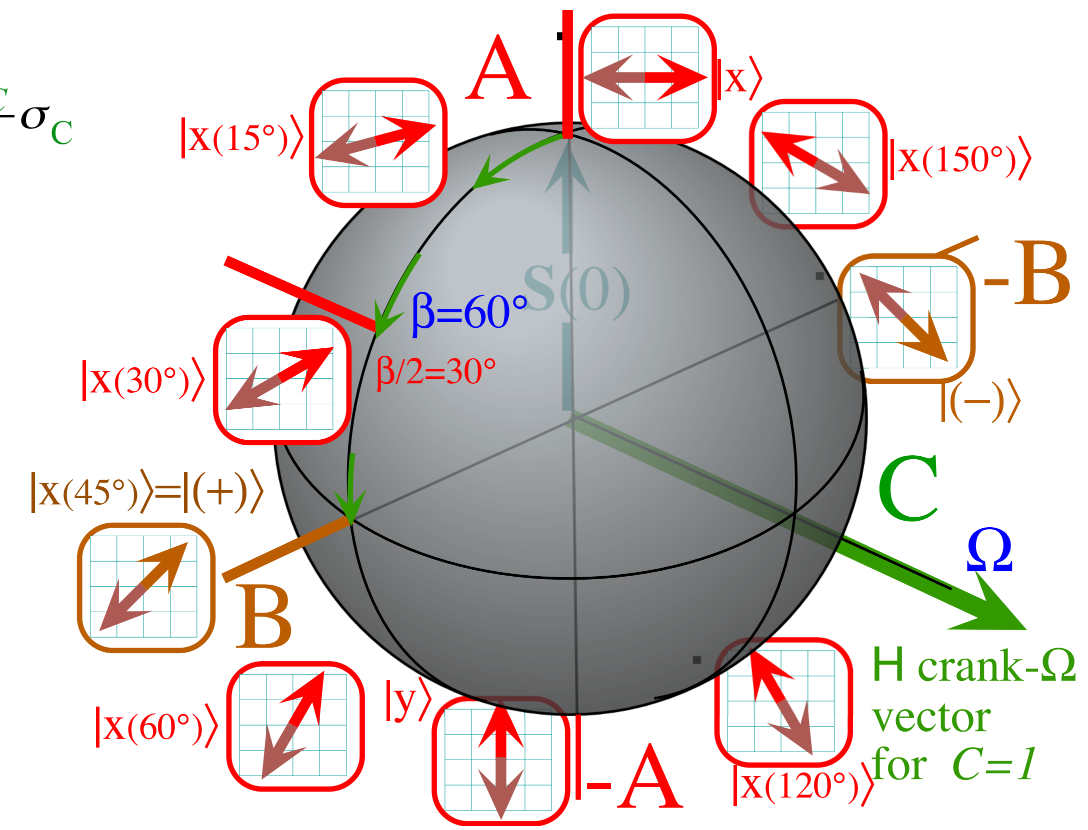
$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Circular-Coriolis... C-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^C|1\rangle & \langle 1|\mathbf{H}^C|2\rangle \\ \langle 2|\mathbf{H}^C|1\rangle & \langle 2|\mathbf{H}^C|2\rangle \end{pmatrix} = \begin{pmatrix} \Omega_0 & -iC \\ iC & \Omega_0 \end{pmatrix} = \Omega_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_C}{2}\sigma_C$$

$$Crank: \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2C \end{pmatrix} \qquad Eigen-Spin: \vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \pm S \end{pmatrix}$$



BoxIt (C-Type)
Web Simulation

The ABC's of U(2) dynamics

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\sigma$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\sigma$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{A+D}{2}\,\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\,\sigma_A$$

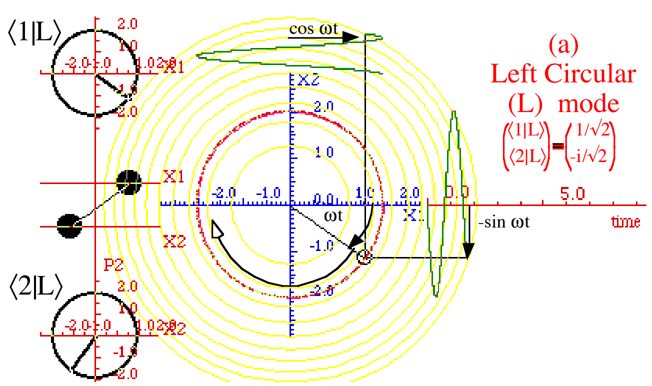
$$= \frac{A+D}{2}\,\sigma_0 + \frac{\Omega_B}{2}\,\sigma_B + \frac{\Omega_C}{2}\,\sigma_C + \frac{\Omega_A}{2}\,\sigma_A$$

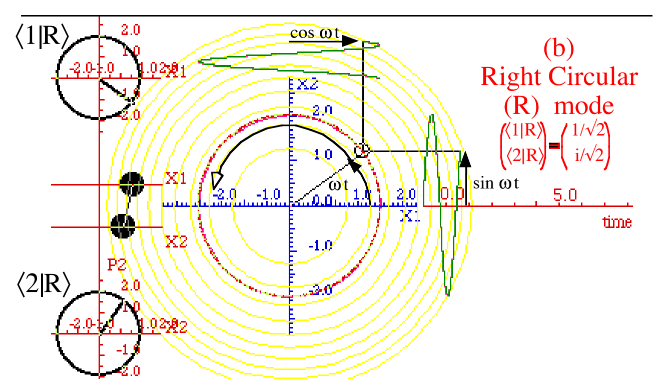
$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

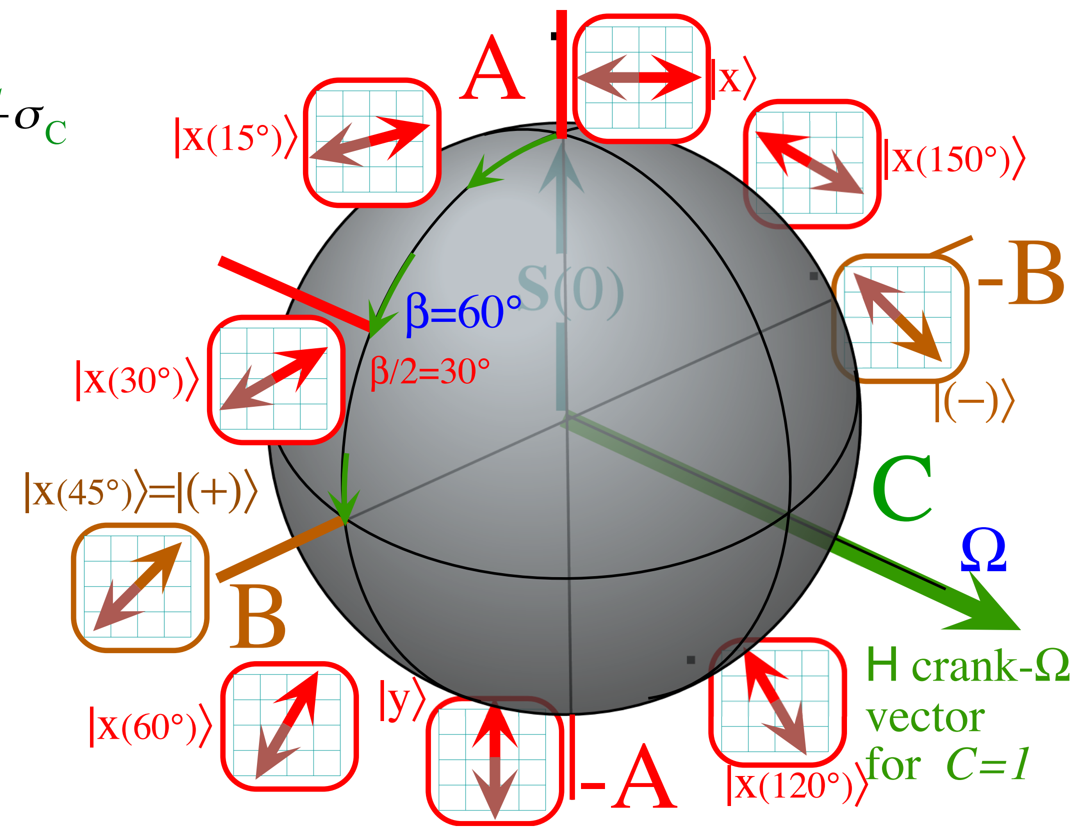
Circular-Coriolis... C-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^C|1\rangle & \langle 1|\mathbf{H}^C|2\rangle \\ \langle 2|\mathbf{H}^C|1\rangle & \langle 2|\mathbf{H}^C|2\rangle \end{pmatrix} = \begin{pmatrix} \Omega_0 & -iC \\ iC & \Omega_0 \end{pmatrix} = \Omega_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \Omega_0\sigma_0 + \frac{\Omega_C}{2}\sigma_C$$

$$Crank:\ \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2C \end{pmatrix} \qquad Eigen-Spin:\vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \pm S \end{pmatrix}$$







BoxIt (
Web Simulation

The ABC's of U(2) dynamics

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

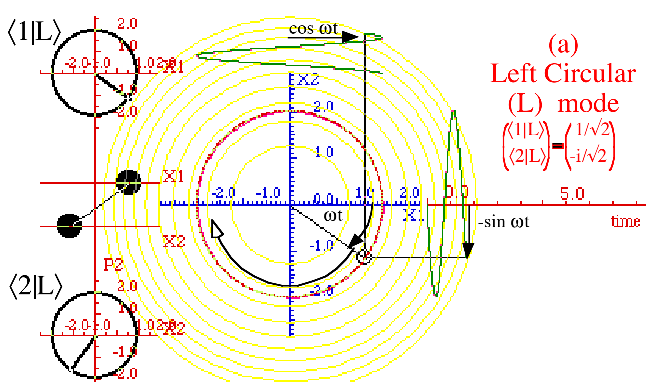
$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

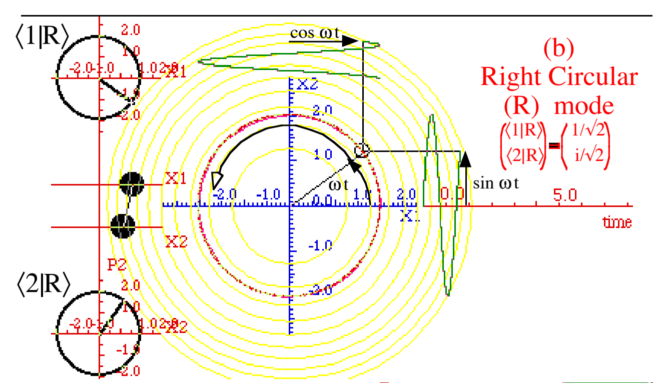
$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

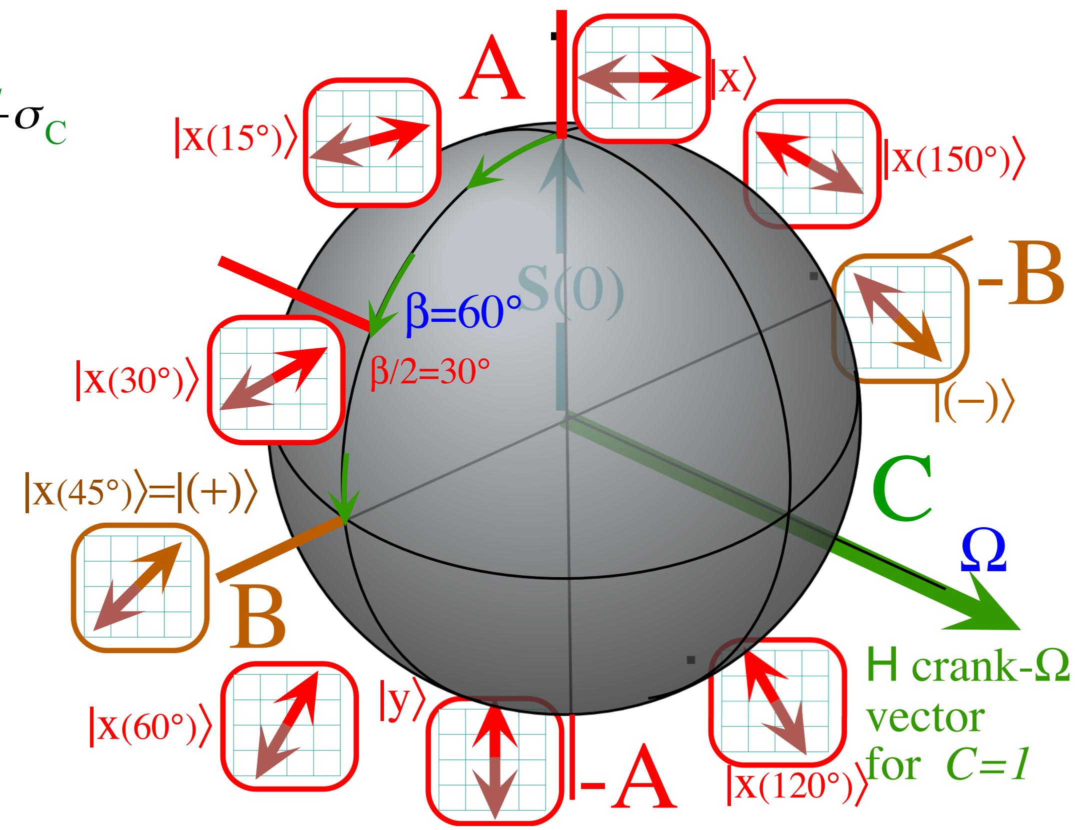
Circular-Coriolis... C-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^C|1\rangle & \langle 1|\mathbf{H}^C|2\rangle \\ \langle 2|\mathbf{H}^C|1\rangle & \langle 2|\mathbf{H}^C|2\rangle \end{pmatrix} = \begin{pmatrix} \Omega_0 & -iC \\ iC & \Omega_0 \end{pmatrix} = \Omega_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_C}{2}\sigma_C$$

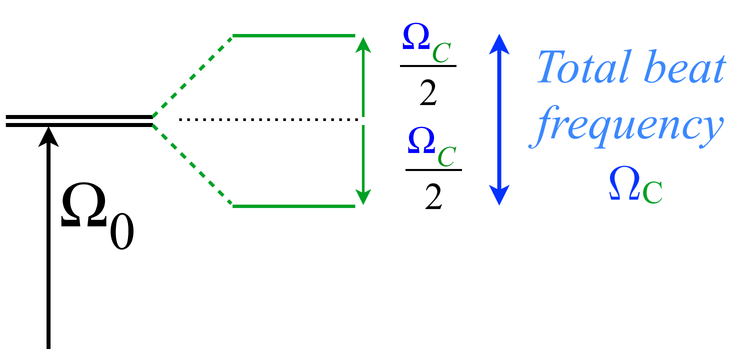
Crank : $\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2C \end{pmatrix}$ *Eigen-Spin :* $\vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \pm S \end{pmatrix}$

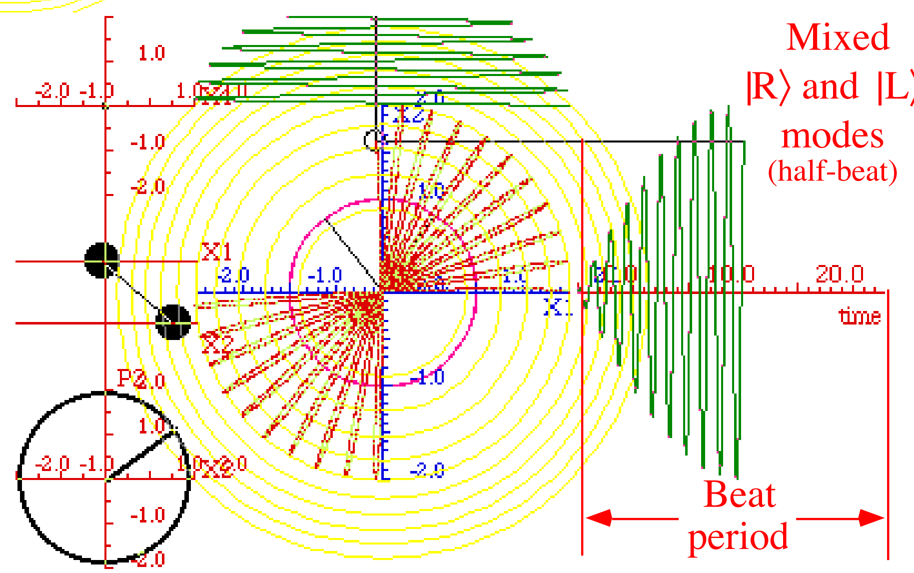






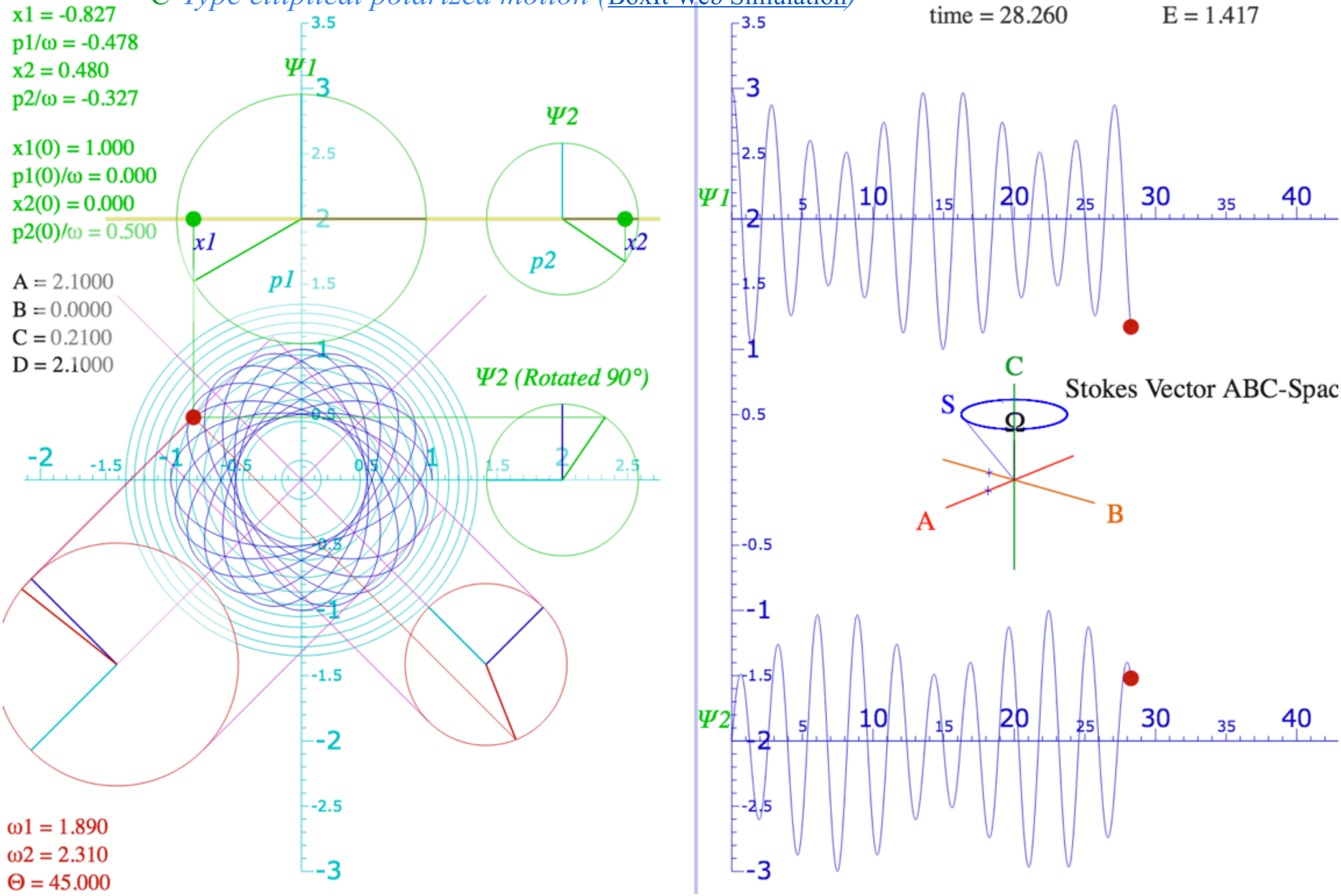
Beat dynamics:





BoxIt (
Web Simulation

C-Type elliptical polarized motion (BoxIt Web Simulation)



BoxIt Web Simulation:
C-Type with A, D=2.1; C=-0.21

Review: Fundamental Euler **R***(αβγ) and Darboux* **R**[φϑΘ] *representations of U(2) and R(3)*

Euler **R***(αβγ) derived from Darboux* **R**[φϑΘ] *and vice versa*

Euler **R***(αβγ) rotation* Θ=0−4π*-sequence* [φϑ] *fixed*

R(3)-U(2) slide rule for converting **R***(αβγ)* ↔ **R**[φϑΘ] *and Sundial*

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes

→ *AB-Type motion and Wigner's Avoided-Symmetry-Crossings*

ABC-Type elliptical polarized motion

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates

Euler Angle (αβγ) ellipse coordinates

The ABC's of U(2) dynamics-Mixed modes

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$
$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{A+D}{2}\,\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\,\sigma_A$$

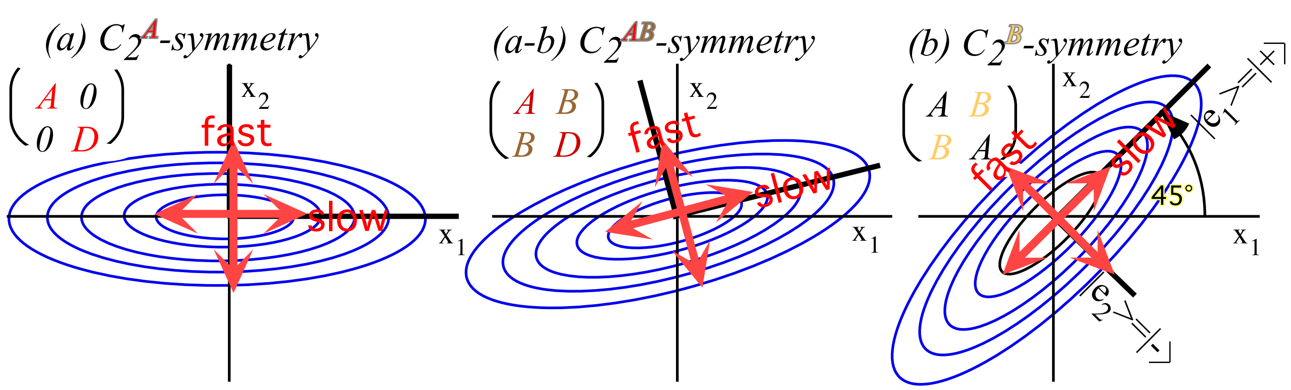
$$= \frac{A+D}{2}\,\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\,\sigma_B + \frac{\Omega_C}{2}\,\sigma_C + \frac{\Omega_A}{2}\,\sigma_A$$

$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

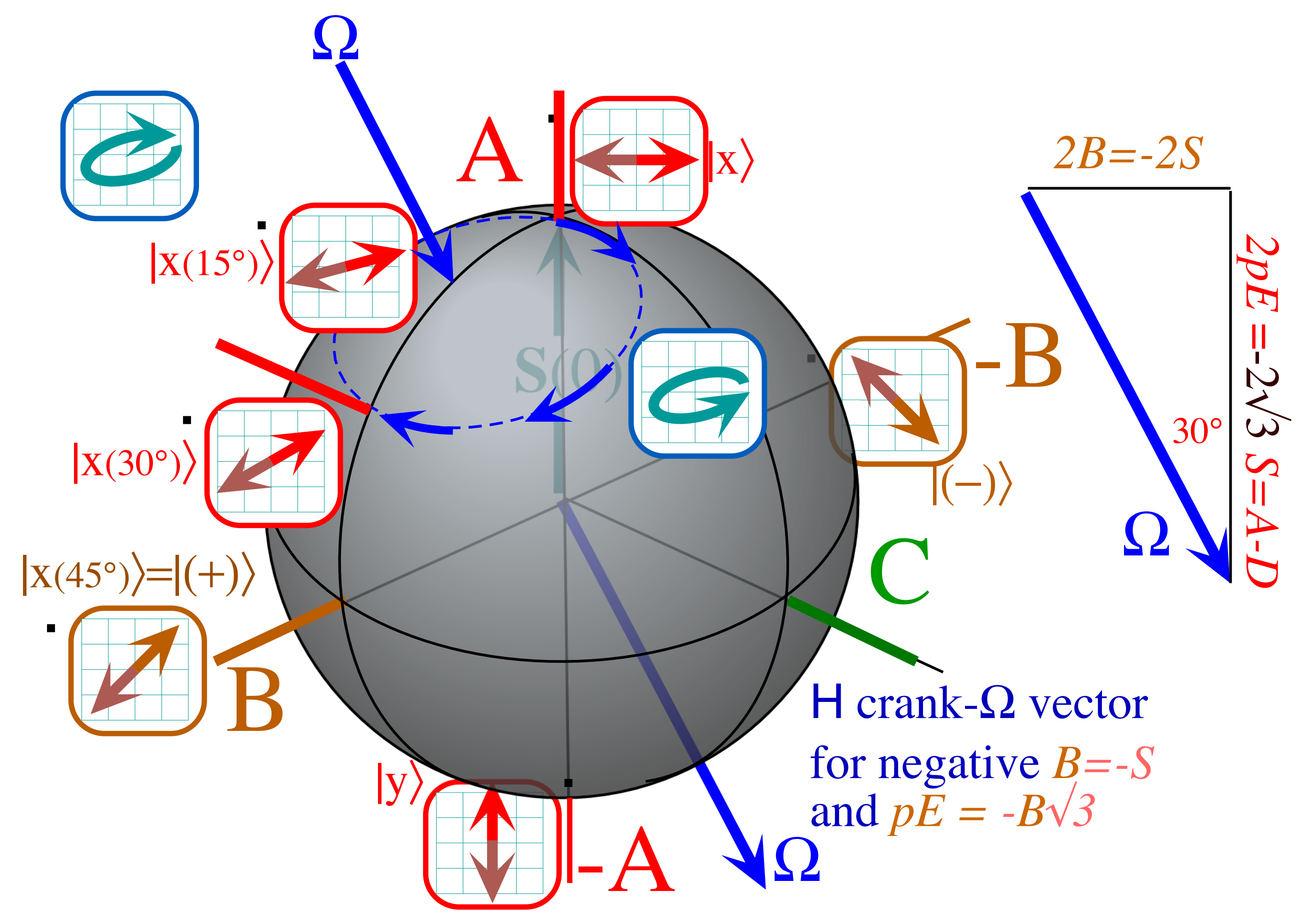
Tilted-plane polarization AB-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^{AB}|1\rangle & \langle 1|\mathbf{H}^{AB}|2\rangle \\ \langle 2|\mathbf{H}^{AB}|1\rangle & \langle 2|\mathbf{H}^{AB}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B \\ B & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_A}{2}\sigma_A + \frac{\Omega_B}{2}\sigma_B$$

Crank : $\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 0 \end{pmatrix}$ *Eigen – Spin* : $\vec{\mathbf{S}} = \pm S\hat{\Omega}$

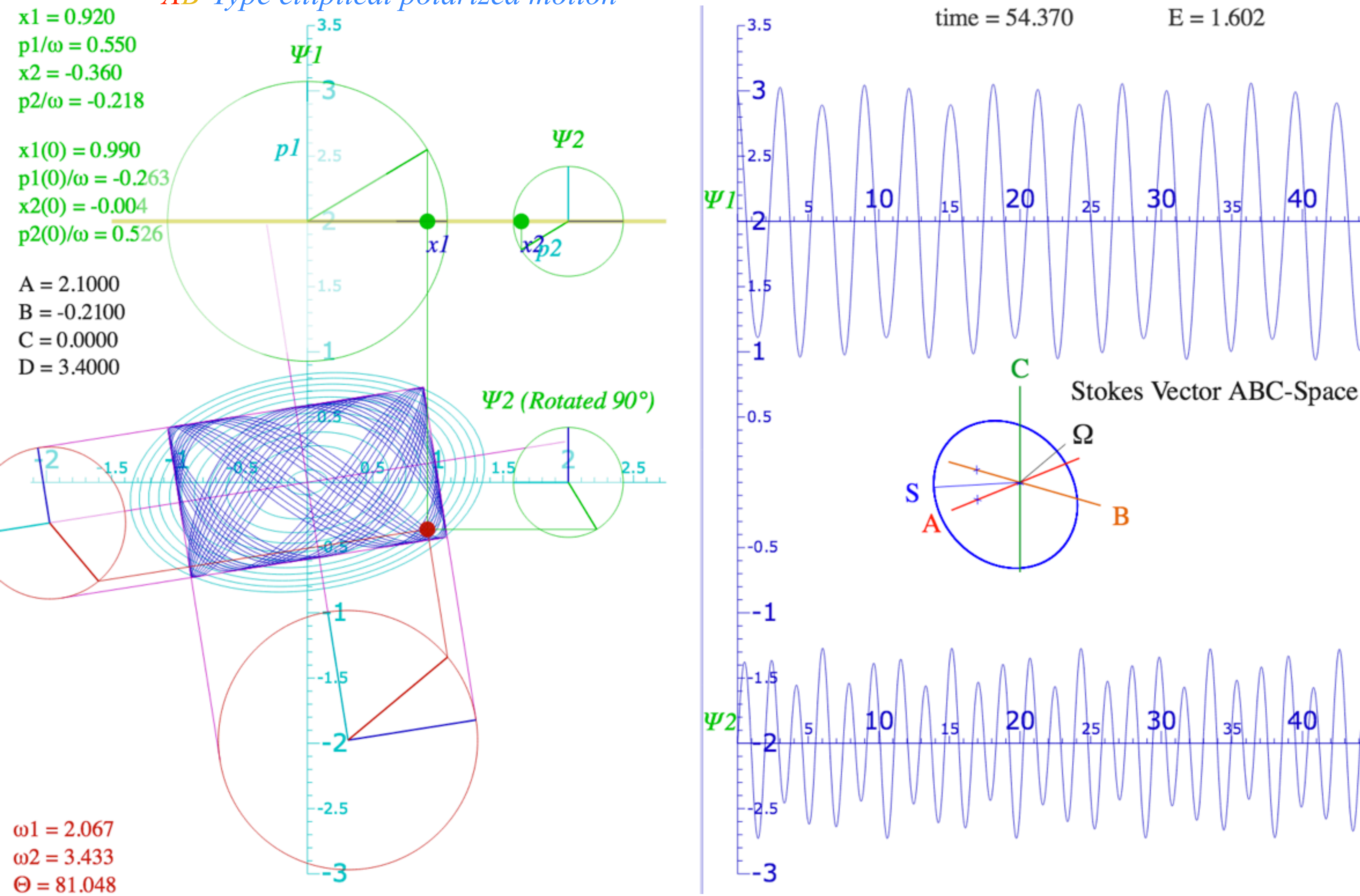


Beat dynamics:



[BoxIt (AB-Type Motion)
Web Simulation]

AB-Type elliptical polarized motion



[BoxIt Web Simulation:
AB-Type with A=2.1; B=-0.21; D=3.4]

A to B to A Symmetry breaking described by hyperbolic eigenvalues of $A\boldsymbol{\sigma}_A + B\boldsymbol{\sigma}_B = \mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$

$\mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$ Secular equation: $\varepsilon^2 - 0 \cdot \varepsilon - (A^2 + B^2)$ gives *hyperbolic* energy levels: $\varepsilon = \pm\sqrt{A^2 + B^2}$

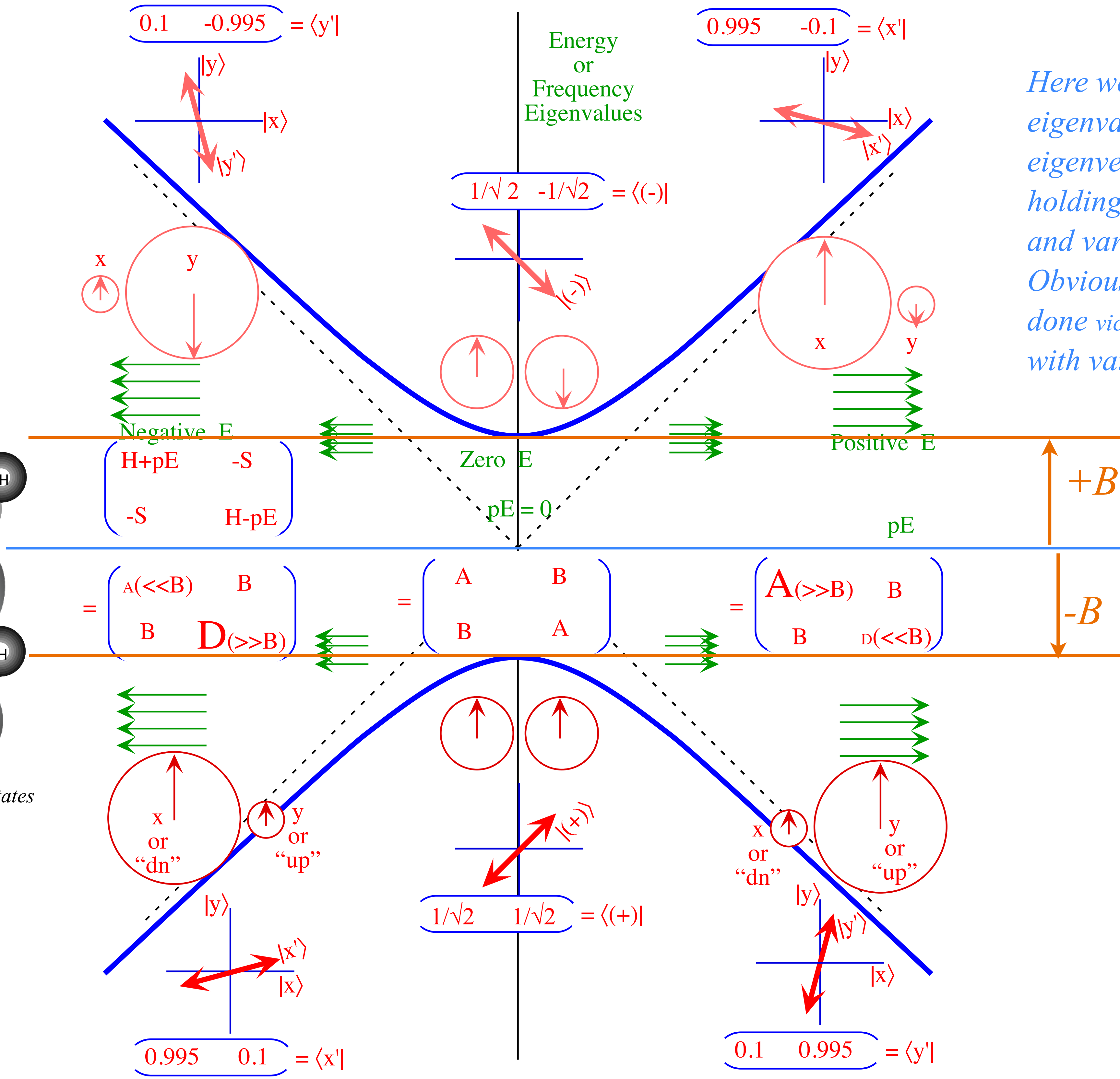


Fig. 10.3.1 (b) Wigner avoided level crossing. (Fixed tunneling B=-S and variable A-D=pE field.)

(A=pE)-Axis
(Applied field)

Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa and with varying C, too.

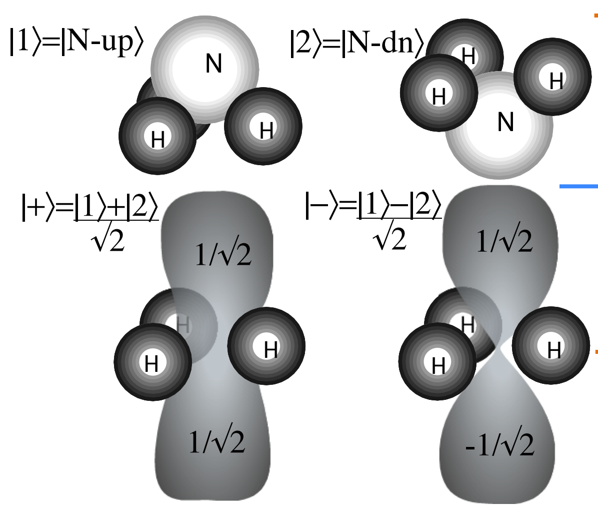


Fig. 10.3.2 Ammonia (NH_3) inversion states
(a) Base states (b) C_2-Eigenstates

A to B to A Symmetry breaking described by hyperbolic eigenvalues of $A\boldsymbol{\sigma}_A + B\boldsymbol{\sigma}_B = \mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$

$\mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$ Secular equation: $\varepsilon^2 - 0\cdot\varepsilon - (A^2 + B^2)$ gives *hyperbolic* energy levels: $\varepsilon = \pm\sqrt{A^2 + B^2}$

$\mathbf{H}(B-basis)$ $\mathbf{H}(A-basis)$

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}} \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}}$$

Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa and with varying C, too.

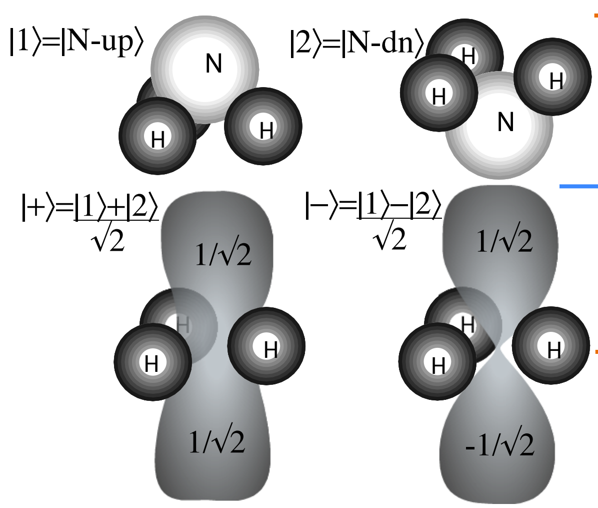


Fig. 10.3.2 Ammonia (NH_3) inversion states (a) Base states (b) C_2-Eigenstates

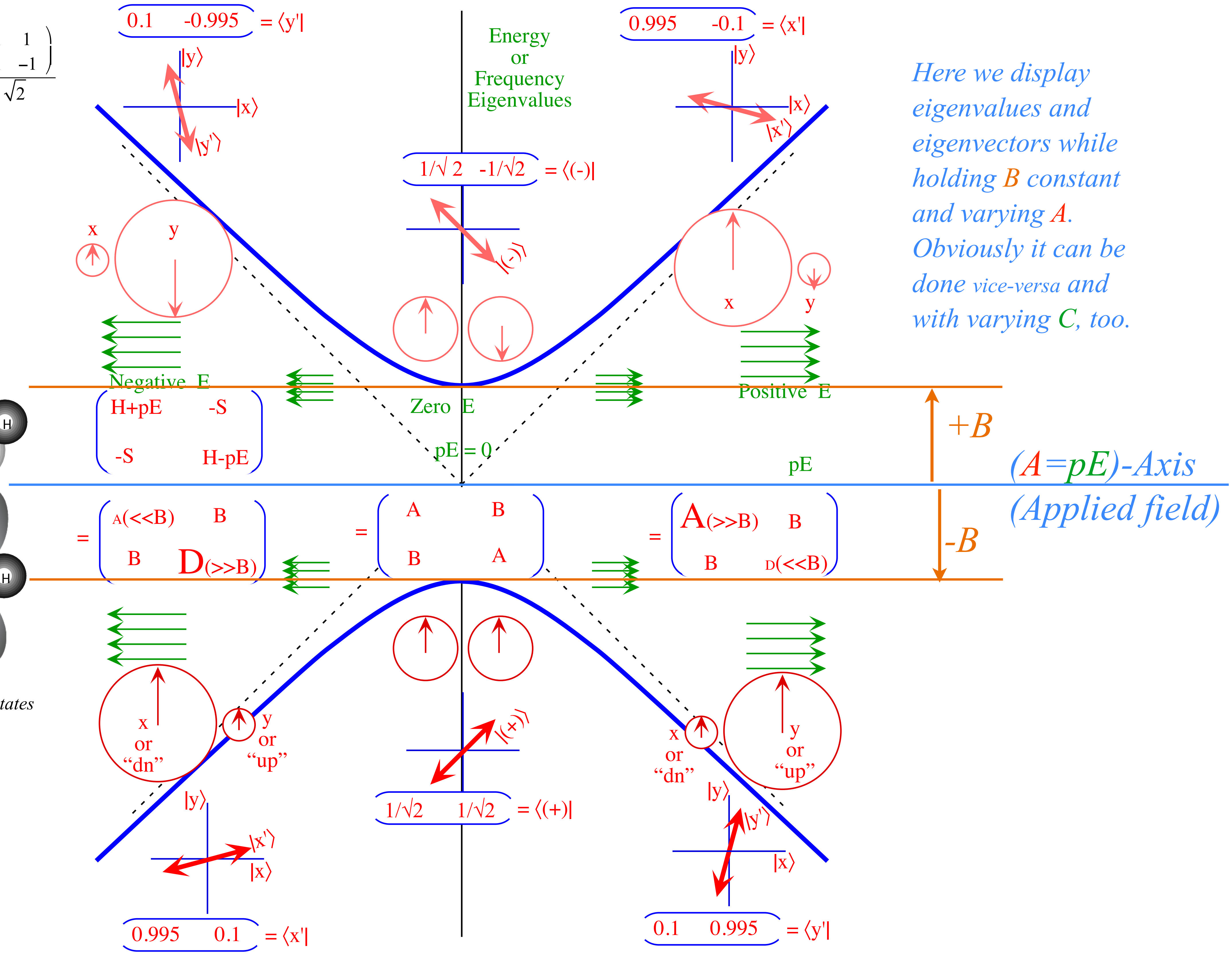


Fig. 10.3.1 (b) Wigner avoided level crossing. (Fixed tunneling B=-S and variable A-D=pE field.)

A to B to A Symmetry breaking described by hyperbolic eigenvalues of $A\boldsymbol{\sigma}_A + B\boldsymbol{\sigma}_B = \mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$

$$\mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \quad \text{Secular equation: } \varepsilon^2 - 0\cdot\varepsilon - (A^2 + B^2) \quad \text{gives } \textit{hyperbolic} \text{ energy levels: } \varepsilon = \pm\sqrt{A^2 + B^2}$$

$\mathbf{H}(B-basis)$ $\quad\quad$ $\mathbf{H}(A-basis)$

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}} \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} +A & B \\ B & -A \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} +A+B & B-A \\ +A-B & B+A \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa and with varying C, too.

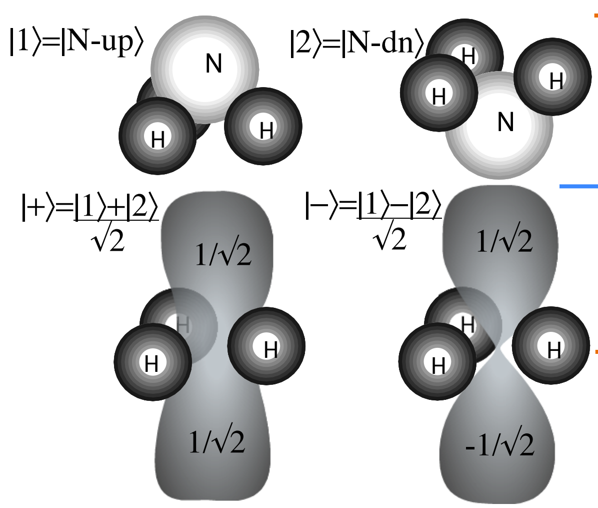


Fig. 10.3.2 Ammonia (NH_3) inversion states
(a) Base states (b) C_2-Eigenstates

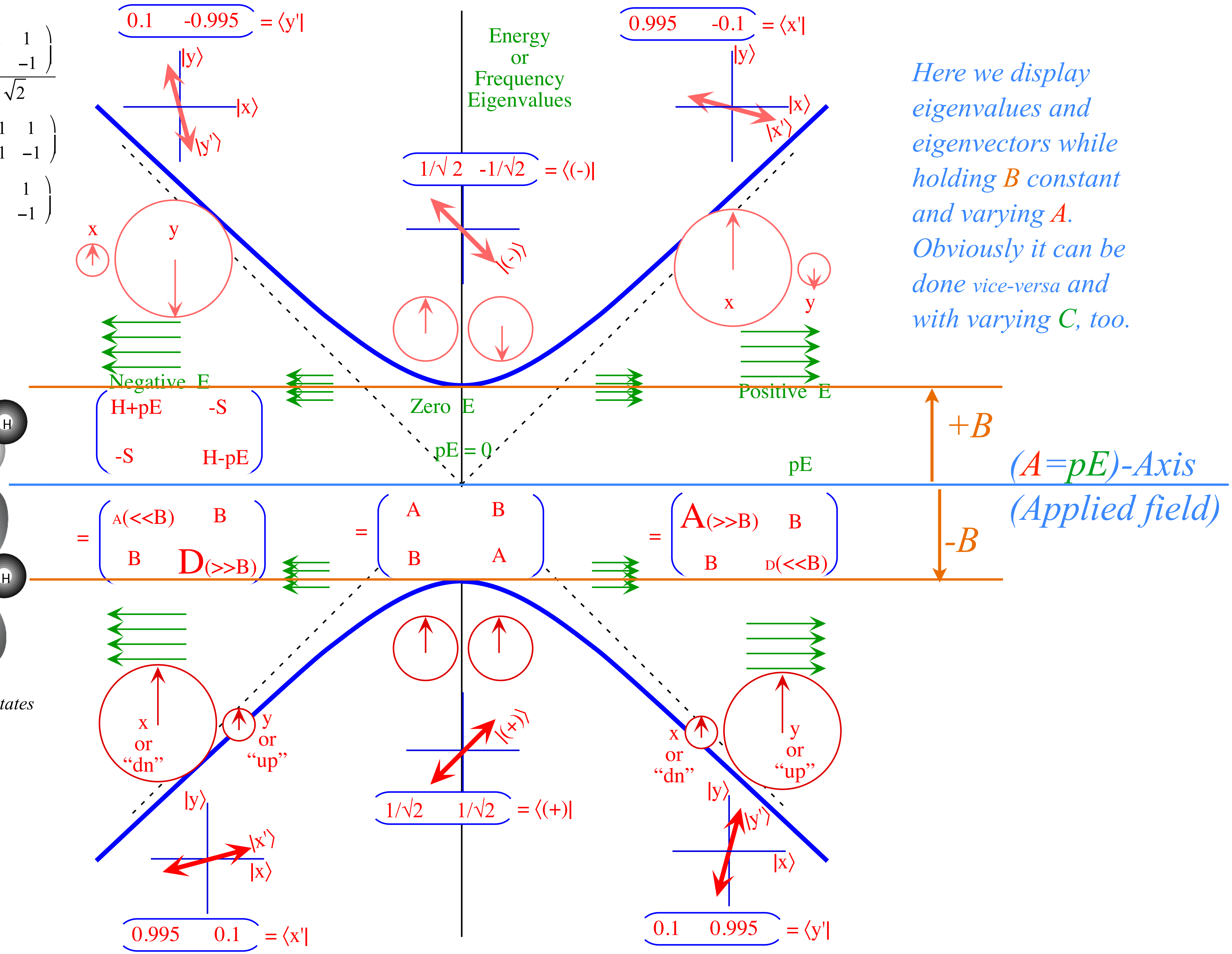


Fig. 10.3.1 (b) Wigner avoided level crossing. (Fixed tunneling B=-S and variable A-D=pE field.)

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$\mathbf{H}(B-basis)$ $\qquad$ $\mathbf{H}(A-basis)$

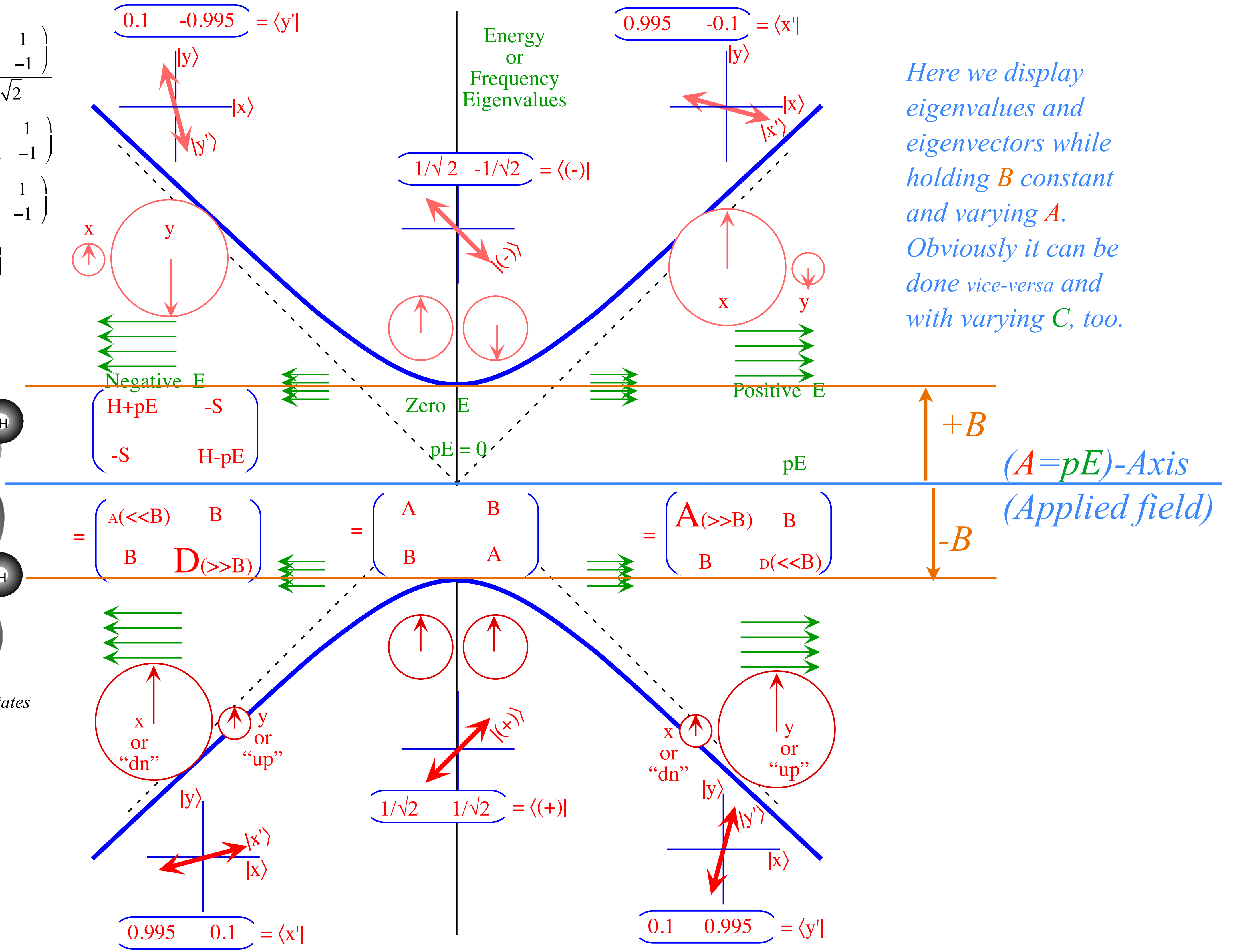
$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}} \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} +A & B \\ B & -A \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} +A+B & B-A \\ +A-B & B+A \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2B & 2A \\ 2A & -2B \end{pmatrix}$$

Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa and with varying C, too.



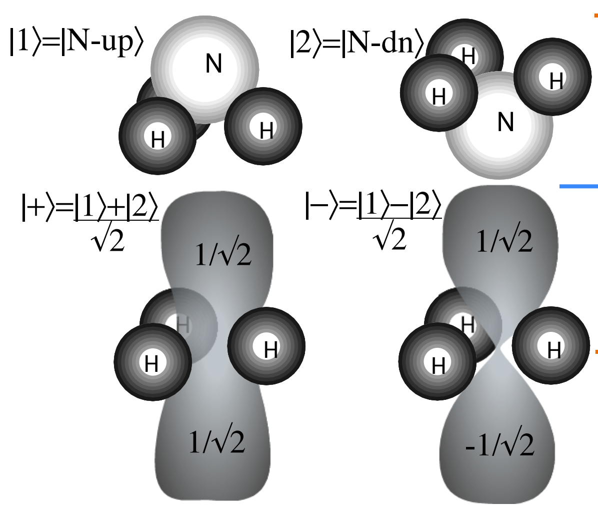


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Fig. 10.3.1 (b) Wigner avoided level crossing. (Fixed tunneling B=-S and variable A-D=pE field.)

A to B to A Symmetry breaking described by hyperbolic eigenvalues of $A\boldsymbol{\sigma}_A + B\boldsymbol{\sigma}_B = \mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$

$\mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$ Secular equation: $\varepsilon^2 - 0 \cdot \varepsilon - (A^2 + B^2)$ gives *hyperbolic* energy levels: $\varepsilon = \pm\sqrt{A^2 + B^2}$

$\mathbf{H}(B-basis)$ $\mathbf{H}(A-basis)$

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}} \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} +A & B \\ B & -A \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} +A+B & B-A \\ +A-B & B+A \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2B & 2A \\ 2A & -2B \end{pmatrix}$$

$$= \begin{pmatrix} +B & A \\ A & -B \end{pmatrix}$$

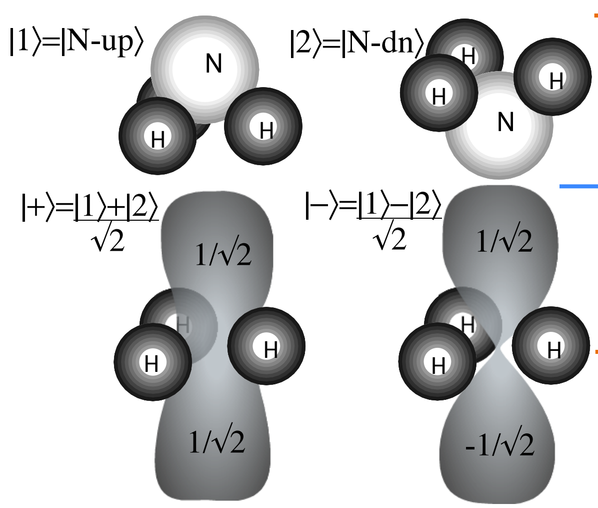
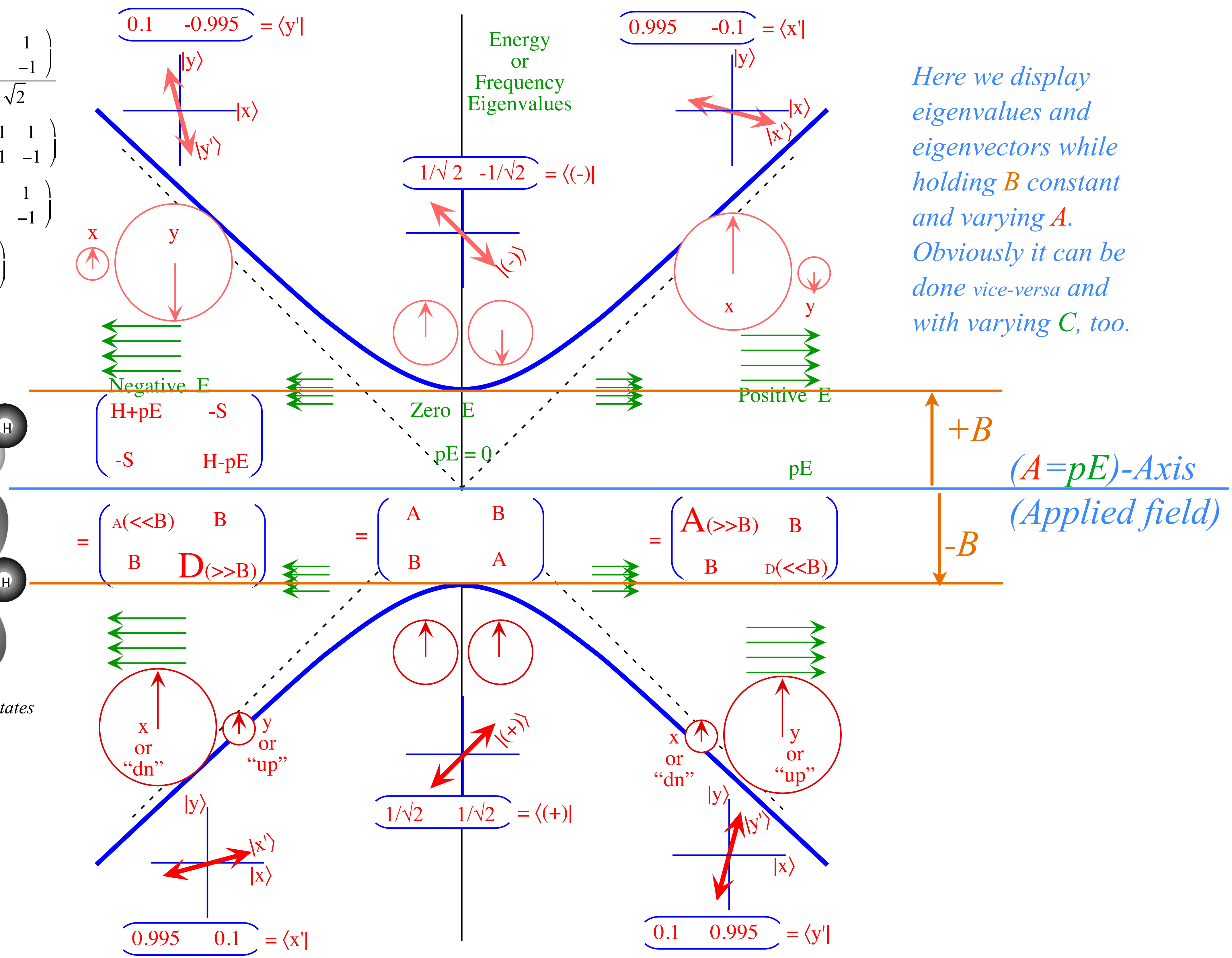


Fig. 10.3.2 Ammonia (NH_3) inversion states
(a) Base states (b) C_2-Eigenstates



Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa and with varying C, too.

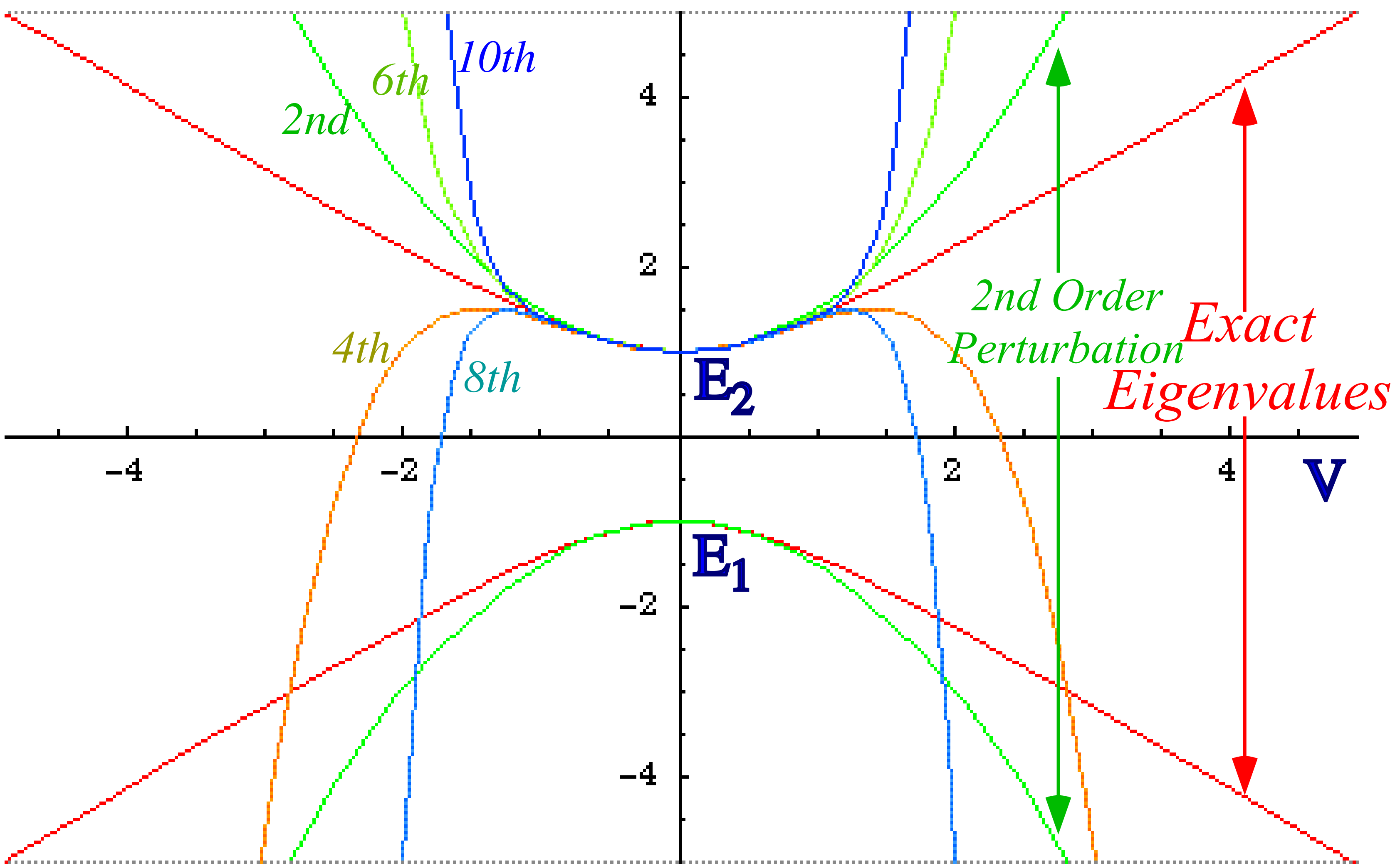
Fig. 10.3.1 (b) Wigner avoided level crossing. (Fixed tunneling B=-S and variable A-D=pE field.)

$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} E_1 & V \\ V & E_2 \end{pmatrix}$$

2nd order perturbation terms

$$\lambda_1 = E_1 + \frac{V^2}{E_1 - E_2} \ ,$$

$$\lambda_2 = E_2 + \frac{V^2}{E_2 - E_1} \ .$$



$$\lambda^2 - (Trace\mathbf{H})\lambda + \det|\mathbf{H}| = 0 = \lambda^2 - (E_1 + E_2)\lambda + (E_1 E_2 - V^2)$$

$$\lambda_{1,2} = \frac{E_1 + E_2 \pm \sqrt{(E_1 + E_2)^2 - 4E_1 E_2 + 4V^2}}{2} = \frac{E_1 + E_2 \pm \sqrt{(E_1 - E_2)^2 + 4V^2}}{2} \ ,$$

Fig. 3.2.2 Comparison of exact vs. 2nd-order thru 10th-order perturbation approximations

$$E_2 = \frac{\Delta}{2} + \frac{V^2}{\Delta} - \frac{V^4}{\Delta^3} + \frac{V^6}{\Delta^5} - \frac{V^8}{\Delta^7} + \frac{V^{10}}{\Delta^9} \cdots \text{ , where: } \Delta = |E_1 - E_2|$$

A view of a conical intersection:

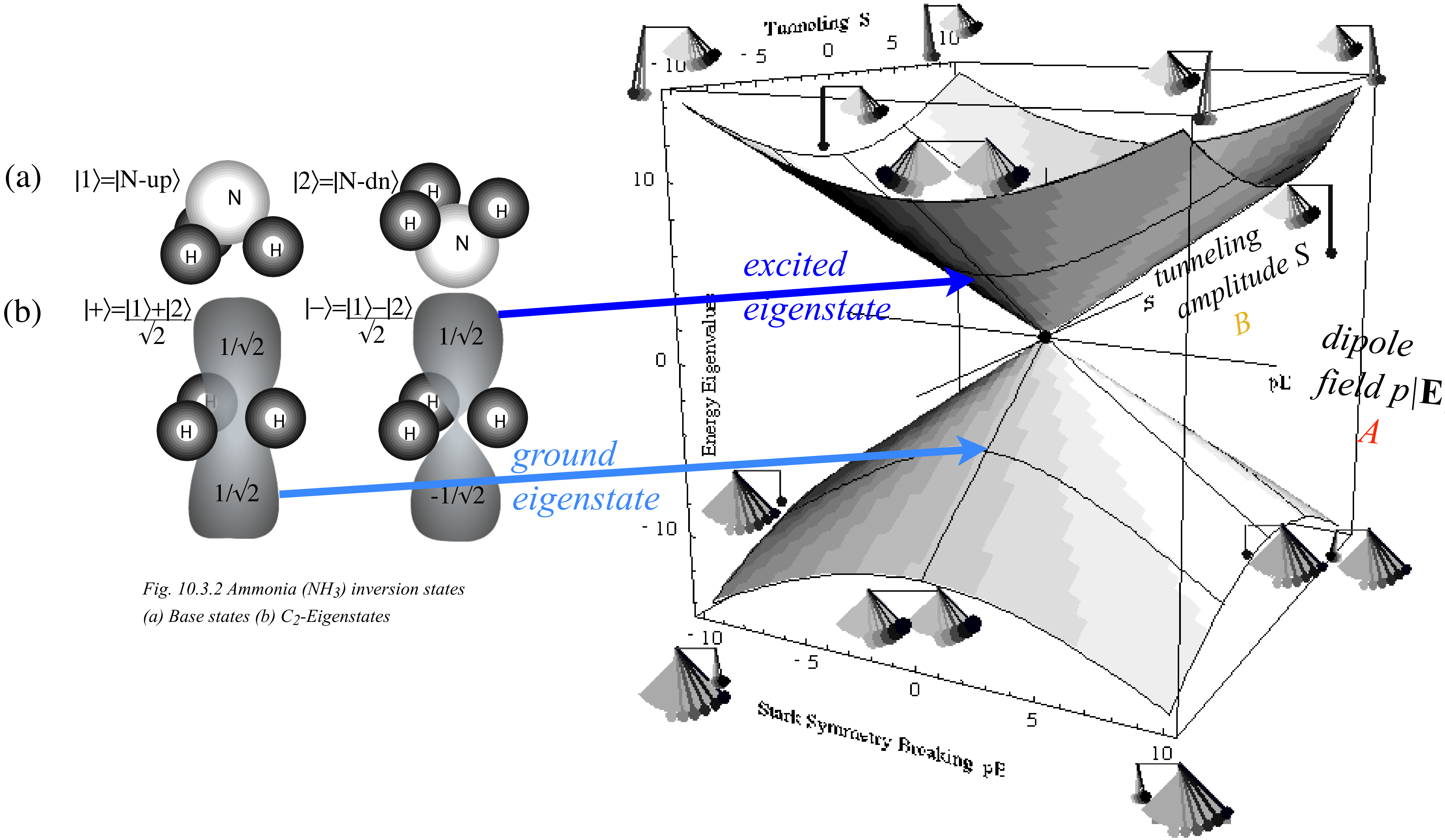


Fig. 10.3.2 Ammonia (NH_3) inversion states
(a) Base states (b) C_2-Eigenstates

10.3.1 (a) Two state eigenvalue "diablo" surfaces and conical intersection and pendulum eigenstates.

A view of a conical intersection: Any vertical cross-section is hyperbolic avoided-crossing

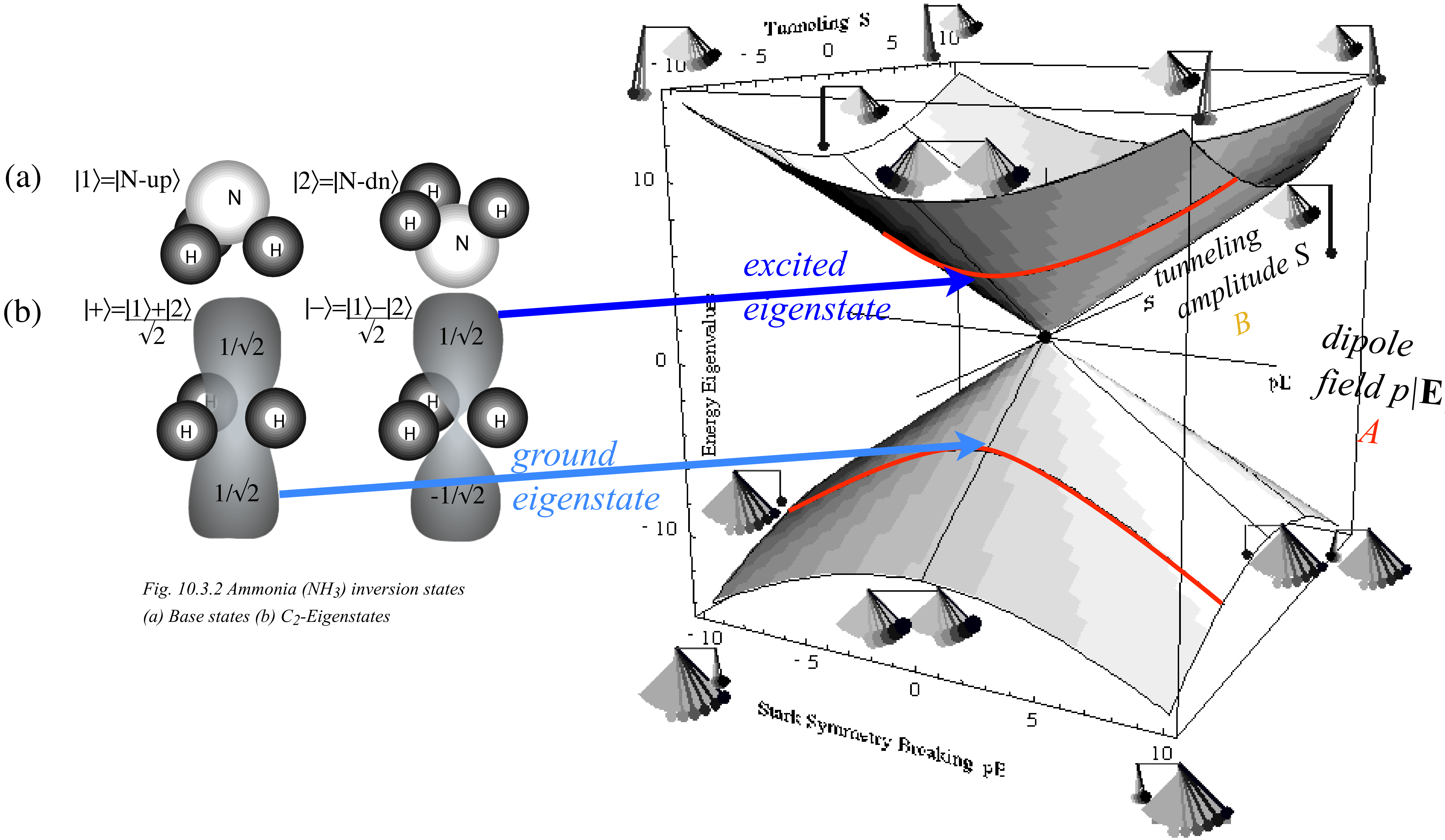


Fig. 10.3.2 Ammonia (NH_3) inversion states
(a) Base states (b) C_2-Eigenstates

10.3.1 (a) Two state eigenvalue "diablo" surfaces and conical intersection and pendulum eigenstates.

Review: Fundamental Euler **R***(αβγ) and Darboux* **R***[φϑΘ] representations of U(2) and R(3)*

Euler **R***(αβγ) derived from Darboux* **R***[φϑΘ] and vice versa*

Euler **R***(αβγ) rotation Θ=0−4π-sequence [φϑ] fixed*

R(3)-U(2) slide rule for converting **R***(αβγ) ↔* **R***[φϑΘ] and Sundial*

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes

AB-Type motion and Wigner's Avoided-Symmetry-Crossings

➡ *ABC-Type elliptical polarized motion*

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates

Euler Angle (αβγ) ellipse coordinates

ABC-Type elliptical polarized motion

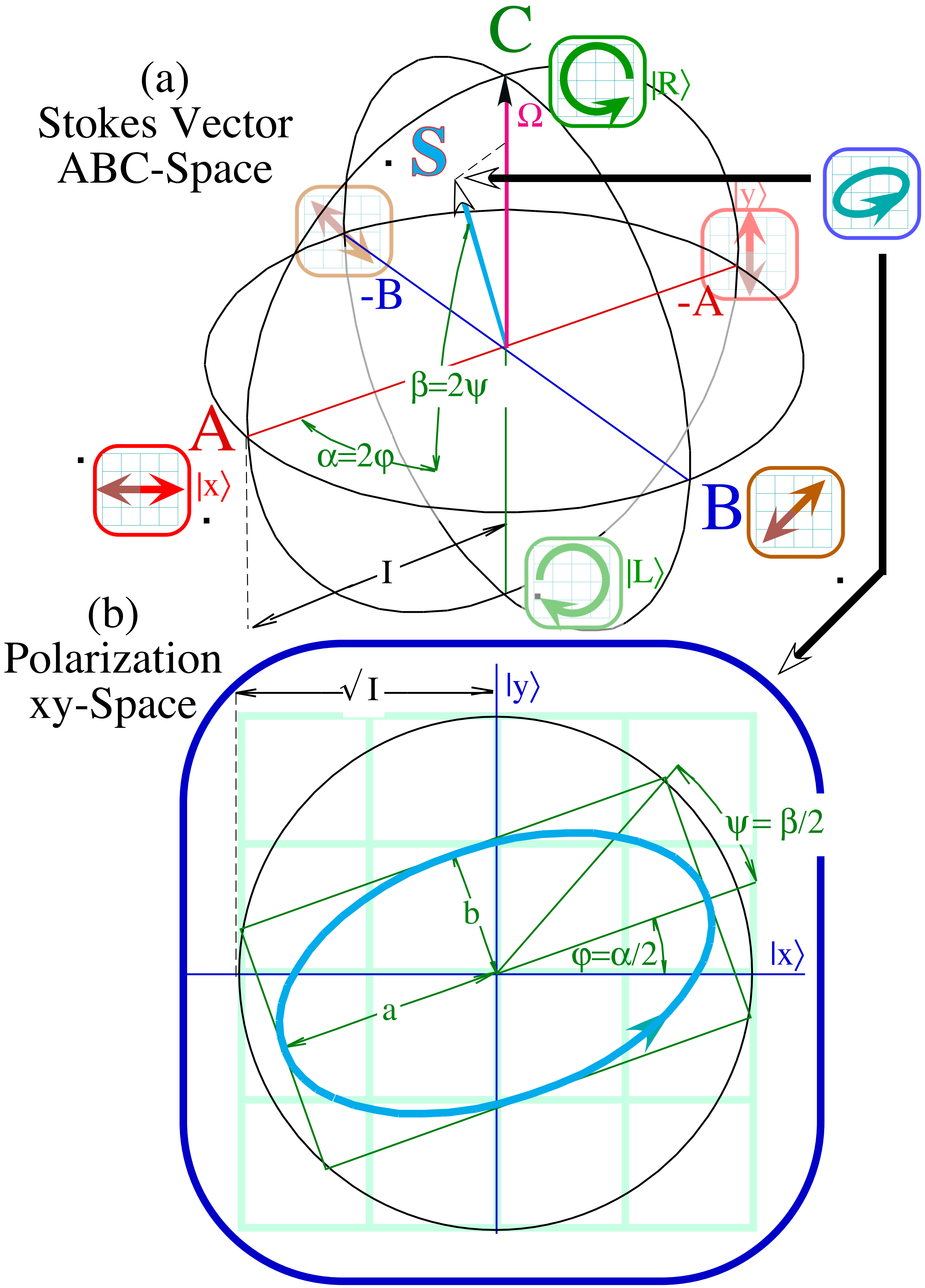


Fig. 10.B.3

Euler-like coordinates for
(a) R(3) spin vector
(b) U(2) polarization ellipse

ABC-Type elliptical polarized motion

(from Principles of Symmetry, Dynamics, and Spectroscopy)

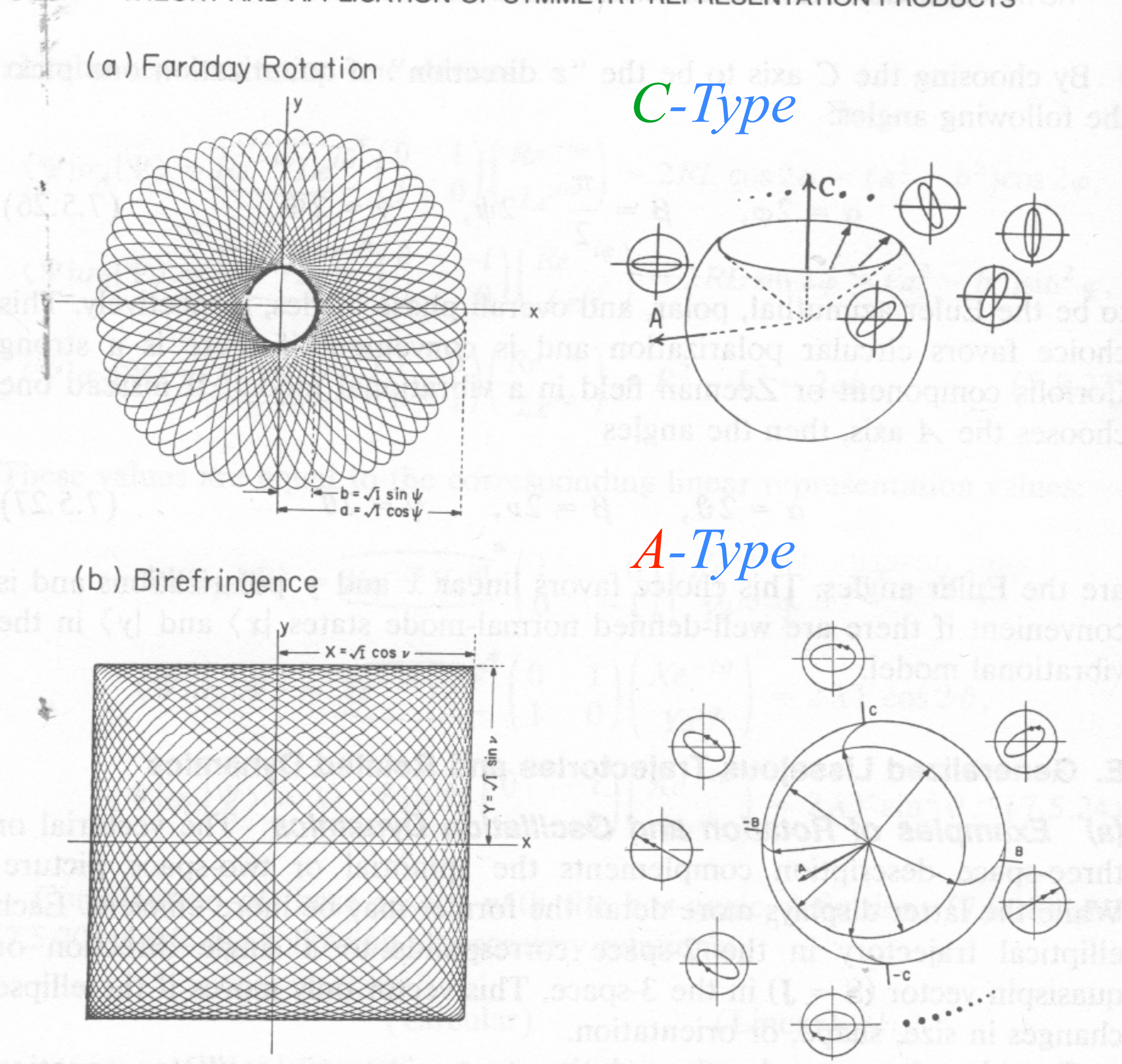
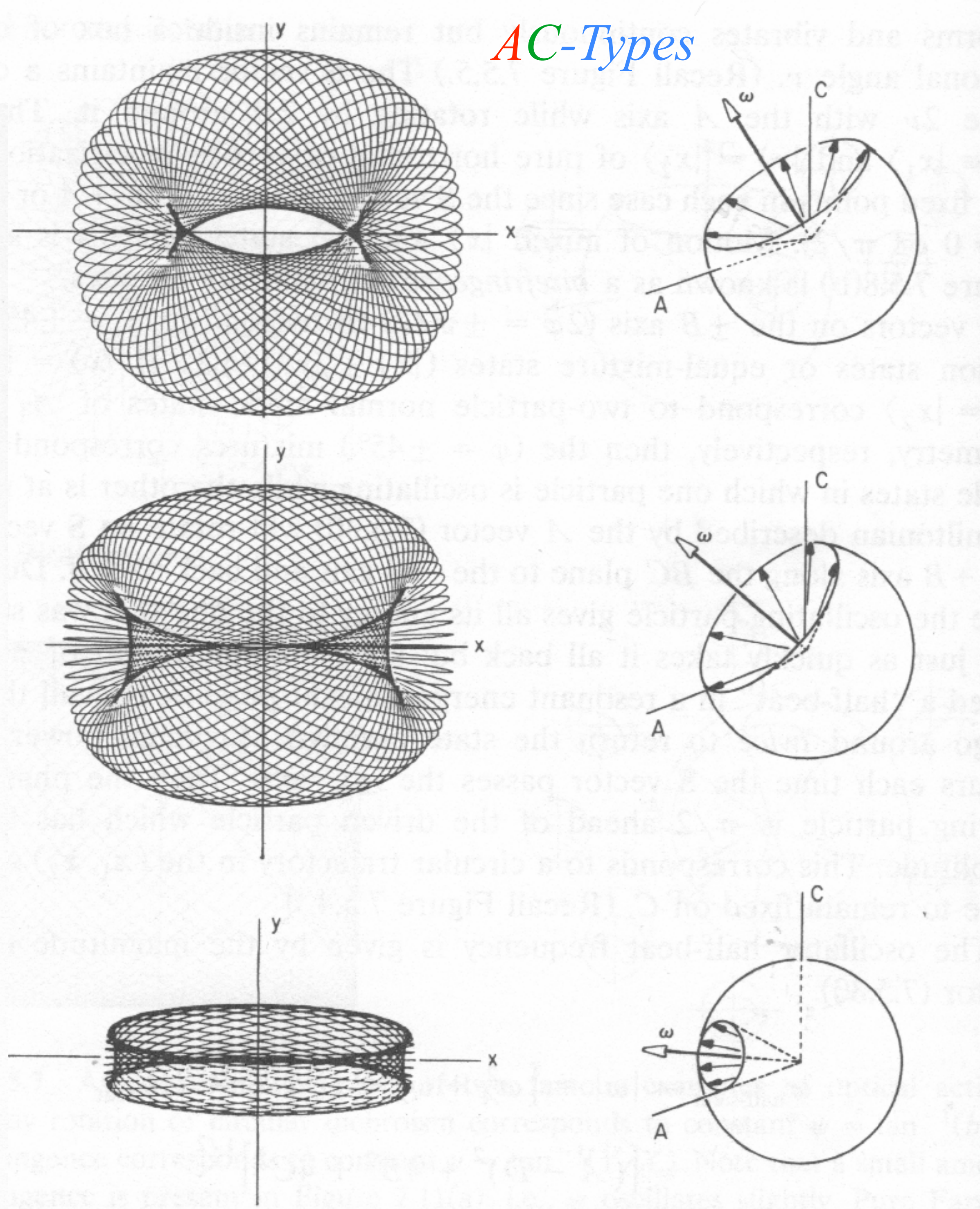
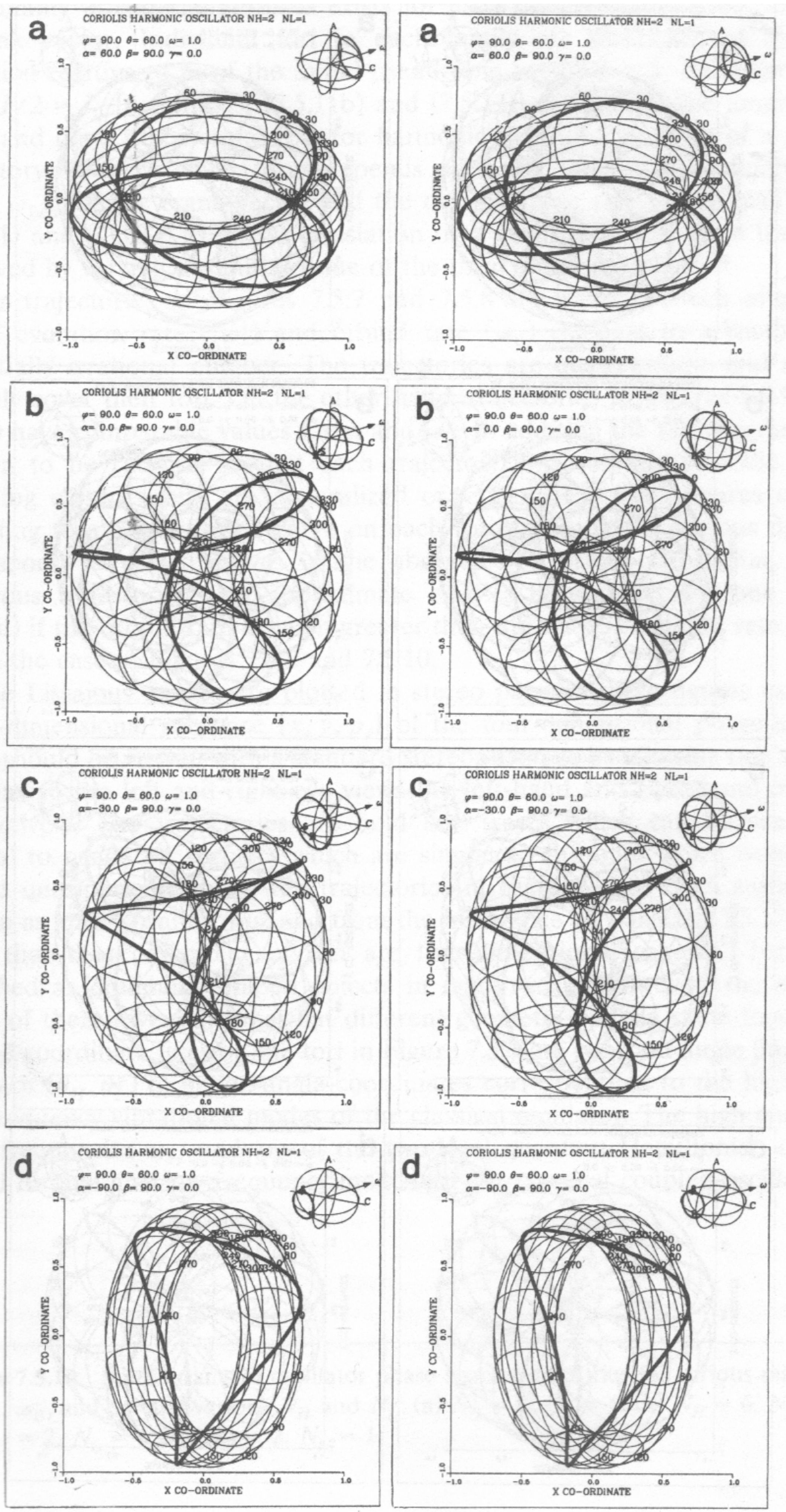


Figure 7.5.7 Analog computer plots of two famous examples of optical activity. (a) Faraday rotation or circular dichroism corresponds to constant $\psi = \tan^{-1}(b/a)$. (b) Birefringence corresponds to constant $\nu = \tan^{-1}(Y/X)$. Note that a small amount of birefringence is present in Figure 7.11(a); i.e., ψ oscillates slightly. Pure Faraday rotation is difficult to achieve on an analog computer.

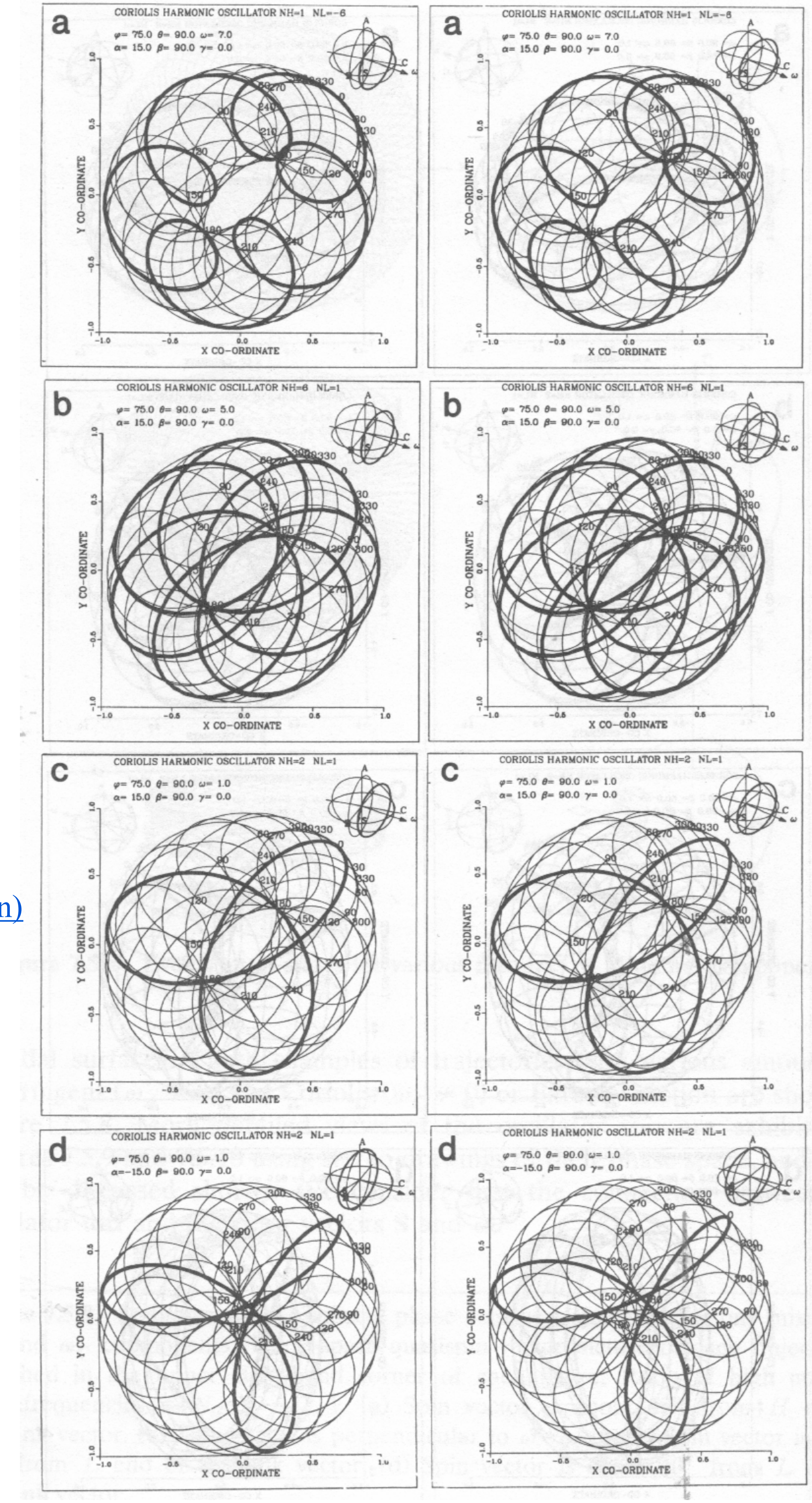


7.5.8 Evolution of states for various mixtures of A and C components.

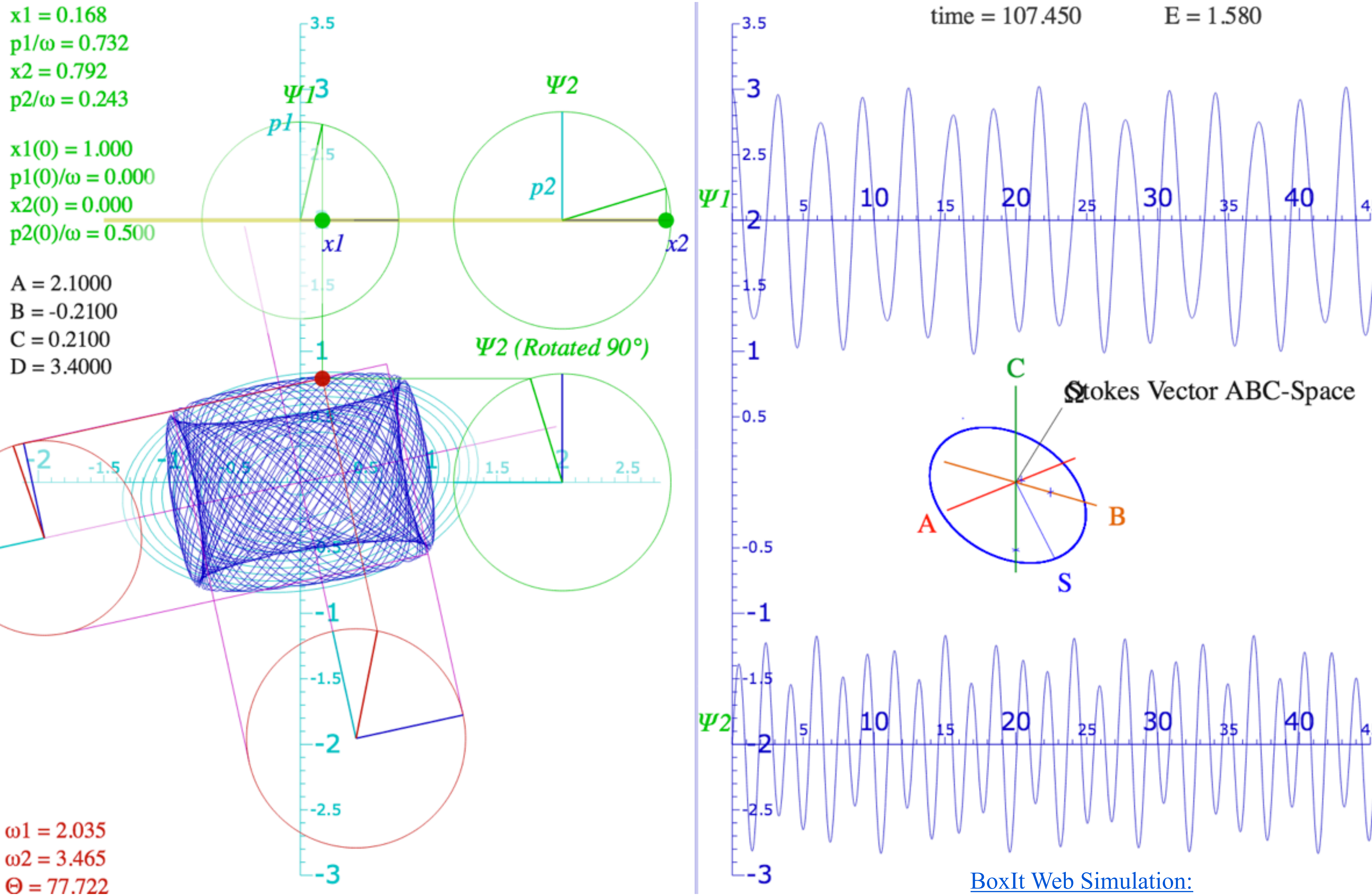


ABC-Type elliptical polarized dynamics

BoxIt (ABC-Motion) Web Simulation



ABC-Type elliptical polarized motion



[BoxIt Web Simulation: ABC-Type with A=2.1; B=-0.21; C=0.21; D=3.4]()

Review: Fundamental Euler **R***(αβγ) and Darboux* **R***[φϑΘ] representations of U(2) and R(3)*

Euler **R***(αβγ) derived from Darboux* **R***[φϑΘ] and vice versa*

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Ellipsometry using U(2) symmetry coordinates

→ *Conventional amp-phase ellipse coordinates*

Euler Angle (αβγ) ellipse coordinates

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates and related to Euler Angles (αβγ)

2D elliptic frequency ω orbit has amplitudes

A_1 and A_2, and phase shifts ρ_1 and $\rho_2 = -\rho_1$.

$$x_1 = A_1 \cos(\omega t + \rho_1)$$

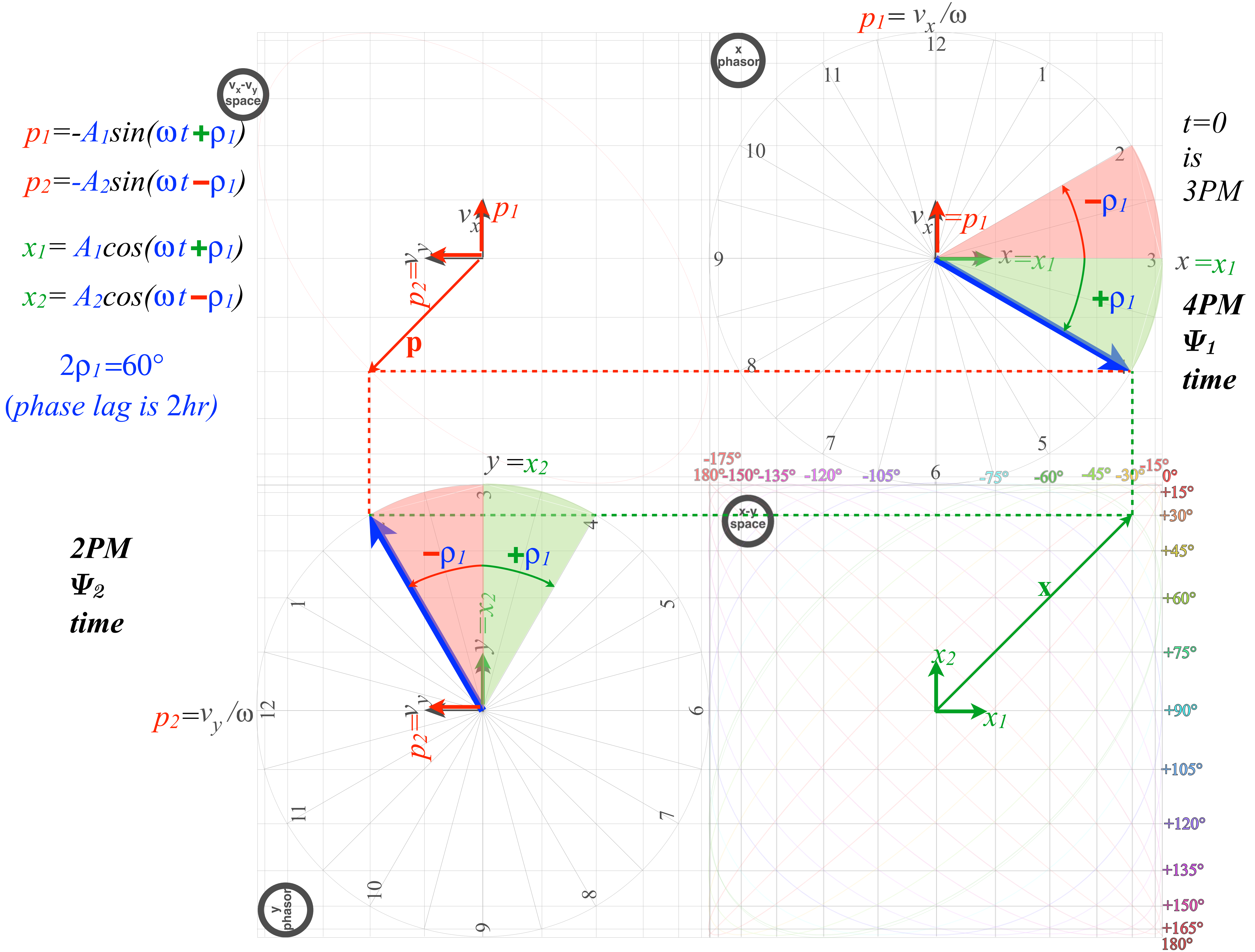
$$-p_1 = A_1 \sin(\omega t + \rho_1)$$

$$x_2 = A_2 \cos(\omega t - \rho_1)$$

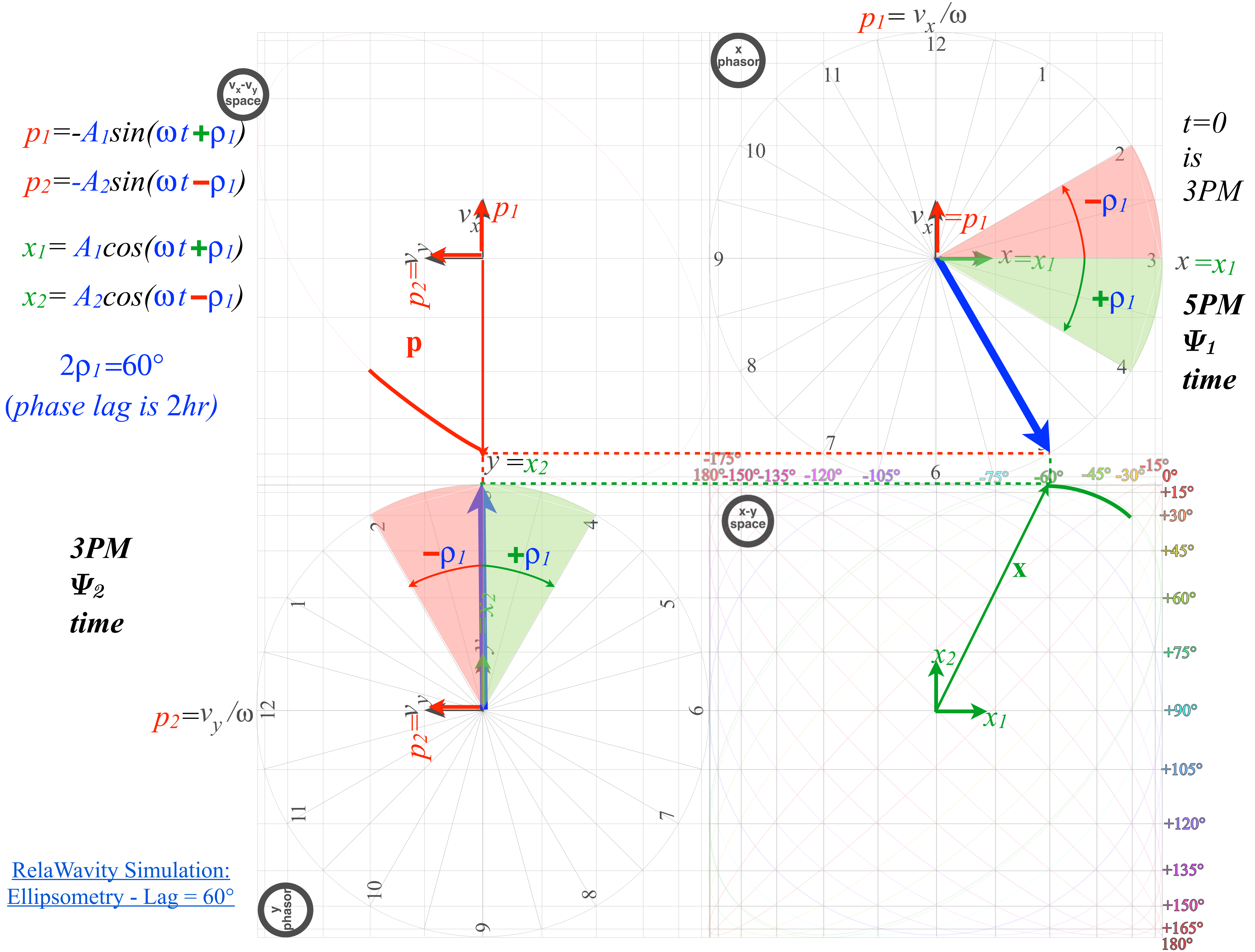
$$-p_2 = A_2 \sin(\omega t - \rho_1)$$

Amp-phase parameters $(A_1, A_2, \omega t, \rho_1)$

$$\begin{pmatrix} A_1 e^{-i(\omega t + \rho_1)} \\ A_2 e^{-i(\omega t - \rho_1)} \end{pmatrix} = \begin{pmatrix} x_1 + i p_1 \\ \\ x_2 + i p_2 \end{pmatrix}$$



p1=-A1sin(ωt+ρ1)
p2=-A2sin(ωt−ρ1)
x1= A1cos(ωt+ρ1)
x2= A2cos(ωt−ρ1)
2ρ1=60°
(phase lag is 2hr)
vx-vy space
x phasor
y phasor
x-y space
p1= vx/ω
vx=p1
x=x1
−ρ1
+ρ1
t=0
is
3PM
4PM
Ψ1
time
2PM
Ψ2
time
p
p2=vy
y=x2
p2=vy/ω
x
x1
x2
-175°
180°
-150°
-135°
-120°
-105°
-75°
-60°
-45°
-30°
-15°
0°
+15°
+30°
+45°
+60°
+75°
+90°
+105°
+120°
+135°
+150°
+165°
180°



p1=-A1sin(ωt+ρ1)
p2=-A2sin(ωt−ρ1)
x1= A1cos(ωt+ρ1)
x2= A2cos(ωt−ρ1)
2ρ1=60°
(phase lag is 2hr)
vx-vy space
x phasor
y phasor
x-y space
p1= vx/ω
p2=vy/ω
t=0
is
3PM
5PM
Ψ1
time
3PM
Ψ2
time
−ρ1
+ρ1
x =x1
y =x2
p
x
x1
x2
RelaWavity Simulation:
Ellipsometry - Lag = 60°

$p_1 = -A_1 \sin(\omega t + \rho_1)$

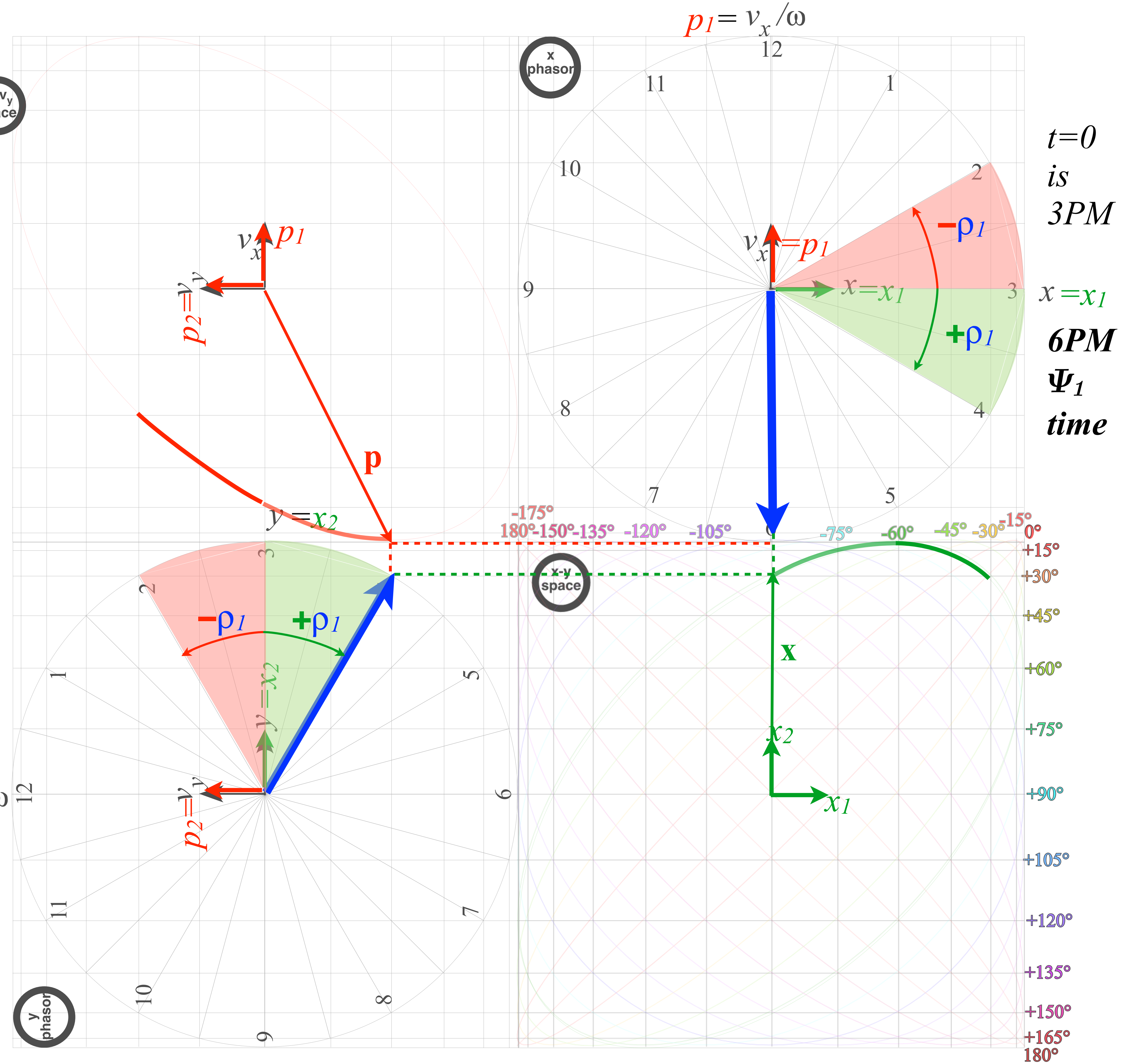
$p_2 = -A_2 \sin(\omega t - \rho_1)$

$x_1 = A_1 \cos(\omega t + \rho_1)$

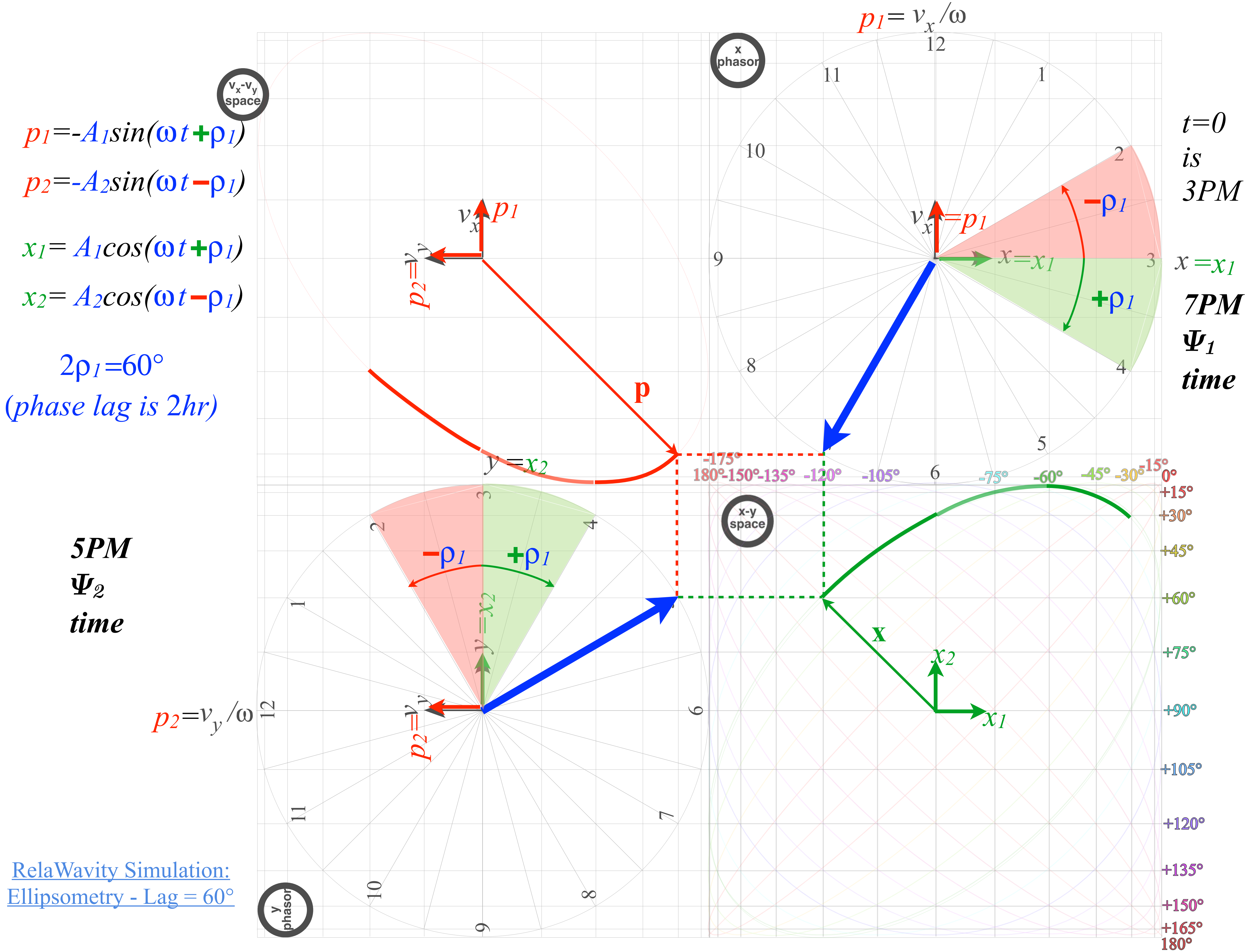
$x_2 = A_2 \cos(\omega t - \rho_1)$

$2\rho_1 = 60°$

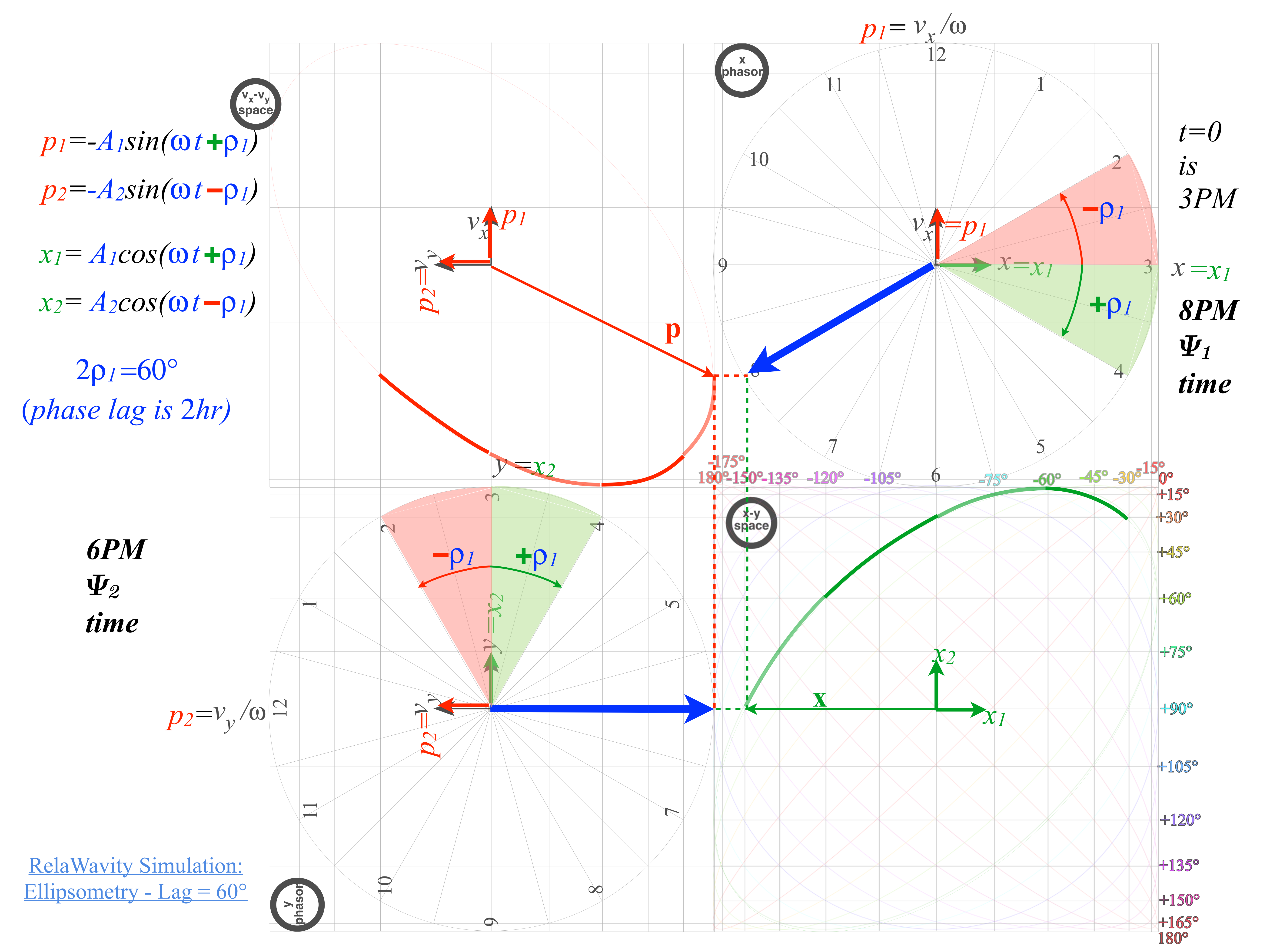
(phase lag is 2hr)



[RelaWavity Simulation: Ellipsometry - Lag = 60°]



$p_1 = v_x/\omega$
12
11
1
10
2
9
3
8
4
5
6
x phasor
v_x-v_y space
p_1=-A_1sin(ωt+ρ_1)
p_2=-A_2sin(ωt−ρ_1)
x_1= A_1cos(ωt+ρ_1)
x_2= A_2cos(ωt−ρ_1)
2ρ_1=60°
(phase lag is 2hr)
t=0
is
3PM
7PM
Ψ_1
time
v_x p_1
p_2=v_y
v_x=p_1
−ρ_1
+ρ_1
x=x_1
x =x_1
p
y=x_2
-175°
180° -150° -135° -120° -105° -75° -60° -45° -30° -15° 0°
+15°
+30°
+45°
+60°
+75°
+90°
+105°
+120°
+135°
+150°
+165°
180°
x-y space
5PM
Ψ_2
time
y=x_2
x
x_2
x_1
p_2=v_y/ω
y phasor
7
RelaWavity Simulation:
Ellipsometry - Lag = 60°



p1=-A1sin(ωt+ρ1)
p2=-A2sin(ωt−ρ1)
x1= A1cos(ωt+ρ1)
x2= A2cos(ωt−ρ1)
2ρ1=60°
(phase lag is 2hr)
vx-vy space
x phasor
p1= vx/ω
t=0
is
3PM
x =x1
8PM
Ψ1
time
−ρ1
+ρ1
p
y=x2
6PM
Ψ2
time
x-y space
p2=vy/ω
x
x1
x2
y phasor
RelaWavity Simulation:
Ellipsometry - Lag = 60°

$p_1 = -A_1 \sin(\omega t + \rho_1)$

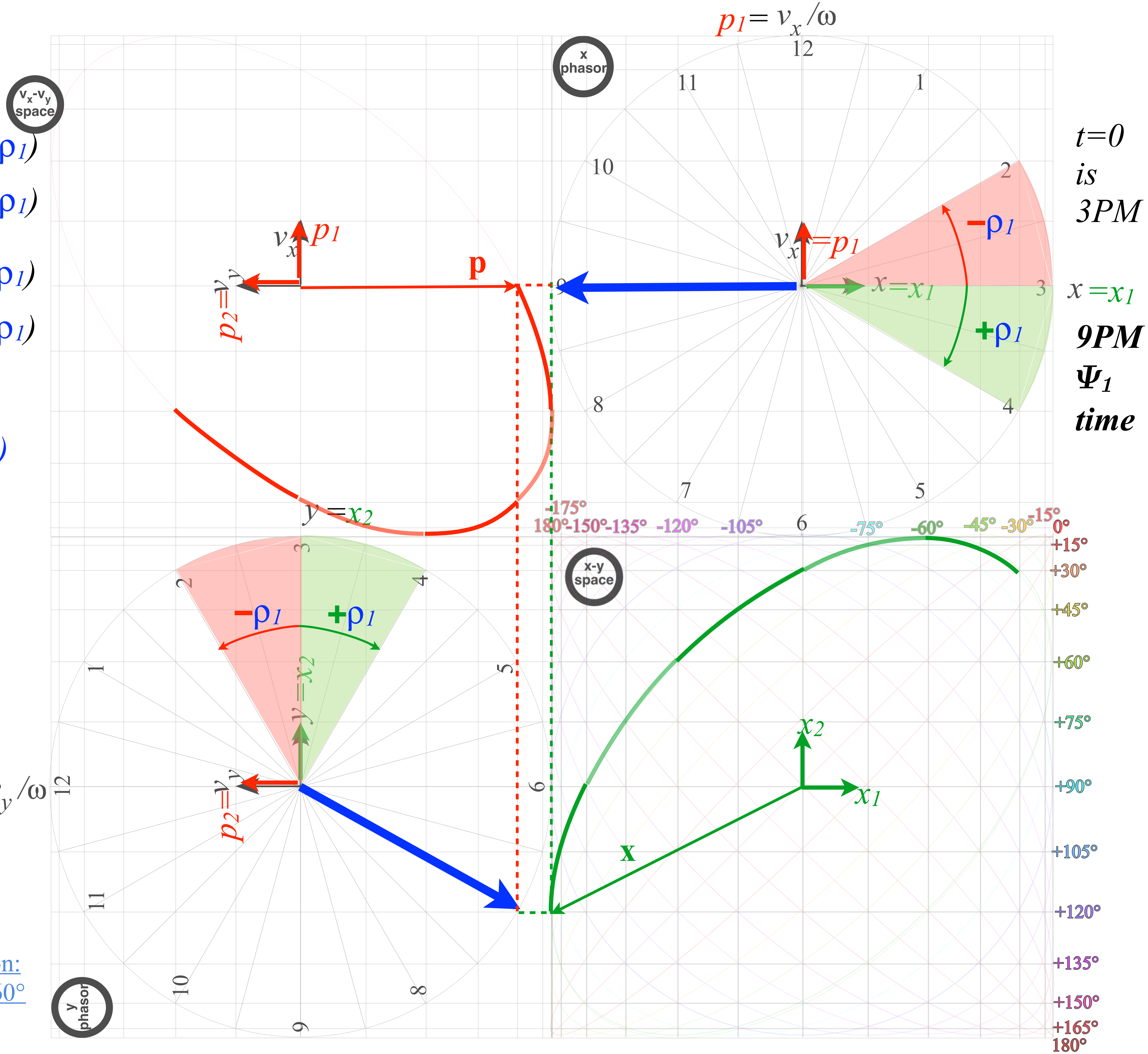
$p_2 = -A_2 \sin(\omega t - \rho_1)$

$x_1 = A_1 \cos(\omega t + \rho_1)$

$x_2 = A_2 \cos(\omega t - \rho_1)$

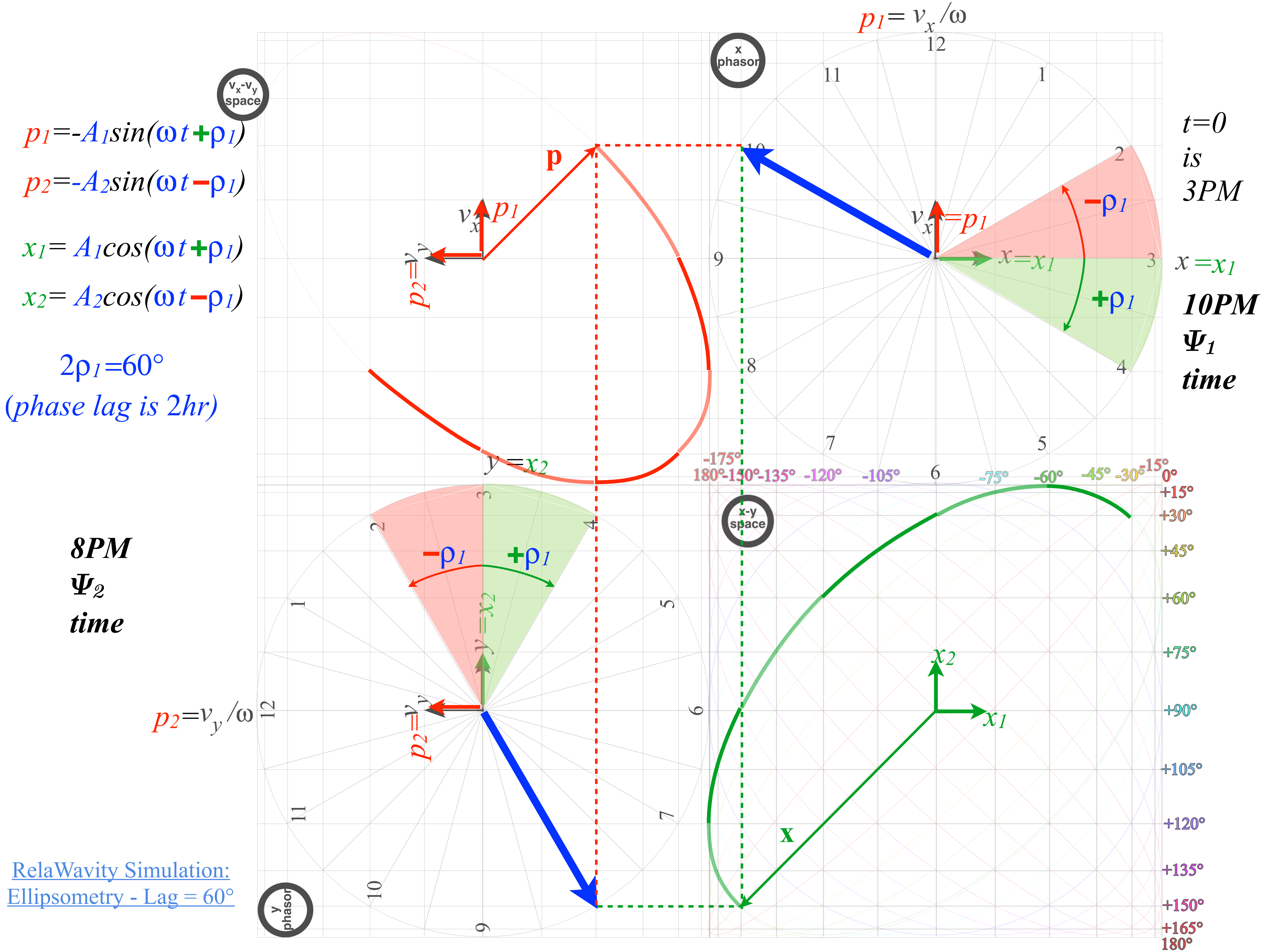
$2\rho_1 = 60°$

(phase lag is 2hr)

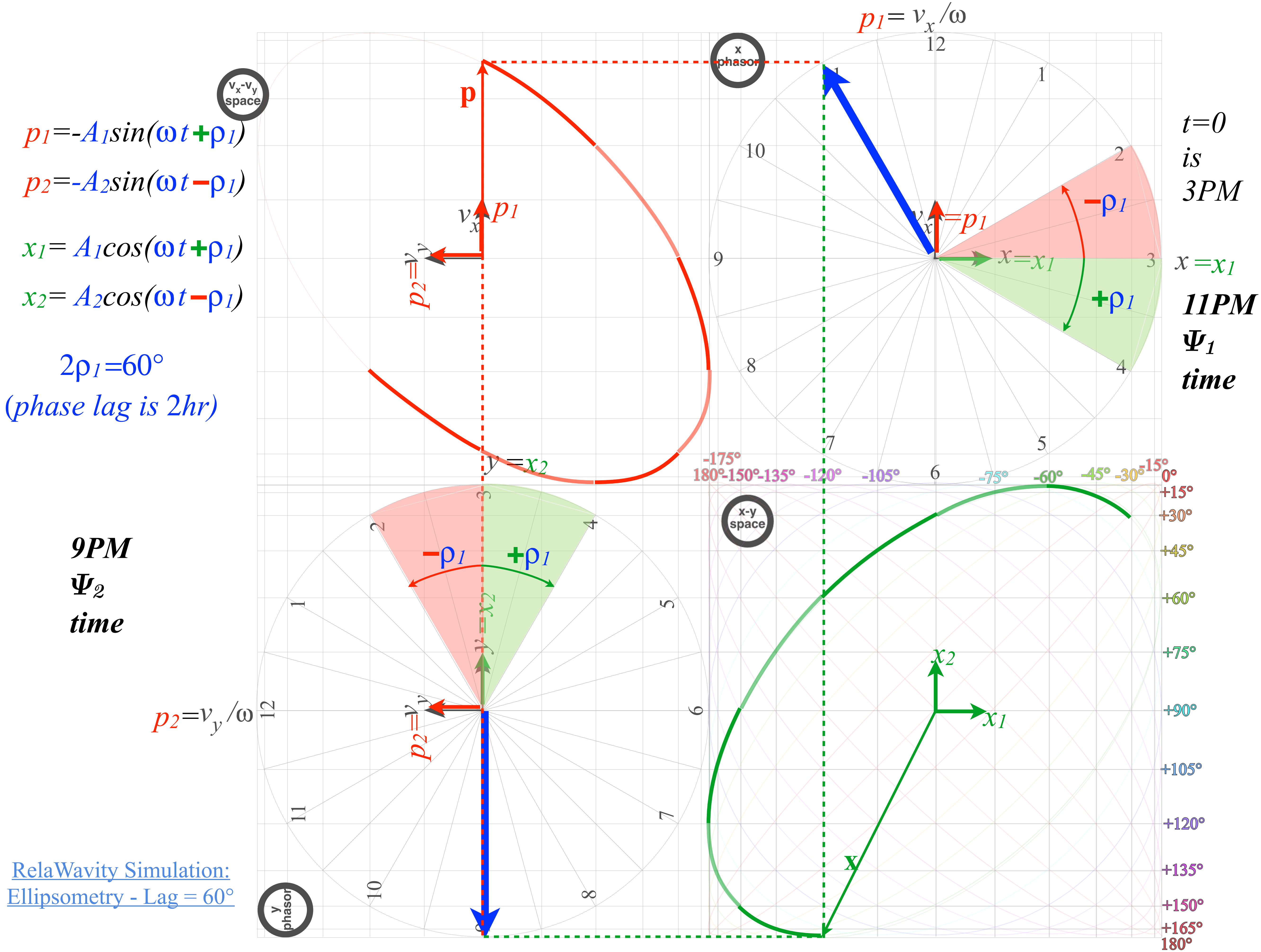


7PM
Ψ₂
time

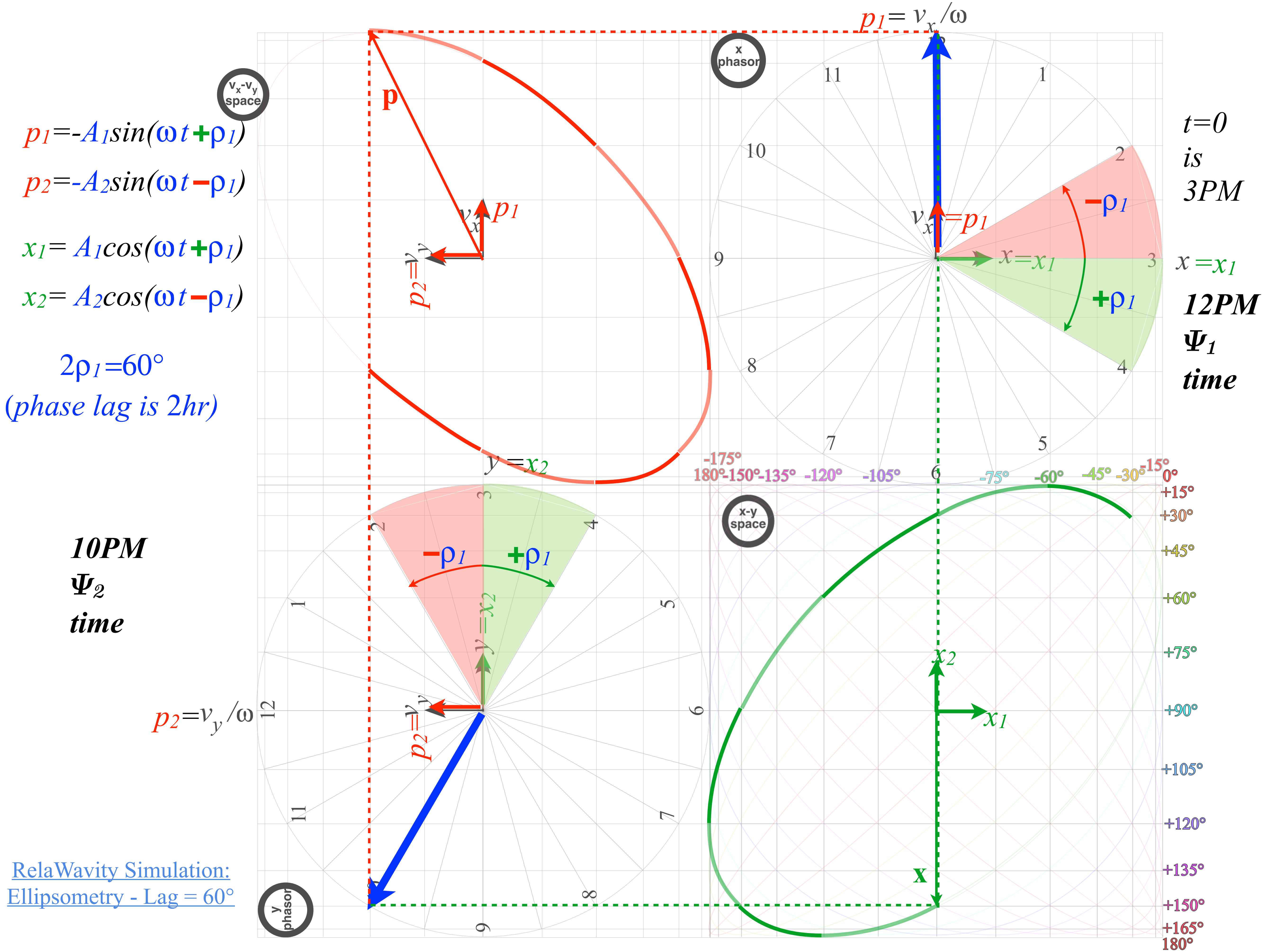
[RelaWavity Simulation: Ellipsometry - Lag = 60°]



[RelaWavity Simulation: Ellipsometry - Lag = 60°]



p_1=-A_1sin(ωt+$ρ_1$)
p_2=-A_2sin(ωt−$ρ_1$)
x_1= A_1cos(ωt+$ρ_1$)
x_2= A_2cos(ωt−$ρ_1$)
2$ρ_1$=60°
(phase lag is 2hr)
v_x-v_y space
p
p_1= v_x/ω
x phasor
t=0 is 3PM
11PM
$Ψ_1$ time
9PM
$Ψ_2$ time
p_2=v_y/ω
y phasor
x-y space
−$ρ_1$
+$ρ_1$
x =x_1
y =x_2
x_1
x_2
x
RelaWavity Simulation:
Ellipsometry - Lag = 60°



p_1=-$A_1 sin(\omega t + \rho_1)$
p_2=-$A_2 sin(\omega t - \rho_1)$
x_1= $A_1 cos(\omega t + \rho_1)$
x_2= $A_2 cos(\omega t - \rho_1)$
$2\rho_1$=60°
(phase lag is 2hr)
v_x-v_y space
x phasor
y phasor
x-y space
p
$p_1 = v_x/\omega$
$p_2 = v_y/\omega$
$x = x_1$
$y = x_2$
$-\rho_1$
$+\rho_1$
t=0
is
3PM
12PM
Ψ_1
time
10PM
Ψ_2
time
x
x_1
x_2
-175°
180°
-150°
-135°
-120°
-105°
-75°
-60°
-45°
-30°
-15°
0°
+15°
+30°
+45°
+60°
+75°
+90°
+105°
+120°
+135°
+150°
+165°
180°
RelaWavity Simulation:
Ellipsometry - Lag = 60°

$p_1 = -A_1 \sin(\omega t + \rho_1)$

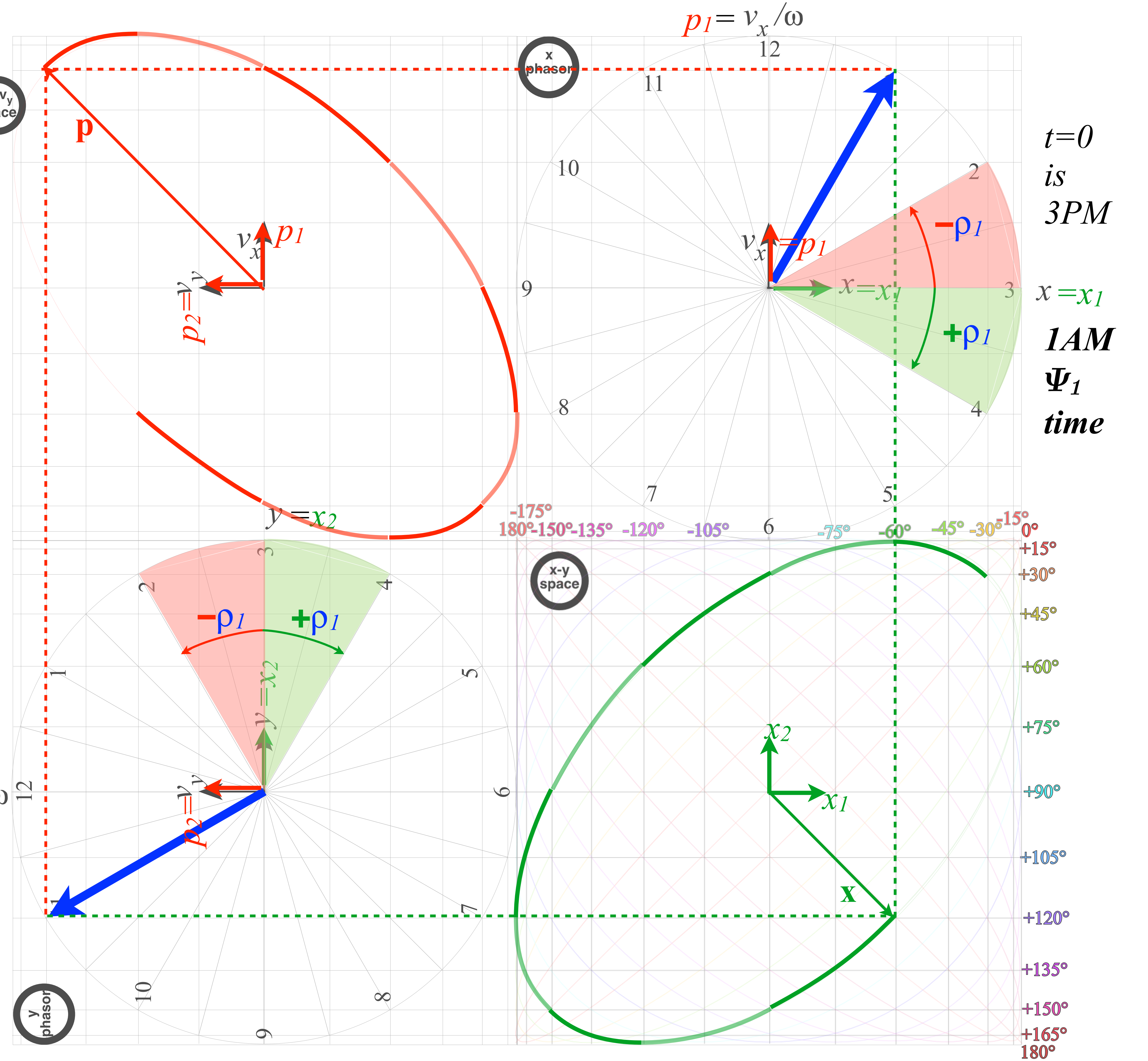
$p_2 = -A_2 \sin(\omega t - \rho_1)$

$x_1 = A_1 \cos(\omega t + \rho_1)$

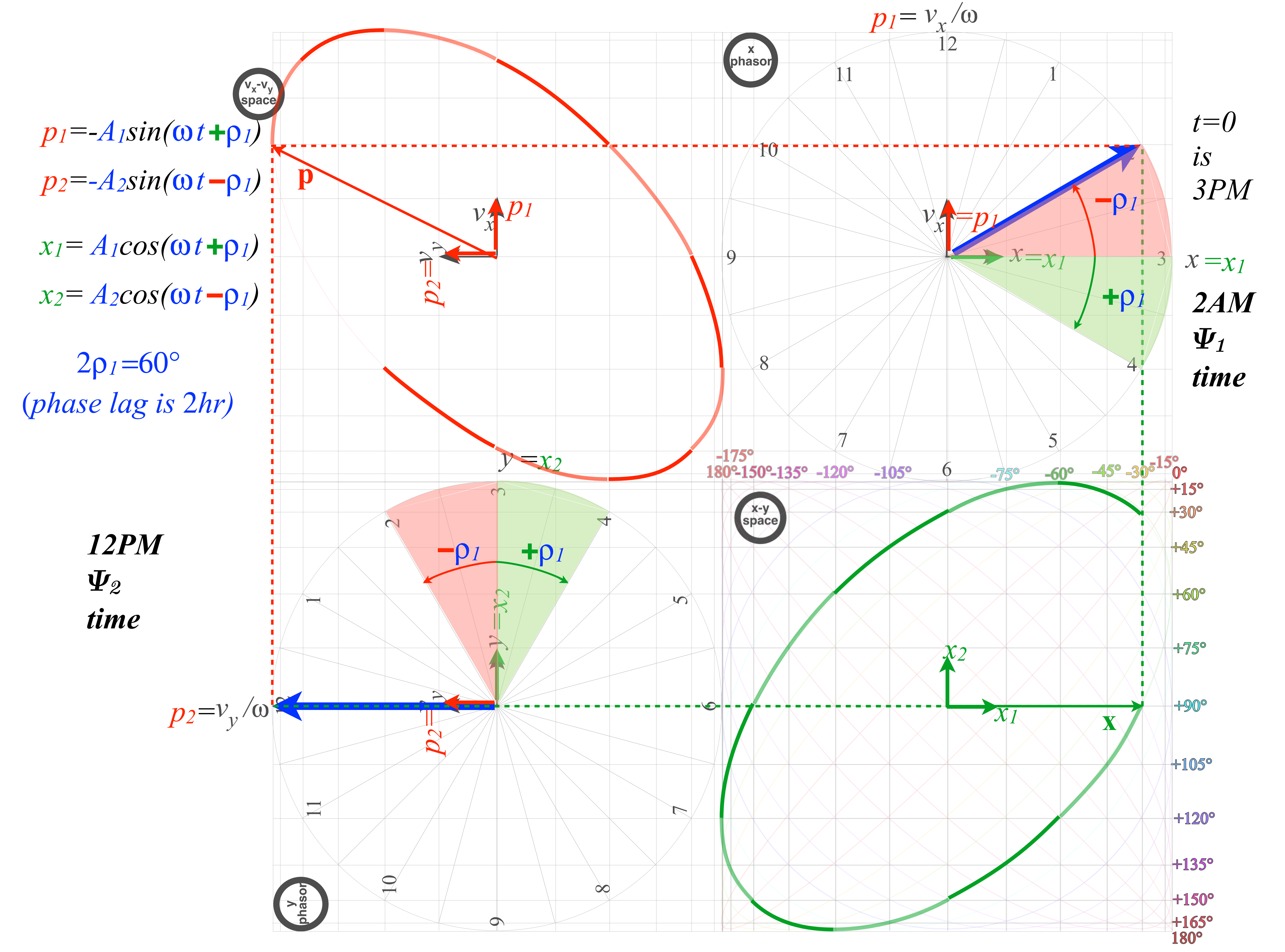
$x_2 = A_2 \cos(\omega t - \rho_1)$

$2\rho_1 = 60°$

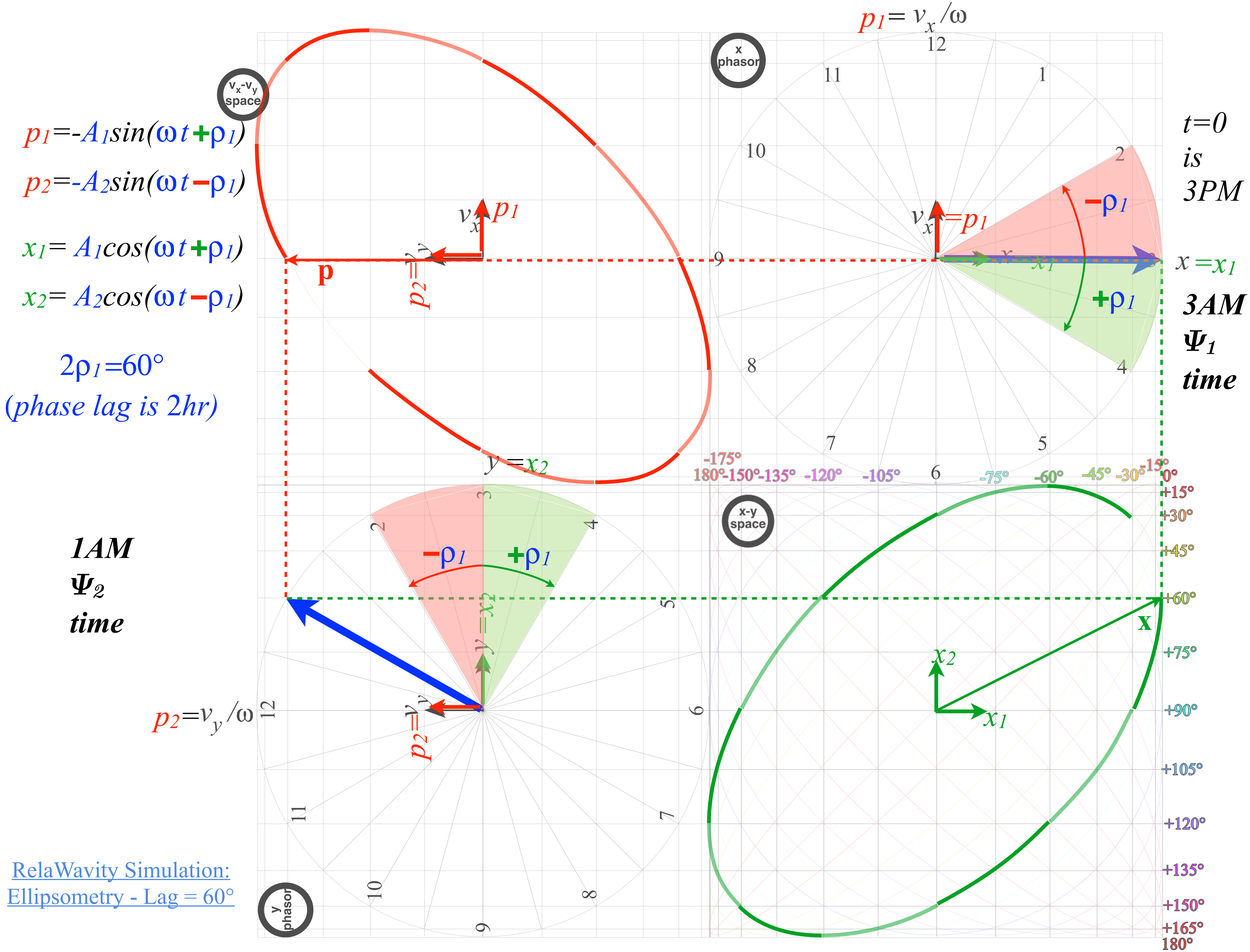
(phase lag is 2hr)



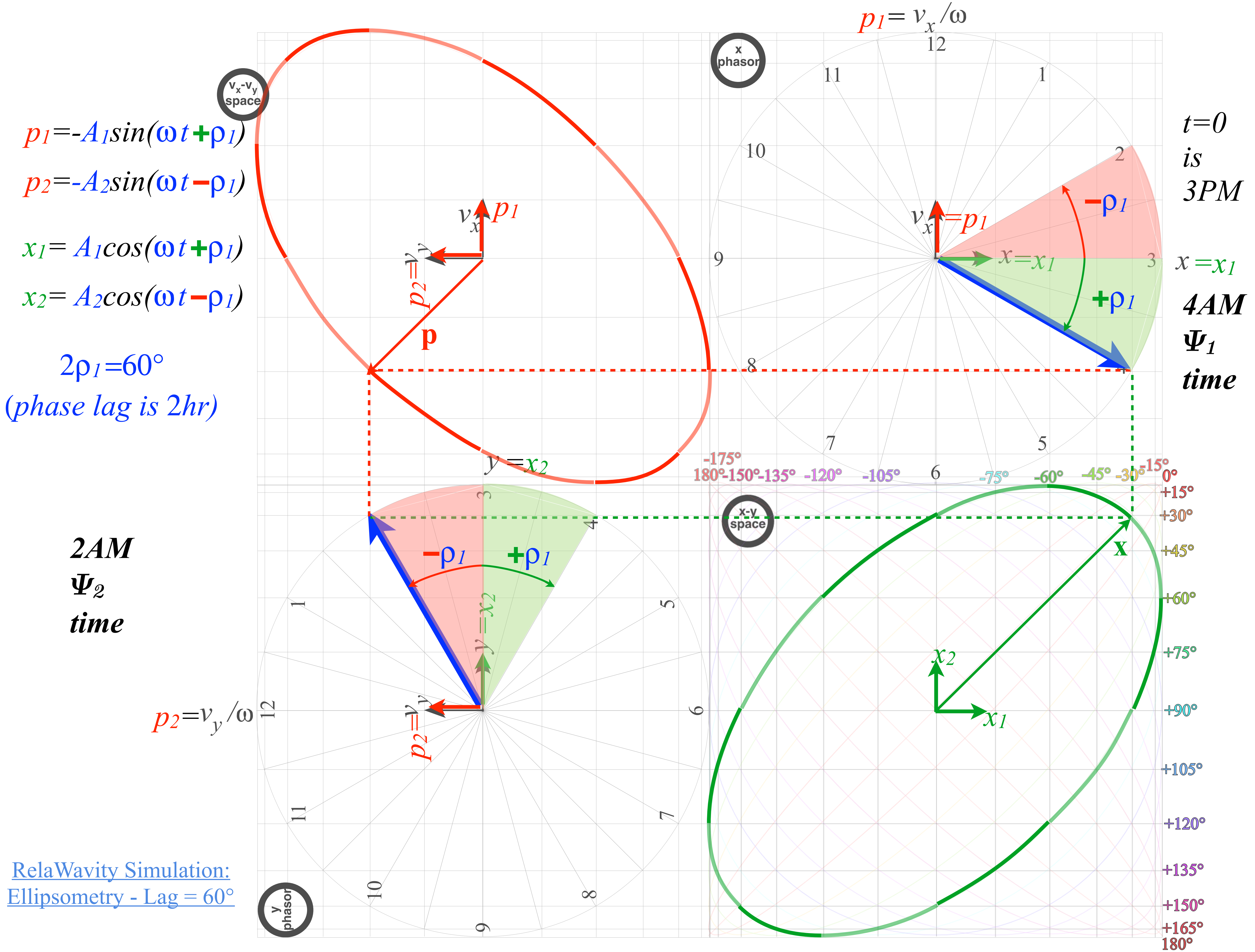
RelaWavity Simulation: Ellipsometry - Lag = 60°



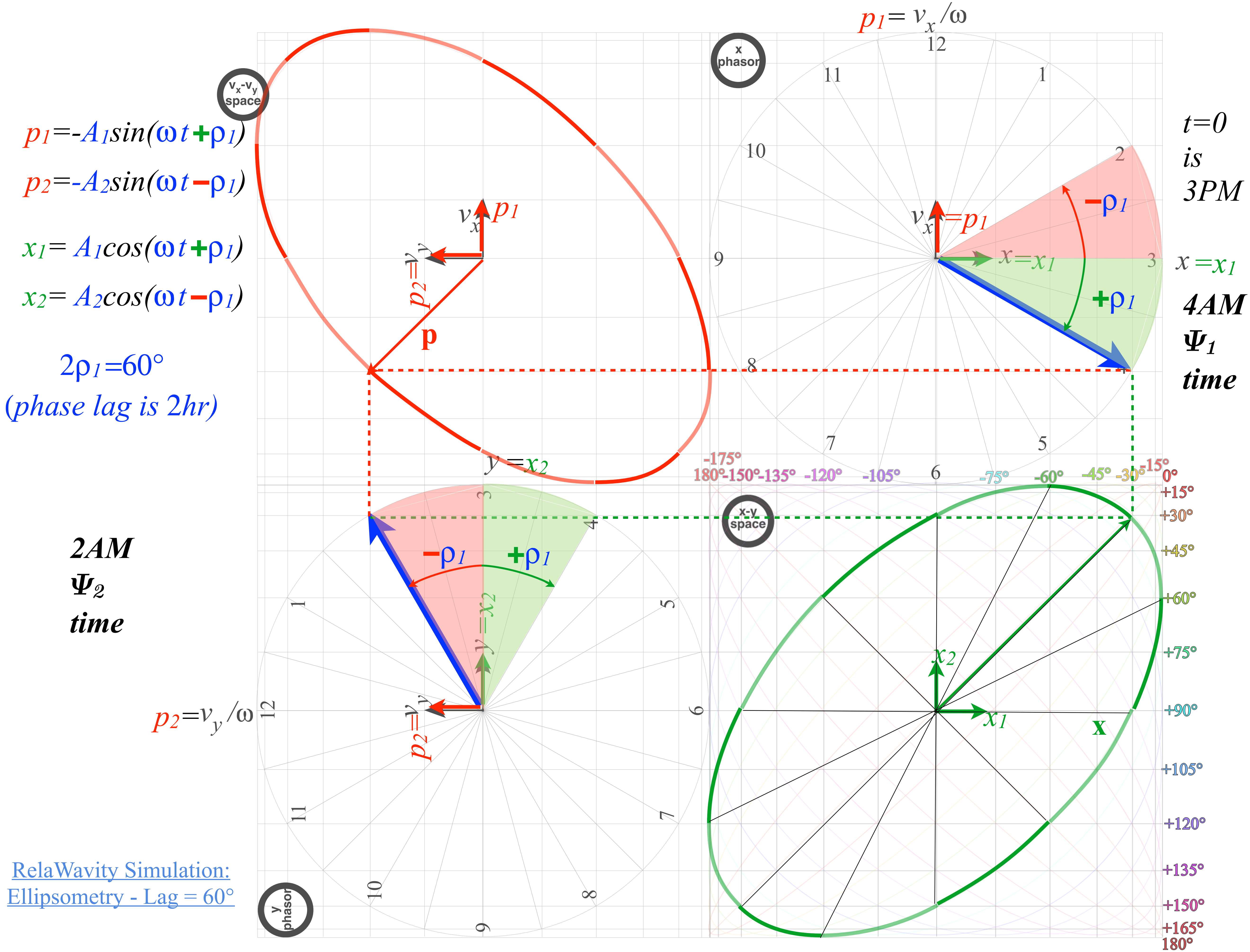
$p_1 = -A_1 \sin(\omega t + \rho_1)$
$p_2 = -A_2 \sin(\omega t - \rho_1)$
$x_1 = A_1 \cos(\omega t + \rho_1)$
$x_2 = A_2 \cos(\omega t - \rho_1)$
$2\rho_1 = 60°$
(phase lag is 2hr)
v_x-v_y space
x phasor
y phasor
x-y space
$p_1 = v_x/\omega$
$p_2 = v_y/\omega$
p
$v_x = p_1$
$v_y = p_2$
$x = x_1$
$y = x_2$
$-\rho_1$
$+\rho_1$
t=0 is 3PM
2AM Ψ_1 time
12PM Ψ_2 time
x_1
x_2
x
-175°
180° -150° -135° -120° -105° -75° -60° -45° -30° -15° 0°
+15° +30° +45° +60° +75° +90° +105° +120° +135° +150° +165° 180°



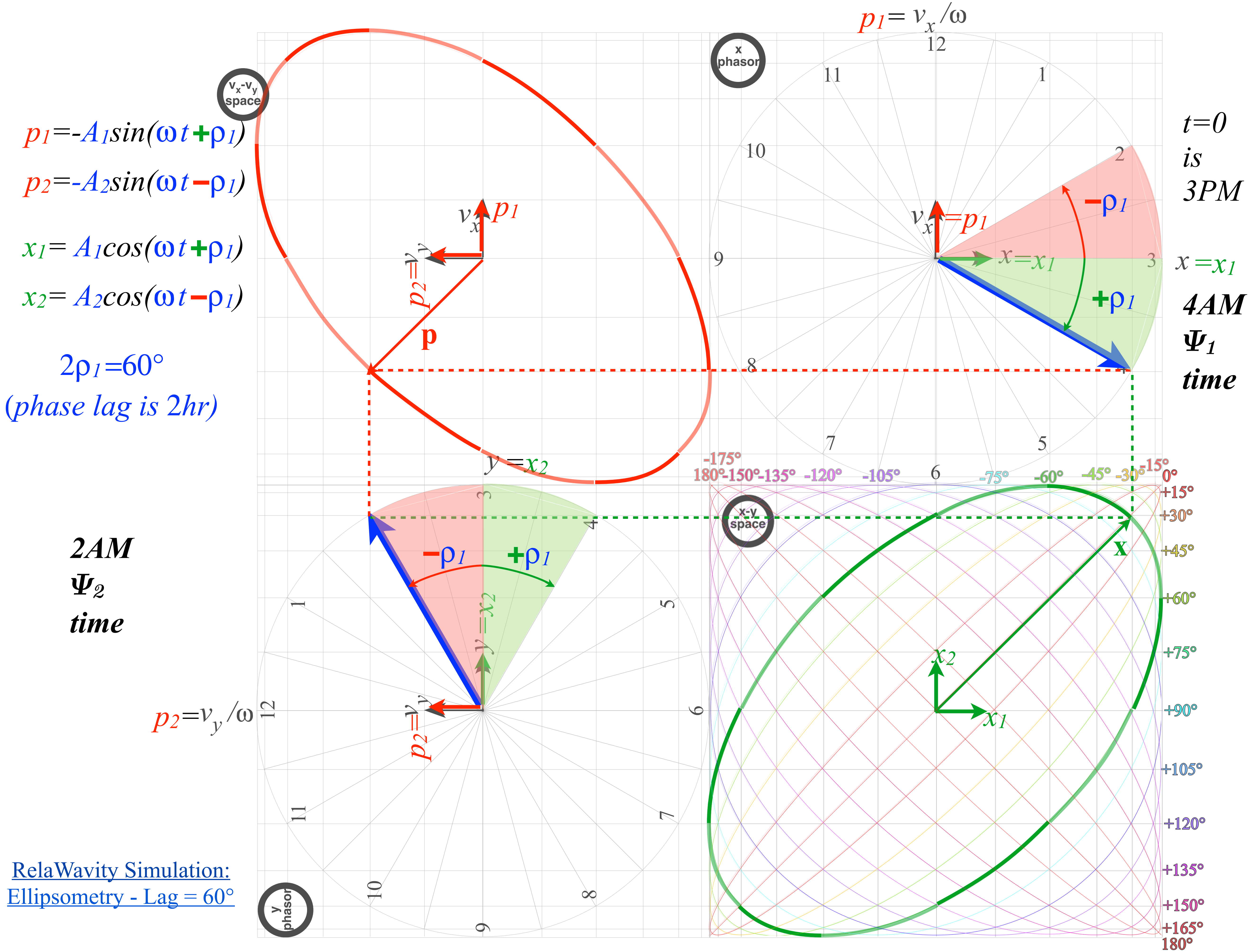
RelaWavity Simulation: Ellipsometry - Lag = 60°



[RelaWavity Simulation: Ellipsometry - Lag = 60°]



RelaWavity Simulation: Ellipsometry - Lag = 60°



RelaWavity Simulation: Ellipsometry - Lag = 60°

Review: Fundamental Euler **R**$(\alpha\beta\gamma)$ *and Darboux* **R**$[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

Euler **R**$(\alpha\beta\gamma)$ *derived from Darboux* **R**$[\varphi\vartheta\Theta]$ *and vice versa*

Euler **R**$(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*

R(3)-U(2) slide rule for converting **R**$(\alpha\beta\gamma) \leftrightarrow$ **R**$[\varphi\vartheta\Theta]$ *and Sundial*

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ABC-Type elliptical polarized motion

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates

→ *Euler Angle* $(\alpha\beta\gamma)$ *ellipse coordinates*

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates related to Euler Angles (αβγ)

2D elliptic frequency ω orbit has amplitudes A_1 and A_2, and phase shifts ρ_1 and $\rho_2 = -\rho_1$.

$$\begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

$$x_1 = A_1 \cos(\omega t + \rho_1)$$
$$-p_1 = A_1 \sin(\omega t + \rho_1)$$
$$x_2 = A_2 \cos(\omega t - \rho_1)$$
$$-p_2 = A_2 \sin(\omega t - \rho_1)$$

Real x_k and imaginary p_k parts of phasor amplitudes $a_k = x_k + ip_k$ depend on Euler angles $(\alpha\beta\gamma)$ and A.

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates related to Euler Angles ($\alpha\beta\gamma$)

2D elliptic frequency ω orbit has amplitudes A_1 and A_2, and phase shifts ρ_1 and $\rho_2 = -\rho_1$.

$$\begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = A_1\cos(\omega t+\rho_1)$$
$$-p_1 = A_1\sin(\omega t+\rho_1)$$
$$x_2 = A_2\cos(\omega t-\rho_1)$$
$$-p_2 = A_2\sin(\omega t-\rho_1)$$

Real x_k and imaginary p_k parts of phasor amplitudes $a_k = x_k + ip_k$ depend on Euler angles ($\alpha\beta\gamma$) and A.

$$x_1 = A\cos\beta/2\cos[(\gamma+\alpha)/2]$$
$$-p_1 = A\cos\beta/2\sin[(\gamma+\alpha)/2]$$
$$x_2 = A\sin\beta/2\cos[(\gamma-\alpha)/2]$$
$$-p_2 = A\sin\beta/2\sin[(\gamma-\alpha)/2]$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates related to Euler Angles $(\alpha\beta\gamma)$

2D elliptic frequency ω orbit has amplitudes A_1 and A_2, and phase shifts ρ_1 and $\rho_2=-\rho_1$.

$$\begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = A_1 \cos(\omega t+\rho_1)$$
$$-p_1 = A_1 \sin(\omega t+\rho_1)$$
$$x_2 = A_2 \cos(\omega t-\rho_1)$$
$$-p_2 = A_2 \sin(\omega t-\rho_1)$$

Let: $A_1 = A\cos\beta/2$

Real x_k and imaginary p_k parts of phasor amplitudes $a_k=x_k+ip_k$ depend on Euler angles $(\alpha\beta\gamma)$ and A.

$$x_1 = A\cos\beta/2\cos[(\gamma+\alpha)/2]$$
$$-p_1 = A\cos\beta/2\sin[(\gamma+\alpha)/2]$$
$$x_2 = A\sin\beta/2\cos[(\gamma-\alpha)/2]$$
$$-p_2 = A\sin\beta/2\sin[(\gamma-\alpha)/2]$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates related to Euler Angles $(\alpha\beta\gamma)$

2D elliptic frequency ω orbit has amplitudes A_1 and A_2, and phase shifts ρ_1 and $\rho_2=-\rho_1$.

$$\begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = A_1 \cos(\omega t+\rho_1)$$
$$-p_1 = A_1 \sin(\omega t+\rho_1)$$
$$x_2 = A_2 \cos(\omega t-\rho_1)$$
$$-p_2 = A_2 \sin(\omega t-\rho_1)$$

Let: $A_1 = A\cos\beta/2$

$A_2 = A\sin\beta/2$

Real x_k and imaginary p_k parts of phasor amplitudes $a_k=x_k+ip_k$ depend on Euler angles $(\alpha\beta\gamma)$ and A.

$$x_1 = A\cos\beta/2\cos[(\gamma+\alpha)/2]$$
$$-p_1 = A\cos\beta/2\sin[(\gamma+\alpha)/2]$$
$$x_2 = A\sin\beta/2\cos[(\gamma-\alpha)/2]$$
$$-p_2 = A\sin\beta/2\sin[(\gamma-\alpha)/2]$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates related to Euler Angles ($\alpha\beta\gamma$)

2D elliptic frequency ω orbit has amplitudes A_1 and A_2, and phase shifts ρ_1 and $\rho_2 = -\rho_1$.

Real x_k and imaginary p_k parts of phasor amplitudes $a_k = x_k + ip_k$ depend on Euler angles ($\alpha\beta\gamma$) and A.

$$\begin{pmatrix} A_1 e^{-i(\omega t + \rho_1)} \\ A_2 e^{-i(\omega t - \rho_1)} \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

$$\begin{aligned} x_1 &= A_1 \cos(\omega t + \rho_1) \\ -p_1 &= A_1 \sin(\omega t + \rho_1) \\ x_2 &= A_2 \cos(\omega t - \rho_1) \\ -p_2 &= A_2 \sin(\omega t - \rho_1) \end{aligned}$$

Let: $A_1 = A\cos\beta/2$

$A_2 = A\sin\beta/2$

$$\begin{aligned} x_1 &= A\cos\beta/2 \cos[(\gamma+\alpha)/2] \\ -p_1 &= A\cos\beta/2 \sin[(\gamma+\alpha)/2] \\ x_2 &= A\sin\beta/2 \cos[(\gamma-\alpha)/2] \\ -p_2 &= A\sin\beta/2 \sin[(\gamma-\alpha)/2] \end{aligned}$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

Let: $\omega t + \rho_1 = (\gamma + \alpha)/2$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t + \rho_1)} \\ A_2 e^{-i(\omega t - \rho_1)} \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates related to Euler Angles $(\alpha\beta\gamma)$

2D elliptic frequency ω orbit has amplitudes A_1 and A_2, and phase shifts ρ_1 and $\rho_2=-\rho_1$.

$$\begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = A_1 \cos(\omega t+\rho_1)$$
$$-p_1 = A_1 \sin(\omega t+\rho_1)$$
$$x_2 = A_2 \cos(\omega t-\rho_1)$$
$$-p_2 = A_2 \sin(\omega t-\rho_1)$$

Let: $A_1 = A\cos\beta/2$

$A_2 = A\sin\beta/2$

Real x_k and imaginary p_k parts of phasor amplitudes $a_k=x_k+ip_k$ depend on Euler angles $(\alpha\beta\gamma)$ and A.

$$x_1 = A\cos\beta/2\cos[(\gamma+\alpha)/2]$$
$$-p_1 = A\cos\beta/2\sin[(\gamma+\alpha)/2]$$
$$x_2 = A\sin\beta/2\cos[(\gamma-\alpha)/2]$$
$$-p_2 = A\sin\beta/2\sin[(\gamma-\alpha)/2]$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

Let: $\omega t+\rho_1 = (\gamma+\alpha)/2$

$\omega t-\rho_1 = (\gamma-\alpha)/2$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates related to Euler Angles $(\alpha\beta\gamma)$

2D elliptic frequency ω orbit has amplitudes A_1 and A_2, and phase shifts ρ_1 and $\rho_2=-\rho_1$.

$$\begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = A_1 \cos(\omega t+\rho_1)$$
$$-p_1 = A_1 \sin(\omega t+\rho_1)$$
$$x_2 = A_2 \cos(\omega t-\rho_1)$$
$$-p_2 = A_2 \sin(\omega t-\rho_1)$$

Let: $A_1 = A\cos\beta/2$

$A_2 = A\sin\beta/2$

$$\tan\beta/2 = A_2/A_1 \qquad A^2 = A_1^2 + A_2^2$$

Real x_k and imaginary p_k parts of phasor amplitudes $a_k = x_k + ip_k$ depend on Euler angles $(\alpha\beta\gamma)$ and A.

$$x_1 = A\cos\beta/2\cos[(\gamma+\alpha)/2]$$
$$-p_1 = A\cos\beta/2\sin[(\gamma+\alpha)/2]$$
$$x_2 = A\sin\beta/2\cos[(\gamma-\alpha)/2]$$
$$-p_2 = A\sin\beta/2\sin[(\gamma-\alpha)/2]$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

Let: $\omega t+\rho_1 = (\gamma+\alpha)/2$

$\omega t-\rho_1 = (\gamma-\alpha)/2$

$$\alpha = 2\rho_1 \qquad \gamma = 2\omega\cdot t$$

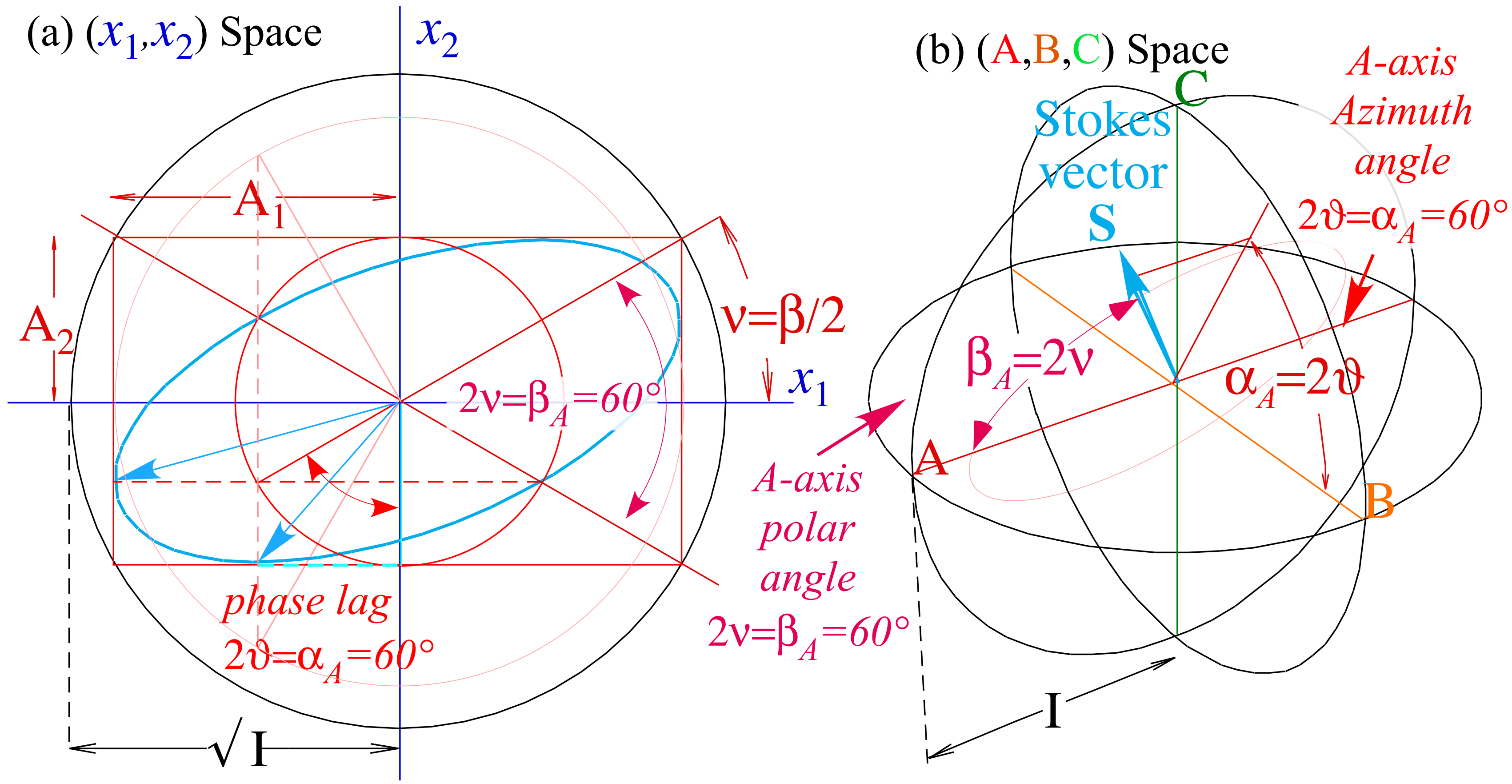
Euler parameters (α,β,γ, A) in terms of *amp-phase parameters* $(A_1, A_2, \omega t, \rho_1)$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

The A-view in {x1,x2}-basis

Angles $\alpha_A = \rho_1 - \rho_2 = 2\rho_1$, $\beta_A = 2\tan^{-1} A_2/A_1$, $\gamma_A = 2\omega \cdot t$ define ellipses with intensity $I = A^2 = A_1^2 + A_2^2$.

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = A \begin{pmatrix} e^{-i\alpha_A/2} \cos\frac{\beta_A}{2} \\ e^{+i\alpha_A/2} \sin\frac{\beta_A}{2} \end{pmatrix} e^{-i\omega t} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$



A or *Z*-axis Euler angles

$\alpha = \alpha_A = \rho_1 - \rho_2 = 2\rho_1 = 60°$

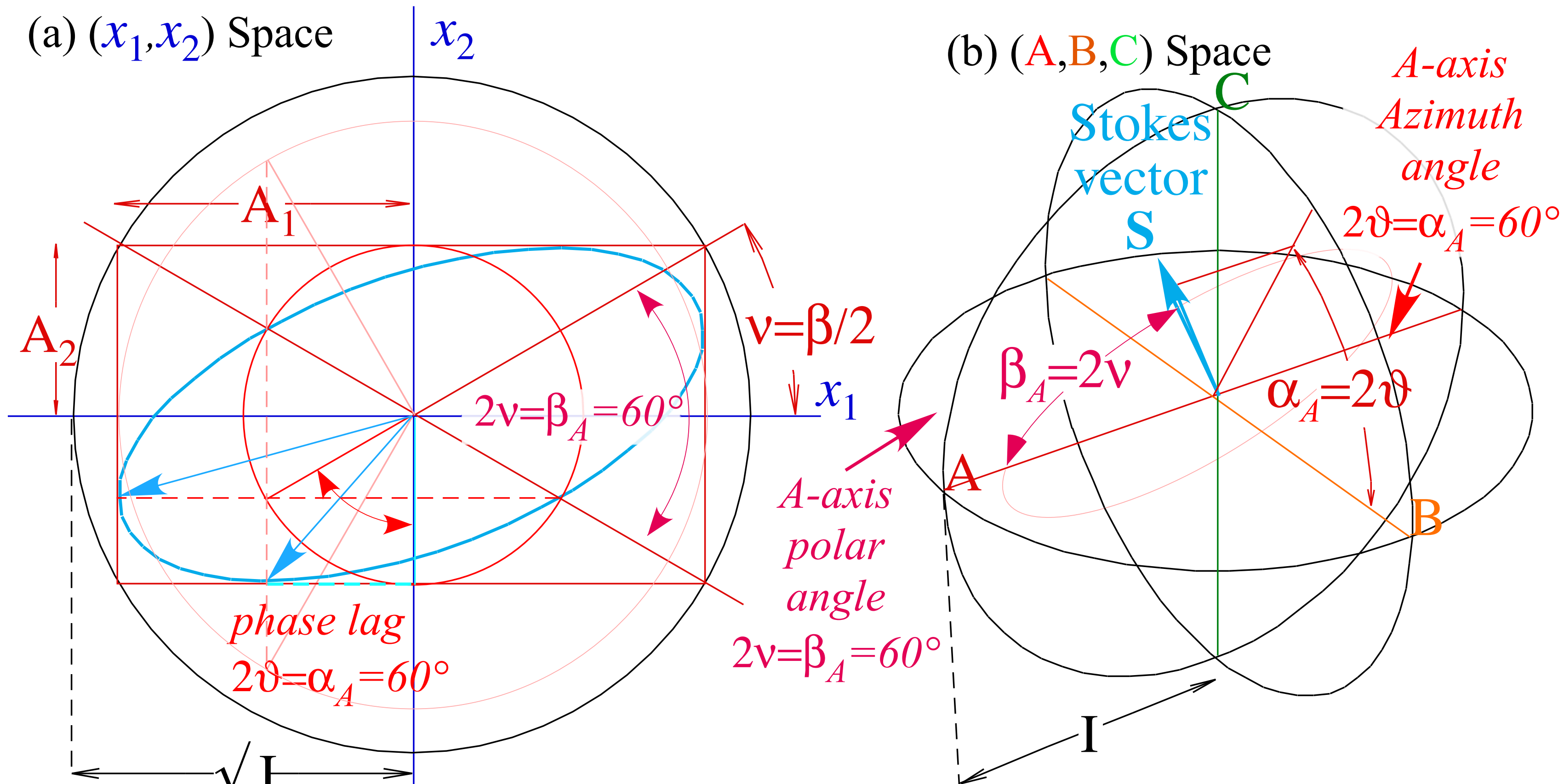
$\beta = \beta_A = 2\tan^{-1} A_2/A_1 = 60°$

$\gamma_A = 2\omega \cdot t$

The A-view in {x₁,x₂}-basis

Angles $\alpha_A = \rho_1 - \rho_2 = 2\rho_1$, $\beta_A = 2\tan^{-1} A_2/A_1$, $\gamma_A = 2\omega \cdot t$ define ellipses with intensity $I = A^2 = A_1^2 + A_2^2$.

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = A \begin{pmatrix} e^{-i\alpha_A/2}\cos\frac{\beta_A}{2} \\ e^{+i\alpha_A/2}\sin\frac{\beta_A}{2} \end{pmatrix} e^{-i\omega t} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$



A or *Z*-axis Euler angles

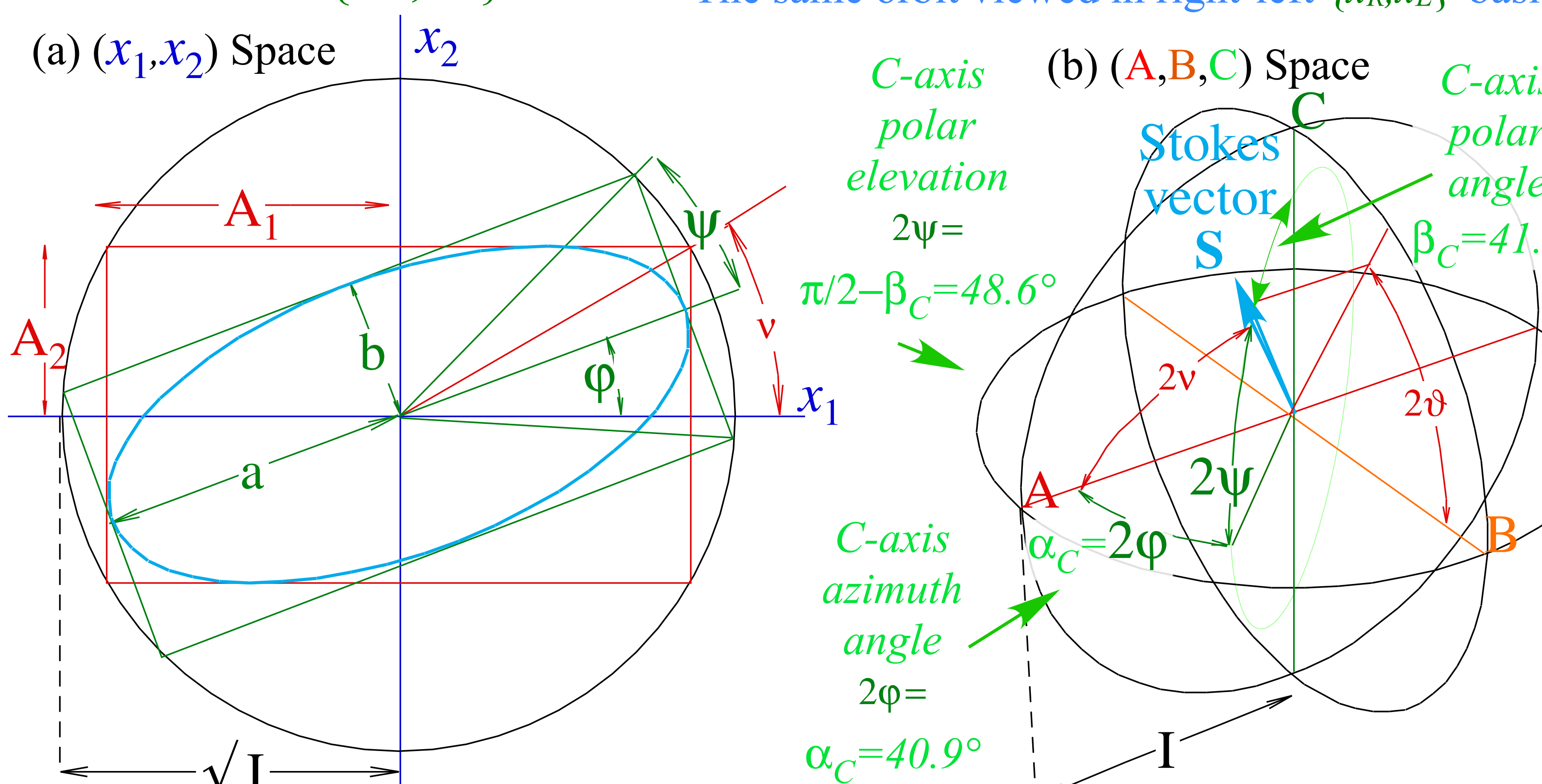
$\alpha = \alpha_A = \rho_1 - \rho_2 = 2\rho_1 = 60°$

$\beta = \beta_A = 2\tan^{-1} A_2/A_1 = 60°$

$\gamma_A = 2\omega \cdot t$

The C-view in {x_R,x_L}-basis

The same orbit viewed in right-left $\{x_R, x_L\}$-basis of circular polarization with angles $(\alpha_C, \beta_C, \gamma_C)$.



$$\begin{pmatrix} a_R \\ a_L \end{pmatrix} = A \begin{pmatrix} e^{-i\alpha_C/2}\cos\frac{\beta_C}{2} \\ e^{+i\alpha_C/2}\sin\frac{\beta_C}{2} \end{pmatrix} e^{-i\frac{\gamma_C}{2}} = \begin{pmatrix} x_R + ip_R \\ x_R + ip_R \end{pmatrix}$$

Converting an A-based set of Stokes parameters into a C-based set or a B-based set involves cyclic permutation of A, B, and C polar formulas

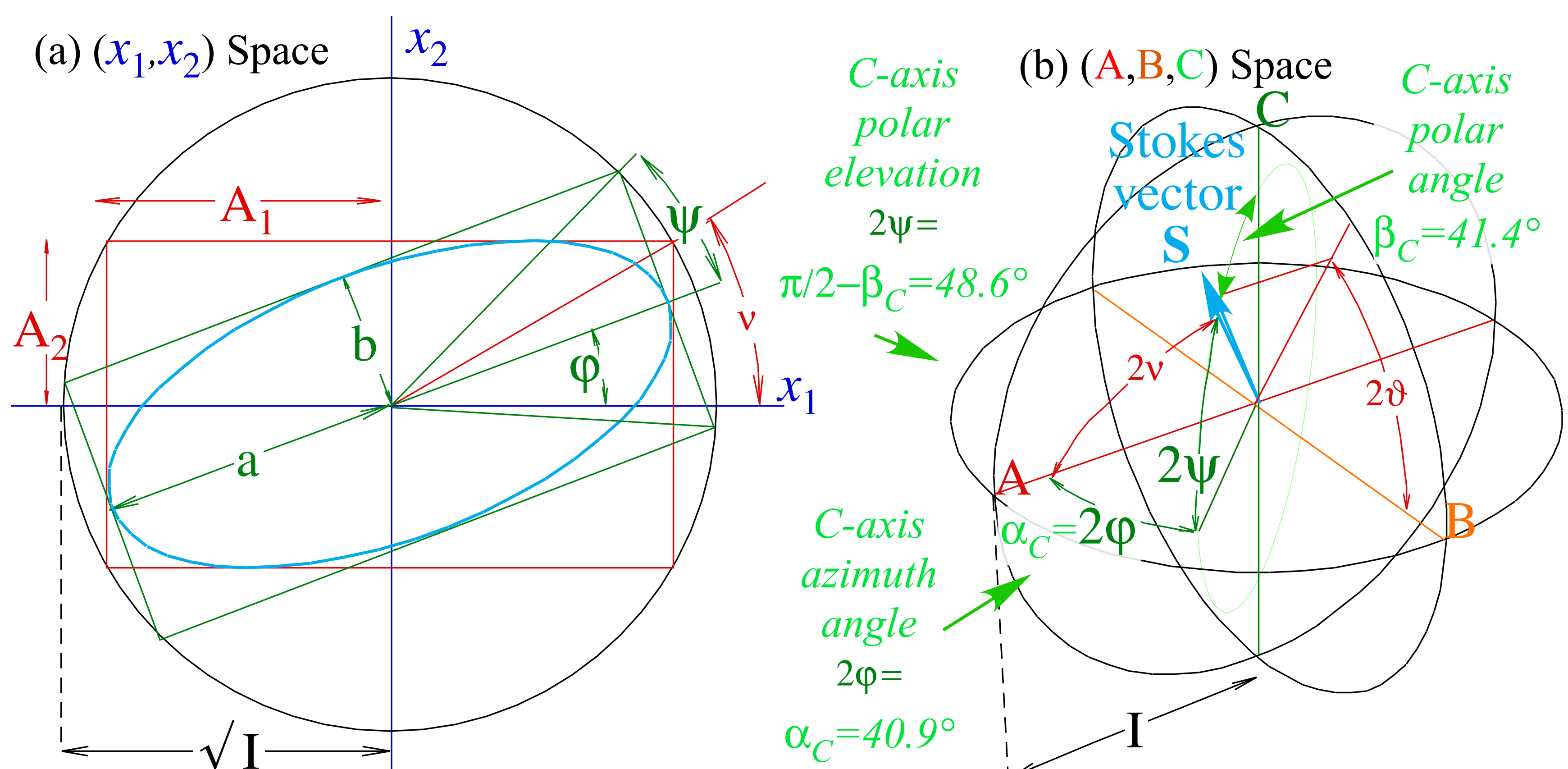
$$\text{Asymmetry } S_A = \frac{I}{2}\cos\beta_A = \frac{I}{2}\sin\alpha_B \sin\beta_B = \frac{I}{2}\cos\alpha_C \sin\beta_C$$

$$\text{Balance } S_B = \frac{I}{2}\cos\alpha_A \sin\beta_A = \frac{I}{2}\cos\beta_B = \frac{I}{2}\sin\alpha_C \sin\beta_C$$

$$\text{Chirality } S_C = \frac{I}{2}\sin\alpha_A \sin\beta_A = \frac{I}{2}\cos\alpha_B \sin\beta_B = \frac{I}{2}\cos\beta_C$$

The C-view in $\{x_R, x_L\}$-basis

The same orbit viewed in right and left circular polarization $\{x_R, x_L\}$-bases using angles $(\alpha_C, \beta_C, \gamma_C)$.



Converting an A-based set of Stokes parameters into a C-based set or a B-based set involves cyclic permutation of A, B, and C polar formulas

$$\textit{Asymmetry}\quad S_A = \frac{I}{2}\cos\beta_A = \frac{I}{2}\sin\alpha_B \sin\beta_B = \frac{I}{2}\cos\alpha_C \sin\beta_C$$

$$\textit{Balance}\quad S_B = \frac{I}{2}\cos\alpha_A \sin\beta_A = \frac{I}{2}\cos\beta_B = \frac{I}{2}\sin\alpha_C \sin\beta_C$$

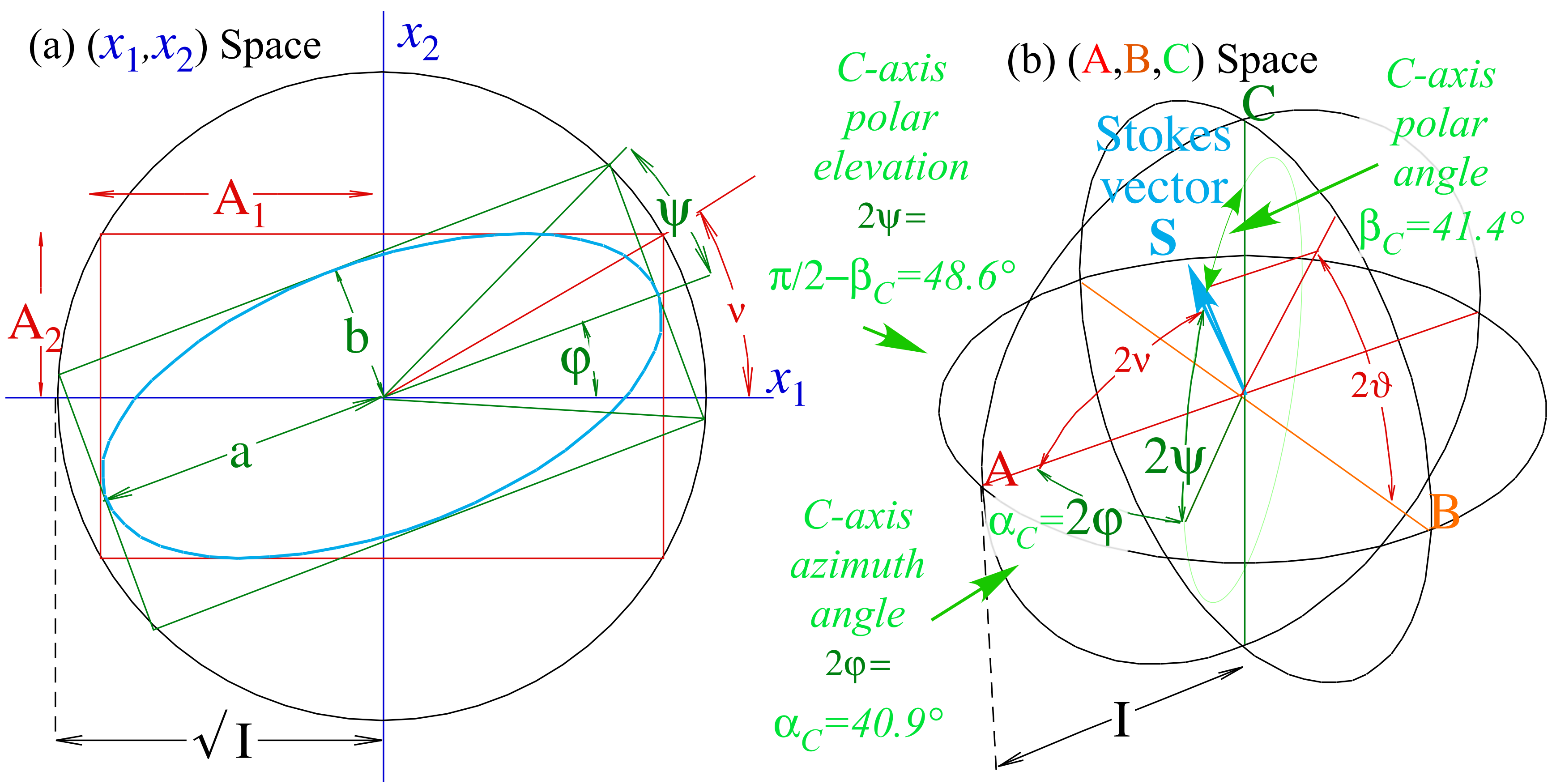
$$\textit{Chirality}\quad S_C = \frac{I}{2}\sin\alpha_A \sin\beta_A = \frac{I}{2}\cos\alpha_B \sin\beta_B = \frac{I}{2}\cos\beta_C$$

The C-view in $\{x_R, x_L\}$-basis

The same orbit viewed in right and left circular polarization $\{x_R, x_L\}$-bases using angles $(\alpha_C, \beta_C, \gamma_C)$.

Angles (α_C, β_C): C-axial polar angle β_C from above.

$$\sin\alpha_A \sin\beta_A = \cos\beta_C \qquad \text{or:}\quad \beta_C = \cos^{-1}(\sin\alpha_A \sin\beta_A) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\right) = 41.4°$$



Converting an A-based set of Stokes parameters into a C-based set or a B-based set involves cyclic permutation of A, B, and C polar formulas

$$\text{Asymmetry } S_A = \frac{I}{2}\cos\beta_A = \frac{I}{2}\sin\alpha_B\sin\beta_B = \frac{I}{2}\cos\alpha_C\sin\beta_C$$

$$\text{Balance } S_B = \frac{I}{2}\cos\alpha_A\sin\beta_A = \frac{I}{2}\cos\beta_B = \frac{I}{2}\sin\alpha_C\sin\beta_C$$

$$\text{Chirality } S_C = \frac{I}{2}\sin\alpha_A\sin\beta_A = \frac{I}{2}\cos\alpha_B\sin\beta_B = \frac{I}{2}\cos\beta_C$$

The C-view in $\{x_R,x_L\}$-basis

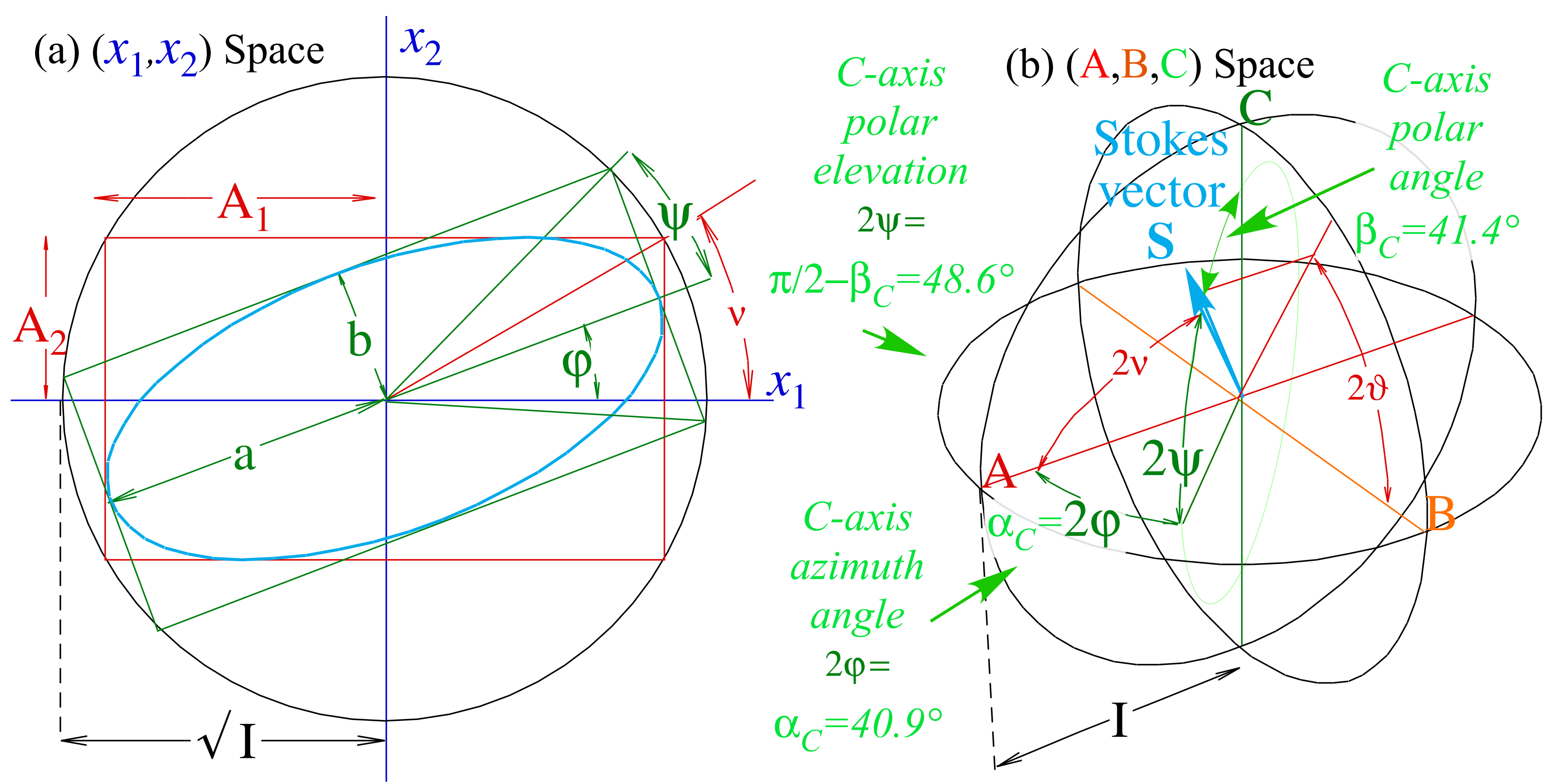
The same orbit viewed in right and left circular polarization $\{x_R,x_L\}$-bases using angles $(\alpha_C,\beta_C,\gamma_C)$.

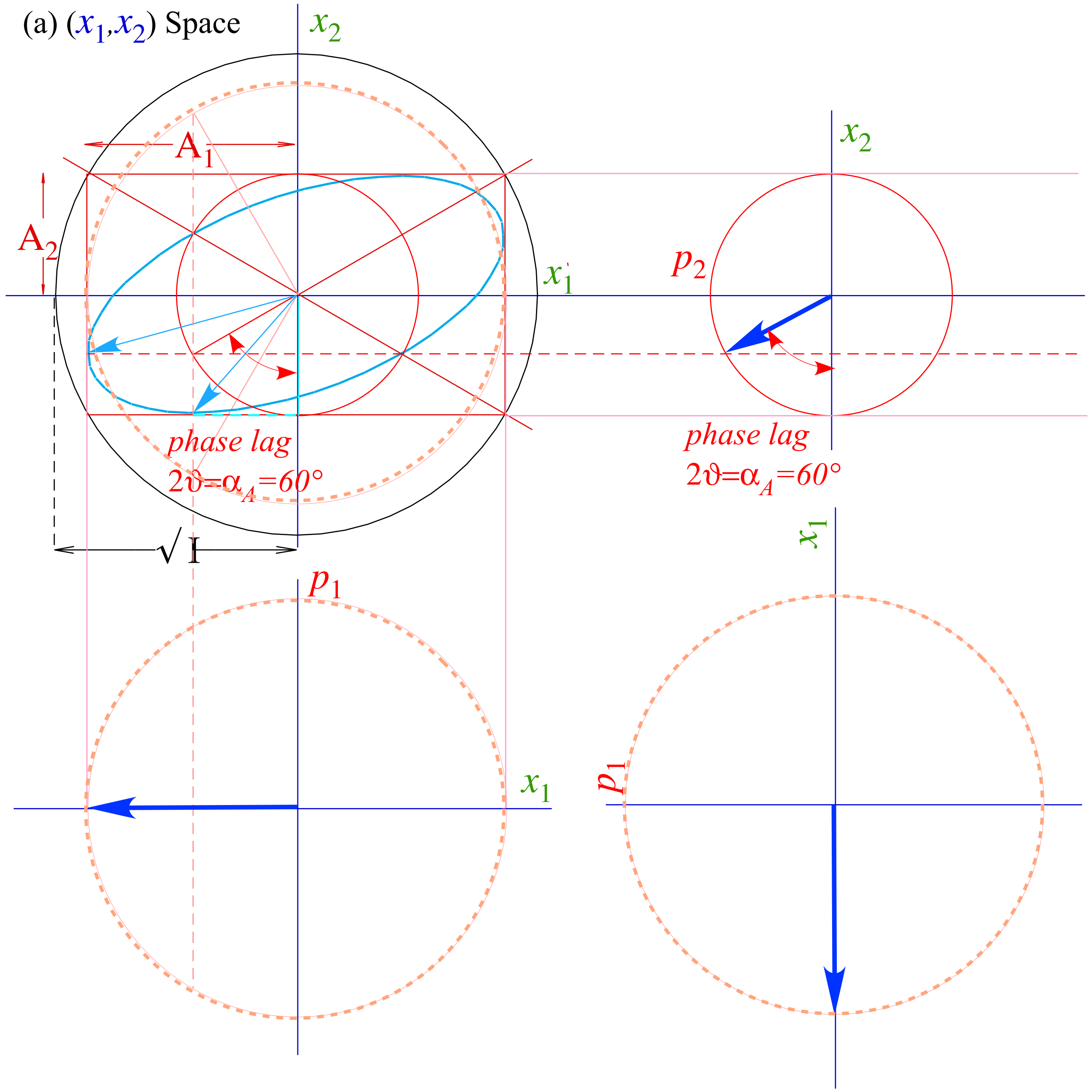
Angles (α_C,β_C): C-axial polar angle β_C from above.

$$\sin\alpha_A\sin\beta_A = \cos\beta_C \qquad \text{or:}\quad \beta_C = \cos^{-1}(\sin\alpha_A\sin\beta_A) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\right) = 41.4°$$

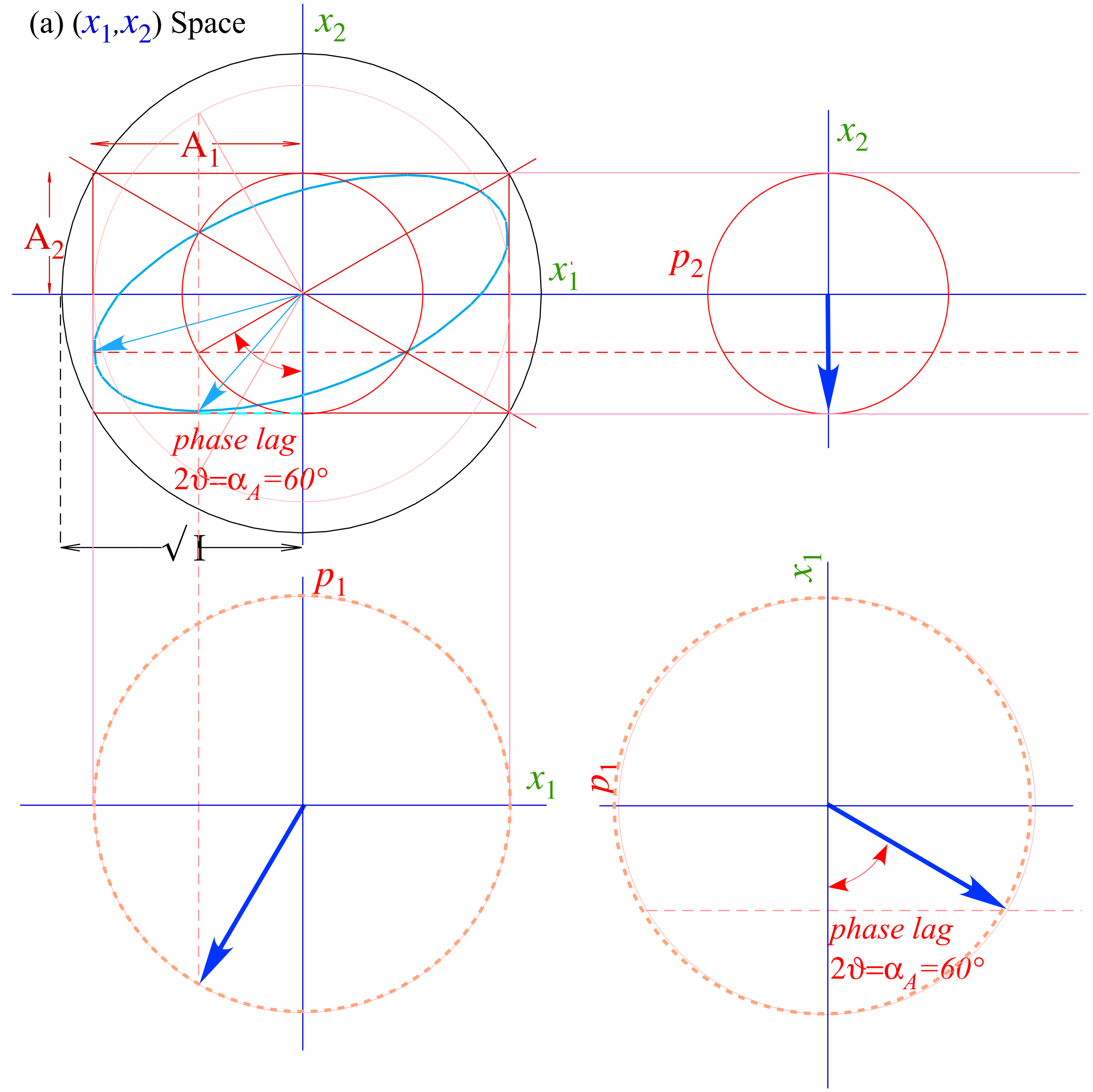
C-axis azimuth angle α_C relates to A-axis angles α_A and β_A . See $\alpha_C = 2\varphi$ below.

$$\frac{\cos\alpha_A\sin\beta_A}{\cos\beta_A} = \tan\alpha_C \qquad \text{or:}\quad \alpha_C = ATN2(\cos\alpha_A\sin\beta_A/\cos\beta_A) = ATN2\left(\frac{1}{2}\cdot\frac{\sqrt{3}}{2}/\frac{1}{2}\right) = 40.9°$$

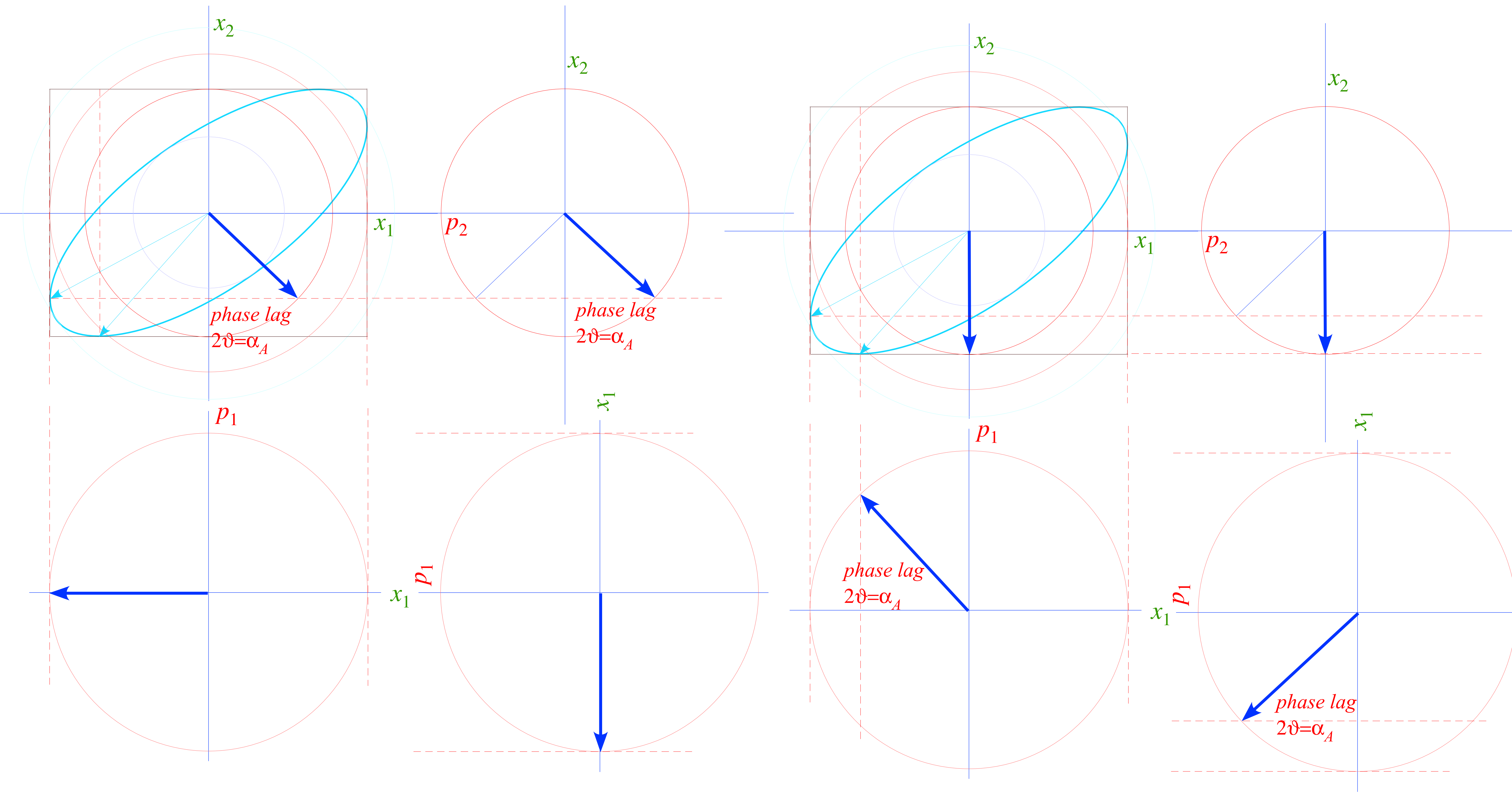




(a) (x1,x2) Space
x2
A1
A2
x1
p2
x2
phase lag
2ϑ=αA=60°
phase lag
2ϑ=αA=60°
√I
p1
x1
x1
p1



(a) (x1,x2) Space
x2
A1
A2
x1
p2
x2
phase lag
2ϑ=αA=60°
√I
p1
x1
x1
p1
phase lag
2ϑ=αA=60°

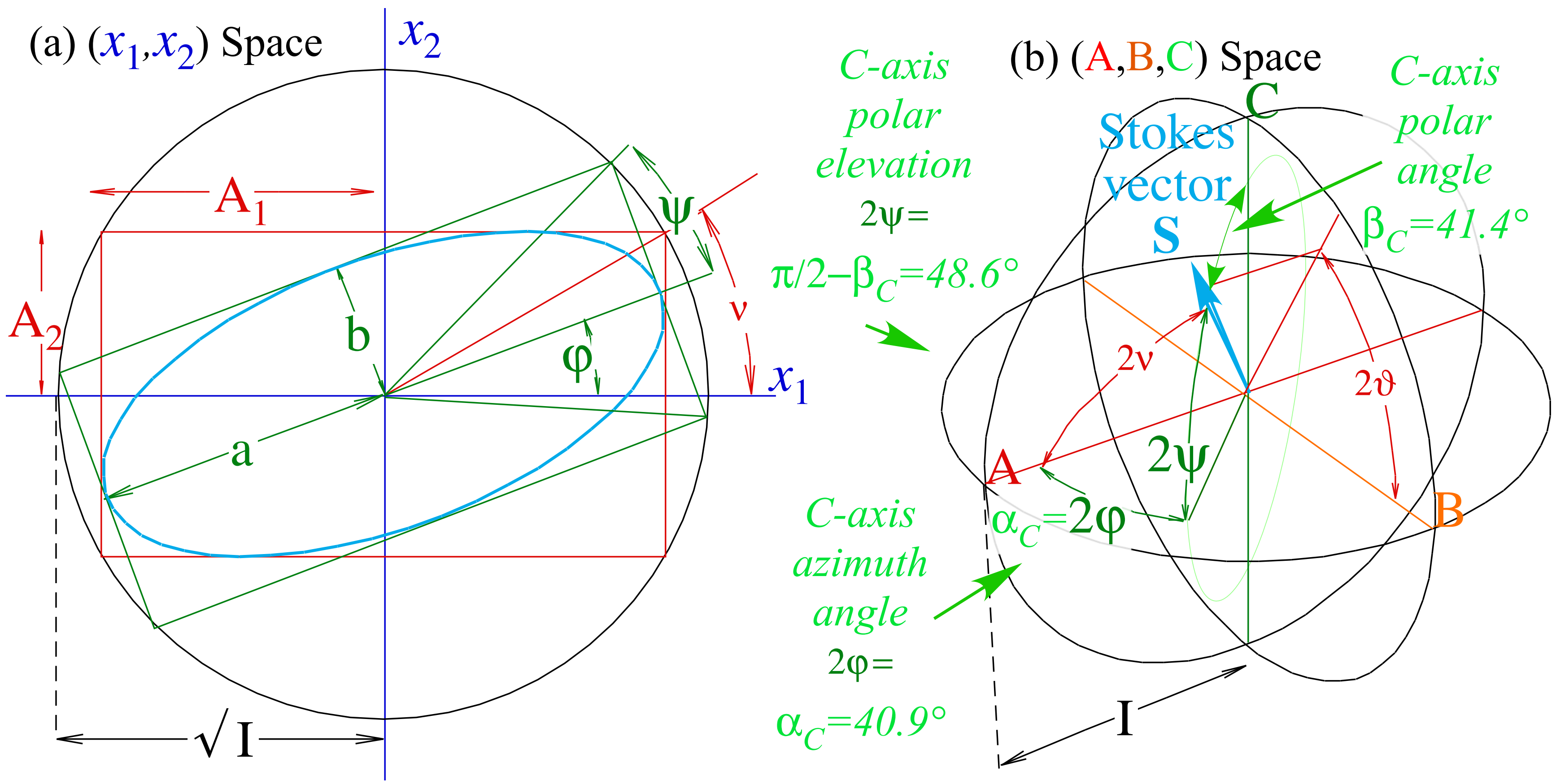


x2
x1
phase lag
2ϑ=αA
p1
x1
x2
p2
phase lag
2ϑ=αA
x1
p1
x2
x1
p1
phase lag
2ϑ=αA
x1
x2
p2
x1
p1
phase lag
2ϑ=αA

The C*-view in* {x_R,x_L}*-basis*

The same orbit viewed in right and left circular polarization $\{x_R, x_L\}$-bases using angles $(\alpha_C, \beta_C, \gamma_C)$.

$$\begin{pmatrix} a_R \\ a_L \end{pmatrix} = A \begin{pmatrix} e^{-i\alpha_C/2}\cos\frac{\beta_C}{2} \\ e^{+i\alpha_C/2}\sin\frac{\beta_C}{2} \end{pmatrix} e^{-i\frac{\gamma_C}{2}} = \begin{pmatrix} x_R + ip_R \\ x_R + ip_R \end{pmatrix}$$



A 90° *B* –rotation $\mathbf{R}(\pi/4)\left|x_1\right\rangle = \left|x_R\right\rangle$ of axis *A* into *C* gets $(\alpha_C, \beta_C, \gamma_C)$ from $(\alpha_A, \beta_A, \gamma_A)$ all at once.

$$\begin{pmatrix} \cos\frac{\pi}{4} & i\sin\frac{\pi}{4} \\ i\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} Ae^{-i\alpha_A/2}\cos\frac{\beta_A}{2} \\ Ae^{+i\alpha_A/2}\sin\frac{\beta_A}{2} \end{pmatrix} e^{-i\frac{\gamma_A}{2}} = \begin{pmatrix} Ae^{-i\alpha_C/2}\cos\frac{\beta_C}{2} \\ Ae^{+i\alpha_C/2}\sin\frac{\beta_C}{2} \end{pmatrix} e^{-i\frac{\gamma_C}{2}} = \begin{pmatrix} x_R + ip_R \\ x_R + ip_R \end{pmatrix}$$

Polarization ellipse and spinor state dynamics

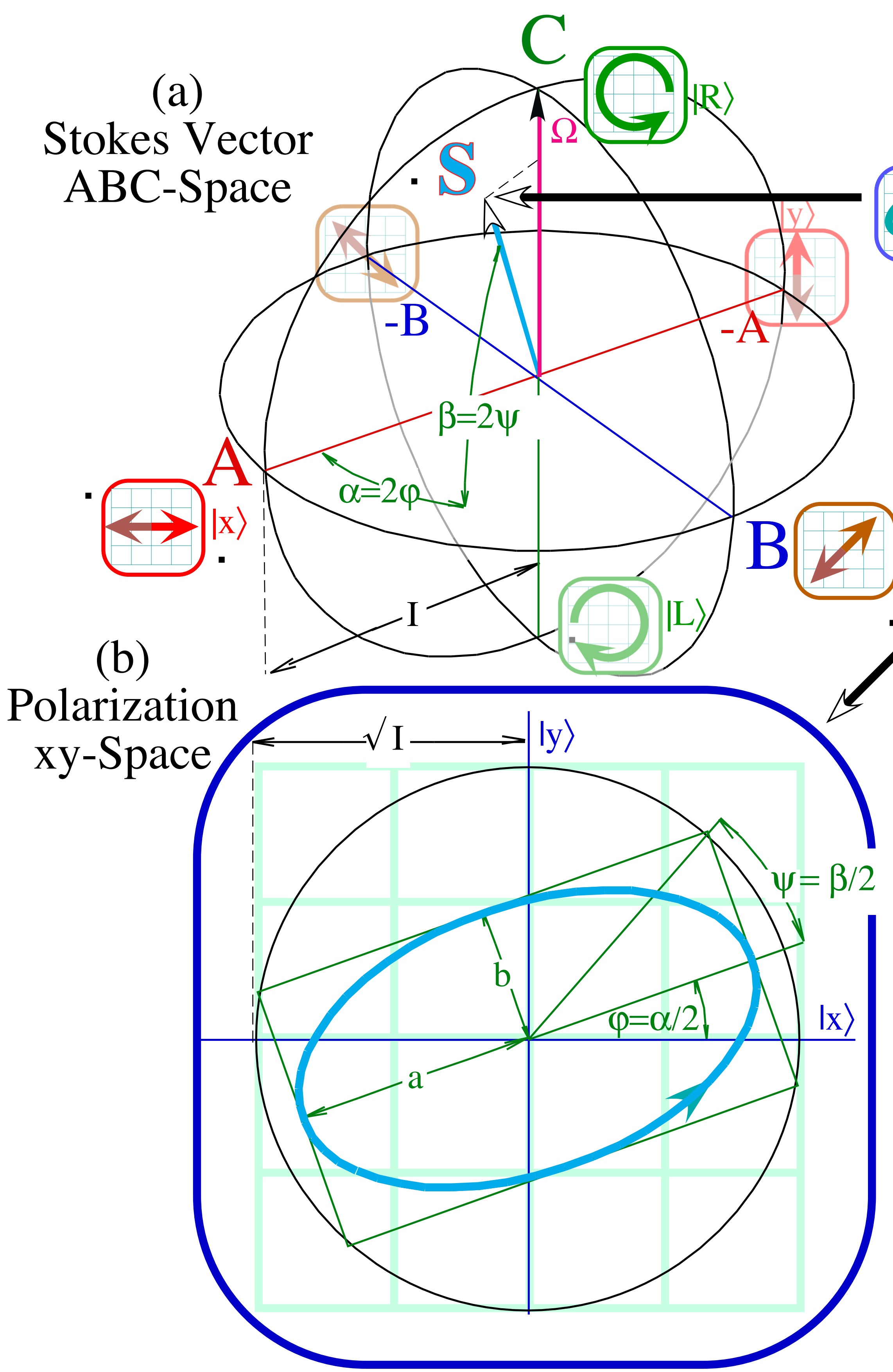


Fig. 3.4.5 Polarization variables (a) Stokes real-vector space (ABC) (b) Complex xy-spinor-space (x_1,x_2).

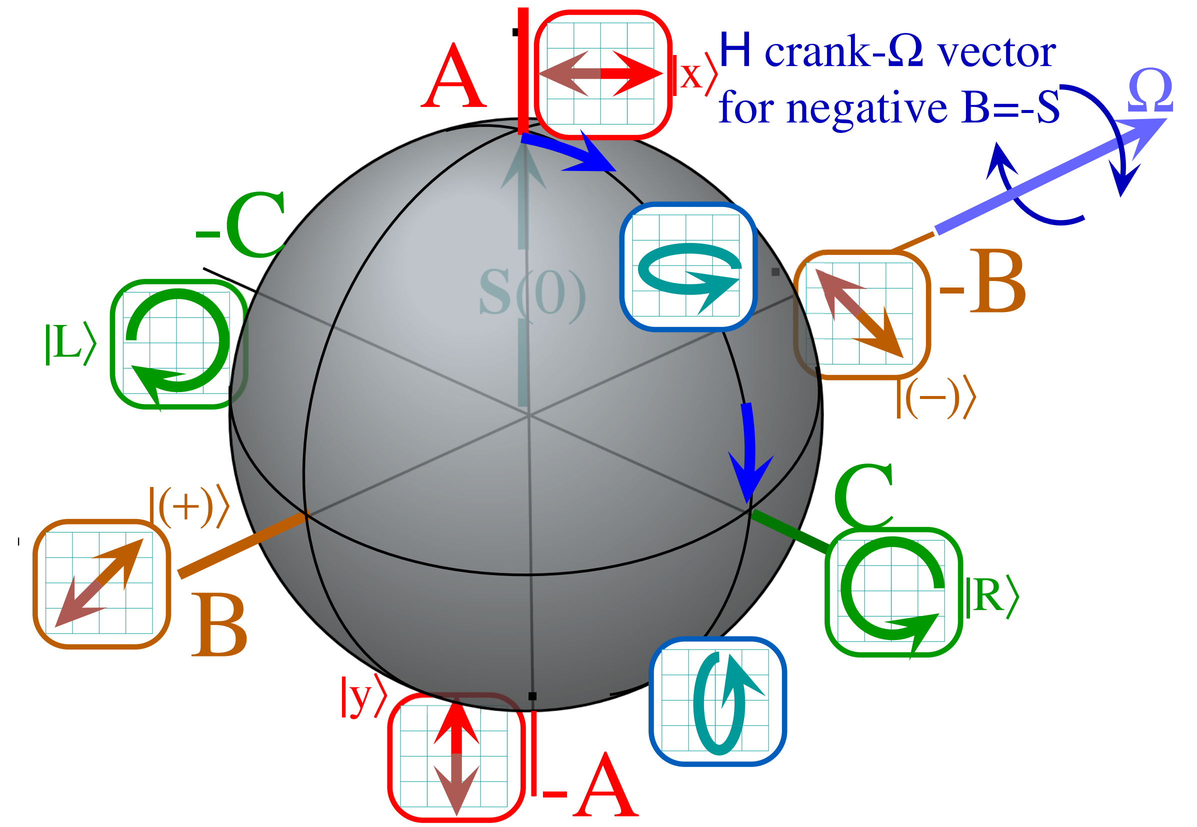
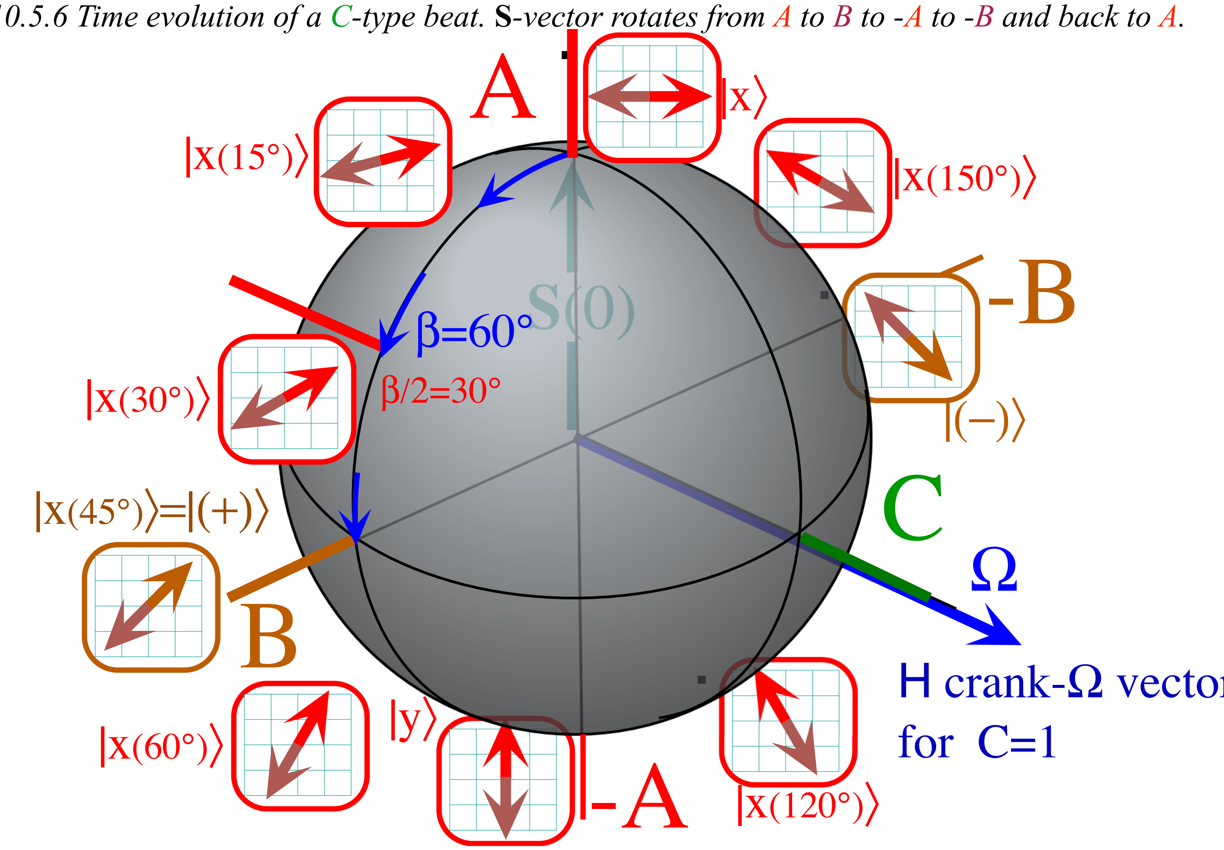


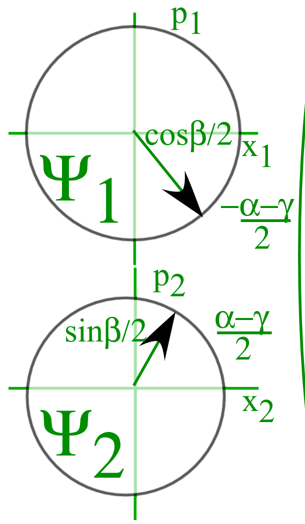
Fig. 10.5.5 Time evolution of a B-type beat. **S**-*vector rotates from A to C to -A to -C and back to A.*

Fig. 10.5.6 Time evolution of a C-type beat. **S**-*vector rotates from A to B to -A to -B and back to A.*

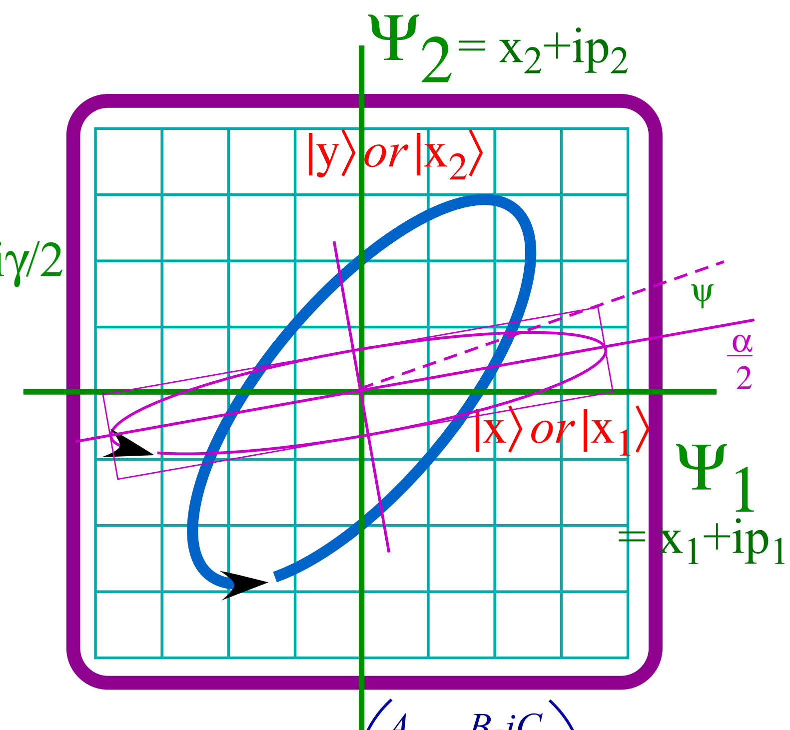


U(2) World : Complex 2D Spinors

2-State ket $|\Psi\rangle =$



$$\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = \begin{pmatrix}\sqrt{N}e^{-i\alpha/2}\cos\beta/2\\ \sqrt{N}e^{i\alpha/2}\sin\beta/2\end{pmatrix} e^{-i\gamma/2}$$



R(3) World : Real 3D Vectors

|Ψ⟩ State

Spin Vector S

$$\begin{pmatrix}S_B\\ S_C\\ S_A\end{pmatrix} = \begin{pmatrix}N\sin\beta\cos\alpha\\ N\sin\beta\sin\alpha\\ N\cos\beta\end{pmatrix}\frac{1}{2}$$

(for α=15° β=45°)

H-Operator $\begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix}$

Angular velocity

$$\Omega = \begin{pmatrix}\Omega_B\\ \Omega_C\\ \Omega_A\end{pmatrix} = \begin{pmatrix}2B\\ 2C\\ A-D\end{pmatrix} = \begin{pmatrix}\Omega\sin\vartheta\cos\varphi\\ \Omega\sin\vartheta\sin\varphi\\ \Omega\cos\vartheta\end{pmatrix}$$