Group Theory in Quantum Mechanics
Lecture 4 (1.27.17)
Matrix Eigensolutions and Spectral Decompositions

(Quantum Theory for Computer Age - Ch. 3 of Unit 1)
(Principles of Symmetry, Dynamics, and Spectroscopy - Sec. 1-3 of Ch. 1)

Unitary operators and matrices that change state vectors
- ...and **eigen**states (“**own**states) that are mostly immune

Geometric visualization of real symmetric matrices and eigenvectors
- Circle-to-ellipse mapping (and I’m Ba-aaack!)
- Ellipse-to-ellipse mapping (Normal space vs. tangent space)
- Eigensolutions as stationary extreme-values (Lagrange λ-multipliers)

Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$
- Secular equation
- Hamilton-Cayley equation and projectors
- Idempotent projectors (how eigenvalues⇒eigenvectors)
- Operator orthonormality and completeness

Spectral Decompositions
- Functional spectral decomposition
- Orthonormality vs. Completeness vis-a`-vis Operator vs. State
- Lagrange functional interpolation formula
  - Proof that completeness relation is “Truer-than-true”

Diagonalizing Transformations (D-Ttran) from projectors
- Eigensolutions for active analyzers

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Unitary operators and matrices that change state vectors

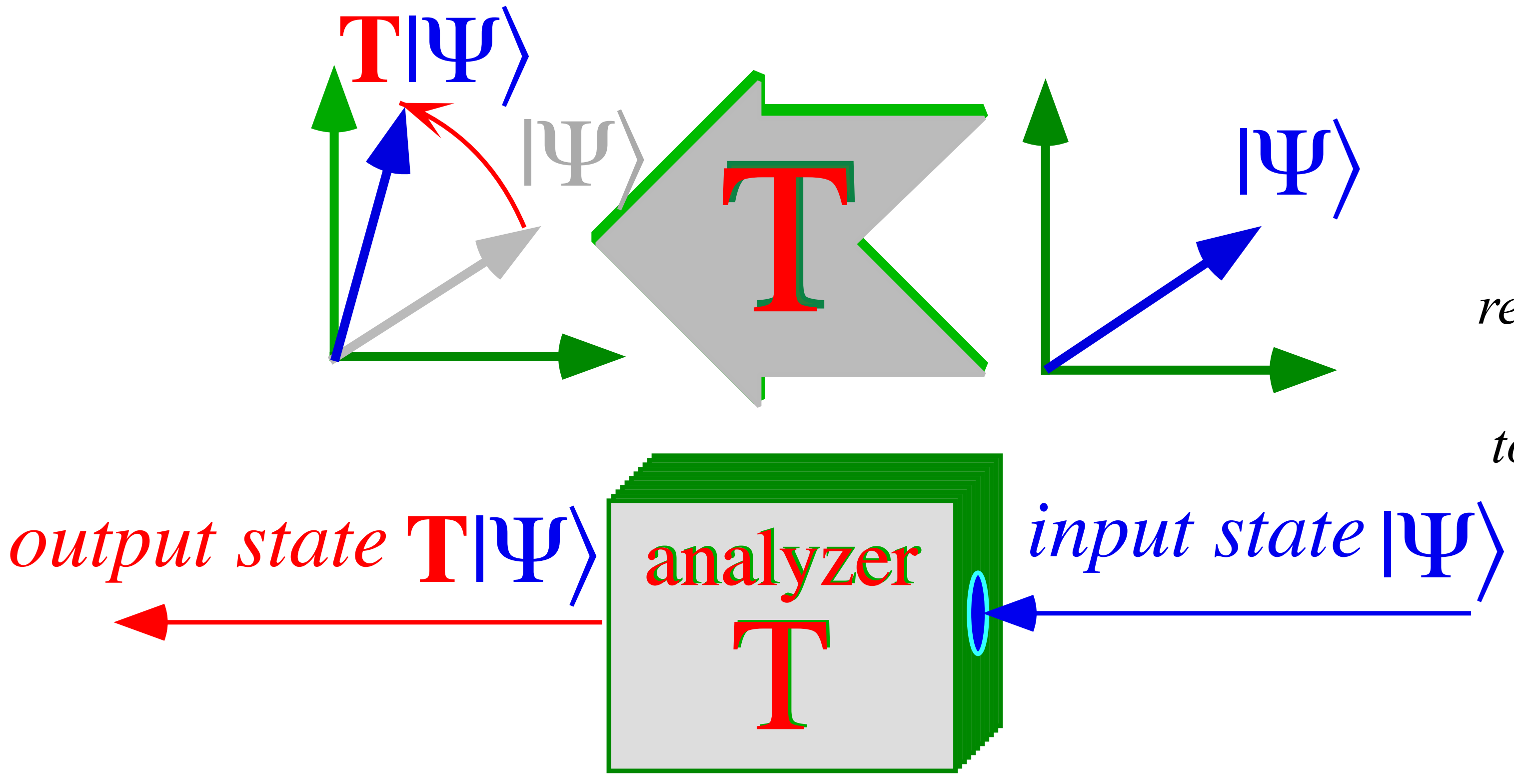
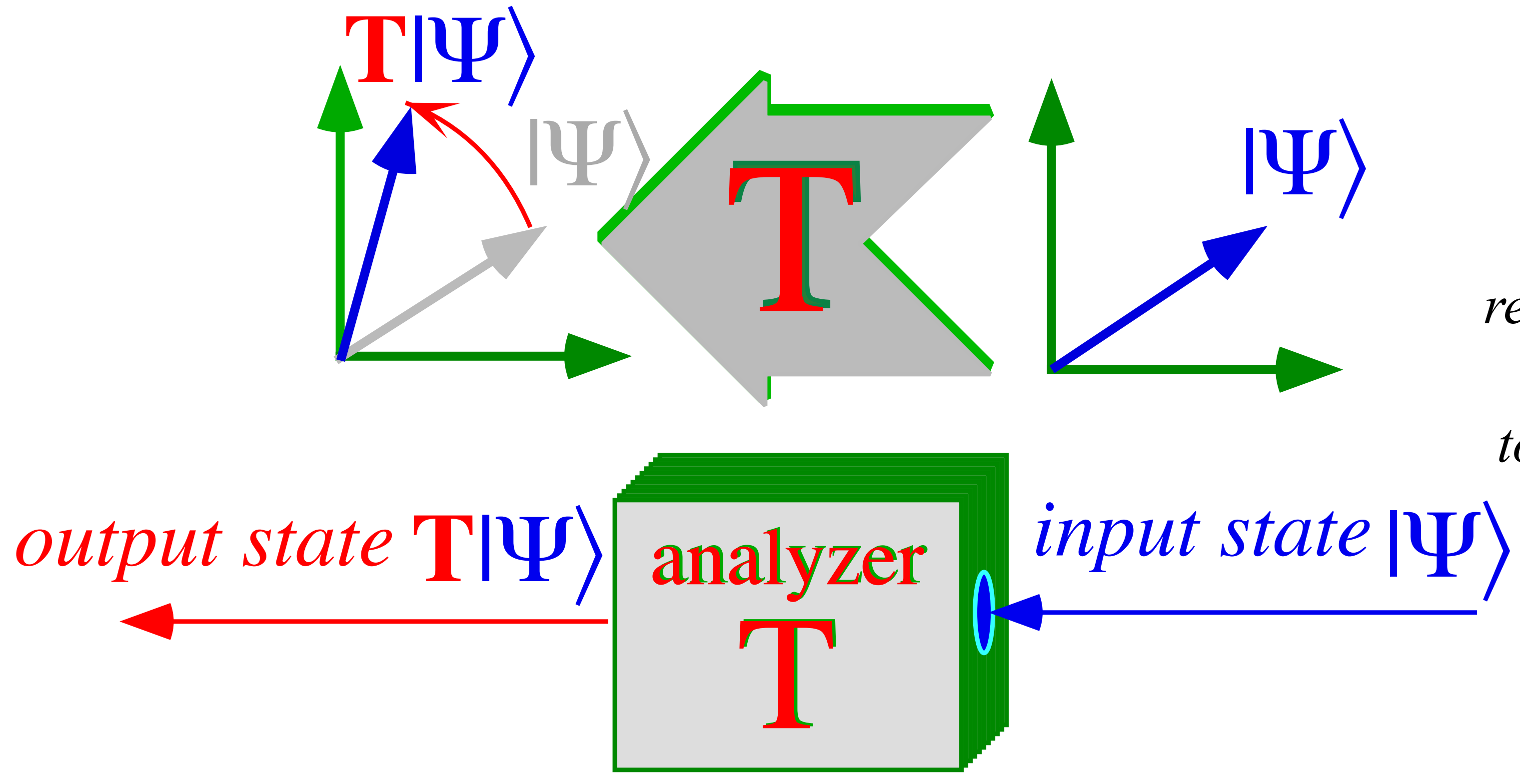


Fig. 3.1.1 Effect of analyzer represented by ket vector transformation of $|\Psi\rangle$ *to new ket vector* $\mathbf{T}|\Psi\rangle$.

Unitary operators and matrices that change state vectors...



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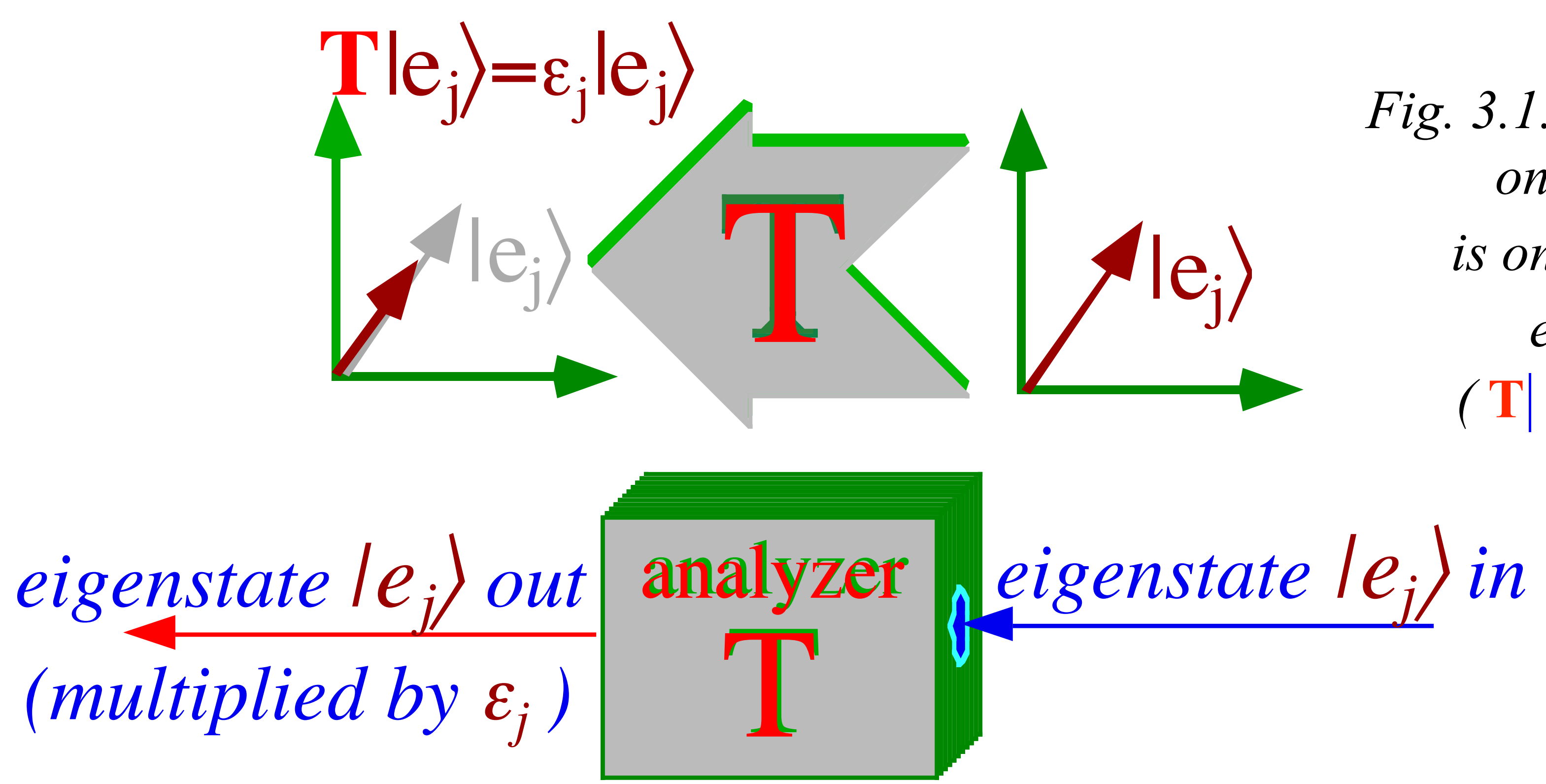

Fig. 3.1.2 Effect of analyzer on eigenket $|\varepsilon_j\rangle$ *is only to multiply by eigenvalue* ε_j *(* $\mathbf{T}|\varepsilon_j\rangle = \varepsilon_j|\varepsilon_j\rangle$ *).*

For Unitary operators $\mathbf{T}=\mathbf{U}$*, the eigenvalues must be phase factors* $\varepsilon_k = e^{i\alpha_k}$

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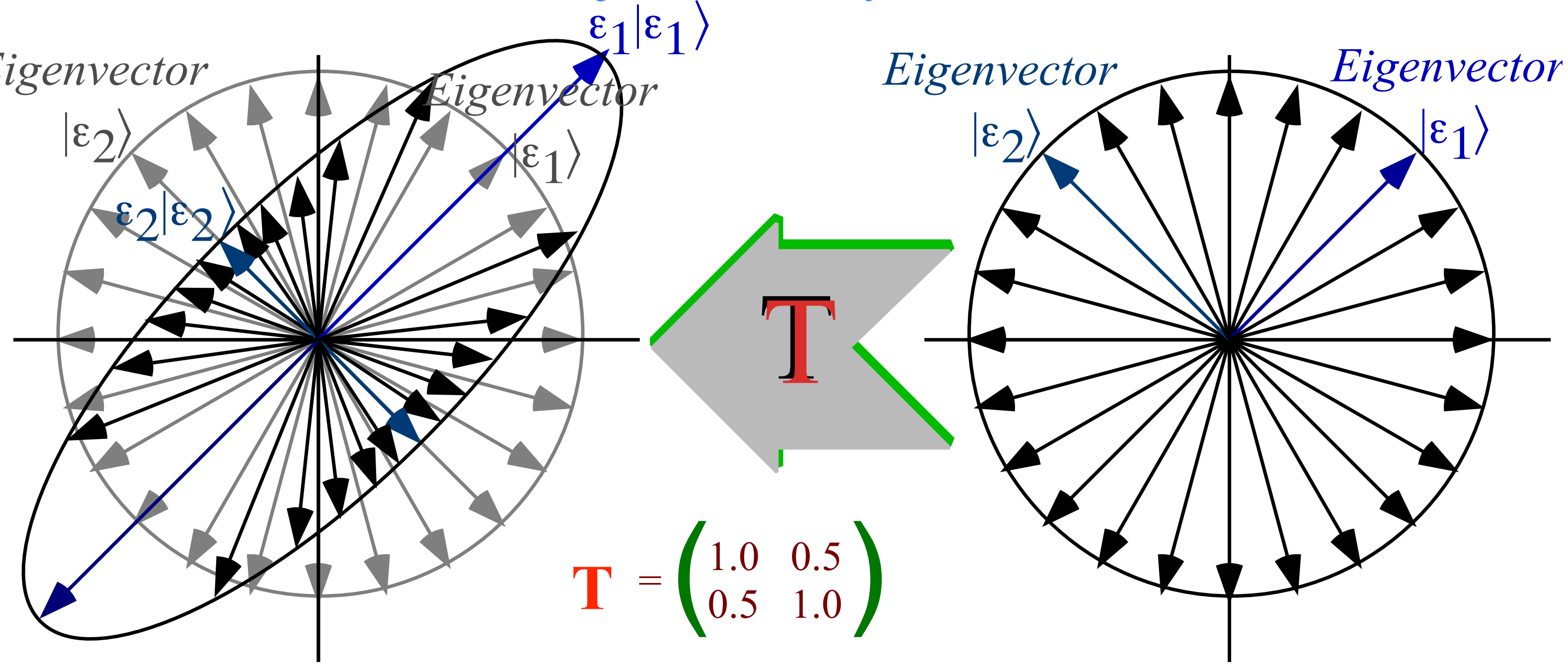
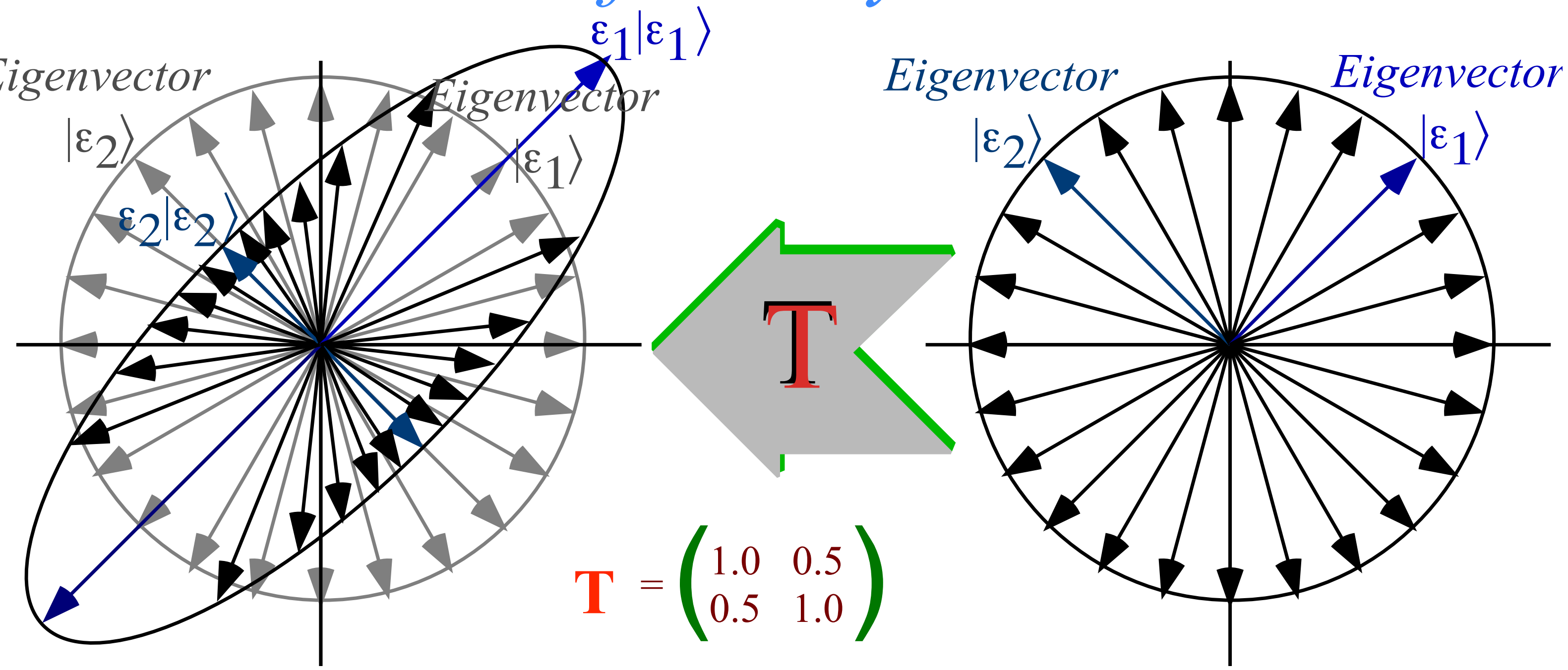


Circle-to-ellipse mapping

Study a real symmetric matrix **T** by applying it to a circular array of unit vectors **c**.

A matrix $\mathbf{T} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$ maps the circular array into an elliptical one.

Geometric visualization of real symmetric matrices and eigenvectors



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Two vectors in the upper half plane survive **T** without changing direction.
These lucky vectors are the *eigenvectors of matrix* **T**.

$$|\varepsilon_1\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix} / \sqrt{2}\,, \qquad |\varepsilon_2\rangle = \begin{pmatrix} -1 \\ 1 \end{pmatrix} / \sqrt{2}$$

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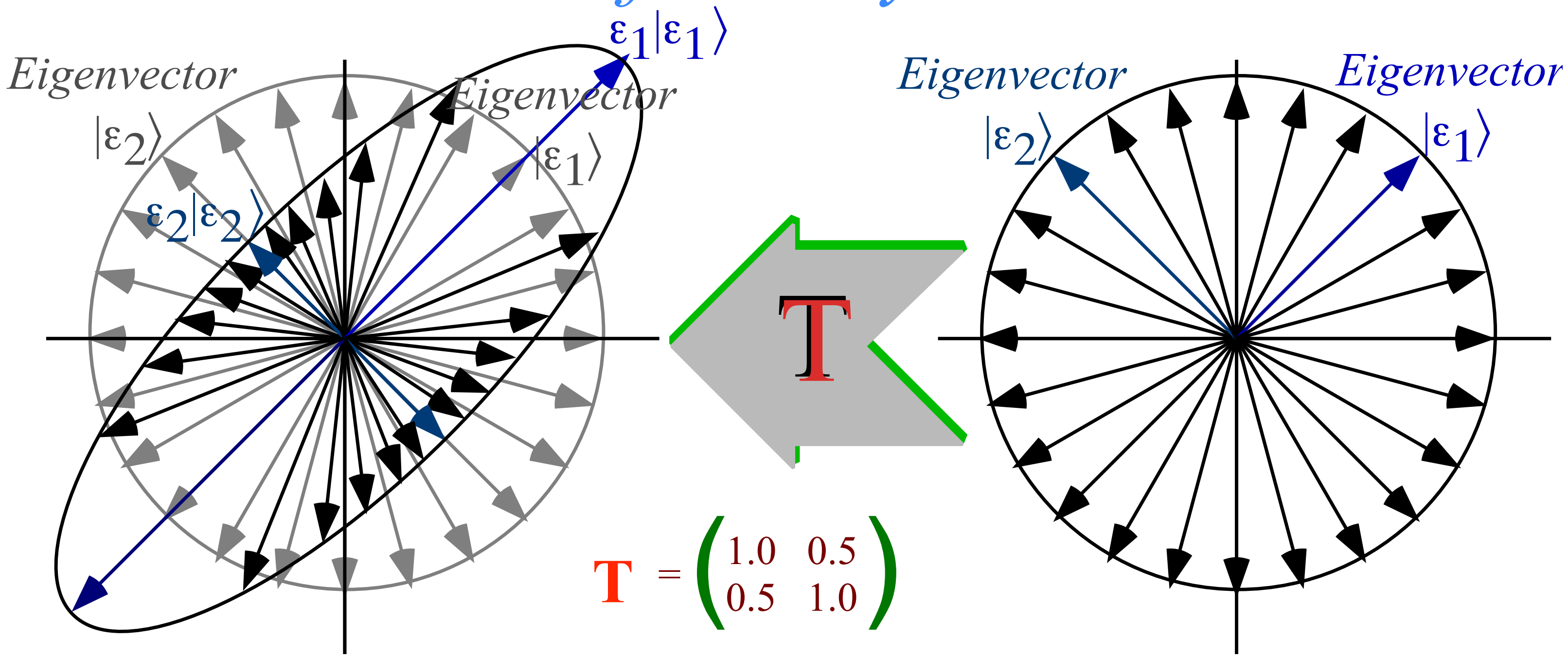

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They transform as follows: $\mathbf{T}|\varepsilon_1\rangle = \varepsilon_1|\varepsilon_1\rangle = 1.5|\varepsilon_1\rangle$, and $\mathbf{T}|\varepsilon_2\rangle = \varepsilon_2|\varepsilon_2\rangle = 0.5|\varepsilon_2\rangle$

to only suffer length change given by *eigenvalues* $\varepsilon_1 = 1.5$ and $\varepsilon_2 = 0.5$

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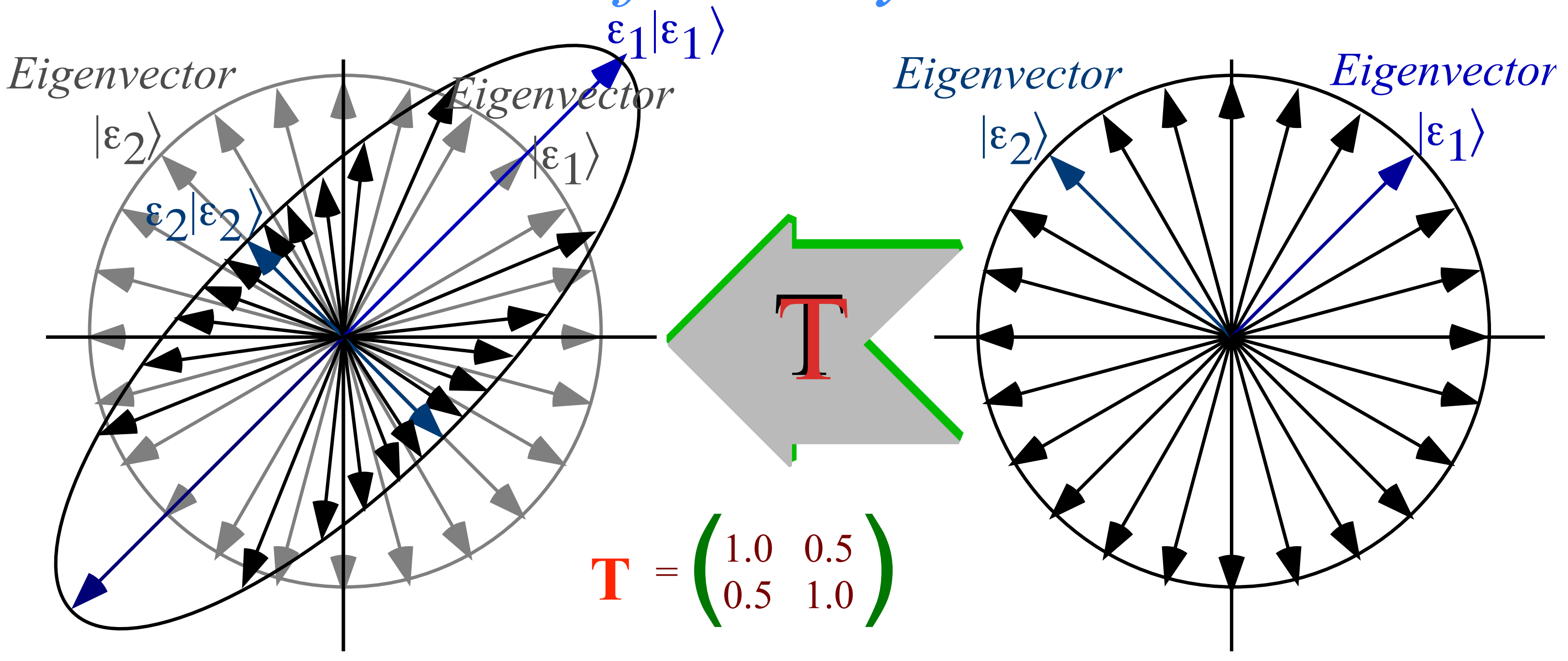

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Normalization ($\langle \mathbf{c}|\mathbf{c}\rangle = 1$) is a condition separate from eigen-relations $\mathbf{T}\left|\varepsilon_k\right\rangle = \varepsilon_k\left|\varepsilon_k\right\rangle$

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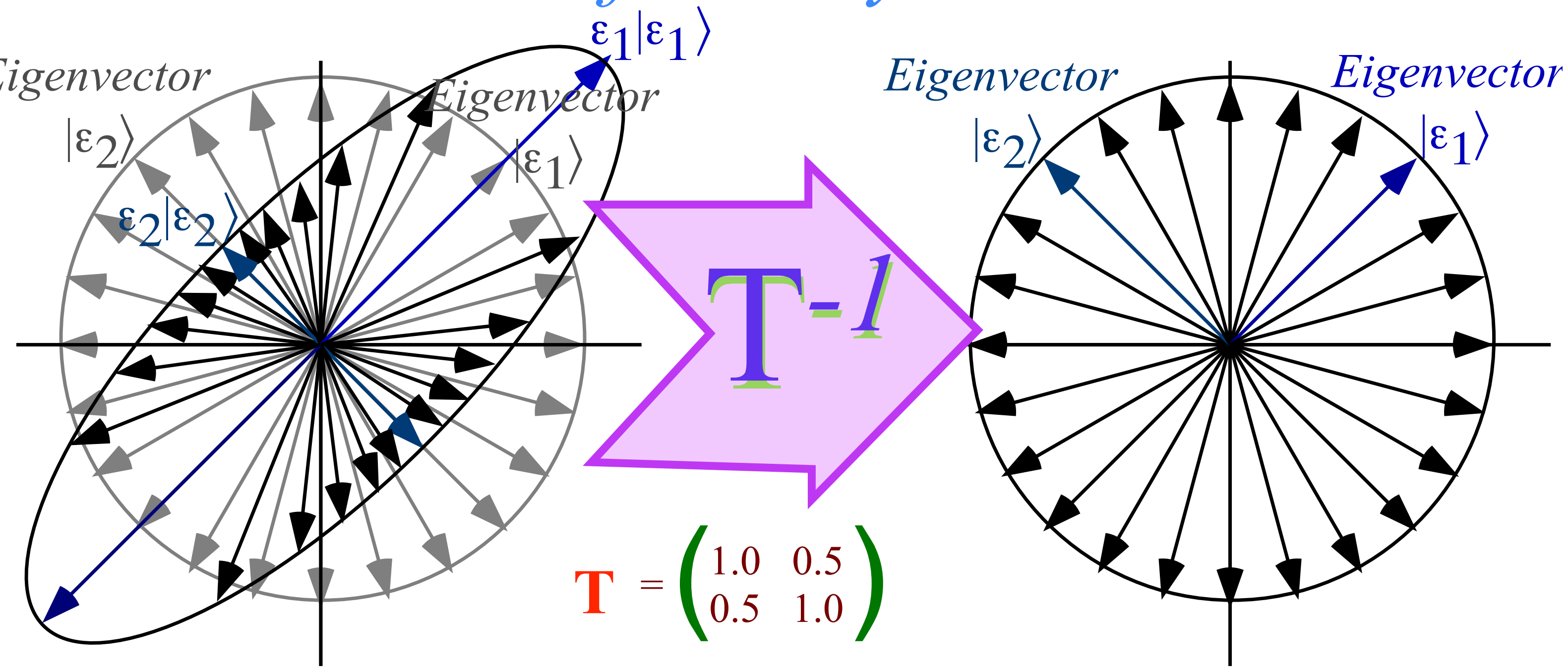


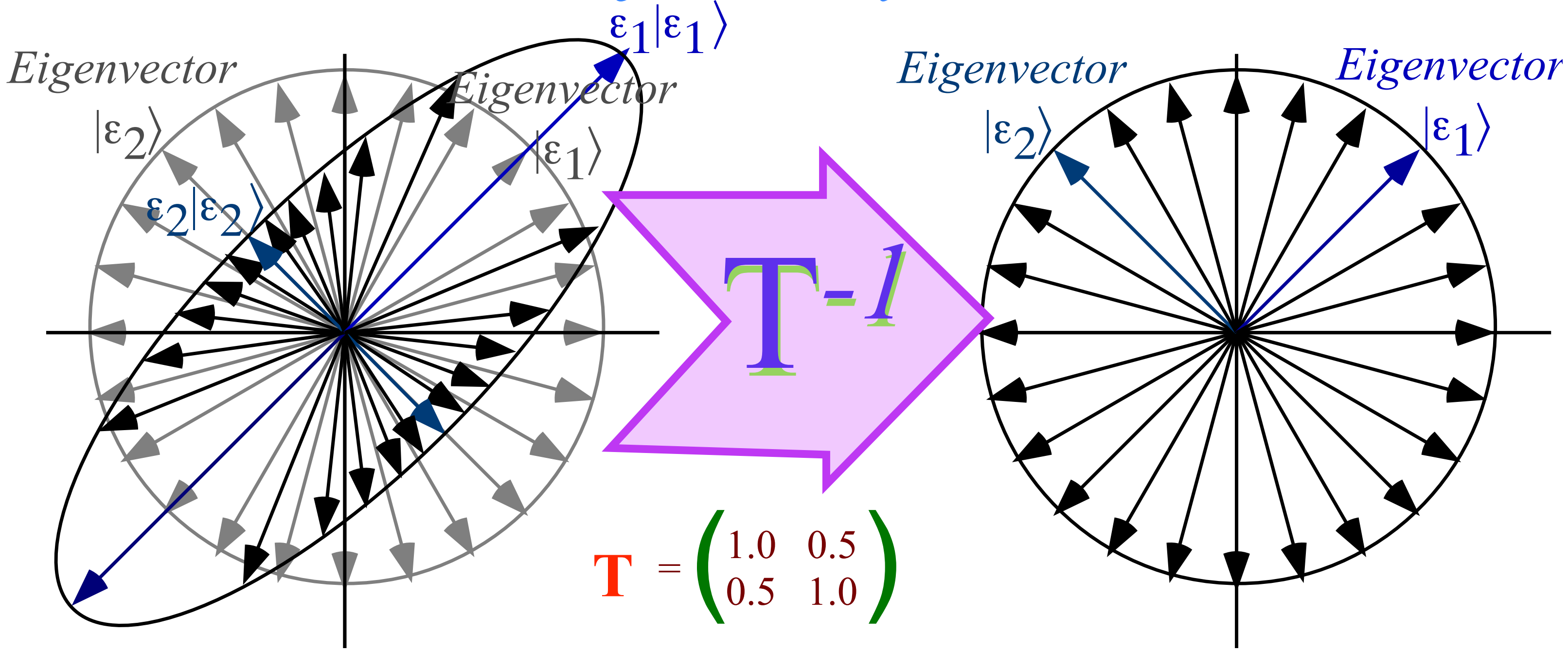
Circle-to-ellipse mapping (and I'm Ba-aaack!)

Each vector $|\mathbf{r}\rangle$ on left ellipse maps back to vector $|\mathbf{c}\rangle = \mathbf{T}^{-1}|\mathbf{r}\rangle$ on right unit circle.

Each $|\mathbf{c}\rangle$ has unit length: $\langle\mathbf{c}|\mathbf{c}\rangle = 1 = \langle\mathbf{r}|\mathbf{T}^{-1}\,\mathbf{T}^{-1}|\mathbf{r}\rangle = \langle\mathbf{r}|\mathbf{T}^{-2}|\mathbf{r}\rangle$. (**T** is real-symmetric: $\mathbf{T}^{\dagger}=\mathbf{T}=\mathbf{T}^{T}$.)

$$\mathbf{c} \bullet \mathbf{c} = 1 = \mathbf{r} \bullet \mathbf{T}^{-2} \bullet \mathbf{r} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} T_{xx} & T_{xy} \\ T_{yx} & T_{y} \end{pmatrix}^{-2} \begin{pmatrix} x \\ y \end{pmatrix}$$

Geometric visualization of real symmetric matrices and eigenvectors



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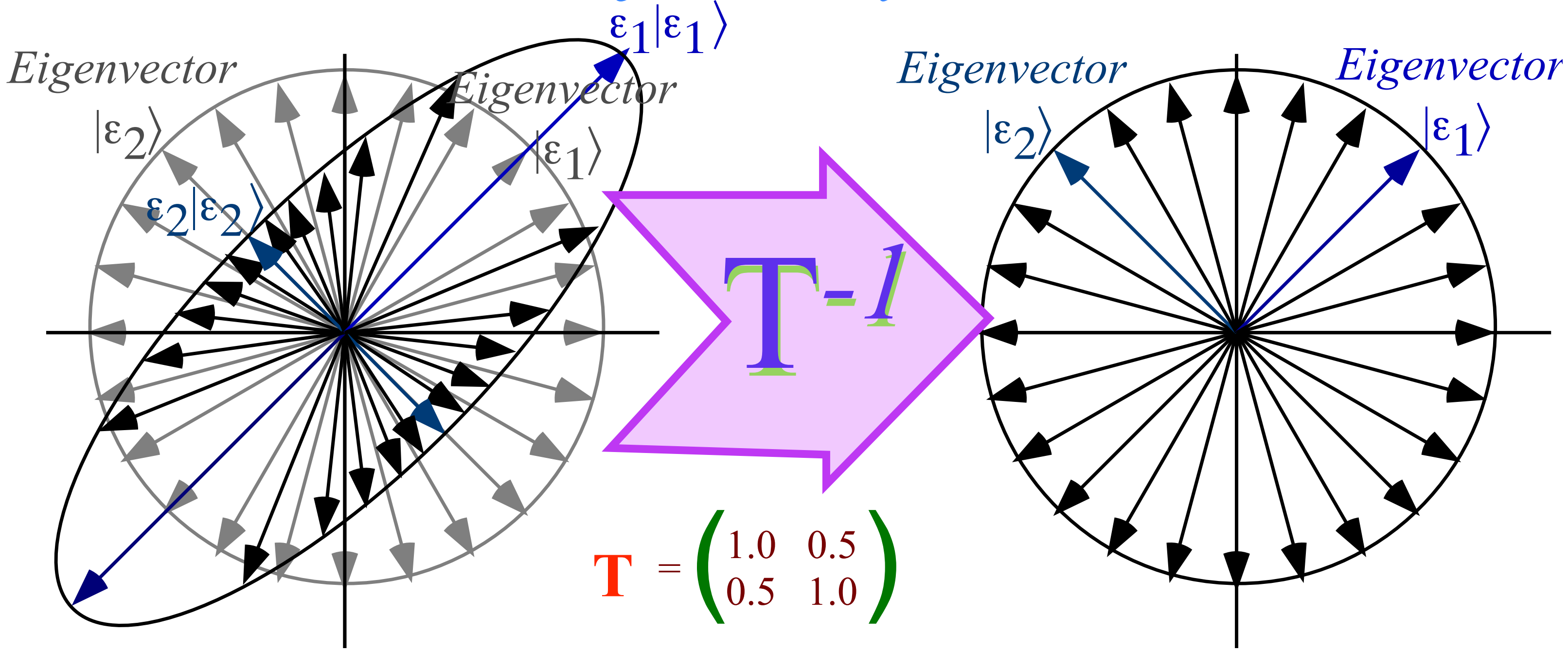
This simplifies if rewritten in a coordinate system (x_1,x_2) of eigenvectors $|\varepsilon_1\rangle$ and $|\varepsilon_2\rangle$

where $\mathbf{T}^{-2}|\varepsilon_1\rangle = \varepsilon_1^{-2}|\varepsilon_1\rangle$ and $\mathbf{T}^{-2}|\varepsilon_2\rangle = \varepsilon_2^{-2}|\varepsilon_2\rangle$, that is, $\mathbf{T}$, $\mathbf{T}^{-1}$, and $\mathbf{T}^{-2}$ are each diagonal.

$$\begin{pmatrix} \langle\varepsilon_1|\mathbf{T}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{T}|\varepsilon_2\rangle \\ \langle\varepsilon_2|\mathbf{T}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{T}|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix}, \text{ and } \begin{pmatrix} \langle\varepsilon_1|\mathbf{T}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{T}|\varepsilon_2\rangle \\ \langle\varepsilon_2|\mathbf{T}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{T}|\varepsilon_2\rangle \end{pmatrix}^{-2} = \begin{pmatrix} \varepsilon_1^{-2} & 0 \\ 0 & \varepsilon_2^{-2} \end{pmatrix}$$

Geometric visualization of real symmetric matrices and eigenvectors



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Matrix equation simplifies to an elementary ellipse equation of the form $(x/a)^2+(y/b)^2=1$.

$$\mathbf{c}\bullet\mathbf{c} = 1 = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} \varepsilon_1^{-2} & 0 \\ 0 & \varepsilon_2^{-2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left(\frac{x_1}{\varepsilon_1}\right)^2 + \left(\frac{x_2}{\varepsilon_2}\right)^2$$

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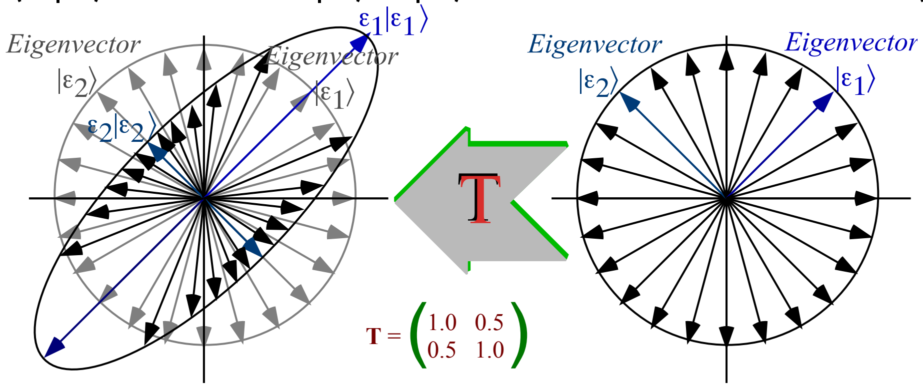

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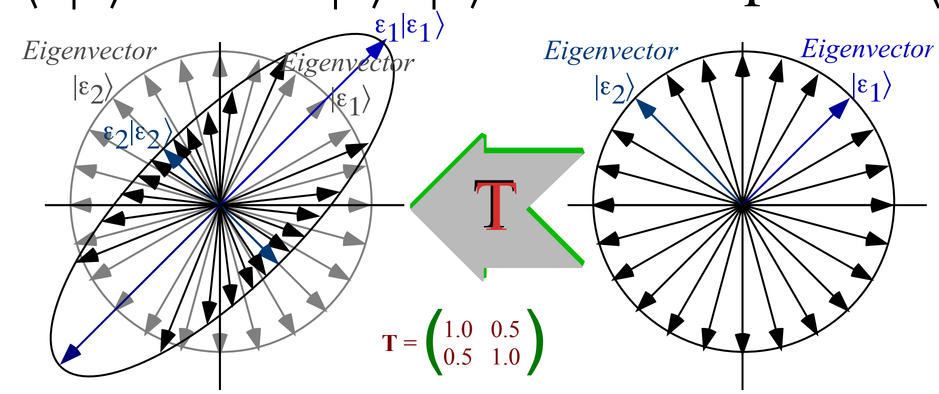

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Now **M** maps vector $|\mathbf{q}\rangle$ from a *quadratic form* $1=\langle\mathbf{q}|\mathbf{M}|\mathbf{q}\rangle$ to vector $|\mathbf{p}\rangle=\mathbf{M}|\mathbf{q}\rangle$ on surface $1=\langle\mathbf{p}|\mathbf{M}^{-1}|\mathbf{p}\rangle$.

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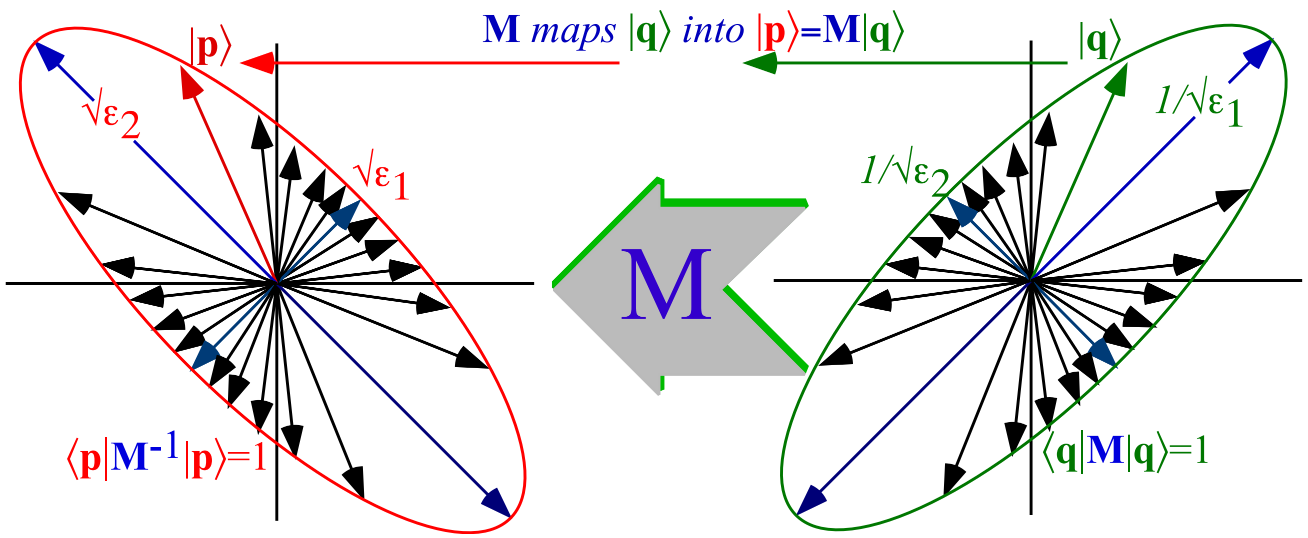

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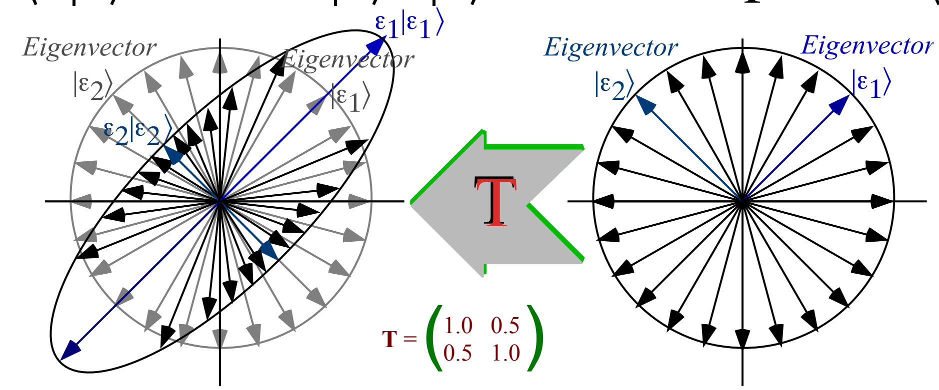

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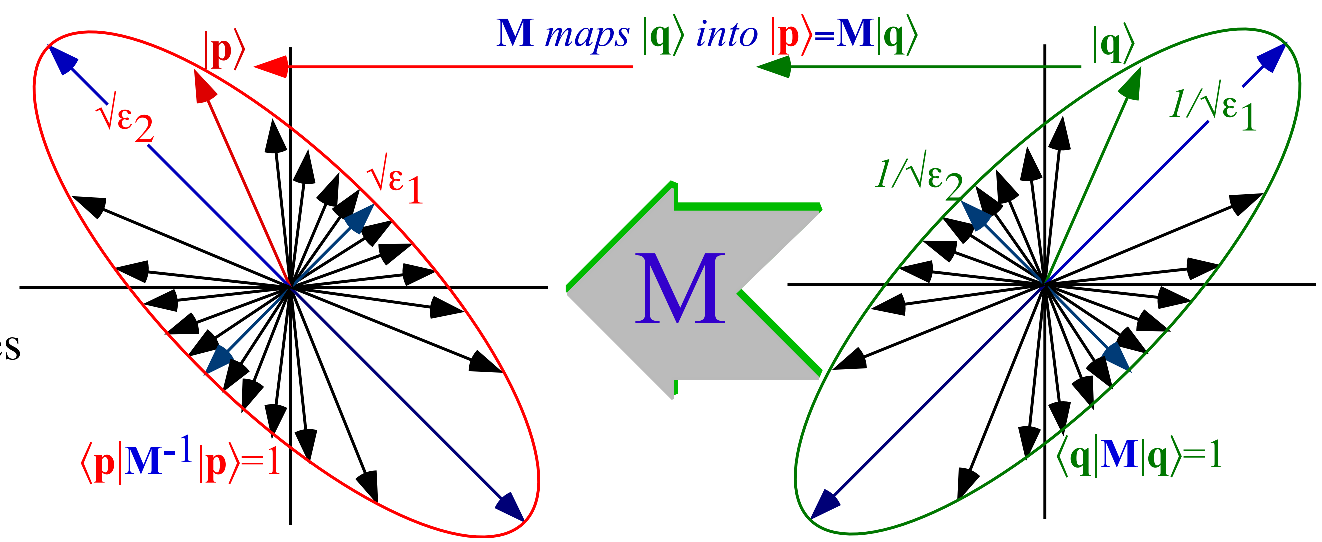
Radii of $|\mathbf{p}\rangle$ ellipse are <u>square</u> <u>roots</u> of eigenvalues $\sqrt{\varepsilon_1}$ and $\sqrt{\varepsilon_2}$



Radii of $|\mathbf{q}\rangle$ ellipse axes are inverse eigenvalue roots $1/\sqrt{\varepsilon_1}$ and $1/\sqrt{\varepsilon_2}$.

Geometric visualization of real symmetric matrices and eigenvectors

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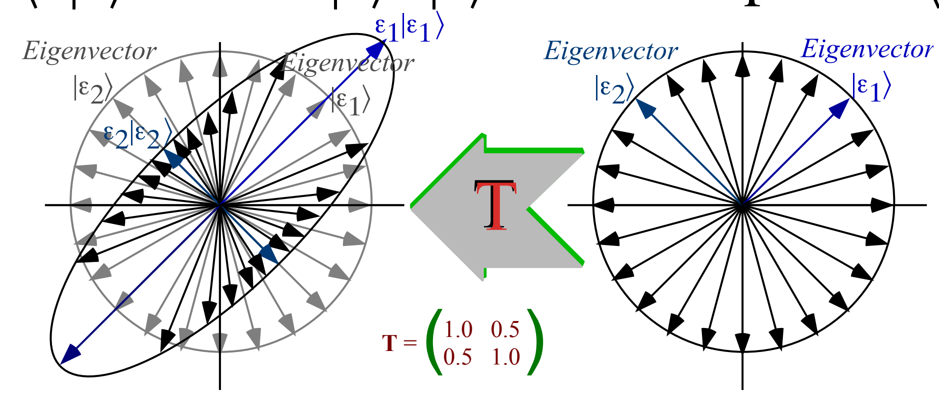

Ellipse-to-ellipse mapping (Normal vs. tangent space)

Now **M** maps vector |**q**⟩ from a *quadratic form* *1*=⟨**q**|**M**|**q**⟩ to vector |**p**⟩=**M**|**q**⟩ on surface *1*=⟨**p**|**M**$^{-1}$|**p**⟩.

$$1 = \langle \mathbf{q}|\mathbf{M}|\mathbf{q}\rangle = \langle \mathbf{q}|\mathbf{p}\rangle = \langle \mathbf{p}|\mathbf{M}^{-1}|\mathbf{p}\rangle$$

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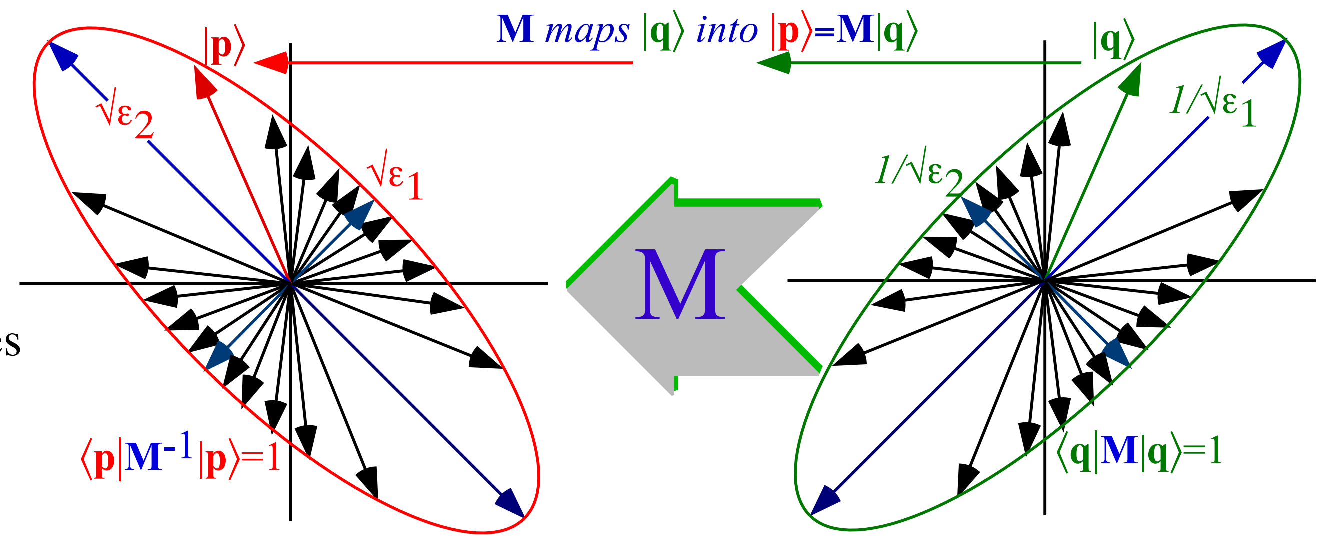

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Tangent-normal geometry of mapping is found by using gradient ∇of quadratic curve *1*=⟨**q**|**M**|**q**⟩ .

$$\nabla(\langle \mathbf{q}|\mathbf{M}|\mathbf{q}\rangle)=\langle \mathbf{q}|\mathbf{M} + \mathbf{M}|\mathbf{q}\rangle = 2\,\mathbf{M}|\mathbf{q}\rangle = 2\,|\mathbf{p}\rangle$$

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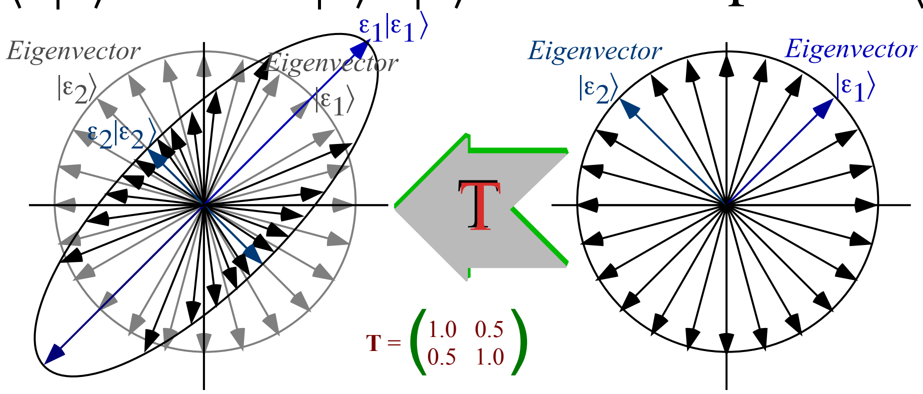

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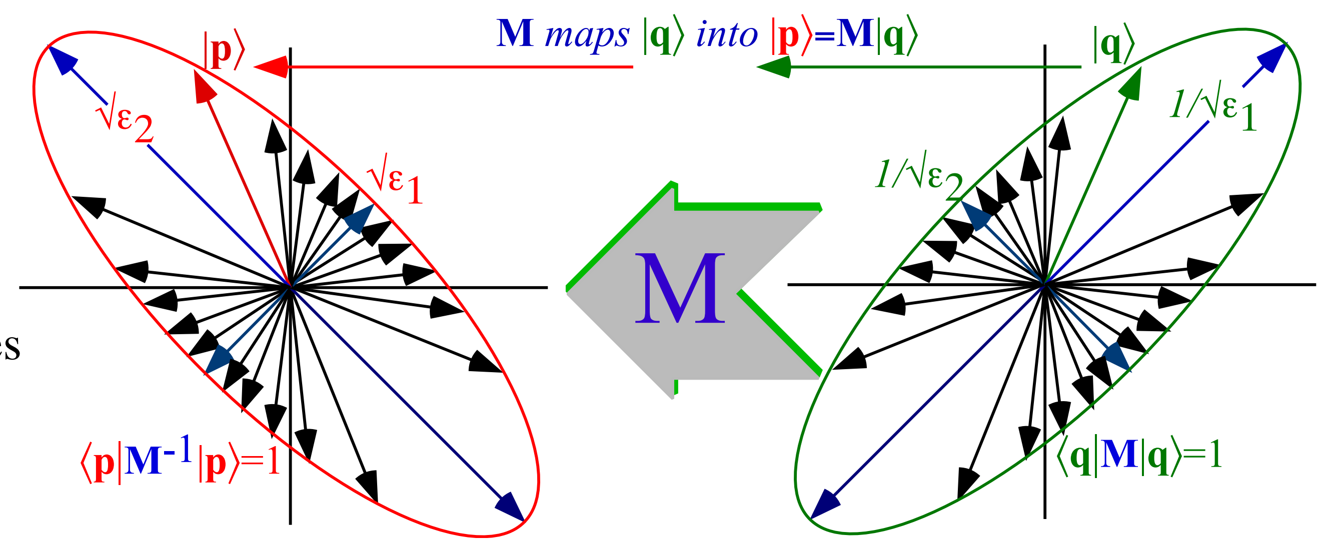

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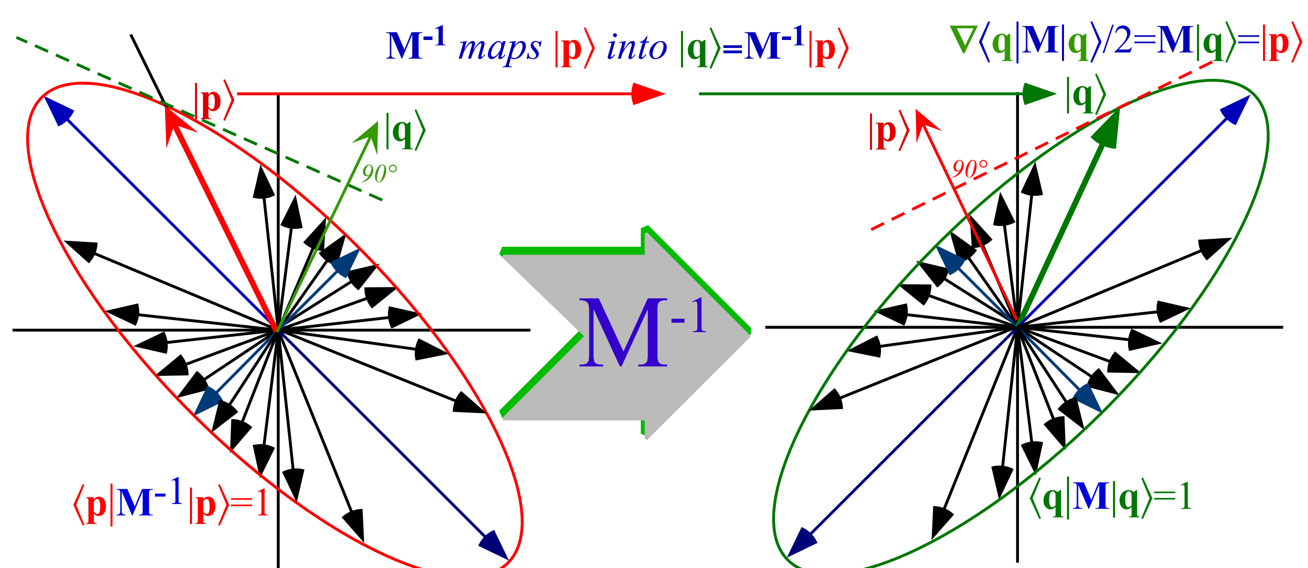

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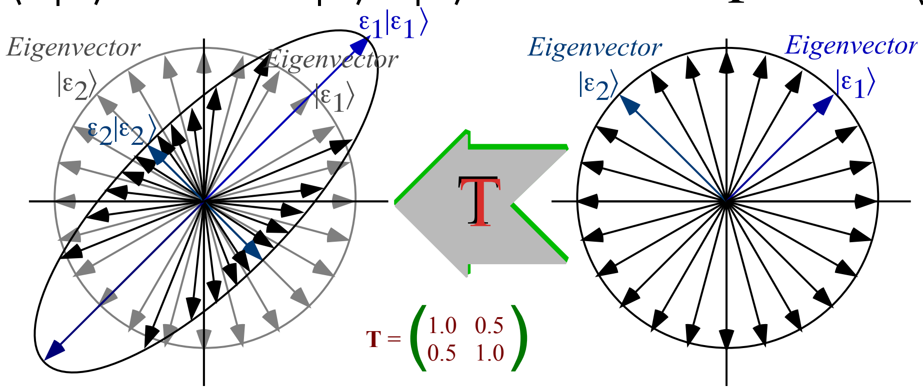

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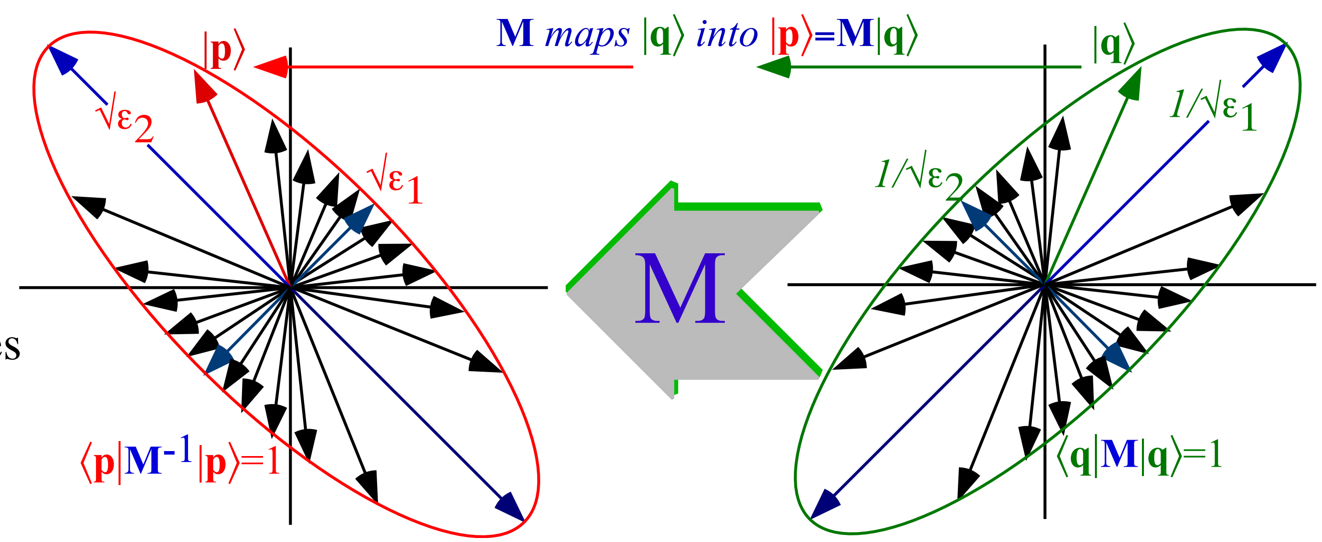
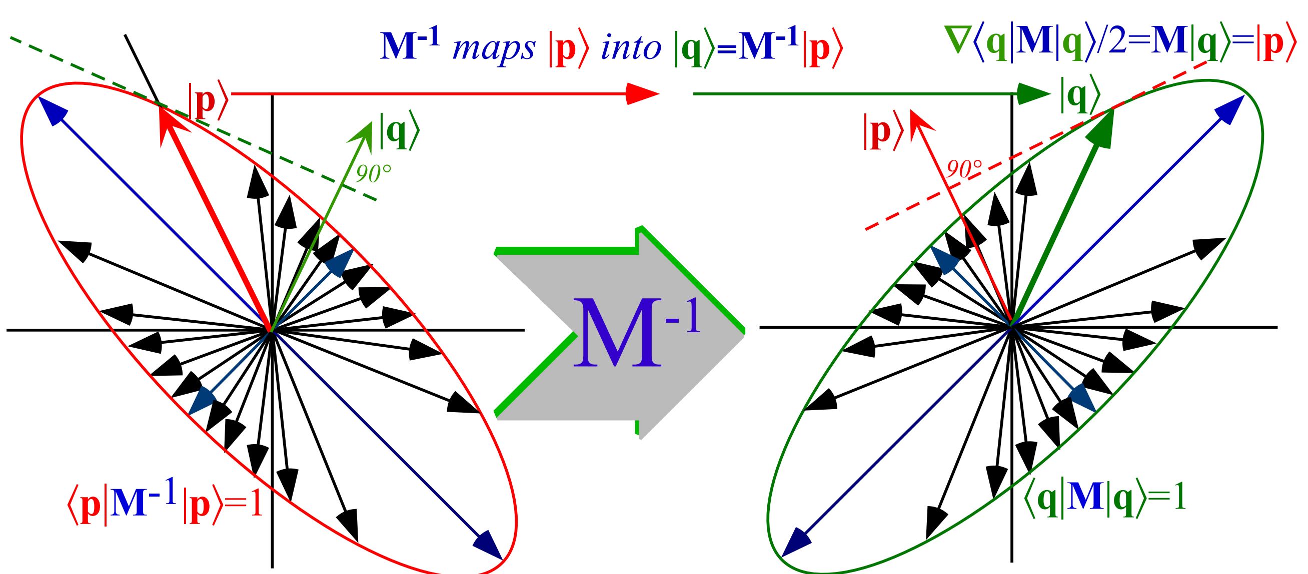


Radii of $|\mathbf{q}\rangle$ ellipse axes are inverse eigenvalue roots $1/\sqrt{\varepsilon_1}$ and $1/\sqrt{\varepsilon_2}$.

Tangent-normal geometry of mapping is found by using gradient ∇ of quadratic curve $1 = \langle \mathbf{q}|\mathbf{M}|\mathbf{q}\rangle$.

$$\nabla(\langle \mathbf{q}|\mathbf{M}|\mathbf{q}\rangle) = \langle \mathbf{q}|\mathbf{M} + \mathbf{M}|\mathbf{q}\rangle = 2\,\mathbf{M}|\mathbf{q}\rangle = 2\,|\mathbf{p}\rangle$$

Mapped vector $|\mathbf{p}\rangle$ lies on gradient $\nabla(\langle \mathbf{q}|\mathbf{M}|\mathbf{q}\rangle)$ that is normal to tangent to original curve at $|\mathbf{q}\rangle$.



Original vector $|\mathbf{q}\rangle$ lies on gradient $\nabla(\langle \mathbf{p}|\mathbf{M}^{-1}|\mathbf{p}\rangle)$ that is normal to tangent to mapped curve at $|\mathbf{p}\rangle$.

Unitary operators and matrices that change state vectors

...and eigenstates ("ownstates) that are mostly immune

Geometric visualization of real symmetric matrices and eigenvectors

Circle-to-ellipse mapping

Ellipse-to-ellipse mapping (Normal space vs. tangent space)

→ *Eigensolutions as stationary extreme-values (Lagrange λ-multipliers)*

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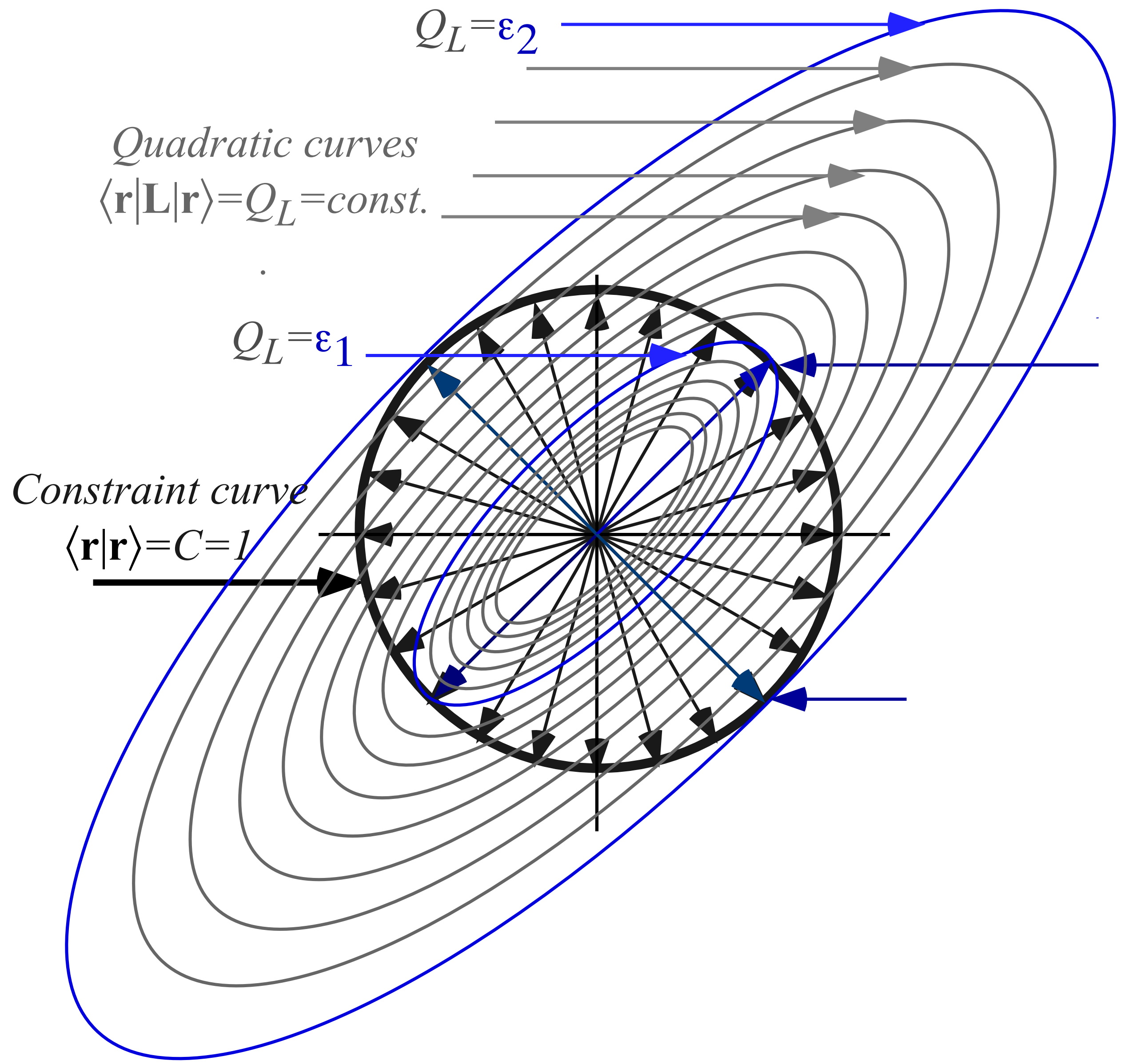
Eigenvalues λ of a matrix **L** can be viewed as stationary-values of its *quadratic form* $Q_L = L(\mathbf{r}) = \langle \mathbf{r}|\mathbf{L}|\mathbf{r}\rangle$



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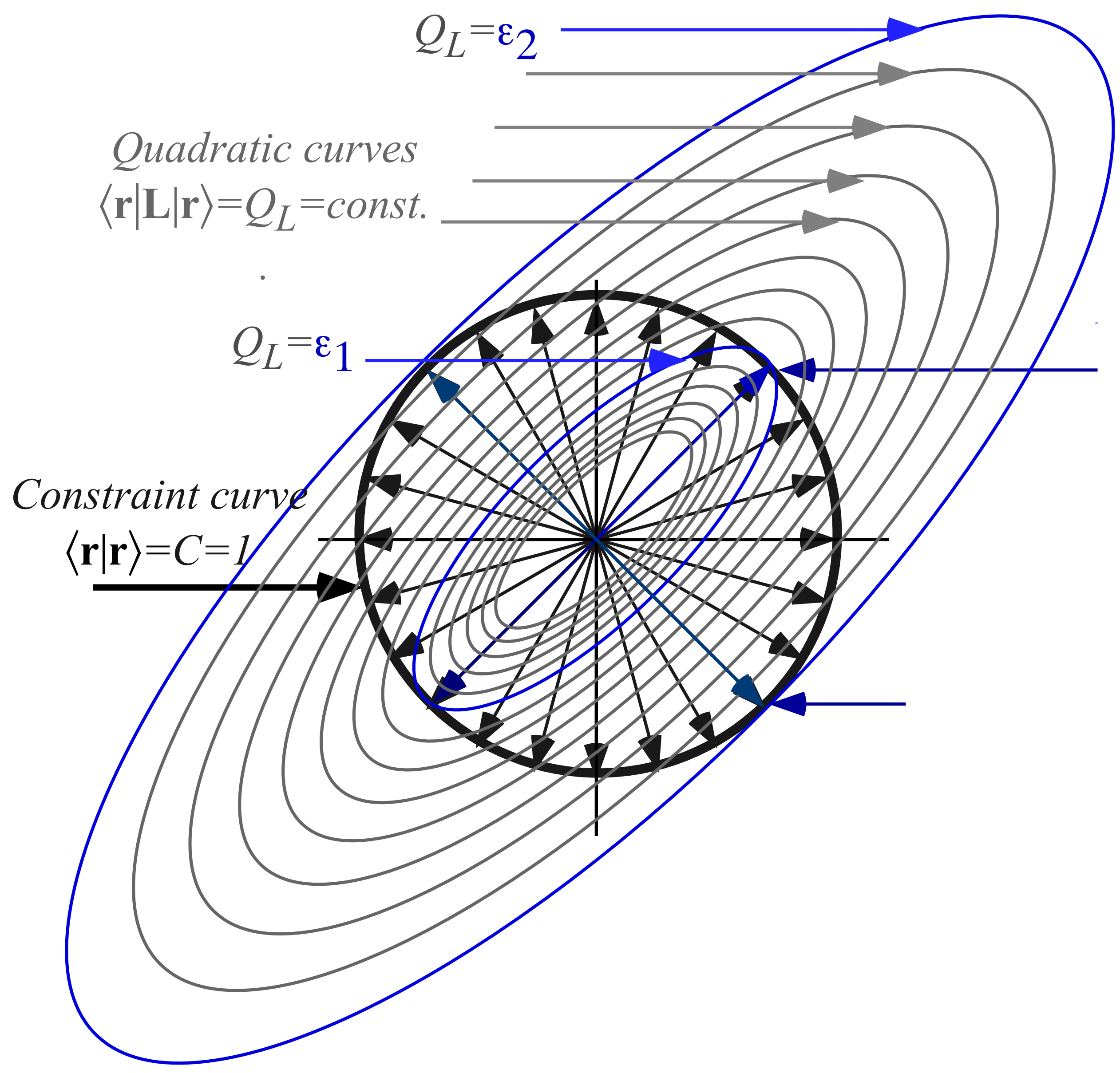

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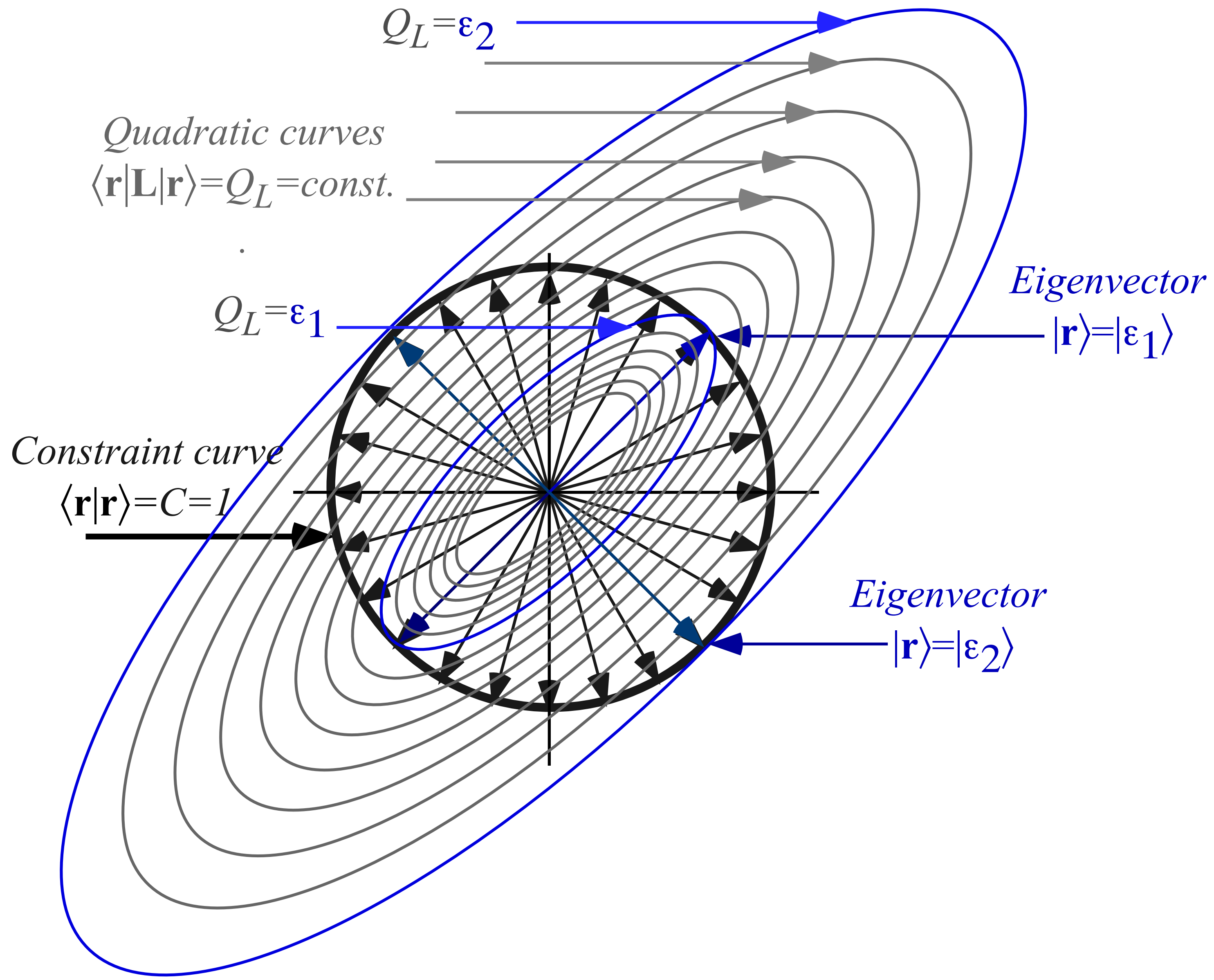

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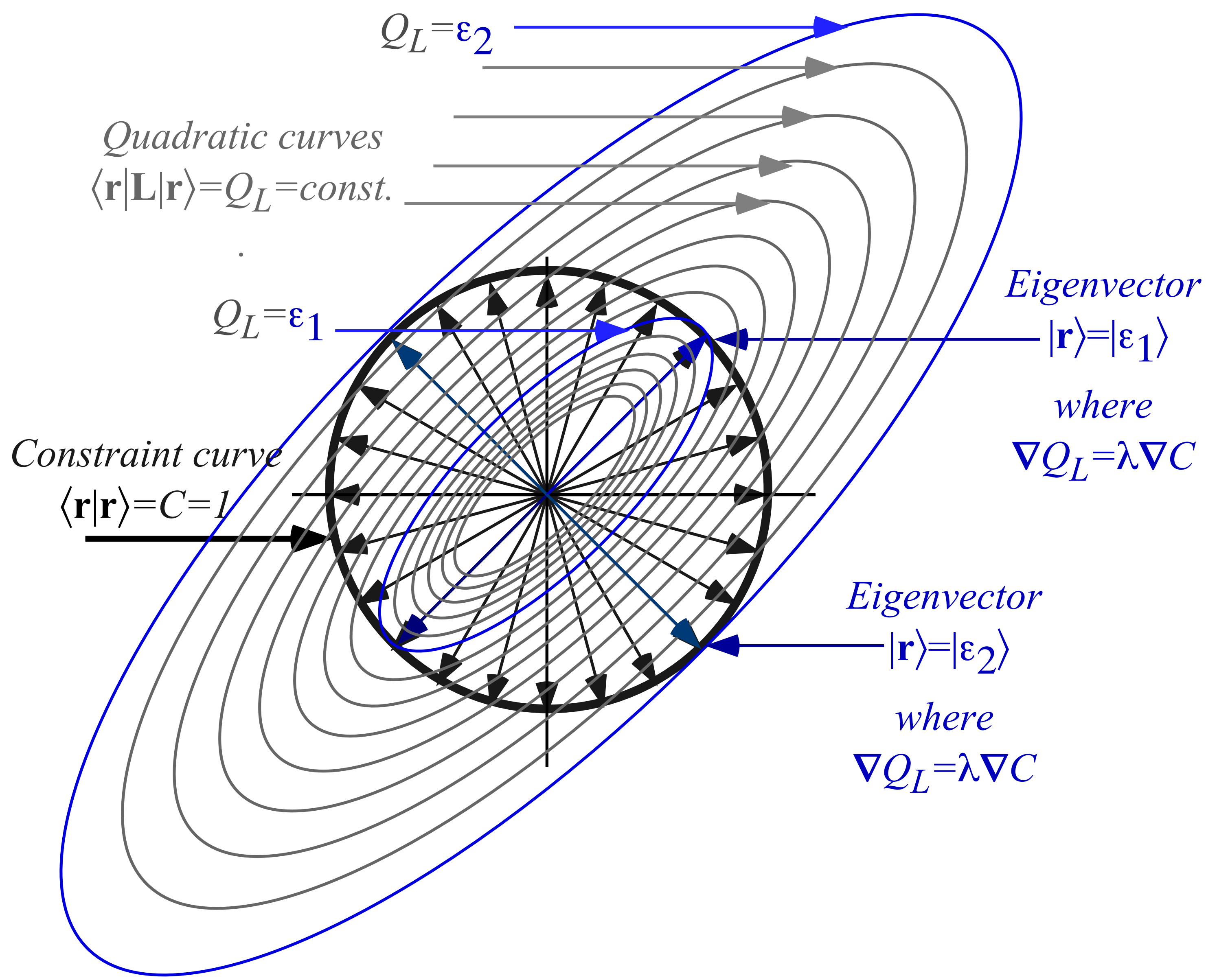
$$\nabla Q_L = \lambda \, \nabla C,$$



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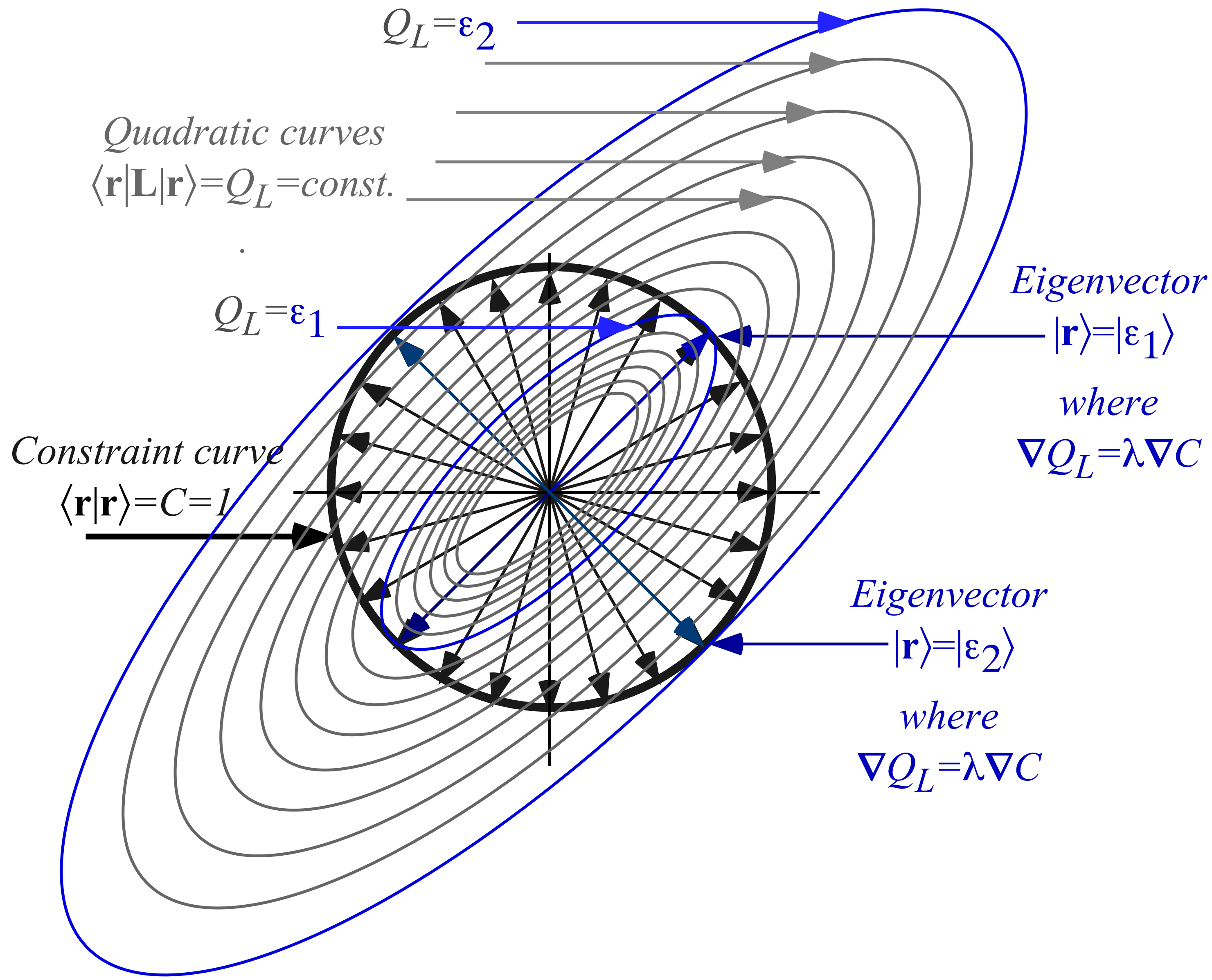

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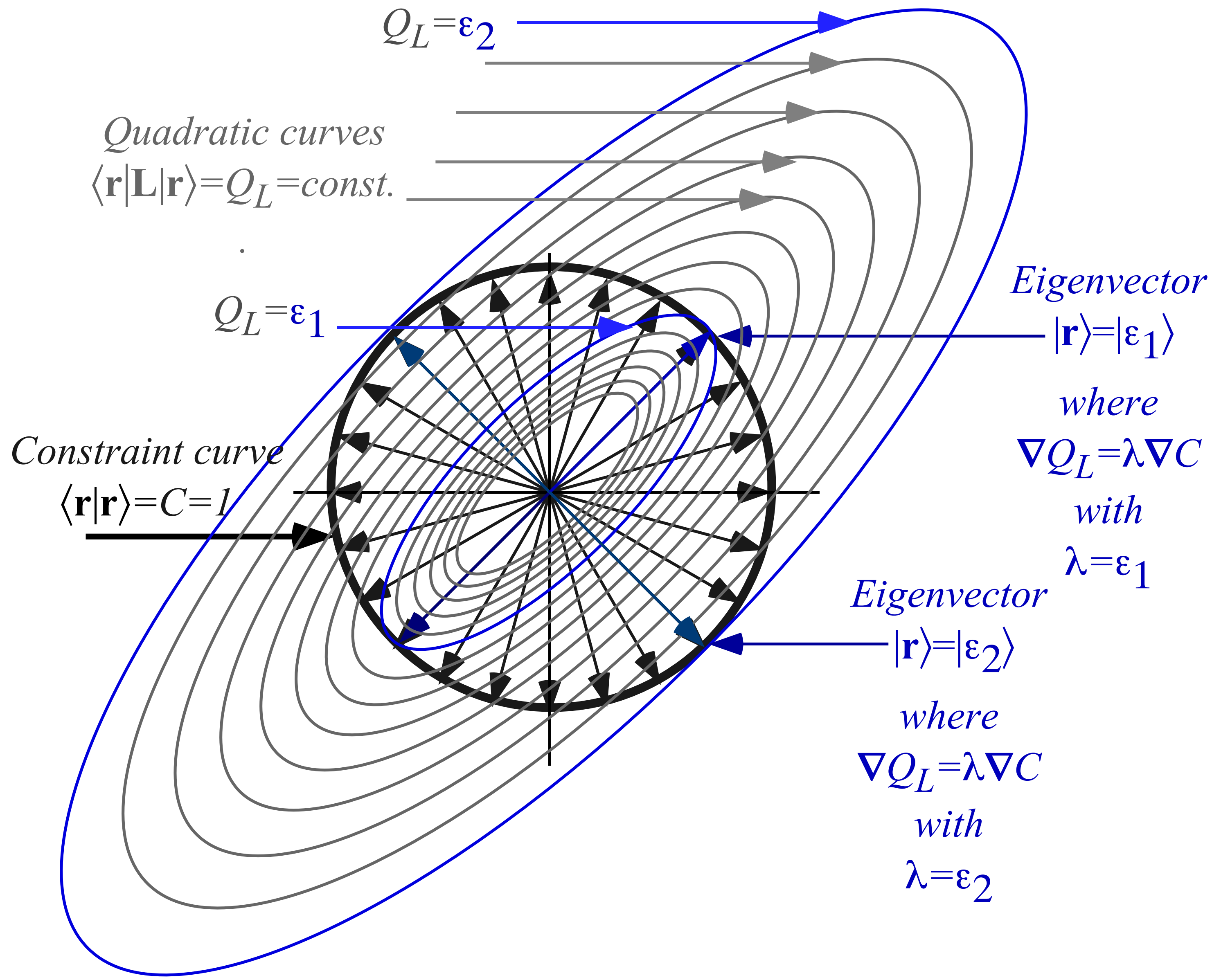

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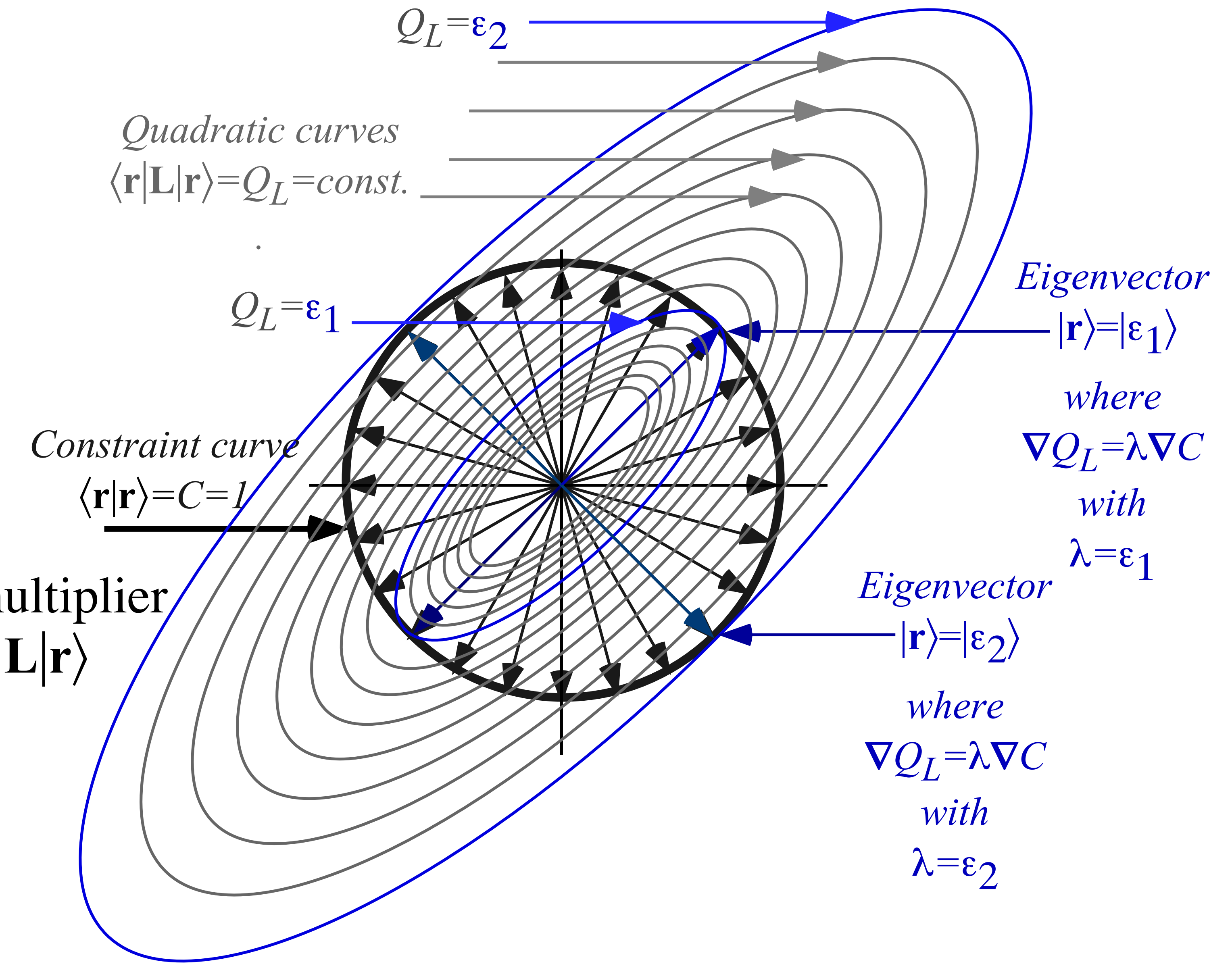

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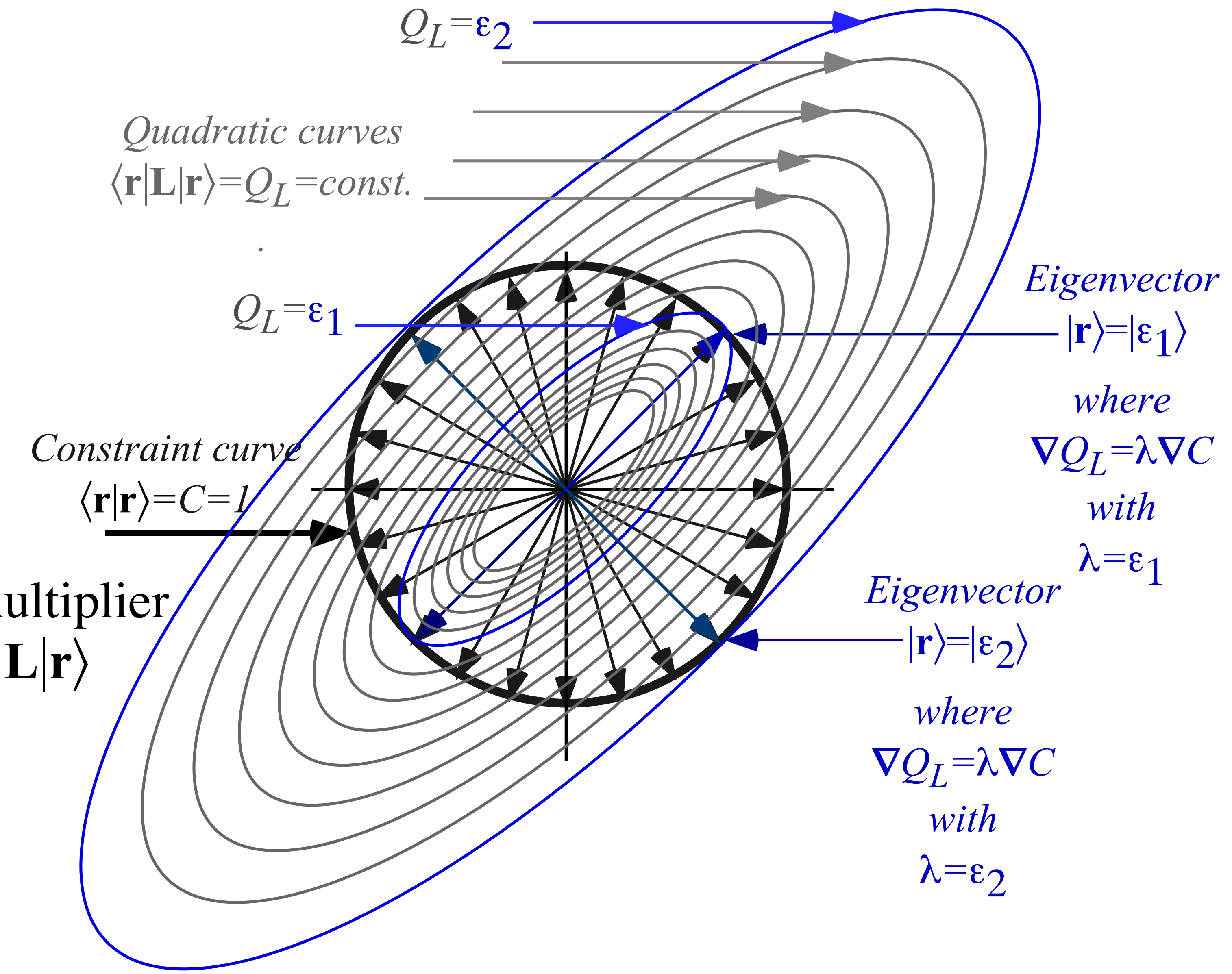

$\langle\mathbf{r}|\mathbf{L}|\mathbf{r}\rangle$ is called a quantum *expectation value* of operator $\mathbf{L}$ at $\mathbf{r}$.
Eigenvalues are extreme expectation values.

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An *eigenvector* $|\varepsilon_k\rangle$ of **M** is in a direction that is left unchanged by **M**.

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First step in finding eigenvalues: Solve *secular equation*

$$\det|\mathbf{M}-\varepsilon\mathbf{1}|=0=(-1)^n\left(\varepsilon^n+a_1\varepsilon^{n-1}+a_2\varepsilon^{n-2}+\ldots+a_{n-1}\varepsilon\ +a_n\right)$$

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Only possible non-zero $\{x,y\}$ if denominator is zero, too!

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$$0=(4-\varepsilon)(2-\varepsilon)-1\cdot3=8-6\varepsilon+\varepsilon^2-1\cdot3=\varepsilon^2-6\varepsilon+5$$

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Secular equation has n-factors, one for each eigenvalue.

$$\det|\mathbf{M} - \varepsilon\mathbf{1}| = 0 = (-1)^n(\varepsilon - \varepsilon_1)(\varepsilon - \varepsilon_2)\cdots(\varepsilon - \varepsilon_n)$$

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Only possible non-zero $\{x, y\}$ if denominator is zero, too!

$$0 = \det|\mathbf{M} - \varepsilon\cdot\mathbf{1}| = \det\left|\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} - \varepsilon\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| = \det\left|\begin{pmatrix} 4-\varepsilon & 1 \\ 3 & 2-\varepsilon \end{pmatrix}\right|$$

$$0 = (4-\varepsilon)(2-\varepsilon) - 1\cdot 3 = 8 - 6\varepsilon + \varepsilon^2 - 1\cdot 3 = \varepsilon^2 - 6\varepsilon + 5$$

$$0 = \varepsilon^2 - Trace(\mathbf{M})\varepsilon + \det(\mathbf{M}) = \varepsilon^2 - 6\varepsilon + 5$$

$$0 = (\varepsilon - 1)(\varepsilon - 5) \quad \text{so let:} \quad \varepsilon_1 = 1 \quad \text{and:} \quad \varepsilon_2 = 5$$

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→ *Hamilton-Cayley equation and projectors*

Idempotent projectors (how eigenvalues⇒eigenvectors)

Operator orthonormality and Completeness

Spectral Decompositions

Functional spectral decomposition

Orthonormality vs. Completeness vis-a`-vis Operator vs. State

Lagrange functional interpolation formula

Proof that completeness relation is "Truer-than-true"

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An *eigenvector* $|\varepsilon_k\rangle$ of **M** is in a direction that is left unchanged by **M**.

$$\mathbf{M}|\varepsilon_k\rangle=\varepsilon_k|\varepsilon_k\rangle,\ \text{ or: }\ (\mathbf{M}-\varepsilon_k\mathbf{1})|\varepsilon_k\rangle=\mathbf{0}$$

ε_k is *eigenvalue* associated with eigenvector $|\varepsilon_k\rangle$ direction.

A change of basis to $\{|\varepsilon_1\rangle,|\varepsilon_2\rangle,\cdots|\varepsilon_n\rangle\}$ called *diagonalization* gives

$$\begin{pmatrix}\langle\varepsilon_1|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_1|\mathbf{M}|\varepsilon_n\rangle\\ \langle\varepsilon_2|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_2|\mathbf{M}|\varepsilon_n\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle\varepsilon_n|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_n|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_n|\mathbf{M}|\varepsilon_n\rangle\end{pmatrix}=\begin{pmatrix}\varepsilon_1&0&\cdots&0\\0&\varepsilon_2&\cdots&0\\ \vdots&\vdots&\ddots&\vdots\\0&0&\cdots&\varepsilon_n\end{pmatrix}$$

First step in finding eigenvalues: Solve *secular equation*

$$\det|\mathbf{M}-\varepsilon\mathbf{1}|=0=(-1)^n\left(\varepsilon^n+a_1\varepsilon^{n-1}+a_2\varepsilon^{n-2}+\ldots+a_{n-1}\varepsilon\ +a_n\right)$$

where:

$a_1=-Trace\mathbf{M},\cdots,\ a_k=(-1)^k\sum\text{diagonal k-by-k minors of }\mathbf{M},\cdots,\ \ a_n=(-1)^n\det|\mathbf{M}|$

Secular equation has n-factors, one for each eigenvalue.

$$\det|\mathbf{M}-\varepsilon\mathbf{1}|=0=(-1)^n(\varepsilon-\varepsilon_1)(\varepsilon-\varepsilon_2)\cdots(\varepsilon-\varepsilon_n)$$

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Only possible non-zero $\{x,y\}$ if denominator is zero, too!

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$$0=(4-\varepsilon)(2-\varepsilon)-1\cdot3=8-6\varepsilon+\varepsilon^2-1\cdot3=\varepsilon^2-6\varepsilon+5$$

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Each $\mathbf{p}_k$ contains *eigen-bra-kets* since: $(\mathbf{M}-\varepsilon_k\mathbf{1})\mathbf{p}_k=0$ or: $\mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$.

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$$\mathbf{M}\mathbf{p}_2=\begin{pmatrix}4&1\\3&2\end{pmatrix}\cdot\begin{pmatrix}3&1\\3&1\end{pmatrix}=5\cdot\begin{pmatrix}3&1\\3&1\end{pmatrix}=5\cdot\mathbf{p}_2$$

Unitary operators and matrices that change state vectors

...and eigenstates ("ownstates) that are mostly immune

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$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix}$$

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"Gauge" scale factors that only affect plots

$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix}3 & 1\\ 3 & 1\end{pmatrix} = k_2\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{3}{2} & \frac{1}{2}\end{pmatrix}}{k_2} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

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$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}) \qquad \mathbf{M}\mathbf{p}_k = \varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$$

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With example matrix $\mathbf{M} = \begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix}$

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Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix}1 & -1\\ -3 & 3\end{pmatrix} = k_1\begin{pmatrix}\frac{1}{2}\\ -\frac{3}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

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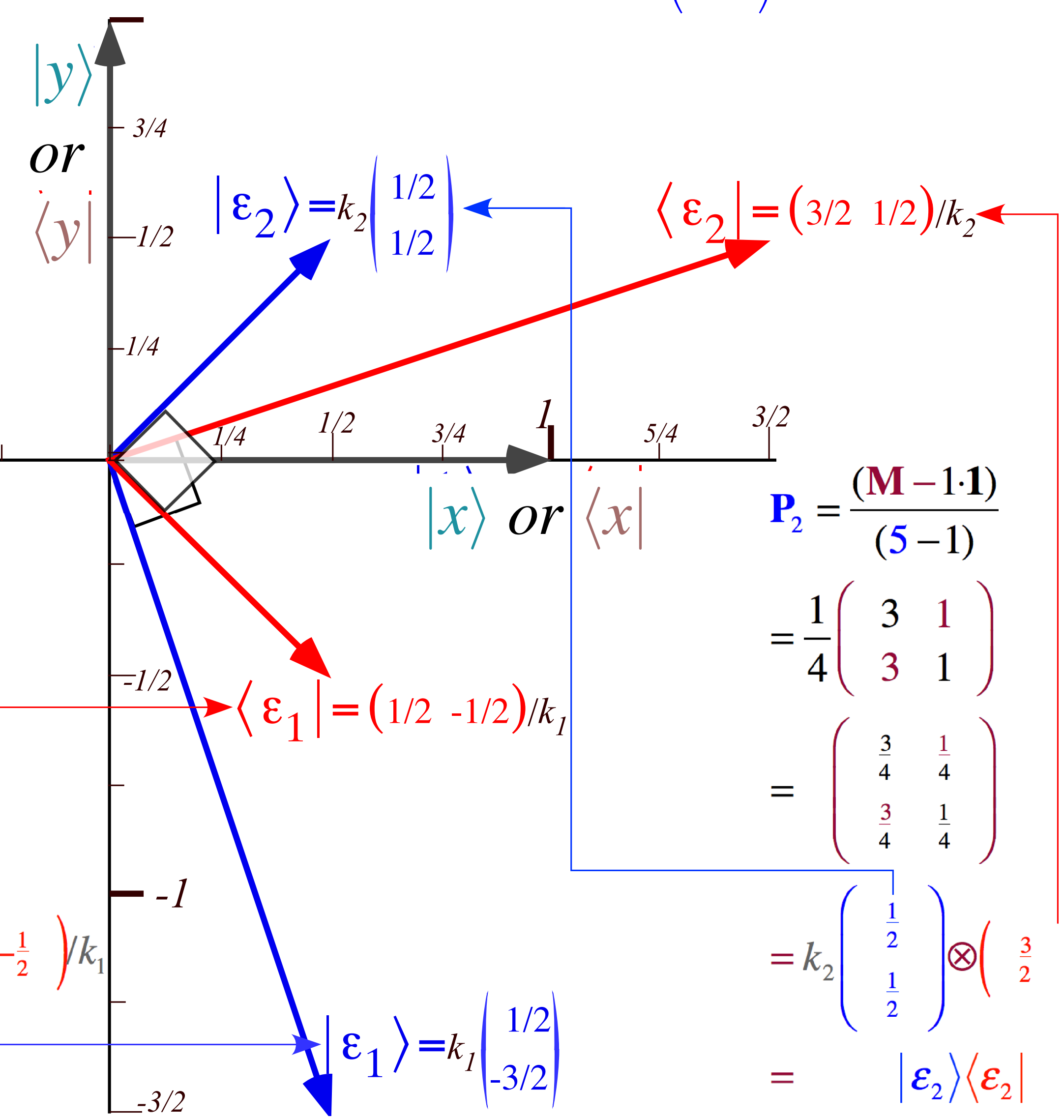

Unitary operators and matrices that change state vectors

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"Gauge" scale factors that only affect plots

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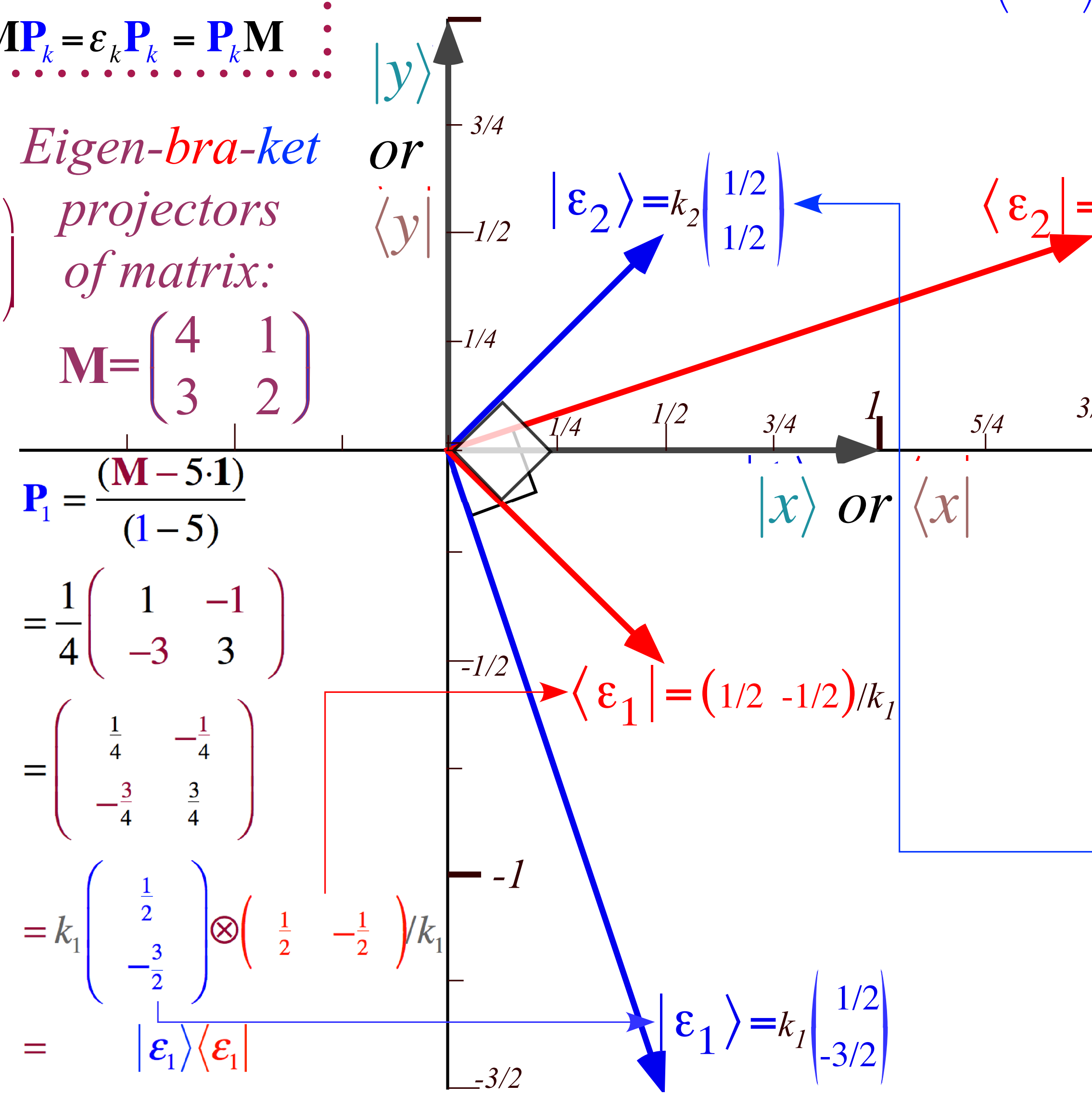

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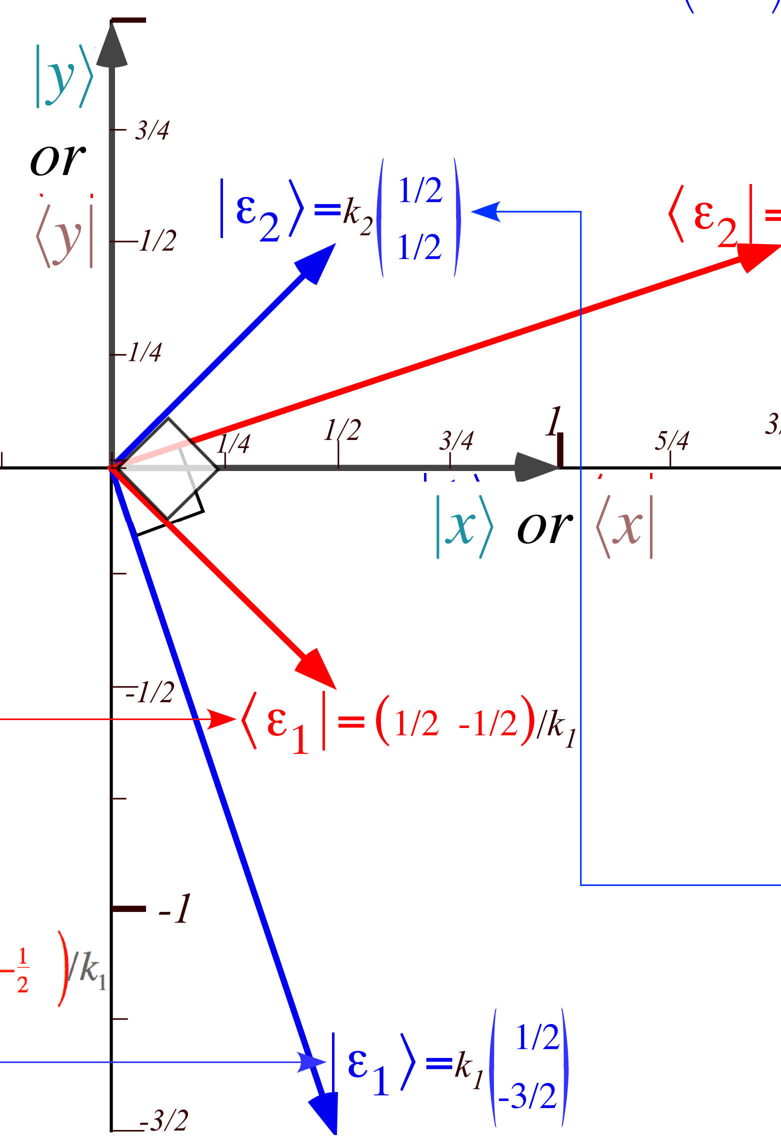

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$$\mathbf{P}_1+\mathbf{P}_2 = \begin{pmatrix}1&0\\0&1\end{pmatrix} = |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigen-operators $\mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k$ then give *Spectral Decomposition* of operator **M**

$$\mathbf{M} = \mathbf{M}\mathbf{P}_1+\mathbf{M}\mathbf{P}_2+\ldots+\mathbf{M}\mathbf{P}_n = \varepsilon_1\mathbf{P}_1+\varepsilon_2\mathbf{P}_2+\ldots+\varepsilon_n\mathbf{P}_n$$

With example matrix $\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$

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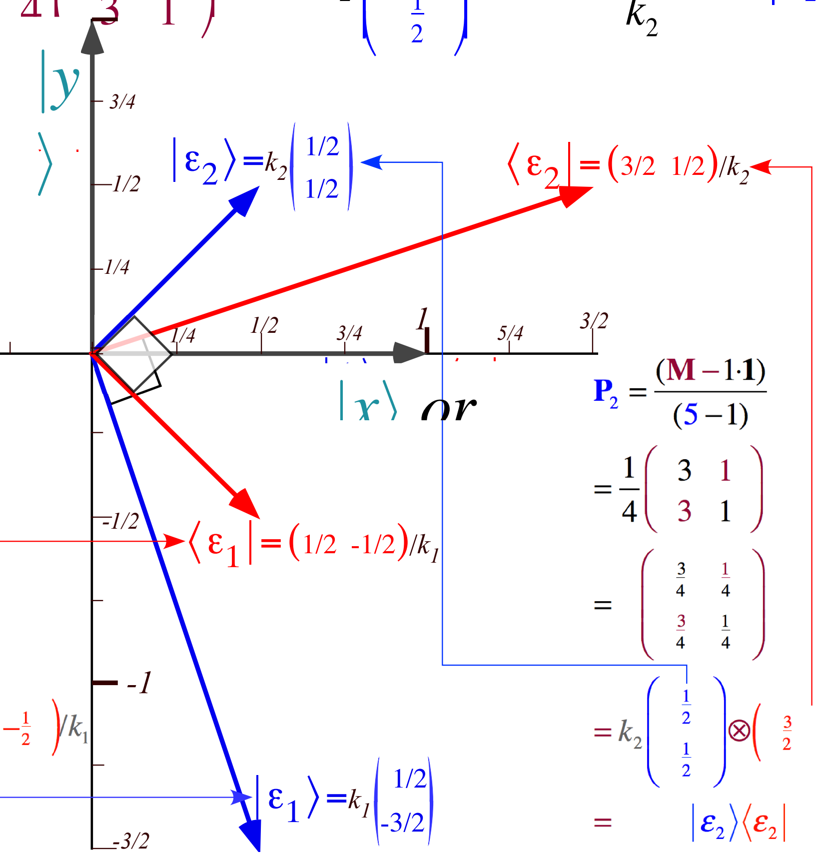

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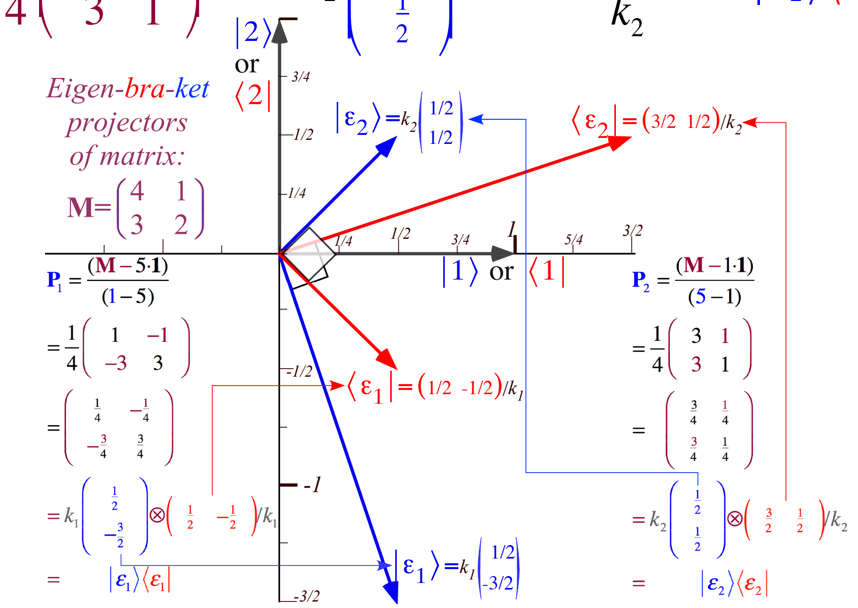

$$\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = 1\mathbf{P}_1 + 5\mathbf{P}_2 = 1|1\rangle\langle 1| + 5|2\rangle\langle 2| = 1\begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} + 5\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

Eigen-operators $\mathbf{M}\mathbf{P}_k = \varepsilon_k\mathbf{P}_k$ then give *Spectral Decomposition* of operator $\mathbf{M}$

$$\mathbf{M} = \mathbf{M}\mathbf{P}_1 + \mathbf{M}\mathbf{P}_2 + \ldots + \mathbf{M}\mathbf{P}_n = \varepsilon_1\mathbf{P}_1 + \varepsilon_2\mathbf{P}_2 + \ldots + \varepsilon_n\mathbf{P}_n$$

Matrix and operator Spectral Decompositons

$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}) \qquad \mathbf{M}\mathbf{p}_k = \varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$$

$$\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$$

$$\mathbf{p}_1 = (\mathbf{M}-5\cdot\mathbf{1}) = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \qquad \mathbf{p}_1\mathbf{p}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{p}_2 = (\mathbf{M}-1\cdot\mathbf{1}) = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k = \prod_{m\neq k}(\varepsilon_j\mathbf{p}_j-\varepsilon_m\mathbf{p}_j) = \mathbf{p}_j\prod_{m\neq k}(\varepsilon_j-\varepsilon_m) = \begin{cases} \mathbf{0} & if : j\neq k \\ \mathbf{p}_k\prod_{m\neq k}(\varepsilon_k-\varepsilon_m) & if : j = k \end{cases}$$

Last step:

make *Idempotent Projectors:* $\mathbf{P}_k = \frac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} = \frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$

(*Idempotent means:* **P·P=P**)

$$\mathbf{P}_j\mathbf{P}_k = \begin{cases} \mathbf{0} & if : j\neq k \\ \mathbf{P}_k & if : j = k \end{cases}$$

$\mathbf{M}\mathbf{p}_k = \varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$

implies :

$\mathbf{M}\mathbf{P}_k = \varepsilon_k\mathbf{P}_k = \mathbf{P}_k\mathbf{M}$

Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1\begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}}{k_2} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

The $\mathbf{P}_j$ are *Mutually Ortho-Normal*
as are bra-ket $\langle\varepsilon_j|$ and $|\varepsilon_j\rangle$ inside $\mathbf{P}_j$'s

$$\begin{pmatrix} \langle\varepsilon_1|\varepsilon_1\rangle & \langle\varepsilon_1|\varepsilon_2\rangle \\ \langle\varepsilon_2|\varepsilon_1\rangle & \langle\varepsilon_2|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

...and the $\mathbf{P}_j$ satisfy a
Completeness Relation:

$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$

$= |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2| + \ldots + |\varepsilon_n\rangle\langle\varepsilon_n|$

$$\mathbf{P}_1 + \mathbf{P}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2|$$

$$\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = 1\mathbf{P}_1 + 5\mathbf{P}_2 = 1|1\rangle\langle 1| + 5|2\rangle\langle 2| = 1\begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} + 5\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

Eigen-operators $\mathbf{M}\mathbf{P}_k = \varepsilon_k\mathbf{P}_k$ then give *Spectral Decomposition* of operator **M**

$$\mathbf{M} = \mathbf{M}\mathbf{P}_1 + \mathbf{M}\mathbf{P}_2 + \ldots + \mathbf{M}\mathbf{P}_n = \varepsilon_1\mathbf{P}_1 + \varepsilon_2\mathbf{P}_2 + \ldots + \varepsilon_n\mathbf{P}_n$$

...and *Functional Spectral Decomposition* of any function *f*(**M**) of **M**

$$f(\mathbf{M}) == f(\varepsilon_1)\mathbf{P}_1 + f(\varepsilon_2)\mathbf{P}_2 + \ldots + f(\varepsilon_n)\mathbf{P}_n$$

Matrix and operator Spectral Decompositons

$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j \prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}) \qquad \mathbf{M}\mathbf{p}_k = \varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k = \prod_{m\neq k}(\varepsilon_j\mathbf{p}_j-\varepsilon_m\mathbf{p}_j) = \mathbf{p}_j\prod_{m\neq k}(\varepsilon_j-\varepsilon_m) = \begin{cases} \mathbf{0} & if: j\neq k \\ \mathbf{p}_k\prod_{m\neq k}(\varepsilon_k-\varepsilon_m) & if: j=k \end{cases}$$

Last step:
make *Idempotent Projectors:* $\mathbf{P}_k = \dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} = \dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$
(*Idempotent means:* **P·P=P**)

$$\mathbf{P}_j\mathbf{P}_k = \begin{cases} \mathbf{0} & if: j\neq k \\ \mathbf{P}_k & if: j=k \end{cases}$$

$\mathbf{M}\mathbf{p}_k = \varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$
implies:
$\mathbf{M}\mathbf{P}_k = \varepsilon_k\mathbf{P}_k = \mathbf{P}_k\mathbf{M}$

$$\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$$

$$\mathbf{p}_1 = (\mathbf{M}-5\cdot\mathbf{1}) = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \qquad \mathbf{p}_1\mathbf{p}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{p}_2 = (\mathbf{M}-1\cdot\mathbf{1}) = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1\begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}}{k_2} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

The $\mathbf{P}_j$ are *Mutually Ortho-Normal*
as are bra-ket $\langle\varepsilon_j|$ and $|\varepsilon_j\rangle$ inside $\mathbf{P}_j$'s

$$\begin{pmatrix} \langle\varepsilon_1|\varepsilon_1\rangle & \langle\varepsilon_1|\varepsilon_2\rangle \\ \langle\varepsilon_2|\varepsilon_1\rangle & \langle\varepsilon_2|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

...and the $\mathbf{P}_j$ satisfy a
Completeness Relation:
$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$
$= |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2| + \ldots + |\varepsilon_n\rangle\langle\varepsilon_n|$

$$\mathbf{P}_1 + \mathbf{P}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2|$$

$$\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = 1\mathbf{P}_1 + 5\mathbf{P}_2 = 1|1\rangle\langle 1| + 5|2\rangle\langle 2| = 1\begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} + 5\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

Eigen-operators $\mathbf{M}\mathbf{P}_k = \varepsilon_k\mathbf{P}_k$ then give *Spectral Decomposition* of operator **M**

$$\mathbf{M} = \mathbf{M}\mathbf{P}_1 + \mathbf{M}\mathbf{P}_2 + \ldots + \mathbf{M}\mathbf{P}_n = \varepsilon_1\mathbf{P}_1 + \varepsilon_2\mathbf{P}_2 + \ldots + \varepsilon_n\mathbf{P}_n$$

...and *Functional Spectral Decomposition* of any function *f*(**M**) of **M**

$$f(\mathbf{M}) == f(\varepsilon_1)\mathbf{P}_1 + f(\varepsilon_2)\mathbf{P}_2 + \ldots + f(\varepsilon_n)\mathbf{P}_n$$

Example:

$$\mathbf{M}^{50} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = 1^{50}\begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} + 5^{50}\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1+3\cdot 5^{50} & 5^{50}-1 \\ 3\cdot 5^{50}-3 & 5^{50}+3 \end{pmatrix}$$

Matrix and operator Spectral Decompositons

$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}) \qquad \mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$$

Multiplication properties of $\mathbf{p}_j$:

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Last step:
make *Idempotent Projectors:* $\mathbf{P}_k = \dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} = \dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$
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$\mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$
implies :
$\mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k=\mathbf{P}_k\mathbf{M}$

$$\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$$

$$\mathbf{p}_1=(\mathbf{M}-5\cdot\mathbf{1})=\begin{pmatrix}-1&1\\3&-3\end{pmatrix} \qquad \mathbf{p}_1\mathbf{p}_2=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$$\mathbf{p}_2=(\mathbf{M}-1\cdot\mathbf{1})=\begin{pmatrix}3&1\\3&1\end{pmatrix}$$

Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1=\frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)}=\frac{1}{4}\begin{pmatrix}1&-1\\-3&3\end{pmatrix}=k_1\begin{pmatrix}\frac{1}{2}\\-\frac{3}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\end{pmatrix}}{k_1}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)}=\frac{1}{4}\begin{pmatrix}3&1\\3&1\end{pmatrix}=k_2\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{3}{2}&\frac{1}{2}\end{pmatrix}}{k_2}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

The $\mathbf{P}_j$ are *Mutually Ortho-Normal*
as are bra-ket $\langle\varepsilon_j|$ and $|\varepsilon_j\rangle$ inside $\mathbf{P}_j$'s

$$\begin{pmatrix}\langle\varepsilon_1|\varepsilon_1\rangle&\langle\varepsilon_1|\varepsilon_2\rangle\\\langle\varepsilon_2|\varepsilon_1\rangle&\langle\varepsilon_2|\varepsilon_2\rangle\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

...and the $\mathbf{P}_j$ satisfy a
Completeness Relation:
$\mathbf{1}= \mathbf{P}_1 + \mathbf{P}_2 +\ldots+ \mathbf{P}_n$
$=|\varepsilon_1\rangle\langle\varepsilon_1|+|\varepsilon_2\rangle\langle\varepsilon_2|+\ldots+|\varepsilon_n\rangle\langle\varepsilon_n|$

$$\mathbf{P}_1+\mathbf{P}_2=\begin{pmatrix}1&0\\0&1\end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|+|\varepsilon_2\rangle\langle\varepsilon_2|$$

$$\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}=1\mathbf{P}_1+5\mathbf{P}_2=1|1\rangle\langle1|+5|2\rangle\langle2|=1\begin{pmatrix}\frac{1}{4}&-\frac{1}{4}\\-\frac{3}{4}&\frac{3}{4}\end{pmatrix}+5\begin{pmatrix}\frac{3}{4}&\frac{1}{4}\\\frac{3}{4}&\frac{1}{4}\end{pmatrix}$$

Eigen-operators $\mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k$ then give *Spectral Decomposition* of operator **M**

$$\mathbf{M}=\mathbf{M}\mathbf{P}_1+\mathbf{M}\mathbf{P}_2+\ldots+\mathbf{M}\mathbf{P}_n=\varepsilon_1\mathbf{P}_1+\varepsilon_2\mathbf{P}_2+\ldots+\varepsilon_n\mathbf{P}_n$$

...and *Functional Spectral Decomposition* of any function *f*(**M**) of **M**

$$f(\mathbf{M})==f(\varepsilon_1)\mathbf{P}_1+f(\varepsilon_2)\mathbf{P}_2+\ldots+f(\varepsilon_n)\mathbf{P}_n$$

Examples:

$$\mathbf{M}^{50}=\begin{pmatrix}4&1\\3&2\end{pmatrix}=1^{50}\begin{pmatrix}\frac{1}{4}&-\frac{1}{4}\\-\frac{3}{4}&\frac{3}{4}\end{pmatrix}+5^{50}\begin{pmatrix}\frac{3}{4}&\frac{1}{4}\\\frac{3}{4}&\frac{1}{4}\end{pmatrix}=\frac{1}{4}\begin{pmatrix}1+3\cdot5^{50}&5^{50}-1\\3\cdot5^{50}-3&5^{50}+3\end{pmatrix}$$

$$\sqrt{\mathbf{M}}=\begin{pmatrix}4&1\\3&2\end{pmatrix}=\pm\sqrt{1}\begin{pmatrix}\frac{1}{4}&-\frac{1}{4}\\-\frac{3}{4}&\frac{3}{4}\end{pmatrix}\pm\sqrt{5}\begin{pmatrix}\frac{3}{4}&\frac{1}{4}\\\frac{3}{4}&\frac{1}{4}\end{pmatrix}$$

Unitary operators and matrices that change state vectors

...and eigenstates ("ownstates) that are mostly immune

Geometric visualization of real symmetric matrices and eigenvectors

Circle-to-ellipse mapping

Ellipse-to-ellipse mapping (Normal space vs. tangent space)

Eigensolutions as stationary extreme-values (Lagrange λ-multipliers)

Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular equation

Hamilton-Cayley equation and projectors

Idempotent projectors (how eigenvalues⇒eigenvectors)

Operator orthonormality and Completeness

Factoring bra-kets into "Ket-Bras:

Spectral Decompositions

Functional spectral decomposition

→ *Orthonormality vs. Completeness vis-a`-vis Operator vs. State* ←

Lagrange functional interpolation formula

Proof that completeness relation is "Truer-than-true"

Spectral Decompositions with degeneracy

Functional spectral decomposition

Orthonormality vs. Completeness

$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1}) \qquad \mathbf{M}\mathbf{p}_k = \varepsilon_k\mathbf{p}_k = \mathbf{p}_k\mathbf{M}$$

Multiplication properties of $\mathbf{p}_j$:

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Last step:

make *Idempotent Projectors:* $\mathbf{P}_k = \dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} = \dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$

(*Idempotent means:* **P·P=P**)

$$\mathbf{P}_j\mathbf{P}_k = \begin{cases}\mathbf{0} & if: j\neq k\\ \mathbf{P}_k & if: j=k\end{cases}$$

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implies:

$\mathbf{M}\mathbf{P}_k = \varepsilon_k\mathbf{P}_k = \mathbf{P}_k\mathbf{M}$

$$\mathbf{M} = \begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix}$$

$$\mathbf{p}_1 = (\mathbf{M}-5\cdot\mathbf{1}) = \begin{pmatrix}-1 & 1\\ 3 & -3\end{pmatrix} \qquad \mathbf{p}_1\mathbf{p}_2 = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

$$\mathbf{p}_2 = (\mathbf{M}-1\cdot\mathbf{1}) = \begin{pmatrix}3 & 1\\ 3 & 1\end{pmatrix}$$

Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix}1 & -1\\ -3 & 3\end{pmatrix} = k_1\begin{pmatrix}\frac{1}{2}\\ -\frac{3}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

"Gauge" scale factors that only affect plots

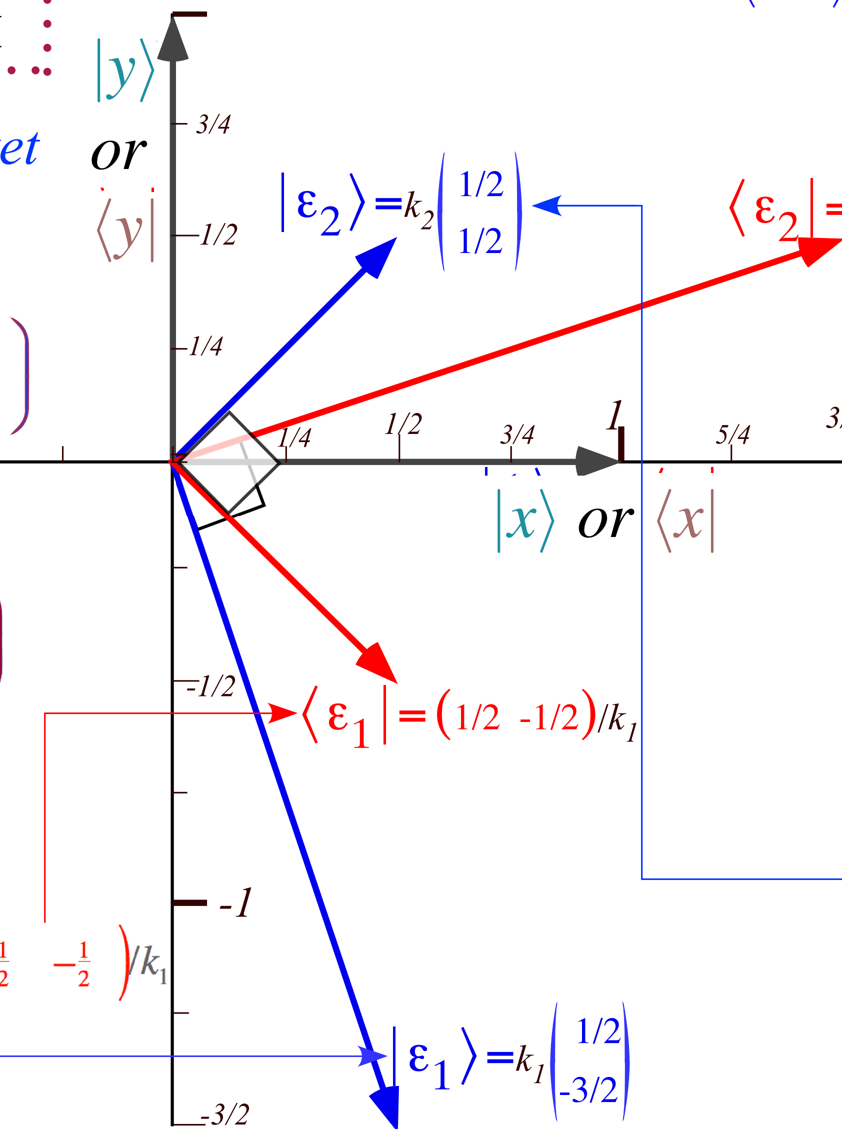
$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix}3 & 1\\ 3 & 1\end{pmatrix} = k_2\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix}\otimes\frac{\begin{pmatrix}\frac{3}{2} & \frac{1}{2}\end{pmatrix}}{k_2} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

The $\mathbf{P}_j$ are *Mutually Ortho-Normal* as are bra-ket $\langle\varepsilon_j|$ and $|\varepsilon_j\rangle$ inside $\mathbf{P}_j$'s

$$\begin{pmatrix}\langle\varepsilon_1|\varepsilon_1\rangle & \langle\varepsilon_1|\varepsilon_2\rangle\\ \langle\varepsilon_2|\varepsilon_1\rangle & \langle\varepsilon_2|\varepsilon_2\rangle\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Eigen-bra-ket projectors of matrix:

$$\mathbf{M} = \begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix}$$



$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix}1 & -1\\ -3 & 3\end{pmatrix} = \begin{pmatrix}\frac{1}{4} & -\frac{1}{4}\\ -\frac{3}{4} & \frac{3}{4}\end{pmatrix} = k_1\begin{pmatrix}\frac{1}{2}\\ -\frac{3}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\end{pmatrix}/k_1 = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix}3 & 1\\ 3 & 1\end{pmatrix} = \begin{pmatrix}\frac{3}{4} & \frac{1}{4}\\ \frac{3}{4} & \frac{1}{4}\end{pmatrix} = k_2\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{3}{2} & \frac{1}{2}\end{pmatrix}/k_2 = |\varepsilon_2\rangle\langle\varepsilon_2|$$

...and the $\mathbf{P}_j$ satisfy a *Completeness Relation:*

$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$

$= |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2| + \ldots + |\varepsilon_n\rangle\langle\varepsilon_n|$

$$\mathbf{P}_1 + \mathbf{P}_2 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2|$$

$\{|x\rangle,|y\rangle\}$-orthonormality with $\{|\varepsilon_1\rangle,|\varepsilon_2\rangle\}$-completeness

$$\langle x|y\rangle = \delta_{x,y} = \langle x|\mathbf{1}|y\rangle = \langle x|\varepsilon_1\rangle\langle\varepsilon_1|y\rangle + \langle x|\varepsilon_2\rangle\langle\varepsilon_2|y\rangle.$$

$\{|\varepsilon_1\rangle,|\varepsilon_2\rangle\}$-orthonormality with $\{|x\rangle,|y\rangle\}$-completeness

$$\langle\varepsilon_i|\varepsilon_j\rangle = \delta_{i,j} = \langle\varepsilon_i|\mathbf{1}|\varepsilon_j\rangle = \langle\varepsilon_i|x\rangle\langle x|\varepsilon_j\rangle + \langle\varepsilon_i|y\rangle\langle y|\varepsilon_j\rangle$$

Orthonormality vs. Completeness vis-a`-vis Operator vs. State

Operator expressions for orthonormality appear quite different from expressions for completeness.

$$\mathbf{P}_j\mathbf{P}_k = \delta_{jk}\mathbf{P}_k = \begin{cases} \mathbf{0} & if : j \neq k \\ \mathbf{P}_k & if : j = k \end{cases}$$

$$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$$

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$$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$$

$$|\varepsilon_j\rangle\langle\varepsilon_j|\varepsilon_k\rangle\langle\varepsilon_k| = \delta_{jk}|\varepsilon_k\rangle\langle\varepsilon_k| \quad \text{or:} \quad \langle\varepsilon_j|\varepsilon_k\rangle = \delta_{jk}$$

$$\mathbf{1} = |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2| + \ldots + |\varepsilon_n\rangle\langle\varepsilon_n|$$

Orthonormality vs. Completeness vis-a`-vis Operator vs. State

*Operator expressions for orthonormality appear quite **different** from expressions for completeness.*

$$\mathbf{P}_j\mathbf{P}_k = \delta_{jk}\mathbf{P}_k = \begin{cases} \mathbf{0} & if: j \neq k \\ \mathbf{P}_k & if: j = k \end{cases}$$

$$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$$

$$|\varepsilon_j\rangle\langle\varepsilon_j|\varepsilon_k\rangle\langle\varepsilon_k| = \delta_{jk}|\varepsilon_k\rangle\langle\varepsilon_k| \quad \text{or:} \quad \langle\varepsilon_j|\varepsilon_k\rangle = \delta_{jk}$$

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*State vector representations of orthonormality are quite **similar** to representations of completeness.*
Like 2-sides of the same coin.

$\{|x\rangle, |y\rangle\}$-orthonormality with $\{|\varepsilon_1\rangle, |\varepsilon_2\rangle\}$-completeness

$$\langle x|y\rangle = \delta_{x,y} = \langle x|\mathbf{1}|y\rangle = \langle x|\varepsilon_1\rangle\langle\varepsilon_1|y\rangle + \langle x|\varepsilon_2\rangle\langle\varepsilon_2|y\rangle.$$

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Orthonormality vs. Completeness vis-a`-vis Operator vs. State

Operator expressions for orthonormality appear quite ***different*** *from expressions for completeness.*

$$\mathbf{P}_j\mathbf{P}_k = \delta_{jk}\mathbf{P}_k = \begin{cases} \mathbf{0} & if : j \neq k \\ \mathbf{P}_k & if : j = k \end{cases}$$

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$$\langle x|y\rangle = \delta(x,y) = \psi_1(x)\psi_1^*(y) + \psi_2(x)\psi_2^*(y) + \ldots$$

Dirac δ-function

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However Schrodinger wavefunction notation $\boldsymbol{\psi}(x) = \langle x|\boldsymbol{\psi}\rangle$ *shows quite a difference...*

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…particularly in the orthonormality integral.

Unitary operators and matrices that change state vectors

...and eigenstates ("ownstates) that are mostly immune

Geometric visualization of real symmetric matrices and eigenvectors

Circle-to-ellipse mapping

Ellipse-to-ellipse mapping (Normal space vs. tangent space)

Eigensolutions as stationary extreme-values (Lagrange λ-multipliers)

Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Secular equation

Hamilton-Cayley equation and projectors

Idempotent projectors (how eigenvalues⇒eigenvectors)

Operator orthonormality and Completeness

Factoring bra-kets into "Ket-Bras:

Spectral Decompositions

Functional spectral decomposition

Orthonormality vs. Completeness vis-a`-vis Operator vs. State

Lagrange functional interpolation formula

Proof that completeness relation is "Truer-than-true"

Spectral Decompositions with degeneracy

Functional spectral decomposition

A Proof of Projector Completeness *(Truer-than-true by Lagrange interpolation)*

Compare matrix *completeness relation* and *functional spectral decompositions*

$$\mathbf{1}=\mathbf{P}_1+\mathbf{P}_2+\ldots+\mathbf{P}_n=\sum_{\varepsilon_k}\mathbf{P}_k=\sum_{\varepsilon_k}\frac{\prod_{m\neq k}\left(\mathbf{M}-\varepsilon_m\mathbf{1}\right)}{\prod_{m\neq k}\left(\varepsilon_k-\varepsilon_m\right)}$$

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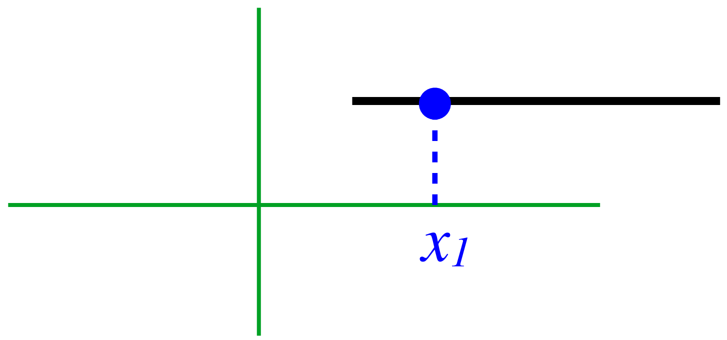

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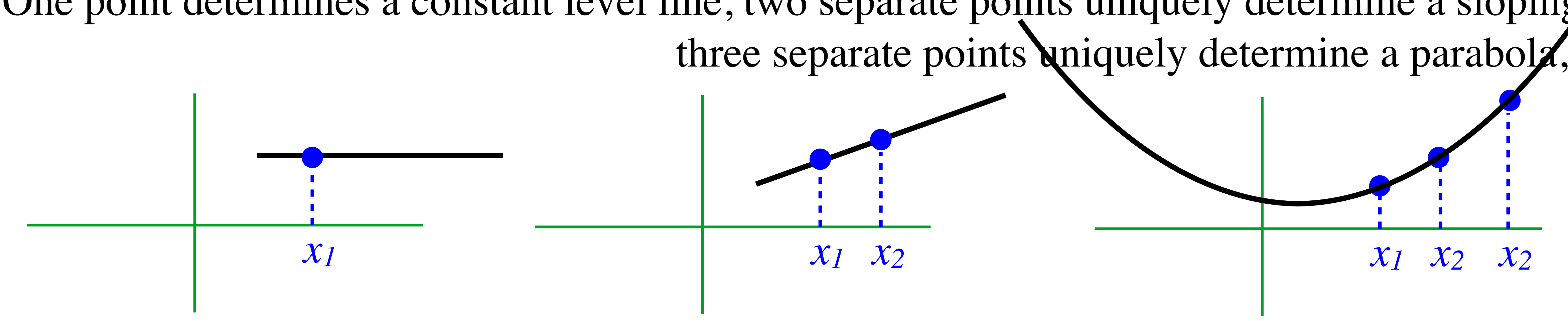

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All distinct values $\varepsilon_1\neq\varepsilon_2\neq\ldots\neq\varepsilon_N$ satisfy $\Sigma\mathbf{P}_k=\mathbf{1}$. Completeness is *truer than true* as is seen for $N=2$.

$$\mathbf{P}_1+\mathbf{P}_2=\frac{\prod_{j\neq1}(\mathbf{M}-\varepsilon_j\mathbf{1})}{\prod_{j\neq1}(\varepsilon_1-\varepsilon_j)}+\frac{\prod_{j\neq1}(\mathbf{M}-\varepsilon_j\mathbf{1})}{\prod_{j\neq1}(\varepsilon_2-\varepsilon_j)}=\frac{(\mathbf{M}-\varepsilon_2\mathbf{1})}{(\varepsilon_1-\varepsilon_2)}+\frac{(\mathbf{M}-\varepsilon_1\mathbf{1})}{(\varepsilon_2-\varepsilon_1)}=\frac{(\mathbf{M}-\varepsilon_2\mathbf{1})-(\mathbf{M}-\varepsilon_1\mathbf{1})}{(\varepsilon_1-\varepsilon_2)}=\frac{-\varepsilon_2\mathbf{1}+\varepsilon_1\mathbf{1}}{(\varepsilon_1-\varepsilon_2)}=\mathbf{1}\ \text{(for } all\ \varepsilon_j)$$

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$$\mathbf{1}=\mathbf{P}_1+\mathbf{P}_2+\ldots+\mathbf{P}_n=\sum_{\varepsilon_k}\mathbf{P}_k=\sum_{\varepsilon_k}\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} \qquad f(\mathbf{M})=f(\varepsilon_1)\mathbf{P}_1+f(\varepsilon_2)\mathbf{P}_2+\ldots+f(\varepsilon_n)\mathbf{P}_n=\sum_{\varepsilon_k}f(\varepsilon_k)\mathbf{P}_k=\sum_{\varepsilon_k}f(\varepsilon_k)\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$$

with *Lagrange interpolation formula* of function *f(x)* approximated by its value at *N* points $x_1, x_2, \ldots x_N$.

$$L\left(f(x)\right)=\sum_{k=1}^{N}f(x_k)\cdot P_k(x) \qquad \text{where: } P_k(x)=\frac{\prod_{j\neq k}^{N}(x-x_j)}{\prod_{j\neq k}^{N}(x_k-x_j)}$$

Each polynomial term $P_m(x)$ has zeros at each point $x=x_j$ *except* where $x=x_m$. Then $P_m(x_m)=1$.
So at each of these points this L-approximation becomes exact: $L(f(x_j))=f(x_j)$.

If $f(x)$ happens to be a polynomial of degree $N-1$ or less, then $L(f(x))=f(x)$ may be exact everywhere.

$$1=\sum_{m=1}^{N}P_m(x) \qquad x=\sum_{m=1}^{N}x_m P_m(x) \qquad x^2=\sum_{m=1}^{N}x_m^2 P_m(x)$$

One point determines a constant level line, two separate points uniquely determine a sloping line, three separate points uniquely determine a parabola, etc.

Lagrange interpolation formula→*Completeness formula* as $x\rightarrow\mathbf{M}$ and as $x_k\rightarrow\varepsilon_k$ and as $P_k(x_k)\rightarrow\mathbf{P}_k$

All distinct values $\varepsilon_1\neq\varepsilon_2\neq\ldots\neq\varepsilon_N$ satisfy $\Sigma\mathbf{P}_k=\mathbf{1}$. Completeness is *truer than true* as is seen for $N=2$.

$$\mathbf{P}_1+\mathbf{P}_2=\frac{\prod_{j\neq1}(\mathbf{M}-\varepsilon_j\mathbf{1})}{\prod_{j\neq1}(\varepsilon_1-\varepsilon_j)}+\frac{\prod_{j\neq1}(\mathbf{M}-\varepsilon_j\mathbf{1})}{\prod_{j\neq1}(\varepsilon_2-\varepsilon_j)}=\frac{(\mathbf{M}-\varepsilon_2\mathbf{1})}{(\varepsilon_1-\varepsilon_2)}+\frac{(\mathbf{M}-\varepsilon_1\mathbf{1})}{(\varepsilon_2-\varepsilon_1)}=\frac{(\mathbf{M}-\varepsilon_2\mathbf{1})-(\mathbf{M}-\varepsilon_1\mathbf{1})}{(\varepsilon_1-\varepsilon_2)}=\frac{-\varepsilon_2\mathbf{1}+\varepsilon_1\mathbf{1}}{(\varepsilon_1-\varepsilon_2)}=\mathbf{1}\ (\text{for } all\ \varepsilon_j)$$

However, only *select* values ε_k work for eigen-forms $\mathbf{MP}_k=\varepsilon_k\mathbf{P}_k$ or orthonormality $\mathbf{P}_j\mathbf{P}_k=\delta_{jk}\mathbf{P}_k$.

Unitary operators and matrices that change state vectors

...and eigenstates ("ownstates) that are mostly immune

Geometric visualization of real symmetric matrices and eigenvectors

Circle-to-ellipse mapping

Ellipse-to-ellipse mapping (Normal space vs. tangent space)

Eigensolutions as stationary extreme-values (Lagrange λ-multipliers)

Matrix-algebraic eigensolutions with example $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

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Hamilton-Cayley equation and projectors

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→ *Diagonalizing Transformations (D-Ttran) from projectors* ←

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Diagonalizing Transformations (D-Ttran) from projectors

Given our eigenvectors and their Projectors.

$$\mathbf{P}_1 = \frac{(\mathbf{M} - 5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1 \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M} - 1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}}{k_2} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

Diagonalizing Transformations (D-Ttran) from projectors

Given our eigenvectors and their Projectors.

$$\mathbf{P}_1 = \frac{(\mathbf{M} - 5 \cdot \mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1 \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

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Load distinct bras $\langle\varepsilon_1|$ and $\langle\varepsilon_2|$ into d-tran rows, kets $|\varepsilon_1\rangle$ and $|\varepsilon_2\rangle$ into <u>inverse</u> d-tran columns.

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$$\mathbf{P}_1 = \frac{(\mathbf{M} - 5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1 \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

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$(\varepsilon_1, \varepsilon_2) \leftarrow (1,2)$ *d–Tran matrix* $\qquad (1,2) \leftarrow (\varepsilon_1, \varepsilon_2)$ INVERSE *d–Tran matrix*

$$\begin{pmatrix} \langle\varepsilon_1|x\rangle & \langle\varepsilon_1|y\rangle \\ \langle\varepsilon_2|x\rangle & \langle\varepsilon_2|y\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}, \quad \begin{pmatrix} \langle x|\varepsilon_1\rangle & \langle x|\varepsilon_2\rangle \\ \langle y|\varepsilon_1\rangle & \langle y|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

Diagonalizing Transformations (D-Ttran) from projectors

Given our eigenvectors and their Projectors.

$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1 \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}}{k_2} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

Load distinct bras $\langle\varepsilon_1|$ and $\langle\varepsilon_2|$ into d-tran rows, kets $|\varepsilon_1\rangle$ and $|\varepsilon_2\rangle$ into <u>inverse</u> d-tran columns.

$$\left\{\langle\varepsilon_1| = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \langle\varepsilon_2| = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}\right\} \quad , \quad \left\{|\varepsilon_1\rangle = \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix}, |\varepsilon_2\rangle = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}\right\}$$

$(\varepsilon_1,\varepsilon_2) \leftarrow (1,2)$ *d–Tran matrix* $\qquad$ $(1,2) \leftarrow (\varepsilon_1,\varepsilon_2)$ INVERSE *d–Tran matrix*

$$\begin{pmatrix} \langle\varepsilon_1|x\rangle & \langle\varepsilon_1|y\rangle \\ \langle\varepsilon_2|x\rangle & \langle\varepsilon_2|y\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix} \quad , \quad \begin{pmatrix} \langle x|\varepsilon_1\rangle & \langle x|\varepsilon_2\rangle \\ \langle y|\varepsilon_1\rangle & \langle y|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

Use *Dirac labeling for all components* so transformation is OK

$$\begin{pmatrix} \langle\varepsilon_1|x\rangle & \langle\varepsilon_1|y\rangle \\ \langle\varepsilon_2|x\rangle & \langle\varepsilon_2|y\rangle \end{pmatrix} \cdot \begin{pmatrix} \langle x|\mathbf{K}|x\rangle & \langle x|\mathbf{K}|y\rangle \\ \langle y|\mathbf{K}|x\rangle & \langle y|\mathbf{K}|y\rangle \end{pmatrix} \cdot \begin{pmatrix} \langle x|\varepsilon_1\rangle & \langle x|\varepsilon_2\rangle \\ \langle y|\varepsilon_1\rangle & \langle y|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \langle\varepsilon_1|\mathbf{K}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{K}|\varepsilon_2\rangle \\ \langle\varepsilon_2|\mathbf{K}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{K}|\varepsilon_2\rangle \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$$

Diagonalizing Transformations (D-Ttran) from projectors

Given our eigenvectors and their Projectors.

$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1 \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}}{k_2} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

Load distinct bras $\langle\varepsilon_1|$ and $\langle\varepsilon_2|$ into d-tran rows, kets $|\varepsilon_1\rangle$ and $|\varepsilon_2\rangle$ into <u>inverse</u> d-tran columns.

$$\left\{\langle\varepsilon_1| = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \langle\varepsilon_2| = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}\right\}, \quad \left\{|\varepsilon_1\rangle = \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix}, |\varepsilon_2\rangle = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}\right\}$$

$(\varepsilon_1,\varepsilon_2) \leftarrow (1,2)$ *d–Tran matrix* $\qquad (1,2) \leftarrow (\varepsilon_1,\varepsilon_2)$ INVERSE *d–Tran matrix*

$$\begin{pmatrix} \langle\varepsilon_1|x\rangle & \langle\varepsilon_1|y\rangle \\ \langle\varepsilon_2|x\rangle & \langle\varepsilon_2|y\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}, \quad \begin{pmatrix} \langle x|\varepsilon_1\rangle & \langle x|\varepsilon_2\rangle \\ \langle y|\varepsilon_1\rangle & \langle y|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

Use *Dirac labeling for all components* so transformation is OK

$$\begin{pmatrix} \langle\varepsilon_1|x\rangle & \langle\varepsilon_1|y\rangle \\ \langle\varepsilon_2|x\rangle & \langle\varepsilon_2|y\rangle \end{pmatrix} \cdot \begin{pmatrix} \langle x|\mathbf{K}|x\rangle & \langle x|\mathbf{K}|y\rangle \\ \langle y|\mathbf{K}|x\rangle & \langle y|\mathbf{K}|y\rangle \end{pmatrix} \cdot \begin{pmatrix} \langle x|\varepsilon_1\rangle & \langle x|\varepsilon_2\rangle \\ \langle y|\varepsilon_1\rangle & \langle y|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \langle\varepsilon_1|\mathbf{K}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{K}|\varepsilon_2\rangle \\ \langle\varepsilon_2|\mathbf{K}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{K}|\varepsilon_2\rangle \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$$

Check inverse-d-tran is really inverse of *your* d-tran.

$$\begin{pmatrix} \langle\varepsilon_1|1\rangle & \langle\varepsilon_1|2\rangle \\ \langle\varepsilon_2|1\rangle & \langle\varepsilon_2|2\rangle \end{pmatrix} \cdot \begin{pmatrix} \langle 1|\varepsilon_1\rangle & \langle 1|\varepsilon_2\rangle \\ \langle 2|\varepsilon_1\rangle & \langle 2|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \langle\varepsilon_1|\mathbf{1}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{1}|\varepsilon_2\rangle \\ \langle\varepsilon_2|\mathbf{1}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{1}|\varepsilon_2\rangle \end{pmatrix}$$

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Diagonalizing Transformations (D-Ttran) from projectors

Given our eigenvectors and their Projectors.

$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = k_1 \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}{k_1} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = k_2 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \frac{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}}{k_2} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

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$(\varepsilon_1,\varepsilon_2) \leftarrow (1,2)$ *d–Tran matrix* $\qquad$ $(1,2) \leftarrow (\varepsilon_1,\varepsilon_2)$ INVERSE *d–Tran matrix*

$$\begin{pmatrix} \langle\varepsilon_1|x\rangle & \langle\varepsilon_1|y\rangle \\ \langle\varepsilon_2|x\rangle & \langle\varepsilon_2|y\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix} \quad , \quad \begin{pmatrix} \langle x|\varepsilon_1\rangle & \langle x|\varepsilon_2\rangle \\ \langle y|\varepsilon_1\rangle & \langle y|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

Use *Dirac labeling for all components* so transformation is OK

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Check inverse-d-tran is really inverse of *your* d-tran. In standard quantum matrices inverses are “easy”

$$\begin{pmatrix} \langle\varepsilon_1|x\rangle & \langle\varepsilon_1|y\rangle \\ \langle\varepsilon_2|x\rangle & \langle\varepsilon_2|y\rangle \end{pmatrix} \cdot \begin{pmatrix} \langle x|\varepsilon_1\rangle & \langle x|\varepsilon_2\rangle \\ \langle y|\varepsilon_1\rangle & \langle y|\varepsilon_2\rangle \end{pmatrix} = \begin{pmatrix} \langle\varepsilon_1|\mathbf{1}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{1}|\varepsilon_2\rangle \\ \langle\varepsilon_2|\mathbf{1}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{1}|\varepsilon_2\rangle \end{pmatrix}$$

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Unitary operators and matrices that change state vectors

...and eigenstates ("ownstates) that are mostly immune

Geometric visualization of real symmetric matrices and eigenvectors

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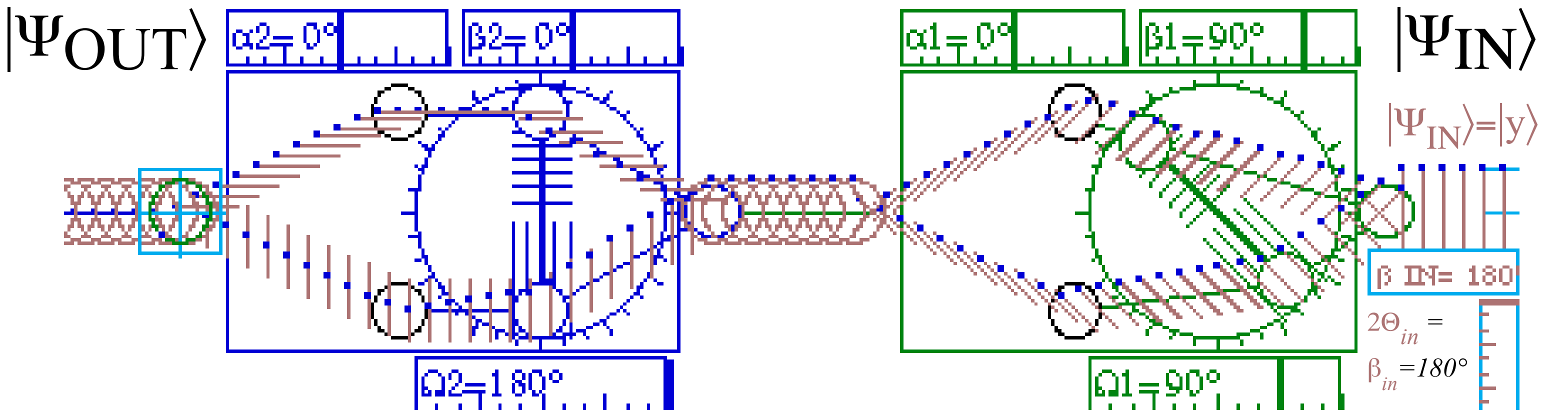
Eigensolutions for active analyzers

Spectral Decompositions with degeneracy

Functional spectral decomposition

Matrix products and eigensolutions for active analyzers

Consider a $45°$ tilted ($\theta_1=\beta_1/2=\pi/4$ or $\beta_1=90°$) analyzer followed by a untilted ($\beta_2=0$) analyzer.
Active analyzers have both paths open and a phase shift $e^{-i\Omega}$ between each path.
Here the first analyzer has $\Omega_1=90°$. The second has $\Omega_2=180°$.



The transfer matrix for each analyzer is a sum of projection operators for each open path multiplied by the phase factor that is active at that path. Apply phase factor $e^{-i\Omega 1}=e^{-i\pi/2}$ to top path in the first analyzer and the factor $e^{-i\Omega 2}=e^{-i\pi}$ to the top path in the second analyzer.

$$T(2)=e^{-i\pi}|x\rangle\langle x|+|y\rangle\langle y|=\begin{pmatrix} e^{-i\pi} & 0 \\ 0 & 1\end{pmatrix}$$

$$T(1)=e^{-i\pi/2}|x'\rangle\langle x'|+|y'\rangle\langle y'|=e^{-i\pi/2}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}+\begin{pmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2}\end{pmatrix}=\begin{pmatrix} \frac{1-i}{2} & \frac{-1-i}{2} \\ \frac{-1-i}{2} & \frac{1-i}{2}\end{pmatrix}$$

The matrix product $T(total)=T(2)T(1)$ relates input states $|\Psi_{IN}\rangle$ to output states: $|\Psi_{OUT}\rangle=T(total)|\Psi_{IN}\rangle$

$$T(total)=T(2)T(1)=\begin{pmatrix} -1 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} \frac{1-i}{2} & \frac{-1-i}{2} \\ \frac{-1-i}{2} & \frac{1-i}{2}\end{pmatrix}=\begin{pmatrix} \frac{-1+i}{2} & \frac{1+i}{2} \\ \frac{-1-i}{2} & \frac{1-i}{2}\end{pmatrix}=e^{-i\pi/4}\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\sim\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$$

We drop the overall phase $e^{-i\pi/4}$ since it is unobservable. $T(total)$ yields two eigenvalues and projectors.

$$\lambda^2-0\lambda-1=0,\text{ or: }\lambda=+1,\ -1\text{, gives projectors}\quad P_{+1}=\frac{\begin{pmatrix} \frac{-1}{\sqrt{2}}+1 & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}}+1\end{pmatrix}}{1-(-1)}=\frac{\begin{pmatrix} -1+\sqrt{2} & i \\ -i & 1+\sqrt{2}\end{pmatrix}}{2\sqrt{2}},\quad P_{-1}=\frac{\begin{pmatrix} 1+\sqrt{2} & -i \\ i & -1+\sqrt{2}\end{pmatrix}}{2\sqrt{2}}$$