Group Theory in Quantum Mechanics
Lecture 30 (5.10.17)
Symmetry product analysis U(m)*Sn tensors

(Int.J.Mol.Sci, 14, 714(2013) p.755-774 , QTCA Unit 7 Ch. 23-26)
(PSDS - Ch. 5, 7)

Review : *2-D* **a†a** *algebra of U(2) representations*

Spin-spin $(1/2)^2$ *product states: Hydrogen hyperfine structure*
- *Kronecker product states and operators*
- *Spin-spin interaction reduces symmetry* $U(2)^{proton}\times U(2)^{electron}$ *to* $U(2)^{e+p}$
  - *Clebsch-Gordan Coefficients*
  - *Hydrogen hyperfine structure: Fermi-contact interaction*
    - *plus B-field gives avoided crossing*

Higher-J product states
- $(J=1)\otimes(J=1)=2\oplus 1\oplus 0$ *case*
- *General U(2) case*

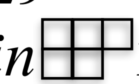
Multi-spin $(1/2)^N$ *product states*
- *Magic squares - Intro to Young Tableaus*

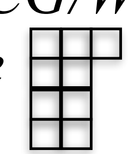
Tensor operators for spin-1/2 states:
Outer products give Hamilton-Pauli-spinors
Tensor operators for spin-1 states:
U(3) generalization of Pauli spinors

<u>*Multi-Lecture Selections*</u>

5.02.17 Lecture 27
Asymm-rotor RES&clusters
O-SymmSF_6 *RES&clusters*
T_d*-Symm*CF_4

5.03.17 Lecture 28
Gyro-rotor REES&levels
(Lect.30 $C^{j_1 j_2 k}_{n_1 n_2 q}$ *coupling intro.)*

5.04.17 Lecture 29
SiF_4 *and* SF_6 *Spin*  *Tableau*
& hyperfine effects
C_{60} *"Buckyball"*

Lecture 30 *R(3)~U(2)Tensors and CG/Wigner coefficients*
More  *Tableau theory*

U(2) and U(3) tensor expansions

2^k-pole expansion of an N-by-N matrix $\mathbf{H}$

2-by-2 case: $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

$$= \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{A+D}{2}\,\mathbf{1} + B\,\boldsymbol{\sigma}_x + C\,\boldsymbol{\sigma}_y + \frac{A-D}{2}\,\boldsymbol{\sigma}_z$$

$$= \frac{A+D}{2}\,\mathbf{T}_0 + (B-iC)\underset{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}{\mathbf{T}_1} + (B+iC)\underset{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}{\mathbf{T}_{-1}} + \frac{A-D}{2}\,\mathbf{T}_0$$

U(2) generators *(spin J=1/2)*

$\mathbf{u}^1_{+1}=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\mathbf{u}^1_{0}=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}$ $\mathbf{u}^1_{-1}=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ rank-1 (vector)

$\mathbf{u}^0_{0}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}$ rank-0 (scalar)

3-by-3 case: $\mathbf{H}=\begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix}$

U(3) generators *(spin J=1)*

$\mathbf{u}^2_{+2}=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\mathbf{u}^2_{+1}=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}$ $\mathbf{u}^2_{0}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\frac{1}{\sqrt{6}}$ $\mathbf{u}^2_{-1}=\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}$ $\mathbf{u}^2_{-2}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ rank-2 (tensor)

$\mathbf{u}^1_{+1}=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}$ $\mathbf{u}^1_{0}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}$ $\mathbf{u}^1_{-1}=\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}$ rank-1 (vector)

$\mathbf{u}^0_{0}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\frac{1}{\sqrt{3}}$ rank-0 (scalar)

Mutually commuting diagonal operators

Wigner-Clebsch-Gordan expressions for Tensor $\left\langle \mathbf{T}^k_q \right\rangle$

$$\left\langle \begin{matrix} J' \\ M' \end{matrix} \middle| \mathbf{T}^k_q \middle| \begin{matrix} J \\ M \end{matrix} \right\rangle = \begin{pmatrix} J' & k & J \\ M' & q & -M \end{pmatrix}(J'\|k\|J) = C^{kJ\ J'}_{q\,M\,M'}\left\langle J'\|k\|J \right\rangle$$

Spin-spin (1/2)^2 product states: Hydrogen hyperfine structure

electron-proton spin-spin interaction gives a simple example of *hyperfine* spectra

Ket-kets for spin-up and spin-dn states and column matrix representations..

$$\left|\uparrow\right\rangle\left|\uparrow\right\rangle=\left|\begin{matrix}\frac{1}{2}\\ \frac{1}{2}\end{matrix}\right\rangle^{proton}\left|\begin{matrix}\frac{1}{2}\\ \frac{1}{2}\end{matrix}\right\rangle^{electron},\left|\uparrow\right\rangle\left|\downarrow\right\rangle=\left|\begin{matrix}\frac{1}{2}\\ \frac{1}{2}\end{matrix}\right\rangle^{proton}\left|\begin{matrix}\frac{1}{2}\\ -\frac{1}{2}\end{matrix}\right\rangle^{electron},\left|\downarrow\right\rangle\left|\uparrow\right\rangle=\left|\begin{matrix}\frac{1}{2}\\ -\frac{1}{2}\end{matrix}\right\rangle^{proton}\left|\begin{matrix}\frac{1}{2}\\ \frac{1}{2}\end{matrix}\right\rangle^{electron},\left|\downarrow\right\rangle\left|\downarrow\right\rangle=\left|\begin{matrix}\frac{1}{2}\\ -\frac{1}{2}\end{matrix}\right\rangle^{proton}\left|\begin{matrix}\frac{1}{2}\\ -\frac{1}{2}\end{matrix}\right\rangle^{electron}$$

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Same spin-1/2 representation applies to either proton or electron kets.

$$D^{1/2}(\alpha\beta\gamma)=\begin{pmatrix}D^{1/2}_{+1/2,+1/2} & D^{1/2}_{+1/2,-1/2}\\ D^{1/2}_{-1/2,+1/2} & D^{1/2}_{-1/2,-1/2}\end{pmatrix}=\begin{pmatrix}e^{\frac{-i(\alpha+\gamma)}{2}}\cos\frac{\beta}{2} & -e^{\frac{-i(\alpha-\gamma)}{2}}\sin\frac{\beta}{2}\\ e^{\frac{i(\alpha-\gamma)}{2}}\sin\frac{\beta}{2} & e^{\frac{i(\alpha+\gamma)}{2}}\cos\frac{\beta}{2}\end{pmatrix}$$

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Kronecker product $D^{\frac{1}{2}}\otimes D^{\frac{1}{2}}$

Applies to *outer product symmetry* $U(2)^{proton}\times U(2)^{electron}$ for NO interaction.

$$\begin{pmatrix}\cos\frac{\beta_p}{2} & -\sin\frac{\beta_p}{2}\\ \sin\frac{\beta_p}{2} & \cos\frac{\beta_p}{2}\end{pmatrix}\otimes\begin{pmatrix}\cos\frac{\beta_e}{2} & -\sin\frac{\beta_e}{2}\\ \sin\frac{\beta_e}{2} & \cos\frac{\beta_e}{2}\end{pmatrix}=\begin{pmatrix}\cos\frac{\beta_p}{2}\cos\frac{\beta_e}{2} & -\cos\frac{\beta_p}{2}\sin\frac{\beta_e}{2} & -\sin\frac{\beta_p}{2}\cos\frac{\beta_e}{2} & \sin\frac{\beta_p}{2}\sin\frac{\beta_e}{2}\\ \cos\frac{\beta_p}{2}\sin\frac{\beta_e}{2} & \cos\frac{\beta_p}{2}\cos\frac{\beta_e}{2} & -\sin\frac{\beta_p}{2}\sin\frac{\beta_e}{2} & -\sin\frac{\beta_p}{2}\cos\frac{\beta_e}{2}\\ \sin\frac{\beta_p}{2}\cos\frac{\beta_e}{2} & -\sin\frac{\beta_p}{2}\sin\frac{\beta_e}{2} & \cos\frac{\beta_p}{2}\cos\frac{\beta_e}{2} & -\cos\frac{\beta_p}{2}\sin\frac{\beta_e}{2}\\ \sin\frac{\beta_p}{2}\sin\frac{\beta_e}{2} & \sin\frac{\beta_p}{2}\cos\frac{\beta_e}{2} & \cos\frac{\beta_p}{2}\sin\frac{\beta_e}{2} & \cos\frac{\beta_p}{2}\cos\frac{\beta_e}{2}\end{pmatrix}$$

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Interaction reduces symmetry:

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Spin-spin interaction reduces symmetry $U(2)^{proton}\times U(2)^{electron}$ *to* $U(2)^{e+p}$

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(Only $(\alpha_e,\beta_e,\gamma_e)=(\alpha_p,\beta_p,\gamma_p)$

is allowed!

Spin-spin interaction reduces symmetry $U(2)^{proton}\times U(2)^{electron}$ *to* $U(2)^{e+p}$

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 0 & 1\\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0\end{pmatrix}\cdot\begin{pmatrix}\cos^2\frac{\beta}{2} & -\sin\frac{\beta}{2}\cos\frac{\beta}{2} & -\sin\frac{\beta}{2}\cos\frac{\beta}{2} & \sin^2\frac{\beta}{2}\\ \sin\frac{\beta}{2}\cos\frac{\beta}{2} & \cos^2\frac{\beta}{2} & -\sin^2\frac{\beta}{2} & -\sin\frac{\beta}{2}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\cos\frac{\beta}{2} & -\sin^2\frac{\beta}{2} & \cos^2\frac{\beta}{2} & -\sin\frac{\beta}{2}\cos\frac{\beta}{2}\\ \sin^2\frac{\beta}{2} & \sin\frac{\beta}{2}\cos\frac{\beta}{2} & \sin\frac{\beta}{2}\cos\frac{\beta}{2} & \cos^2\frac{\beta}{2}\end{pmatrix}\cdot\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}}\\ 0 & 0 & 1 & 0\end{pmatrix}=\begin{pmatrix}\sin^2\frac{\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \sin^2\frac{\beta}{2} & 0\\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} & 0\\ \sin^2\frac{\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \cos^2\frac{\beta}{2} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}=\left(\begin{array}{c|c}D^{J=1}_{(0\beta0)} & \\ \hline & D^{J=0}\end{array}\right)$$

...and "irreducible" becomes "reducible"...

Spin-spin interaction reduces symmetry $U(2)^{proton}\times U(2)^{electron}$ to $U(2)^{e+p}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix} \cdot \begin{pmatrix} \cos^2\frac{\beta}{2} & -\sin\frac{\beta}{2}\cos\frac{\beta}{2} & -\sin\frac{\beta}{2}\cos\frac{\beta}{2} & \sin^2\frac{\beta}{2} \\ \sin\frac{\beta}{2}\cos\frac{\beta}{2} & \cos^2\frac{\beta}{2} & -\sin^2\frac{\beta}{2} & -\sin\frac{\beta}{2}\cos\frac{\beta}{2} \\ \sin\frac{\beta}{2}\cos\frac{\beta}{2} & -\sin^2\frac{\beta}{2} & \cos^2\frac{\beta}{2} & -\sin\frac{\beta}{2}\cos\frac{\beta}{2} \\ \sin^2\frac{\beta}{2} & \sin\frac{\beta}{2}\cos\frac{\beta}{2} & \sin\frac{\beta}{2}\cos\frac{\beta}{2} & \cos^2\frac{\beta}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \sin^2\frac{\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \sin^2\frac{\beta}{2} & 0 \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} & 0 \\ \sin^2\frac{\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \cos^2\frac{\beta}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} D^{J=1}_{(0\beta 0)} & \\ & D^{J=0} \end{pmatrix}$$

Clebsch-Gordan coefficients(CGC)

$$C^{\frac{1}{2}\ \frac{1}{2}\ J}_{m_p\ m_e\ M} \equiv \left\langle \begin{matrix} \frac{1}{2} & \frac{1}{2} \\ m_p & m_e \end{matrix} \middle| \begin{matrix} J \\ M \end{matrix} \right\rangle$$

reduce $D^{1/2}\otimes D^{1/2}$ to $D^1\oplus D^0$

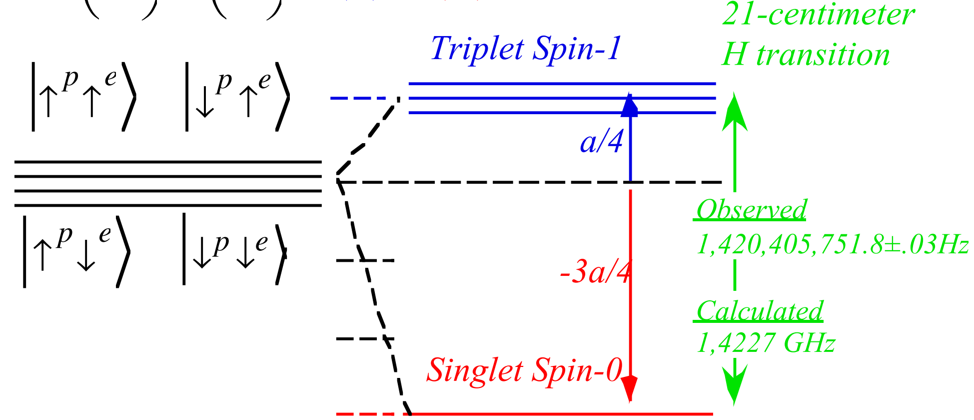
$\frac{1}{2}\otimes\frac{1}{2}$	$J=1$, $M=1$	1, 0	1, -1	0, 0
$\frac{1}{2},\frac{1}{2}$	1	0	0	0
$\frac{1}{2},\frac{-1}{2}$	0	$\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$
$\frac{-1}{2},\frac{1}{2}$	0	$\frac{1}{\sqrt{2}}$	0	$\frac{-1}{\sqrt{2}}$
$\frac{-1}{2},\frac{-1}{2}$	0	0	1	0

$$= \left\langle C^{\frac{1}{2}\ \frac{1}{2}}_{m_p\ m_e} \middle| \begin{matrix} J \\ M \end{matrix} \right\rangle$$

$$\sum_{m_1 m_1'}\sum_{m_2 m_2'} C^{\frac{1}{2}\ \frac{1}{2}\ J}_{m_1\ m_1'\ M} D^{\frac{1}{2}}_{m_1 m_2} D^{\frac{1}{2}}_{m_1' m_2'} C^{\frac{1}{2}\ \frac{1}{2}\ J'}_{m_2\ m_2'\ M'} = \delta^{JJ'} D^J_{M\,M'}$$

$$\left| \begin{matrix} J \\ M \end{matrix} \right.^{(1/2\otimes 1/2)} \Big\rangle = \sum_{m_1,m_2} C^{1/2\ 1/2\ J}_{m_1\ m_2\ M} \left| \begin{matrix} 1/2 \\ m_1 \end{matrix} \right\rangle \left| \begin{matrix} 1/2 \\ m_2 \end{matrix} \right\rangle$$

$$\left(\frac{1}{2}\right)\otimes\left(\frac{1}{2}\right) = (1)\oplus(0)$$



$$\left|\begin{matrix} 1 \\ +1 \end{matrix}\right\rangle = \left|\uparrow^p\uparrow^e\right\rangle$$

$$\left|\begin{matrix} 1 \\ 0 \end{matrix}\right\rangle = \left(\left|\uparrow^p\downarrow^e\right\rangle + \left|\downarrow^p\uparrow^e\right\rangle\right)/\sqrt{2}$$

$$\left|\begin{matrix} 1 \\ -1 \end{matrix}\right\rangle = \left|\downarrow^p\downarrow^e\right\rangle$$

$$\left|\begin{matrix} 0 \\ 0 \end{matrix}\right\rangle = \left(\left|\uparrow^p\downarrow^e\right\rangle - \left|\downarrow^p\uparrow^e\right\rangle\right)/\sqrt{2}$$

Hydrogen hyperfine structure: Fermi-contact interaction

Racah's trick for energy eigenvalues

$$a_{ep}\mathbf{J}^{proton}\bullet\mathbf{J}^{electron}=\frac{a_{ep}}{2}\left[\left(\mathbf{J}^{proton}+\mathbf{J}^{electron}\right)^2-\left(\mathbf{J}^{proton}\right)^2-\left(\mathbf{J}^{electron}\right)^2\right]$$

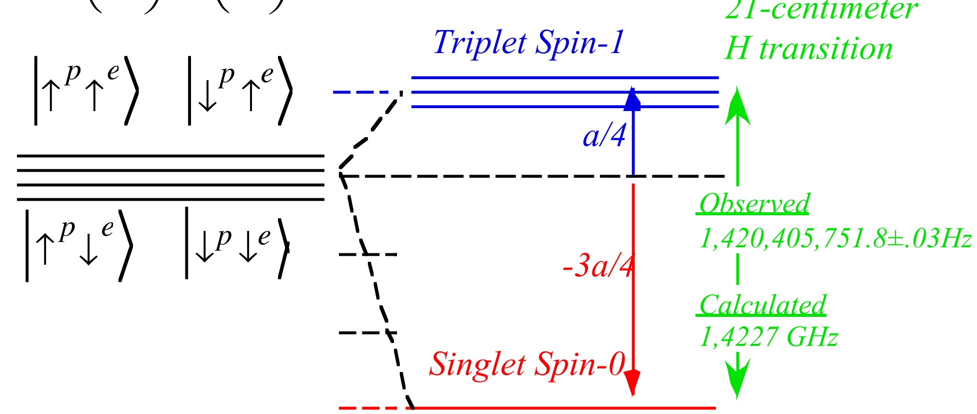
$$=\frac{a_{ep}}{2}\left[\left(\mathbf{J}^{total}\right)^2-\left(\mathbf{J}^{proton}\right)^2-\left(\mathbf{J}^{electron}\right)^2\right].$$

$$\left\langle {J \atop M}^{(1/2\otimes 1/2)}\right| H_{contact}\left| {J \atop M}^{(1/2\otimes 1/2)}\right\rangle=\frac{a_{ep}}{2}\left[J(J+1)-\tfrac{1}{2}\left(\tfrac{1}{2}+1\right)-\tfrac{1}{2}\left(\tfrac{1}{2}+1\right)\right]$$

$$=\begin{cases} a_{ep}/4 \ \textit{for the}\ (J=1) & \text{triplet state,}\\ -3a_{ep}/4 \ \textit{for the}\ (J=0) & \text{singlet state.}\end{cases}$$

$$\left| {J \atop M}^{(1/2\otimes 1/2)}\right\rangle=\sum_{m_1,m_2} C^{1/2\ 1/2\ J}_{m_1\ m_2\ M}\left| {1/2 \atop m_1}\right\rangle\left| {1/2 \atop m_2}\right\rangle$$

$$\left(\frac{1}{2}\right)\otimes\left(\frac{1}{2}\right)=(1)\oplus(0)$$



$$\left| {1 \atop +1}\right\rangle=\left|\uparrow^p\uparrow^e\right\rangle$$

$$\left| {1 \atop 0}\right\rangle=\left(\left|\uparrow^p\downarrow^e\right\rangle+\left|\downarrow^p\uparrow^e\right\rangle\right)/\sqrt{2}$$

$$\left| {1 \atop -1}\right\rangle=\left|\downarrow^p\downarrow^e\right\rangle$$

$$\left| {0 \atop 0}\right\rangle=\left(\left|\uparrow^p\downarrow^e\right\rangle-\left|\downarrow^p\uparrow^e\right\rangle\right)/\sqrt{2}$$

Hydrogen hyperfine structure: Fermi-contact interaction + B-field

$$H_{1s-B-field} = -a_p B_z J_z^{proton} + a_e B_z J_z^{electron} + a_{ep} \mathbf{J}^{proton} \bullet \mathbf{J}^{electron}$$

	$g-factor$	$Bohr-magneton$	$gyromagnetic\ factor$
$electron$	$g_e = 2.0023$	$\mu_e = \frac{e\hbar}{2m_e} = 9.27401 \cdot 10^{-24} \frac{J}{T}$	$a_e = g_e \mu_e = 1.8570 \cdot 10^{-23} \frac{J}{T}$
$proton$	$g_p = 5.585$	$\mu_p = \frac{e\hbar}{2m_p} = 5.05078 \cdot 10^{-27} \frac{J}{T}$	$a_p = g_p \mu_p = 2.8209 \cdot 10^{-26} \frac{J}{T}$

$Magnetic\ constant:\ \mu_0 / 4\pi = 10^{-7} N/A^2$

$Fermi-contact\ factor$
$a_{ep} = \mu_0 \frac{2}{3} \frac{1}{\pi a_0^3} a_e a_p = 9.427 \cdot 10^{-25} J$
$\mu_0 \frac{2}{3} \frac{1}{\pi a_0^3} \frac{a_e a_p}{h} = 1.4227 \cdot 10^9 Hz$
$\mu_0 \frac{2}{3} \frac{1}{\pi a_0^3} \frac{a_e a_p}{hc} = 4.746 m^{-1}$
$= \frac{1}{21.1} cm^{-1}$

$$H_{1s-B-field} = -a_p B_z J_z^{proton} + a_e B_z J_z^{electron} + a_{ep} \mathbf{J}^{proton} \bullet \mathbf{J}^{electron}$$

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Magnetic constant : $\mu_0 / 4\pi = 10^{-7} N / A^2$

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$\mu_0 \frac{2}{3} \frac{1}{\pi a_0^3} \frac{a_e a_p}{hc} = 4.746 m^{-1} = \frac{1}{21.1} cm^{-1}$

$$\left\langle -a_p B_z J_z^{proton} + a_e B_z J_z^{electron} \right\rangle =$$

	$\lvert\uparrow^p\uparrow^e\rangle$	$\lvert\uparrow^p\downarrow^e\rangle$	$\lvert\downarrow^p\uparrow^e\rangle$	$\lvert\downarrow^p\downarrow^e\rangle$
$\langle\uparrow^p\uparrow^e\rvert$	$\frac{1}{2}(a_e - a_p)B_z$	.	.	.
$\langle\uparrow^p\downarrow^e\rvert$	.	$\frac{-1}{2}(a_e + a_p)B_z$	0	.
$\langle\downarrow^p\uparrow^e\rvert$	.	0	$\frac{1}{2}(a_e + a_p)B_z$	.
$\langle\downarrow^p\downarrow^e\rvert$	.	.	.	$\frac{-1}{2}(a_e - a_p)B_z$

$$\left\langle a_{ep} \mathbf{J}^{proton} \bullet \mathbf{J}^{electron} \right\rangle =$$

	$\left\lvert \begin{smallmatrix}1\\1\end{smallmatrix} \right\rangle$	$\left\lvert \begin{smallmatrix}1\\0\end{smallmatrix} \right\rangle$	$\left\lvert \begin{smallmatrix}0\\0\end{smallmatrix} \right\rangle$	$\left\lvert \begin{smallmatrix}1\\-1\end{smallmatrix} \right\rangle$
$\left\langle \begin{smallmatrix}1\\1\end{smallmatrix} \right\rvert$	$\frac{a_{ep}}{4}$	.	.	.
$\left\langle \begin{smallmatrix}1\\0\end{smallmatrix} \right\rvert$	.	$\frac{a_{ep}}{4}$	0	.
$\left\langle \begin{smallmatrix}0\\0\end{smallmatrix} \right\rvert$	.	0	$\frac{-3a_{ep}}{4}$	.
$\left\langle \begin{smallmatrix}1\\-1\end{smallmatrix} \right\rvert$	.	.	.	$\frac{a_{ep}}{4}$

$$H_{1s-B-field} = -a_p B_z J_z^{proton} + a_e B_z J_z^{electron} + a_{ep} \mathbf{J}^{proton} \bullet \mathbf{J}^{electron}$$

	$g-factor$	$Bohr-magneton$	$gyromagnetic\ factor$
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$\mu_0 \dfrac{2}{3} \dfrac{1}{\pi a_0^3} \dfrac{a_e a_p}{hc} = 4.746 m^{-1} = \dfrac{1}{21.1} cm^{-1}$

$$\begin{array}{c|ccc|c} \frac{1}{2}\otimes\frac{1}{2} & \begin{smallmatrix} J=1 \\ M=1 \end{smallmatrix} & \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} & \begin{smallmatrix} 1 \\ -1 \end{smallmatrix} & \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \\ \hline \frac{1}{2},\frac{1}{2} & 1 & 0 & 0 & 0 \\ \frac{1}{2},\frac{-1}{2} & 0 & \frac{1}{\sqrt{2}} & & \frac{1}{\sqrt{2}} \\ \frac{-1}{2},\frac{1}{2} & 0 & \frac{1}{\sqrt{2}} & & \frac{-1}{\sqrt{2}} \\ \frac{-1}{2},\frac{-1}{2} & 0 & 0 & 1 & 0 \\ \hline \end{array} = \left\langle C^{\frac{1}{2}\ \frac{1}{2}}_{m_p\ m_e} \middle| \begin{smallmatrix} J \\ M \end{smallmatrix} \right\rangle$$

$$\left\langle -a_p B_z J_z^{proton} + a_e B_z J_z^{electron} \right\rangle =$$

	$\lvert\uparrow^p\uparrow^e\rangle$	$\lvert\uparrow^p\downarrow^e\rangle$	$\lvert\downarrow^p\uparrow^e\rangle$	$\lvert\downarrow^p\downarrow^e\rangle$
$\langle\uparrow^p\uparrow^e\rvert$	$\frac{1}{2}(a_e - a_p)B_z$	.	.	.
$\langle\uparrow^p\downarrow^e\rvert$	.	$\frac{-1}{2}(a_e + a_p)B_z$	0	.
$\langle\downarrow^p\uparrow^e\rvert$	.	0	$\frac{1}{2}(a_e + a_p)B_z$	.
$\langle\downarrow^p\downarrow^e\rvert$	.	.	.	$\frac{-1}{2}(a_e - a_p)B_z$

$$\left\langle a_{ep} \mathbf{J}^{proton} \bullet \mathbf{J}^{electron} \right\rangle =$$

	$\lvert\begin{smallmatrix}1\\1\end{smallmatrix}\rangle$	$\lvert\begin{smallmatrix}1\\0\end{smallmatrix}\rangle$	$\lvert\begin{smallmatrix}0\\0\end{smallmatrix}\rangle$	$\lvert\begin{smallmatrix}1\\-1\end{smallmatrix}\rangle$
$\langle\begin{smallmatrix}1\\1\end{smallmatrix}\rvert$	$\frac{a_{ep}}{4}$	.	.	.
$\langle\begin{smallmatrix}1\\0\end{smallmatrix}\rvert$	.	$\frac{a_{ep}}{4}$	0	.
$\langle\begin{smallmatrix}0\\0\end{smallmatrix}\rvert$	.	0	$\frac{-3a_{ep}}{4}$	.
$\langle\begin{smallmatrix}1\\-1\end{smallmatrix}\rvert$	.	.	.	$\frac{a_{ep}}{4}$

$$H_{1s-B-field} = -a_p B_z J_z^{proton} + a_e B_z J_z^{electron} + a_{ep}\mathbf{J}^{proton} \bullet \mathbf{J}^{electron}$$

	$g-factor$	$Bohr-magneton$	$gyromagnetic\ factor$
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$\mu_0\frac{2}{3}\frac{1}{\pi a_0^3}\frac{a_e a_p}{hc} = 4.746m^{-1} = \frac{1}{21.1}cm^{-1}$

$\frac{1}{2}\otimes\frac{1}{2}$	$J=1$ $M=1$	1 0	1 -1	0 0
$\frac{1}{2},\frac{1}{2}$	1	0	0	0
$\frac{1}{2},\frac{-1}{2}$	0	$\frac{1}{\sqrt{2}}$		$\frac{1}{\sqrt{2}}$
$\frac{-1}{2},\frac{1}{2}$	0	$\frac{1}{\sqrt{2}}$		$\frac{-1}{\sqrt{2}}$
$\frac{-1}{2},\frac{-1}{2}$	0	0	1	0

$= \left\langle C^{\frac{1}{2}\ \frac{1}{2}}_{m_p\ m_e} \middle| {J \atop M} \right\rangle$

$\left\langle -a_p B_z J_z^{proton} + a_e B_z J_z^{electron} \right\rangle =$

	$\lvert\uparrow^p\uparrow^e\rangle$	$\lvert\uparrow^p\downarrow^e\rangle$	$\lvert\downarrow^p\uparrow^e\rangle$	$\lvert\downarrow^p\downarrow^e\rangle$
$\langle\uparrow^p\uparrow^e\rvert$	$\frac{1}{2}(a_e-a_p)B_z$	.	.	.
$\langle\uparrow^p\downarrow^e\rvert$	.	$\frac{-1}{2}(a_e+a_p)B_z$	0	.
$\langle\downarrow^p\uparrow^e\rvert$	.	0	$\frac{1}{2}(a_e+a_p)B_z$	.
$\langle\downarrow^p\downarrow^e\rvert$	.	.	.	$\frac{-1}{2}(a_e-a_p)B_z$

$\left\langle a_{ep}\mathbf{J}^{proton}\bullet\mathbf{J}^{electron}\right\rangle =$

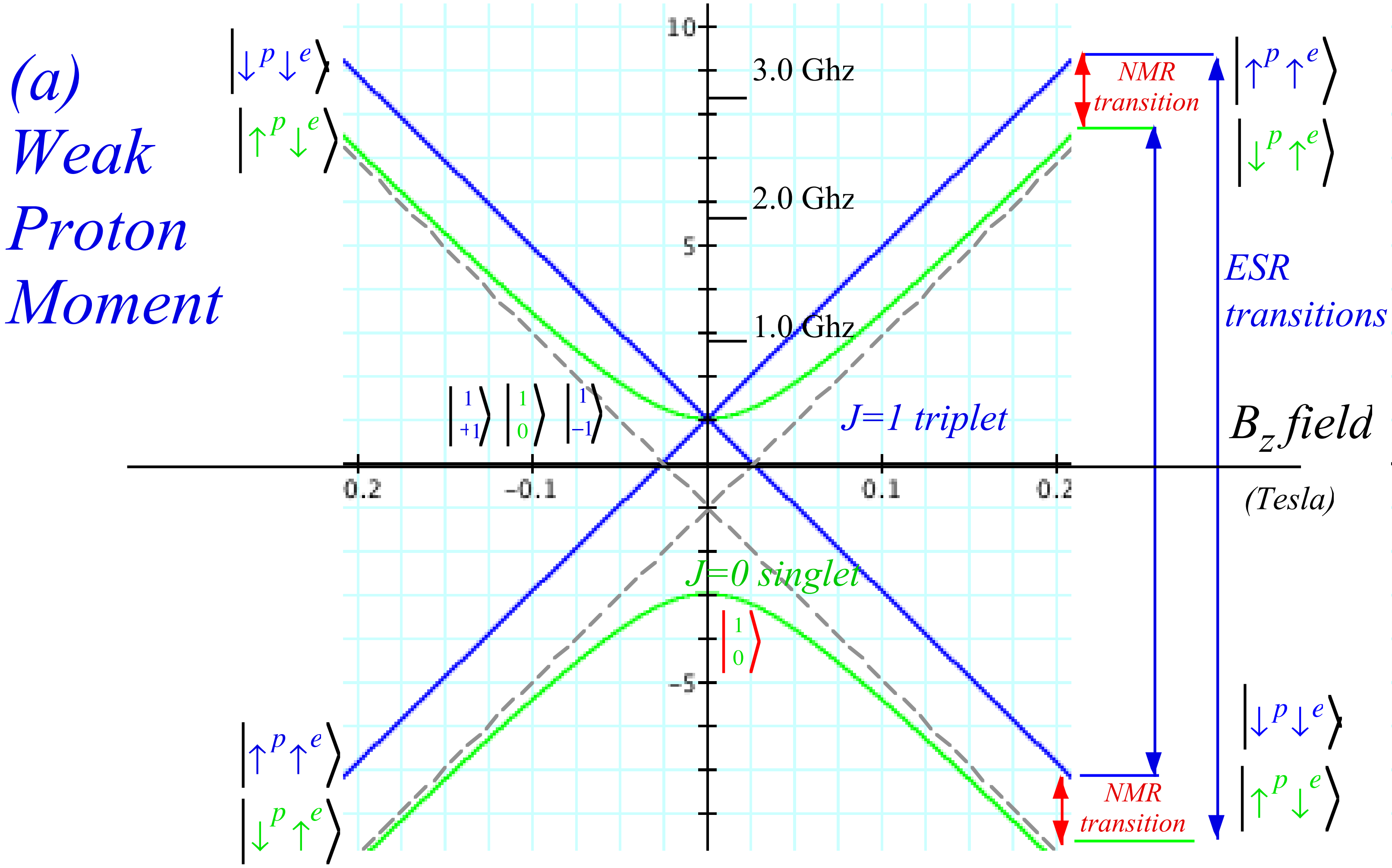
	$\lvert\uparrow^p\uparrow^e\rangle$	$\lvert\uparrow^p\downarrow^e\rangle$	$\lvert\downarrow^p\uparrow^e\rangle$	$\lvert\downarrow^p\downarrow^e\rangle$
$\langle\uparrow^p\uparrow^e\rvert$	$\frac{a_{ep}}{4}$	.	.	.
$\langle\uparrow^p\downarrow^e\rvert$	.	$\frac{-a_{ep}}{4}$	$\frac{a_{ep}}{2}$	.
$\langle\downarrow^p\uparrow^e\rvert$	.	$\frac{a_{ep}}{2}$	$\frac{-a_{ep}}{4}$	.
$\langle\downarrow^p\downarrow^e\rvert$	.	.	.	$\frac{a_{ep}}{4}$

$\left\langle -a_p B_z J_z^{proton} + a_e B_z J_z^{electron} \right\rangle =$

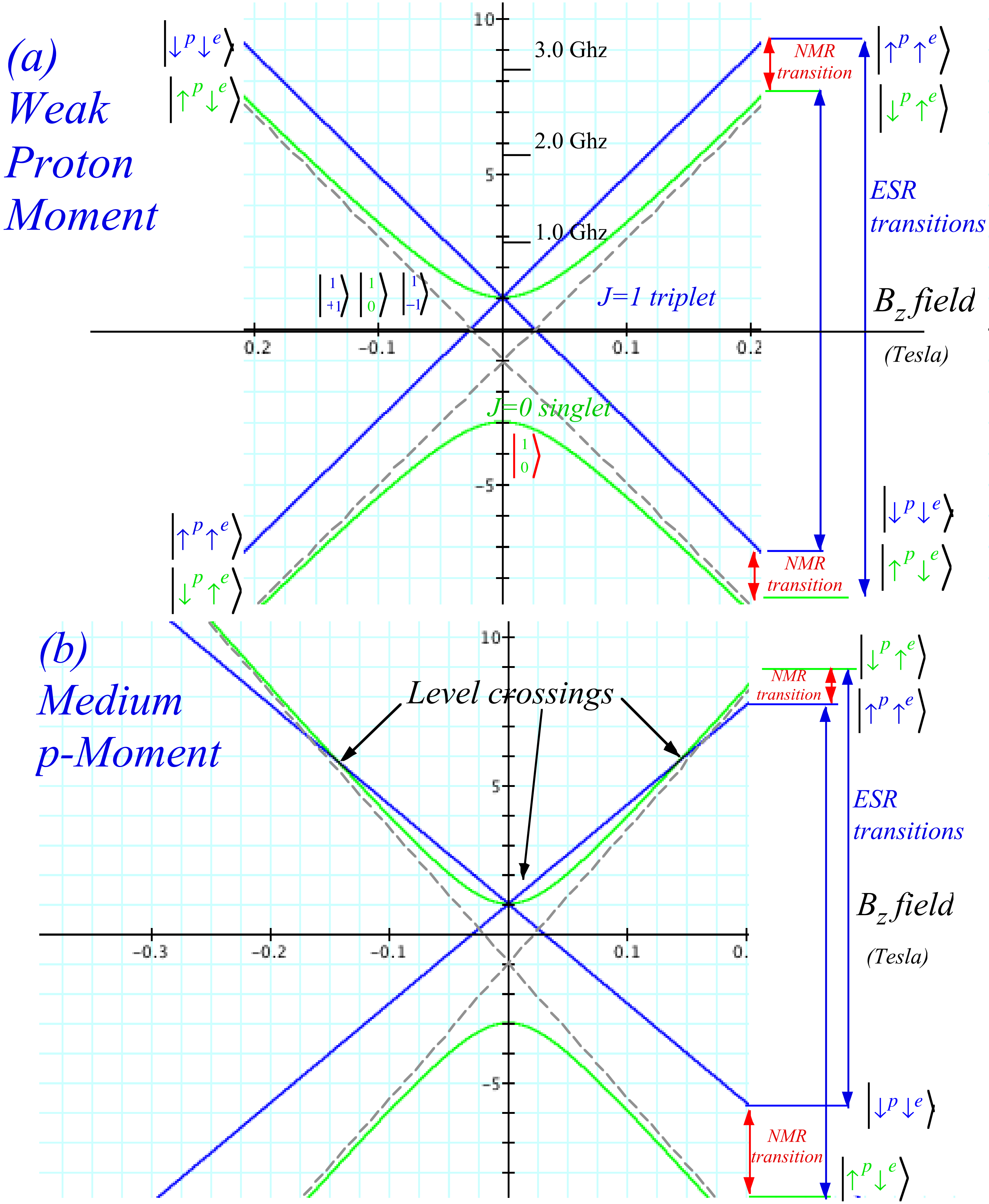
	$\lvert{1 \atop 1}\rangle$	$\lvert{1 \atop 0}\rangle$	$\lvert{0 \atop 0}\rangle$	$\lvert{1 \atop -1}\rangle$
$\langle{1 \atop 1}\rvert$	$\frac{1}{2}(a_e-a_p)B_z$	.	.	.
$\langle{1 \atop 0}\rvert$	.	0	$\frac{-1}{2}(a_e+a_p)B_z$	.
$\langle{0 \atop 0}\rvert$	.	$\frac{-1}{2}(a_e+a_p)B_z$	0	.
$\langle{1 \atop -1}\rvert$	.	.	.	$\frac{-1}{2}(a_e-a_p)B_z$

$\left\langle a_{ep}\mathbf{J}^{proton}\bullet\mathbf{J}^{electron}\right\rangle =$

	$\lvert{1 \atop 1}\rangle$	$\lvert{1 \atop 0}\rangle$	$\lvert{0 \atop 0}\rangle$	$\lvert{1 \atop -1}\rangle$
$\langle{1 \atop 1}\rvert$	$\frac{a_{ep}}{4}$	.	.	.
$\langle{1 \atop 0}\rvert$	.	$\frac{a_{ep}}{4}$	0	.
$\langle{0 \atop 0}\rvert$	.	0	$\frac{-3a_{ep}}{4}$	.
$\langle{1 \atop -1}\rvert$	.	.	.	$\frac{a_{ep}}{4}$



(a)
Weak
Proton
Moment
3.0 Ghz
2.0 Ghz
1.0 Ghz
J=1 triplet
J=0 singlet
NMR transition
NMR transition
ESR transitions
B_z field
(Tesla)



(a)
Weak Proton Moment
3.0 Ghz
2.0 Ghz
1.0 Ghz
10
5
-5
0.2
-0.1
0.1
0.2
$|\downarrow^p\downarrow^e\rangle$
$|\uparrow^p\downarrow^e\rangle$
$|\uparrow^p\uparrow^e\rangle$
$|\downarrow^p\uparrow^e\rangle$
$\left|{1 \atop +1}\right\rangle$ $\left|{1 \atop 0}\right\rangle$ $\left|{1 \atop -1}\right\rangle$
J=1 triplet
J=0 singlet
$\left|{1 \atop 0}\right\rangle$
NMR transition
ESR transitions
B_z field
(Tesla)
NMR transition
(b)
Medium p-Moment
Level crossings
10
5
-5
-0.3
-0.2
-0.1
0.1
0.
NMR transition
$|\downarrow^p\uparrow^e\rangle$
$|\uparrow^p\uparrow^e\rangle$
ESR transitions
B_z field
(Tesla)
$|\downarrow^p\downarrow^e\rangle$
NMR transition
$|\uparrow^p\downarrow^e\rangle$

Higher-J product states

$(J=1)\otimes(J=1)=2\oplus 1\oplus 0$ case

$\left|C^{\,1\ \ 1\ \ L}_{\,m_1\, m_2\, M}\right\rangle =$

		2	2	2	2	2	1	1	1	0
1 $\otimes$	1	2	1	0	−1	−2	1	0	−1	0
1	1	1	·	·	·	·	·	·	·	·
1	0	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·
1	−1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
0	1	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	·	·
0	0	·	·	$\sqrt{\frac{2}{3}}$	·	·	·	·	·	$-\frac{1}{\sqrt{3}}$
0	−1	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·
−1	1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
−1	0	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·
−1	−1	·	·	·	·	1	·	·	·	·

..

Higher-J product states

(J=1)⊗(J=1)=2⊕1⊕0 case

$$\left|C^{1\ \ 1\ \ L}_{m_1 m_2 M}\right\rangle =$$

$1 \otimes 1$		$L=2$, $M=2$	2, 1	2, 0	2, -1	2, -2	1, 1	1, 0	1, -1	0, 0
1	1	1	·	·	·	·	·	·	·	·
1	0	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·
1	-1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
0	1	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	·	·
0	0	·	·	$\sqrt{\frac{2}{3}}$	·	·	·	·	·	$-\frac{1}{\sqrt{3}}$
0	-1	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·
-1	1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
-1	0	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·
-1	-1	·	·	·	·	1	·	·	·	·

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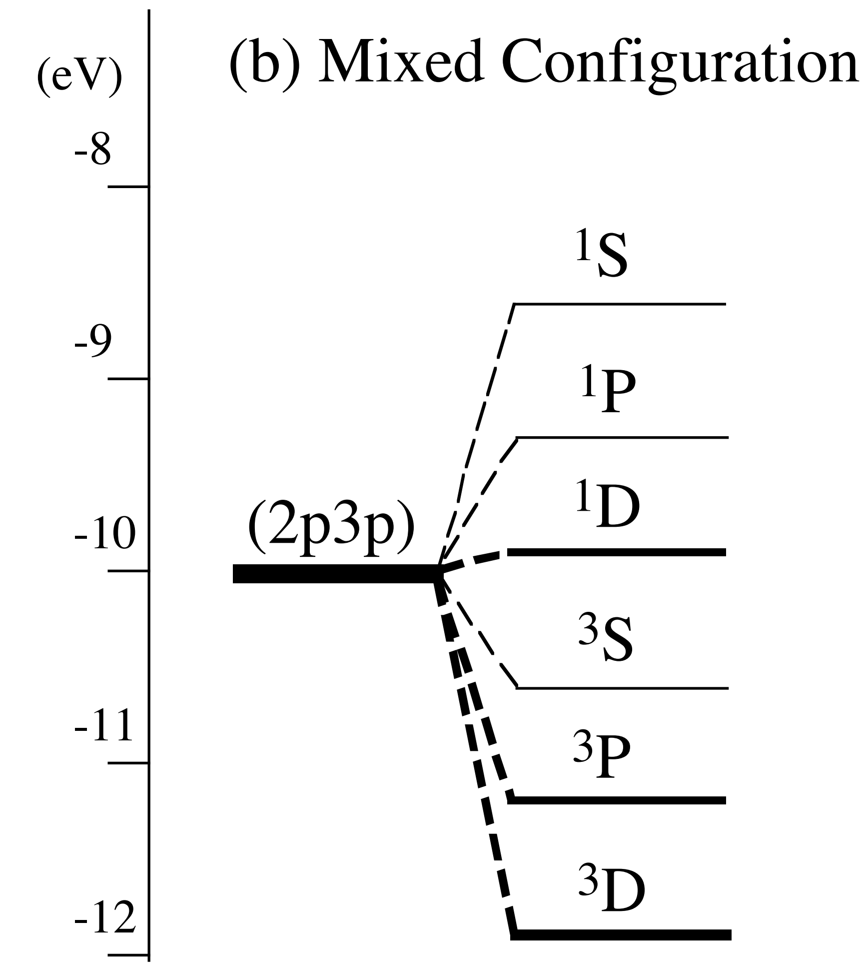

Figure 24.1.3 *Atomic* ^{2S+1}L *multiplet levels for two* ($l = 1$) p *electrons.*

$(J=1)\otimes(J=1)=2\oplus1\oplus0$ case

$\left|C^{1\ \ 1\ \ L}_{m_1 m_2 M}\right\rangle =$ ...

$1\ \otimes\ 1$		2, 2	2, 1	2, 0	2, −1	2, −2	1, 1	1, 0	1, −1	0, 0
1	1	1	·	·	·	·	·	·	·	·
1	0	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·
1	−1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
0	1	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	·	·
0	0	·	·	$\sqrt{\frac{2}{3}}$	·	·	·	·	·	$-\frac{1}{\sqrt{3}}$
0	−1	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·
−1	1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
−1	0	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·
−1	−1	·	·	·	·	1	·	·	·	·

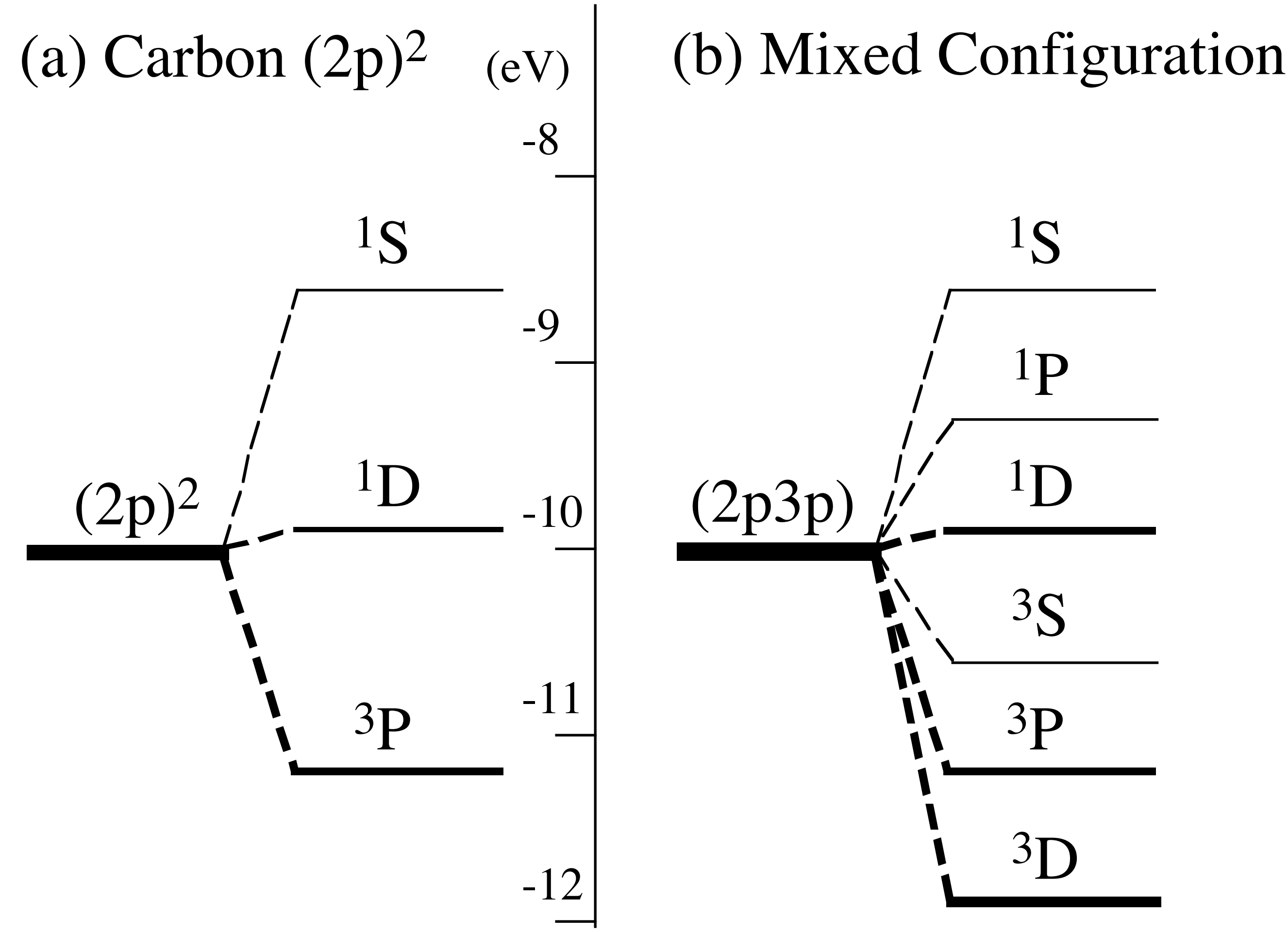

Figure 24.1.3 *Atomic* ^{2S+1}L *multiplet levels for two* ($l = 1$) p *electrons.*

Higher-J product states

(J=1)⊗(J=1)=2⊕1⊕0 case

$\left| C^{\,1\ \ 1\ \ L}_{m_1\, m_2\, M} \right\rangle =$

$1 \otimes 1$		2 2	2 1	2 0	2 -1	2 -2	1 1	1 0	1 -1	0 0
1	1	1	·	·	·	·	·	·	·	·
1	0	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·
1	−1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
0	1	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	·	·
0	0	·	·	$\sqrt{\frac{2}{3}}$	·	·	·	·	·	$-\frac{1}{\sqrt{3}}$
0	−1	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·
−1	1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
−1	0	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·
−1	−1	·	·	·	·	1	·	·	·	·

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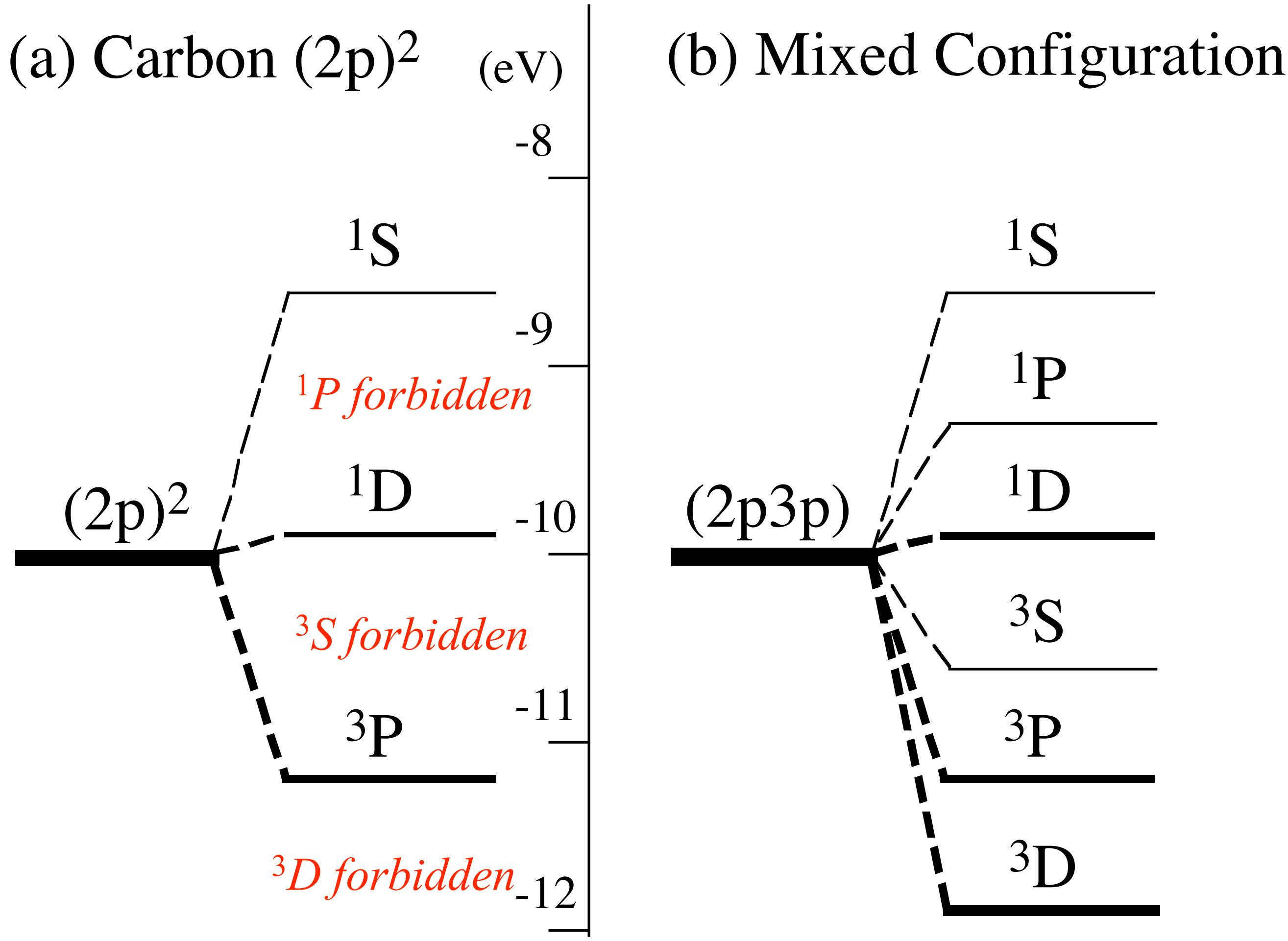


Figure 24.1.3 *Atomic* $^{2S+1}$L *multiplet levels for two (*l *= 1)* p *electrons.*

Pauli-Fermi *selection rules*
requires total anti-symmetry

Higher-J product states

(J=1)⊗(J=1)=2⊕1⊕0 case

$\left|C^{1\;\,1\;\,L}_{m_1 m_2 M}\right\rangle =$

$1 \otimes 1$		$J=2$, $M=2$	2, 1	2, 0	2, -1	2, -2	1, 1	1, 0	1, -1	0, 0
1	1	1	·	·	·	·	·	·	·	·
1	0	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·
1	−1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
0	1	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	·	·
0	0	·	·	$\sqrt{\frac{2}{3}}$	·	·	·	·	·	$-\frac{1}{\sqrt{3}}$
0	−1	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·
−1	1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
−1	0	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·
−1	−1	·	·	·	·	1	·	·	·	·

..

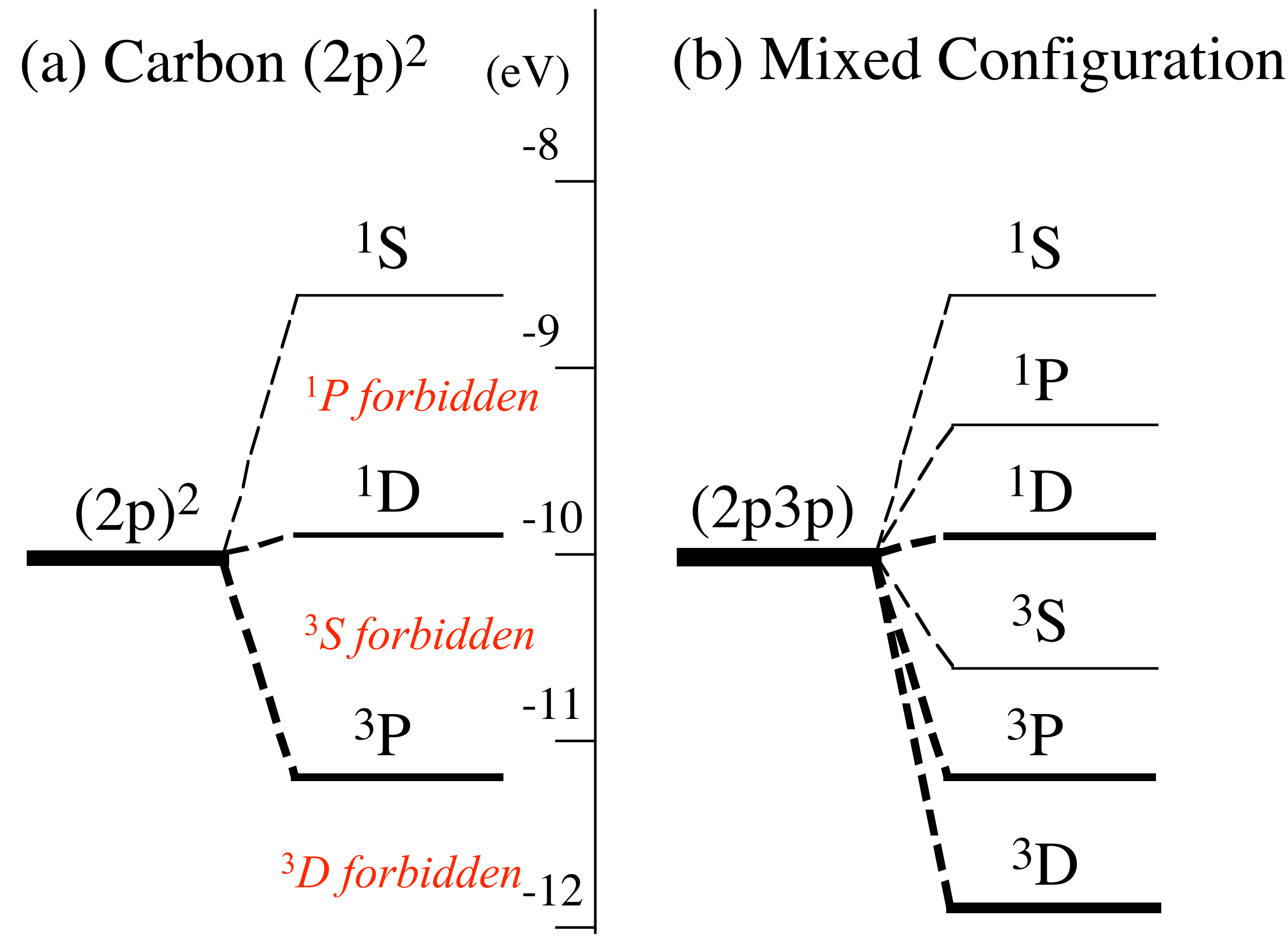
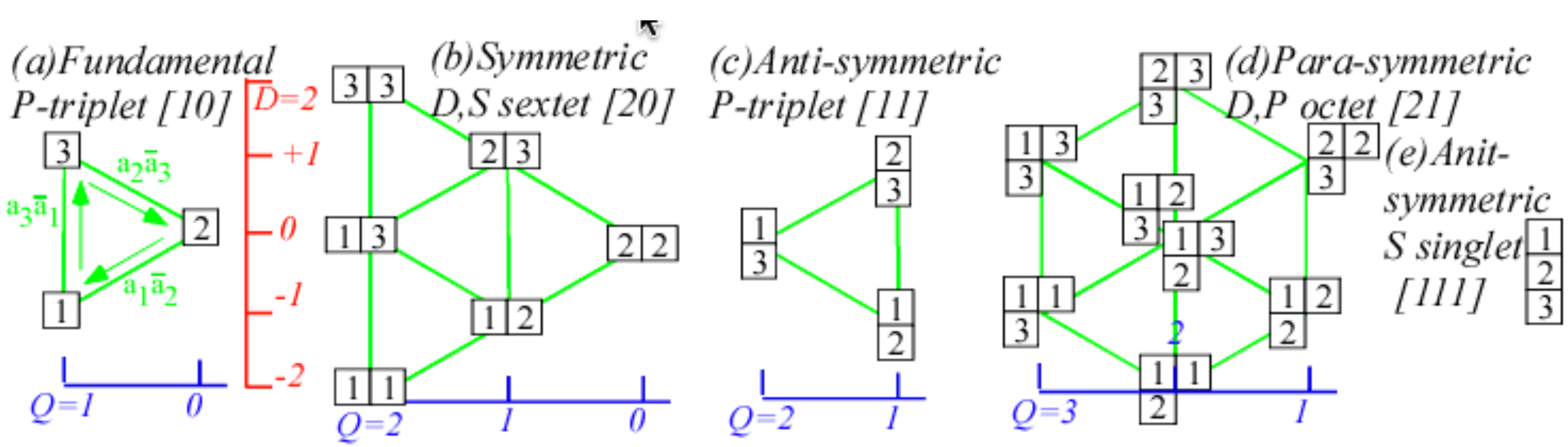
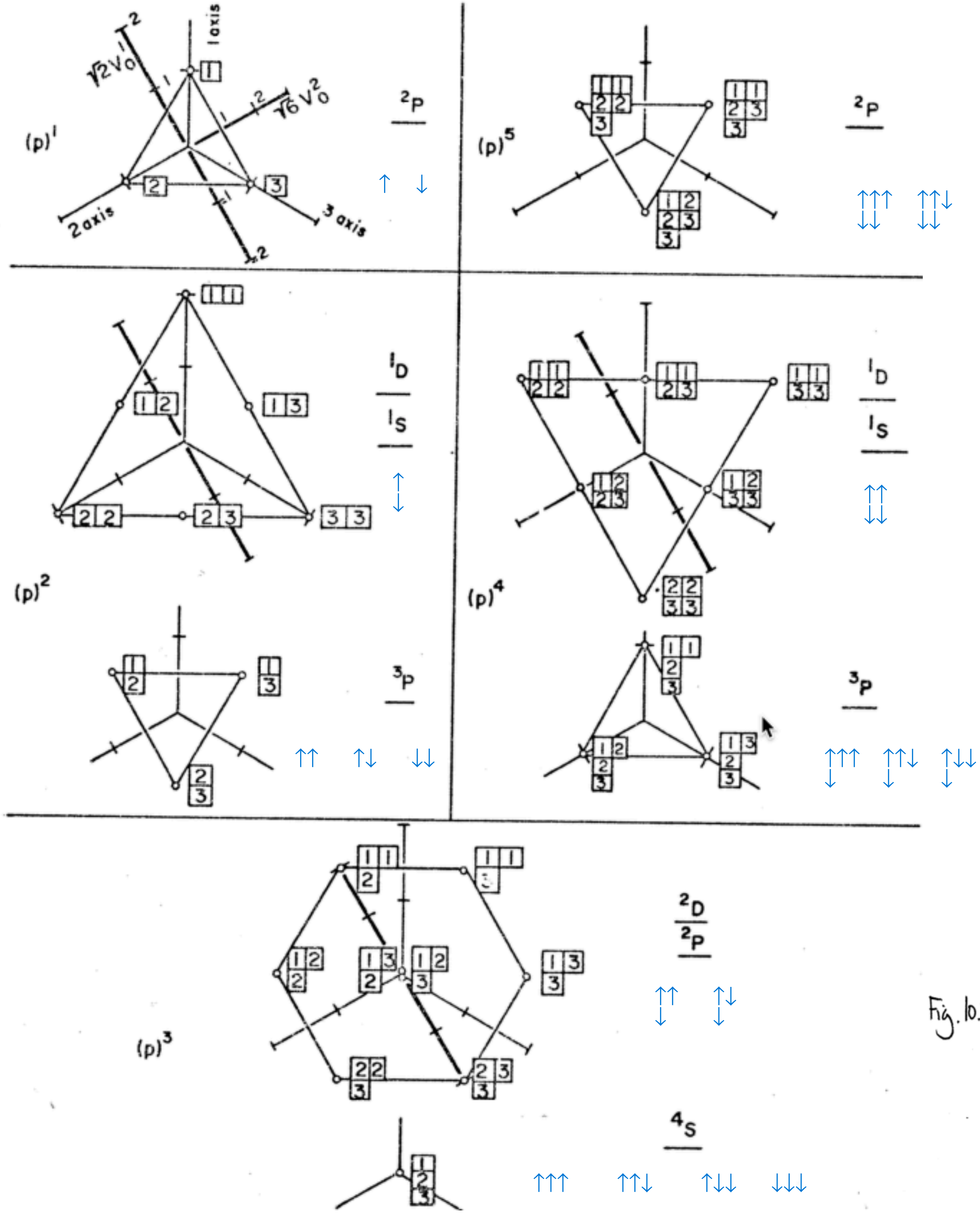


Figure 24.1.3 *Atomic* $^{2S+1}$L *multiplet levels for two (l = 1)* p *electrons.*

Pauli-Fermi *selection rules*
requires total anti-symmetry





From unpublished Ch.10 for
Principles of Symmetry, Dynamics & Spectroscopy

Fig. 10.2.2 Weight or Moment Diagrams of Atomic $(p)^n$ States
Each tableau is located at point $(x_1\ x_2\ x_3)$ in a cartesian co-ordinate system for which x_n is the number of n's in the tableau. An alternative co-ordinate system is $(v_0^2,\ v_0^1,\ v_0^0)$ defined by (10.2.19) which gives the zz-quadrupole moment, z-magnetic dipole moment, and number of particles, respectively. The last axis (v_0^0) would be pointing straight out of the figure, and each family of states lies in a plane perpendicular to it.

Higher-J product states

(J=1)⊗(J=1)=2⊕1⊕0 case

$\left| C^{1\ \ 1\ \ L}_{m_1\ m_2\ M} \right\rangle =$

$1 \otimes 1$	$L=2$, $M=2$	2, 1	2, 0	2, -1	2, -2	1, 1	1, 0	1, -1	0, 0
$1\ \ 1$	1	·	·	·	·	·	·	·	·
$1\ \ 0$	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·
$1\ \ -1$	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
$0\ \ 1$	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	·	·
$0\ \ 0$	·	·	$\sqrt{\frac{2}{3}}$	·	·	·	·	·	$-\frac{1}{\sqrt{3}}$
$0\ \ -1$	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·
$-1\ \ 1$	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
$-1\ \ 0$	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·
$-1\ \ -1$	·	·	·	·	1	·	·	·	·

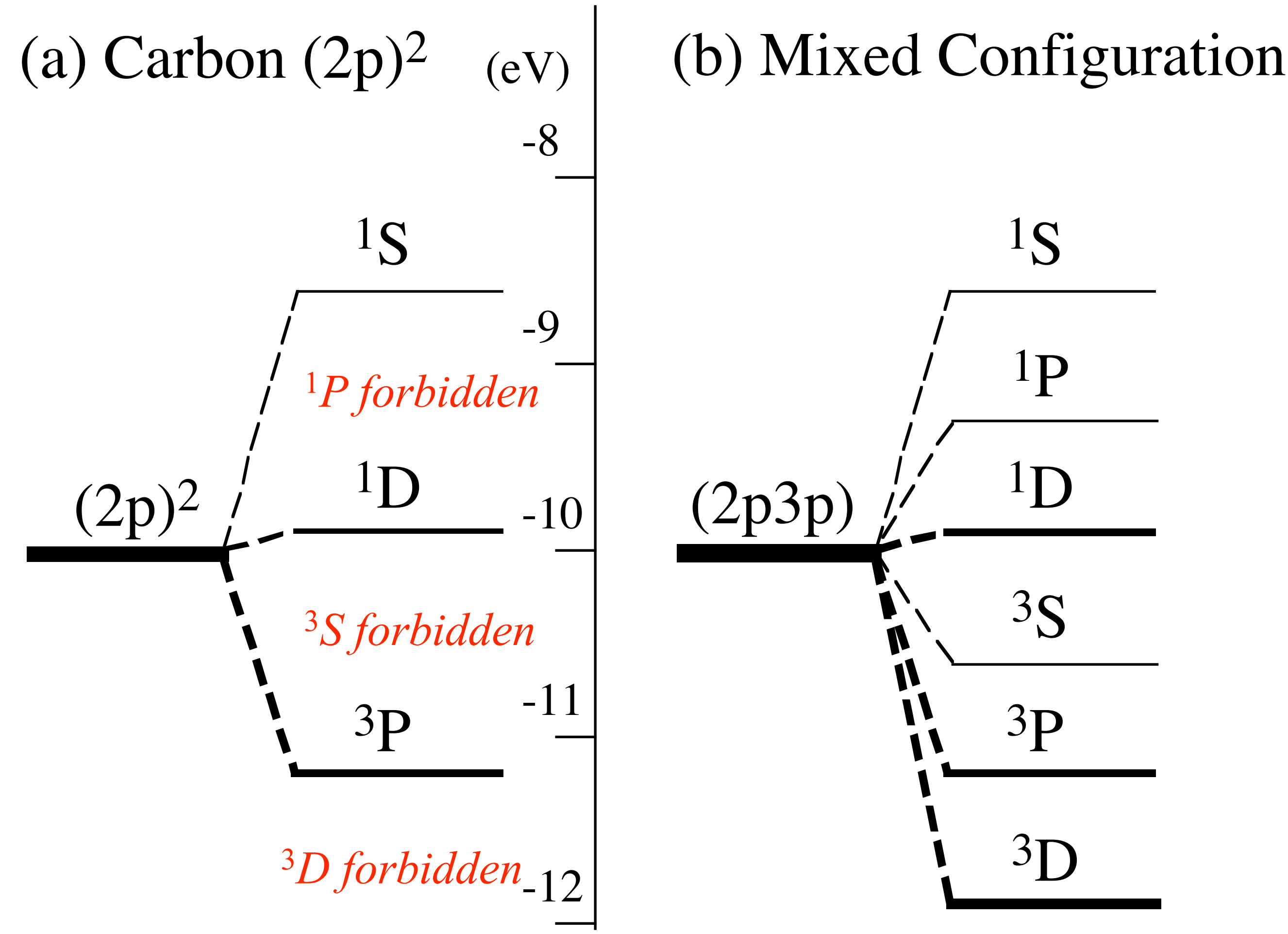


Figure 24.1.3 *Atomic* $^{2S+1}$L *multiplet levels for two (*l *= 1)* p *electrons.*

Pauli-Fermi *selection rules requires total anti-symmetry*

General U(2) case

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1 - j_2 - m_3} C^{j_1\ \ j_2\ \ j_3}_{m_1\ m_2\ m_3} \Big/ (2j_3+1)^{\frac{1}{2}}$$

Wigner 3j vs. Clebsch-Gordon (CGC)

Higher-J product states

(J=1)⊗(J=1)=2⊕1⊕0 case

$\left|C^{1\ \ 1\ \ L}_{m_1\, m_2\, M}\right\rangle =$

1 ⊗	1	2 2	2 1	2 0	2 −1	2 −2	1 1	1 0	1 −1	0 0
1	1	1	·	·	·	·	·	·	·	·
1	0	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·
1	−1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
0	1	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	·	·
0	0	·	·	$\sqrt{\frac{2}{3}}$	·	·	·	·	·	$-\frac{1}{\sqrt{3}}$
0	−1	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·
−1	1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
−1	0	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·
−1	−1	·	·	·	·	1	·	·	·	·

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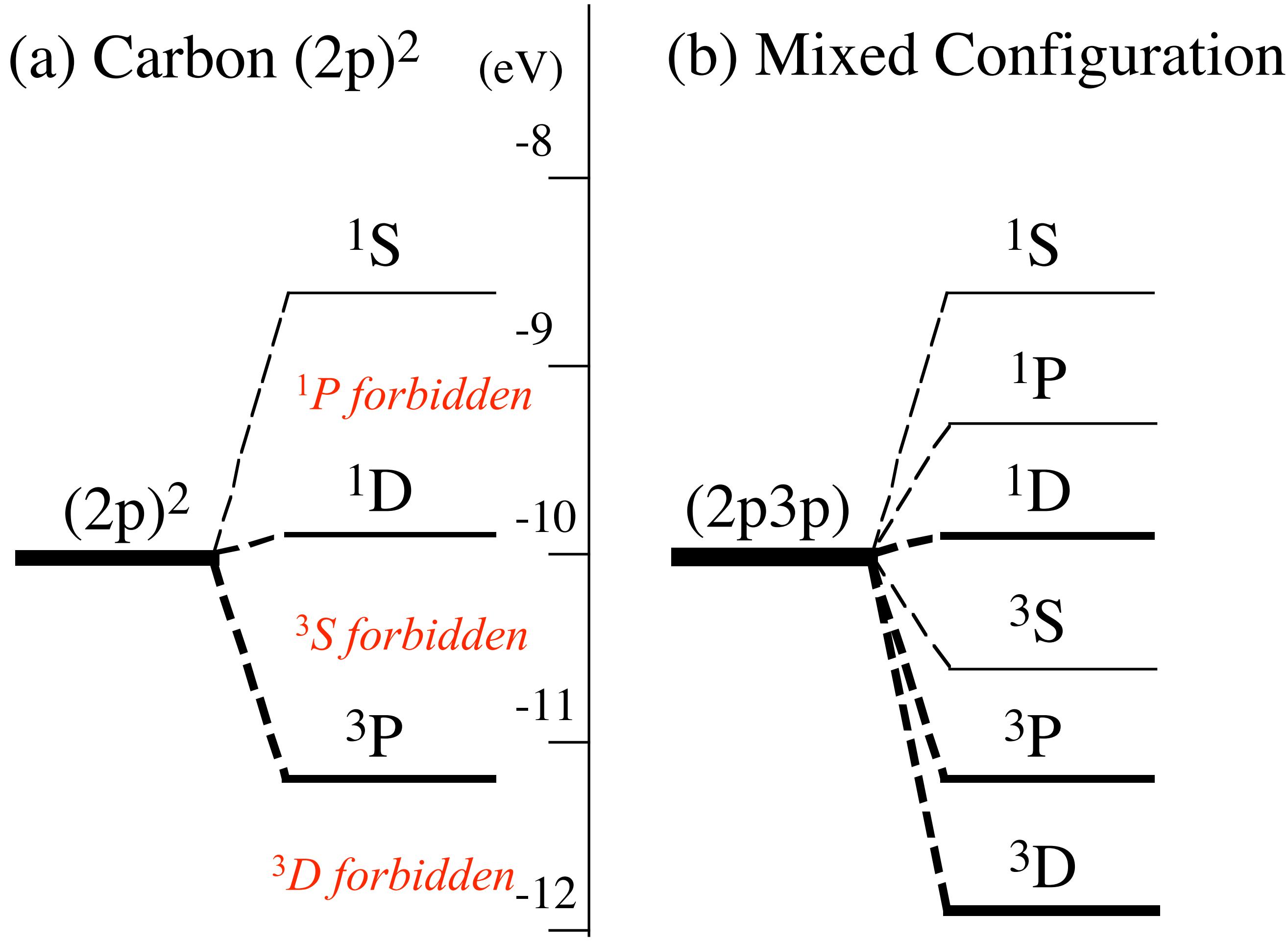


Figure 24.1.3 *Atomic* $^{2S+1}$L *multiplet levels for two (l = 1)* p *electrons.*

Pauli-Fermi *selection rules*
requires total anti-symmetry

General U(2) case

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1-j_2-m_3}\, C^{j_1\ \ j_2\ \ j_3}_{m_1\, m_2\, m_3} \Big/ (2j_3+1)^{\frac{1}{2}}$$

Wigner 3j vs. Clebsch-Gordon (CGC)

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1-j_2-n_3} \sqrt{\frac{(j_1+j_2-j_3)!(j_1-j_2+j_3)(-j_1+j_2+j_3)}{(j_1+j_2+j_3+1)!}}$$

$$\sum_k \frac{(-1)^k}{k!} \frac{\sqrt{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(j_3+m_3)!(j_3-m_3)!}}{(j_1-m_1-k)!(j_2-m_2-k)!(j_1+j_2-j_3-k)!(j_3-j_2-m_1+k)!(j_3-j_1-m_2+k)!}$$

Higher-J product states

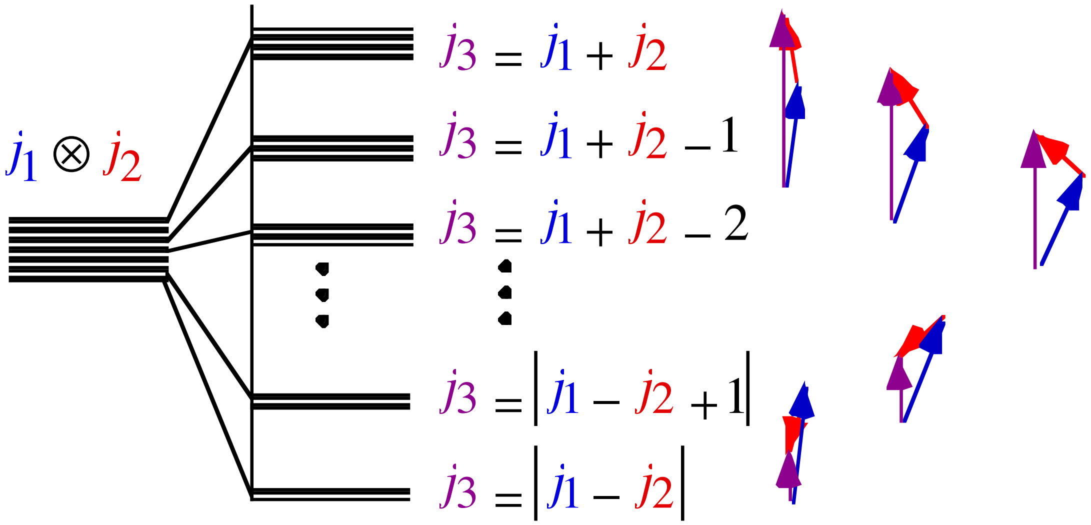


Figure 24.1.6 Level-splitting and vector-addition picture of angular-momentum coupling.

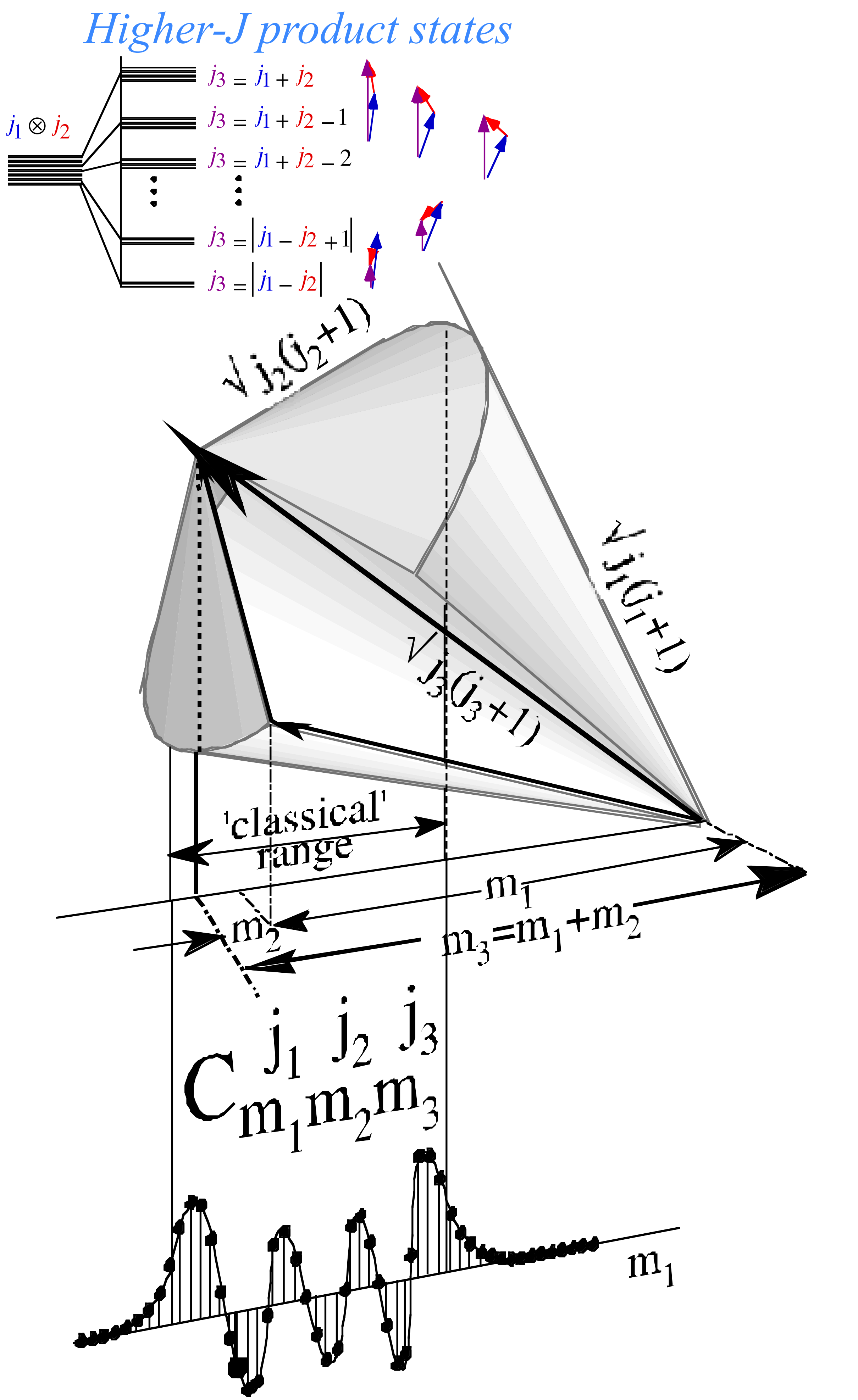


Figure 24.1.7 *Angular-momentum cone picture of Clebsch-Gordan coupling amplitudes.*

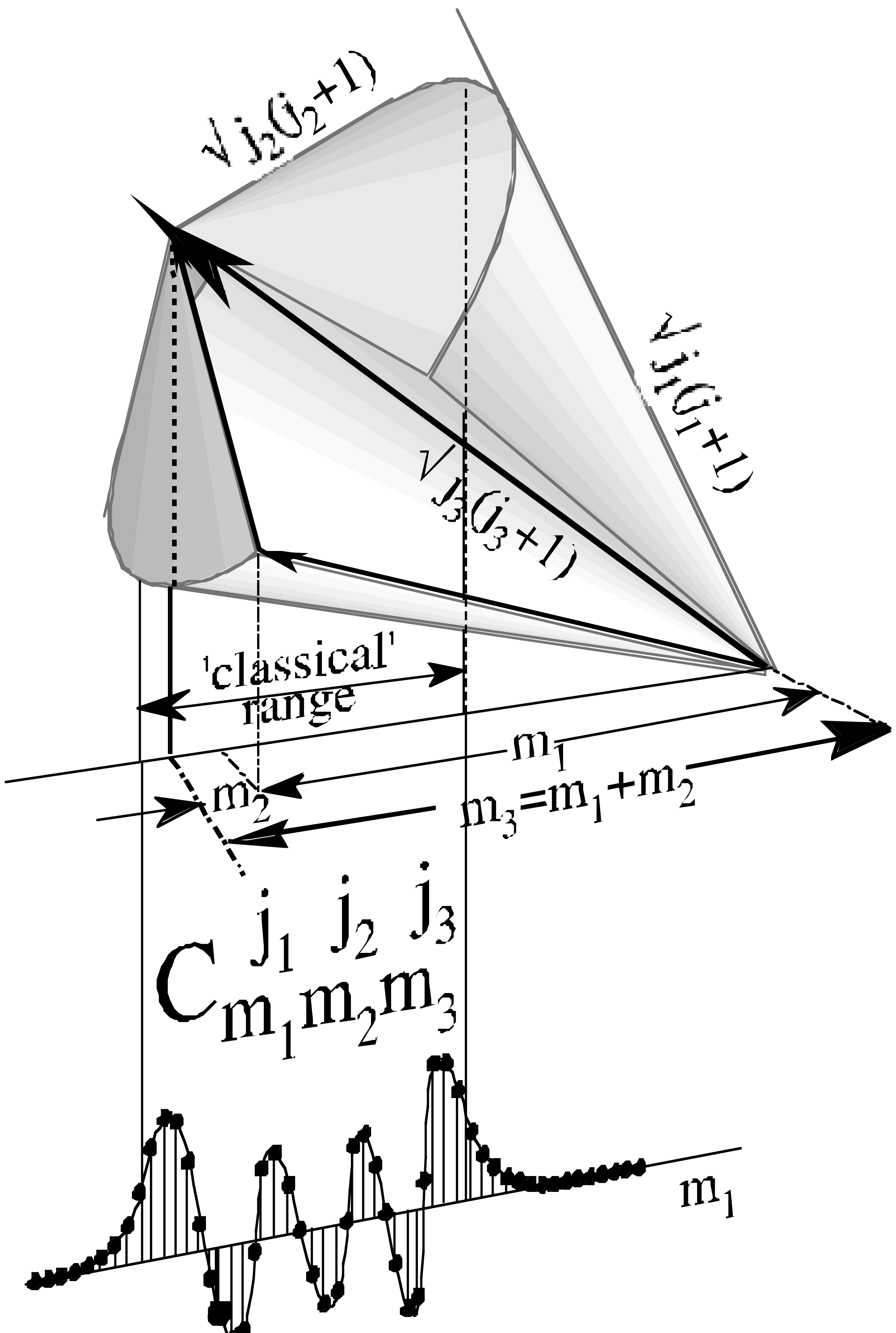


Figure 24.1.7 *Angular-momentum cone picture of Clebsch-Gordan coupling amplitudes.*

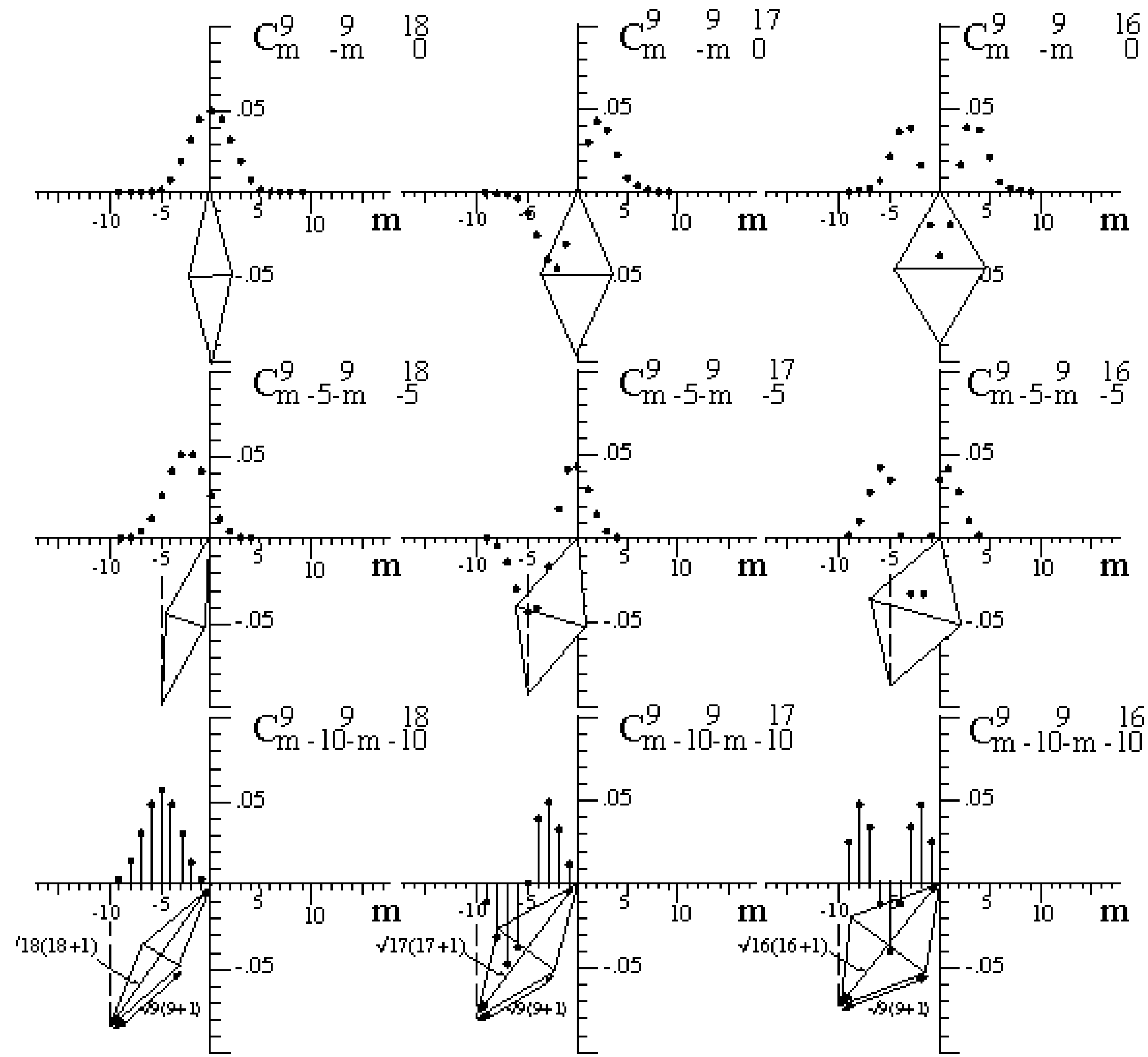
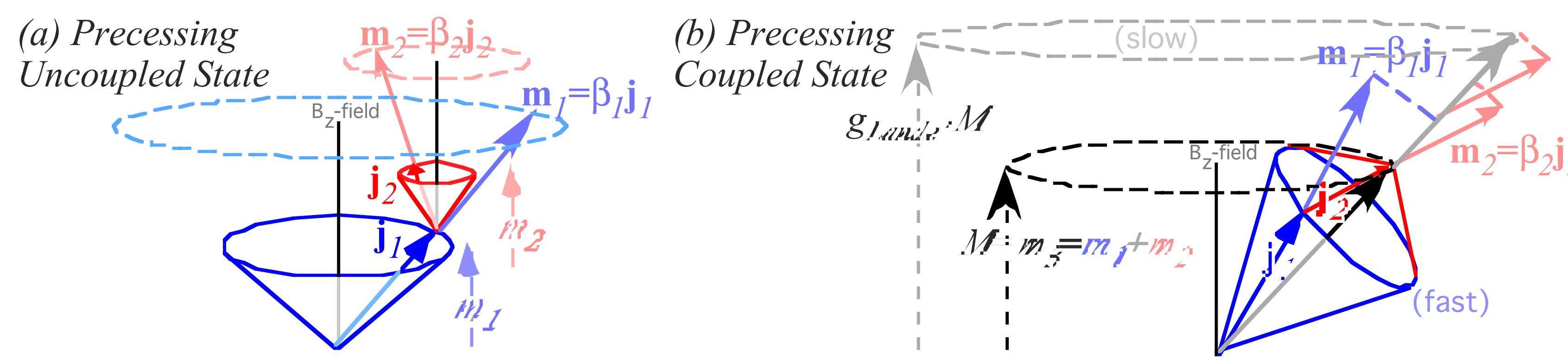


Figure 24.1.8 *Clebsch-Gordan coefficients plotted next to their angular-momentum cones.*

Higher-J product states



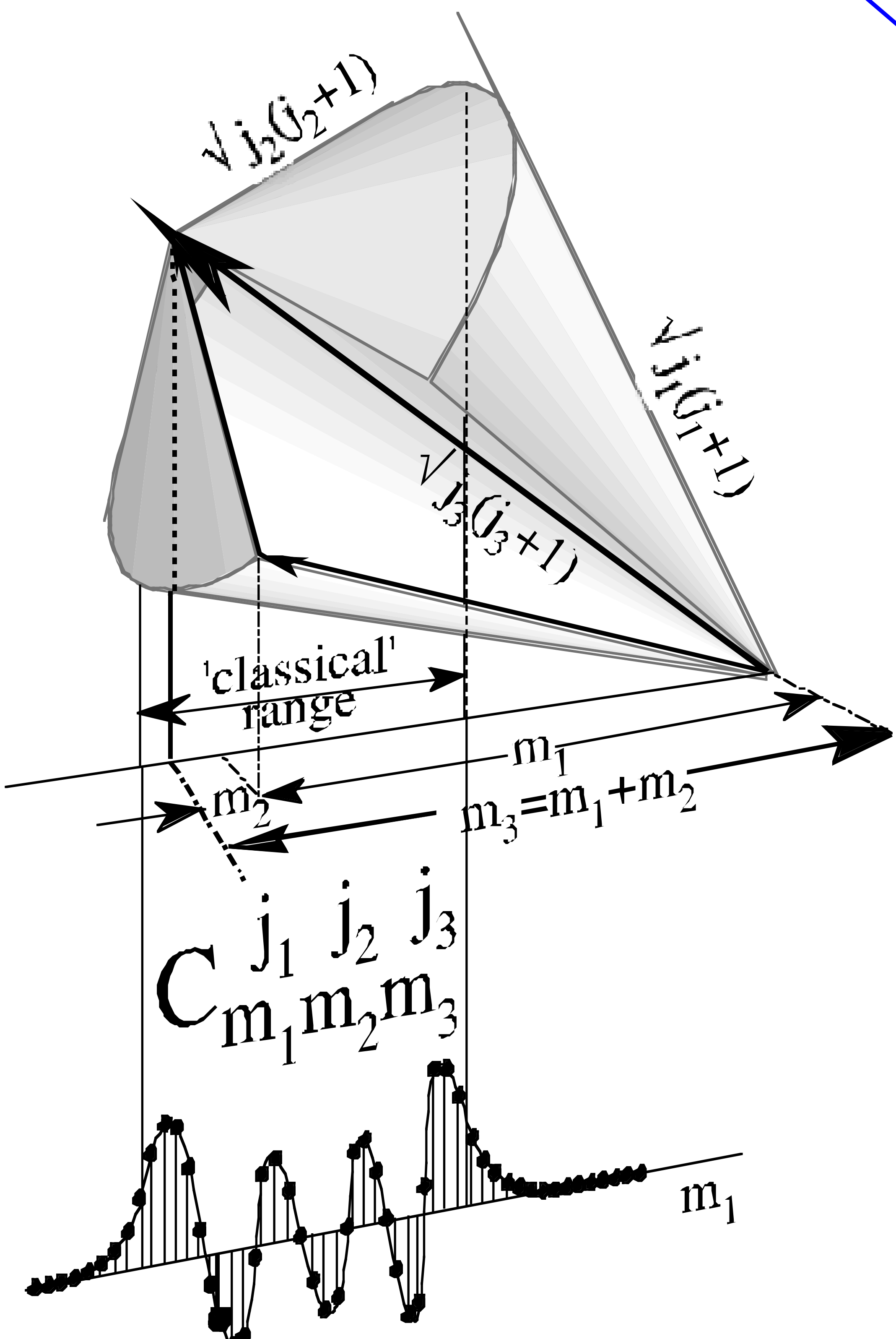


Figure 24.1.7 Angular-momentum cone picture of Clebsch-Gordan coupling amplitudes.

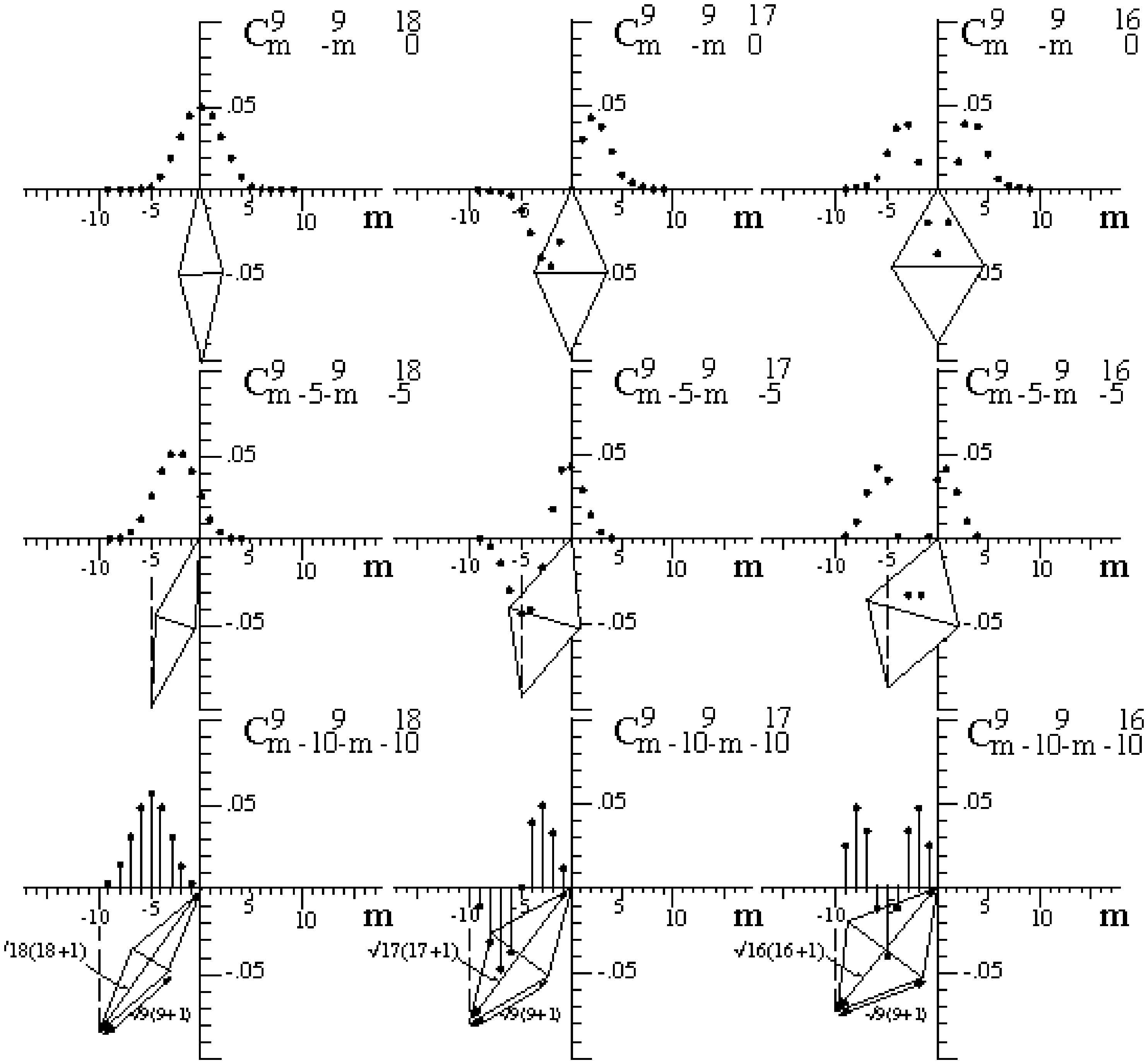


Figure 24.1.8 Clebsch-Gordan coefficients plotted next to their angular-momentum cones.

Multi-spin $(1/2)^N$ product states

$$\left(\frac{1}{2}\otimes\frac{1}{2}\right)\otimes\frac{1}{2}$$

Multi-spin $(1/2)^N$ product states

$$\left(\frac{1}{2}\otimes\frac{1}{2}\right)\otimes\frac{1}{2}=(0\oplus 1)\otimes\frac{1}{2}$$

Multi-spin $(1/2)^N$ product states

$$\left(\frac{1}{2}\otimes\frac{1}{2}\right)\otimes\frac{1}{2}=(0\oplus 1)\otimes\frac{1}{2}=\left(0\otimes\frac{1}{2}\right)\oplus\quad\left(1\otimes\frac{1}{2}\right)$$

Multi-spin $(1/2)^N$ product states

$$\left(\frac{1}{2}\otimes\frac{1}{2}\right)\otimes\frac{1}{2}=(0\oplus 1)\otimes\frac{1}{2}=\left(0\otimes\frac{1}{2}\right)\oplus\left(1\otimes\frac{1}{2}\right)$$

$$=\left(\frac{1}{2}\right)\oplus\left(\left(\frac{1}{2}\right)\oplus\left(\frac{3}{2}\right)\right)$$

Multi-spin $(1/2)^N$ product states

$$\left(\frac{1}{2}\otimes\frac{1}{2}\right)\otimes\frac{1}{2}=(0\oplus 1)\otimes\frac{1}{2}=\left(0\otimes\frac{1}{2}\right)\oplus\left(1\otimes\frac{1}{2}\right)$$

$$=\left(\frac{1}{2}\right)\oplus\left(\left(\frac{1}{2}\right)\oplus\left(\frac{3}{2}\right)\right)=\frac{1}{2}\oplus\frac{1}{2}\oplus\frac{3}{2}$$

Multi-spin $(1/2)^N$ product states

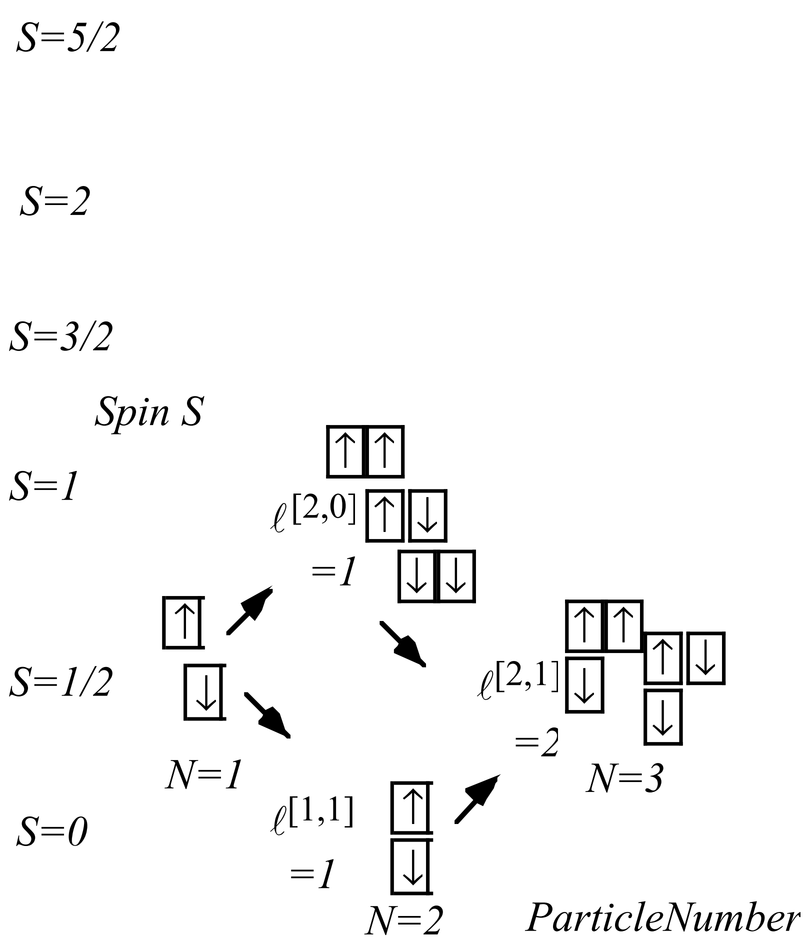
$$\left(\frac{1}{2}\otimes\frac{1}{2}\right)\otimes\frac{1}{2}=(0\oplus 1)\otimes\frac{1}{2}=\left(0\otimes\frac{1}{2}\right)\oplus\left(1\otimes\frac{1}{2}\right)$$

$$=\left(\frac{1}{2}\right)\oplus\left(\left(\frac{1}{2}\right)\oplus\left(\frac{3}{2}\right)\right)=\frac{1}{2}\oplus\frac{1}{2}\oplus\frac{3}{2}=2\left(\frac{1}{2}\right)\oplus 1\left(\frac{3}{2}\right)$$

Multi-spin $(1/2)^N$ product states

$$\left(\frac{1}{2}\otimes\frac{1}{2}\right)\otimes\frac{1}{2}=(0\oplus 1)\otimes\frac{1}{2}=\left(0\otimes\frac{1}{2}\right)\oplus\left(1\otimes\frac{1}{2}\right)$$

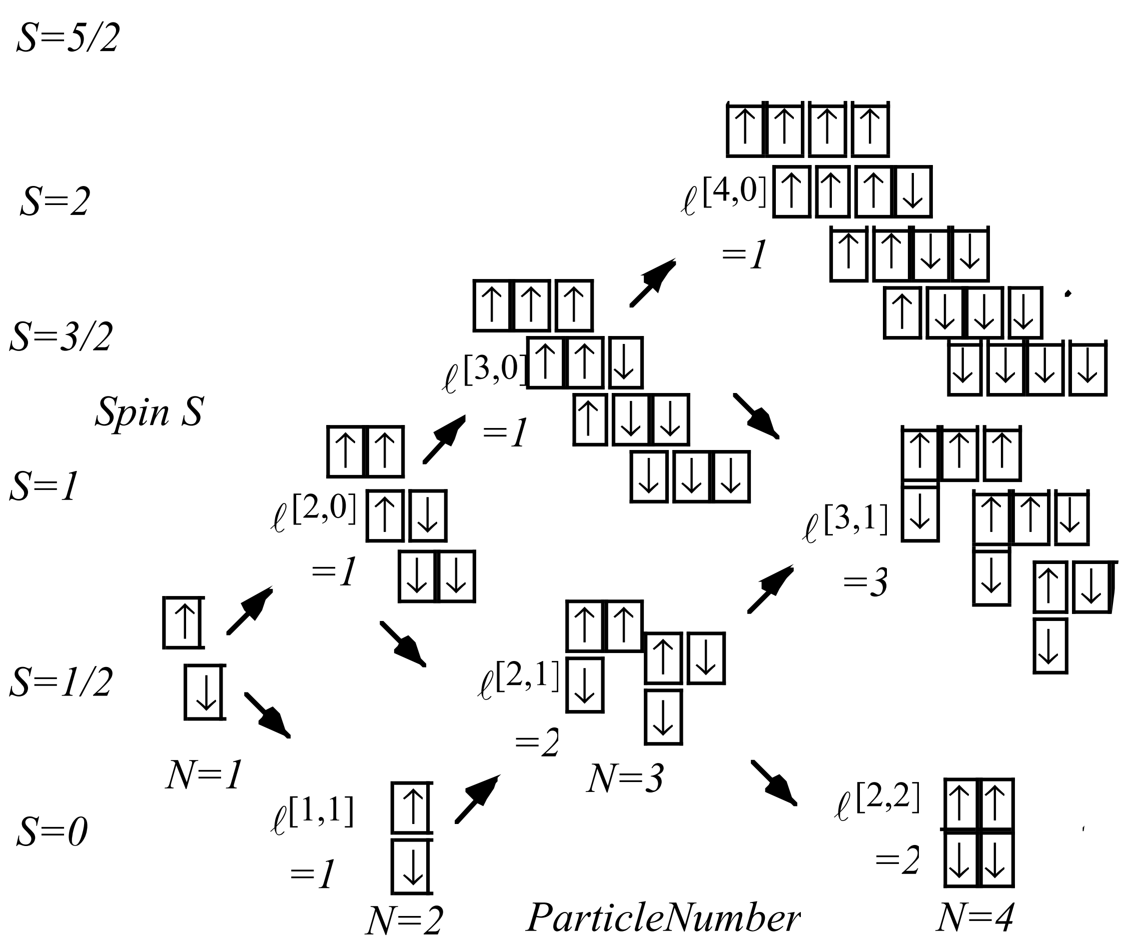
$$=\left(\frac{1}{2}\right)\oplus\left(\left(\frac{1}{2}\right)\oplus\left(\frac{3}{2}\right)\right)=\frac{1}{2}\oplus\frac{1}{2}\oplus\frac{3}{2}=2\left(\frac{1}{2}\right)\oplus 1\left(\frac{3}{2}\right)$$



Multi-spin $(1/2)^N$ product states

$$\left(\frac{1}{2}\otimes\frac{1}{2}\right)\otimes\frac{1}{2}=(0\oplus 1)\otimes\frac{1}{2}=\left(0\otimes\frac{1}{2}\right)\oplus\quad\left(1\otimes\frac{1}{2}\right)$$

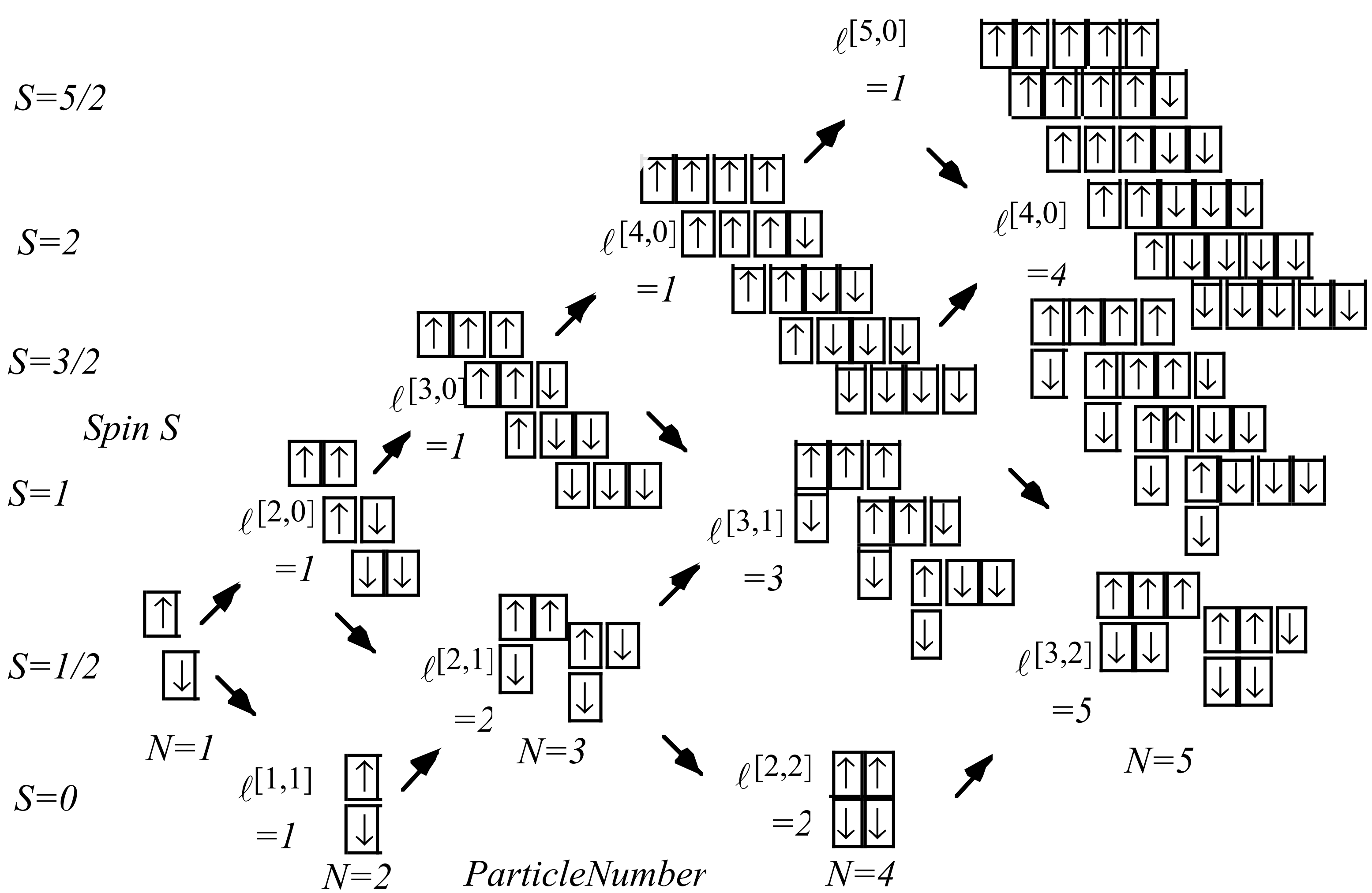
$$=\quad\left(\frac{1}{2}\right)\quad\oplus\left(\left(\frac{1}{2}\right)\oplus\left(\frac{3}{2}\right)\right)=\frac{1}{2}\oplus\ \frac{1}{2}\oplus\frac{3}{2}=2\left(\frac{1}{2}\right)\oplus 1\left(\frac{3}{2}\right)$$



Multi-spin $(1/2)^N$ product states

$$\left(\frac{1}{2}\otimes\frac{1}{2}\right)\otimes\frac{1}{2}=(0\oplus 1)\otimes\frac{1}{2}=\left(0\otimes\frac{1}{2}\right)\oplus\left(1\otimes\frac{1}{2}\right)$$

$$=\left(\frac{1}{2}\right)\oplus\left(\left(\frac{1}{2}\right)\oplus\left(\frac{3}{2}\right)\right)=\frac{1}{2}\oplus\frac{1}{2}\oplus\frac{3}{2}=2\left(\frac{1}{2}\right)\oplus 1\left(\frac{3}{2}\right)$$

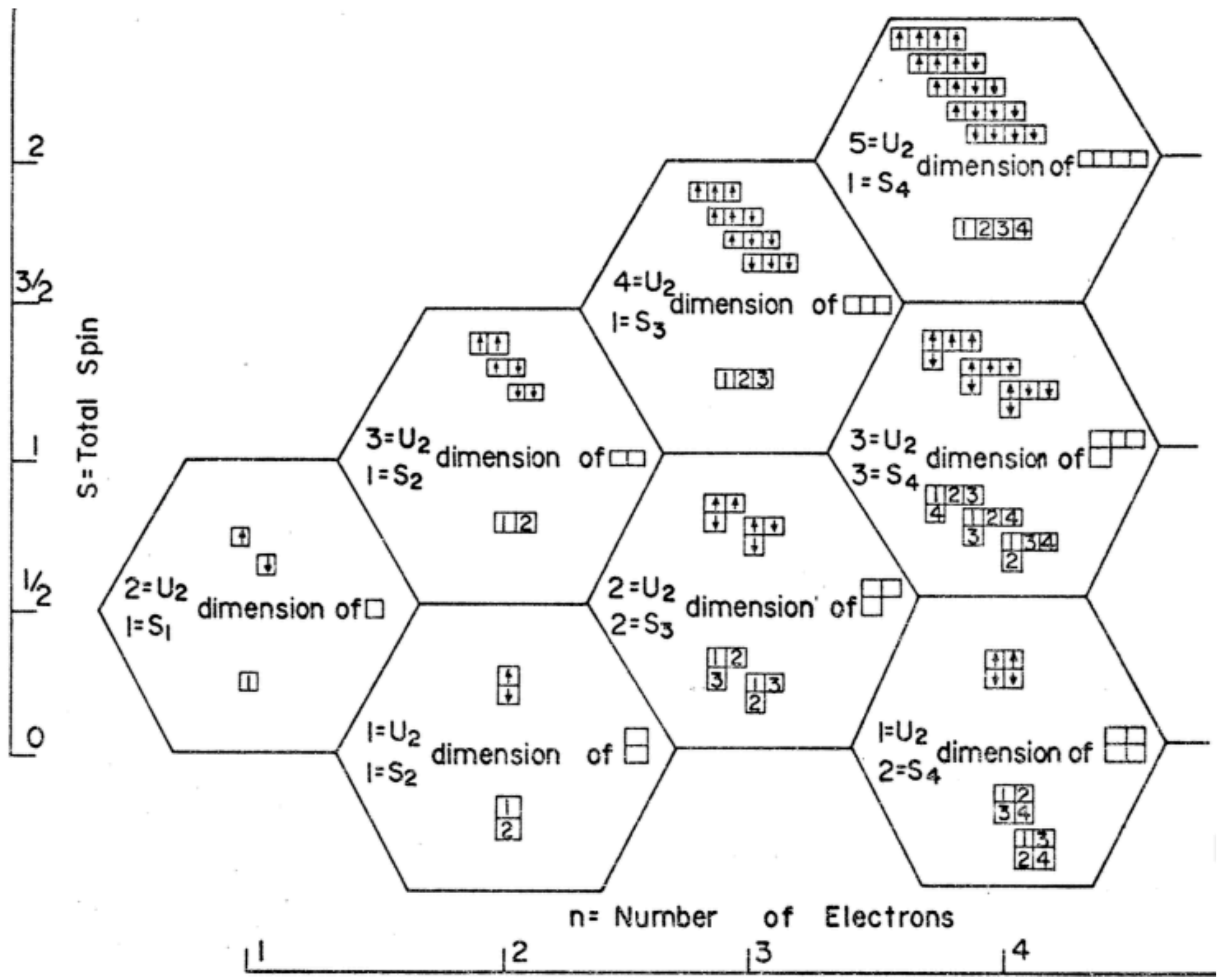


Multi-spin $(1/2)^N$ product states

$$\left|\begin{array}{cccccc}\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\downarrow}\\\boxed{\downarrow}&\boxed{\downarrow}&&&&\end{array}\;\begin{array}{c}S=2\\M_S=1\end{array}\right\rangle = C^{5/2\ 1/2\ 2}_{1/2\ 1/2\ 1}\left|\begin{array}{cccccc}\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\downarrow}&\boxed{\downarrow}\\\boxed{\downarrow}&\boxed{\cdot\cdot}&&&&\end{array}\begin{array}{c}5/2\\1/2\end{array}\right\rangle\left|\boxed{\uparrow}^{1/2}_{1/2}\right\rangle + C^{5/2\ 1/2\ 2}_{3/2\ -1/2\ 1}\left|\begin{array}{cccccc}\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\downarrow}\\\boxed{\downarrow}&\boxed{\cdot\cdot}&&&&\end{array}\begin{array}{c}5/2\\1/2\end{array}\right\rangle\left|\boxed{\downarrow}^{1/2}_{-1/2}\right\rangle$$

$$+C^{3/2\ 1/2\ 2}_{1/2\ 1/2\ 1}\left|\begin{array}{cccccc}\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\downarrow}&\boxed{\cdot\cdot}\\\boxed{\downarrow}&\boxed{\downarrow}&&&&\end{array}\begin{array}{c}3/2\\1/2\end{array}\right\rangle\left|\boxed{\uparrow}^{1/2}_{1/2}\right\rangle + C^{3/2\ 1/2\ 2}_{3/2\ -1/2\ 1}\left|\begin{array}{cccccc}\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\cdot\cdot}\\\boxed{\downarrow}&\boxed{\downarrow}&&&&\end{array}\begin{array}{c}3/2\\1/2\end{array}\right\rangle\left|\boxed{\downarrow}^{1/2}_{-1/2}\right\rangle$$

$$\left(\begin{array}{cc}C^{j\ \ 1/2\ \ j+1/2}_{m\ 1/2\ m+1/2}=\sqrt{\dfrac{j+m+1}{2j+1}} & C^{j\ \ \ 1/2\ \ j+1/2}_{m+1\ -1/2\ m+1/2}=\sqrt{\dfrac{j-m}{2j+1}}\\[2ex] C^{j+1\ 1/2\ \ j+1/2}_{m\ \ \ 1/2\ m+1/2}=-\sqrt{\dfrac{j-m+1}{2j+3}} & C^{j+1\ \ 1/2\ \ j+1/2}_{m+1\ -1/2\ m+1/2}=\sqrt{\dfrac{j+m+2}{2j+3}}\end{array}\right)\quad \text{example:}\quad\left(\begin{array}{cc}C^{3/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=\sqrt{\dfrac{2}{3}} & C^{3/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=\sqrt{\dfrac{1}{3}}\\[2ex] C^{5/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=-\sqrt{\dfrac{1}{3}} & C^{5/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=\sqrt{\dfrac{2}{3}}\end{array}\right)$$

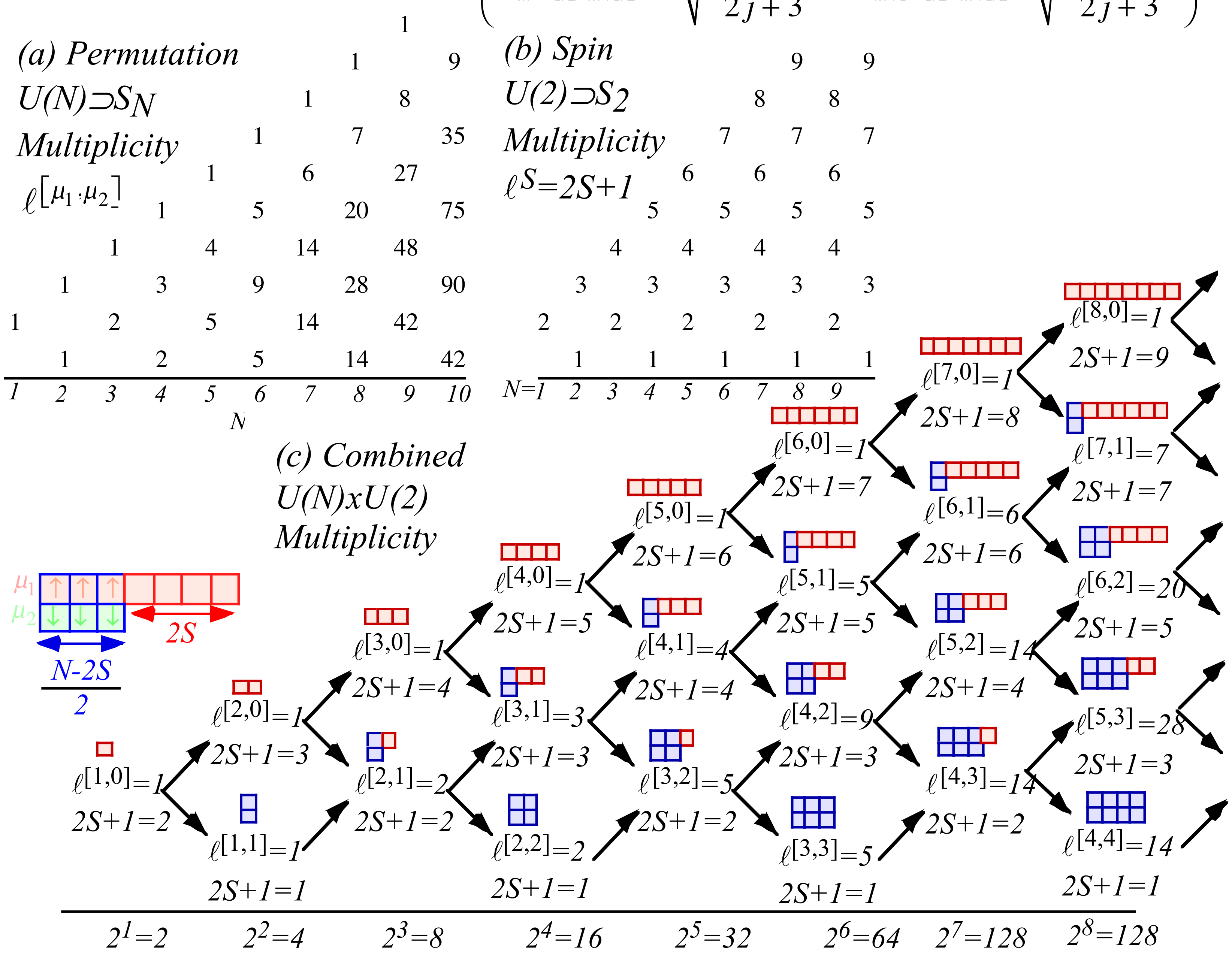


Multi-spin $(1/2)^N$ product states

$$\left|\begin{array}{cccccc}\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\downarrow}\\ \boxed{\downarrow}&\boxed{\downarrow}&&&&\end{array}\;\begin{array}{c}S=2\\ M_S=1\end{array}\right\rangle = C^{5/2\;1/2\;2}_{1/2\;\;1/2\;1}\left|\begin{array}{cccccc}\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\downarrow}&\boxed{\downarrow}\\ \boxed{\downarrow}&\boxed{\cdot\cdot}&&&&\end{array}\begin{array}{c}5/2\\ 1/2\end{array}\right\rangle\left|\boxed{\uparrow}^{1/2}_{1/2}\right\rangle + C^{5/2\;1/2\;\;2}_{3/2\;-1/2\;1}\left|\begin{array}{cccccc}\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\downarrow}\\ \boxed{\downarrow}&\boxed{\cdot\cdot}&&&&\end{array}\begin{array}{c}5/2\\ 1/2\end{array}\right\rangle\left|\boxed{\downarrow}^{1/2}_{-1/2}\right\rangle$$

$$+C^{3/2\;1/2\;2}_{1/2\;\;1/2\;1}\left|\begin{array}{cccccc}\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\downarrow}&\boxed{\cdot\cdot}\\ \boxed{\downarrow}&\boxed{\downarrow}&&&&\end{array}\begin{array}{c}3/2\\ 1/2\end{array}\right\rangle\left|\boxed{\uparrow}^{1/2}_{1/2}\right\rangle + C^{3/2\;1/2\;\;2}_{3/2\;-1/2\;1}\left|\begin{array}{cccccc}\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\cdot\cdot}\\ \boxed{\downarrow}&\boxed{\downarrow}&&&&\end{array}\begin{array}{c}3/2\\ 1/2\end{array}\right\rangle\left|\boxed{\downarrow}^{1/2}_{-1/2}\right\rangle$$

$$\left(\begin{array}{cc} C^{j\;\;1/2\;\;j+1/2}_{m\;1/2\;\;m+1/2}=\sqrt{\dfrac{j+m+1}{2j+1}} & C^{j\;\;\;\;1/2\;\;\;j+1/2}_{m+1\;-1/2\;\;m+1/2}=\sqrt{\dfrac{j-m}{2j+1}}\\ C^{j+1\;1/2\;\;j+1/2}_{m\;\;\;\;1/2\;\;m+1/2}=-\sqrt{\dfrac{j-m+1}{2j+3}} & C^{j+1\;\;\;1/2\;\;\;j+1/2}_{m+1\;-1/2\;\;m+1/2}=\sqrt{\dfrac{j+m+2}{2j+3}}\end{array}\right)\quad \text{example:}\quad \left(\begin{array}{cc} C^{3/2\;1/2\;2}_{1/2\;\;1/2\;1}=\sqrt{\dfrac{2}{3}} & C^{3/2\;1/2\;2}_{1/2\;\;1/2\;1}=\sqrt{\dfrac{1}{3}}\\ C^{5/2\;1/2\;2}_{1/2\;\;1/2\;1}=-\sqrt{\dfrac{1}{3}} & C^{5/2\;1/2\;2}_{1/2\;\;1/2\;1}=\sqrt{\dfrac{2}{3}}\end{array}\right)$$



$$2^N=\sum_S^{N/2}\ell^{[S]}\ell^{[\mu_1,\mu_2]}$$

$$=\sum_S^{N/2}(2S+1)\,\ell^{\left[\frac{N+2S}{2},\frac{N-2S}{2}\right]}$$

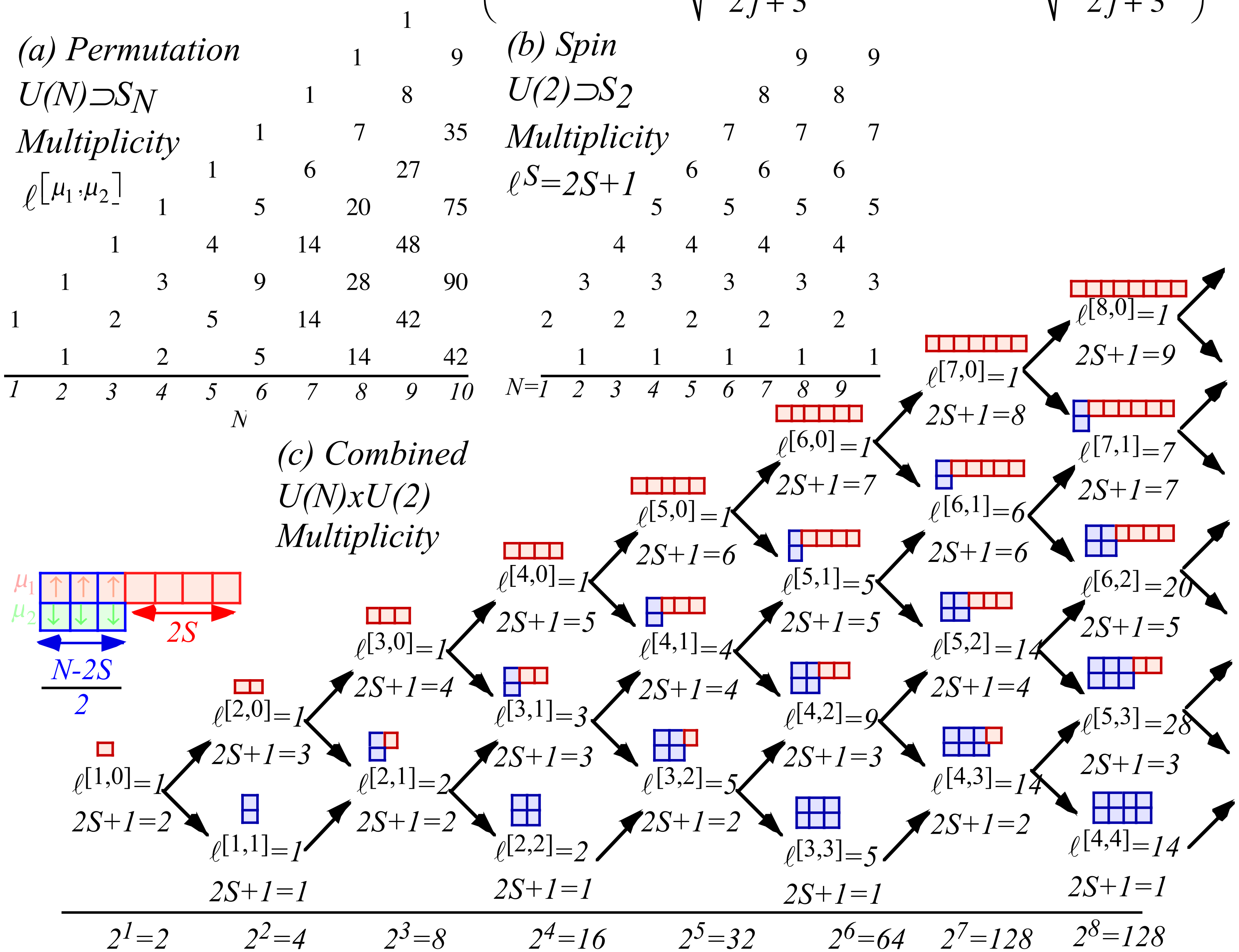
***Fig. 23.3.2** Spin-1/2 and U(2) Tableau branching diagrams*

Magic squares - Intro to Young Tableaus

$$\left|\begin{array}{cccccc}\uparrow&\uparrow&\uparrow&\uparrow&\uparrow&\downarrow\\ \downarrow&\downarrow&&&&\end{array}\ \begin{array}{c}S=2\\ M_S=1\end{array}\right\rangle = C^{5/2\ 1/2\ 2}_{1/2\ 1/2\ 1}\left|\begin{array}{cccccc}\uparrow&\uparrow&\uparrow&\uparrow&\downarrow&\downarrow\\ \downarrow&\cdot\cdot&&&&\end{array}\begin{array}{c}5/2\\ 1/2\end{array}\right\rangle\left|\uparrow^{1/2}_{1/2}\right\rangle + C^{5/2\ 1/2\ 2}_{3/2\ -1/2\ 1}\left|\begin{array}{cccccc}\uparrow&\uparrow&\uparrow&\uparrow&\uparrow&\downarrow\\ \downarrow&\cdot\cdot&&&&\end{array}\begin{array}{c}5/2\\ 1/2\end{array}\right\rangle\left|\downarrow^{1/2}_{-1/2}\right\rangle$$

$$+C^{3/2\ 1/2\ 2}_{1/2\ 1/2\ 1}\left|\begin{array}{cccccc}\uparrow&\uparrow&\uparrow&\uparrow&\downarrow&\cdot\cdot\\ \downarrow&\downarrow&&&&\end{array}\begin{array}{c}3/2\\ 1/2\end{array}\right\rangle\left|\uparrow^{1/2}_{1/2}\right\rangle + C^{3/2\ 1/2\ 2}_{3/2\ -1/2\ 1}\left|\begin{array}{cccccc}\uparrow&\uparrow&\uparrow&\uparrow&\uparrow&\cdot\cdot\\ \downarrow&\downarrow&&&&\end{array}\begin{array}{c}3/2\\ 1/2\end{array}\right\rangle\left|\downarrow^{1/2}_{-1/2}\right\rangle$$

$$\left(\begin{array}{cc} C^{j\ \ 1/2\ \ j+1/2}_{m\ 1/2\ \ m+1/2}=\sqrt{\dfrac{j+m+1}{2j+1}} & C^{j\ \ \ \ 1/2\ \ \ j+1/2}_{m+1\ -1/2\ \ m+1/2}=\sqrt{\dfrac{j-m}{2j+1}} \\ C^{j+1\ 1/2\ \ j+1/2}_{m\ \ \ \ 1/2\ \ m+1/2}=-\sqrt{\dfrac{j-m+1}{2j+3}} & C^{j+1\ \ \ 1/2\ \ \ j+1/2}_{m+1\ -1/2\ \ m+1/2}=\sqrt{\dfrac{j+m+2}{2j+3}} \end{array}\right)\quad \text{example:}\quad \left(\begin{array}{cc} C^{3/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=\sqrt{\dfrac{2}{3}} & C^{3/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=\sqrt{\dfrac{1}{3}} \\ C^{5/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=-\sqrt{\dfrac{1}{3}} & C^{5/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=\sqrt{\dfrac{2}{3}} \end{array}\right)$$



2	3	4	5
1	2	3	4

5	4	3	2
4	3	2	1

$$2^N = \sum_S^{N/2} \ell^{[S]}\ell^{[\mu_1,\mu_2]}$$

$$= \sum_S^{N/2}(2S+1)\ell^{\left[\frac{N+2S}{2},\frac{N-2S}{2}\right]}$$

$8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1$

5	4	3	2
4	3	2	1

***Fig. 23.3.2** Spin-1/2 and U(2) Tableau branching diagrams*

$$\left|\begin{array}{cccccc}\uparrow&\uparrow&\uparrow&\uparrow&\uparrow&\downarrow\\ \downarrow&\downarrow&&&&\end{array}\;\begin{array}{c}S=2\\ M_S=1\end{array}\right\rangle = C^{5/2\ 1/2\ 2}_{1/2\ 1/2\ 1}\left|\begin{array}{ccccccc}\uparrow&\uparrow&\uparrow&\uparrow&\downarrow&\downarrow&5/2\\ \downarrow&\cdot\cdot&&&&&1/2\end{array}\right\rangle\left|\uparrow^{1/2}_{1/2}\right\rangle + C^{5/2\ 1/2\ 2}_{3/2\ -1/2\ 1}\left|\begin{array}{ccccccc}\uparrow&\uparrow&\uparrow&\uparrow&\uparrow&\downarrow&5/2\\ \downarrow&\cdot\cdot&&&&&1/2\end{array}\right\rangle\left|\downarrow^{1/2}_{-1/2}\right\rangle$$

$$+C^{3/2\ 1/2\ 2}_{1/2\ 1/2\ 1}\left|\begin{array}{ccccccc}\uparrow&\uparrow&\uparrow&\uparrow&\downarrow&\cdot\cdot&3/2\\ \downarrow&\downarrow&&&&&1/2\end{array}\right\rangle\left|\uparrow^{1/2}_{1/2}\right\rangle + C^{3/2\ 1/2\ 2}_{3/2\ -1/2\ 1}\left|\begin{array}{ccccccc}\uparrow&\uparrow&\uparrow&\uparrow&\uparrow&\cdot\cdot&3/2\\ \downarrow&\downarrow&&&&&1/2\end{array}\right\rangle\left|\downarrow^{1/2}_{-1/2}\right\rangle$$

$$\left(\begin{array}{cc} C^{j\ \ 1/2\ \ j+1/2}_{m\ 1/2\ m+1/2}=\sqrt{\dfrac{j+m+1}{2j+1}} & C^{j\ \ \ 1/2\ \ j+1/2}_{m+1\ -1/2\ m+1/2}=\sqrt{\dfrac{j-m}{2j+1}}\\ C^{j+1\ 1/2\ j+1/2}_{m\ \ \ 1/2\ m+1/2}=-\sqrt{\dfrac{j-m+1}{2j+3}} & C^{j+1\ \ 1/2\ \ j+1/2}_{m+1\ -1/2\ m+1/2}=\sqrt{\dfrac{j+m+2}{2j+3}}\end{array}\right)$$

example:

$$\left(\begin{array}{cc} C^{3/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=\sqrt{\dfrac{2}{3}} & C^{3/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=\sqrt{\dfrac{1}{3}}\\ C^{5/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=-\sqrt{\dfrac{1}{3}} & C^{5/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=\sqrt{\dfrac{2}{3}}\end{array}\right)$$

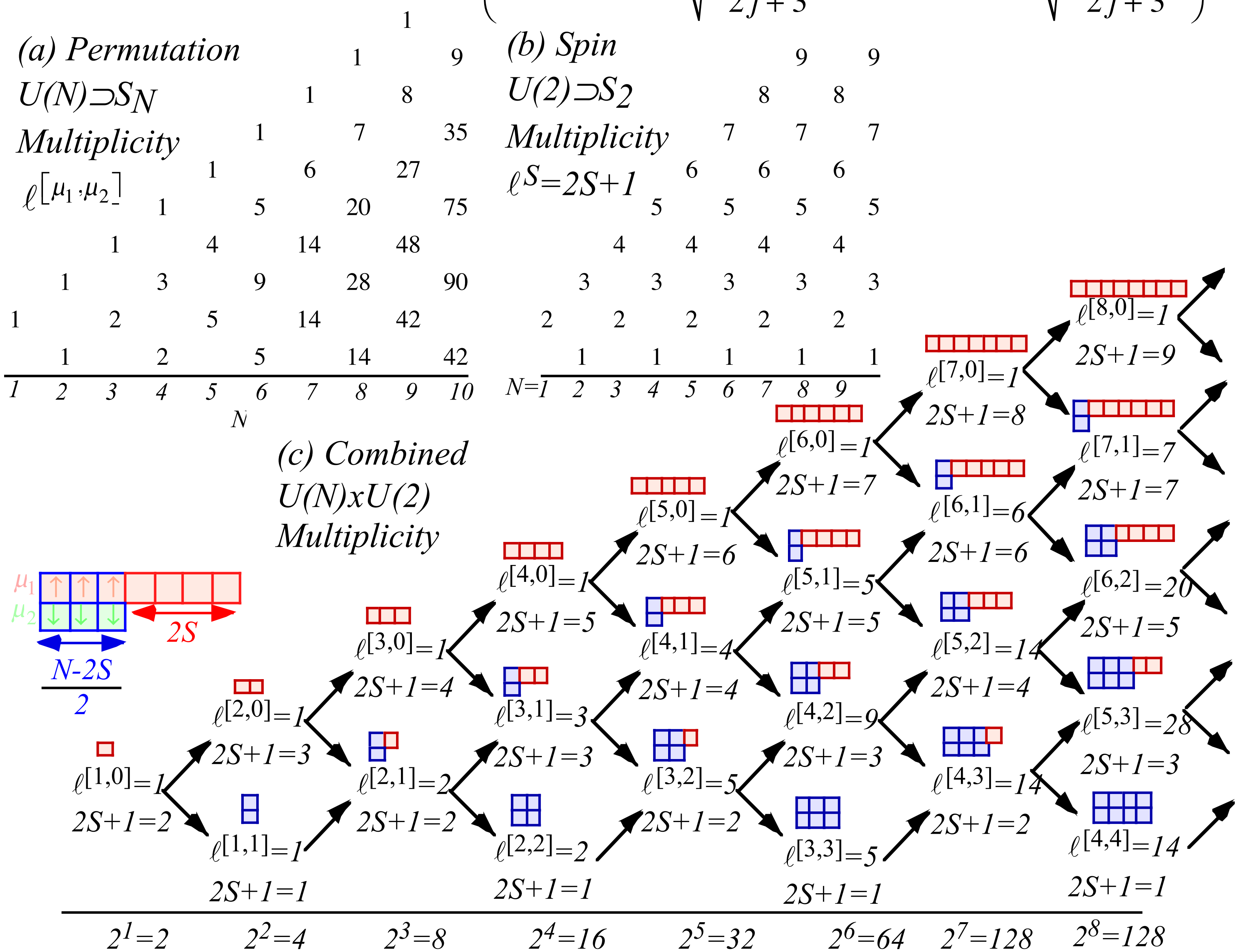


Fig. 23.3.2 Spin-1/2 and U(2) Tableau branching diagrams

2	3	4	5
1	2	3	4

5	4	3	2
4	3	2	1

$$2^N = \sum_S^{N/2} \ell^{[S]}\ell^{[\mu_1,\mu_2]}$$

$$= \sum_S^{N/2}(2S+1)\ell^{\left[\frac{N+2S}{2},\frac{N-2S}{2}\right]}$$

$8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1$

5	4	3	2
4	3	2	1

$$\left|\begin{array}{cccccc}\uparrow&\uparrow&\uparrow&\uparrow&\uparrow&\downarrow\\ \downarrow&\downarrow&&&&\end{array}\quad \begin{array}{c}S=2\\ M_S=1\end{array}\right\rangle = C^{5/2\ 1/2\ 2}_{1/2\ 1/2\ 1}\left|\begin{array}{cccccc}\uparrow&\uparrow&\uparrow&\uparrow&\downarrow&\downarrow\\ \downarrow&\cdot\cdot&&&&\end{array}\ \begin{array}{c}5/2\\ 1/2\end{array}\right\rangle\left|\uparrow^{1/2}_{1/2}\right\rangle + C^{5/2\ 1/2\ 2}_{3/2\ -1/2\ 1}\left|\begin{array}{cccccc}\uparrow&\uparrow&\uparrow&\uparrow&\uparrow&\downarrow\\ \downarrow&\cdot\cdot&&&&\end{array}\ \begin{array}{c}5/2\\ 1/2\end{array}\right\rangle\left|\downarrow^{1/2}_{-1/2}\right\rangle$$

$$+C^{3/2\ 1/2\ 2}_{1/2\ 1/2\ 1}\left|\begin{array}{cccccc}\uparrow&\uparrow&\uparrow&\uparrow&\downarrow&\cdot\cdot\\ \downarrow&\downarrow&&&&\end{array}\ \begin{array}{c}3/2\\ 1/2\end{array}\right\rangle\left|\uparrow^{1/2}_{1/2}\right\rangle + C^{3/2\ 1/2\ 2}_{3/2\ -1/2\ 1}\left|\begin{array}{cccccc}\uparrow&\uparrow&\uparrow&\uparrow&\uparrow&\cdot\cdot\\ \downarrow&\downarrow&&&&\end{array}\ \begin{array}{c}3/2\\ 1/2\end{array}\right\rangle\left|\downarrow^{1/2}_{-1/2}\right\rangle$$

$$\left(\begin{array}{cc} C^{j\ \ 1/2\ \ j+1/2}_{m\ \ 1/2\ \ m+1/2}=\sqrt{\dfrac{j+m+1}{2j+1}} & C^{j\ \ \ 1/2\ \ j+1/2}_{m+1\ -1/2\ \ m+1/2}=\sqrt{\dfrac{j-m}{2j+1}} \\ C^{j+1\ 1/2\ \ j+1/2}_{m\ \ \ 1/2\ \ m+1/2}=-\sqrt{\dfrac{j-m+1}{2j+3}} & C^{j+1\ \ 1/2\ \ j+1/2}_{m+1\ -1/2\ \ m+1/2}=\sqrt{\dfrac{j+m+2}{2j+3}}\end{array}\right)$$

example:

$$\left(\begin{array}{cc} C^{3/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=\sqrt{\dfrac{2}{3}} & C^{3/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=\sqrt{\dfrac{1}{3}} \\ C^{5/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=-\sqrt{\dfrac{1}{3}} & C^{5/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=\sqrt{\dfrac{2}{3}}\end{array}\right)$$

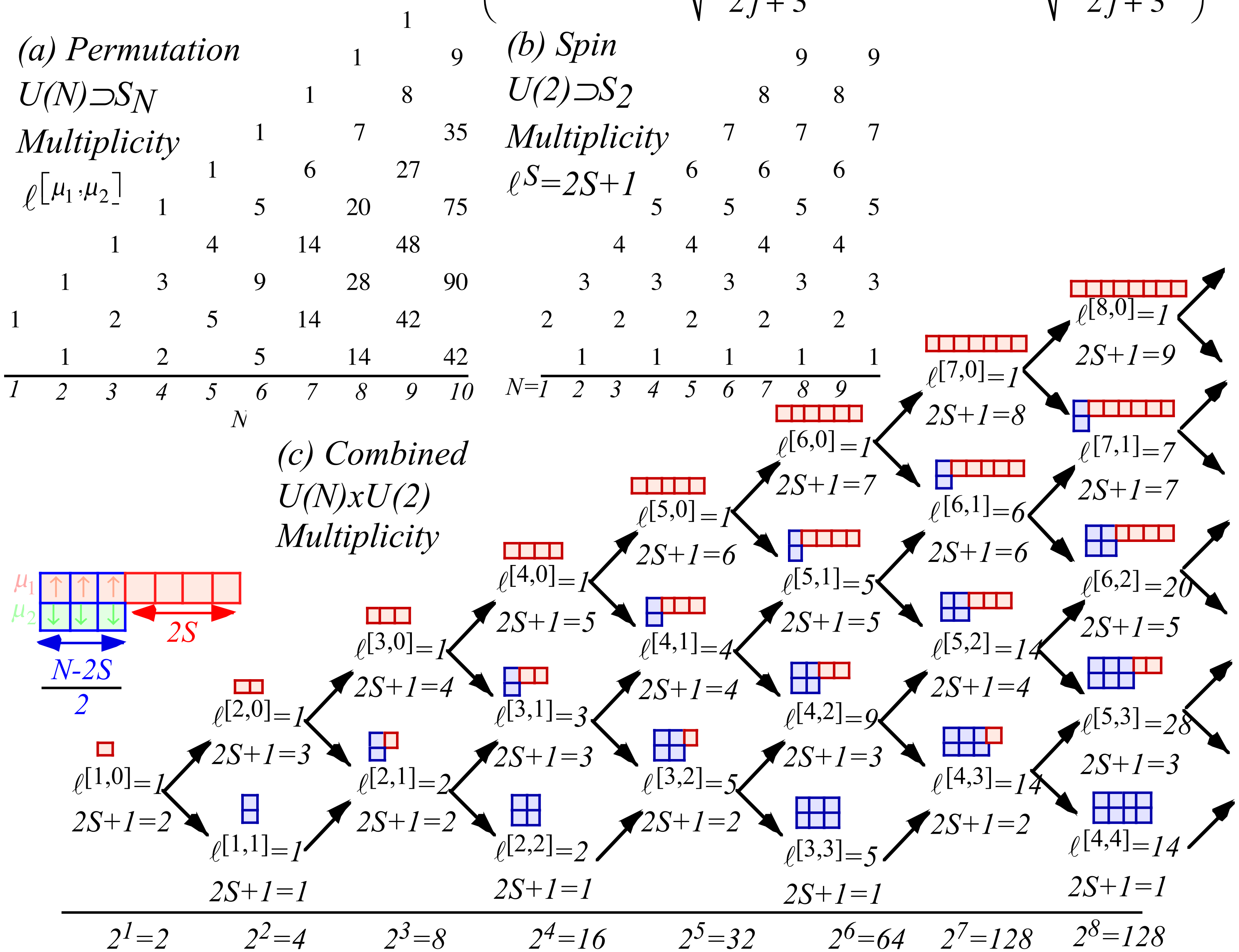


Fig. 23.3.2 Spin-1/2 and U(2) Tableau branching diagrams

$$\frac{\begin{array}{|c|c|c|c|}\hline 2&3&4&5\\\hline 1&2&3&4\\\hline\end{array}}{\begin{array}{|c|c|c|c|}\hline 5&4&3&2\\\hline 4&3&2&1\\\hline\end{array}}=1$$

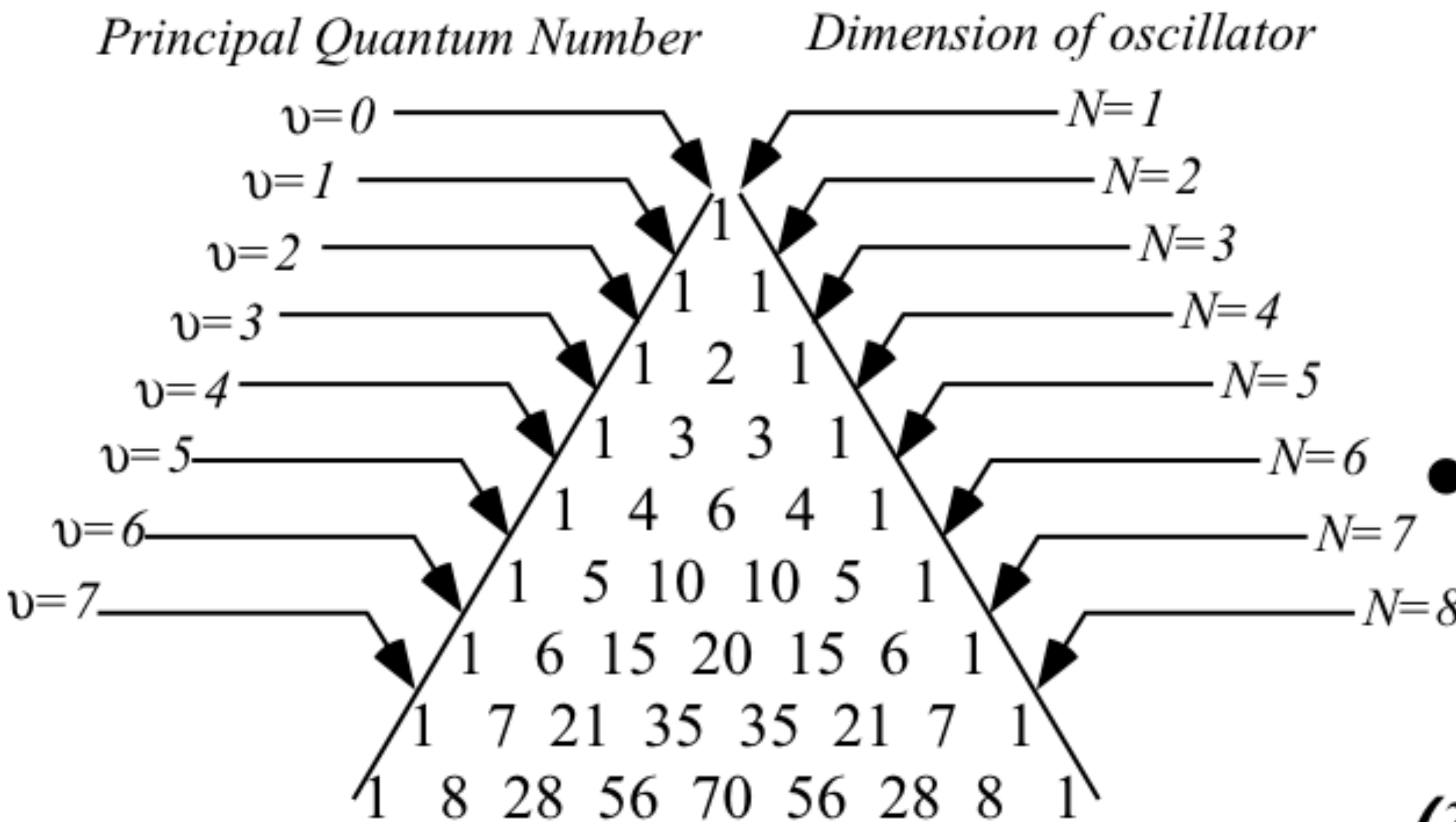
$$2^N=\sum_S^{N/2}\ell^{[S]}\ell^{[\mu_1,\mu_2]}$$

$$=\sum_S^{N/2}(2S+1)\ell^{\left[\frac{N+2S}{2},\frac{N-2S}{2}\right]}$$

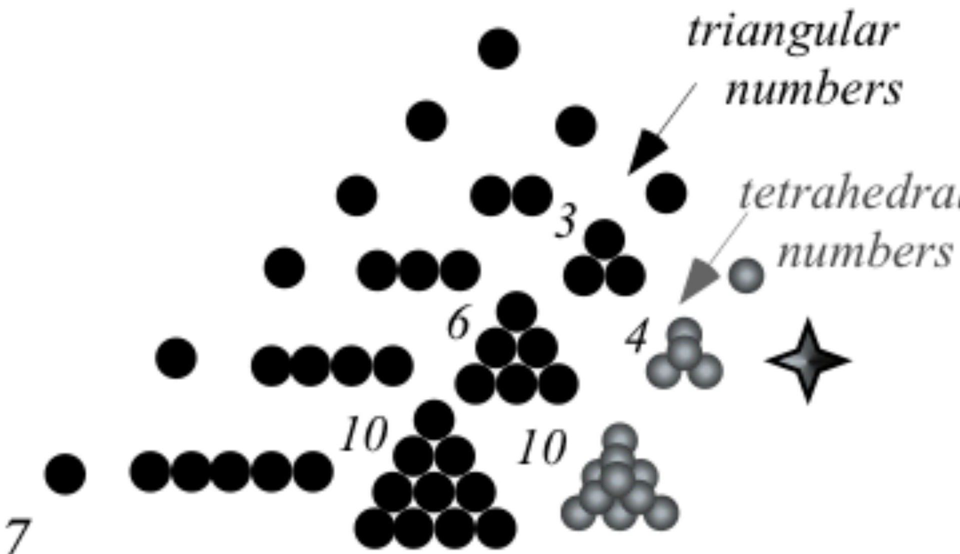
$$\frac{\begin{array}{c}2\\ \cancel{8}\cdot\cancel{7}\cdot\cancel{6}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}\end{array}}{\begin{array}{|c|c|c|c|}\hline \cancel{5}&\cancel{4}&\cancel{3}&\cancel{2}\\\hline \cancel{4}&\cancel{3}&\cancel{2}&1\\\hline\end{array}}=14$$

Introducing U(N)

(a) N-D Oscillator Degeneracy ℓ of quamtum level υ

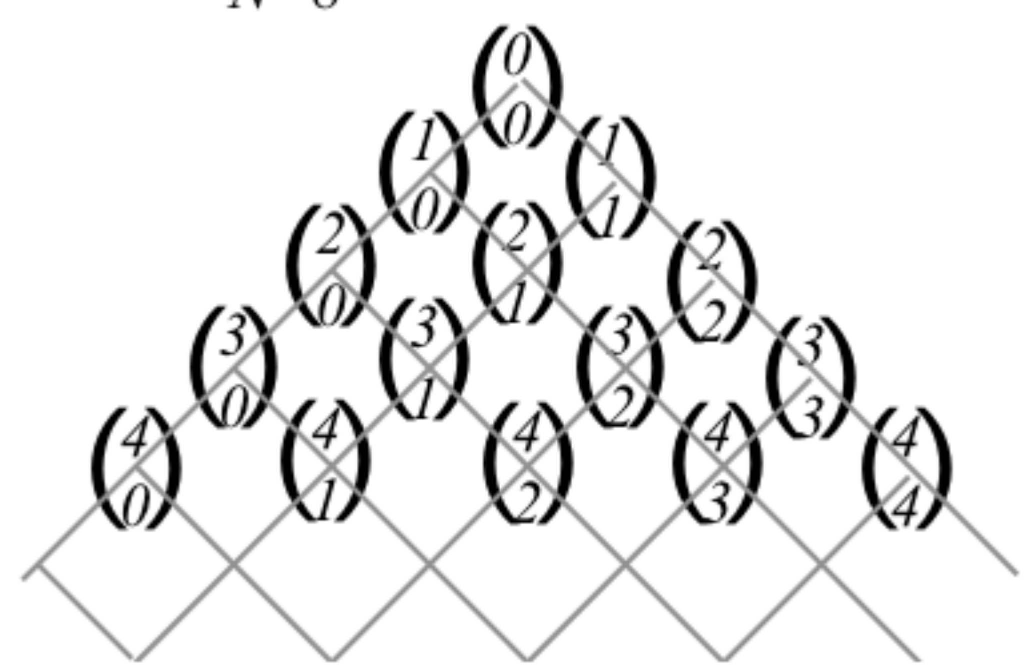


(b) Stacking numbers



(c) Binomial coefficients

$$\frac{(N-1+\upsilon)!}{(N-1)!\,\upsilon!} = \binom{N-1+\upsilon}{\upsilon} = \binom{N-1+\upsilon}{N-1}$$



Introducing U(3)

(b) N-particle 3-level states ...or spin-1 states

$\boxed{1} = |1\ 0\ 0\rangle = a_1^\dagger\, |0\ 0\ 0\rangle$

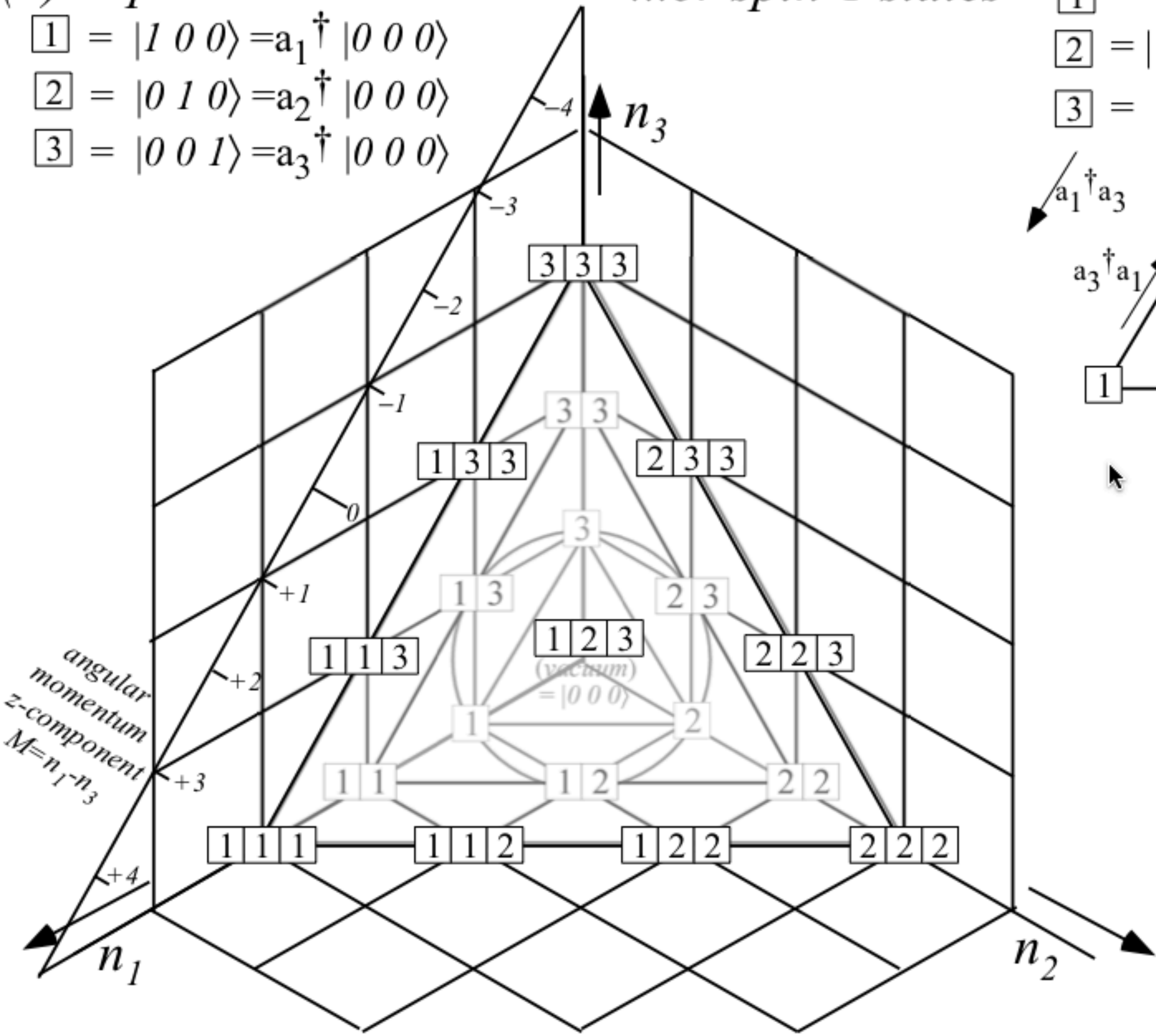
$\boxed{2} = |0\ 1\ 0\rangle = a_2^\dagger\, |0\ 0\ 0\rangle$

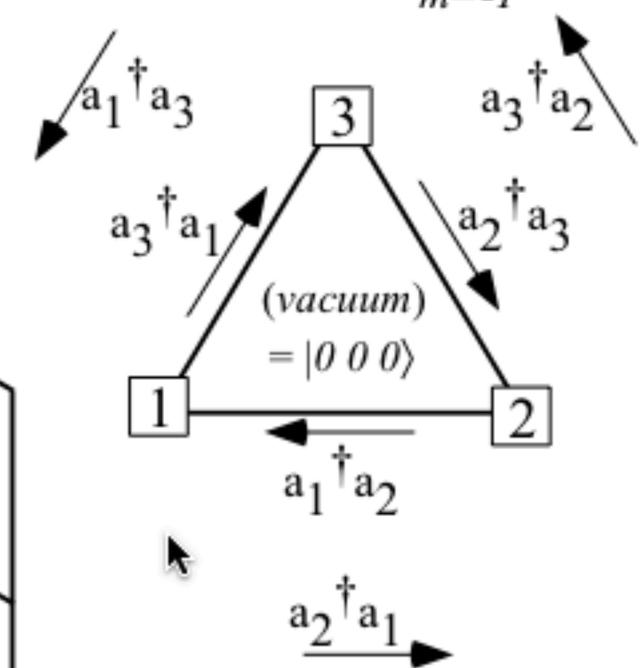
$\boxed{3} = |0\ 0\ 1\rangle = a_3^\dagger\, |0\ 0\ 0\rangle$

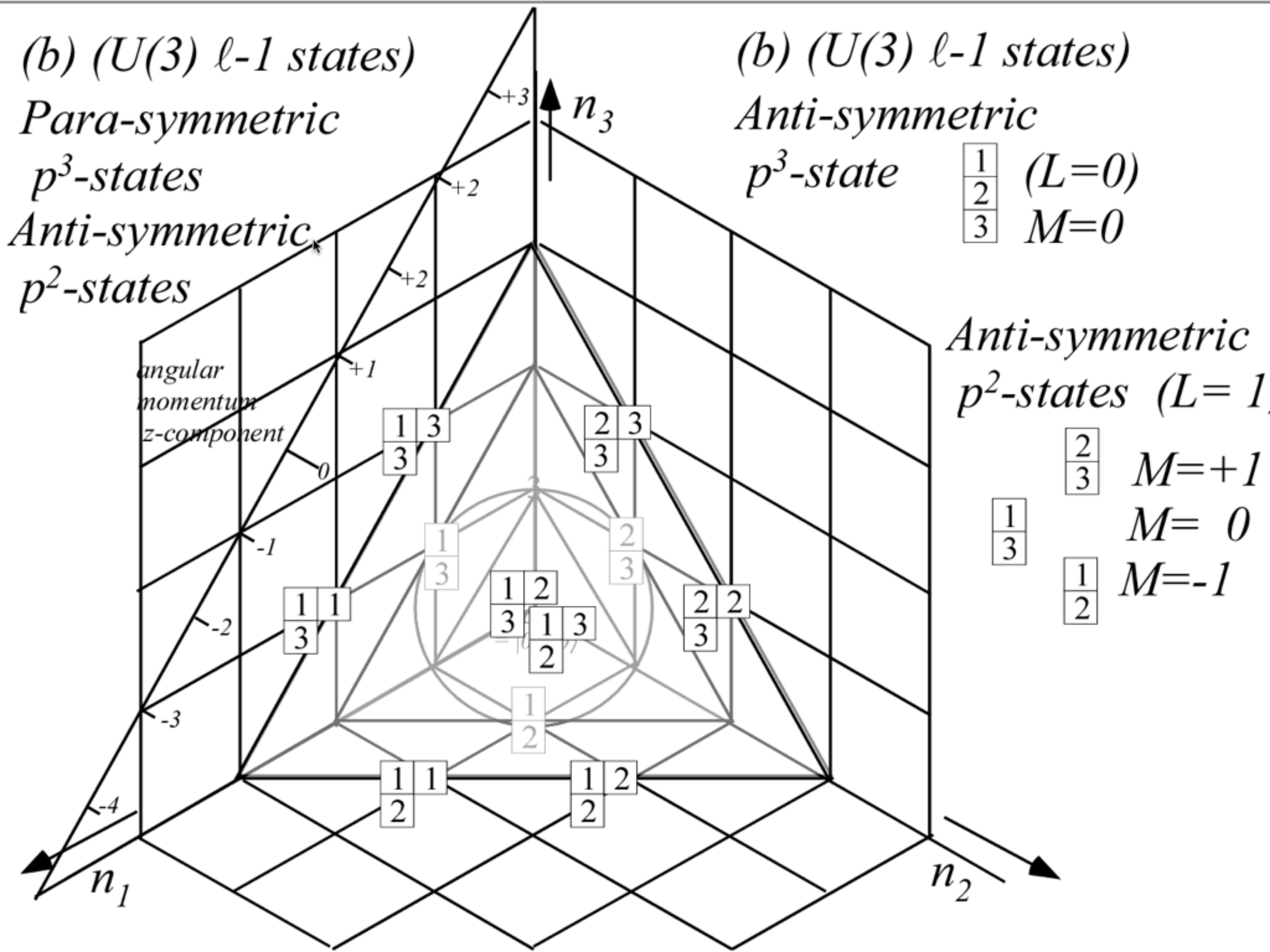
$\boxed{1} = |\uparrow\rangle = \left|{j=1 \atop m=+1}\right\rangle$

$\boxed{2} = |\leftrightarrow\rangle = \left|{j=1 \atop m=0}\right\rangle$

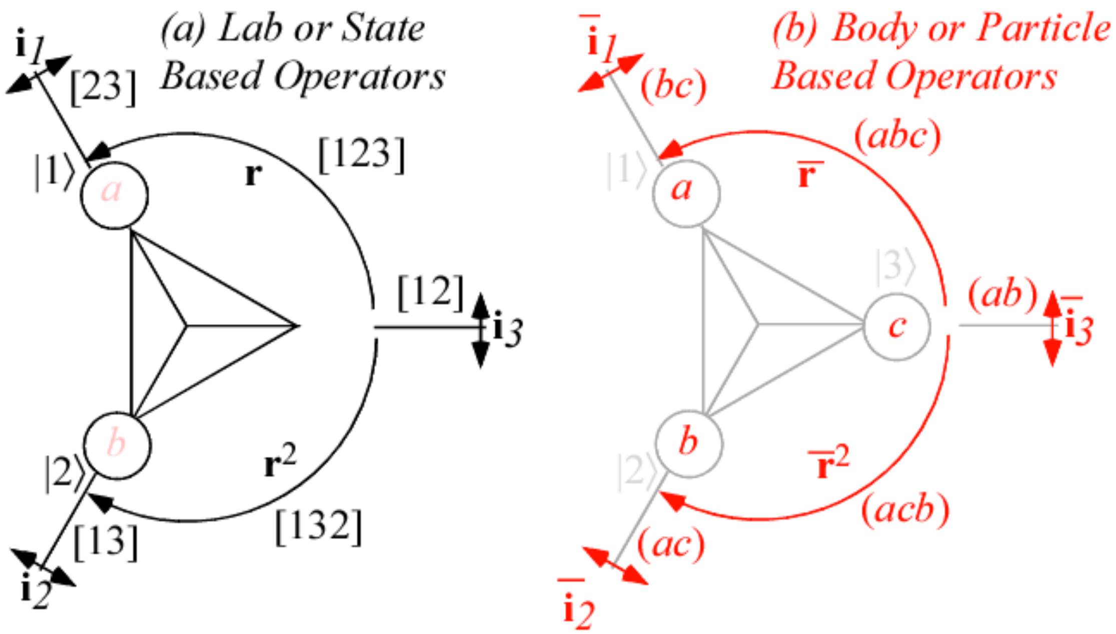
$\boxed{3} = |\downarrow\rangle = \left|{j=1 \atop m=-1}\right\rangle$

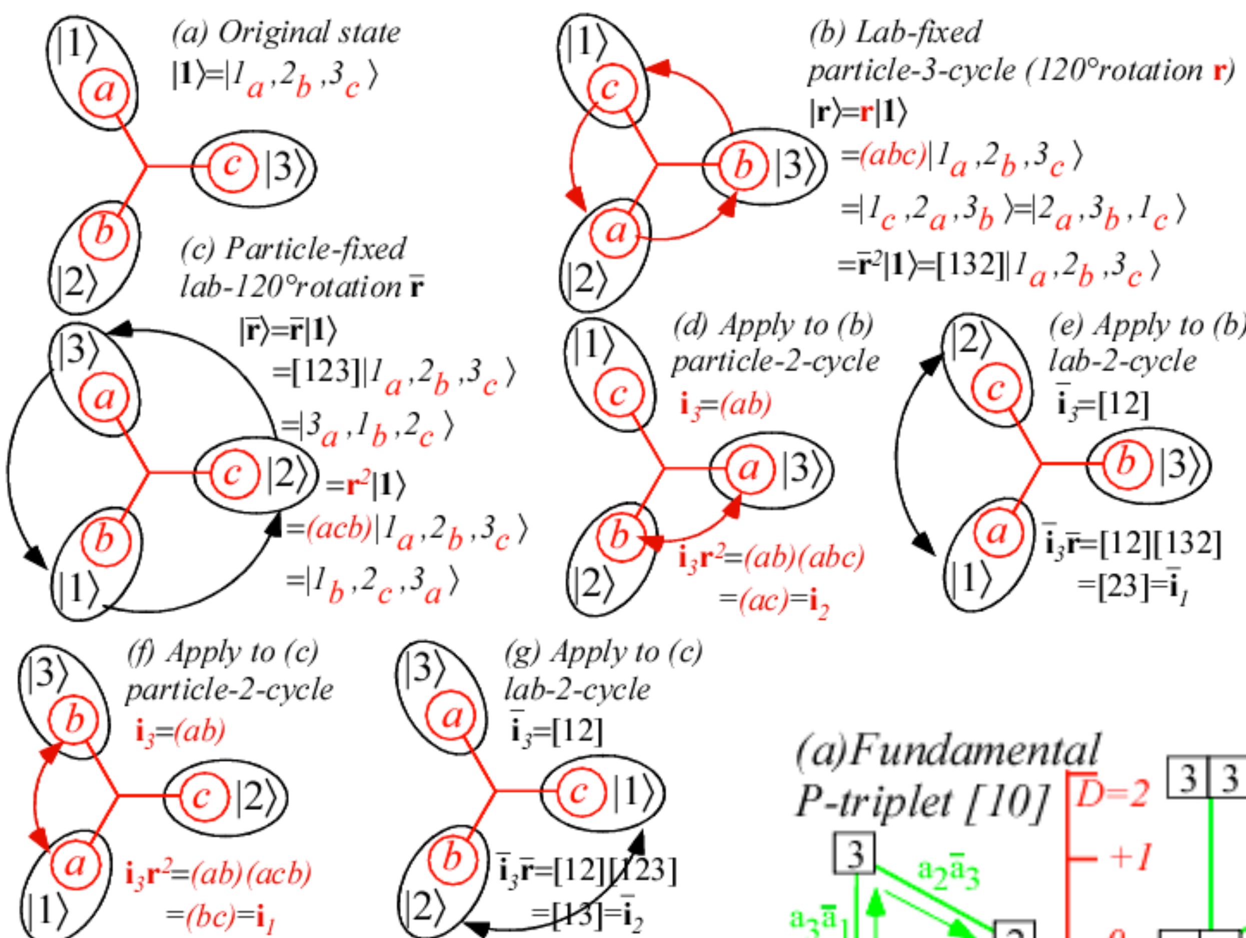


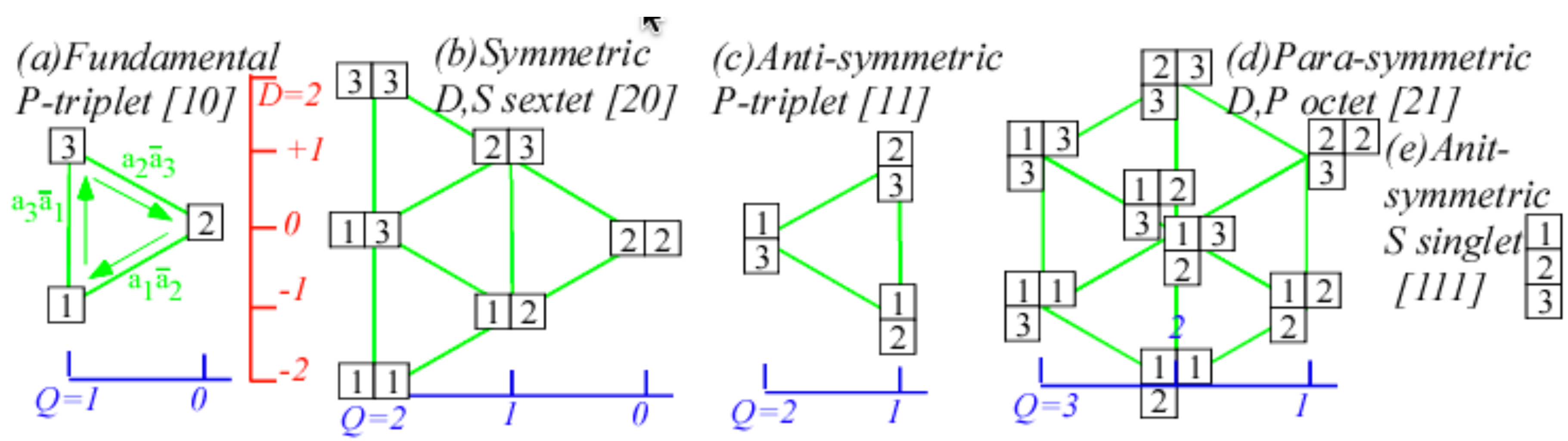




(b) (U(3) ℓ-1 states)
Para-symmetric
p^3-states
Anti-symmetric
p^2-states
angular
momentum
z-component
+3
+2
+2
+1
0
-1
-2
-3
-4
n_3
n_1
n_2
(b) (U(3) ℓ-1 states)
Anti-symmetric
p^3-state
(L=0)
M=0
Anti-symmetric
p^2-states (L= 1)
M=+1
M= 0
M=-1


(a) Lab or State Based Operators
i1
[23]
[123]
r
|1⟩
a
[12]
i3
b
|2⟩
r2
[13]
[132]
i2
(b) Body or Particle Based Operators
ī1
(bc)
(abc)
r̄
|1⟩
a
|3⟩
c
(ab)
ī3
b
|2⟩
r̄2
(acb)
(ac)
ī2


(a) Original state
|1⟩=|1a,2b,3c⟩
|1⟩ a
c |3⟩
b |2⟩
(b) Lab-fixed particle-3-cycle (120°rotation r)
|r⟩=r|1⟩
=(abc)|1a,2b,3c⟩
=|1c,2a,3b⟩=|2a,3b,1c⟩
=r̄2|1⟩=[132]|1a,2b,3c⟩
(c) Particle-fixed lab-120°rotation r̄
|r̄⟩=r̄|1⟩
=[123]|1a,2b,3c⟩
=|3a,1b,2c⟩
=r2|1⟩
=(acb)|1a,2b,3c⟩
=|1b,2c,3a⟩
(d) Apply to (b) particle-2-cycle
i3=(ab)
i3r2=(ab)(abc)
=(ac)=i2
(e) Apply to (b) lab-2-cycle
ī3=[12]
ī3r̄=[12][132]
=[23]=ī1
(f) Apply to (c) particle-2-cycle
i3=(ab)
i3r2=(ab)(acb)
=(bc)=i1
(g) Apply to (c) lab-2-cycle
ī3=[12]
ī3r̄=[12][123]
=[13]=ī2


(a)Fundamental P-triplet [10]
a2ā3
a3ā1
a1ā2
Q=1
0
D=2
+1
0
-1
-2
(b)Symmetric D,S sextet [20]
Q=2
1
0
(c)Anti-symmetric P-triplet [11]
Q=2
1
(d)Para-symmetric D,P octet [21]
Q=3
2
1
(e)Anit-symmetric S singlet [111]

Tensor operators for spin-1/2 states: Outer products give Hamilton-Pauli-spinors

CG-Products of spin-$^1/_2$ **ket-bras** $\left\{ \left| \begin{smallmatrix} 1/2 \\ m_1 \end{smallmatrix} \right\rangle, \left\langle \begin{smallmatrix} 1/2 \\ m_2 \end{smallmatrix} \right| \right\}$ give scalar/vector operators analogous to: **ket-kets**

$$T_q^k = \sum_{m_1} C_{m_1\ m_2\ q}^{1/2\ 1/2\ k} \left| \begin{matrix} 1/2 \\ m_1 \end{matrix} \right\rangle \left\langle \begin{matrix} 1/2 \\ -m_2 \end{matrix} \right| (-1)^{\frac{1}{2}-m_2}$$

analogous to:

$$\left| \begin{matrix} J \\ M \end{matrix} (1/2 \otimes 1/2) \right\rangle = \sum_{m_1, m_2} C_{m_1\ m_2\ M}^{1/2\ 1/2\ J} \left| \begin{matrix} 1/2 \\ m_1 \end{matrix} \right\rangle \left| \begin{matrix} 1/2 \\ m_2 \end{matrix} \right\rangle$$

$$T_{-1}^1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = -\left| \begin{matrix} 1/2 \\ -1/2 \end{matrix} \right\rangle \left\langle \begin{matrix} 1/2 \\ 1/2 \end{matrix} \right|, \quad T_0^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\frac{1}{\sqrt{2}} \left[\left| \begin{matrix} 1/2 \\ 1/2 \end{matrix} \right\rangle \left\langle \begin{matrix} 1/2 \\ 1/2 \end{matrix} \right| - \left| \begin{matrix} 1/2 \\ -1/2 \end{matrix} \right\rangle \left\langle \begin{matrix} 1/2 \\ -1/2 \end{matrix} \right| \right], \quad T_1^1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \left| \begin{matrix} 1/2 \\ 1/2 \end{matrix} \right\rangle \left\langle \begin{matrix} 1/2 \\ -1/2 \end{matrix} \right|,$$

analogous to:

$$\left| \begin{matrix} 1 \\ 1 \end{matrix} (1/2 \otimes 1/2) \right\rangle = \left| \begin{matrix} 1/2 \\ 1/2 \end{matrix} \right\rangle \left| \begin{matrix} 1/2 \\ 1/2 \end{matrix} \right\rangle$$

$$\left| \begin{matrix} 1 \\ 0 \end{matrix} (1/2 \otimes 1/2) \right\rangle = \frac{1}{\sqrt{2}} \left| \begin{matrix} 1/2 \\ 1/2 \end{matrix} \right\rangle \left| \begin{matrix} 1/2 \\ -1/2 \end{matrix} \right\rangle + \frac{1}{\sqrt{2}} \left| \begin{matrix} 1/2 \\ -1/2 \end{matrix} \right\rangle \left| \begin{matrix} 1/2 \\ 1/2 \end{matrix} \right\rangle$$

$$\left| \begin{matrix} 1 \\ -1 \end{matrix} (1/2 \otimes 1/2) \right\rangle = \left| \begin{matrix} 1/2 \\ -1/2 \end{matrix} \right\rangle \left| \begin{matrix} 1/2 \\ -1/2 \end{matrix} \right\rangle$$

$$T_0^0 = -\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\frac{1}{\sqrt{2}} \left[\left| \begin{matrix} 1/2 \\ 1/2 \end{matrix} \right\rangle \left\langle \begin{matrix} 1/2 \\ 1/2 \end{matrix} \right| + \left| \begin{matrix} 1/2 \\ -1/2 \end{matrix} \right\rangle \left\langle \begin{matrix} 1/2 \\ -1/2 \end{matrix} \right| \right].$$

analogous to:

$$\left| \begin{matrix} 0 \\ 0 \end{matrix} (1/2 \otimes 1/2) \right\rangle = \frac{1}{\sqrt{2}} \left| \begin{matrix} 1/2 \\ 1/2 \end{matrix} \right\rangle \left| \begin{matrix} 1/2 \\ -1/2 \end{matrix} \right\rangle + \frac{-1}{\sqrt{2}} \left| \begin{matrix} 1/2 \\ -1/2 \end{matrix} \right\rangle \left| \begin{matrix} 1/2 \\ 1/2 \end{matrix} \right\rangle$$

1st three operators are a *vector* set with following Cartesian combinations:

$$T_x \equiv -\frac{T_{-1}^1 - T_1^1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \equiv \frac{1}{\sqrt{2}} \sigma_x \equiv \sqrt{2} J_x$$

$$T_y \equiv -i\frac{T_{-1}^1 + T_1^1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \equiv \frac{1}{\sqrt{2}} \sigma_y \equiv \sqrt{2} J_y$$

$$T_z \equiv -T_0^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv \frac{1}{\sqrt{2}} \sigma_z \equiv \sqrt{2} J_z$$

(Some old friends!)

$$\sigma_X \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_Y \to \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_Z \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

Spherical vs. Cartesian operators

$$T_{-1}^1 = J_-/2 = \left(J_x - iJ_y\right)/\sqrt{2}, \qquad T_0^1 = J_z/\sqrt{2}, \qquad T_{-1}^1 = J_+/2 = \left(J_x + iJ_y\right)/2.$$

Tensor operators for spin-1/2 states: Outer products give Hamilton-Pauli-spinors

CG-Products of spin-$^1/_2$ **ket-bras** $\left\{ \left|\begin{matrix}1/2\\ m_1\end{matrix}\right\rangle, \left\langle\begin{matrix}1/2\\ m_2\end{matrix}\right| \right\}$ give scalar/vector operators analogous to: **ket-kets**

$$T_q^k = \sum_{m_1} C_{m_1\ m_2\ q}^{1/2\ 1/2\ k} \left|\begin{matrix}1/2\\ m_1\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\ -m_2\end{matrix}\right| (-1)^{\frac{1}{2}-m_2} \quad \text{analogous to:} \quad \left|\begin{matrix}J\\ M\end{matrix}(1/2\otimes 1/2)\right\rangle = \sum_{m_1,m_2} C_{m_1\ m_2\ M}^{1/2\ 1/2\ J} \left|\begin{matrix}1/2\\ m_1\end{matrix}\right\rangle\left|\begin{matrix}1/2\\ m_2\end{matrix}\right\rangle$$

$$T_{-1}^1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = -\left|\begin{matrix}1/2\\ -1/2\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\ 1/2\end{matrix}\right|, \quad T_0^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\frac{1}{\sqrt{2}}\left[\left|\begin{matrix}1/2\\ 1/2\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\ 1/2\end{matrix}\right| - \left|\begin{matrix}1/2\\ -1/2\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\ -1/2\end{matrix}\right|\right], \quad T_1^1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \left|\begin{matrix}1/2\\ 1/2\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\ -1/2\end{matrix}\right|,$$

analogous to:

$$\left|\begin{matrix}1\\ 1\end{matrix}(1/2\otimes 1/2)\right\rangle = \left|\begin{matrix}1/2\\ 1/2\end{matrix}\right\rangle\left|\begin{matrix}1/2\\ 1/2\end{matrix}\right\rangle$$

$$\left|\begin{matrix}1\\ 0\end{matrix}(1/2\otimes 1/2)\right\rangle = \frac{1}{\sqrt{2}}\left|\begin{matrix}1/2\\ 1/2\end{matrix}\right\rangle\left|\begin{matrix}1/2\\ -1/2\end{matrix}\right\rangle + \frac{1}{\sqrt{2}}\left|\begin{matrix}1/2\\ -1/2\end{matrix}\right\rangle\left|\begin{matrix}1/2\\ 1/2\end{matrix}\right\rangle$$

$$\left|\begin{matrix}1\\ -1\end{matrix}(1/2\otimes 1/2)\right\rangle = \left|\begin{matrix}1/2\\ -1/2\end{matrix}\right\rangle\left|\begin{matrix}1/2\\ -1/2\end{matrix}\right\rangle$$

$$T_0^0 = -\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\frac{1}{\sqrt{2}}\left[\left|\begin{matrix}1/2\\ 1/2\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\ 1/2\end{matrix}\right| + \left|\begin{matrix}1/2\\ -1/2\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\ -1/2\end{matrix}\right|\right].$$

analogous to:

$$\left|\begin{matrix}0\\ 0\end{matrix}(1/2\otimes 1/2)\right\rangle = \frac{1}{\sqrt{2}}\left|\begin{matrix}1/2\\ 1/2\end{matrix}\right\rangle\left|\begin{matrix}1/2\\ -1/2\end{matrix}\right\rangle + \frac{-1}{\sqrt{2}}\left|\begin{matrix}1/2\\ -1/2\end{matrix}\right\rangle\left|\begin{matrix}1/2\\ 1/2\end{matrix}\right\rangle$$

1st three operators are a *vector* set that *transform like a vector set*

$$R(0\beta 0) \quad T_0^1 \quad R^\dagger(0\beta 0) \quad = \quad T_0'$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & \sin\frac{\beta}{2} \\ -\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} = -\frac{1}{\sqrt{2}}\begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{pmatrix}$$

$$= D_{10}^1(0\beta 0)T_1^1 \quad + D_{00}^1(0\beta 0)T_0^1 \quad + D_{-10}^1(0\beta 0)T_{-1}^1$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$= \frac{-\sin\beta}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \cos\beta\begin{pmatrix} -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix} + \frac{\sin\beta}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

Tensor operators for spin-1/2 states: Outer products give Hamilton-Pauli-spinors

CG-Products of spin-$^1/_2$ **ket-bras** $\left\{ \left|\begin{smallmatrix}1/2\\ m_1\end{smallmatrix}\right\rangle, \left\langle\begin{smallmatrix}1/2\\ m_2\end{smallmatrix}\right| \right\}$ give scalar/vector operators analogous to: **ket-kets**

$$T^k_q = \sum_{m_1} C^{1/2\ 1/2\ k}_{m_1\ m_2\ q} \left|\begin{matrix}1/2\\ m_1\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\ -m_2\end{matrix}\right| (-1)^{\frac{1}{2}-m_2}$$

analogous to:

$$\left|\begin{matrix}J\\ M\end{matrix}(1/2\otimes 1/2)\right\rangle = \sum_{m_1,m_2} C^{1/2\ 1/2\ J}_{m_1\ m_2\ M} \left|\begin{matrix}1/2\\ m_1\end{matrix}\right\rangle\left|\begin{matrix}1/2\\ m_2\end{matrix}\right\rangle$$

$$T^1_{-1} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = -\left|\begin{matrix}1/2\\ -1/2\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\ 1/2\end{matrix}\right|,$$

$$T^1_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\frac{1}{\sqrt{2}}\left[\left|\begin{matrix}1/2\\ 1/2\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\ 1/2\end{matrix}\right| - \left|\begin{matrix}1/2\\ -1/2\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\ -1/2\end{matrix}\right|\right],$$

$$T^1_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \left|\begin{matrix}1/2\\ 1/2\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\ -1/2\end{matrix}\right|,$$

analogous to:

$$\left|\begin{matrix}1\\ 1\end{matrix}(1/2\otimes 1/2)\right\rangle = \left|\begin{matrix}1/2\\ 1/2\end{matrix}\right\rangle\left|\begin{matrix}1/2\\ 1/2\end{matrix}\right\rangle$$

$$\left|\begin{matrix}1\\ 0\end{matrix}(1/2\otimes 1/2)\right\rangle = \frac{1}{\sqrt{2}}\left|\begin{matrix}1/2\\ 1/2\end{matrix}\right\rangle\left|\begin{matrix}1/2\\ -1/2\end{matrix}\right\rangle + \frac{1}{\sqrt{2}}\left|\begin{matrix}1/2\\ -1/2\end{matrix}\right\rangle\left|\begin{matrix}1/2\\ 1/2\end{matrix}\right\rangle$$

$$\left|\begin{matrix}1\\ -1\end{matrix}(1/2\otimes 1/2)\right\rangle = \left|\begin{matrix}1/2\\ -1/2\end{matrix}\right\rangle\left|\begin{matrix}1/2\\ -1/2\end{matrix}\right\rangle$$

$$T^0_0 = -\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\frac{1}{\sqrt{2}}\left[\left|\begin{matrix}1/2\\ 1/2\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\ 1/2\end{matrix}\right| + \left|\begin{matrix}1/2\\ -1/2\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\ -1/2\end{matrix}\right|\right].$$

analogous to:

$$\left|\begin{matrix}0\\ 0\end{matrix}(1/2\otimes 1/2)\right\rangle = \frac{1}{\sqrt{2}}\left|\begin{matrix}1/2\\ 1/2\end{matrix}\right\rangle\left|\begin{matrix}1/2\\ -1/2\end{matrix}\right\rangle + \frac{-1}{\sqrt{2}}\left|\begin{matrix}1/2\\ -1/2\end{matrix}\right\rangle\left|\begin{matrix}1/2\\ 1/2\end{matrix}\right\rangle$$

1st three operators are a *vector* set that *transform like a vector set*

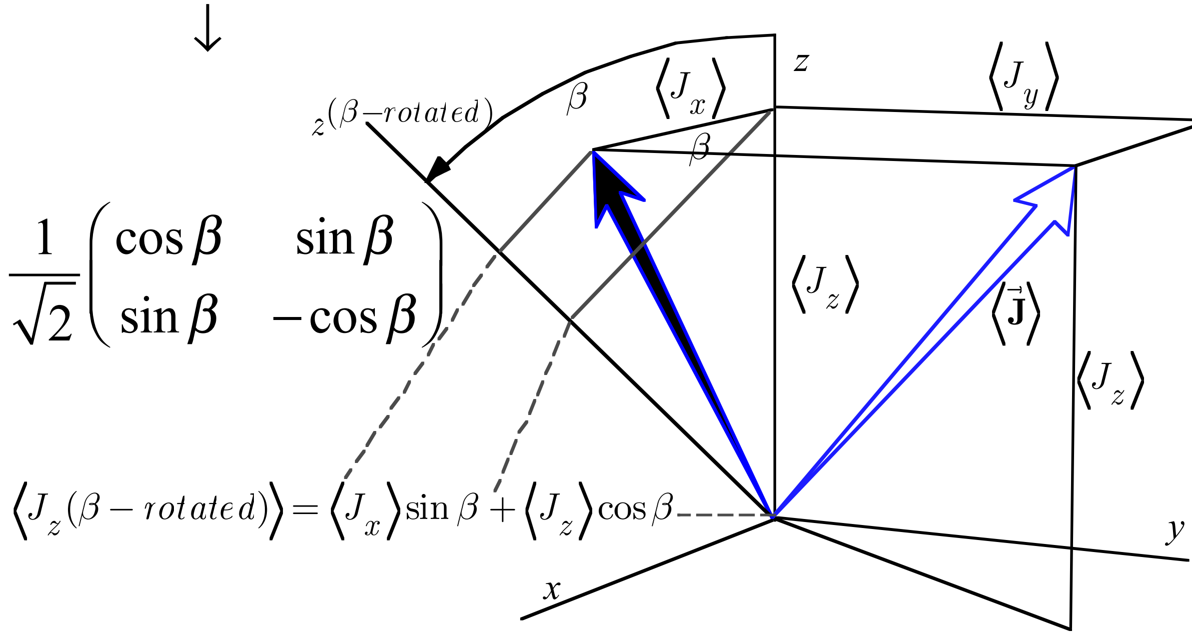
so do expectation values

$$R(0\beta 0) \quad T^1_0 \quad R^\dagger(0\beta 0) \quad = \quad T'_0$$

$$\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & \sin\frac{\beta}{2} \\ -\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} = -\frac{1}{\sqrt{2}}\begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{pmatrix}$$

$$= D^1_{10}(0\beta 0)T^1_1 + D^1_{00}(0\beta 0)T^1_0 + D^1_{-10}(0\beta 0)T^1_{-1}$$

$$= \frac{-\sin\beta}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \cos\beta\begin{pmatrix} -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix} + \frac{\sin\beta}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$



Tensor operators for spin-1 states: U(1) generalization of Pauli spinors

CGC definition:

$$\mathbf{v}^k_q = \sum_{m,m'} C^{j\ j\ \ \ k}_{m\ -m'\ q}(-1)^{j-m'}\left|{j \atop m}\right\rangle\left\langle{j \atop m'}\right| = (-1)^{2j}\,T^k_q .$$

Wigner 3jm definition:

$$\mathbf{v}^k_q = \sum_{m,m'} (-1)^{j-m}\sqrt{2k+1}\begin{pmatrix} k & j & j \\ q & m' & -m \end{pmatrix}\left|{j \atop m}\right\rangle\left\langle{j \atop m'}\right|$$

$$T^2_{-2} = \left|{1 \atop -1}\right\rangle\left\langle{1 \atop 1}\right|,\quad T^2_{-1} = \frac{\left|{1 \atop 0}\right\rangle\left\langle{1 \atop 1}\right| - \left|{1 \atop -1}\right\rangle\left\langle{1 \atop 0}\right|}{\sqrt{2}},\quad T^2_0 = \frac{\left|{1 \atop 1}\right\rangle\left\langle{1 \atop 1}\right| - 2\left|{1 \atop 0}\right\rangle\left\langle{1 \atop 0}\right| + \left|{1 \atop -1}\right\rangle\left\langle{1 \atop -1}\right|}{\sqrt{6}},\quad T^2_1 = \frac{-\left|{1 \atop 1}\right\rangle\left\langle{1 \atop 0}\right| + \left|{1 \atop 0}\right\rangle\left\langle{1 \atop -1}\right|}{\sqrt{2}},\quad T^2_2 = \left|{1 \atop 1}\right\rangle\left\langle{1 \atop -1}\right|$$

$$\rightarrow\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\quad \rightarrow\begin{pmatrix} 0 & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 \\ 0 & -1/\sqrt{2} & 0 \end{pmatrix}\quad \rightarrow\begin{pmatrix} 1/\sqrt{6} & 0 & 0 \\ 0 & -2/\sqrt{6} & 0 \\ 0 & 0 & 1/\sqrt{6} \end{pmatrix}\quad \rightarrow\begin{pmatrix} 0 & -1/\sqrt{2} & 0 \\ 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}\quad \rightarrow\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T^1_{-1} = \frac{\left|{1 \atop 0}\right\rangle\left\langle{1 \atop 1}\right| + \left|{1 \atop -1}\right\rangle\left\langle{1 \atop 0}\right|}{\sqrt{2}},\quad T^1_0 = \frac{\left|{1 \atop 1}\right\rangle\left\langle{1 \atop 1}\right| - \left|{1 \atop -1}\right\rangle\left\langle{1 \atop -1}\right|}{\sqrt{2}},\quad T^1_1 = \frac{-\left|{1 \atop 1}\right\rangle\left\langle{1 \atop 0}\right| - \left|{1 \atop 0}\right\rangle\left\langle{1 \atop -1}\right|}{\sqrt{2}}$$

$$\rightarrow\begin{pmatrix} 0 & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \end{pmatrix}\quad \rightarrow\begin{pmatrix} 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1/\sqrt{2} \end{pmatrix}\quad \rightarrow\begin{pmatrix} 0 & -1/\sqrt{2} & 0 \\ 0 & 0 & -1/\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$T^0_0 = \frac{\left|{1 \atop 1}\right\rangle\left\langle{1 \atop 1}\right| + \left|{1 \atop 0}\right\rangle\left\langle{1 \atop 0}\right| + \left|{1 \atop -1}\right\rangle\left\langle{1 \atop -1}\right|}{\sqrt{2}}$$

$$\rightarrow\begin{pmatrix} 1/\sqrt{3} & 0 & 0 \\ 0 & 1/\sqrt{3} & 0 \\ 0 & 0 & 1/\sqrt{3} \end{pmatrix}$$

These T^k_q operators are combinations of Elementary operators E_{mn}

Example:

$$T^2_1 = \frac{-1}{\sqrt{2}}E_{12} + \frac{1}{\sqrt{2}}E_{23}$$

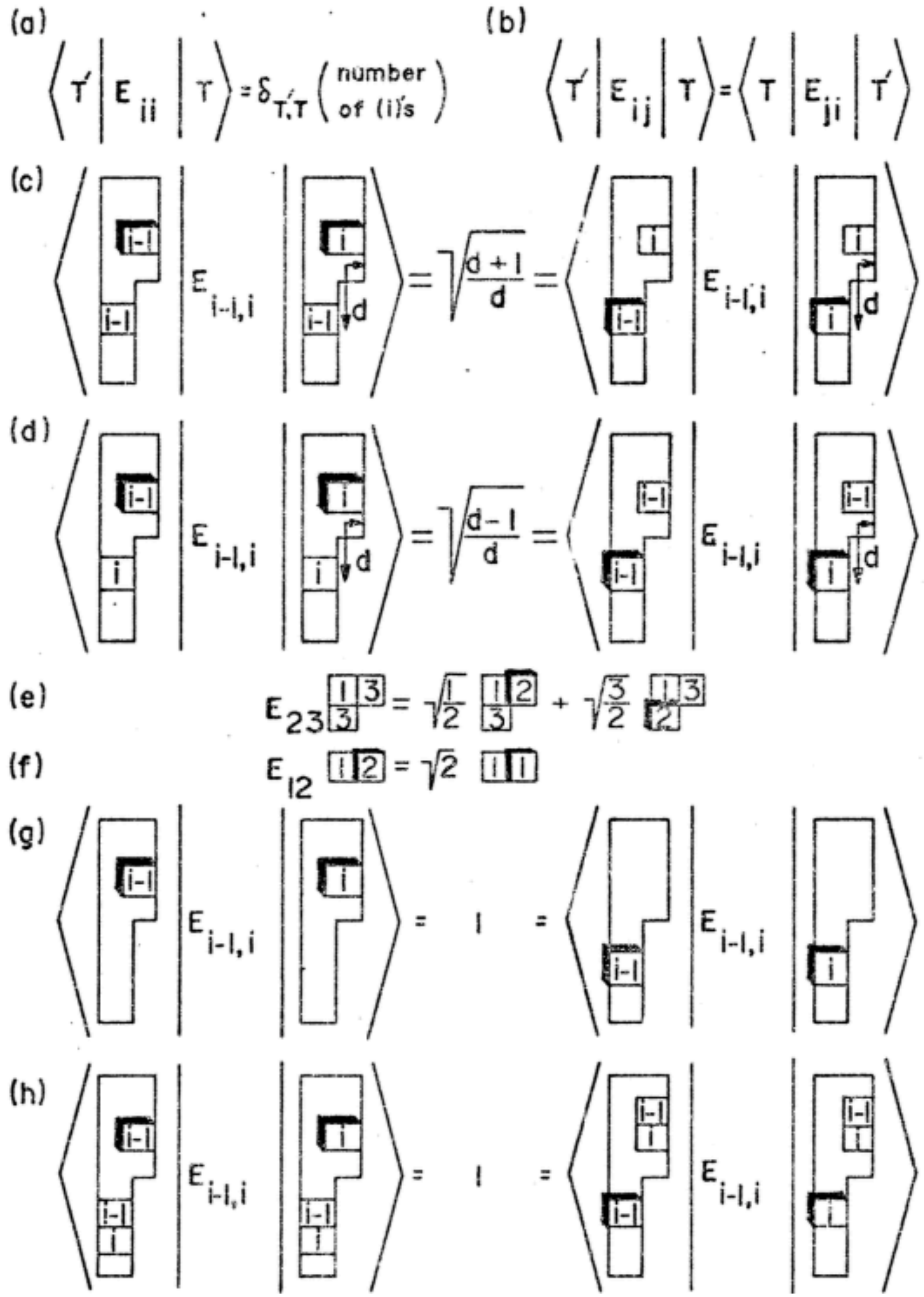


Fig.10.1.6

Tableau Formulas for Electronic Orbital Operators

(a) Number operators E_{ii} are diagonal.(The only eigen-values for orbitalstates are 0,1,and 2.)
(b) Raising and lowering operators are simply transposes of each other.
(c-h) $E_{i-1,i}$ acting on a tableau state gives zero unless there is an (i) in a column of the tableau that doesn't already have an (i-1),too. Then it gives back a new state with the (i) changed to(i-1) and a factor (matrix element) that depends on where the other (i)'s and (i-1)'s are located. (Boxes not outlined in the figure contain numbers not equal to (i) or (i-1).) Cases (c) and (d) involved the "city block" distance d (See Fig.7) which is the denominator of the matrix element. The numerator is one larger (d+1) or smaller (d-1) depending on whether the involved tableaus favor the larger or smaller state number (i or i-1) with a higher position. The special cases of (d=1) shown in (f) always pick the larger (and non-zero) choice of d+1=2. All other non-zero matrix elements are equal to unity.

These T^k_q operators are combinations of Elementary operators E_{mn}
Elementary operators have tableau hook length formula above.

Example:

$$T_1^2 = \frac{-1}{\sqrt{2}} E_{12} + \frac{1}{\sqrt{2}} E_{23}$$

For applications of Tableaus and Tensors to Molecular physics
Go back to Lect. 29 p. 50

Tensor operators for spin-J states: U(2J+1) generalization of Pauli spinors

$$\mathbf{v}^k_q = \sum_{m,m'} (-1)^{j-m}\sqrt{2k+1}\begin{pmatrix} k & j & j \\ q & m' & -m \end{pmatrix} \left| \begin{matrix} j \\ m \end{matrix} \right\rangle \left\langle \begin{matrix} j \\ m' \end{matrix} \right|$$

for $j = 1,2,3.$

$q = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$

-1

$$v^6_q = \begin{array}{|ccccccc|} \hline 1 & -\sqrt{2} & 1 & -\sqrt{2} & \sqrt{5} & -1 & 1 \\ \sqrt{2} & -6 & \sqrt{30} & -\sqrt{8} & 3 & -\sqrt{12} & 1 \\ 1 & -\sqrt{30} & 15 & -10 & \sqrt{15} & -3 & \sqrt{5} \\ \sqrt{2} & -\sqrt{8} & 10 & -20 & 10 & -\sqrt{8} & \sqrt{2} \\ \sqrt{5} & -3 & \sqrt{15} & -10 & 15 & -\sqrt{30} & 1 \\ 1 & -\sqrt{12} & 3 & -\sqrt{8} & \sqrt{30} & -6 & \sqrt{2} \\ 1 & -1 & \sqrt{5} & -\sqrt{2} & 1 & -\sqrt{2} & 1 \\ \hline \end{array} \qquad 1,\ \sqrt{2},\ \sqrt{22},\ \sqrt{22},\ \sqrt{33},\ \sqrt{264},\ \sqrt{924}$$

$$v^5_q = \begin{array}{|ccccccc|} \hline 1 & -\sqrt{5} & 1 & -\sqrt{2} & 1 & -1 & \cdot \\ \sqrt{5} & -4 & \sqrt{27} & -\sqrt{2} & 1 & 0 & -1 \\ 1 & -\sqrt{27} & 5 & -\sqrt{10} & 0 & 1 & -1 \\ \sqrt{2} & -\sqrt{2} & \sqrt{10} & 0 & -\sqrt{10} & \sqrt{2} & \sqrt{2} \\ 1 & -1 & 0 & \sqrt{10} & -5 & \sqrt{27} & -1 \\ 1 & 0 & -1 & \sqrt{2} & -\sqrt{27} & 4 & -\sqrt{5} \\ \cdot & 1 & -1 & \sqrt{2} & -1 & \sqrt{5} & -1 \\ \hline \end{array} \qquad \sqrt{2},\ \sqrt{2},\ \sqrt{6},\ \sqrt{6},\ \sqrt{84},\ \sqrt{84}$$

$$v^4_q = \begin{array}{|ccccccc|} \hline 3 & -\sqrt{30} & \sqrt{54} & -3 & \sqrt{3} & \cdot & \cdot \\ \sqrt{30} & -7 & \sqrt{32} & -\sqrt{3} & -\sqrt{2} & \sqrt{5} & \cdot \\ \sqrt{54} & -\sqrt{32} & 1 & \sqrt{15} & -\sqrt{40} & \sqrt{2} & \sqrt{3} \\ 3 & -\sqrt{3} & -\sqrt{15} & 6 & -\sqrt{15} & -\sqrt{3} & 3 \\ \sqrt{3} & \sqrt{2} & -\sqrt{40} & \sqrt{15} & 1 & -\sqrt{32} & \sqrt{54} \\ \cdot & \sqrt{5} & -\sqrt{2} & -\sqrt{3} & \sqrt{32} & -7 & \sqrt{30} \\ \cdot & \cdot & \sqrt{3} & -3 & \sqrt{54} & -\sqrt{30} & 3 \\ \hline \end{array} \qquad \sqrt{11},\ \sqrt{22},\ \sqrt{154},\ \sqrt{154},\ \sqrt{154}$$

$$v^3_q = \begin{array}{|ccccccc|} \hline 1 & -\sqrt{2} & \sqrt{2} & -1 & \cdot & \cdot & \cdot \\ \sqrt{2} & -1 & 0 & 1 & -\sqrt{2} & \cdot & \cdot \\ \sqrt{2} & 0 & -1 & 1 & 0 & -\sqrt{2} & \cdot \\ 1 & 1 & -1 & 0 & 1 & -1 & -1 \\ \cdot & \sqrt{2} & 0 & -1 & 1 & 0 & -\sqrt{2} \\ \cdot & \cdot & \sqrt{2} & -1 & 0 & 1 & -\sqrt{2} \\ \cdot & \cdot & \cdot & 1 & -\sqrt{2} & \sqrt{2} & -1 \\ \hline \end{array} \qquad \sqrt{6},\ \sqrt{6},\ \sqrt{6},\ \sqrt{6}$$

$$v^2_q = \begin{array}{|ccccccc|} \hline 5 & -5 & \sqrt{5} & \cdot & \cdot & \cdot & \cdot \\ 5 & 0 & -\sqrt{15} & \sqrt{10} & \cdot & \cdot & \cdot \\ \sqrt{5} & \sqrt{15} & -3 & -\sqrt{2} & \sqrt{12} & \cdot & \cdot \\ \cdot & \sqrt{10} & \sqrt{2} & -4 & \sqrt{2} & \sqrt{10} & \cdot \\ \cdot & \cdot & \sqrt{12} & -\sqrt{2} & -3 & \sqrt{15} & \sqrt{5} \\ \cdot & \cdot & \cdot & \sqrt{10} & -\sqrt{15} & 0 & 5 \\ \cdot & \cdot & \cdot & \cdot & \sqrt{5} & -5 & 5 \\ \hline \end{array} \qquad \sqrt{42},\ \sqrt{84},\ \sqrt{84}$$

$$v^1_q = \begin{array}{|ccccccc|} \hline 3 & -\sqrt{3} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \sqrt{3} & 2 & -\sqrt{5} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \sqrt{5} & 1 & -\sqrt{6} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \sqrt{6} & 0 & -\sqrt{6} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \sqrt{6} & -1 & -\sqrt{5} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \sqrt{5} & -2 & -\sqrt{3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{3} & -3 \\ \hline \end{array} \qquad \sqrt{28},\ \sqrt{28}$$

(f) $l = 3$

$q = 0 \quad 1 \quad 2 \quad 3 \quad 4$

$$\begin{array}{|ccccc|} \hline 1 & -1 & \sqrt{3} & -1 & 1 \\ 1 & -4 & \sqrt{6} & -\sqrt{8} & 1 \\ \sqrt{3} & -\sqrt{6} & 6 & -\sqrt{6} & \sqrt{3} \\ 1 & -\sqrt{8} & \sqrt{6} & -4 & 1 \\ 1 & -1 & \sqrt{3} & -1 & 1 \\ \hline \end{array} \qquad 1,\ \sqrt{2},\ \sqrt{14},\ \sqrt{14},\ \sqrt{70}$$

$$\begin{array}{|ccccc|} \hline 1 & -\sqrt{3} & 1 & -1 & \cdot \\ \sqrt{3} & -2 & \sqrt{2} & 0 & -1 \\ 1 & -\sqrt{2} & 0 & \sqrt{2} & -1 \\ 1 & 0 & -\sqrt{2} & 2 & -\sqrt{3} \\ \cdot & 1 & -1 & \sqrt{3} & -1 \\ \hline \end{array} \qquad \sqrt{2},\ \sqrt{2},\ \sqrt{10},\ \sqrt{10}$$

$$\begin{array}{|ccccc|} \hline 2 & -\sqrt{6} & \sqrt{2} & \cdot & \cdot \\ \sqrt{6} & -1 & -1 & \sqrt{3} & \cdot \\ \sqrt{2} & 1 & -2 & 1 & \sqrt{2} \\ \cdot & \sqrt{3} & -1 & -1 & \sqrt{6} \\ \cdot & \cdot & \sqrt{2} & -\sqrt{6} & 2 \\ \hline \end{array} \qquad \sqrt{7},\ \sqrt{14},\ \sqrt{14}$$

$$\begin{array}{|ccccc|} \hline 2 & -\sqrt{2} & \cdot & \cdot & \cdot \\ \sqrt{2} & 1 & -\sqrt{3} & \cdot & \cdot \\ \cdot & \sqrt{3} & 0 & -\sqrt{3} & \cdot \\ \cdot & \cdot & \sqrt{3} & -1 & -\sqrt{2} \\ \cdot & \cdot & \cdot & \sqrt{2} & -2 \\ \hline \end{array} \qquad \sqrt{10},\ \sqrt{10}$$

(d) $l = 2$

$q = 0 \quad 1 \quad 2$

$$\begin{array}{|ccc|} \hline 1 & -1 & 1 \\ 1 & -2 & 1 \\ 1 & -1 & -1 \\ \hline \end{array}$$

$$\begin{array}{|ccc|} \hline 1 & -1 & \cdot \\ 1 & 0 & -1 \\ \cdot & 1 & -1 \\ \hline \end{array}$$

(p) $l = 1$

Tensor operators for spin-J states: U(2J+1) generalization of Pauli spinors

$$\mathrm{v}^k_q = \sum_{m,m'} (-1)^{j-m}\sqrt{2k+1}\begin{pmatrix} k & j & j \\ q & m' & -m \end{pmatrix} \left| \begin{matrix} j \\ m \end{matrix} \right\rangle \left\langle \begin{matrix} j \\ m' \end{matrix} \right|$$

for $j = 1,2,3.$

$q = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$

-1

$$v^6_q = \begin{array}{|ccccccc|c}
1 & -\sqrt{2} & 1 & -\sqrt{2} & \sqrt{5} & -1 & 1 & 1 \\
\sqrt{2} & -6 & \sqrt{30} & -\sqrt{8} & 3 & -\sqrt{12} & 1 & \sqrt{2} \\
1 & -\sqrt{30} & 15 & -10 & \sqrt{15} & -3 & \sqrt{5} & \sqrt{22} \\
\sqrt{2} & -\sqrt{8} & 10 & -20 & 10 & -\sqrt{8} & \sqrt{2} & \sqrt{22} \\
\sqrt{5} & -3 & \sqrt{15} & -10 & 15 & -\sqrt{30} & 1 & \sqrt{33} \\
1 & -\sqrt{12} & 3 & -\sqrt{8} & \sqrt{30} & -6 & \sqrt{2} & \sqrt{264} \\
1 & -1 & \sqrt{5} & -\sqrt{2} & 1 & -\sqrt{2} & 1 & \sqrt{924}
\end{array}$$

$$v^5_q = \begin{array}{|ccccccc|c}
1 & -\sqrt{5} & 1 & -\sqrt{2} & 1 & -1 & \cdot & \\
\sqrt{5} & -4 & \sqrt{27} & -\sqrt{2} & 1 & 0 & -1 & \sqrt{2} \\
1 & -\sqrt{27} & 5 & -\sqrt{10} & 0 & 1 & -1 & \sqrt{2} \\
\sqrt{2} & -\sqrt{2} & \sqrt{10} & 0 & -\sqrt{10} & \sqrt{2} & \sqrt{2} & \sqrt{6} \\
1 & -1 & 0 & \sqrt{10} & -5 & \sqrt{27} & -1 & \sqrt{6} \\
1 & 0 & -1 & \sqrt{2} & -\sqrt{27} & 4 & -\sqrt{5} & \sqrt{84} \\
\cdot & 1 & -1 & \sqrt{2} & -1 & \sqrt{5} & -1 & \sqrt{84}
\end{array}$$

$$v^4_q = \begin{array}{|ccccccc|c}
3 & -\sqrt{30} & \sqrt{54} & -3 & \sqrt{3} & \cdot & \cdot & \\
\sqrt{30} & -7 & \sqrt{32} & -\sqrt{3} & -\sqrt{2} & \sqrt{5} & \cdot & \\
\sqrt{54} & -\sqrt{32} & 1 & \sqrt{15} & -\sqrt{40} & \sqrt{2} & \sqrt{3} & \sqrt{11} \\
3 & -\sqrt{3} & -\sqrt{15} & 6 & -\sqrt{15} & -\sqrt{3} & 3 & \sqrt{22} \\
\sqrt{3} & \sqrt{2} & -\sqrt{40} & \sqrt{15} & 1 & -\sqrt{32} & \sqrt{54} & \sqrt{154} \\
\cdot & \sqrt{5} & -\sqrt{2} & -\sqrt{3} & \sqrt{32} & -7 & \sqrt{30} & \sqrt{154} \\
\cdot & \cdot & \sqrt{3} & -3 & \sqrt{54} & -\sqrt{30} & 3 & \sqrt{154}
\end{array}$$

$$v^3_q = \begin{array}{|ccccccc|c}
1 & -\sqrt{2} & \sqrt{2} & -1 & \cdot & \cdot & \cdot & \\
\sqrt{2} & -1 & 0 & 1 & -\sqrt{2} & \cdot & \cdot & \\
\sqrt{2} & 0 & -1 & 1 & 0 & -\sqrt{2} & \cdot & \\
1 & 1 & -1 & 0 & 1 & -1 & -1 & \sqrt{6} \\
\cdot & \sqrt{2} & 0 & -1 & 1 & 0 & -\sqrt{2} & \sqrt{6} \\
\cdot & \cdot & \sqrt{2} & -1 & 0 & 1 & -\sqrt{2} & \sqrt{6} \\
\cdot & \cdot & \cdot & 1 & -\sqrt{2} & \sqrt{2} & -1 & \sqrt{6}
\end{array}$$

$$v^2_q = \begin{array}{|ccccccc|c}
5 & -5 & \sqrt{5} & \cdot & \cdot & \cdot & \cdot & \\
5 & 0 & -\sqrt{15} & \sqrt{10} & \cdot & \cdot & \cdot & \\
\sqrt{5} & \sqrt{15} & -3 & -\sqrt{2} & \sqrt{12} & \cdot & \cdot & \\
\cdot & \sqrt{10} & \sqrt{2} & -4 & \sqrt{2} & \sqrt{10} & \cdot & \\
\cdot & \cdot & \sqrt{12} & -\sqrt{2} & -3 & \sqrt{15} & \sqrt{5} & \sqrt{42} \\
\cdot & \cdot & \cdot & \sqrt{10} & -\sqrt{15} & 0 & 5 & \sqrt{84} \\
\cdot & \cdot & \cdot & \cdot & \sqrt{5} & -5 & 5 & \sqrt{84}
\end{array}$$

$$v^1_q = \begin{array}{|ccccccc|c}
3 & -\sqrt{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\sqrt{3} & 2 & -\sqrt{5} & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \sqrt{5} & 1 & -\sqrt{6} & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \sqrt{6} & 0 & -\sqrt{6} & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \sqrt{6} & -1 & -\sqrt{5} & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \sqrt{5} & -2 & -\sqrt{3} & \sqrt{28} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{3} & -3 & \sqrt{28}
\end{array}$$

(f) $l = 3$

$q = 0 \quad 1 \quad 2 \quad 3 \quad 4$

$$\begin{array}{|ccccc|c}
1 & -1 & \sqrt{3} & -1 & 1 & 1 \\
1 & -4 & \sqrt{6} & -\sqrt{8} & 1 & \sqrt{2} \\
\sqrt{3} & -\sqrt{6} & 6 & -\sqrt{6} & \sqrt{3} & \sqrt{14} \\
1 & -\sqrt{8} & \sqrt{6} & -4 & 1 & \sqrt{14} \\
1 & -1 & \sqrt{3} & -1 & 1 & \sqrt{70}
\end{array}$$

$$\begin{array}{|ccccc|c}
1 & -\sqrt{3} & 1 & -1 & \cdot & \\
\sqrt{3} & -2 & \sqrt{2} & 0 & -1 & \sqrt{2} \\
1 & -\sqrt{2} & 0 & \sqrt{2} & -1 & \sqrt{2} \\
1 & 0 & -\sqrt{2} & 2 & -\sqrt{3} & \sqrt{10} \\
\cdot & 1 & -1 & \sqrt{3} & -1 & \sqrt{10}
\end{array}$$

$$\begin{array}{|ccccc|c}
2 & -\sqrt{6} & \sqrt{2} & \cdot & \cdot & \\
\sqrt{6} & -1 & -1 & \sqrt{3} & \cdot & \\
\sqrt{2} & 1 & -2 & 1 & \sqrt{2} & \sqrt{7} \\
\cdot & \sqrt{3} & -1 & -1 & \sqrt{6} & \sqrt{14} \\
\cdot & \cdot & \sqrt{2} & -\sqrt{6} & 2 & \sqrt{14}
\end{array}$$

$$\begin{array}{|ccccc|c}
2 & -\sqrt{2} & \cdot & \cdot & \cdot & \\
\sqrt{2} & 1 & -\sqrt{3} & \cdot & \cdot & \\
\cdot & \sqrt{3} & 0 & -\sqrt{3} & \cdot & \\
\cdot & \cdot & \sqrt{3} & -1 & -\sqrt{2} & \sqrt{10} \\
\cdot & \cdot & \cdot & \sqrt{2} & -2 & \sqrt{10}
\end{array}$$

(d) $l = 2$

$q = 0 \quad 1 \quad 2$

$$\begin{array}{|ccc|}
1 & -1 & 1 \\
1 & -2 & 1 \\
1 & -1 & -1
\end{array}$$

$$\begin{array}{|ccc|}
1 & -1 & \cdot \\
1 & 0 & -1 \\
\cdot & 1 & -1
\end{array}$$

(p) $l = 1$

(a) $j = \frac{1}{2}$

$q = 0 \quad 1$

$$v^1_q = \begin{array}{|cc|c}
1 & -1 & 1 \\
1 & -1 & \sqrt{2}
\end{array}$$

(b) $j = \frac{3}{2}$

$q = 0 \quad 1 \quad 2 \quad 3$

$$v^1_q = \begin{array}{|cccc|c}
3 & -\sqrt{3} & \cdot & \cdot & \\
\sqrt{3} & 1 & -2 & \cdot & \\
\cdot & 2 & -1 & -\sqrt{3} & \sqrt{10} \\
\cdot & \cdot & \sqrt{3} & -3 & \sqrt{20}
\end{array}$$

$$v^2_q = \begin{array}{|cccc|c}
1 & -1 & 1 & \cdot & \\
1 & -1 & 0 & 1 & \sqrt{2} \\
1 & 0 & -1 & 1 & \sqrt{2} \\
\cdot & 1 & -1 & 1 & \sqrt{4}
\end{array}$$

$$v^3_q = \begin{array}{|cccc|c}
1 & -1 & 1 & -1 & 1 \\
1 & -3 & \sqrt{3} & -1 & \sqrt{2} \\
1 & -\sqrt{3} & 3 & -1 & \sqrt{5} \\
1 & -1 & 1 & -1 & \sqrt{20}
\end{array}$$

(c) $j = \frac{5}{2}$

$q = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5$

$$v^1q = \begin{array}{|cccccc|c}
5 & -\sqrt{5} & \cdot & \cdot & \cdot & \cdot & \\
\sqrt{5} & 3 & -\sqrt{8} & \cdot & \cdot & \cdot & \\
\cdot & \sqrt{8} & 1 & -3 & \cdot & \cdot & \\
\cdot & \cdot & 3 & -1 & -\sqrt{8} & \cdot & \\
\cdot & \cdot & \cdot & \sqrt{8} & -3 & -\sqrt{5} & \sqrt{35} \\
\cdot & \cdot & \cdot & \cdot & \sqrt{5} & -5 & \sqrt{70}
\end{array}$$

$$v^2_q = \begin{array}{|cccccc|c}
5 & -\sqrt{5} & \sqrt{5} & \cdot & \cdot & \cdot & \\
\sqrt{5} & -1 & -\sqrt{2} & 3 & \cdot & \cdot & \\
\sqrt{5} & \sqrt{2} & -4 & 0 & 3 & \cdot & \\
\cdot & 3 & 0 & -4 & \sqrt{2} & \sqrt{5} & \sqrt{28} \\
\cdot & \cdot & 3 & -\sqrt{2} & -1 & \sqrt{5} & \sqrt{14} \\
\cdot & \cdot & \cdot & \sqrt{5} & -\sqrt{5} & 5 & \sqrt{84}
\end{array}$$

$$v^3_q = \begin{array}{|cccccc|c}
5 & -\sqrt{10} & \sqrt{5} & -\sqrt{5} & \cdot & \cdot & \\
\sqrt{10} & -7 & 1 & 1 & -\sqrt{8} & \cdot & \\
\sqrt{5} & -1 & -4 & \sqrt{8} & -1 & -\sqrt{5} & \sqrt{18} \\
\sqrt{5} & 1 & -\sqrt{8} & 4 & 1 & -\sqrt{5} & \sqrt{12} \\
\cdot & \sqrt{8} & -1 & -1 & 7 & -\sqrt{10} & \sqrt{30} \\
\cdot & \cdot & \sqrt{5} & -\sqrt{5} & \sqrt{10} & -5 & \sqrt{180}
\end{array}$$

$$v^4_q = \begin{array}{|cccccc|c}
1 & -\sqrt{2} & 3 & -1 & 1 & \cdot & \\
\sqrt{2} & -3 & \sqrt{5} & -\sqrt{5} & 0 & 1 & \sqrt{2} \\
3 & -\sqrt{5} & 2 & 0 & -\sqrt{5} & 1 & \sqrt{2} \\
1 & -\sqrt{5} & 0 & 2 & -\sqrt{5} & 3 & \sqrt{28} \\
1 & 0 & -\sqrt{5} & \sqrt{5} & -3 & \sqrt{2} & \sqrt{14} \\
\cdot & 1 & -1 & 3 & -\sqrt{2} & 1 & \sqrt{28}
\end{array}$$

$$v^5_q = \begin{array}{|cccccc|c}
1 & -1 & 1 & -\sqrt{2} & 1 & -1 & 1 \\
1 & -5 & \sqrt{10} & -\sqrt{5} & \sqrt{5} & -1 & \sqrt{2} \\
1 & -\sqrt{10} & 10 & -\sqrt{20} & \sqrt{5} & -\sqrt{2} & \sqrt{9} \\
\sqrt{2} & -\sqrt{5} & \sqrt{20} & -10 & \sqrt{10} & -1 & \sqrt{12} \\
1 & -\sqrt{5} & \sqrt{5} & -\sqrt{10} & 5 & -1 & \sqrt{42} \\
1 & -1 & \sqrt{2} & -1 & 1 & -1 & \sqrt{252}
\end{array}$$

Tensor operators for spin-J states: Application to splitting

$$V^{(4)} = D\left[x^4 + y^4 + z^4 - \frac{3}{4}r^4\right] = D\left[\frac{2}{\sqrt{70}}\left(X_4^4 + X_{-4}^4\right) + \frac{2}{5}X_0^4\right]$$

$$\left\langle V^{(4)}\right\rangle_{j=2} = D\left\langle\frac{2}{\sqrt{70}}\left(v_4^4 + v_{-4}^4\right) + \frac{2}{5}v_0^4\right\rangle_{j=2}\frac{\sqrt{5}}{3}\left\langle 2\left|\left|X^4\right|\right|2\right\rangle.$$

$$\left\langle V^{(4)}\right\rangle_{j=2} = \frac{D}{\sqrt{70}}\begin{pmatrix} \frac{2}{5} & \cdot & \cdot & \cdot & 2 \\ \cdot & -\frac{8}{5} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \frac{12}{5} & \cdot & \cdot \\ \cdot & \cdot & \cdot & -\frac{8}{5} & \cdot \\ 2 & \cdot & \cdot & \cdot & \frac{2}{5} \end{pmatrix}\frac{\sqrt{5}}{3}\left\langle 2\left\|X^4\right\|2\right\rangle.$$

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-1	1	1
$-\sqrt{12}$	1	$\sqrt{2}$
-3	$\sqrt{5}$	$\sqrt{22}$
$-\sqrt{8}$	$\sqrt{2}$	$\sqrt{22}$
$-\sqrt{30}$	1	$\sqrt{33}$
-6	$\sqrt{2}$	$\sqrt{264}$
$-\sqrt{2}$	1	$\sqrt{924}$

-1	$\cdot$	
0	-1	$\sqrt{2}$
1	-1	$\sqrt{2}$
$\sqrt{2}$	$\sqrt{2}$	$\sqrt{6}$
$\sqrt{27}$	-1	$\sqrt{6}$
4	$-\sqrt{5}$	$\sqrt{84}$
$\sqrt{5}$	-1	$\sqrt{84}$

$\cdot$	$\cdot$	
$\sqrt{5}$	$\cdot$	
$\sqrt{2}$	$\sqrt{3}$	$\sqrt{11}$
$-\sqrt{3}$	3	$\sqrt{22}$
$-\sqrt{32}$	$\sqrt{54}$	$\sqrt{154}$
-7	$\sqrt{30}$	$\sqrt{154}$
$-\sqrt{30}$	3	$\sqrt{154}$

$\cdot$	$\cdot$	
$\cdot$	$\cdot$	
$-\sqrt{2}$	$\cdot$	
-1	-1	$\sqrt{6}$
0	$-\sqrt{2}$	$\sqrt{6}$
1	$-\sqrt{2}$	$\sqrt{6}$
$\sqrt{2}$	-1	$\sqrt{6}$

$\cdot$	$\cdot$	
$\cdot$	$\cdot$	
$\cdot$	$\cdot$	
$\sqrt{10}$	$\cdot$	
$\sqrt{15}$	$\sqrt{5}$	$\sqrt{42}$
0	5	$\sqrt{84}$
-5	5	$\sqrt{84}$

$\cdot$	$\cdot$	
$\cdot$	$\cdot$	
$\cdot$	$\cdot$	
$\cdot$	$\cdot$	
$-\sqrt{5}$	$\cdot$	
-2	$-\sqrt{3}$	$\sqrt{28}$
$\sqrt{3}$	-3	$\sqrt{28}$

$q = 0$	1	2	3	4	
1	-1	$\sqrt{3}$	-1	1	1
1	-4	$\sqrt{6}$	$-\sqrt{8}$	1	$\sqrt{2}$
$\sqrt{3}$	$-\sqrt{6}$	6	$-\sqrt{6}$	$\sqrt{3}$	$\sqrt{14}$
1	$-\sqrt{8}$	$\sqrt{6}$	-4	1	$\sqrt{14}$
1	-1	$\sqrt{3}$	-1	1	$\sqrt{70}$

1	$-\sqrt{3}$	1	-1	$\cdot$	
$\sqrt{3}$	-2	$\sqrt{2}$	0	-1	$\sqrt{2}$
1	$-\sqrt{2}$	0	$\sqrt{2}$	-1	$\sqrt{2}$
1	0	$-\sqrt{2}$	2	$-\sqrt{3}$	$\sqrt{10}$
$\cdot$	1	-1	$\sqrt{3}$	-1	$\sqrt{10}$

2	$-\sqrt{6}$	$\sqrt{2}$	$\cdot$	$\cdot$	
$\sqrt{6}$	-1	-1	$\sqrt{3}$	$\cdot$	
$\sqrt{2}$	1	-2	1	$\sqrt{2}$	$\sqrt{7}$
$\cdot$	$\sqrt{3}$	-1	-1	$\sqrt{6}$	$\sqrt{14}$
$\cdot$	$\cdot$	$\sqrt{2}$	$-\sqrt{6}$	2	$\sqrt{14}$

2	$-\sqrt{2}$	$\cdot$	$\cdot$	$\cdot$	
$\sqrt{2}$	1	$-\sqrt{3}$	$\cdot$	$\cdot$	
$\cdot$	$\sqrt{3}$	0	$-\sqrt{3}$	$\cdot$	
$\cdot$	$\cdot$	$\sqrt{3}$	-1	$-\sqrt{2}$	$\sqrt{10}$
$\cdot$	$\cdot$	$\cdot$	$\sqrt{2}$	-2	$\sqrt{10}$

(d) $l = 2$

$q = 0$	1	2
1	-1	1
1	-2	1
1	-1	-1

1	-1	$\cdot$
1	0	-1
$\cdot$	1	-1

(p) $l = 1$