Group Theory in Quantum Mechanics
Lecture 24 *(4.25.17)*
Rotational symmetry U(2)⊂U(3) and O(3)

(Int.J.Mol.Sci, 14, 714(2013) p.755-774 , QTCA Unit 7 Ch. 21-22)
(PSDS - Ch. 5, 7)

Review : *2-D* $\mathbf{a^\dagger a}$ *algebra of U(2) representations*

Angular momentum generators by U(2) analysis
Review : *Angular momentum raise-n-lower operators* $\mathbf{s}_+$ and $\mathbf{s}_-$
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Angular momentum commutation relations
Key Lie theorems

Angular momentum magnitude and uncertainty
Angular momentum uncertainty angle

Generating R(3) rotation and U(2) representations
Applications of R(3) rotation and U(2) representations
Molecular and nuclear wavefunctions
Molecular and nuclear eigenlevels
Generalized Stern-Gerlach and transformation matrices
Angular momentum cones and high J properties

Partial listing of the Harter-Soft/Heyoka LearnIt Web Apps as of April 24, 2017
(Apps are being upgraded as time permits)

Production Links - ***For the students & general public***

BohrIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/BohrItWeb.html"
BounceIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/BounceItWeb.html"
BoxIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/BoxItWeb.html"
CoulIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/CoulItWeb.html"
Cycloidulum - Production; URL is "http://www.uark.edu/ua/modphys/markup/CycloidulumWeb.html"
LearnIt - Production; URL is "**http://www.uark.edu/ua/modphys**" or "http://www.uark.edu/ua/modphys/markup/LearnItWeb.html"
JerkIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/JerkItWeb.html"
Pendulum - Production; URL is "http://www.uark.edu/ua/modphys/markup/PendulumWeb.html"
QuantIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/QuantItWeb.html"
Relativity - Pirelli Entrant: URL is "http://www.uark.edu/ua/pirelli" or "http://www.uark.edu/ua/pirelli/html/default.html"
Trebuchet Production; URL is "http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html"

Testing Links - ***For internal use and testing by Harter & Heyoka***

BohrIt - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/BohrItWeb.html"
BounceIt - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/BounceItWeb.html"
BounceIt Title Page - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/BounceItTitlePage.html"
BoxIt - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/BoxItWeb.html"
CoulIt - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/CoulItWeb.html"
Cycloidulum - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/CycloidulumWeb.html"
Harter-Soft Web Apps - Quick Reference - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/JerkItWeb.html"
JerkIt - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/JerkItWeb.html"
ModernPhysics - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/IntroCover.html"
Pendulum - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/PendulumWeb.html"
QuantIt - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/QuantItWeb.html"
Trebuchet Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/TrebuchetWeb.html"

Link to the complete listing of Harter-Soft LearnIt Web Apps and resources for Physics

Review : *2-D* $\mathbf{a}^{\dagger}\mathbf{a}$ *algebra of U(2) representations*

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U(2)-2D-HO Hamiltonian and irreducible representations

Review Lect.23 p74-76

"Little-Endian" indexing
(...01,02,03..10,11,12,13 ... 20,21,22,23,...)

$$\mathbf{H} = A\left(\mathbf{a}_1^\dagger\mathbf{a}_1+\mathbf{1}/2\right)+\left(B-iC\right)\mathbf{a}_1^\dagger\mathbf{a}_2 +\left(B+iC\right)\mathbf{a}_2^\dagger\mathbf{a}_1+D\left(\mathbf{a}_2^\dagger\mathbf{a}_2+\mathbf{1}/2\right)$$

$$\langle\mathbf{H}\rangle = A(\mathbf{1}/2)+D(\mathbf{1}/2)+$$

	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 02\rangle$	$\cdots$	$\vert 10\rangle$	$\vert 11\rangle$	$\vert 12\rangle$	$\cdots$	$\vert 20\rangle$	$\vert 21\rangle$	$\vert 22\rangle$	$\cdots$
$\langle 00\vert$	0			$\cdots$	$\cdot$			$\cdots$				$\cdots$
$\langle 01\vert$		D		$\cdots$	$B+iC$	$\cdot$		$\cdots$				$\cdots$
$\langle 02\vert$			$2D$	$\cdots$		$\sqrt{2}(B+iC)$	$\cdot$	$\cdots$				$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$				$\cdots$
$\langle 10\vert$	$\cdot$	$B-iC$		$\cdots$	A			$\cdots$	$\cdot$			$\cdots$
$\langle 11\vert$		$\cdot$	$\sqrt{2}(B-iC)$	$\cdots$		$A+D$		$\cdots$	$\sqrt{2}(B+iC)$	$\cdot$		$\cdots$
$\langle 12\vert$		*Example:*	$\cdot$	$\cdots$			$A+2D$	$\cdots$		$\sqrt{4}(B+iC)$	$\cdot$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\langle 20\vert$	$\mathbf{a}_1^\dagger\mathbf{a}_2\vert 02\rangle=\sqrt{0+1}\sqrt{2}\vert 0{+}1\ 2{-}1\rangle=\sqrt{2}\vert 1\,1\rangle$				$\cdot$	$\sqrt{2}(B-iC)$		$\cdots$	$2A$			$\cdots$
$\langle 21\vert$	$\mathbf{a}_1^\dagger\mathbf{a}_2\vert n_1n_2\rangle=\sqrt{n_1+1}\sqrt{n_2}\vert n_1{+}1\ n_2{-}1\rangle$					$\cdot$	$\sqrt{4}(B-iC)$	$\cdots$		$2A+D$		$\cdots$
$\langle 22\vert$							$\cdot$	$\cdots$			$2A+2D$	$\cdots$
$\vdots$					$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$$\mathbf{a}_1^\dagger\mathbf{a}_1\vert n_1n_2\rangle=n_1\vert n_1n_2\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1\vert n_1n_2\rangle=\sqrt{n_1}\sqrt{n_2+1}\,\vert n_1{-}1\ n_2{+}1\rangle$$

$$\mathbf{a}_1^\dagger\mathbf{a}_2\vert n_1n_2\rangle=\sqrt{n_1+1}\sqrt{n_2}\,\vert n_1{+}1\ n_2{-}1\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_2\vert n_1n_2\rangle=n_2\vert n_1n_2\rangle$$

Rearrangement of rows and columns brings the matrix to a block-diagonal form.

U(2)-2D-HO Hamiltonian and irreducible representations

Review Lect.23 p74-76

"Little-Endian" indexing
(*...01,02,03..10,11,12,13 ... 20,21,22,23,...*)

$$\mathbf{H}= A\left(\mathbf{a}_1^\dagger\mathbf{a}_1+\mathbf{1}/2\right)+\left(B-iC\right)\mathbf{a}_1^\dagger\mathbf{a}_2 +\left(B+iC\right)\mathbf{a}_2^\dagger\mathbf{a}_1+D\left(\mathbf{a}_2^\dagger\mathbf{a}_2+\mathbf{1}/2\right)$$

$$\langle\mathbf{H}\rangle = A(\mathbf{1}/2)+D(\mathbf{1}/2)+$$

	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 02\rangle$	$\cdots$	$\vert 10\rangle$	$\vert 11\rangle$	$\vert 12\rangle$	$\cdots$	$\vert 20\rangle$	$\vert 21\rangle$	$\vert 22\rangle$	$\cdots$
$\langle 00\vert$	0			$\cdots$	$\cdot$			$\cdots$				$\cdots$
$\langle 01\vert$		D		$\cdots$	$B+iC$	$\cdot$		$\cdots$				$\cdots$
$\langle 02\vert$			$2D$	$\cdots$		$\sqrt{2}(B+iC)$	$\cdot$	$\cdots$				$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$				$\cdots$
$\langle 10\vert$	$\cdot$	$B-iC$		$\cdots$	A			$\cdots$	$\cdot$			$\cdots$
$\langle 11\vert$		$\cdot$	$\sqrt{2}(B-iC)$	$\cdots$		$A+D$		$\cdots$	$\sqrt{2}(B+iC)$	$\cdot$		$\cdots$
$\langle 12\vert$			$\cdot$	$\cdots$			$A+2D$	$\cdots$		$\sqrt{4}(B+iC)$	$\cdot$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\langle 20\vert$					$\cdot$	$\sqrt{2}(B-iC)$		$\cdots$	$2A$			$\cdots$
$\langle 21\vert$						$\cdot$	$\sqrt{4}(B-iC)$	$\cdots$		$2A+D$		$\cdots$
$\langle 22\vert$							$\cdot$	$\cdots$			$2A+2D$	$\cdots$
$\vdots$					$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Example:

$$\mathbf{a}_1^\dagger\mathbf{a}_2\vert 02\rangle=\sqrt{0+1}\sqrt{2}\vert 0+1\;2-1\rangle=\sqrt{2}\vert 1\,1\rangle$$

$$\mathbf{a}_1^\dagger\mathbf{a}_2\vert n_1 n_2\rangle=\sqrt{n_1+1}\sqrt{n_2}\vert n_1+1\;n_2-1\rangle$$

$$\mathbf{a}_1^\dagger\mathbf{a}_1\vert n_1 n_2\rangle=n_1\vert n_1 n_2\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1\vert n_1 n_2\rangle=\sqrt{n_1}\sqrt{n_2+1}\vert n_1-1\;n_2+1\rangle$$

$$\mathbf{a}_1^\dagger\mathbf{a}_2\vert n_1 n_2\rangle=\sqrt{n_1+1}\sqrt{n_2}\vert n_1+1\;n_2-1\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_2\vert n_1 n_2\rangle=n_2\vert n_1 n_2\rangle$$

Rearrangement of rows and columns brings the matrix to a block-diagonal form.

Base states $|n_1\rangle|n_2\rangle$ with the same *total quantum number* $\nu= n_1 + n_2$ define each block.

U(2)-2D-HO Hamiltonian and irreducible representations

Review Lect.23 p74-76

"Little-Endian" indexing
(...01,02,03..10,11,12,13 ... 20,21,22,23,...)

$\mathbf{H} =$

$$A\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + 1/2\right) + (B - iC)\,\mathbf{a}_1^\dagger\mathbf{a}_2$$

$$+(B + iC)\,\mathbf{a}_2^\dagger\mathbf{a}_1 + D\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + 1/2\right)$$

$\langle\mathbf{H}\rangle = A(1/2) + D(1/2) +$

	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 02\rangle$	$\cdots$	$\vert 10\rangle$	$\vert 11\rangle$	$\vert 12\rangle$	$\cdots$	$\vert 20\rangle$	$\vert 21\rangle$	$\vert 22\rangle$	$\cdots$
$\langle 00\vert$	0			$\cdots$	$\cdot$			$\cdots$				$\cdots$
$\langle 01\vert$		D		$\cdots$	$B+iC$	$\cdot$		$\cdots$				$\cdots$
$\langle 02\vert$			$2D$	$\cdots$		$\sqrt{2}(B+iC)$	$\cdot$	$\cdots$				$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$				$\cdots$
$\langle 10\vert$	$\cdot$	$B-iC$		$\cdots$	A			$\cdots$	$\cdot$			$\cdots$
$\langle 11\vert$		$\cdot$	$\sqrt{2}(B-iC)$	$\cdots$		$A+D$		$\cdots$	$\sqrt{2}(B+iC)$	$\cdot$		$\cdots$
$\langle 12\vert$		*Example:*	$\cdot$	$\cdots$			$A+2D$	$\cdots$		$\sqrt{4}(B+iC)$	$\cdot$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\langle 20\vert$					$\cdot$	$\sqrt{2}(B-iC)$		$\cdots$	$2A$			$\cdots$
$\langle 21\vert$						$\cdot$	$\sqrt{4}(B-iC)$	$\cdots$		$2A+D$		$\cdots$
$\langle 22\vert$							$\cdot$	$\cdots$			$2A+2D$	$\cdots$
$\vdots$					$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Example: $\mathbf{a}_1^\dagger\mathbf{a}_2|02\rangle = \sqrt{0+1}\sqrt{2}\,|0+1\ 2-1\rangle = \sqrt{2}\,|1\,1\rangle$

$\mathbf{a}_1^\dagger\mathbf{a}_2|n_1 n_2\rangle = \sqrt{n_1+1}\sqrt{n_2}\,|n_1+1\ n_2-1\rangle$

$$\mathbf{a}_1^\dagger\mathbf{a}_1|n_1 n_2\rangle = n_1|n_1 n_2\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1|n_1 n_2\rangle = \sqrt{n_1}\sqrt{n_2+1}\,|n_1-1\ n_2+1\rangle$$

$$\mathbf{a}_1^\dagger\mathbf{a}_2|n_1 n_2\rangle = \sqrt{n_1+1}\sqrt{n_2}\,|n_1+1\ n_2-1\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_2|n_1 n_2\rangle = n_2|n_1 n_2\rangle$$

Rearrangement of rows and columns brings the matrix to a block-diagonal form.

Base states $|n_1\rangle|n_2\rangle$ with the same ***total quantum number*** $\upsilon = n_1 + n_2$ define each block.

Group reorganized "Little-Endian" indexing
(...01,02,03..10,11,12,13 ... 20,21,22,23,...)

$\langle\mathbf{H}\rangle = A(1/2) + D(1/2) +$

	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 10\rangle$	$\vert 02\rangle$	$\vert 11\rangle$	$\vert 20\rangle$	$\vert 03\rangle$	$\vert 12\rangle$	$\vert 21\rangle$	$\vert 30\rangle$	$\cdots$
$\langle 00\vert$	0	*Vacuum* ($\nu=0$)									
$\langle 01\vert$		D	$B+iC$	*Fundamental* ($\nu=1$)							
$\langle 10\vert$		$B-iC$	A	vibrational sub-space							
$\langle 02\vert$				$2D$	$\sqrt{2}(B+iC)$						
$\langle 11\vert$				$\sqrt{2}(B-iC)$	$A+D$	$\sqrt{2}(B+iC)$	*Overtone* ($\nu=2$) vibrational sub-space				
$\langle 20\vert$					$\sqrt{2}(B-iC)$	$2A$					
$\langle 03\vert$							$3D$	$\sqrt{3}(B+iC)$			
$\langle 12\vert$							$\sqrt{3}(B-iC)$	$A+2D$	$\sqrt{4}(B+iC)$		*Overtone* ($\nu=3$) vibrational sub-space
$\langle 21\vert$								$\sqrt{4}(B-iC)$	$2A+D$	$\sqrt{3}(B+iC)$	
$\langle 30\vert$									$\sqrt{3}(B-iC)$	$3A$	
$\vdots$											

$$\mathbf{H}^A = A\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + 1/2\right) + D\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + 1/2\right)$$

$$\varepsilon^A_{n_1 n_2} = A\left(n_1 + \frac{1}{2}\right) + D\left(n_2 + \frac{1}{2}\right) = \frac{A+D}{2}(n_1 + n_2 + 1) + \frac{A-D}{2}(n_1 - n_2)$$

Setting $(B=0=C)$ and $(A=\omega_+)$ and $(D=\omega_-)$ gives diagonal block matrices. Review Lect.23 p80-92

$\langle \mathbf{H} \rangle = A(1/2) + D(1/2) +$

	$\|00\rangle$	$\|01\rangle$	$\|10\rangle$	$\|02\rangle$	$\|11\rangle$	$\|20\rangle$	$\|03\rangle$	$\|12\rangle$	$\|21\rangle$	$\|30\rangle$	$\cdots$
$\langle 00\|$	0										
$\langle 01\|$		ω_-									
$\langle 10\|$			ω_+								
$\langle 02\|$				$2\omega_-$							
$\langle 11\|$					$\omega_+ + \omega_-$						
$\langle 20\|$						$2\omega_+$					
$\langle 03\|$							$3\omega_-$				
$\langle 12\|$								$\omega_+ + 2\omega_-$			
$\langle 21\|$									$2\omega_+ + \omega_-$		
$\langle 30\|$										$3\omega_+$	
$\vdots$											

$$\omega_+ - \omega_- = \Omega$$
$$= \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}$$
$$= A - D$$

$$\mathbf{H}^A = A\left(\mathbf{a}_1^\dagger \mathbf{a}_1 + \mathbf{1}/2\right) + D\left(\mathbf{a}_2^\dagger \mathbf{a}_2 + \mathbf{1}/2\right)$$

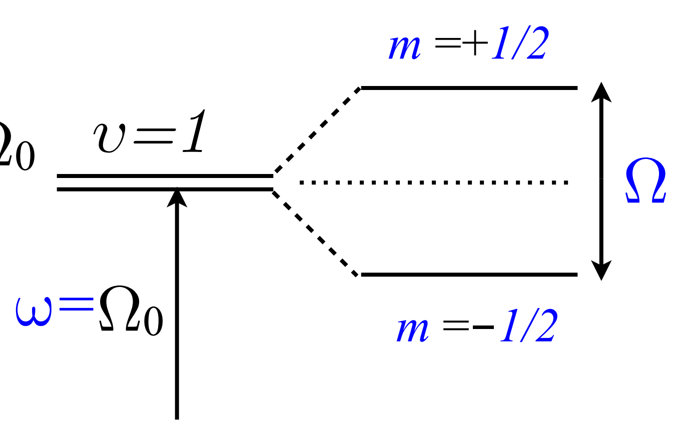
$$\varepsilon^A_{n_1 n_2} = A\left(n_1 + \frac{1}{2}\right) + D\left(n_2 + \frac{1}{2}\right) = \frac{A+D}{2}\left(n_1 + n_2 + 1\right) + \frac{A-D}{2}\left(n_1 - n_2\right)$$
$$= \Omega_0\left(n_1 + n_2 + 1\right) + \frac{\Omega}{2}\left(n_1 - n_2\right) = \Omega_0\left(\upsilon + 1\right) + \Omega\, m$$

Define *total quantum number* $\upsilon = 2j$ and half-difference or *asymmetry quantum number m*

$$\upsilon = n_1 + n_2 = 2j \qquad j = \frac{n_1 + n_2}{2} = \frac{\upsilon}{2} \qquad m = \frac{n_1 - n_2}{2}$$

$\upsilon + 1 = 2j + 1$ multiplies *base frequency* $\omega = \Omega_0$

m multiplies *beat frequency* Ω



$$\omega_+ = \Omega_0 + \Omega\left(+\tfrac{1}{2}\right)$$

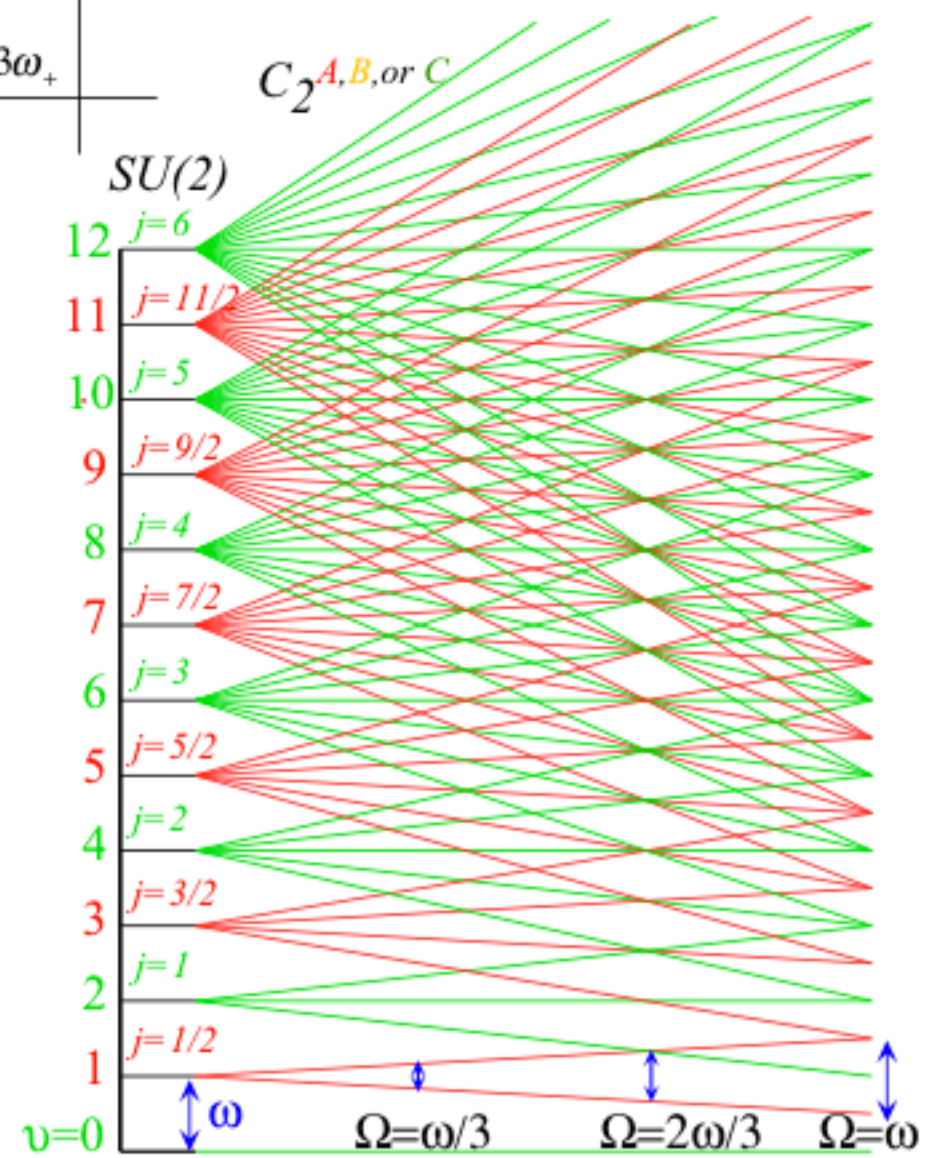
$$\omega_- = \Omega_0 + \Omega\left(-\tfrac{1}{2}\right)$$

Setting $(B=0=C)$ and $(A=\omega_+)$ and $(D=\omega_-)$ gives diagonal block matrices.

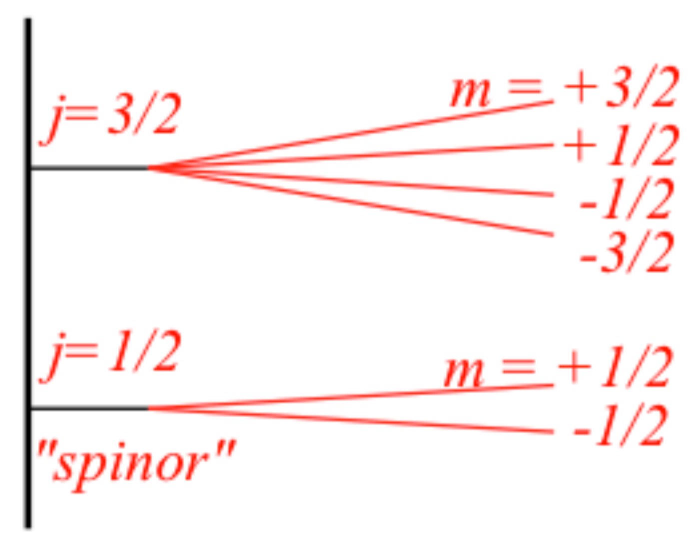
$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 10\rangle$	$\vert 02\rangle$	$\vert 11\rangle$	$\vert 20\rangle$	$\vert 03\rangle$	$\vert 12\rangle$	$\vert 21\rangle$	$\vert 30\rangle$	$\cdots$
$\langle 00\vert$	0										
$\langle 01\vert$		ω_-									
$\langle 10\vert$			ω_+								
$\langle 02\vert$				$2\omega_-$							
$\langle 11\vert$					$\omega_+ + \omega_-$						
$\langle 20\vert$						$2\omega_+$					
$\langle 03\vert$							$3\omega_-$				
$\langle 12\vert$								$\omega_+ + 2\omega_-$			
$\langle 21\vert$									$2\omega_+ + \omega_-$		
$\langle 30\vert$										$3\omega_+$	
$\vdots$											

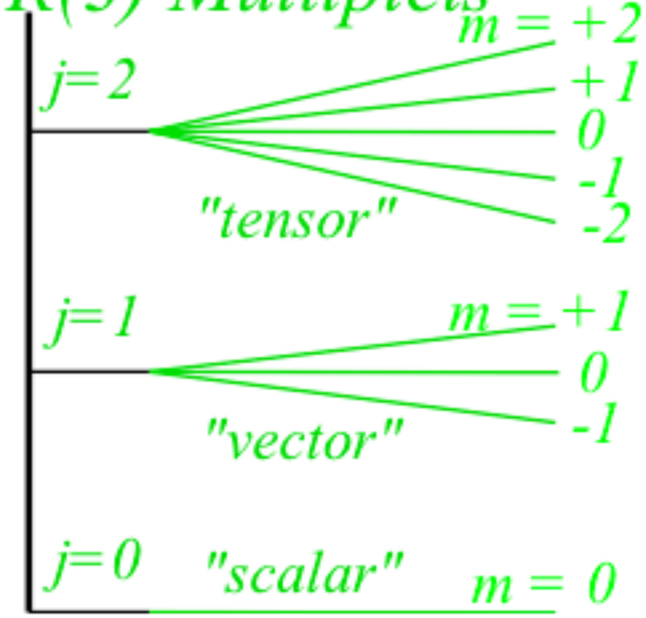
$$\omega_+ - \omega_- = \Omega$$
$$= \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}$$
$$= A - D$$



SU(2) Multiplets



R(3) Multiplets



Review : *2-D* **a†a** *algebra of U(2) representations* Review Lect.23 p99-103

Angular momentum generators by U(2) analysis

Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$

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R(3) Angular momentum generators by U(2) analysis

($\upsilon=1$) or ($j=1/2$) block **H** matrices of U(2) oscillator

Use irreps of unit operator $\mathbf{S}_0 = \mathbf{1}$ and spin operators $\{ \mathbf{S}_X, \mathbf{S}_Y, \mathbf{S}_Z \}$. (also known as: $\{ \mathbf{S}_B, \mathbf{S}_C, \mathbf{S}_A \}$)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + 2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} + (A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

R(3) Angular momentum generators by U(2) analysis

Review Lect.23 p99-103

(υ=*1*) or (*j*=*1/2*) block **H** matrices of U(2) oscillator

Use irreps of unit operator $\mathbf{S}_0 = \mathbf{1}$ and spin operators $\{\mathbf{S}_X, \mathbf{S}_Y, \mathbf{S}_Z\}$. (also known as: $\{\mathbf{S}_B, \mathbf{S}_C, \mathbf{S}_A\}$)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + 2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} + (A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

(υ=*2*) or (*j*=*1*) 3-by-3 block uses their vector irreps.

$$\begin{pmatrix} 2A & \sqrt{2}(B-iC) & \cdot \\ \sqrt{2}(B+iC) & A+D & \sqrt{2}(B-iC) \\ \cdot & \sqrt{2}(B+iC) & 2D \end{pmatrix} = (A+D)\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} + 2B\begin{pmatrix} \cdot & \frac{\sqrt{2}}{2} & \cdot \\ \frac{\sqrt{2}}{2} & \cdot & \frac{\sqrt{2}}{2} \\ \cdot & \frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + 2C\begin{pmatrix} \cdot & -i\frac{\sqrt{2}}{2} & \cdot \\ i\frac{\sqrt{2}}{2} & \cdot & -i\frac{\sqrt{2}}{2} \\ \cdot & i\frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + (A-D)\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

R(3) Angular momentum generators by U(2) analysis

(υ=*1*) or (*j*=*1/2*) block **H** matrices of U(2) oscillator

Use irreps of unit operator $\mathbf{S}_0 = \mathbf{1}$ and spin operators $\{\mathbf{S}_X, \mathbf{S}_Y, \mathbf{S}_Z\}$. (also known as: $\{\mathbf{S}_B, \mathbf{S}_C, \mathbf{S}_A\}$)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + 2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} + (A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

(υ=*2*) or (*j*=*1*) 3-by-3 block uses their vector irreps.

$$\begin{pmatrix} 2A & \sqrt{2}(B-iC) & \cdot \\ \sqrt{2}(B+iC) & A+D & \sqrt{2}(B-iC) \\ \cdot & \sqrt{2}(B+iC) & 2D \end{pmatrix} = (A+D)\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} + 2B\begin{pmatrix} \cdot & \frac{\sqrt{2}}{2} & \cdot \\ \frac{\sqrt{2}}{2} & \cdot & \frac{\sqrt{2}}{2} \\ \cdot & \frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + 2C\begin{pmatrix} \cdot & -i\frac{\sqrt{2}}{2} & \cdot \\ i\frac{\sqrt{2}}{2} & \cdot & -i\frac{\sqrt{2}}{2} \\ \cdot & i\frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + (A-D)\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

(υ=*3*) or (*j*=*3/2*) 4-by-4 block uses Dirac spinor irreps.

$$\begin{pmatrix} 3A & \sqrt{3}(B-iC) & & \\ \sqrt{3}(B+iC) & 2A+D & \sqrt{4}(B-iC) & \\ & \sqrt{4}(B+iC) & A+2D & \sqrt{3}(B-iC) \\ & & \sqrt{3}(B+iC) & 3D \end{pmatrix} = \frac{3(A+D)}{2}\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} + 2B\begin{pmatrix} \cdot & \frac{\sqrt{3}}{2} & \cdot & \cdot \\ \frac{\sqrt{3}}{2} & \cdot & \frac{\sqrt{4}}{2} & \cdot \\ \cdot & \frac{\sqrt{4}}{2} & \cdot & \frac{\sqrt{3}}{2} \\ \cdot & \cdot & \frac{\sqrt{3}}{2} & \cdot \end{pmatrix} + 2C\begin{pmatrix} \cdot & -i\frac{\sqrt{3}}{2} & \cdot & \cdot \\ i\frac{\sqrt{3}}{2} & \cdot & -i\frac{\sqrt{4}}{2} & \cdot \\ \cdot & i\frac{\sqrt{4}}{2} & \cdot & -i\frac{\sqrt{3}}{2} \\ \cdot & \cdot & i\frac{\sqrt{3}}{2} & \cdot \end{pmatrix} + (A-D)\begin{pmatrix} \frac{3}{2} & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{2} & \cdot & \cdot \\ \cdot & \cdot & -\frac{1}{2} & \cdot \\ \cdot & \cdot & \cdot & -\frac{3}{2} \end{pmatrix}$$

R(3) Angular momentum generators by U(2) analysis

(υ=*1*) or (j=*1/2*) block **H** matrices of U(2) oscillator

Use irreps of unit operator $\mathbf{S}_0 = \mathbf{1}$ and spin operators { $\mathbf{S}_X$, $\mathbf{S}_Y$, $\mathbf{S}_Z$ }. (also known as: { $\mathbf{S}_B, \mathbf{S}_C, \mathbf{S}_A$ })

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$$\begin{pmatrix} 3A & \sqrt{3}(B-iC) & & \\ \sqrt{3}(B+iC) & 2A+D & \sqrt{4}(B-iC) & \\ & \sqrt{4}(B+iC) & A+2D & \sqrt{3}(B-iC) \\ & & \sqrt{3}(B+iC) & 3D \end{pmatrix} = \frac{3(A+D)}{2}\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} + 2B\begin{pmatrix} \cdot & \frac{\sqrt{3}}{2} & \cdot & \cdot \\ \frac{\sqrt{3}}{2} & \cdot & \frac{\sqrt{4}}{2} & \cdot \\ \cdot & \frac{\sqrt{4}}{2} & \cdot & \frac{\sqrt{3}}{2} \\ \cdot & \cdot & \frac{\sqrt{3}}{2} & \cdot \end{pmatrix} + 2C\begin{pmatrix} \cdot & -i\frac{\sqrt{3}}{2} & \cdot & \cdot \\ i\frac{\sqrt{3}}{2} & \cdot & -i\frac{\sqrt{4}}{2} & \cdot \\ \cdot & i\frac{\sqrt{4}}{2} & \cdot & -i\frac{\sqrt{3}}{2} \\ \cdot & \cdot & i\frac{\sqrt{3}}{2} & \cdot \end{pmatrix} + (A-D)\begin{pmatrix} \frac{3}{2} & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{2} & \cdot & \cdot \\ \cdot & \cdot & -\frac{1}{2} & \cdot \\ \cdot & \cdot & \cdot & -\frac{3}{2} \end{pmatrix}$$

(υ=*2j*) or (2j+*1*)-by-(2j+*1*) block uses $D^{(j)}(\mathbf{s}_\mu)$ irreps of U(2) or R(3).

$$\langle \mathbf{H} \rangle^{j-block} = 2j\Omega_0 \langle \mathbf{1} \rangle^j + \Omega_X \langle \mathbf{s}_X \rangle^j + \Omega_Y \langle \mathbf{s}_Y \rangle^j + \Omega_Z \langle \mathbf{s}_Z \rangle^j$$

R(3) Angular momentum generators by U(2) analysis

($\upsilon=1$) or ($j=1/2$) block **H** matrices of U(2) oscillator

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All j-block matrix operators factor into *raise-n-lower* operators $\mathbf{s}_\pm = \mathbf{s}_X \pm i\mathbf{s}_Y$ plus the diagonal $\mathbf{s}_Z$

$$\langle \mathbf{H} \rangle^{j-block} = 2j\Omega_0 \langle \mathbf{1} \rangle^j + \left[(\Omega_X - i\Omega_Y)\langle \mathbf{s}_X + i\mathbf{s}_Y \rangle^j + (\Omega_X + i\Omega_Y)\langle \mathbf{s}_X - i\mathbf{s}_Y \rangle^j \right]/2 + \Omega_Z \langle \mathbf{s}_Z \rangle^j$$

Review : *2-D* $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(2) representations*

Review Lect.23 p105-111

Angular momentum generators by U(2) analysis

Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Angular momentum commutation relations

Key Lie theorems

Angular momentum magnitude and uncertainty

Angular momentum uncertainty angle

Generating R(3) rotation and U(2) representations

Applications of R(3) rotation and U(2) representations

Molecular and nuclear wavefunctions

Molecular and nuclear eigenstates

Generalized Stern-Gerlach and transformation matrices

Angular momentum cones and high J properties

Angular momentum raise-n-lower operators $\mathbf{S}_+$ and $\mathbf{S}_-$

$\mathbf{s}_+ = \mathbf{s}_X + i\mathbf{s}_Y$ and $\mathbf{s}_- = \mathbf{s}_X - i\mathbf{s}_Y = \mathbf{s}_+^\dagger$

Starting with *j=1/2* we see that $\mathbf{S}_+$ is an *elementary projection operator* $\mathbf{e}_{12} = |1\rangle\langle 2| = \mathbf{P}_{12}$

$$\langle \mathbf{s}_+ \rangle^{\frac{1}{2}} = D^{\frac{1}{2}}(\mathbf{s}_+) = D^{\frac{1}{2}}(\mathbf{s}_X + i\mathbf{s}_Y) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + i \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{P}_{12}$$

Such operators can be upgraded to *creation-destruction operator* combinations $\mathbf{a}^\dagger\mathbf{a}$

$$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2 = \mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow \quad , \qquad \mathbf{s}_- = (\mathbf{a}_1^\dagger \mathbf{a}_2)^\dagger = \mathbf{a}_2^\dagger \mathbf{a}_1 = \mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow$$

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This suggests an $\mathbf{a}^\dagger\mathbf{a}$ form for $\mathbf{s}_Z$.

$$\mathbf{s}_Z = \tfrac{1}{2}(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2) = \tfrac{1}{2}(\mathbf{a}_\uparrow^\dagger \mathbf{a}_\uparrow - \mathbf{a}_\downarrow^\dagger \mathbf{a}_\downarrow)$$

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Let $\mathbf{a}_2^\dagger = \mathbf{a}_\downarrow^\dagger$ create dn-spin $\downarrow$

$$|2\rangle = |\downarrow\rangle = \left| \begin{matrix} 1/2 \\ -1/2 \end{matrix} \right\rangle = \mathbf{a}_2^\dagger|0\rangle = \mathbf{a}_\downarrow^\dagger|0\rangle$$

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$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2 = \mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow$ destroys dn-spin $\downarrow$

creates up-spin $\uparrow$

to raise angular momentum by one $\hbar$ unit

$$\mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow |\downarrow\rangle = |\uparrow\rangle \quad \text{or:} \quad \mathbf{a}_1^\dagger \mathbf{a}_2 |2\rangle = |1\rangle$$

Angular momentum raise-n-lower operators $\mathbf{S}_+$ and $\mathbf{S}_-$

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This suggests an $\mathbf{a}^\dagger\mathbf{a}$ form for $\mathbf{s}_Z$.

$$\mathbf{s}_Z = \tfrac{1}{2}(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2) = \tfrac{1}{2}(\mathbf{a}_\uparrow^\dagger \mathbf{a}_\uparrow - \mathbf{a}_\downarrow^\dagger \mathbf{a}_\downarrow)$$

Let $\mathbf{a}_1^\dagger = \mathbf{a}_\uparrow^\dagger$ create up-spin $\uparrow$

$$|1\rangle = |\uparrow\rangle = \left| \begin{matrix} 1/2 \\ +1/2 \end{matrix} \right\rangle = \mathbf{a}_1^\dagger|0\rangle = \mathbf{a}_\uparrow^\dagger|0\rangle$$

$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2 = \mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow$ destroys dn-spin $\downarrow$

creates up-spin $\uparrow$

to <u>raise</u> angular momentum by one $\hbar$ unit

$$\mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow |\downarrow\rangle = |\uparrow\rangle \quad \text{or:} \quad \mathbf{a}_1^\dagger \mathbf{a}_2 |2\rangle = |1\rangle$$

Let $\mathbf{a}_2^\dagger = \mathbf{a}_\downarrow^\dagger$ create dn-spin $\downarrow$

$$|2\rangle = |\downarrow\rangle = \left| \begin{matrix} 1/2 \\ -1/2 \end{matrix} \right\rangle = \mathbf{a}_2^\dagger|0\rangle = \mathbf{a}_\downarrow^\dagger|0\rangle$$

$\mathbf{s}_- = \mathbf{a}_2^\dagger \mathbf{a}_1 = \mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow$ destroys up-spin $\uparrow$

creates dn-spin $\downarrow$

to <u>lower</u> angular momentum by one $\hbar$ unit

$$\mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow |\uparrow\rangle = |\downarrow\rangle \quad \text{or:} \quad \mathbf{a}_2^\dagger \mathbf{a}_1 |1\rangle = |2\rangle$$

Review : *2-D* **a†a** *algebra of U(2) representations*

Review Lect.23 p113-125

Angular momentum generators by U(2) analysis
Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Angular momentum commutation relations
Key Lie theorems

Angular momentum magnitude and uncertainty
Angular momentum uncertainty angle

Generating R(3) rotation and U(2) representations
Applications of R(3) rotation and U(2) representations
Molecular and nuclear wavefunctions
Molecular and nuclear eigenstates
Generalized Stern-Gerlach and transformation matrices
Angular momentum cones and high J properties

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

U(2) boson oscillator states $|n_1,n_2\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$

$$|n_1 n_2\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}|0\,0\rangle$$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Review Lect.23 p113-125

U(2) boson oscillator states $|n_1,n_2\rangle$= R(3) spin or rotor states $\left|{j \atop m}\right\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$ Rotor total momenta: $j=\upsilon/2$

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SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

U(2) boson oscillator states $|n_1,n_2\rangle$ = R(3) spin or rotor states $\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$

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1/2-difference of number-ops is $\mathbf{s}_Z$ eigenvalue.

$$\left.\begin{array}{l}\mathbf{a}_1^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle=n_1\left|n_1 n_2\right\rangle \\ \mathbf{a}_2^\dagger\mathbf{a}_2\left|n_1 n_2\right\rangle=n_2\left|n_1 n_2\right\rangle\end{array}\right\}$$

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j=1 vector $\mathbf{s}_+$

$$D^1(\mathbf{s}_+)=D^1(\mathbf{s}_X+i\mathbf{s}_Y)=\begin{pmatrix} \cdot & \frac{\sqrt{2}}{2} & \cdot \\ \frac{\sqrt{2}}{2} & \cdot & \frac{\sqrt{2}}{2} \\ \cdot & \frac{\sqrt{2}}{2} & \cdot \end{pmatrix}+i\begin{pmatrix} \cdot & -i\frac{\sqrt{2}}{2} & \cdot \\ i\frac{\sqrt{2}}{2} & \cdot & -i\frac{\sqrt{2}}{2} \\ \cdot & i\frac{\sqrt{2}}{2} & \cdot \end{pmatrix}=\begin{pmatrix} \cdot & \sqrt{2} & \cdot \\ 0 & \cdot & \sqrt{2} \\ \cdot & 0 & \cdot \end{pmatrix},$$

...and $\mathbf{s}_Z$

$$D^1(\mathbf{s}_Z)=\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

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Oscillator $\mathbf{a}^\dagger\mathbf{a}$ give $\mathbf{s}_+$ and $\mathbf{s}_-$ matrices. 1/2-difference of number-ops is $\mathbf{s}_Z$ eigenvalue.

$$\mathbf{a}_1^\dagger\mathbf{a}_2|n_1 n_2\rangle=\sqrt{n_1+1}\sqrt{n_2}\,|n_1+1\; n_2-1\rangle \Rightarrow \mathbf{s}_+\left|\begin{smallmatrix} j \\ m\end{smallmatrix}\right\rangle=\sqrt{j+m+1}\sqrt{j-m}\left|\begin{smallmatrix} j \\ m+1\end{smallmatrix}\right\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1|n_1 n_2\rangle=\sqrt{n_1}\sqrt{n_2+1}\,|n_1-1\; n_2+1\rangle \Rightarrow \mathbf{s}_-\left|\begin{smallmatrix} j \\ m\end{smallmatrix}\right\rangle=\sqrt{j+m}\sqrt{j-m+1}\left|\begin{smallmatrix} j \\ m-1\end{smallmatrix}\right\rangle$$

$$\left.\begin{matrix}\mathbf{a}_1^\dagger\mathbf{a}_1|n_1 n_2\rangle=n_1|n_1 n_2\rangle \\ \mathbf{a}_2^\dagger\mathbf{a}_2|n_1 n_2\rangle=n_2|n_1 n_2\rangle\end{matrix}\right\}\; \mathbf{s}_Z\left|\begin{smallmatrix} j \\ m\end{smallmatrix}\right\rangle=\tfrac{1}{2}(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2)\left|\begin{smallmatrix} j \\ m\end{smallmatrix}\right\rangle=\frac{n_1-n_2}{2}\left|\begin{smallmatrix} j \\ m\end{smallmatrix}\right\rangle=m\left|\begin{smallmatrix} j \\ m\end{smallmatrix}\right\rangle$$

j=1 vector $\mathbf{s}_+$

$$D^1(\mathbf{s}_+)=D^1(\mathbf{s}_X+i\mathbf{s}_Y)=\begin{pmatrix} \cdot & \frac{\sqrt{2}}{2} & \cdot \\ \frac{\sqrt{2}}{2} & \cdot & \frac{\sqrt{2}}{2} \\ \cdot & \frac{\sqrt{2}}{2} & \cdot \end{pmatrix}+i\begin{pmatrix} \cdot & -i\frac{\sqrt{2}}{2} & \cdot \\ i\frac{\sqrt{2}}{2} & \cdot & -i\frac{\sqrt{2}}{2} \\ \cdot & i\frac{\sqrt{2}}{2} & \cdot \end{pmatrix}=\begin{pmatrix} \cdot & \sqrt{2} & \cdot \\ 0 & \cdot & \sqrt{2} \\ \cdot & 0 & \cdot \end{pmatrix},$$

...and $\mathbf{s}_Z$

$$D^1(\mathbf{s}_Z)=\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

j=3/2 spinor $\mathbf{s}_+$ *...and* $\mathbf{s}_Z$

$$D^{\frac{3}{2}}(\mathbf{s}_+)=\begin{pmatrix} \cdot & \sqrt{3} & \cdot & \cdot \\ 0 & \cdot & \sqrt{4} & \cdot \\ \cdot & 0 & \cdot & \sqrt{3} \\ \cdot & \cdot & 0 & \cdot \end{pmatrix}=\left(D^{\frac{3}{2}}(\mathbf{s}_-)\right)^\dagger,\qquad D^{\frac{3}{2}}(\mathbf{s}_Z)=\begin{pmatrix} \frac{3}{2} & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{2} & \cdot & \cdot \\ \cdot & \cdot & -\frac{1}{2} & \cdot \\ \cdot & \cdot & \cdot & -\frac{3}{2} \end{pmatrix}$$

$SU(2)\subset U(2)$ oscillators vs. $R(3)\subset O(3)$ rotors

$U(2)$ boson oscillator states $|n_1,n_2\rangle$ = R(3) spin or rotor states $\left|{}^{j}_{m}\right\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$ Rotor total momenta: $j=\upsilon/2$ and z-momenta: $m=(n_1-n_2)/2$

$$|n_1 n_2\rangle = \frac{(\mathbf{a}_1^\dagger)^{n_1}(\mathbf{a}_2^\dagger)^{n_2}}{\sqrt{n_1!\,n_2!}}|0\,0\rangle = \frac{(\mathbf{a}_1^\dagger)^{j+m}(\mathbf{a}_2^\dagger)^{j-m}}{\sqrt{(j+m)!(j-m)!}}|0\,0\rangle = \left|{}^{j}_{m}\right\rangle$$

$j=\upsilon/2=(n_1+n_2)/2$, $m=(n_1-n_2)/2$

$n_1=j+m$, $n_2=j-m$

$U(2)$ boson oscillator states = $U(2)$ spinor states

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$$\mathbf{a}_2^\dagger\mathbf{a}_1|n_1 n_2\rangle=\sqrt{n_1}\sqrt{n_2+1}|n_1-1\; n_2+1\rangle \Rightarrow \mathbf{s}_-\left|{}^{j}_{m}\right\rangle=\sqrt{j+m}\sqrt{j-m+1}\left|{}^{j}_{m-1}\right\rangle$$

1/2-difference of number-ops is $\mathbf{s}_Z$ eigenvalue.

$$\mathbf{a}_1^\dagger\mathbf{a}_1|n_1 n_2\rangle=n_1|n_1 n_2\rangle,\quad \mathbf{a}_2^\dagger\mathbf{a}_2|n_1 n_2\rangle=n_2|n_1 n_2\rangle$$

$$\mathbf{s}_Z\left|{}^{j}_{m}\right\rangle=\tfrac{1}{2}(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2)\left|{}^{j}_{m}\right\rangle=\frac{n_1-n_2}{2}\left|{}^{j}_{m}\right\rangle=m\left|{}^{j}_{m}\right\rangle$$

$j=1$ vector $\mathbf{s}_+$ ...and $\mathbf{s}_Z$

$$D^1(\mathbf{s}_+)=D^1(\mathbf{s}_X+i\mathbf{s}_Y)=\begin{pmatrix} \cdot & \frac{\sqrt{2}}{2} & \cdot \\ \frac{\sqrt{2}}{2} & \cdot & \frac{\sqrt{2}}{2} \\ \cdot & \frac{\sqrt{2}}{2} & \cdot \end{pmatrix}+i\begin{pmatrix} \cdot & -i\frac{\sqrt{2}}{2} & \cdot \\ i\frac{\sqrt{2}}{2} & \cdot & -i\frac{\sqrt{2}}{2} \\ \cdot & i\frac{\sqrt{2}}{2} & \cdot \end{pmatrix}=\begin{pmatrix} \cdot & \sqrt{2} & \cdot \\ 0 & \cdot & \sqrt{2} \\ \cdot & 0 & \cdot \end{pmatrix},\qquad D^1(\mathbf{s}_Z)=\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

$j=3/2$ spinor $\mathbf{s}_+$ ...and $\mathbf{s}_Z$

$$D^{\frac{3}{2}}(\mathbf{s}_+)=\begin{pmatrix} \cdot & \sqrt{3} & \cdot & \cdot \\ 0 & \cdot & \sqrt{4} & \cdot \\ \cdot & 0 & \cdot & \sqrt{3} \\ \cdot & \cdot & 0 & \cdot \end{pmatrix}=\left(D^{\frac{3}{2}}(\mathbf{s}_-)\right)^\dagger,\qquad D^{\frac{3}{2}}(\mathbf{s}_Z)=\begin{pmatrix} \frac{3}{2} & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{2} & \cdot & \cdot \\ \cdot & \cdot & -\frac{1}{2} & \cdot \\ \cdot & \cdot & \cdot & -\frac{3}{2} \end{pmatrix}$$

$j=2$ tensor $\mathbf{s}_+$ ...and $\mathbf{s}_Z$

$$D^2(\mathbf{s}_+)=\begin{pmatrix} \cdot & \sqrt{4} & \cdot & \cdot & \cdot \\ 0 & \cdot & \sqrt{3} & \cdot & \cdot \\ \cdot & 0 & \cdot & \sqrt{3} & \cdot \\ \cdot & \cdot & 0 & \cdot & \sqrt{4} \\ \cdot & \cdot & \cdot & 0 & \cdot \end{pmatrix}=\left(D^2(\mathbf{s}_-)\right)^\dagger,\qquad D^2(\mathbf{s}_Z)=\begin{pmatrix} 2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & -2 \end{pmatrix}$$

Review : *2-D* $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(2) representations*

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Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$
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Angular momentum commutation relations
  Key Lie theorems

Angular momentum magnitude and uncertainty
  Angular momentum uncertainty angle

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Given Hamilton-Jordan-Pauli product relations : $\sigma_\alpha \sigma_\beta = \delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma$

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QED

Then there are *up-down commutation relation*: $[\mathbf{s}_+, \mathbf{s}_-] = [\mathbf{e}_{12}, \mathbf{e}_{21}] = \mathbf{e}_{11} - \mathbf{e}_{22} = 2\mathbf{s}_Z$

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General eigen-commutation theorem:

If Hamiltonian $\mathbf{H}$ (or any operator such as $\mathbf{s}_Z$) eigen-commutes with $\mathbf{a}_m$ and $\mathbf{a}^\dagger_n$, that is:

$[\mathbf{H}, \mathbf{a}^\dagger_n] = \omega_n\,\mathbf{a}^\dagger_n$ and $[\mathbf{H}, \mathbf{a}_m] = \omega_m \mathbf{a}_m$, then $\mathbf{H}$ is a combination $\omega_n \mathbf{a}^\dagger_n \mathbf{a}_n$ of number operators.

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$\mathbf{e}_{jk}\,\mathbf{e}_{mn} = \delta_{km}\,\mathbf{e}_{jn}$ gives: $[(\mathbf{e}_{11} - \mathbf{e}_{22})/2, \mathbf{e}_{12}] = +\mathbf{e}_{12}$ and: $[(\mathbf{e}_{11} - \mathbf{e}_{22})/2, \mathbf{e}_{21}] = -\mathbf{e}_{21}$

QED

Then there are *up-down commutation relation*: $[\mathbf{s}_+, \mathbf{s}_-] = [\mathbf{e}_{12}, \mathbf{e}_{21}] = \mathbf{e}_{11} - \mathbf{e}_{22} = 2\mathbf{s}_Z$

General eigen-commutation theorem:

If Hamiltonian $\mathbf{H}$ (or any operator such as $\mathbf{s}_Z$) eigen-commutes with $\mathbf{a}_m$ and $\mathbf{a}^\dagger_n$, that is: $[\mathbf{H},\mathbf{a}^\dagger_n] = \omega_n\,\mathbf{a}^\dagger_n$ and $[\mathbf{H},\mathbf{a}_m] = \omega_m\mathbf{a}_m$, then $\mathbf{H}$ is a combination $\omega_n\mathbf{a}^\dagger_n\mathbf{a}_n$ of number operators.

$$\mathbf{H} = \sum_{n=1}^{2} \omega_n\mathbf{a}^\dagger_n\mathbf{a}_n = \omega_1\mathbf{a}^\dagger_1\mathbf{a}_1 + \omega_2\mathbf{a}^\dagger_2\mathbf{a}_2 \approx \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}$$

Angular momentum commutation relations

Given Hamilton-Jordan-Pauli product relations : $\sigma_\alpha \sigma_\beta = \delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma$ with: $\mathbf{s}_\alpha = \sigma_\alpha/2$

Commutator formulae for $\mathbf{s}_\alpha$: $\mathbf{s}_\alpha \mathbf{s}_\beta - \mathbf{s}_\beta \mathbf{s}_\alpha = [\mathbf{s}_\alpha, \mathbf{s}_\beta] = i\, \varepsilon_{\alpha\beta\gamma} \mathbf{s}_\gamma$

$\sigma_X \sigma_Y = i\sigma_Z$ implies: $[\mathbf{s}_X, \mathbf{s}_Y] = i\mathbf{s}_Z$

$\sigma_Z \sigma_X = i\sigma_Y$ implies: $[\mathbf{s}_Z, \mathbf{s}_X] = i\mathbf{s}_Y$

$\sigma_Y \sigma_Z = i\sigma_X$ implies: $[\mathbf{s}_Y, \mathbf{s}_Z] = i\mathbf{s}_X$

Key Lie theorem:

$\mathbf{s}_Z$ and $\mathbf{s}_\pm = \mathbf{s}_X \pm i\mathbf{s}_Y$ obey *eigen-commutation relations*. $[\mathbf{s}_Z, \mathbf{s}_+] = (+1)\mathbf{s}_+$ and: $[\mathbf{s}_Z, \mathbf{s}_-] = (-1)\mathbf{s}_-$

Proof using elementary matrix operator multiplication: $\mathbf{e}_{jk}\, \mathbf{e}_{mn} = \delta_{km}\, \mathbf{e}_{jn}$ with: $\mathbf{s}_+ = \mathbf{e}_{12}$ and: $\mathbf{s}_- = \mathbf{e}_{21}$

Also: $\mathbf{s}_Z = (\mathbf{e}_{11} - \mathbf{e}_{22})/2 \approx \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$ $\approx \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\approx \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$\mathbf{e}_{jk}\, \mathbf{e}_{mn} = \delta_{km}\, \mathbf{e}_{jn}$ gives: $[(\mathbf{e}_{11} - \mathbf{e}_{22})/2, \mathbf{e}_{12}] = +\mathbf{e}_{12}$ and: $[(\mathbf{e}_{11} - \mathbf{e}_{22})/2, \mathbf{e}_{21}] = -\mathbf{e}_{21}$

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Then there are *up-down commutation relation*: $[\mathbf{s}_+, \mathbf{s}_-] = [\mathbf{e}_{12}, \mathbf{e}_{21}] = \mathbf{e}_{11} - \mathbf{e}_{22} = 2\mathbf{s}_Z$

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$$\mathbf{H} = \sum_{n=1}^{2} \omega_n \mathbf{a}^\dagger_n \mathbf{a}_n = \omega_1 \mathbf{a}^\dagger_1 \mathbf{a}_1 + \omega_2 \mathbf{a}^\dagger_2 \mathbf{a}_2 \approx \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}$$

U(2) Oscillator eigensolutions:

$$\mathbf{H}\left|n_1 n_2\right\rangle = \sum_{n=1}^{2} \omega_n \mathbf{a}^\dagger_n \mathbf{a}_n \left|n_1 n_2\right\rangle = \left(\omega_1 n_1 + \omega_2 n_2\right)\left|n_1 n_2\right\rangle = \left(\omega_1 (j+m) + \omega_2 (j-m)\right)\left|{}^{j}_{m}\right\rangle$$

Review : *2-D* $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(2) representations*

Angular momentum generators by U(2) analysis
Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Angular momentum commutation relations
Key Lie theorems

→ *Angular momentum magnitude and uncertainty* ←
Angular momentum uncertainty angle

Generating R(3) rotation and U(2) representations
Applications of R(3) rotation and U(2) representations
Molecular and nuclear wavefunctions
Molecular and nuclear eigenstates
Generalized Stern-Gerlach and transformation matrices
Angular momentum cones and high J properties

Angular momentum magnitude and uncertainty

$\mathbf{s}_{\pm}=\mathbf{s}_X \pm i\mathbf{s}_Y$

$\mathbf{s}_{+}=\mathbf{a}_1^{\dagger}\mathbf{a}_2$

$\mathbf{s}_{-}=\mathbf{a}_2^{\dagger}\mathbf{a}_1$

$\mathbf{s}_Z=\frac{1}{2}\left(\mathbf{a}_1^{\dagger}\mathbf{a}_1-\mathbf{a}_2^{\dagger}\mathbf{a}_2\right)$

Angular momentum squared $\mathbf{s}\bullet\mathbf{s}$ and Z-component $\mathbf{s}_Z$ share eigenstates

$$\mathbf{s}\bullet\mathbf{s} = \mathbf{s}_X{}^2 + \mathbf{s}_Y{}^2 + \mathbf{s}_Z{}^2 = (\mathbf{s}_+\mathbf{s}_- + \mathbf{s}_-\mathbf{s}_+)/2 + \mathbf{s}_Z{}^2$$

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j=1/2 fundamental matrices square up not to *(1/2)*^{2} *=1/4* but to *3/4*.

$$D^{\frac{1}{2}}\left(\mathbf{S}_X^2 + \mathbf{S}_Y^2 + \mathbf{S}_Z^2\right) = \frac{1}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{3}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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$$\mathbf{s}\bullet\mathbf{s}=\frac{1}{4}\left[2\mathbf{a}_1^\dagger\mathbf{a}_2\mathbf{a}_2^\dagger\mathbf{a}_1+2\mathbf{a}_2^\dagger\mathbf{a}_1\mathbf{a}_1^\dagger\mathbf{a}_2+\left(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2\right)\left(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2\right)\right]$$

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Angular momentum magnitude and uncertainty

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$$\mathbf{s}\bullet\mathbf{s}=\frac{1}{4}\left[2\mathbf{a}_1^\dagger\mathbf{a}_2\mathbf{a}_2^\dagger\mathbf{a}_1+2\mathbf{a}_2^\dagger\mathbf{a}_1\mathbf{a}_1^\dagger\mathbf{a}_2+\left(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2\right)\left(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2\right)\right]$$

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Using $\mathbf{a}_m\mathbf{a}^\dagger_n = \mathbf{a}^\dagger_n\mathbf{a}_m + \delta_{mn}\mathbf{1}$ gives $\mathbf{s}\bullet\mathbf{s}$ as number operators.
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$$\mathbf{s}\bullet\mathbf{s} = \frac{1}{4}\left[2\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + \mathbf{1}\right)\mathbf{a}_1^\dagger\mathbf{a}_1 + 2\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{1}\right)\mathbf{a}_2^\dagger\mathbf{a}_2 + \left(\mathbf{a}_1^\dagger\mathbf{a}_1 - \mathbf{a}_2^\dagger\mathbf{a}_2\right)\left(\mathbf{a}_1^\dagger\mathbf{a}_1 - \mathbf{a}_2^\dagger\mathbf{a}_2\right)\right]$$

Eigenvalue formula is then found. (Replace number-operator $\mathbf{a}^\dagger_k\mathbf{a}_k$ with its number n_k)

$$\mathbf{S}\bullet\mathbf{S}\left|n_1 n_2\right\rangle = \frac{1}{4}\left[2\left(n_2+1\right)n_1 + 2\left(n_1+1\right)n_2 + \left(n_1-n_2\right)\left(n_1-n_2\right)\right]\left|n_1 n_2\right\rangle$$

$$= \frac{1}{4}\left[2n_1 + 2n_2 + 4n_1 n_2 + \left(n_1-n_2\right)\left(n_1-n_2\right)\right]\left|n_1 n_2\right\rangle$$

Angular momentum magnitude and uncertainty

$\mathbf{s}_{\pm}=\mathbf{s}_X \pm i\mathbf{s}_Y$

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$$\mathbf{s}\bullet\mathbf{s}=\frac{1}{4}\left[2\mathbf{a}_1^\dagger\mathbf{a}_2\mathbf{a}_2^\dagger\mathbf{a}_1+2\mathbf{a}_2^\dagger\mathbf{a}_1\mathbf{a}_1^\dagger\mathbf{a}_2+\left(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2\right)\left(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2\right)\right]$$

Using $\mathbf{a}_m\mathbf{a}^\dagger_n = \mathbf{a}^\dagger_n\,\mathbf{a}_m + \delta_{mn}\mathbf{1}$ gives $\mathbf{s}\bullet\mathbf{s}$ as number operators.

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$$\mathbf{s}\bullet\mathbf{s}\left|n_1n_2\right\rangle=\frac{1}{4}\left[2\left(n_2+1\right)n_1+2\left(n_1+1\right)n_2+\left(n_1-n_2\right)\left(n_1-n_2\right)\right]\left|n_1n_2\right\rangle$$

$$=\frac{1}{4}\left[2n_1+2n_2+4n_1n_2+\left(n_1-n_2\right)\left(n_1-n_2\right)\right]\left|n_1n_2\right\rangle$$

$j=\upsilon/2=(n_1+n_2)/2$

$m=(n_1-n_2)/2$

R(3) angular quanta in $n_1=j+m$ and $n_2=j-m$ give R(3) eigenvalue formula.

$n_1=j+m$

$n_2=j-m$

$$\mathbf{s}\bullet\mathbf{s}\left|{}^{j}_{m}\right\rangle=\frac{1}{4}\left[2\left(j+m+1\right)\left(j-m\right)+2\left(j-m+1\right)\left(j+m\right)+4m^2\right]\left|{}^{j}_{m}\right\rangle$$

Has very simple j-formula…

$\mathbf{s}_\pm=\mathbf{s}_X\pm i\mathbf{s}_Y$

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$$=\frac{1}{4}\left[2n_1+2n_2+4n_1n_2+\left(n_1-n_2\right)\left(n_1-n_2\right)\right]\left|n_1n_2\right\rangle$$

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$n_1=j+m$

$n_2=j-m$

$$\mathbf{s}\bullet\mathbf{s}\left|{}^j_m\right\rangle=\frac{1}{4}\left[2\left(j+m+1\right)\left(j-m\right)+2\left(j-m+1\right)\left(j+m\right)+4m^2\right]\left|{}^j_m\right\rangle=j\left(j+1\right)\left|{}^j_m\right\rangle$$

Has very simple j-formula…

Angular momentum magnitude and uncertainty

$\mathbf{s}_\pm = \mathbf{s}_X \pm i\mathbf{s}_Y$

$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2$

$\mathbf{s}_- = \mathbf{a}_2^\dagger \mathbf{a}_1$

$\mathbf{s}_Z = \frac{1}{2}\left(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2\right)$

Angular momentum squared $\mathbf{s}\bullet\mathbf{s}$ and Z-component $\mathbf{s}_Z$ share eigenstates

$$\mathbf{s}\bullet\mathbf{s} = \mathbf{s}_X^2 + \mathbf{s}_Y^2 + \mathbf{s}_Z^2 = (\mathbf{s}_+\mathbf{s}_- + \mathbf{s}_-\mathbf{s}_+)/2 + \mathbf{s}_Z^2$$

$j=1/2$ fundamental matrices square up not to $(1/2)^2 = 1/4$ but to $3/4$.

$$D^{\frac{1}{2}}\left(\mathbf{s}_X^2 + \mathbf{s}_Y^2 + \mathbf{s}_Z^2\right) = \frac{1}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{3}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In terms of $\mathbf{a}$-operators the squared momentum operator is

$$\mathbf{s}\bullet\mathbf{s} = \frac{1}{4}\left[2\mathbf{a}_1^\dagger \mathbf{a}_2 \mathbf{a}_2^\dagger \mathbf{a}_1 + 2\mathbf{a}_2^\dagger \mathbf{a}_1 \mathbf{a}_1^\dagger \mathbf{a}_2 + \left(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2\right)\left(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2\right)\right]$$

Using $\mathbf{a}_m \mathbf{a}^\dagger_n = \mathbf{a}^\dagger_n \mathbf{a}_m + \delta_{mn}\mathbf{1}$ gives $\mathbf{s}\bullet\mathbf{s}$ as number operators.
(Normal order: *left←creation*, *destruct→right*.)

$$\mathbf{s}\bullet\mathbf{s} = \frac{1}{4}\left[2\left(\mathbf{a}_2^\dagger \mathbf{a}_2 + \mathbf{1}\right)\mathbf{a}_1^\dagger \mathbf{a}_1 + 2\left(\mathbf{a}_1^\dagger \mathbf{a}_1 + \mathbf{1}\right)\mathbf{a}_2^\dagger \mathbf{a}_2 + \left(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2\right)\left(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2\right)\right]$$

U(2) eigenvalue formula is then found.

$$\mathbf{s}\bullet\mathbf{s}\left|n_1 n_2\right\rangle = \frac{1}{4}\left[2\left(n_2+1\right)n_1 + 2\left(n_1+1\right)n_2 + \left(n_1-n_2\right)\left(n_1-n_2\right)\right]\left|n_1 n_2\right\rangle$$

$$= \frac{1}{4}\left[2n_1 + 2n_2 + 4n_1 n_2 + \left(n_1-n_2\right)\left(n_1-n_2\right)\right]\left|n_1 n_2\right\rangle$$

$j = \upsilon/2 = (n_1+n_2)/2$

$m = (n_1-n_2)/2$

R(3) angular quanta in $n_1=j+m$ and $n_2=j-m$ give R(3) eigenvalue formula.

$n_1 = j+m$

$n_2 = j-m$

$$\mathbf{s}\bullet\mathbf{s}\left|{}^{j}_{m}\right\rangle = \frac{1}{4}\left[2\left(j+m+1\right)\left(j-m\right) + 2\left(j-m+1\right)\left(j+m\right) + 4m^2\right]\left|{}^{j}_{m}\right\rangle = j\left(j+1\right)\left|{}^{j}_{m}\right\rangle$$

For Large j:

Magnitude of angular momentum $|\mathbf{s}|$ approaches $j+1/2$: $\quad |\mathbf{s}|\left|{}^{j}_{m}\right\rangle = \sqrt{\mathbf{s}\bullet\mathbf{s}}\left|{}^{j}_{m}\right\rangle = \sqrt{j\left(j+1\right)}\left|{}^{j}_{m}\right\rangle \cong \left(j+\frac{1}{2}\right)\left|{}^{j}_{m}\right\rangle$

Review : *2-D* **a†a** *algebra of U(2) representations*

Angular momentum generators by U(2) analysis
Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Angular momentum commutation relations
Key Lie theorems

Angular momentum magnitude and uncertainty
Angular momentum uncertainty angle

Generating R(3) rotation and U(2) representations
Applications of R(3) rotation and U(2) representations
Molecular and nuclear wavefunctions
Molecular and nuclear eigenstates
Generalized Stern-Gerlach and transformation matrices
Angular momentum cones and high J properties

Angular momentum uncertainty angle

The *angular momentum uncertainty angle* Θ_m^j is given by:

$$\Theta_m^j = \arccos\left(\frac{m}{\sqrt{j\left(j+1\right)}}\right)$$

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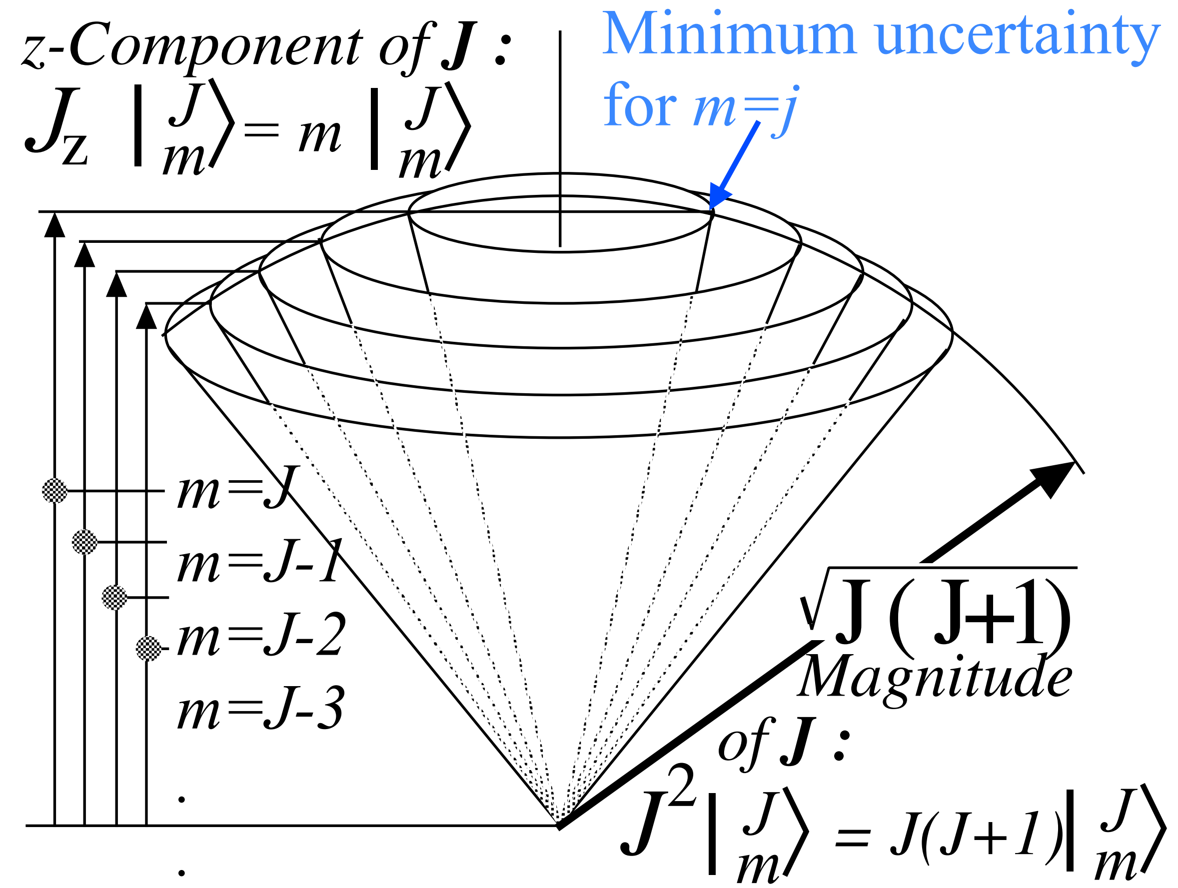

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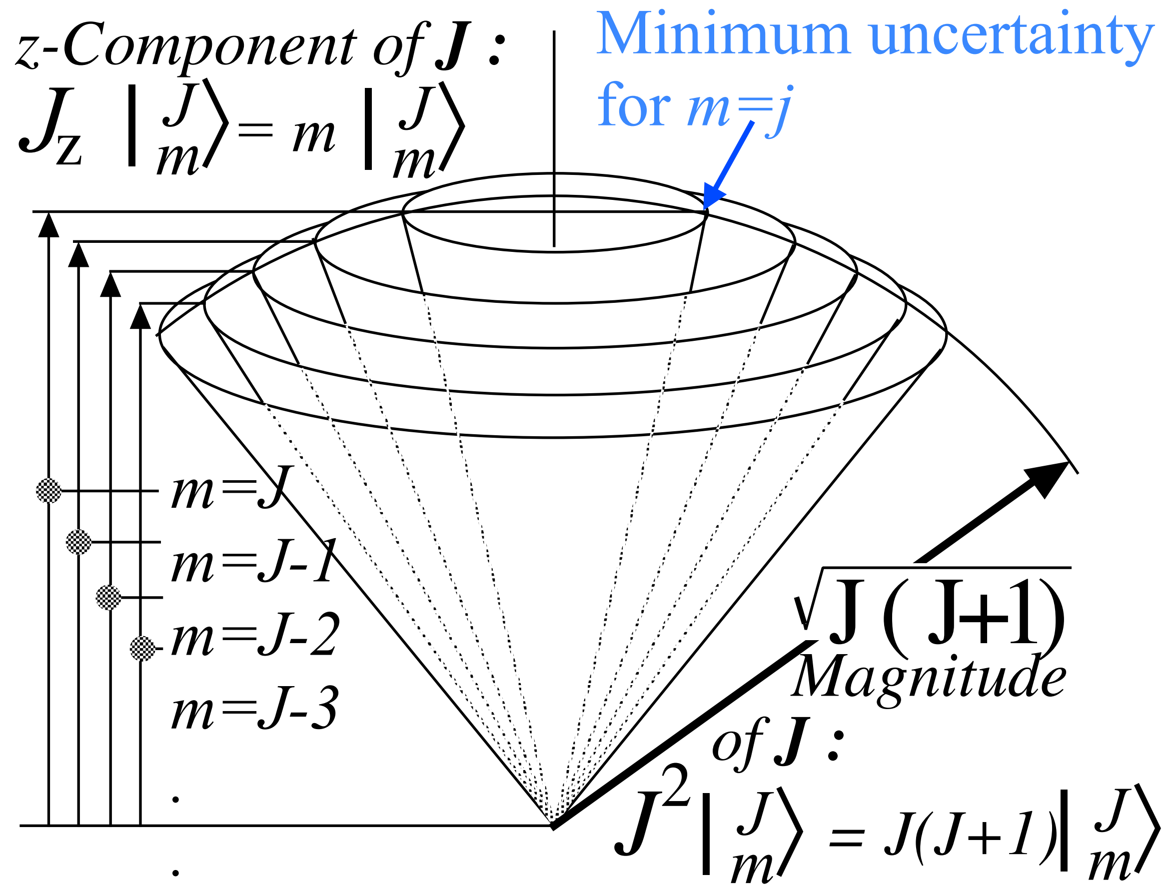

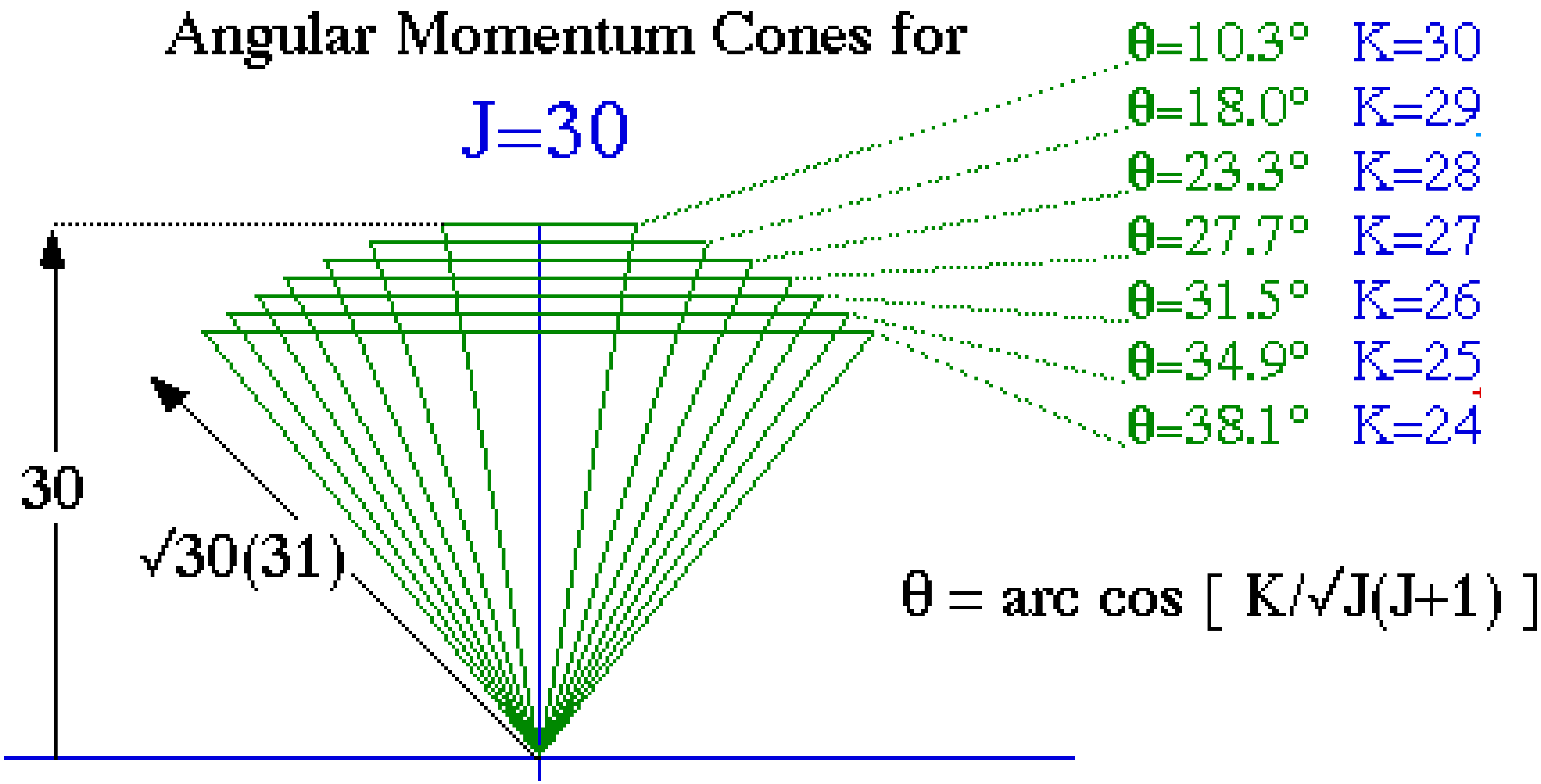

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$$\begin{pmatrix} \langle 1|1'\rangle & \langle 1|2'\rangle \\ \langle 2|1'\rangle & \langle 2|2'\rangle \end{pmatrix} = \begin{pmatrix} \langle 1|\mathbf{R}(\alpha\beta\gamma)|1\rangle & \langle 1|\mathbf{R}(\alpha\beta\gamma)|2\rangle \\ \langle 2|\mathbf{R}(\alpha\beta\gamma)|1\rangle & \langle 2|\mathbf{R}(\alpha\beta\gamma)|2\rangle \end{pmatrix} = \begin{pmatrix} D^{1/2}_{11}(\alpha\beta\gamma) & D^{1/2}_{12}(\alpha\beta\gamma) \\ D^{1/2}_{21}(\alpha\beta\gamma) & D^{1/2}_{22}(\alpha\beta\gamma) \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}$$

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$$\mathbf{a}^\dagger_{1'} = D^{1/2}_{11}(\alpha\beta\gamma)\mathbf{a}^\dagger_1 + D^{1/2}_{21}(\alpha\beta\gamma)\mathbf{a}^\dagger_2 = e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\mathbf{a}^\dagger_1 + e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\mathbf{a}^\dagger_2$$

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Problem: Find corresponding transformation $D^{(j)}(\alpha\beta\gamma)$ matrix for a $(\upsilon=2j)$-oscillator state

$(\upsilon=2j)$-quantum state is rotated to a new "prime" basis.

$$\mathbf{R}(\alpha\beta\gamma)\left|{}^{j}_{n}\right\rangle = \frac{\left(\mathbf{a}^\dagger_{1'}\right)^{j+n}\left(\mathbf{a}^\dagger_{2'}\right)^{j-n}}{\sqrt{(j+n)!(j-n)!}}|0\,0\rangle = \frac{\left(D_{11}\mathbf{a}^\dagger_1 + D_{21}\mathbf{a}^\dagger_2\right)^{j+n}\left(D_{21}\mathbf{a}^\dagger_1 + D_{22}\mathbf{a}^\dagger_2\right)^{j-n}}{\sqrt{(j+n)!(j-n)!}}|0\,0\rangle$$

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Binomial expansion is a double sum over binomial coefficients: $\begin{pmatrix} n \\ k \end{pmatrix} = n!/\,k!(n-k)!$

$$\mathbf{R}(\alpha\beta\gamma)\left|{}^{j}_{n}\right\rangle = \frac{\sum_\ell\sum_k \binom{j+n}{\ell}(D_{11}\mathbf{a}^\dagger_1)^\ell(D_{21}\mathbf{a}^\dagger_2)^{j+n-\ell}\binom{j-n}{k}(D_{12}\mathbf{a}^\dagger_1)^k(D_{22}\mathbf{a}^\dagger_2)^{j-n-k}}{\sqrt{(j+n)!(j-n)!}}|0\,0\rangle \qquad = \sqrt{(j+n)!(j-n)!}\frac{\sum_\ell\sum_k (D_{11}\mathbf{a}^\dagger_1)^\ell(D_{21}\mathbf{a}^\dagger_2)^{j+n-\ell}(D_{12}\mathbf{a}^\dagger_1)^k(D_{22}\mathbf{a}^\dagger_2)^{j-n-k}}{\ell!(j+n-\ell)!k!(j-n-k)!}|0\,0\rangle$$

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Let $\mathbf{a}^\dagger$-operator powers be $j\pm m$ forms : $j+m=\ell+k$, $j-m=2j-\ell-k$ so $\ell=j+m-k$ and $j+n-\ell=n-m+k$

$$= \sqrt{(j+n)!(j-n)!}\frac{\sum_\ell \sum_k (D_{11})^\ell (D_{21})^{j+n-\ell}(D_{12})^k (D_{22})^{j-n-k}}{\ell!(j+n-\ell)!k!(j-n-k)!}(\mathbf{a}^\dagger_1)^{\ell+k}(\mathbf{a}^\dagger_2)^{2j-\ell-k}|0\,0\rangle = \sqrt{(j+n)!(j-n)!}\frac{\sum_m \sum_k (D_{11})^{j+m-k}(D_{21})^{n-m+k}(D_{12})^k (D_{22})^{j-n-k}}{(j+m-k)!(n-m+k)!k!(j-n-k)!}(\mathbf{a}^\dagger_1)^{j+m}(\mathbf{a}^\dagger_2)^{j-m}|0\,0\rangle$$

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This gives general *irreducible representation of U(2)* :

$$\left\langle {}^{j}_{m}\right|\mathbf{R}(\alpha\beta\gamma)\left|{}^{j}_{n}\right\rangle = D^{j}_{m,n}(\alpha\beta\gamma) = \sqrt{(j+n)!(j-n)!}\sqrt{(j+m)!(j-m)!}\,\frac{\sum_k \left(D_{11}\right)^{j+m-k}\left(D_{21}\right)^{n-m+k}\left(D_{12}\right)^{k}\left(D_{22}\right)^{j-n-k}}{(j+m-k)!(n-m+k)!k!(j-n-k)!}$$

And general *SU(2) irreducible representation for Euler angles (αβγ)*.

$$\left\langle {}^{j}_{m}\right|\mathbf{R}(\alpha\beta\gamma)\left|{}^{j}_{n}\right\rangle = D^{j}_{m,n}(\alpha\beta\gamma) = \sqrt{(j+n)!(j-n)!}\sqrt{(j+m)!(j-m)!}\,\frac{\sum_k (-1)^k \left(\cos\frac{\beta}{2}\right)^{2j+m-n-2k}\left(\sin\frac{\beta}{2}\right)^{n-m+2k} e^{-i(m\alpha+n\gamma)}}{(j+m-k)!(n-m+k)!k!(j-n-k)!}$$

k-sum limited by (-*integer*)!=∞ and 0!=1=1!

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Move to Lect. 25 for up-to-date graphics

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Applications of R(3) rotation and U(2) representations

Vector (j=ℓ=1) representation

$$D^1(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha} \end{pmatrix}\begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\gamma} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\gamma} \end{pmatrix}=\begin{pmatrix} e^{-i\alpha}\frac{1+\cos\beta}{2}e^{-i\gamma} & e^{-i\alpha}\frac{-\sin\beta}{\sqrt{2}} & e^{-i\alpha}\frac{1-\cos\beta}{2}e^{i\gamma} \\ \frac{\sin\beta}{\sqrt{2}}e^{-i\gamma} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}}e^{i\gamma} \\ e^{i\alpha}\frac{1-\cos\beta}{2}e^{-i\gamma} & e^{i\alpha}\frac{\sin\beta}{\sqrt{2}} & e^{i\alpha}\frac{1+\cos\beta}{2}e^{i\gamma} \end{pmatrix}$$

Here half-angle identities were used. $\cos^2\frac{\beta}{2}=\frac{1+\cos\beta}{2},\ \sin^2\frac{\beta}{2}=\frac{1-\cos\beta}{2},\ \sin\frac{\beta}{2}\cos\frac{\beta}{2}=\frac{\sin\beta}{2},$

Move to Lect. 25 for up-to-date graphics

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Center (*n=0*) column with the factor $\sqrt{\frac{2\ell+1}{4\pi}}$ gives set of *spherical harmonics* Y^ℓ_m.

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Dipole ($j=\ell=1$) wave functions

$$D^{1*}_{1,0}(\phi\theta 0)=-e^{i\phi}\frac{\sin\theta}{\sqrt{2}}=-\frac{\cos\phi\sin\theta+i\cos\phi\sin\theta}{\sqrt{2}}=-\frac{x+iy}{r\sqrt{2}}$$

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3-D linear-circular polarization T-matrix.

$$\begin{pmatrix} \left\langle {1 \atop 1}\middle| {1 \atop x}\right\rangle & \left\langle {1 \atop 1}\middle| {1 \atop y}\right\rangle & \left\langle {1 \atop 1}\middle| {1 \atop z}\right\rangle \\ \left\langle {1 \atop 0}\middle| {1 \atop x}\right\rangle & \left\langle {1 \atop 0}\middle| {1 \atop y}\right\rangle & \left\langle {1 \atop 0}\middle| {1 \atop z}\right\rangle \\ \left\langle {1 \atop -1}\middle| {1 \atop x}\right\rangle & \left\langle {1 \atop -1}\middle| {1 \atop y}\right\rangle & \left\langle {1 \atop -1}\middle| {1 \atop z}\right\rangle \end{pmatrix}=\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \end{pmatrix}$$

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Applying T-matrix:

$$\begin{pmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & 1 & 0\end{pmatrix}\cdot\begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2}\end{pmatrix}\cdot\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0\end{pmatrix}=\begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \sin\beta\end{pmatrix}$$

Dipole (j=ℓ=1) wave functions

$$D^{1*}_{1,0}(\phi\theta 0)=-e^{i\phi}\frac{\sin\theta}{\sqrt{2}}=-\frac{\cos\phi\sin\theta+i\cos\phi\sin\theta}{\sqrt{2}}=-\frac{x+iy}{r\sqrt{2}}$$

$$D^{1*}_{0,0}(\phi\theta 0)=\cos\theta=\cos\theta=z/r$$

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3-D linear-circular polarization T-matrix:

$$\begin{pmatrix} \left\langle \begin{smallmatrix}1\\1\end{smallmatrix}\middle|\begin{smallmatrix}1\\x\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\1\end{smallmatrix}\middle|\begin{smallmatrix}1\\y\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\1\end{smallmatrix}\middle|\begin{smallmatrix}1\\z\end{smallmatrix}\right\rangle \\ \left\langle \begin{smallmatrix}1\\0\end{smallmatrix}\middle|\begin{smallmatrix}1\\x\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\0\end{smallmatrix}\middle|\begin{smallmatrix}1\\y\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\0\end{smallmatrix}\middle|\begin{smallmatrix}1\\z\end{smallmatrix}\right\rangle \\ \left\langle \begin{smallmatrix}1\\-1\end{smallmatrix}\middle|\begin{smallmatrix}1\\x\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\-1\end{smallmatrix}\middle|\begin{smallmatrix}1\\y\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\-1\end{smallmatrix}\middle|\begin{smallmatrix}1\\z\end{smallmatrix}\right\rangle \end{pmatrix}=\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0\end{pmatrix}$$

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Applying T-matrix:

$$\begin{pmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & 1 & 0\end{pmatrix}\cdot\begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2}\end{pmatrix}\cdot\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0\end{pmatrix}=\begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \sin\beta\end{pmatrix}$$

$$\begin{pmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & 1 & 0\end{pmatrix}\cdot\begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha}\end{pmatrix}\cdot\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0\end{pmatrix}=\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1\end{pmatrix}$$

Dipole (j=ℓ=1) wave functions

$$D^{1*}_{1,0}(\phi\theta 0)=-e^{i\phi}\frac{\sin\theta}{\sqrt{2}}=-\frac{\cos\phi\sin\theta+i\cos\phi\sin\theta}{\sqrt{2}}=-\frac{x+iy}{r\sqrt{2}}$$

$$D^{1*}_{0,0}(\phi\theta 0)=\cos\theta=\cos\theta=z/r$$

$$D^{1*}_{-1,0}(\phi\theta 0)=e^{-i\phi}\frac{\sin\theta}{\sqrt{2}}=\frac{\cos\phi\sin\theta-i\cos\phi\sin\theta}{\sqrt{2}}=\frac{x-iy}{r\sqrt{2}}$$

3-D linear-circular polarization T-matrix:

$$\begin{pmatrix} \left\langle\begin{matrix}1\\1\end{matrix}\middle|\begin{matrix}1\\x\end{matrix}\right\rangle & \left\langle\begin{matrix}1\\1\end{matrix}\middle|\begin{matrix}1\\y\end{matrix}\right\rangle & \left\langle\begin{matrix}1\\1\end{matrix}\middle|\begin{matrix}1\\z\end{matrix}\right\rangle \\ \left\langle\begin{matrix}1\\0\end{matrix}\middle|\begin{matrix}1\\x\end{matrix}\right\rangle & \left\langle\begin{matrix}1\\0\end{matrix}\middle|\begin{matrix}1\\y\end{matrix}\right\rangle & \left\langle\begin{matrix}1\\0\end{matrix}\middle|\begin{matrix}1\\z\end{matrix}\right\rangle \\ \left\langle\begin{matrix}1\\-1\end{matrix}\middle|\begin{matrix}1\\x\end{matrix}\right\rangle & \left\langle\begin{matrix}1\\-1\end{matrix}\middle|\begin{matrix}1\\y\end{matrix}\right\rangle & \left\langle\begin{matrix}1\\-1\end{matrix}\middle|\begin{matrix}1\\z\end{matrix}\right\rangle\end{pmatrix}=\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0\end{pmatrix}$$

$$\begin{pmatrix} D^1_{x,x}(\alpha\beta\gamma) & D^1_{x,y} & D^1_{x,z} \\ D^1_{y,x} & D^1_{y,y} & D^1_{y,z} \\ D^1_{z,x} & D^1_{z,y} & D^1_{z,z}\end{pmatrix}=$$

$$\begin{pmatrix} \cos\alpha\cos\beta\cos\gamma-\sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma-\sin\alpha\cos\gamma & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta\cos\gamma+\cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma+\cos\alpha\cos\gamma & \sin\alpha\sin\beta \\ -\cos\gamma\sin\beta & \sin\gamma\sin\beta & \cos\beta\end{pmatrix}$$

Applications of R(3) rotation and U(2) representations

Vector (j=ℓ=1) representation

$$D^1(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha}\end{pmatrix}\begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\gamma} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\gamma}\end{pmatrix}=\begin{pmatrix} e^{-i\alpha}\frac{1+\cos\beta}{2}e^{-i\gamma} & e^{-i\alpha}\frac{-\sin\beta}{\sqrt{2}} & e^{-i\alpha}\frac{1-\cos\beta}{2}e^{i\gamma} \\ \frac{\sin\beta}{\sqrt{2}}e^{-i\gamma} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}}e^{i\gamma} \\ e^{i\alpha}\frac{1-\cos\beta}{2}e^{-i\gamma} & e^{i\alpha}\frac{\sin\beta}{\sqrt{2}} & e^{i\alpha}\frac{1+\cos\beta}{2}e^{i\gamma}\end{pmatrix}$$

Here half-angle identities were used. $\cos^2\frac{\beta}{2}=\frac{1+\cos\beta}{2},\quad \sin^2\frac{\beta}{2}=\frac{1-\cos\beta}{2},\quad \sin\frac{\beta}{2}\cos\frac{\beta}{2}=\frac{\sin\beta}{2},$

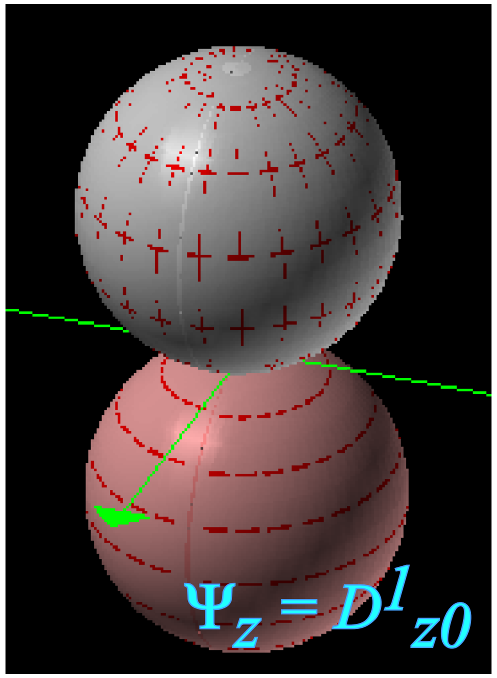
$$Y_1^1(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\,e^{-i\phi}\frac{-\sin\theta}{\sqrt{2}}$$
$$Y_0^1(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\,\cos\beta$$
$$Y_{-1}^1(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\,e^{+i\phi}\frac{\sin\theta}{\sqrt{2}}$$

Center (n=0) column with the factor $\sqrt{\frac{2\ell+1}{4\pi}}$ gives set of *spherical harmonics* Y^ℓ_m.

$$Y^\ell_m(\phi\theta)=D^{\ell *}_{m,n=0}(\phi\theta 0)\sqrt{\frac{2\ell+1}{4\pi}}$$

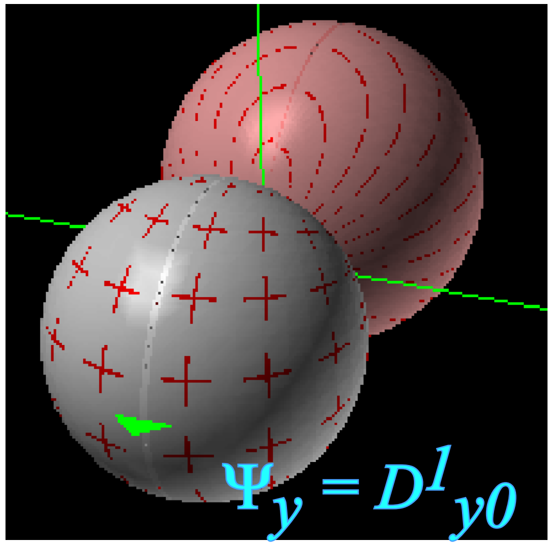
Dipole (j=ℓ=1) wave functions

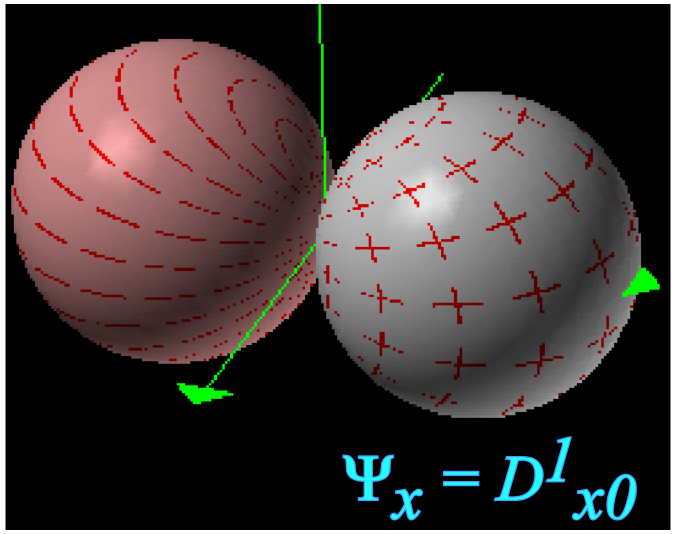
$$D^{1*}_{1,0}(\phi\theta 0)=-e^{i\phi}\frac{\sin\theta}{\sqrt{2}}=-\frac{\cos\phi\sin\theta+i\cos\phi\sin\theta}{\sqrt{2}}=-\frac{x+iy}{r\sqrt{2}}$$
$$D^{1*}_{0,0}(\phi\theta 0)=\cos\theta\quad=\quad\cos\theta\quad=\ z/r$$
$$D^{1*}_{-1,0}(\phi\theta 0)=e^{-i\phi}\frac{\sin\theta}{\sqrt{2}}=\frac{\cos\phi\sin\theta-i\cos\phi\sin\theta}{\sqrt{2}}=\frac{x-iy}{r\sqrt{2}}$$



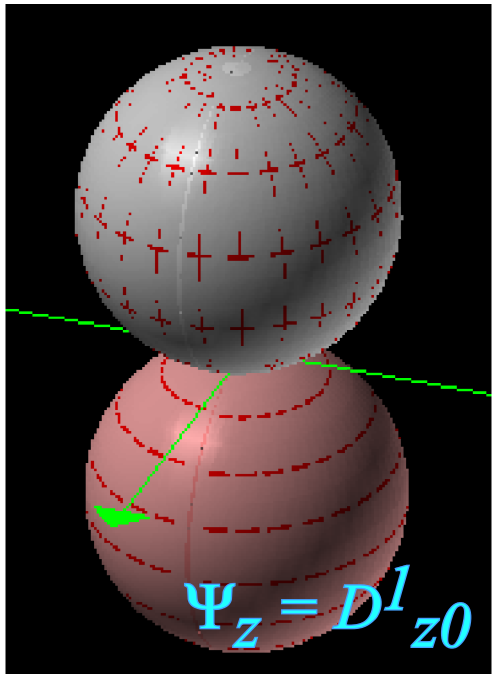
j = 1
Standing
p-Waves

$$\Psi^1_x(\phi,\theta)=D^1_{x,z}(\phi,\theta,0)=\cos\phi\sin\theta$$
$$\Psi^1_y(\phi,\theta)=D^1_{y,z}(\phi,\theta,0)=\sin\phi\sin\theta$$
$$\Psi^1_z(\phi,\theta)=D^1_{z,z}(\phi,\theta,0)=\cos\theta$$

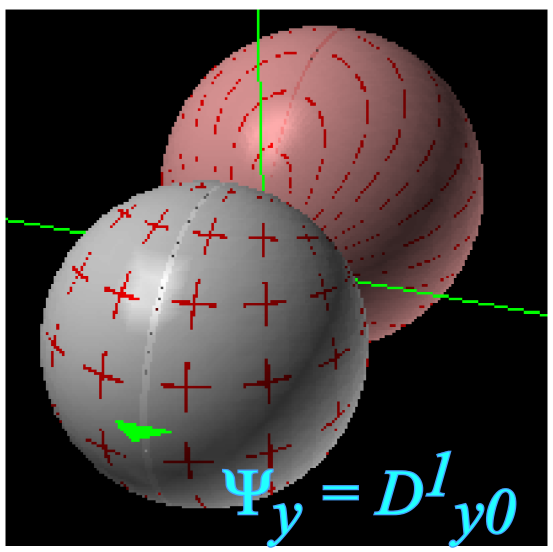


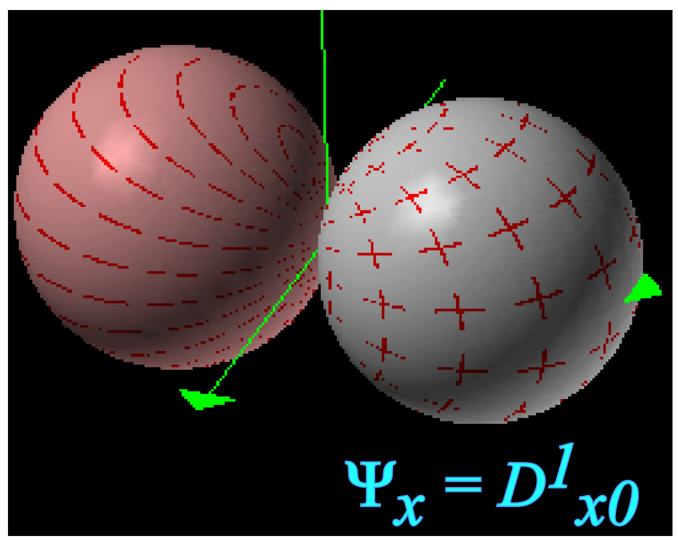


[QuantIt web simulation: Visualizing D representations]



j = 1
Standing p-Waves





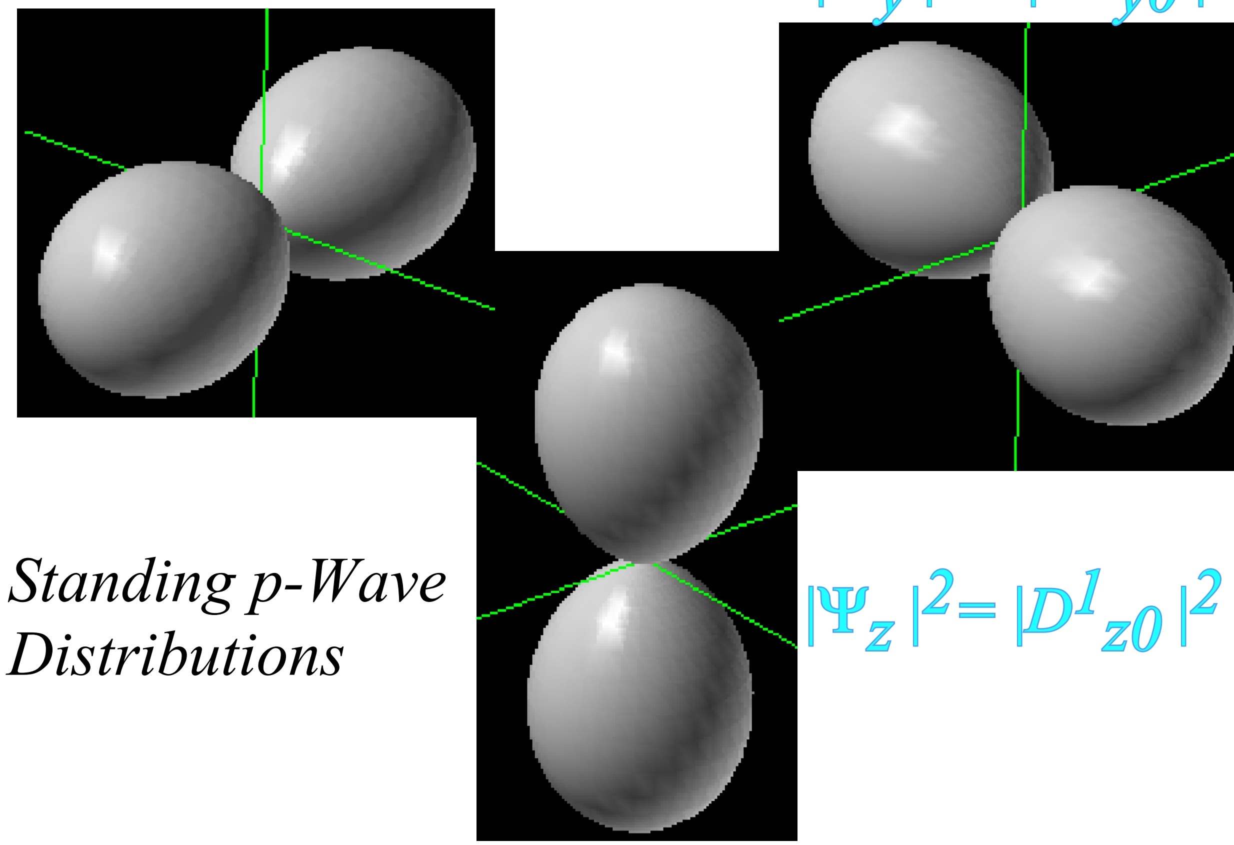
$$\Psi^1_x(\phi,\theta) = D^1_{x,z}(\phi,\theta,0) = \cos\phi\sin\theta$$

$$\Psi^1_y(\phi,\theta) = D^1_{y,z}(\phi,\theta,0) = \sin\phi\sin\theta$$

$$\Psi^1_z(\phi,\theta) = D^1_{z,z}(\phi,\theta,0) = \cos\theta$$

$|\Psi_x|^2 = |D^1_{x0}|^2$

$|\Psi_y|^2 = |D^1_{y0}|^2$

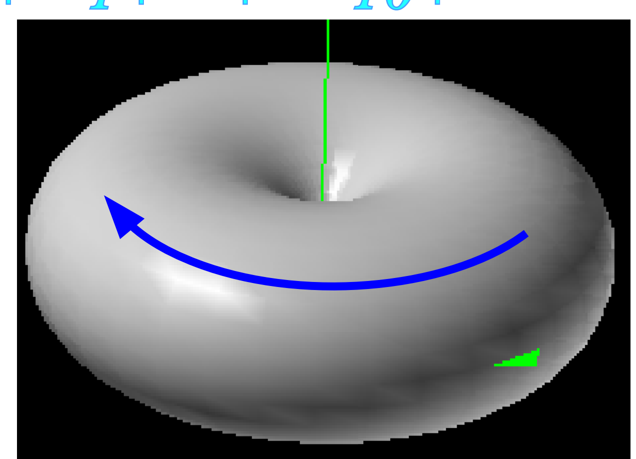


$|\Psi_z|^2 = |D^1_{z0}|^2$

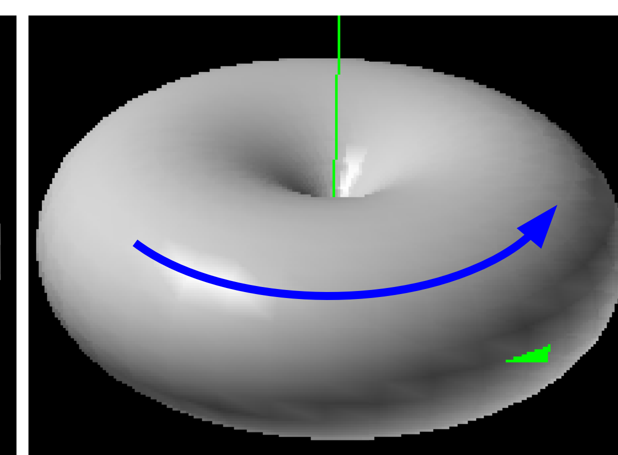
Standing p-Wave Distributions

Moving p-Wave Distributions

$|\Psi_{-1}|^2 = |D^1_{-10}|^2$



$|\Psi_1|^2 = |D^1_{10}|^2$



Applications of R(3) rotation and U(2) representations

Tensor ($j=\ell=2$) representation

$$D^2(\alpha\beta 0)=\left(\begin{array}{cc|c|cc}
e^{-i2\alpha}\left(\frac{1+\cos\beta}{2}\right)^2 & e^{-i2\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & \sqrt{\frac{3}{8}}e^{-i2\alpha}\sin^2\beta & e^{-i2\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{-i2\alpha}\left(\frac{1-\cos\beta}{2}\right)^2 \\
e^{-i\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{-i\alpha}\left(\frac{1+\cos\beta}{2}\right)(2\cos\beta-1) & -\sqrt{\frac{3}{2}}e^{-i\alpha}\sin\beta\cos\beta & e^{-i\alpha}\left(\frac{1-\cos\beta}{2}\right)(2\cos\beta+1) & -e^{-i\alpha}\left(\frac{1-\cos\beta}{2}\right)\sin\beta \\
\sqrt{\frac{3}{8}}\sin^2\beta & \sqrt{\frac{3}{2}}\sin\beta\cos\beta & \frac{3\cos^2\beta-1}{2} & \sqrt{\frac{3}{2}}\sin\beta\cos\beta & \sqrt{\frac{3}{8}}\sin^2\beta \\
e^{i\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{i\alpha}\left(\frac{1-\cos\beta}{2}\right)(2\cos\beta+1) & \sqrt{\frac{3}{2}}e^{i\alpha}\sin\beta\cos\beta & e^{i\alpha}\left(\frac{1+\cos\beta}{2}\right)(2\cos\beta-1) & -e^{i\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta \\
e^{i2\alpha}\left(\frac{1-\cos\beta}{2}\right)^2 & e^{i2\alpha}\left(\frac{1-\cos\beta}{2}\right)\sin\beta & \sqrt{\frac{3}{8}}e^{i2\alpha}\sin^2\beta & e^{i2\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{i2\alpha}\left(\frac{1+\cos\beta}{2}\right)^2
\end{array}\right)$$

Applications of R(3) rotation and U(2) representations

Tensor ($j=\ell=2$) representation

$$D^2(\alpha\beta 0)=\left(\begin{array}{cc|c|cc}
e^{-i2\alpha}\left(\frac{1+\cos\beta}{2}\right)^2 & e^{-i2\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & \sqrt{\frac{3}{8}}e^{-i2\alpha}\sin^2\beta & e^{-i2\alpha}\left(\frac{1-\cos\beta}{2}\right)\sin\beta & e^{-i2\alpha}\left(\frac{1-\cos\beta}{2}\right)^2 \\
e^{-i\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{-i\alpha}\left(\frac{1+\cos\beta}{2}\right)(2\cos\beta-1) & -\sqrt{\frac{3}{2}}e^{-i\alpha}\sin\beta\cos\beta & e^{-i\alpha}\left(\frac{1-\cos\beta}{2}\right)(2\cos\beta+1) & -e^{-i\alpha}\left(\frac{1-\cos\beta}{2}\right)\sin\beta \\
\sqrt{\frac{3}{8}}\sin^2\beta & \sqrt{\frac{3}{2}}\sin\beta\cos\beta & \frac{3\cos^2\beta-1}{2} & \sqrt{\frac{3}{2}}\sin\beta\cos\beta & \sqrt{\frac{3}{8}}\sin^2\beta \\
e^{i\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{i\alpha}\left(\frac{1-\cos\beta}{2}\right)(2\cos\beta+1) & \sqrt{\frac{3}{2}}e^{i\alpha}\sin\beta\cos\beta & e^{i\alpha}\left(\frac{1+\cos\beta}{2}\right)(2\cos\beta-1) & -e^{i\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta \\
e^{i2\alpha}\left(\frac{1-\cos\beta}{2}\right)^2 & e^{i2\alpha}\left(\frac{1-\cos\beta}{2}\right)\sin\beta & \sqrt{\frac{3}{8}}e^{i2\alpha}\sin^2\beta & e^{i2\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{i2\alpha}\left(\frac{1+\cos\beta}{2}\right)^2
\end{array}\right)$$

Spherical 2^k-*multipole functions* X^k_q or *X-functions* are *D**-functions times the k^{th} power of radius (r^k).

$$X^k_q = r^k D^{k*}_{q,0} = \sqrt{\frac{4\pi}{2k+1}}\, r^k Y^k_q$$

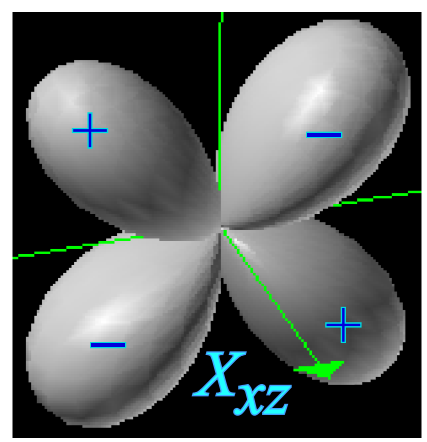
$$\sqrt{4\pi/5}\,Y^{\ell=2}_{m=2}(\phi\theta) = D^{2*}_{2,0}(\phi\theta 0) = \sqrt{\frac{3}{8}}e^{i2\phi}\sin^2\theta = \sqrt{\frac{3}{8}}\frac{(x+iy)^2}{r^2}$$

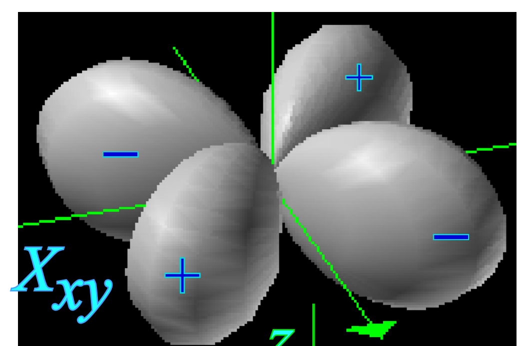
$$\sqrt{4\pi/5}\,Y^{\ell=2}_{m=1}(\phi\theta) = D^{2*}_{1,0}(\phi\theta 0) = -\sqrt{\frac{3}{2}}e^{i\phi}\sin\theta\cos\theta = -\sqrt{\frac{3}{2}}\frac{(x+iy)z}{r^2}$$

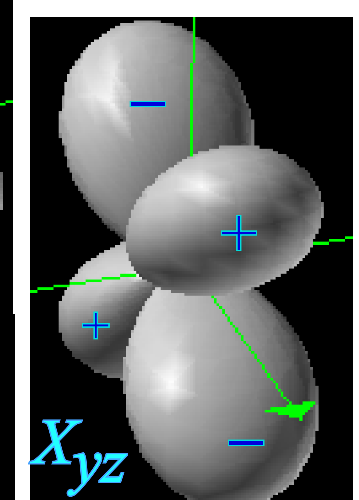
$$\sqrt{4\pi/5}\,Y^{\ell=2}_{m=0}(\phi\theta) = D^{2*}_{0,0}(\phi\theta 0) = \frac{3\cos^2\theta-1}{2} = \frac{3z^2-r^2}{2r^2}$$

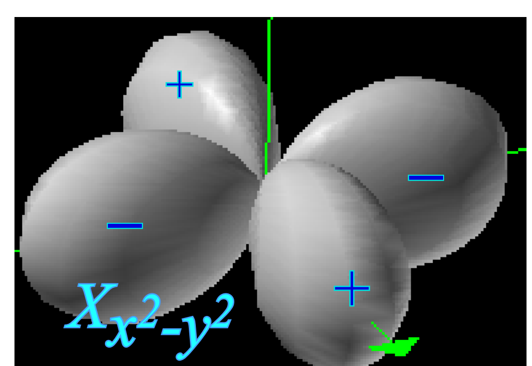
$$\sqrt{4\pi/5}\,Y^{\ell=2}_{m=-1}(\phi\theta) = D^{2*}_{-1,0}(\phi\theta 0) = \sqrt{\frac{3}{2}}e^{-i\phi}\sin\theta\cos\theta = \sqrt{\frac{3}{2}}\frac{(x-iy)z}{r^2}$$

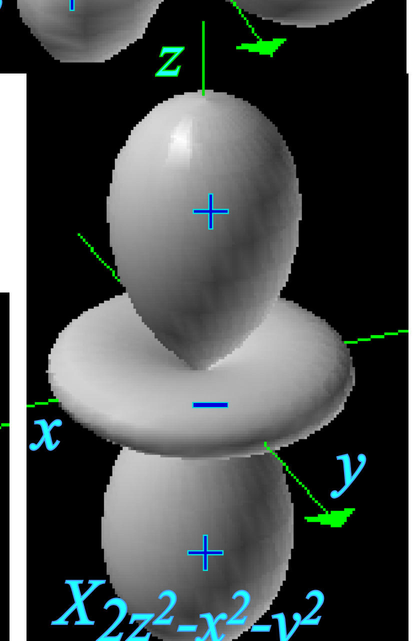
$$\sqrt{4\pi/5}\,Y^{\ell=2}_{m=-2}(\phi\theta) = D^{2*}_{-2,0}(\phi\theta 0) = \sqrt{\frac{3}{8}}e^{-i2\phi}\sin^2\theta = \sqrt{\frac{3}{8}}\frac{(x-iy)^2}{r^2}$$











j = 2
Standing
d-Waves

[QuantIt web simulation: Visualizing D representations]

Applications of R(3) rotation and U(2) representations

Tensor ($j=\ell=2$) representation

Spherical 2^k*-multipole functions* X^k_q or *X-functions* are D^*-functions times the k^{th} power of radius (r^k).

$$\sqrt{4\pi/5}\,Y^{\ell=2}_{m=2}(\phi\theta) = D^{2*}_{2,0}(\phi\theta 0) = \sqrt{\frac{3}{8}}e^{i2\phi}\sin^2\theta = \sqrt{\frac{3}{8}}\frac{(x+iy)^2}{r^2}$$

$$\sqrt{4\pi/5}\,Y^{\ell=2}_{m=1}(\phi\theta) = D^{2*}_{1,0}(\phi\theta 0) = -\sqrt{\frac{3}{2}}e^{i\phi}\sin\theta\cos\theta = -\sqrt{\frac{3}{2}}\frac{(x+iy)z}{r^2}$$

$$\sqrt{4\pi/5}\,Y^{\ell=2}_{m=0}(\phi\theta) = D^{2*}_{0,0}(\phi\theta 0) = \frac{3\cos^2\theta-1}{2} = \frac{3z^2-r^2}{2r^2}$$

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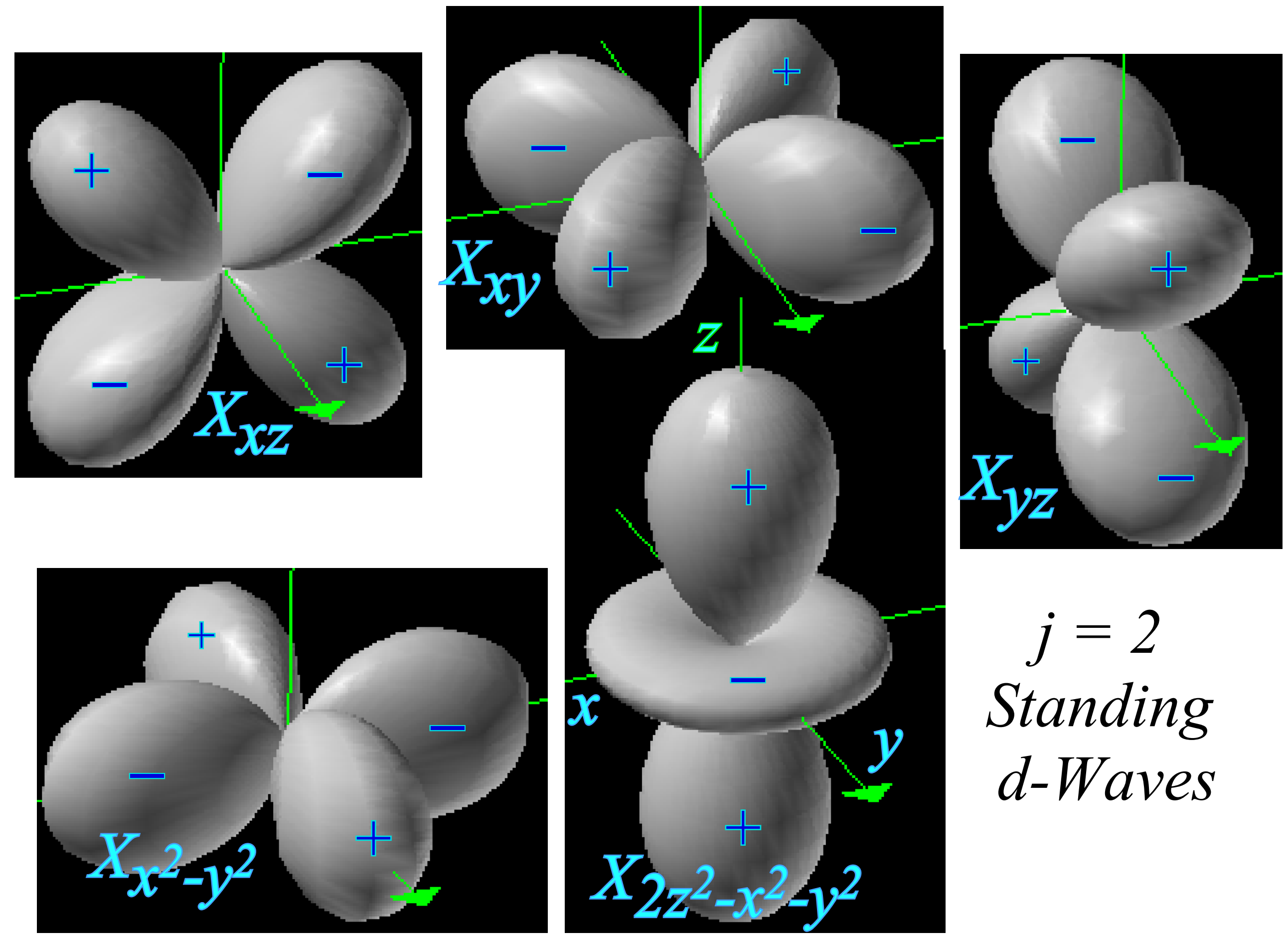
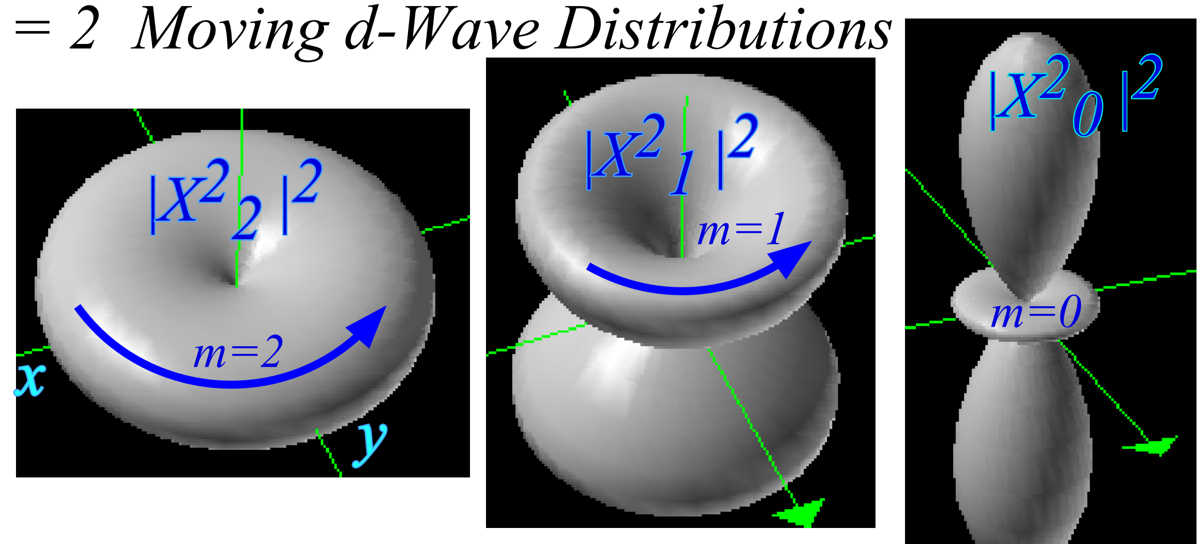
$$\sqrt{4\pi/5}\,Y^{\ell=2}_{m=-2}(\phi\theta) = D^{2*}_{-2,0}(\phi\theta 0) = \sqrt{\frac{3}{8}}e^{-i2\phi}\sin^2\theta = \sqrt{\frac{3}{8}}\frac{(x-iy)^2}{r^2}$$

$$X^k_q = r^k D^{k*}_{q,0} = \sqrt{\frac{4\pi}{2k+1}}\,r^k Y^k_q$$



j = 2
Standing
d-Waves

j = 2 Moving d-Wave Distributions



Review : *2-D* $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(2) representations*

Angular momentum generators by U(2) analysis
Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Angular momentum commutation relations
Key Lie theorems

Angular momentum magnitude and uncertainty
Angular momentum uncertainty angle

Generating R(3) rotation and U(2) representations
Applications of R(3) rotation and U(2) representations
➡ *Molecular and nuclear wavefunctions* ⬅
Molecular and nuclear eigenlevels
Generalized Stern-Gerlach and transformation matrices
Angular momentum cones and high J properties

Applications of R(3) rotation and U(2) representations

Molecular and nuclear wavefunctions

For *SU(2)* and *R(3)*, sum over rotations is an integral over Euler angles *(αβγ)*.

For integral-*j=0, 1, 2,..* the *R(3)* integral over polar angle β ranges from 0 to π.

$$\textit{for } R(3): \frac{\ell^j}{N}\int d(\alpha\beta\gamma) = \frac{2j+1}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma \;= 2j+1$$

N is normalization

Applications of R(3) rotation and U(2) representations

Molecular and nuclear wavefunctions

For *SU(2)* and *R(3)*, sum over rotations is an integral over Euler angles *(αβγ)*.

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N is normalization

For ½-integral-*j=1/2, 3/2,..* the *U(2)* integral over polar angle β ranges from -π to π.

$$\text{for } SU(2): \frac{\ell^j}{N}\int d(\alpha\beta\gamma) = \frac{2j+1}{16\pi^2}\int_0^{2\pi} d\alpha \int_{-\pi}^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma \;= 2j+1 = \ell^j$$

Applications of R(3) rotation and U(2) representations

Molecular and nuclear wavefunctions

For *SU(2)* and *R(3)*, sum over rotations is an integral over Euler angles *(αβγ)*.

For integral-$j=0, 1, 2,..$ the *R(3)* integral over polar angle β ranges from 0 to π.

$$\text{for } R(3): \frac{\ell^j}{N}\int d(\alpha\beta\gamma) = \frac{2j+1}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma = 2j+1 = \ell^j \qquad N \text{ is normalization}$$

For ½-integral-$j=1/2, 3/2,..$ the *U(2)* integral over polar angle β ranges from -π to π.

$$\text{for } SU(2): \frac{\ell^j}{N}\int d(\alpha\beta\gamma) = \frac{2j+1}{16\pi^2}\int_0^{2\pi} d\alpha \int_{-\pi}^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma = 2j+1 = \ell^j$$

Eigenstates of angular momentum are built from projected initial position states |000⟩.

$$\left|{}^{j}_{m,n}\right\rangle = \frac{\mathbf{P}^j_{m,n}|000\rangle}{\sqrt{\ell^j}} = \frac{1}{N}\int d(\alpha\beta\gamma)\, D^{j*}_{m,n}(\alpha\beta\gamma)\mathbf{R}(\alpha\beta\gamma)|000\rangle\sqrt{\ell^j} = \frac{1}{N}\int d(\alpha\beta\gamma)\, D^{j*}_{m,n}(\alpha\beta\gamma)\sqrt{\ell^j}\,|\alpha\beta\gamma\rangle$$

Applications of R(3) rotation and U(2) representations

Molecular and nuclear wavefunctions

For *SU(2)* and *R(3)*, sum over rotations is an integral over Euler angles *(αβγ)*.

For integral-*j=0, 1, 2,..* the *R(3)* integral over polar angle β ranges from 0 to π.

$$for\ R(3): \frac{\ell^j}{N}\int d(\alpha\beta\gamma) = \frac{2j+1}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma = 2j+1 = \ell^j$$

N is normalization

For ½-integral-*j=1/2, 3/2,..* the *U(2)* integral over polar angle β ranges from -π to π.

$$for\ SU(2): \frac{\ell^j}{N}\int d(\alpha\beta\gamma) = \frac{2j+1}{16\pi^2}\int_0^{2\pi} d\alpha \int_{-\pi}^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma = 2j+1 = \ell^j$$

Eigenstates of angular momentum are built from projected initial position states |000⟩.

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Applications of R(3) rotation and U(2) representations

Molecular and nuclear wavefunctions

For *SU(2)* and *R(3)*, sum over rotations is an integral over Euler angles *(αβγ)*.

For integral-*j=0, 1, 2,..* the *R(3)* integral over polar angle β ranges from 0 to π.

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Same applies to the generators $\mathbf{s}_Z$ or $\mathbf{J}_Z$ of *SU(2)* or *R(3)*.

$$\mathbf{s}_Z\left|{}^{j}_{m,n}\right\rangle = m\left|{}^{j}_{m,n}\right\rangle \qquad \bar{\mathbf{s}}_Z\left|{}^{j}_{m,n}\right\rangle = -n\left|{}^{j}_{m,n}\right\rangle$$

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➡ *Molecular and nuclear eigenlevels* ⬅
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The reversed sign is a nuisance, so let us define ***reversed momentum operators*** that give a positive sign.

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Example of a rotor spectrum and Hamiltonian of a symmetric top molecule.

$$\mathbf{H}_{symmetric\ top} = B\mathbf{J}_{\bar{X}}^2 + B\mathbf{J}_{\bar{Y}}^2 + A\mathbf{J}_{\bar{Z}}^2$$

(Molecular spin is labeled **J** instead of **s**)

Applications of R(3) rotation and U(2) representations

Molecular and nuclear eigenlevels

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$$\mathbf{H}_{symmetric\ top}\left|\begin{matrix} j \\ m,n \end{matrix}\right\rangle$$

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$$= \left[BJ(J+1) + (A-B)n^2\right]\left|\begin{matrix} j \\ m,n \end{matrix}\right\rangle$$

Eigenvalue and energy level spectrum is shown next.

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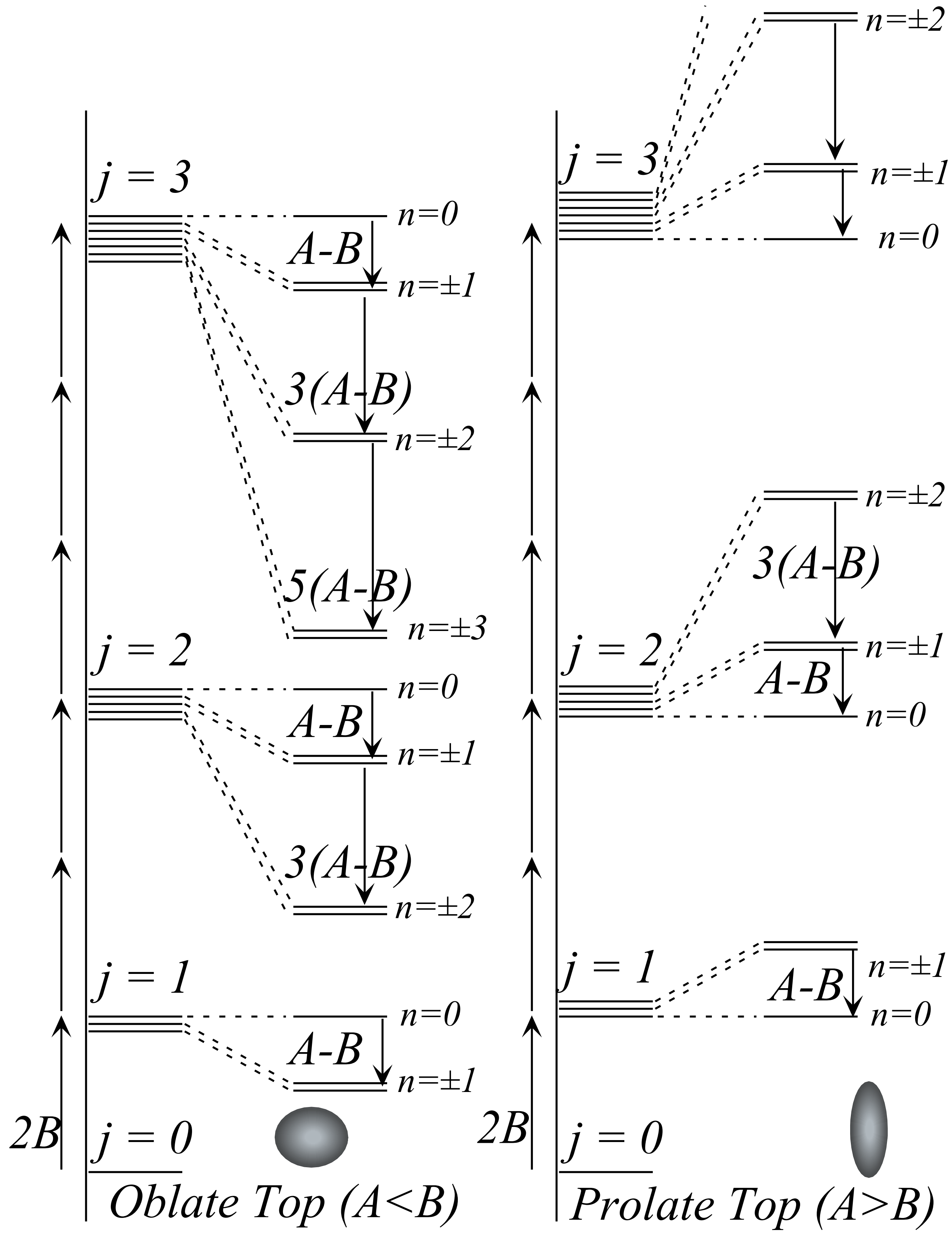

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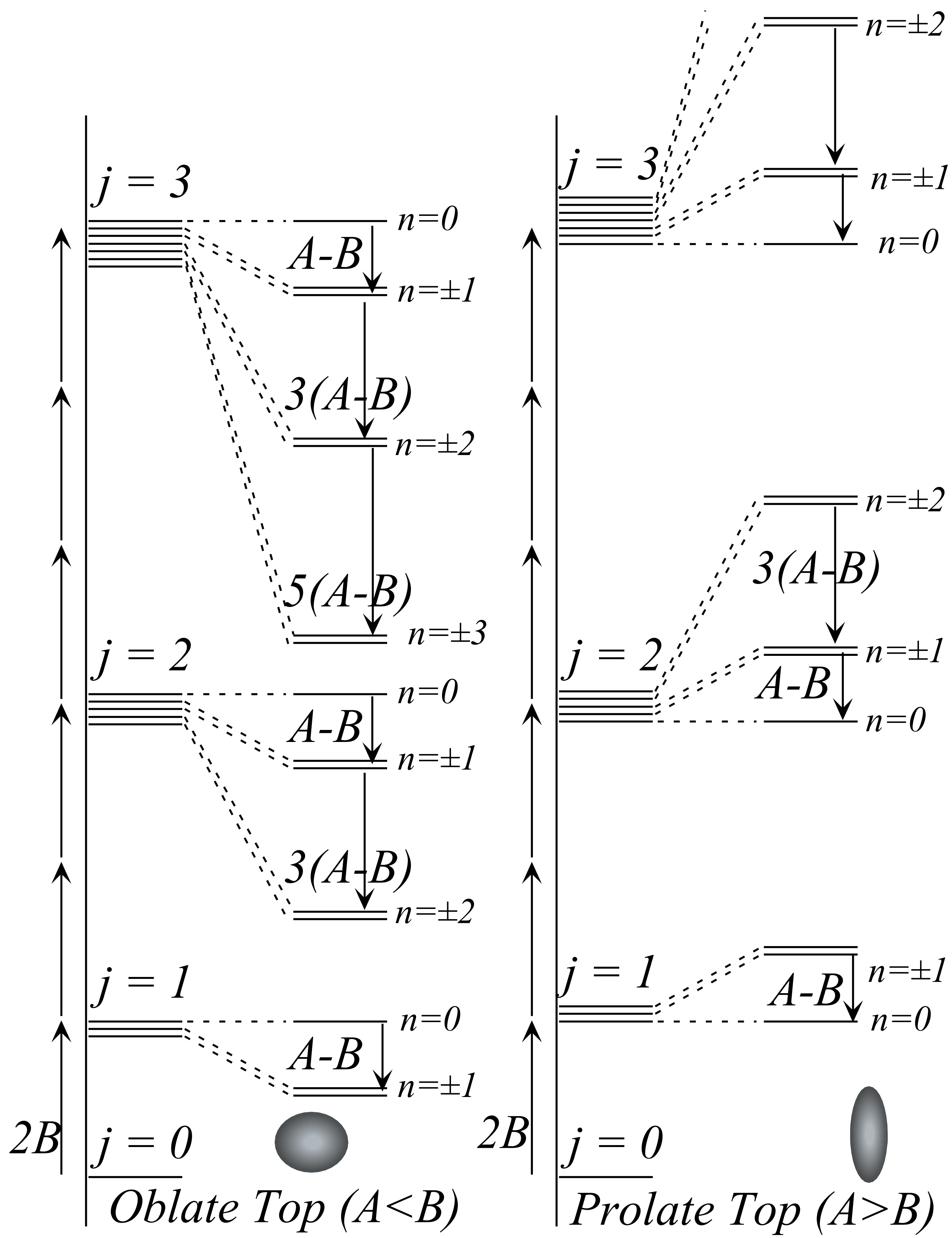
Mock-Mach-Multiplicity is $(2j+1)^2$ for each j



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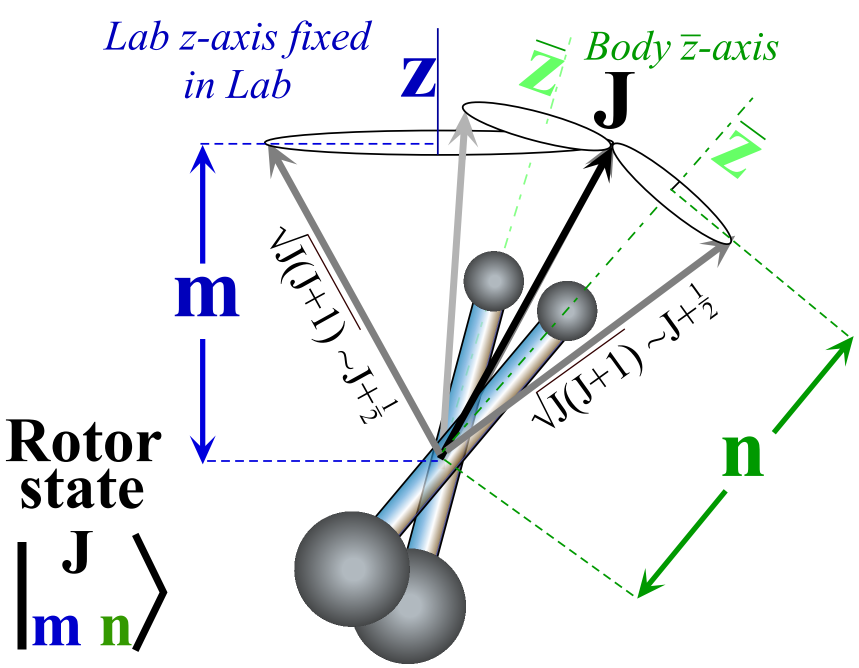

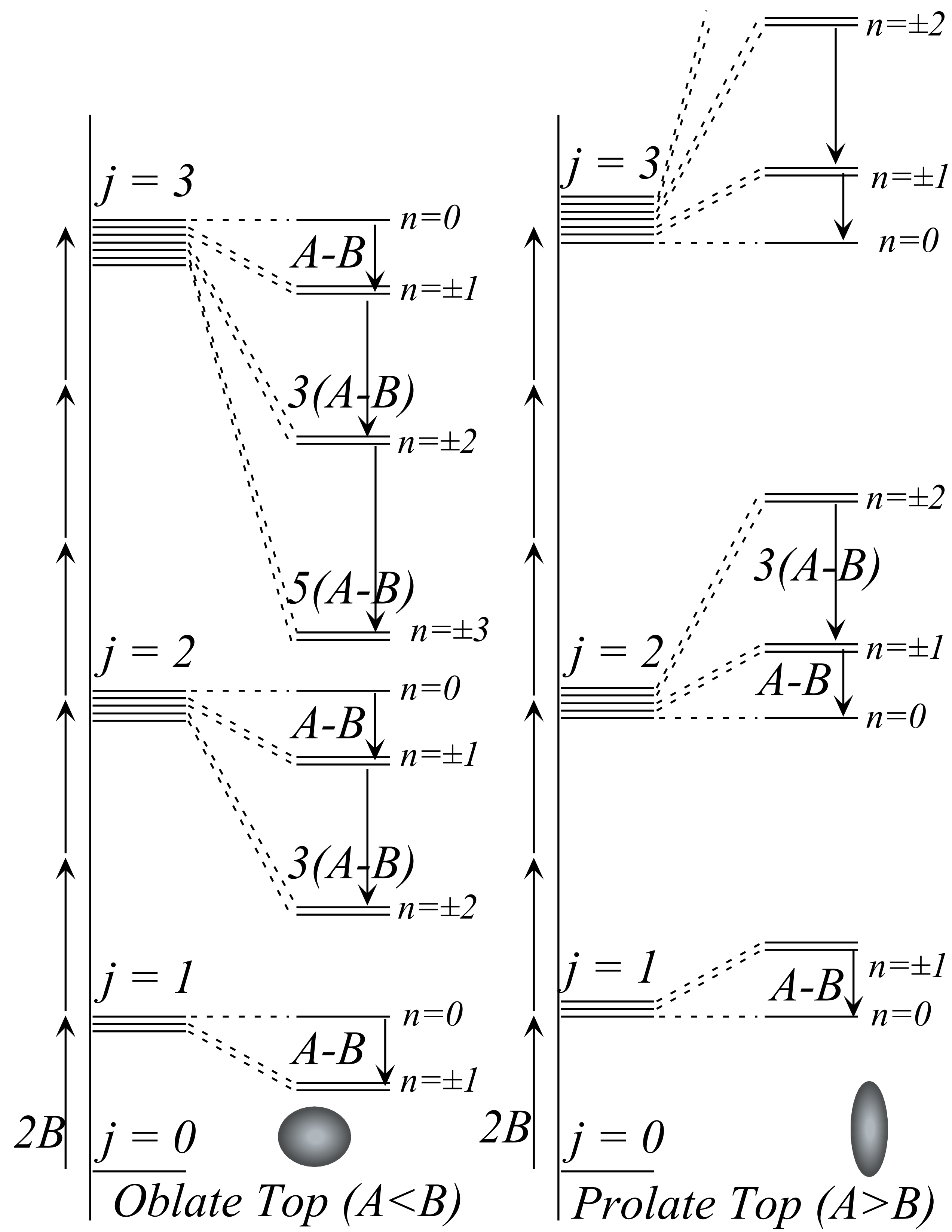

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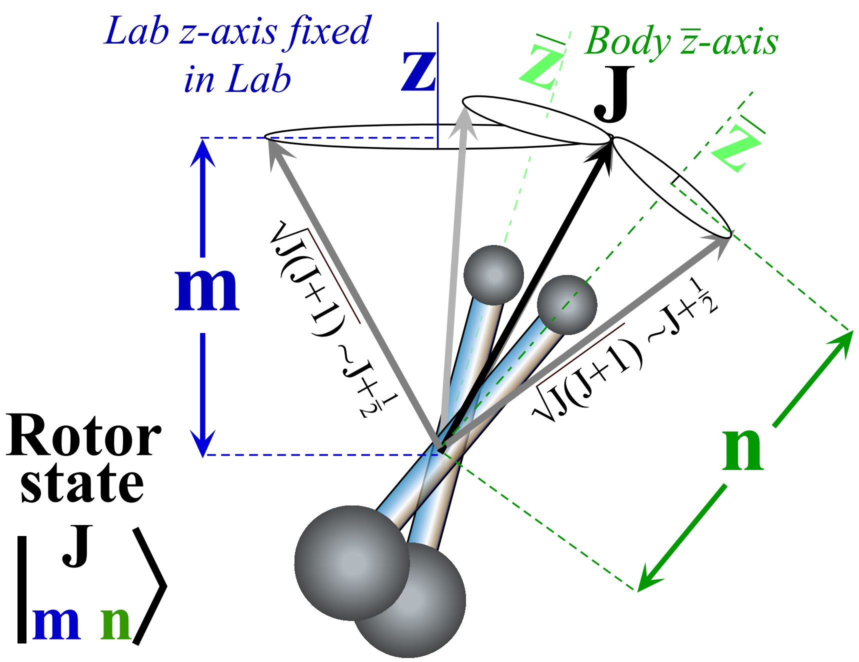
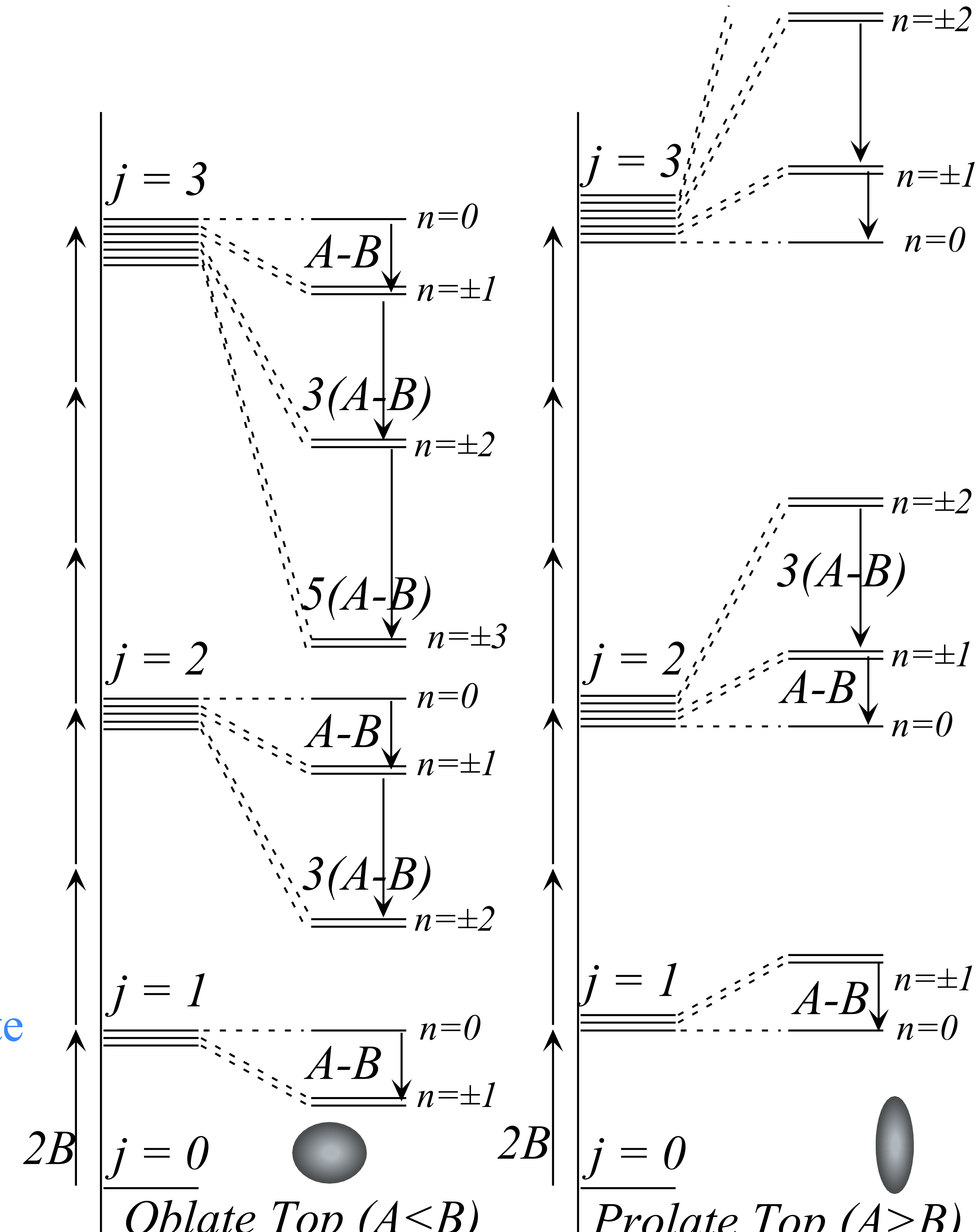


The $n=0$ levels are $2j+1$-fold degenerate

$n\neq 0$ level degeneracy is $4j+2$.



Applications of R(3) rotation and U(2) representations

Molecular and nuclear eigenlevels

Introducing Racah tensor notation

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$$T_0^0 = \mathrm{J}\cdot\mathrm{J} = \langle J\rangle^2 \qquad = \qquad \left(J_x^2 + J_y^2 + J_y^2\right),$$

$$T_0^2 = \tfrac{1}{2}\langle J\rangle^2\left(3\cos^2\beta - 1\right) = \tfrac{1}{2}\left(2J_z^2 - J_x^2 - J_y^2\right),$$

$$H = B\ T_0^0 + \frac{2}{3}(A-B)\ T_0^2$$

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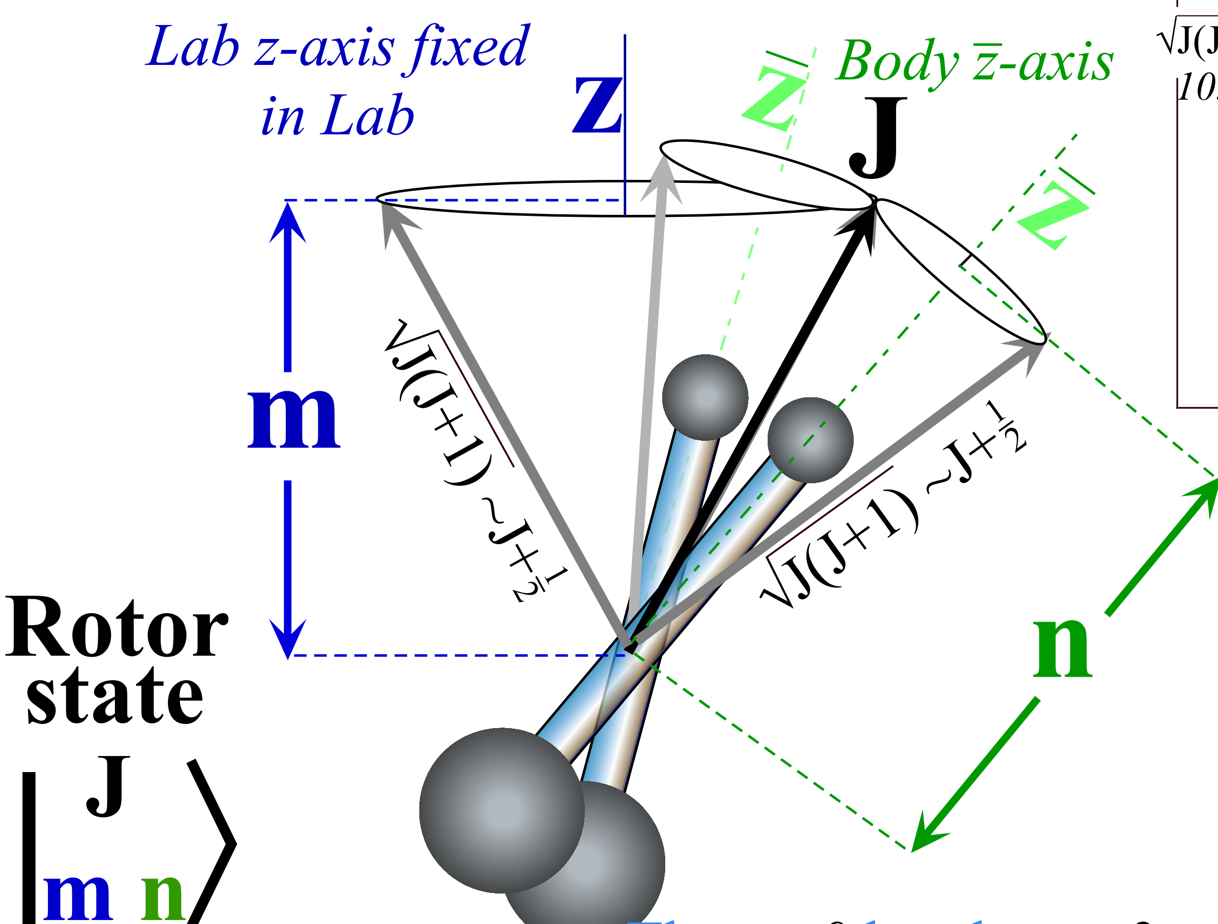

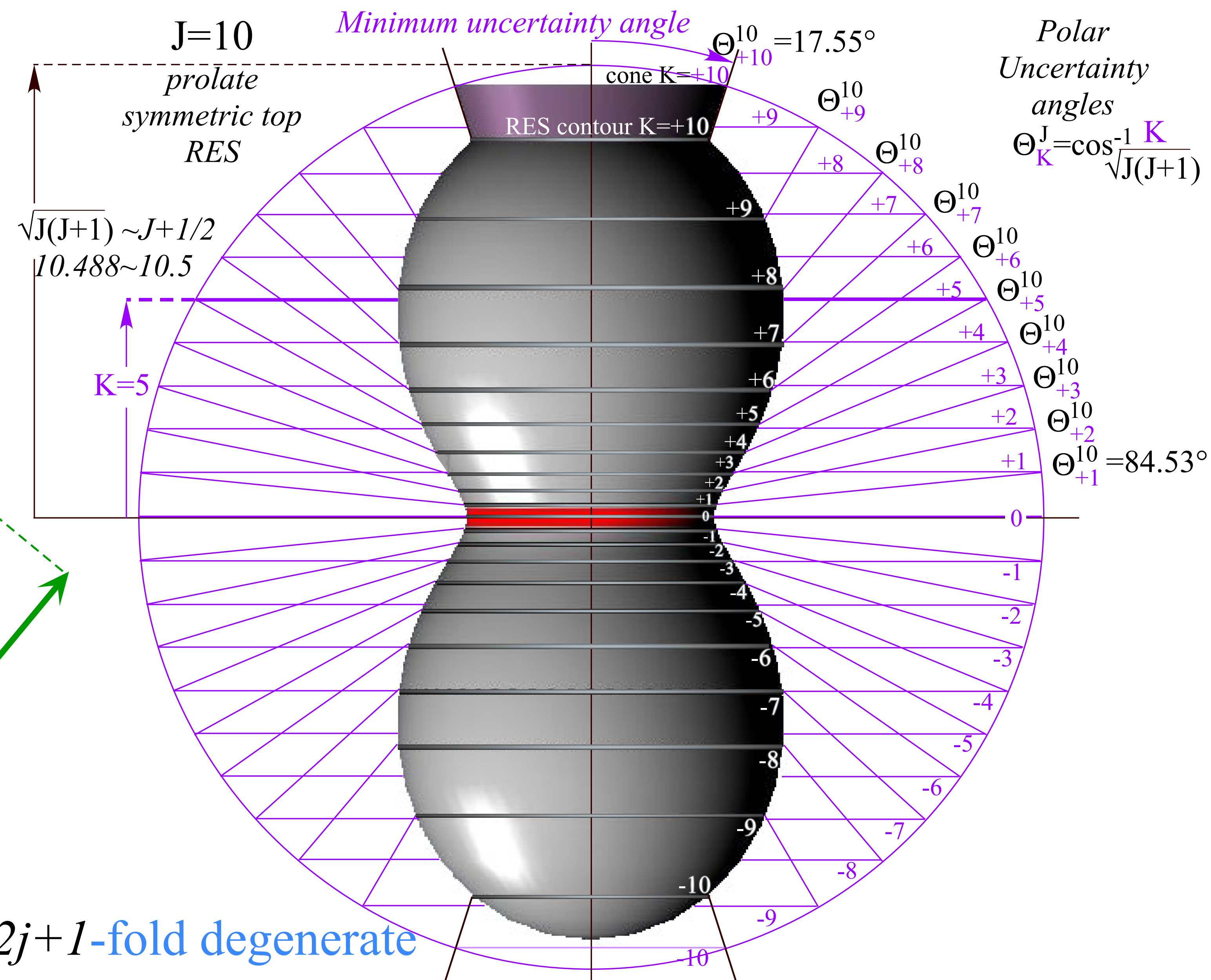

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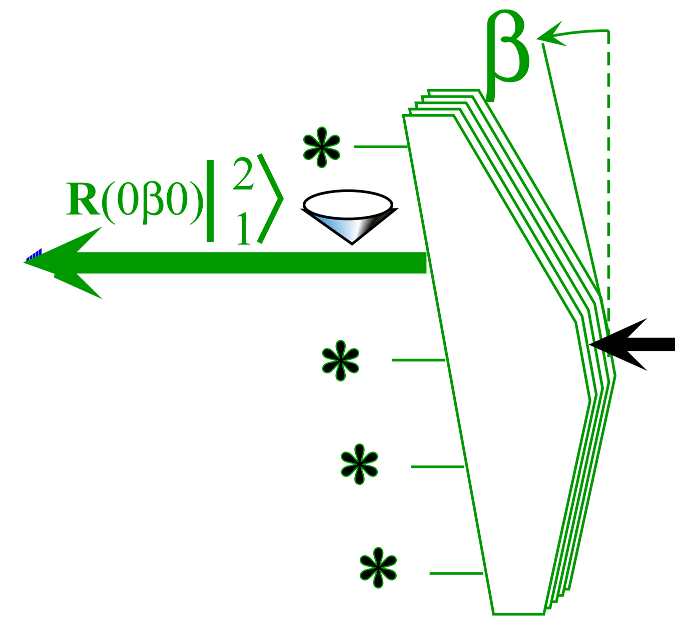
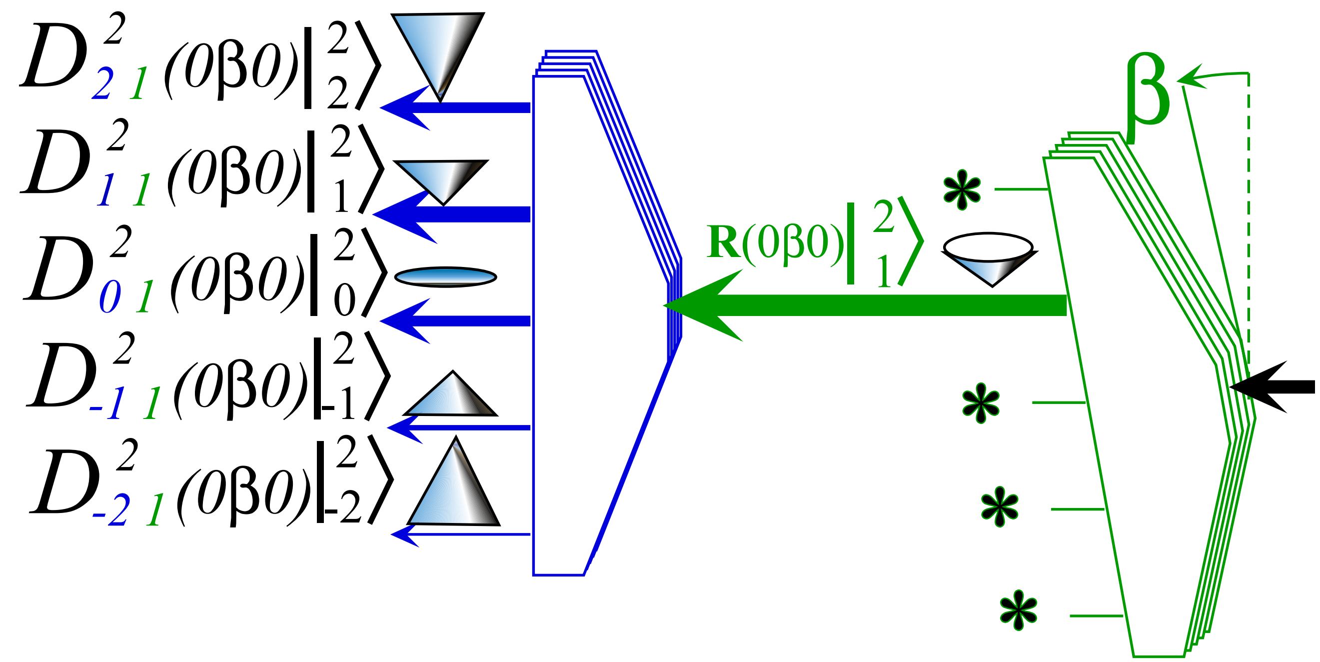
Polarization analysis: Suppose a spin-j state $\mathbf{R}(0\beta 0)\,|^{j=2}{}_{m=1}\rangle$ exits an analyzer rotated by β



Polarization analysis: Suppose a spin-j state $\mathbf{R}(0\beta 0)\left|{}^{j=2}_{m=1}\right\rangle$ exits an analyzer rotated by β and then enters a vertical ($\beta=0$) analyzer that forces it to choose from unrotated states $\left|{}^{j=2}_{m'}\right\rangle$

$$\mathbf{R}(0\beta 0)\left|{}^{j}_{m}\right\rangle = \sum_{m'=-j}^{j}\left|{}^{j}_{m'}\right\rangle\left\langle{}^{j}_{m'}\right|\mathbf{R}(0\beta 0)\left|{}^{j}_{m}\right\rangle$$

$$= \sum_{m'=-j}^{j}\left|{}^{j}_{m'}\right\rangle D^{j}_{m'm}(0\beta 0)$$



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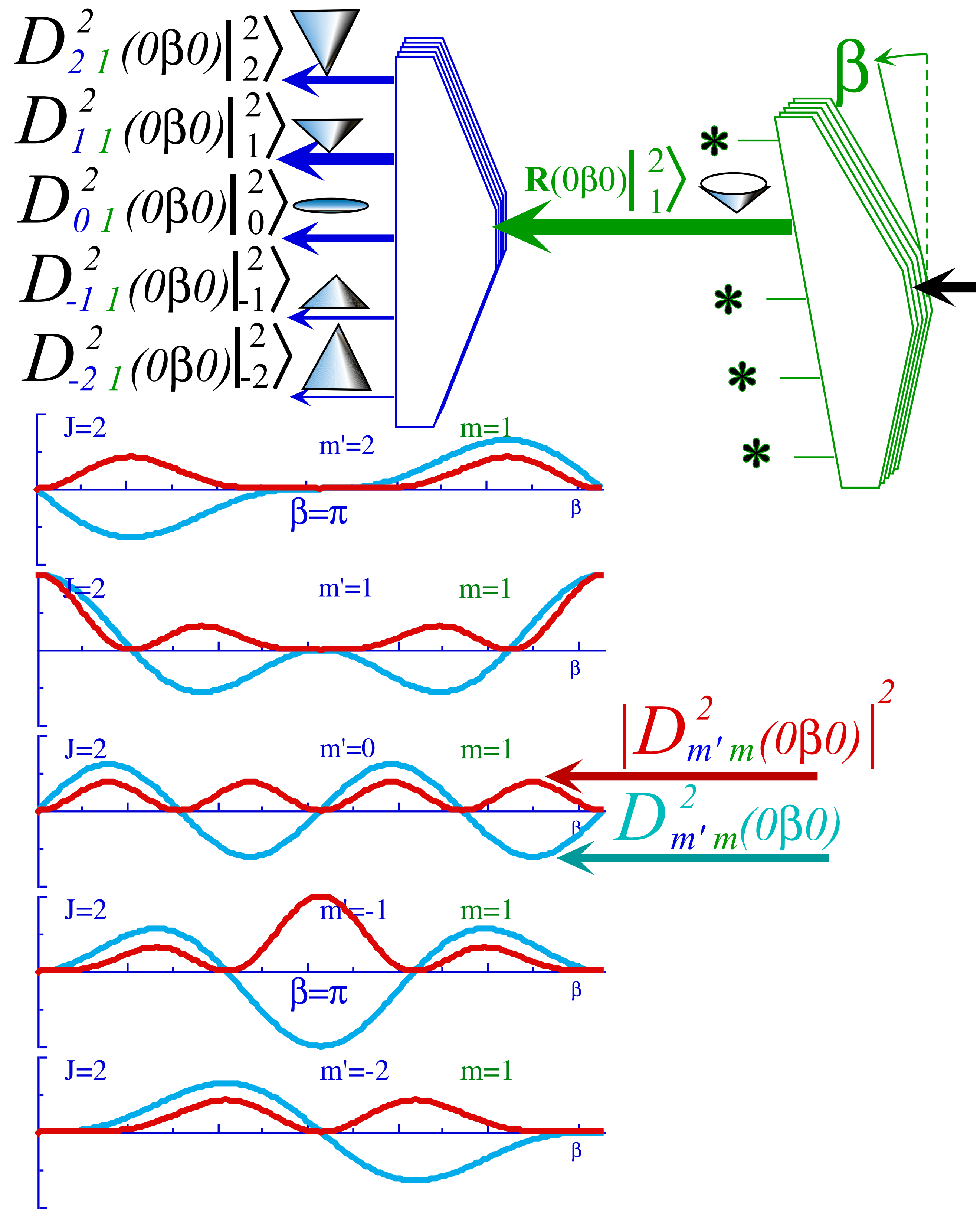


Overlap of state $\mathbf{R}(\alpha\beta\gamma)\,|{}^{2}_{1}\rangle$ with unrotated $|{}^{j=2}_{m'}\rangle$ is the corresponding D-matrix element.

$$\left\langle{}^{j'}_{m'}\right|\mathbf{R}(\alpha\beta\gamma)\left|{}^{2}_{1}\right\rangle = \delta^{j'2}D^{2}_{m'1}(\alpha\beta\gamma) = \left\langle{}^{j'}_{m'}\middle|{}^{2}_{1}\right\rangle_R$$

$D^{j}_{m'n}(0\beta 0)$ plotted vs. β for fixed j,m',n

Polarization analysis: Suppose a spin-j state $\mathbf{R}(0\beta 0)\left|{}^{j=2}{}_{m}\right\rangle$ exits an analyzer rotated by β and then enters a vertical ($\beta=0$) analyzer that forces it to choose from unrotated states $\left|{}^{j=2}{}_{m'}\right\rangle$

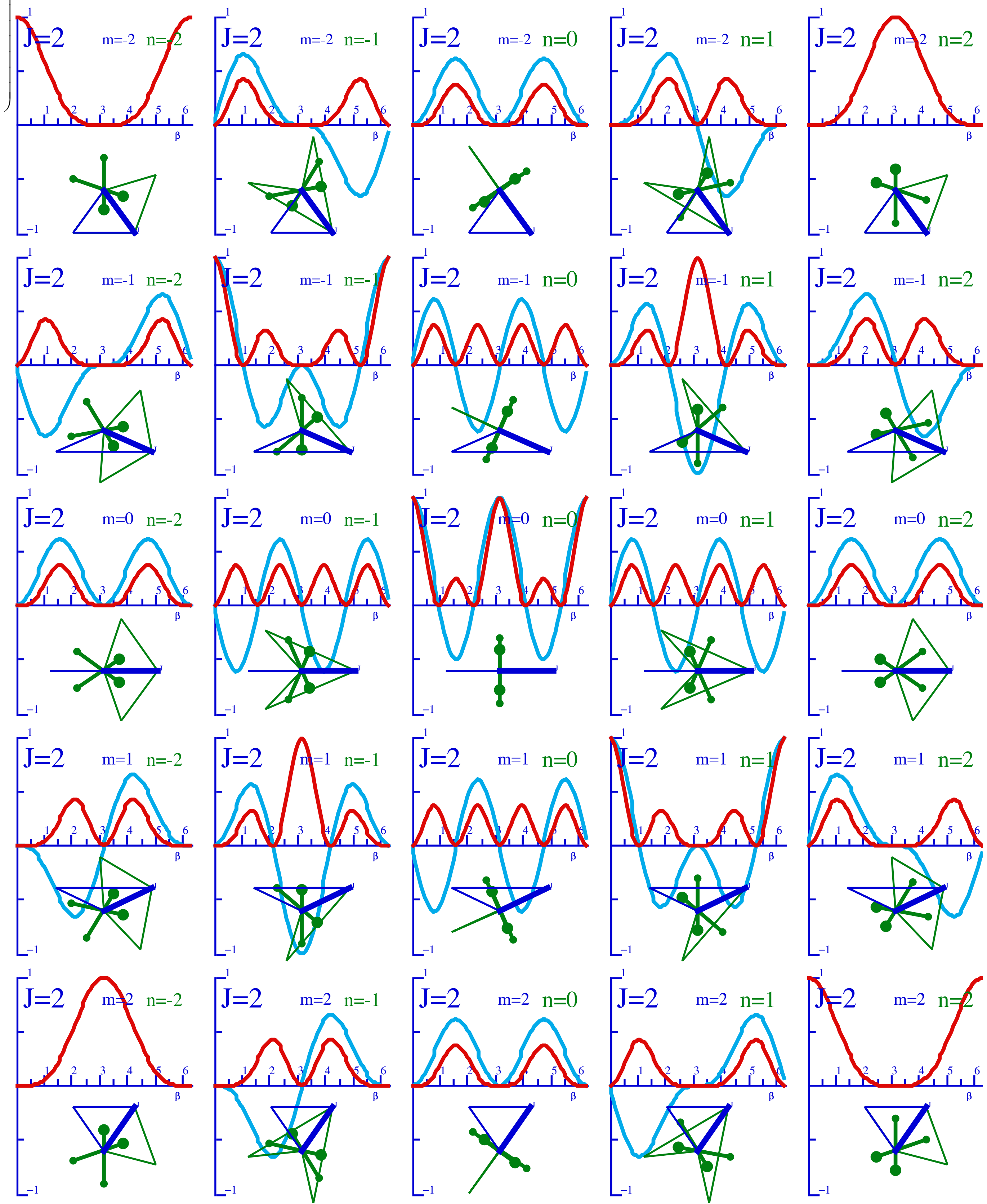
$$D^2(\alpha\beta 0)=\left(\begin{array}{cc|c|cc} e^{-i2\alpha}\left(\frac{1+\cos\beta}{2}\right)^2 & e^{-i2\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & \sqrt{\frac{3}{8}}e^{-i2\alpha}\sin^2\beta & e^{-i2\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{-i2\alpha}\left(\frac{1-\cos\beta}{2}\right)^2 \\ e^{-i\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{-i\alpha}\left(\frac{1+\cos\beta}{2}\right)(2\cos\beta-1) & -\sqrt{\frac{3}{2}}e^{-i\alpha}\sin\beta\cos\beta & e^{-i\alpha}\left(\frac{1-\cos\beta}{2}\right)(2\cos\beta+1) & -e^{-i\alpha}\left(\frac{1-\cos\beta}{2}\right)\sin\beta \\ \sqrt{\frac{3}{8}}\sin^2\beta & \sqrt{\frac{3}{2}}\sin\beta\cos\beta & \frac{3\cos^2\beta-1}{2} & \sqrt{\frac{3}{2}}\sin\beta\cos\beta & \sqrt{\frac{3}{8}}\sin^2\beta \\ e^{i\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{i\alpha}\left(\frac{1-\cos\beta}{2}\right)(2\cos\beta+1) & \sqrt{\frac{3}{2}}e^{i\alpha}\sin\beta\cos\beta & e^{i\alpha}\left(\frac{1+\cos\beta}{2}\right)(2\cos\beta-1) & -e^{i\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta \\ e^{i2\alpha}\left(\frac{1-\cos\beta}{2}\right)^2 & e^{i2\alpha}\left(\frac{1-\cos\beta}{2}\right)\sin\beta & \sqrt{\frac{3}{8}}e^{i2\alpha}\sin^2\beta & e^{i2\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{i2\alpha}\left(\frac{1+\cos\beta}{2}\right)^2 \end{array}\right)$$

$$\mathbf{R}(0\beta 0)\left|{}^{j}_{m}\right\rangle = \sum_{m'=-j}^{j}\left|{}^{j}_{m'}\right\rangle\left\langle{}^{j}_{m'}\right|\mathbf{R}(0\beta 0)\left|{}^{j}_{m}\right\rangle$$

$$= \sum_{m'=-j}^{j}\left|{}^{j}_{m'}\right\rangle D^{j}_{m'm}(0\beta 0)$$

Overlap of state $\mathbf{R}(\alpha\beta\gamma)\left|{}^{2}{}_{1}\right\rangle$ with unrotated $\left|{}^{j=2}{}_{m'}\right\rangle$ is the corresponding D-matrix element.

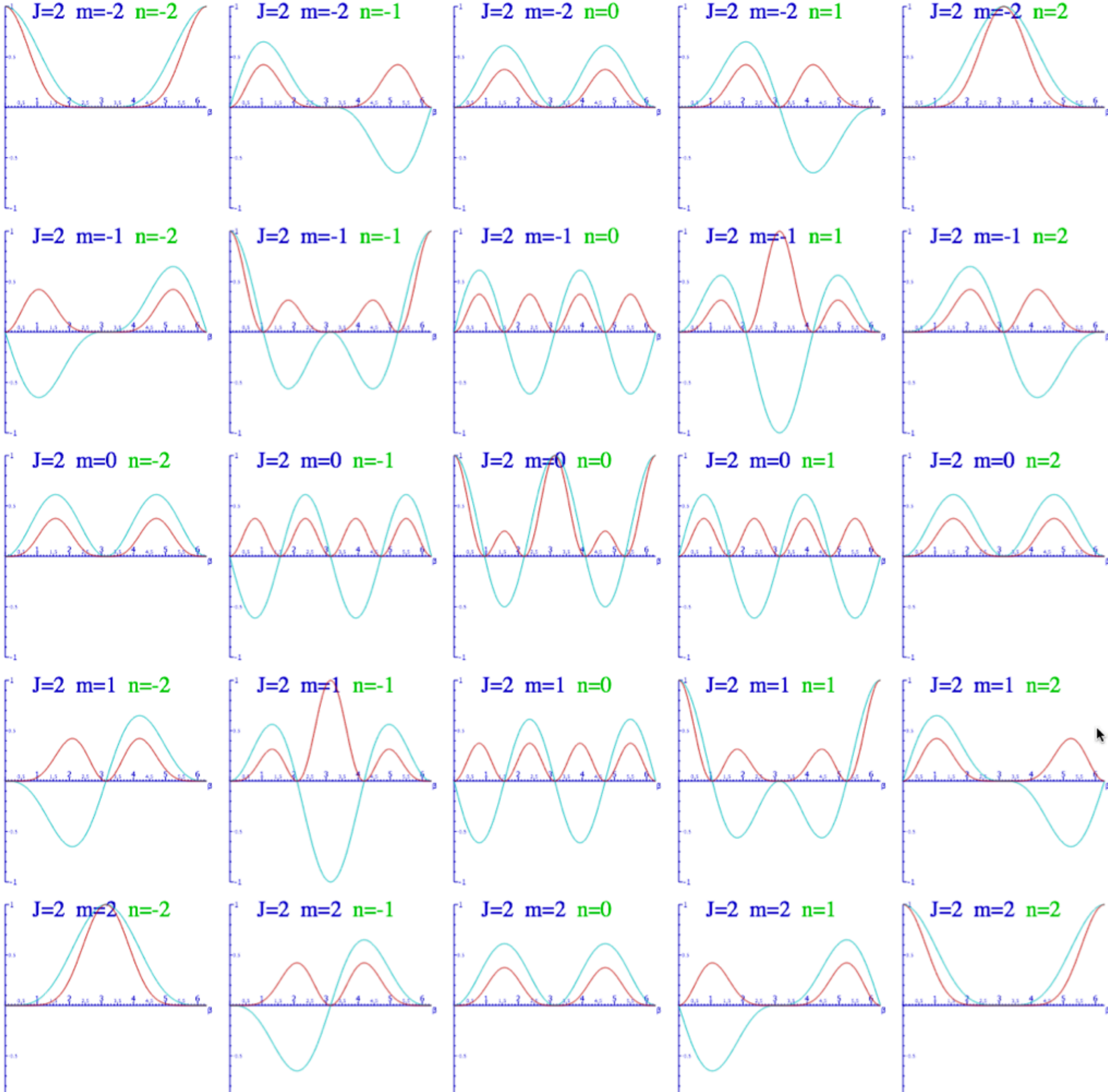
$$\left\langle{}^{j'}_{m'}\right|\mathbf{R}(\alpha\beta\gamma)\left|{}^{2}_{1}\right\rangle = \delta^{j'2}D^{2}_{m'1}(\alpha\beta\gamma) = \left\langle{}^{j'}_{m'}\middle|{}^{2}_{1}\right\rangle_R$$



$D^{j}_{m'n}(0\beta 0)$ plotted vs. β for fixed j,m',n

QuantIt web simulation: Visualizing D representations

$D^{j}_{m'n}(0\beta 0)$
plotted
vs. β
for fixed
j,m',n



QuantIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/QuantItWeb.html"

Review : *2-D* **a†a** *algebra of U(2) representations*

Angular momentum generators by U(2) analysis

Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Angular momentum commutation relations

Key Lie theorems

Angular momentum magnitude and uncertainty

Angular momentum uncertainty angle

Generating R(3) rotation and U(2) representations

Applications of R(3) rotation and U(2) representations

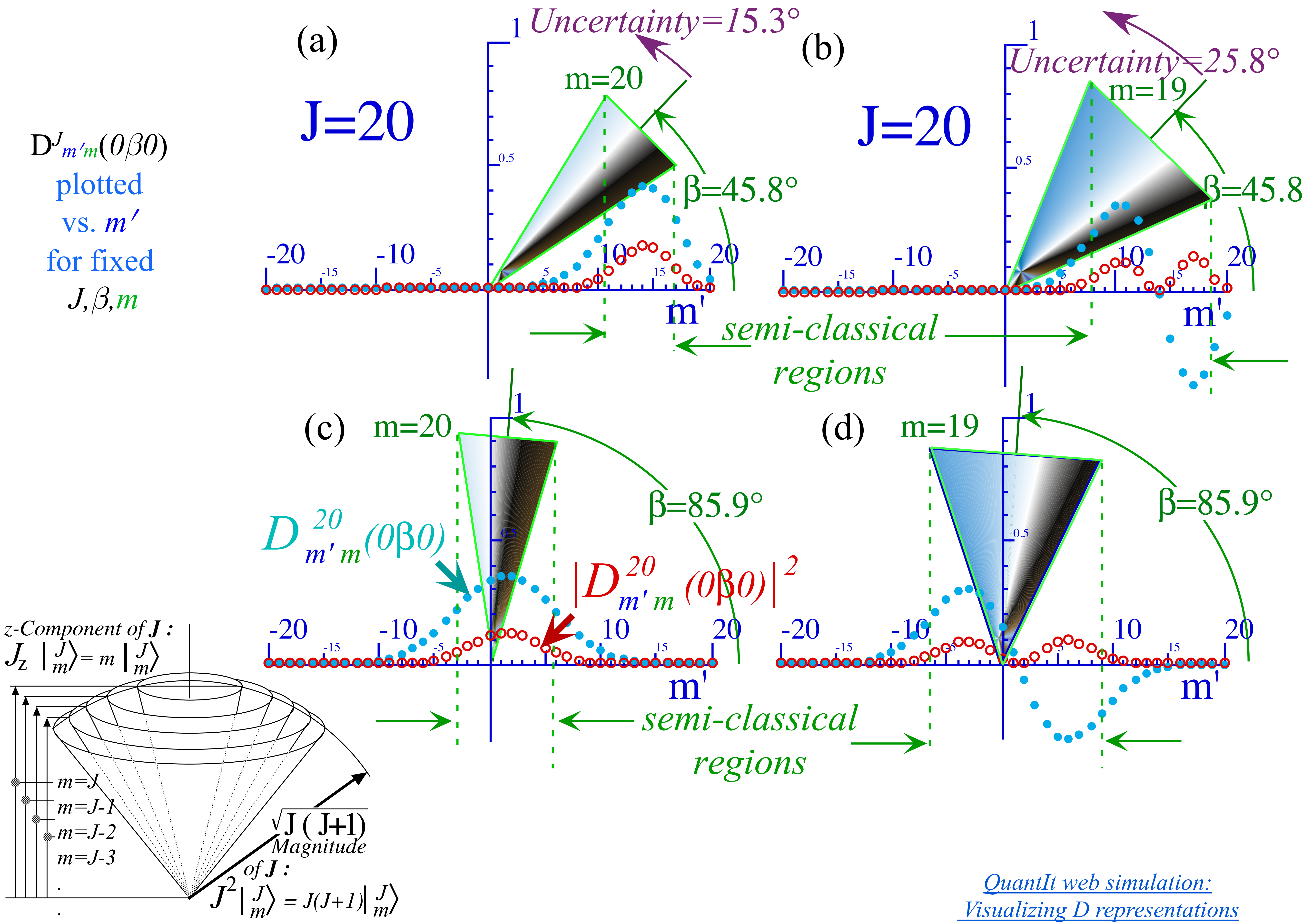
Molecular and nuclear wavefunctions

Molecular and nuclear eigenlevels

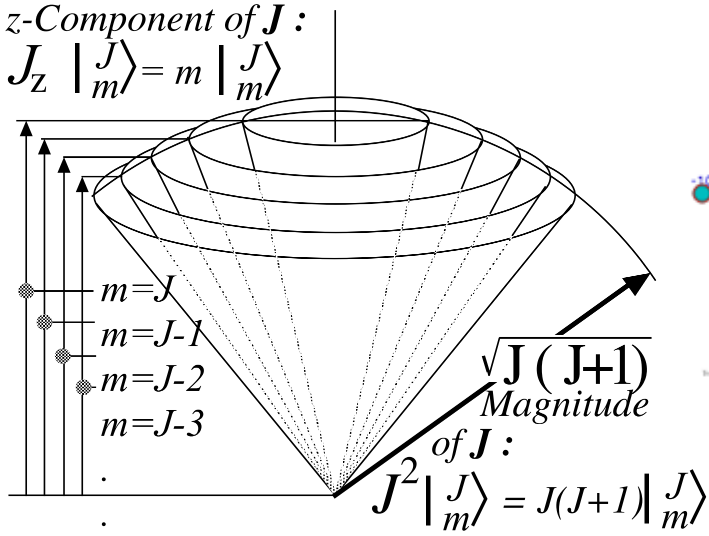
Generalized Stern-Gerlach and transformation matrices

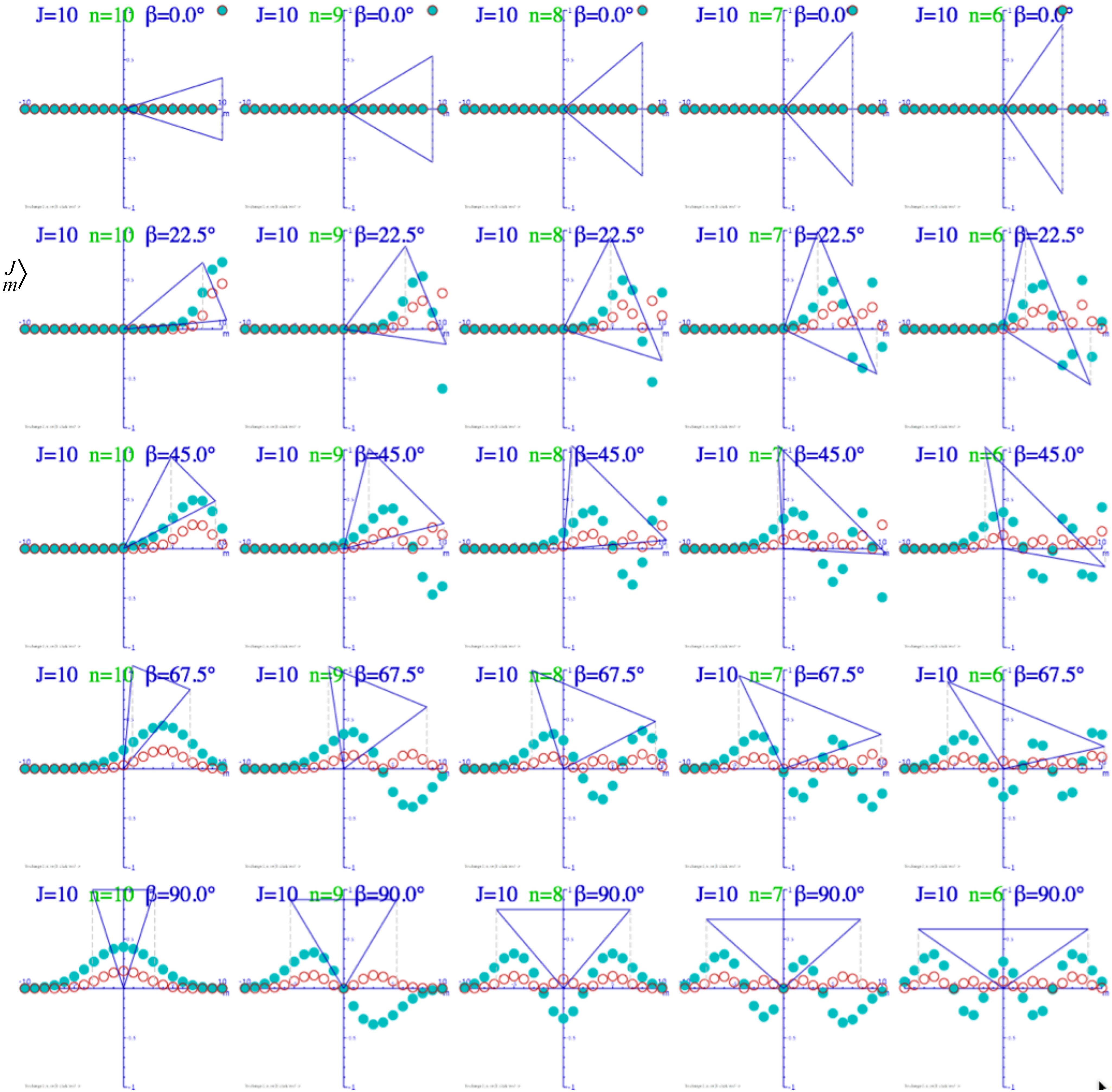
➡ *Angular momentum cones and high J properties* ⬅

Angular momentum cones and high J properties



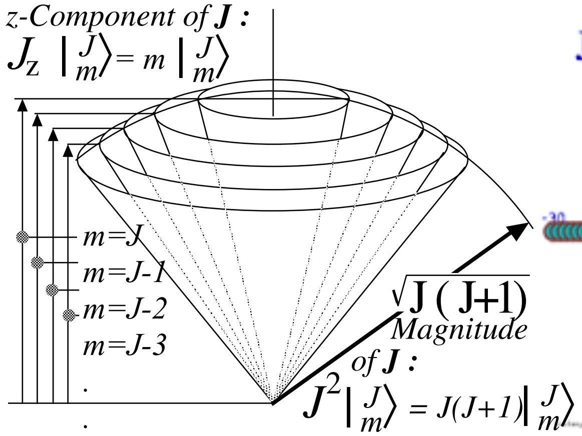
QuantIt web simulation:
Visualizing D representations





$D^{J=10}_{m,n}(0\beta 0)$
plotted
vs. m
for fixed
J=10,β,n

QuantIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/QuantItWeb.html"



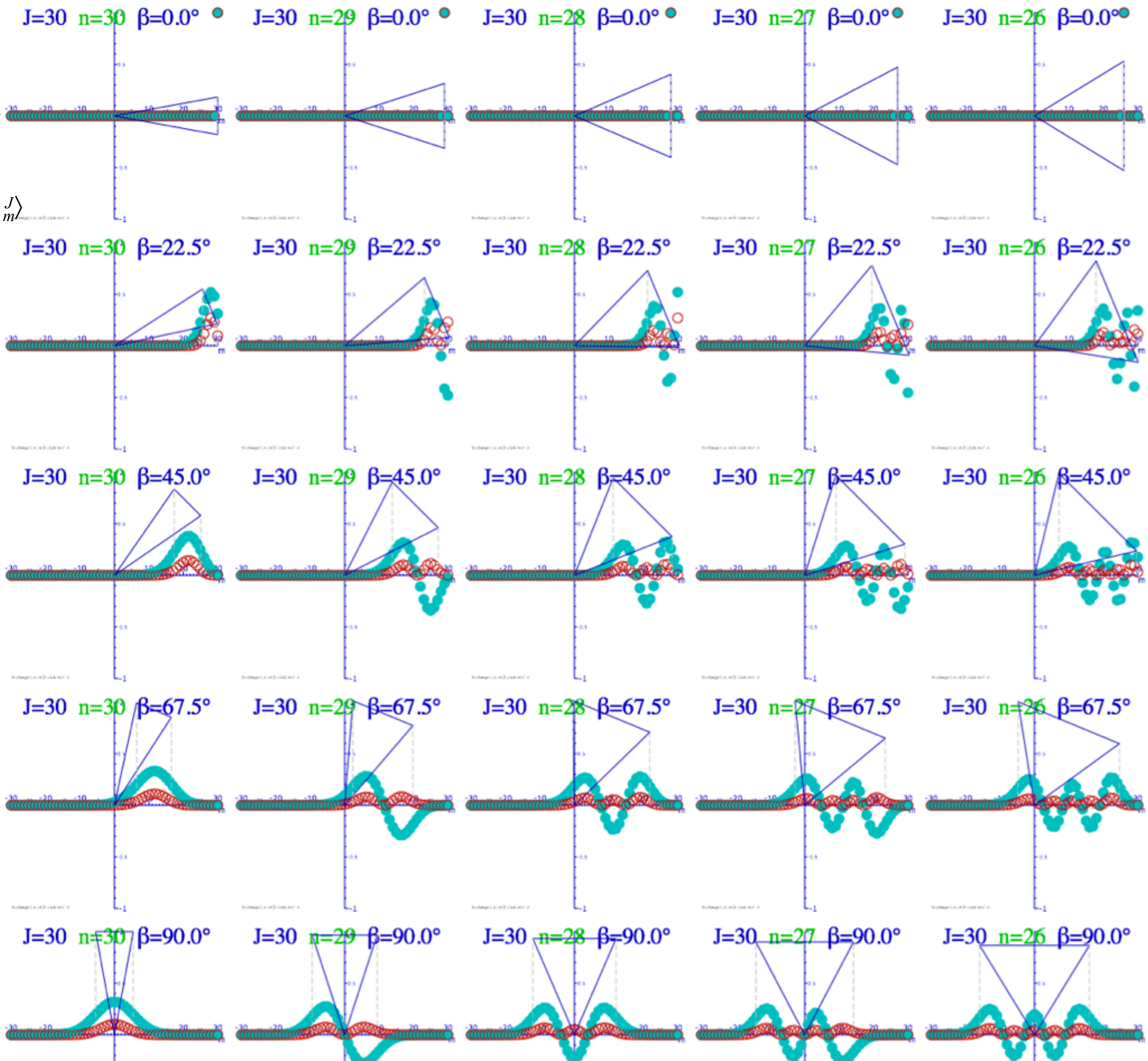
$D^{J=30}_{m,n}(0\beta 0)$

plotted

vs. m

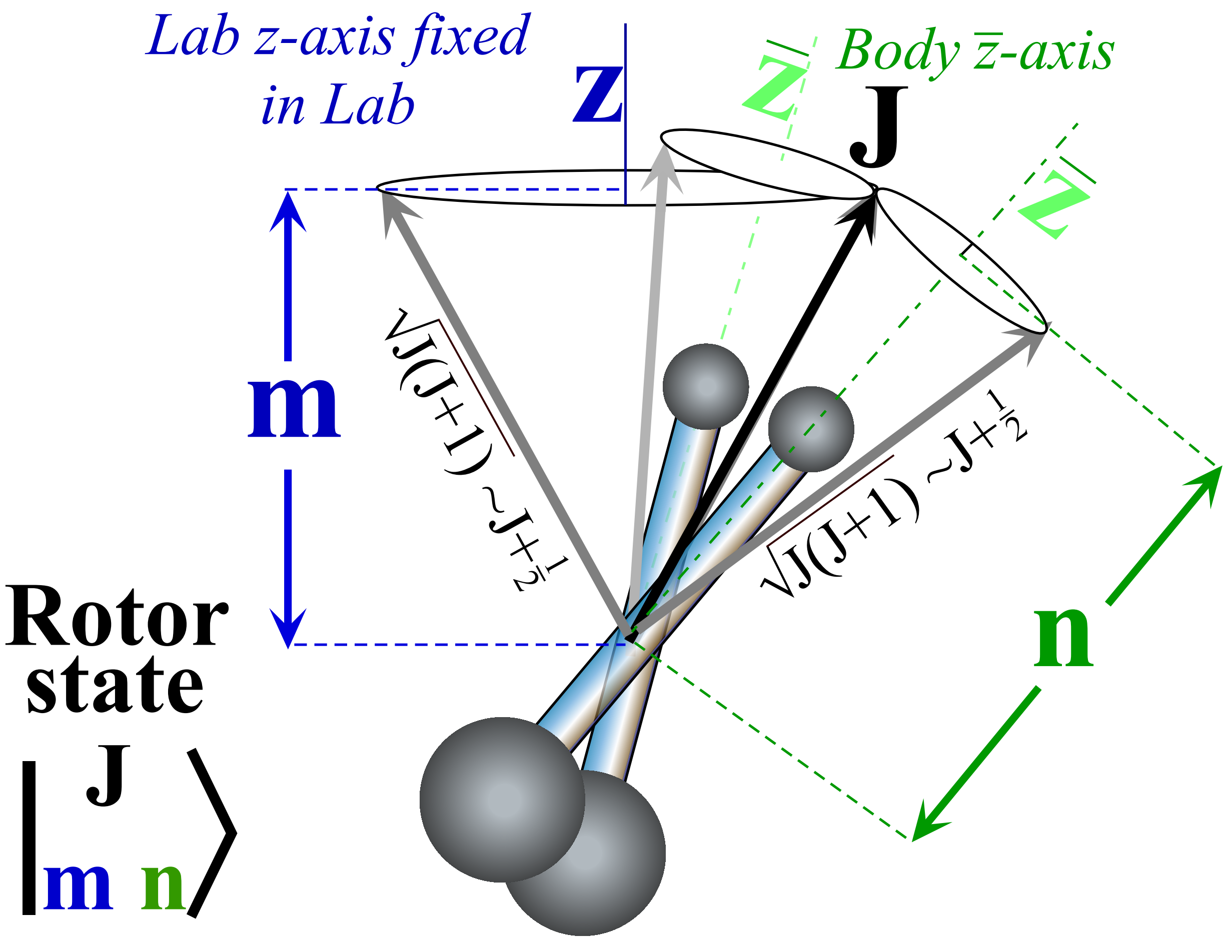
for fixed

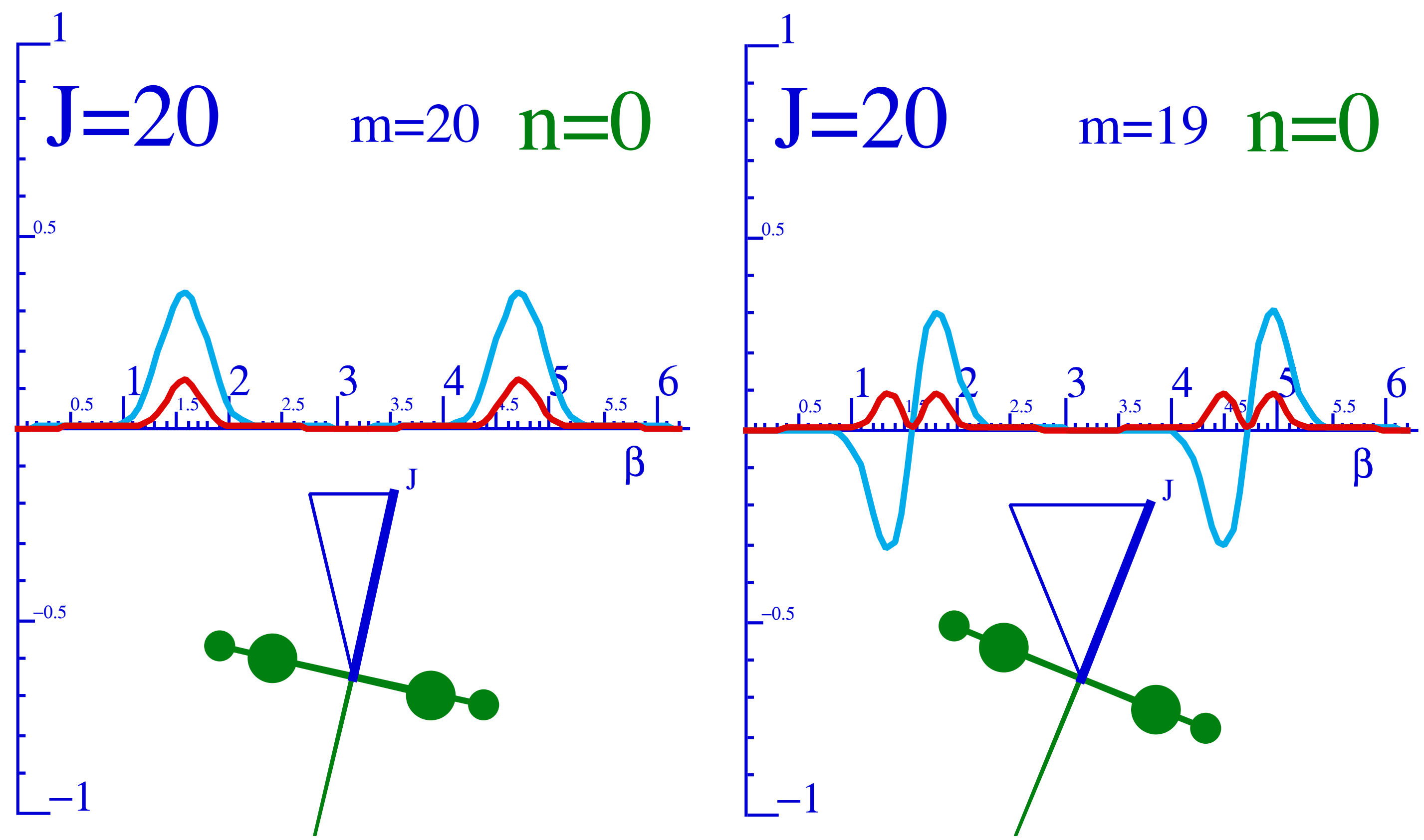
J=30,β,n



QuantIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/QuantItWeb.html"

$D^{J=20}_{m,n}(0\beta 0)$ plotted vs. β for fixed J=20,m,n

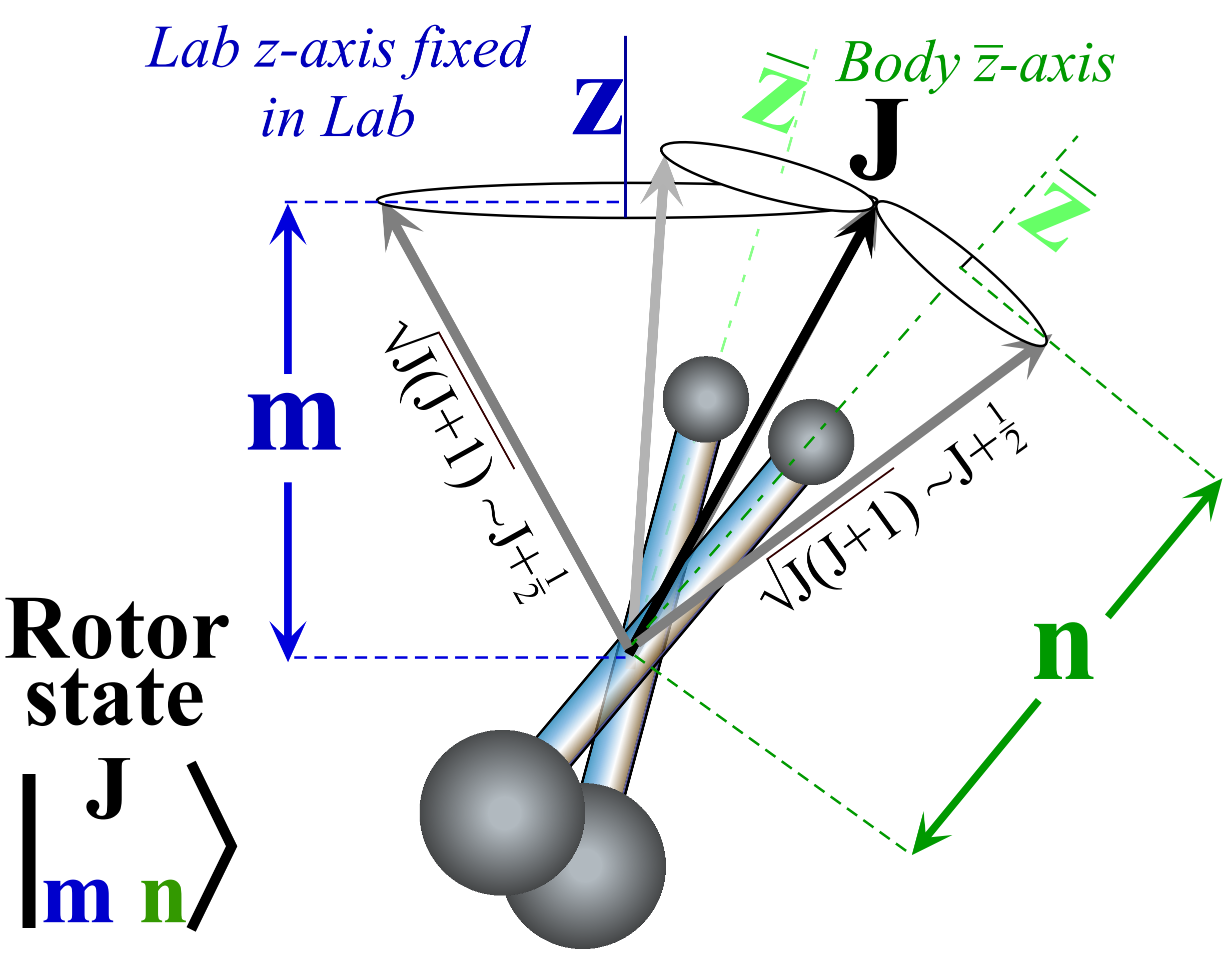


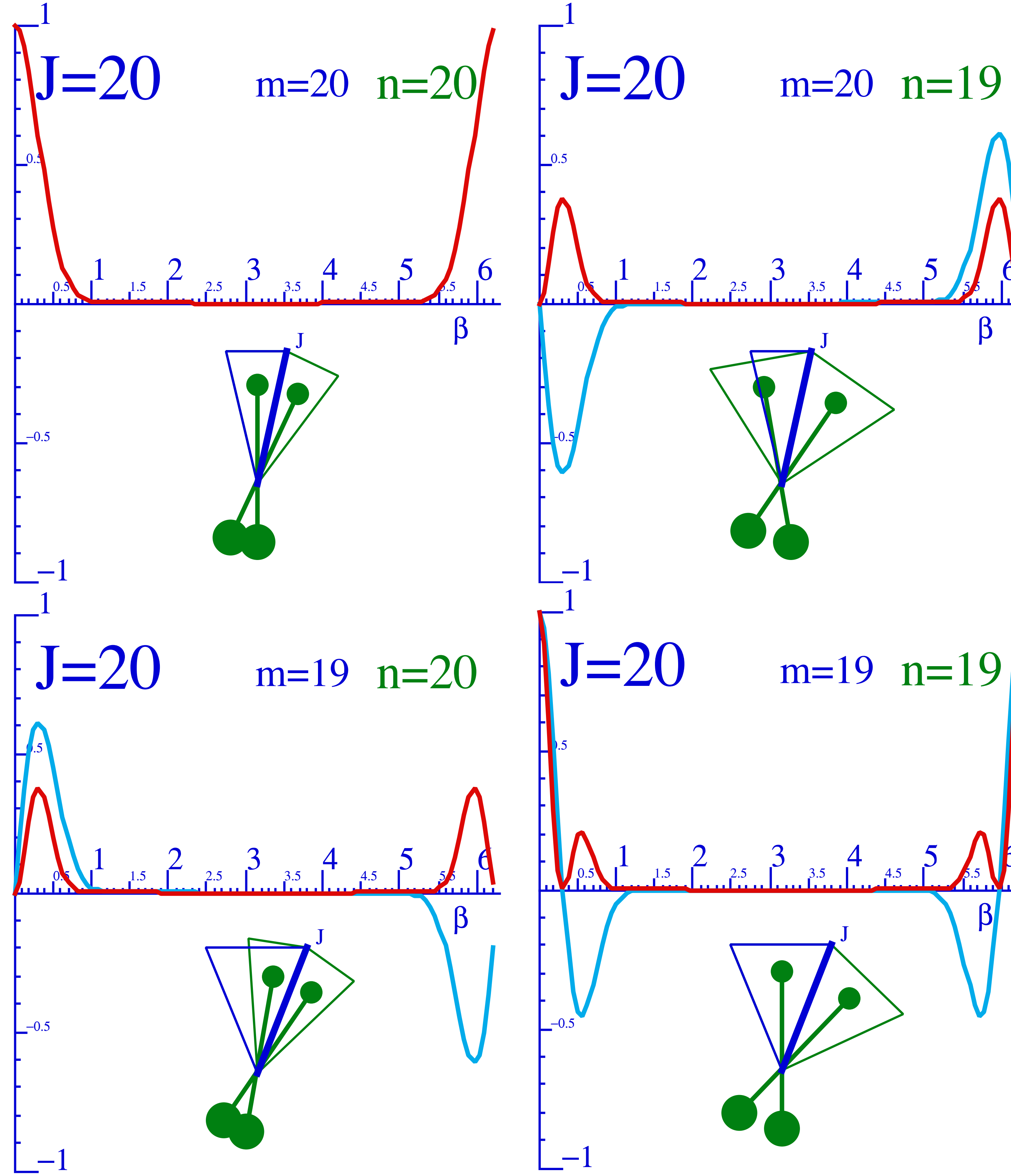


$D^{J}_{m,n}(0\beta 0)$ plotted vs. β for fixed J,m,n

QuantIt web simulation: Visualizing D representations

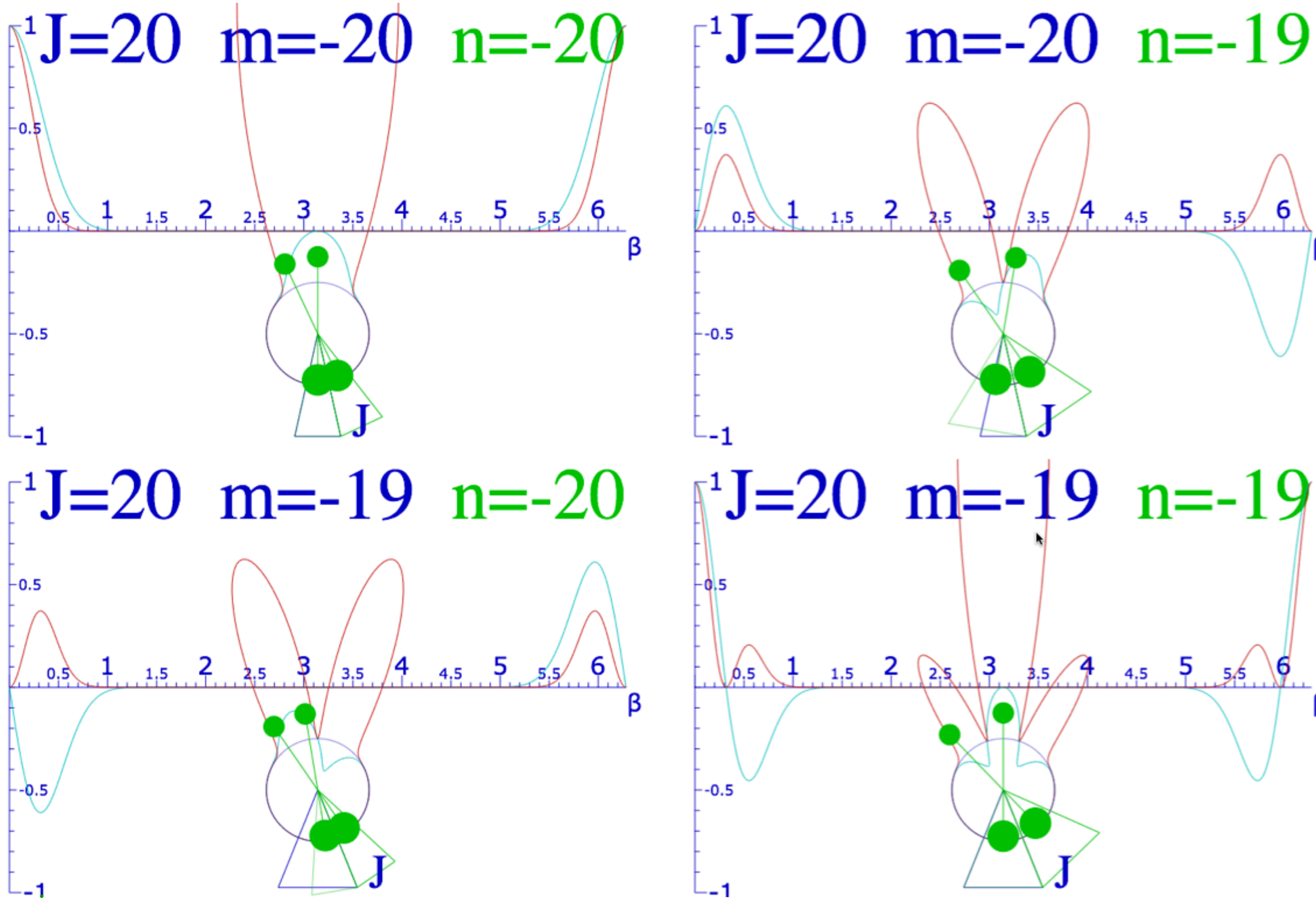
$D^{J=20}_{m,n}(0\beta 0)$
plotted
vs. β
for fixed
J=20,m,n

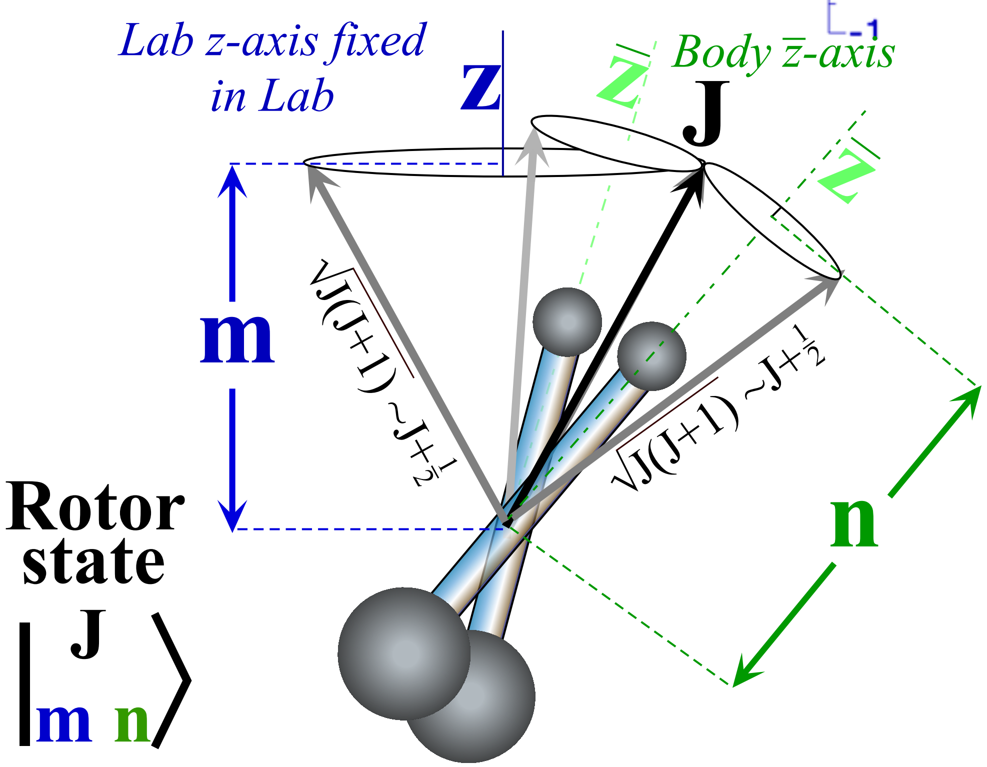




QuantIt web simulation:
Visualizing D representations

$D^{J=20}_{m,n}(0\beta 0)$
plotted
vs. β
for fixed
J=20,m,n





QuantIt web simulation: Visualizing D representations

Partial listing of the Harter-Soft/Heyoka LearnIt Web Apps as of April 24, 2017
(Apps are being upgraded as time permits)

Production Links - ***For the students & general public***

BohrIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/BohrItWeb.html"
BounceIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/BounceItWeb.html"
BoxIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/BoxItWeb.html"
CoulIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/CoulItWeb.html"
Cycloidulum - Production; URL is "http://www.uark.edu/ua/modphys/markup/CycloidulumWeb.html"
LearnIt - Production; URL is "**http://www.uark.edu/ua/modphys**" or "http://www.uark.edu/ua/modphys/markup/LearnItWeb.html"
JerkIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/JerkItWeb.html"
Pendulum - Production; URL is "http://www.uark.edu/ua/modphys/markup/PendulumWeb.html"
QuantIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/QuantItWeb.html"
Relativity - Pirelli Entrant: URL is "http://www.uark.edu/ua/pirelli" or "http://www.uark.edu/ua/pirelli/html/default.html"
Trebuchet Production; URL is "http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html"

Testing Links - ***For internal use and testing by Harter & Heyoka***

BohrIt - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/BohrItWeb.html"
BounceIt - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/BounceItWeb.html"
BounceIt Title Page - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/BounceItTitlePage.html"
BoxIt - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/BoxItWeb.html"
CoulIt - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/CoulItWeb.html"
Cycloidulum - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/CycloidulumWeb.html"
Harter-Soft Web Apps - Quick Reference - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/JerkItWeb.html"
JerkIt - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/JerkItWeb.html"
ModernPhysics - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/IntroCover.html"
Pendulum - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/PendulumWeb.html"
QuantIt - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/QuantItWeb.html"
Trebuchet Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/TrebuchetWeb.html"

Link to the complete listing of Harter-Soft LearnIt Web Apps and resources for Physics