Group Theory in Quantum Mechanics

Lecture 1 *(1.17.17)*

Introduction to quantum amplitudes and analyzers

(Quantum Theory for Computer Age - Ch. 1 of Unit 1)
(Principles of Symmetry, Dynamics, and Spectroscopy - Sec. 1-2 of Ch. 1)

Beam Sorters

2-State Sorters: spin-1/2 vs. optical polarization

Geometry of optical polarization selection and Brewster's angle

Feynman's lever

Beam Sorters in Series and Transformation Matrices
Introducing Dirac bra-ket notation

"Abstraction" of bra and ket vectors from a Transformation Matrix
Introducing scalar and matrix products

Portal pages to the two primary texts: *(Note throughout the course many of the underlined statements are linkable to the WWW)*

Principles of Symmetry, Dynamics, and Spectroscopy - URL is "http://www.uark.edu/ua/modphys/markup/PSDSWeb.html"
Quantum Theory for the Computer Age - URL is "http://www.uark.edu/ua/modphys/markup/QTCAWeb.html"

Beam Sorters

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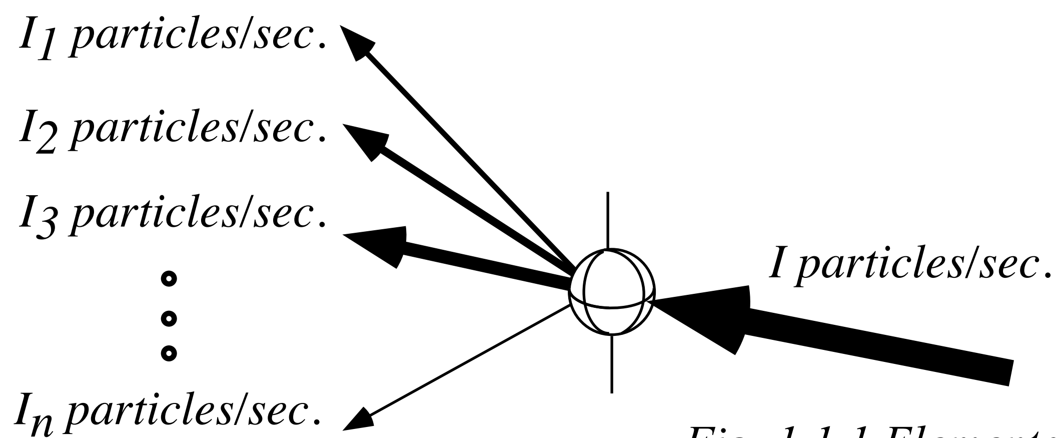
Beam Sorters



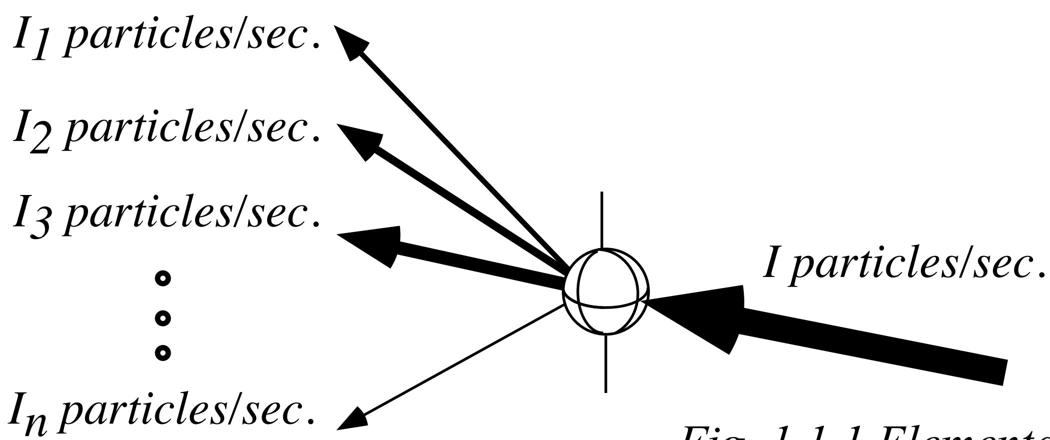
$$I = I_1 + I_2 + I_3 + \ldots + I_n$$

Fig. 1.1.1 Elementary beam sorter for n-state beam

One job of quantum mechanics is to compute *relative intensities* or *probabilities P_k* defined by

$P_k = I_k / I$ where: $1 = P_1 + P_2 + P_3 + \ldots + P_n$

Beam Sorters



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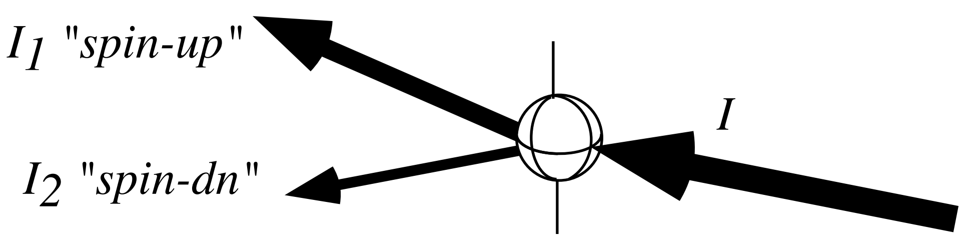
2-State Beam Sorters

Spin-1/2



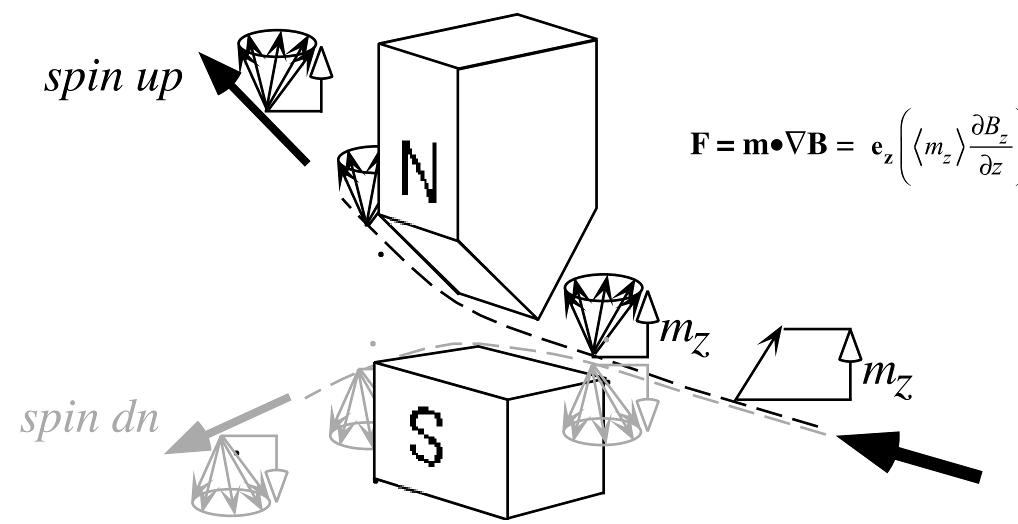
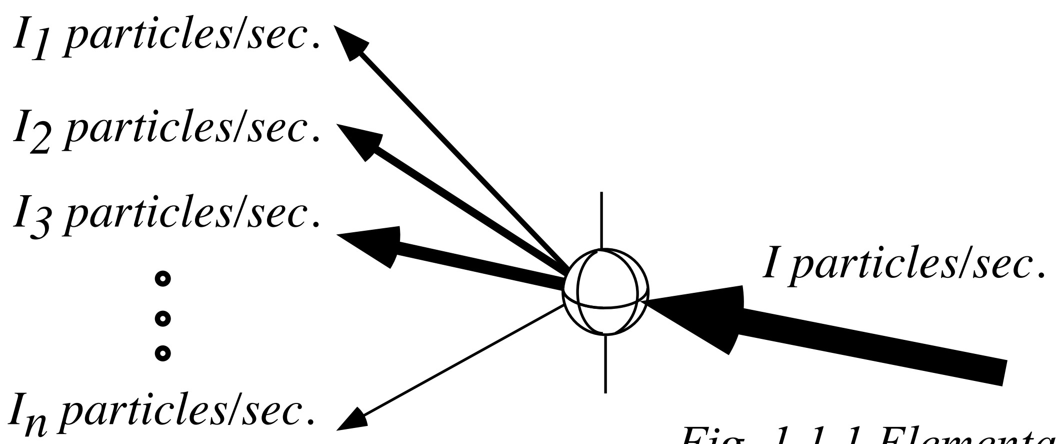


Fig. 1.1.2 Stern-Gerlach beam sorter for 2-state electron spin beam

[Quantum Theory in the Computer Age](#)
[Unit 1 Quantum Amplitudes](#)
[Chapter 1. Amplitudes, Analyzers and Matrices](#)

[Principles of Symmetry and Analysis *at page 4*](#)
[*Chapter 1. A Review of Matrix Algebra and Quantum Mechanics*](#)

Beam Sorters



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2-State Beam Sorters

Spin-1/2

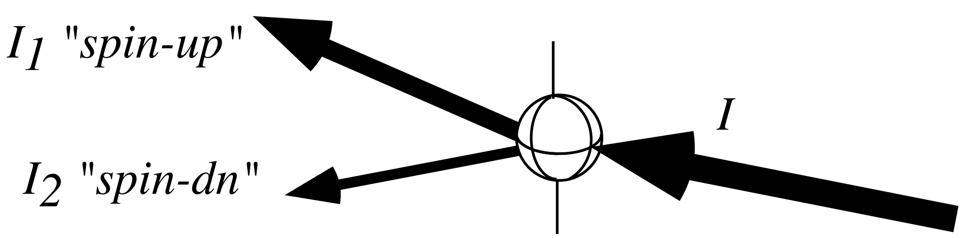

Fig. 1.1.2 Stern-Gerlach beam sorter for 2-state electron spin beam

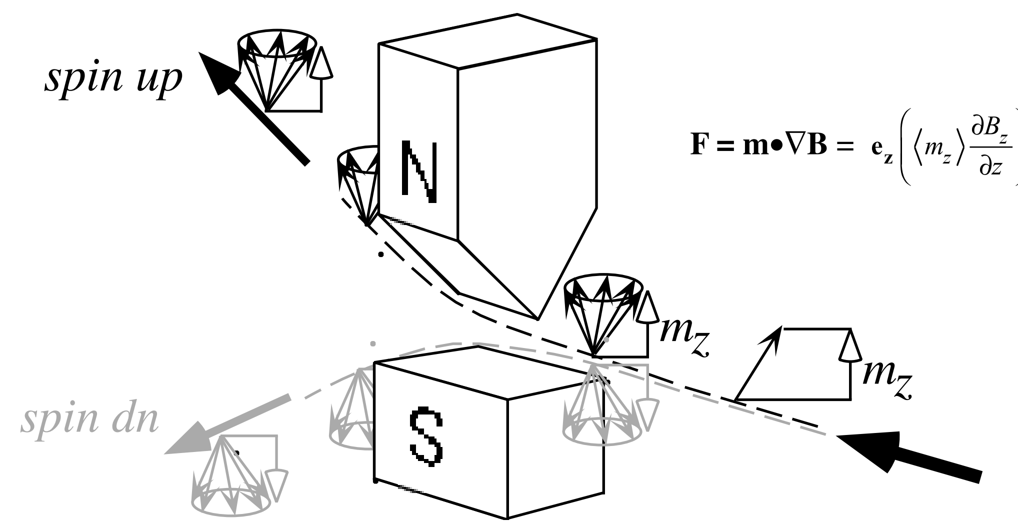

*Fig. 1.1.6 Sketch of electron beam sorting by non-uniform **B**-field: (Stern-Gerlach polarizer)*

Optical polarization

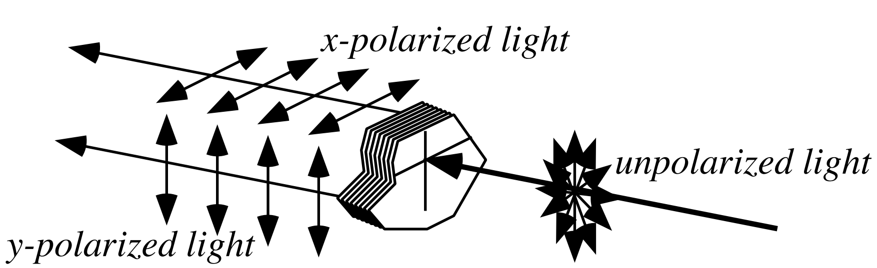

Fig. 1.1.3 Primitive photon beam sorter for 2-state polarization

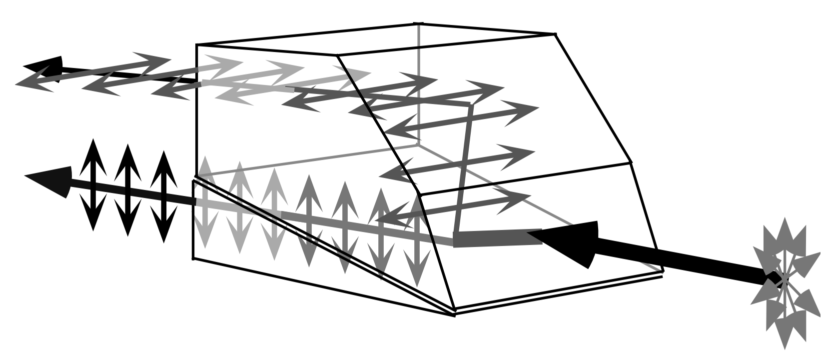

Fig. 1.1.5 Sketch of modern optical polarization sorter: (The Brewster prism)

Beam Sorters

2-State Sorters: spin-1/2 vs. optical polarization

➡ *Geometry of optical polarization selection and Brewster's angle*

Feynman's lever

Beam Sorters in Series and Transformation Matrices

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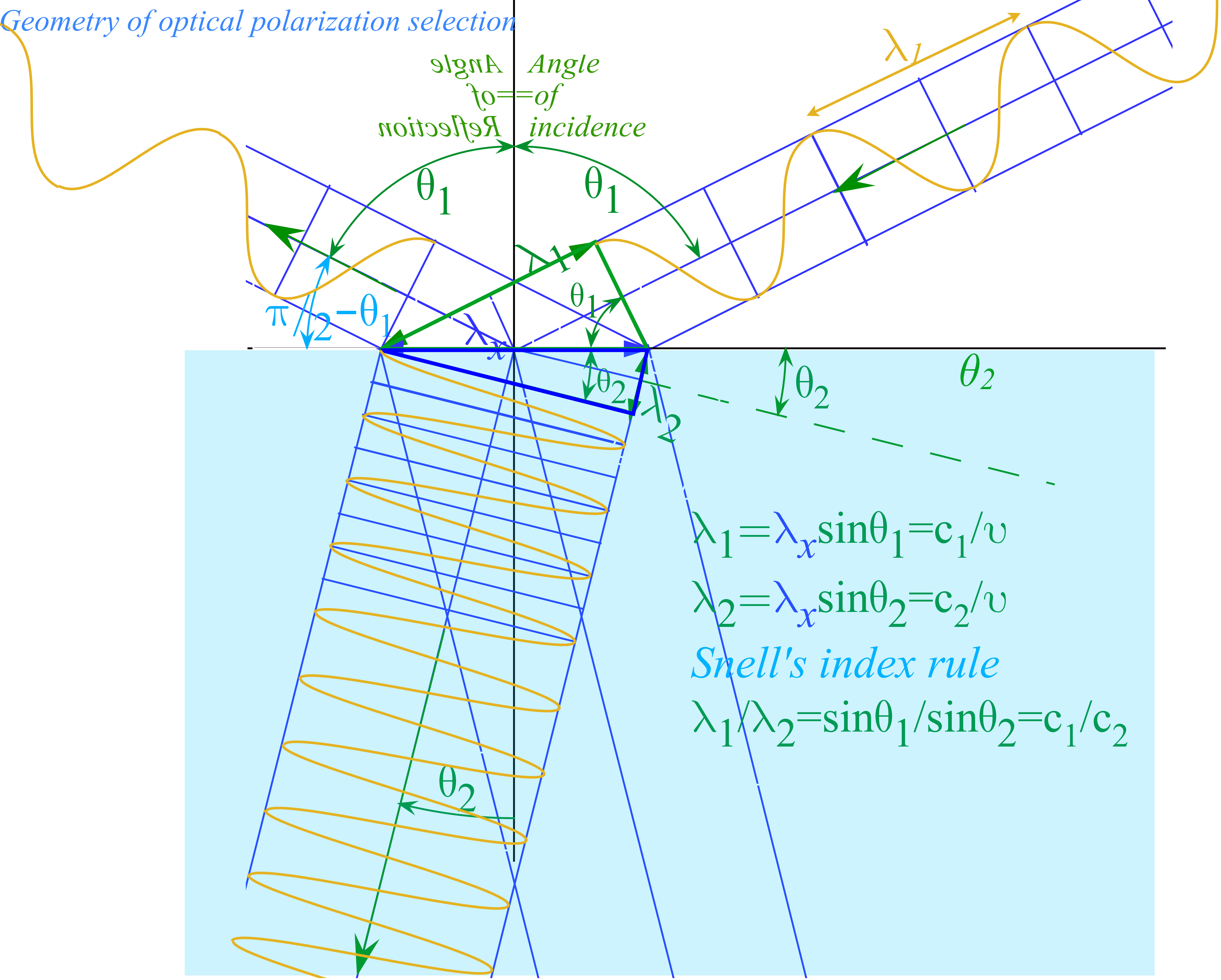
Geometry of optical polarization selection



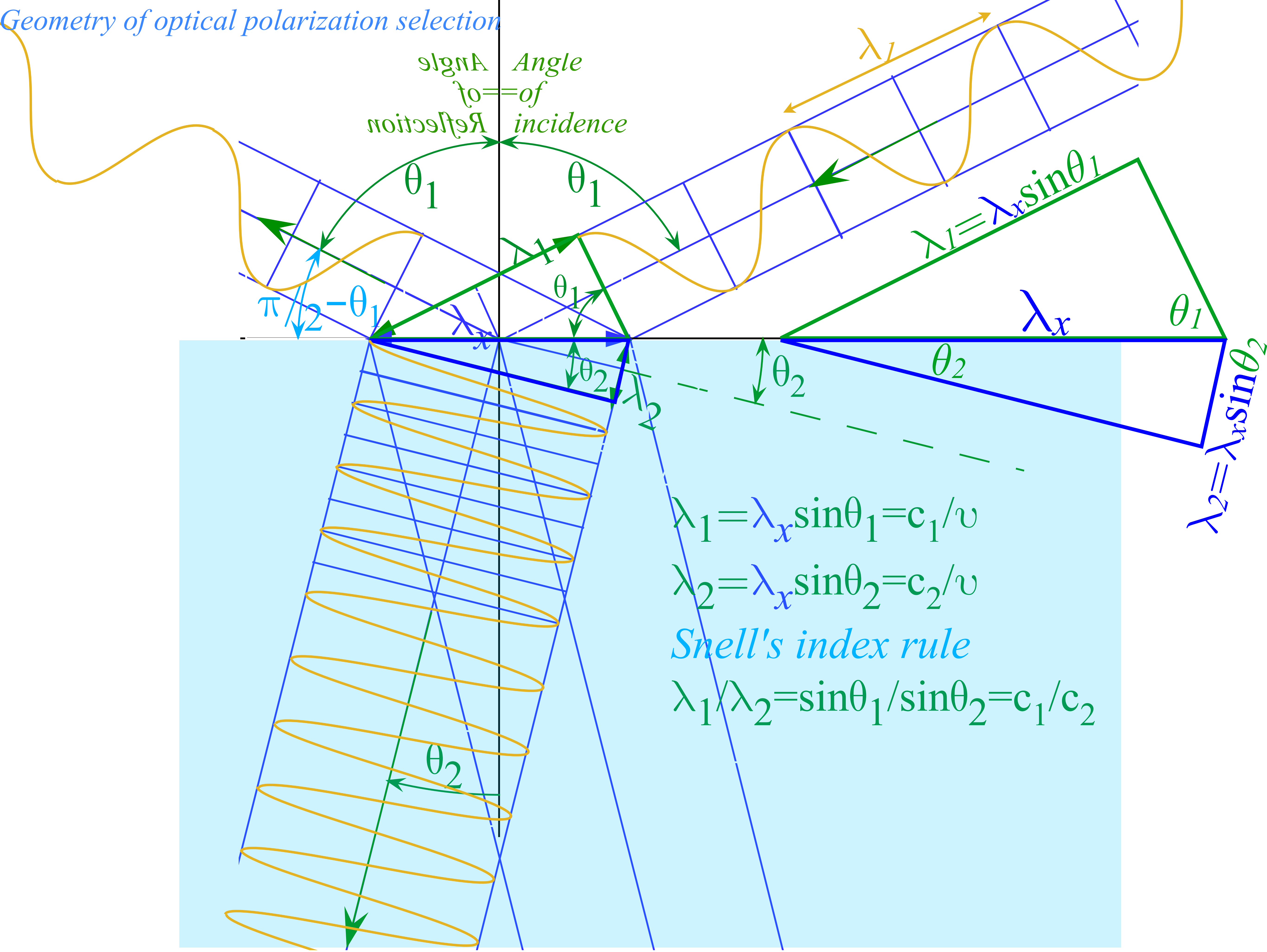
$$\lambda_1 = \lambda_x \sin\theta_1 = c_1/\upsilon$$

$$\lambda_2 = \lambda_x \sin\theta_2 = c_2/\upsilon$$

Snell's index rule

$$\lambda_1/\lambda_2 = \sin\theta_1/\sin\theta_2 = c_1/c_2$$

Geometry of optical polarization selection

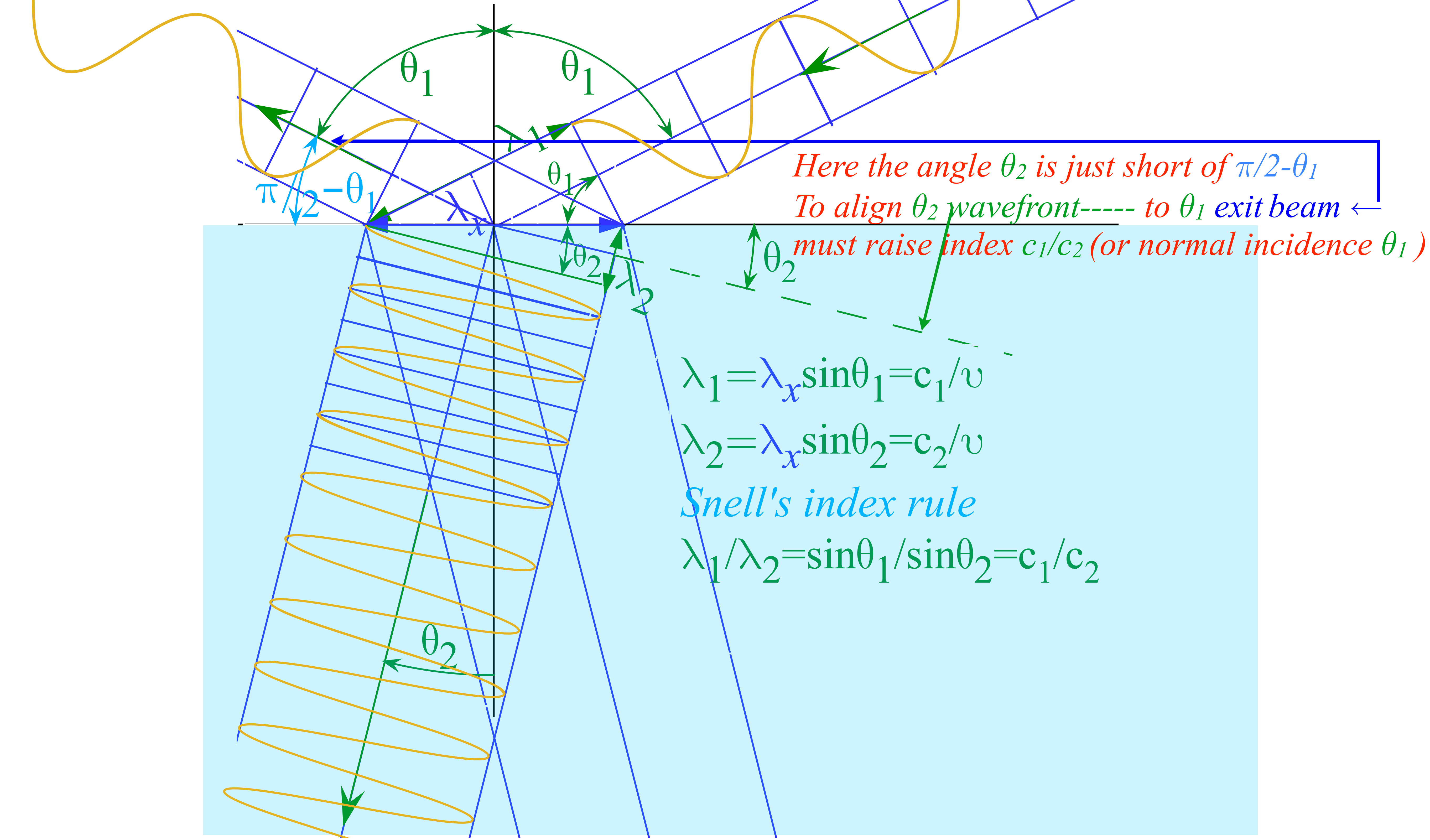


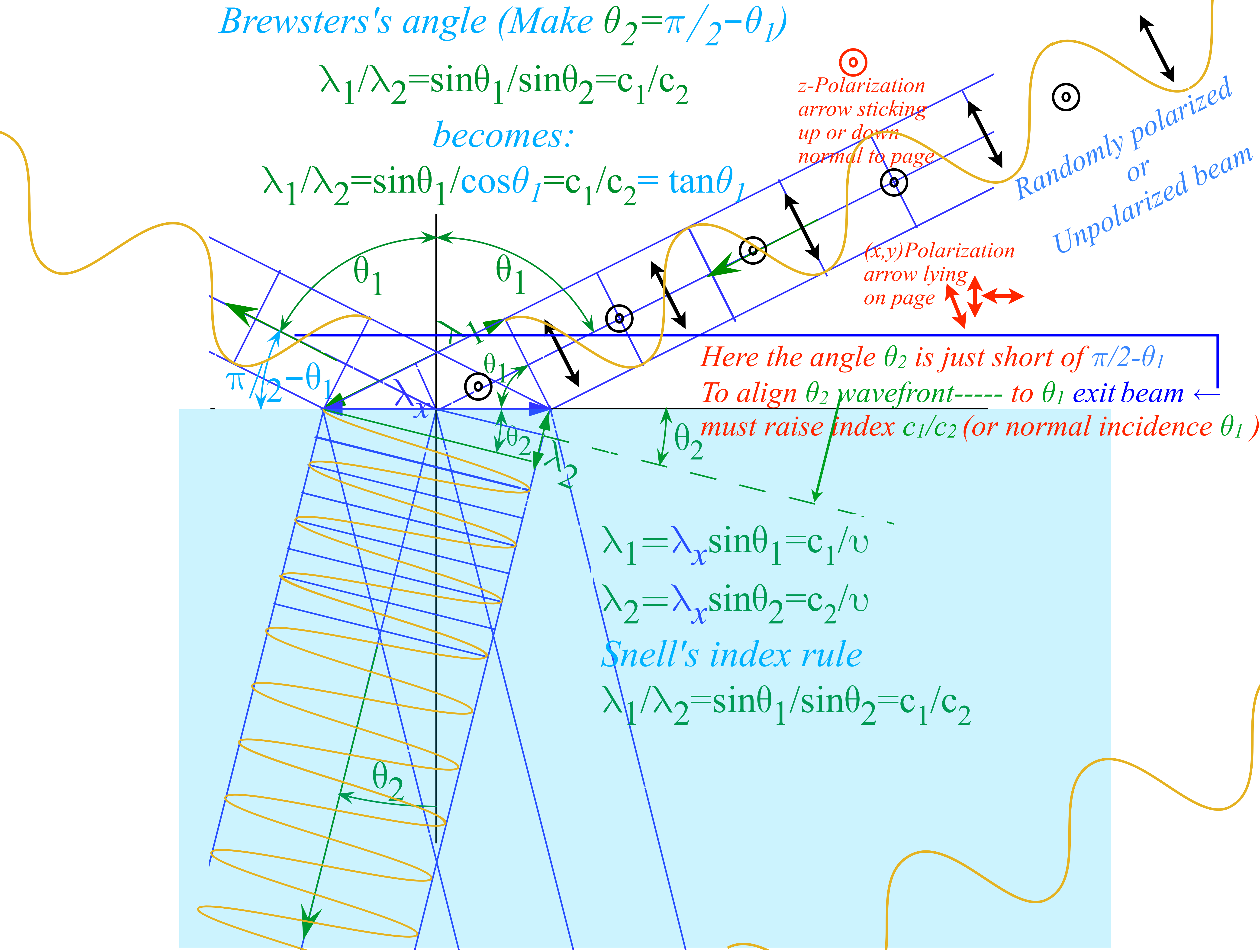
Brewsters's angle (Make $\theta_2=\pi/2-\theta_1$)

$$\lambda_1/\lambda_2=\sin\theta_1/\sin\theta_2=c_1/c_2$$

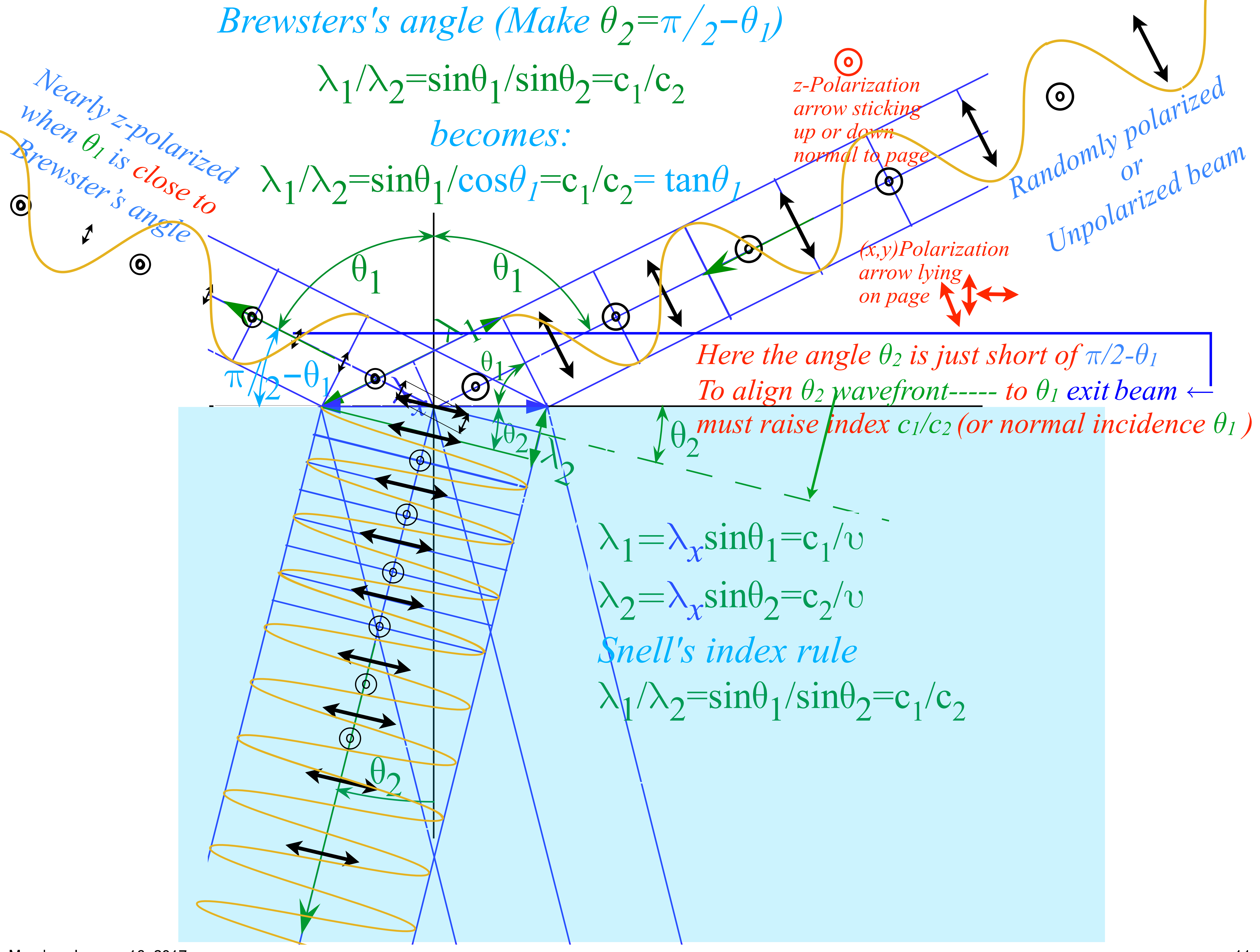
becomes:

$$\lambda_1/\lambda_2=\sin\theta_1/\cos\theta_1=c_1/c_2=\tan\theta_1$$





Brewsters's angle (Make θ2=π/2−θ1)
λ1/λ2=sinθ1/sinθ2=c1/c2
becomes:
λ1/λ2=sinθ1/cosθ1=c1/c2= tanθ1
z-Polarization
arrow sticking
up or down
normal to page
Randomly polarized
or
Unpolarized beam
(x,y)Polarization
arrow lying
on page
θ1
θ1
λ1
θ1
π/2−θ1
λx
θ2
λ2
θ2
Here the angle θ2 is just short of π/2-θ1
To align θ2 wavefront----- to θ1 exit beam ←
must raise index c1/c2 (or normal incidence θ1)
λ1=λx sinθ1=c1/υ
λ2=λx sinθ2=c2/υ
Snell's index rule
λ1/λ2=sinθ1/sinθ2=c1/c2
θ2


Brewsters's angle (Make θ2=π/2−θ1)
λ1/λ2=sinθ1/sinθ2=c1/c2
becomes:
λ1/λ2=sinθ1/cosθ1=c1/c2= tanθ1
Nearly z-polarized when θ1 is close to Brewster's angle
z-Polarization arrow sticking up or down normal to page
Randomly polarized or Unpolarized beam
(x,y)Polarization arrow lying on page
θ1
θ1
λ1
θ1
π/2−θ1
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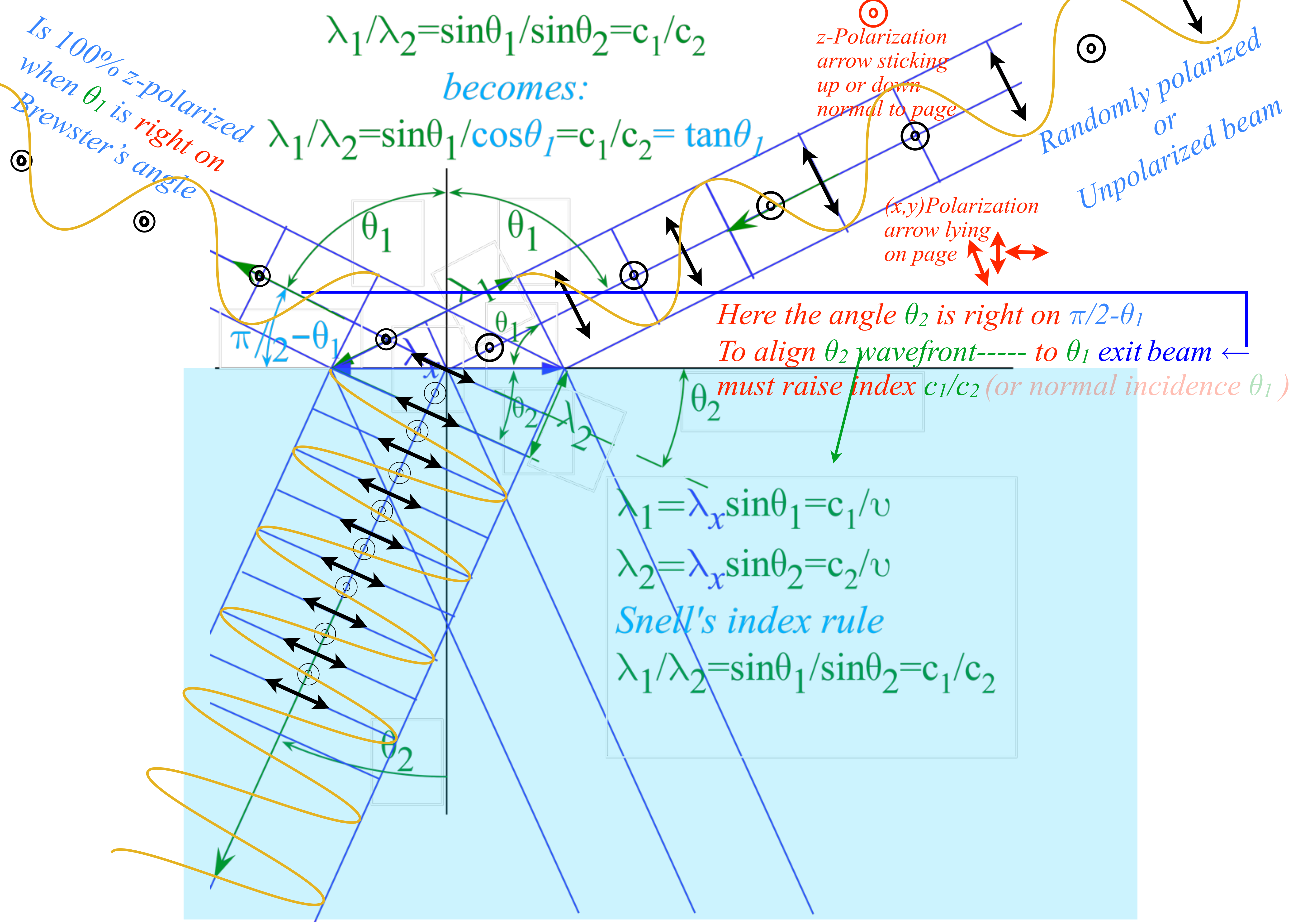
Brewsters's angle (Make $\theta_2=\pi/2-\theta_1$)



Beam Sorters

2-State Sorters: spin-1/2 vs. optical polarization

Geometry of optical polarization selection and Brewster's angle

➡ *Feynman's lever*

Beam Sorters in Series and Transformation Matrices

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(c) Visualizing Radiation Coupling Using Feynman's Lever The detailed solutions of Newton's and Maxwell's equations for coupled particles and em fields are complicated. However, for small numbers of particles there is a graphical construction given in the Feynman Lectures (Section II-21) which is very instructive. It provides a way to tell exactly what the fields will be around an arbitrarily moving charge.

Imagine that you are holding a charge and moving it back and forth. Let the charge be attached to a ring which can slide on a long lever arm as shown in Figure 6.5.6(a). Let the lever have a unit vector $(-\hat{\mathbf{e}}_r)$ or pointer pointing in the opposite direction of the lever $\mathbf{r}$ on the other side of its swivel point (0) at origin. Feynman has shown that the $\mathbf{E}$ field at origin at time t depends on the position of the pointer $\hat{\mathbf{e}}'_r$ and lever $\mathbf{r}'$ at a slightly earlier time $(t' = t - r/c)$. The time delay is just the time it would take a signal traveling at c to propagate from r at t' to origin at t. The $\mathbf{E}$ field is given by

$$\mathbf{E}(0,t) = \frac{q}{4\pi\varepsilon_0}\left\{\frac{-\hat{\mathbf{e}}'_r}{(r')^2} + \frac{r'}{c}\frac{d}{dt}\left[\frac{-\hat{\mathbf{e}}'_r}{(r')^2}\right] + \frac{1}{c^2}\frac{d^2}{dt^2}[-\mathbf{e}'_r]\right\}$$

$$= \underset{\text{term}}{\text{Coulomb}} + \underset{\text{term}}{\text{induction}} + \underset{\text{term}}{\text{radiation}}. \qquad (6.5.25a)$$

The first term is just the usual Coulomb field. The second term gives rise to a magnetic induction field,

$$\mathbf{B}(0,t) = (\hat{\mathbf{e}}'_r \times \mathbf{E})/c, \qquad (6.5.25b)$$

at origin if the charge has velocity transverse to $\mathbf{r}$. Finally, the third radiation term contributes to $\mathbf{E}(0,t)$ and $\mathbf{B}(0,t)$ in (6.5.25) if the charge has acceleration transverse to $\mathbf{r}$. It is interesting to note that in some ways this term is the reverse of Newton's law. For Newton's law one is given a field $\mathbf{E}$ or

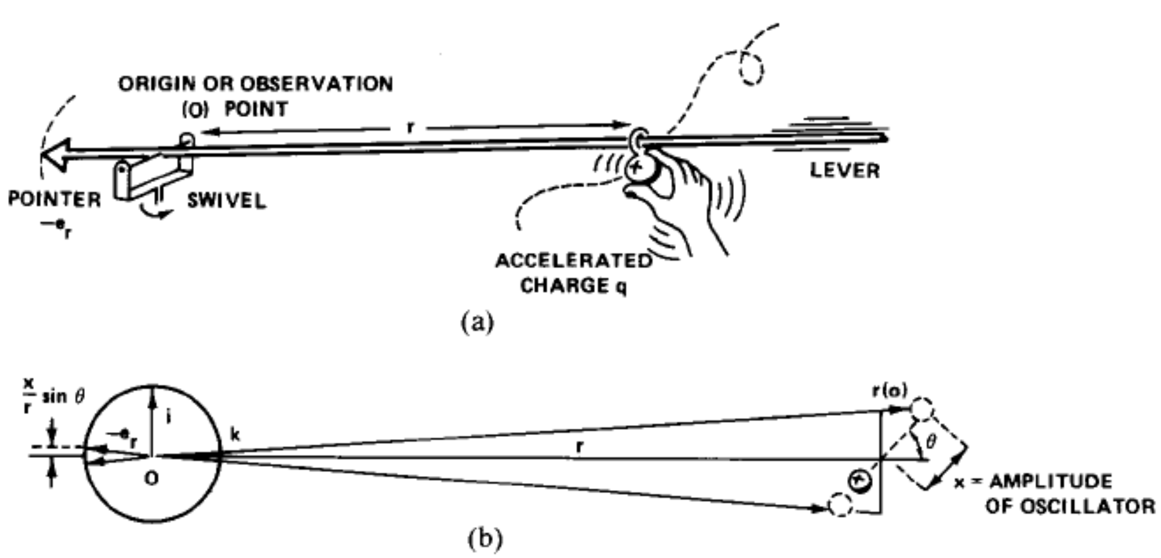

Figure 6.5.6 Feynman's lever. This construction provides a convenient way to visualize the field due to an accelerated or moving charge.

Feynman's Lectures now free online

http://www.feynmanlectures.caltech.edu/

See Volume II Chapter 21 for the lever

Feynman's lever as described in PSDS:

http://www.uark.edu/ua/modphys/pdfs/PSDS_Pdfs/PSDS_Ch.6_(4.20.10).pdf#page=517

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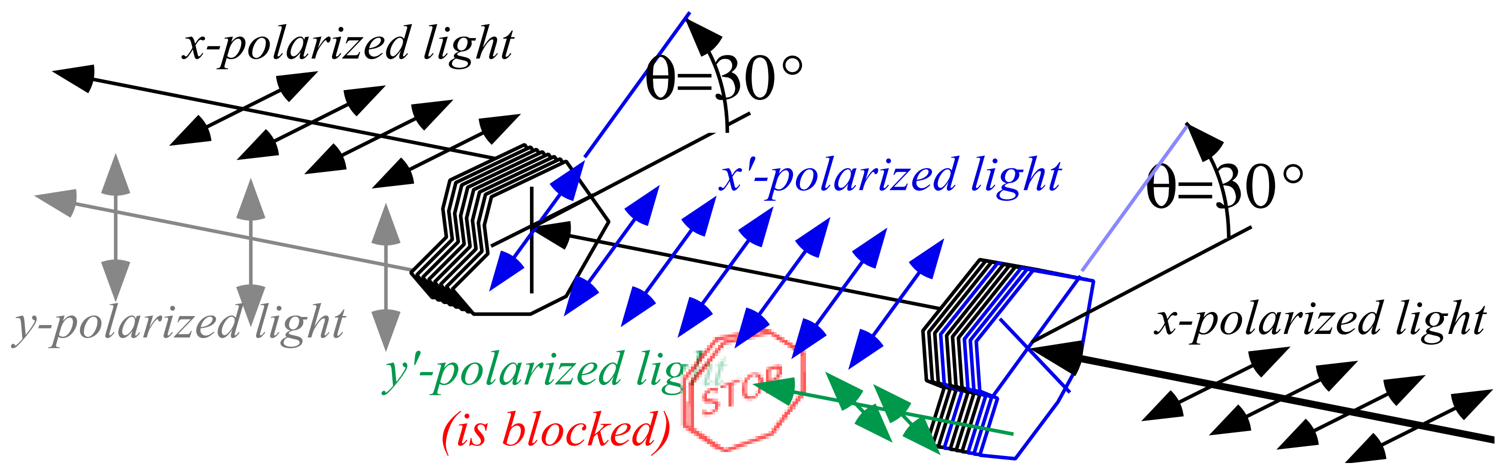


Fig. 1.2.1 Photon beam sorters in series with the first one y-blocked and tilted by angle θ=30°.

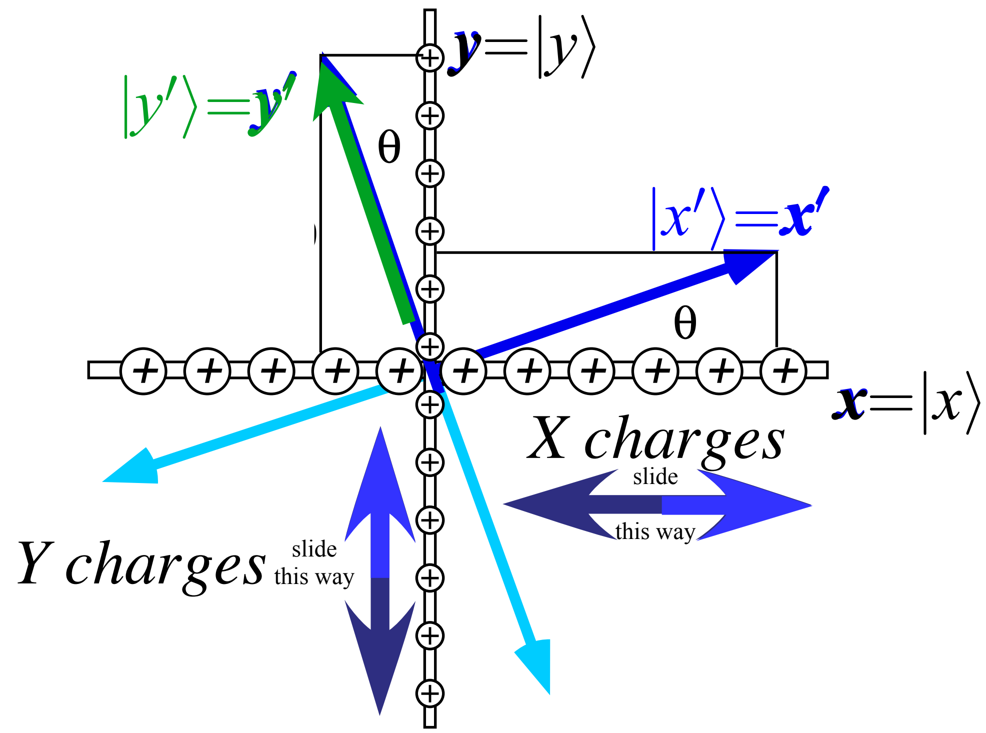
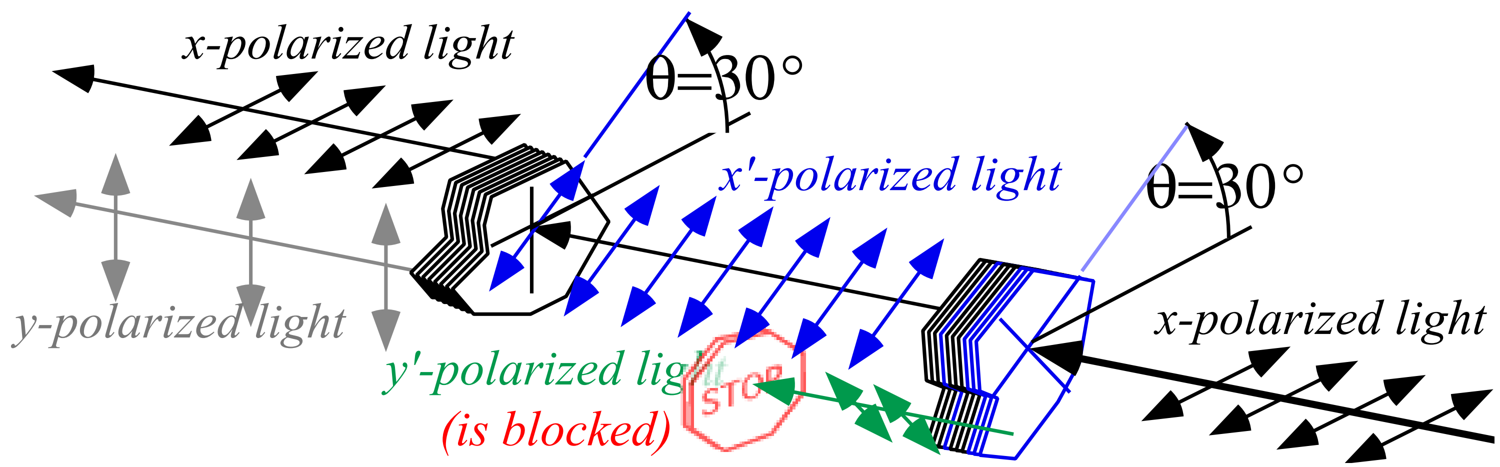


Fig. 1.2.2 Geometry of photon beam sorter for input polarizations (x',y') tilted by angle θ [relative to (*x*,*y*)].

Beam Sorters in Series and Transformation Matrices



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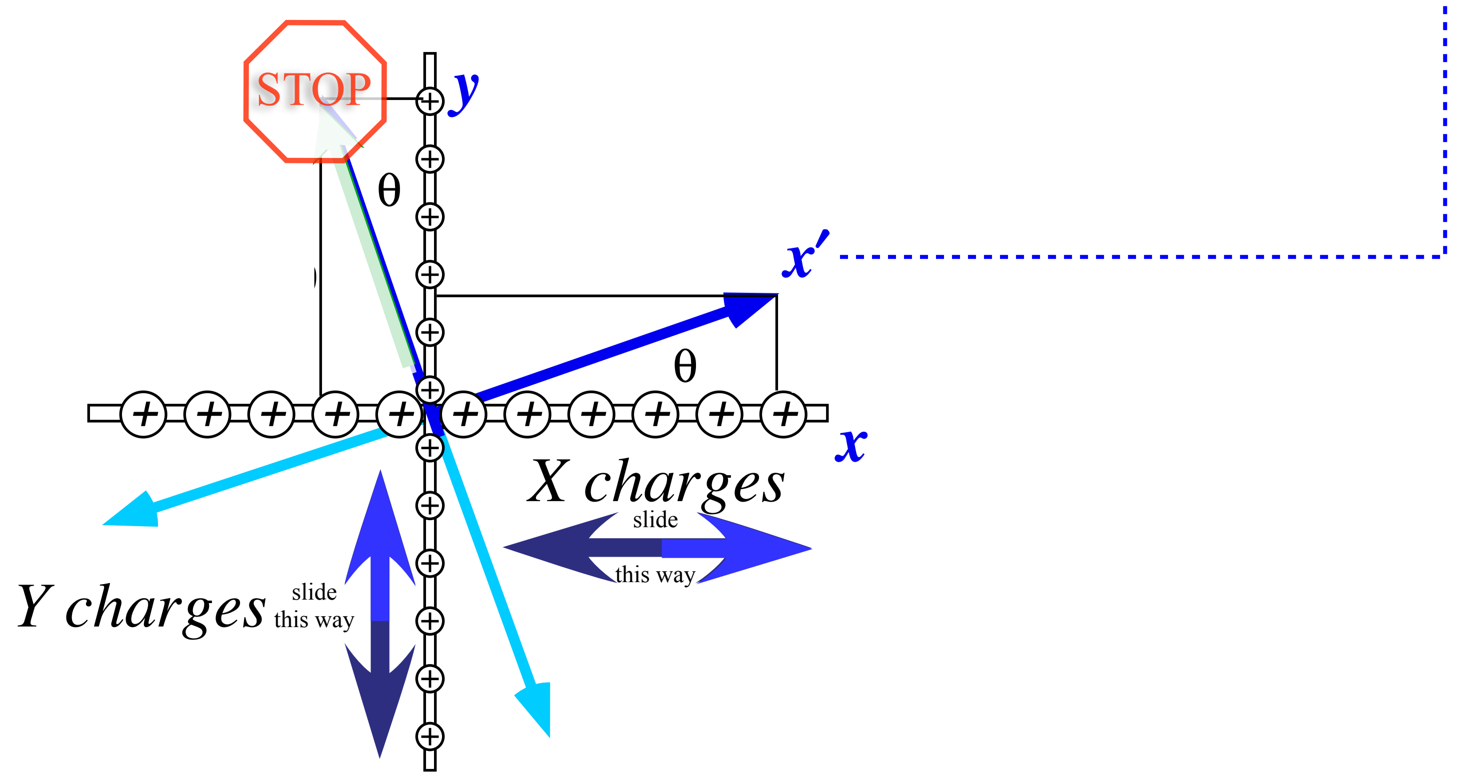
If we y′-blocked and let x′ through: *(to be (x,y)-analyzed*



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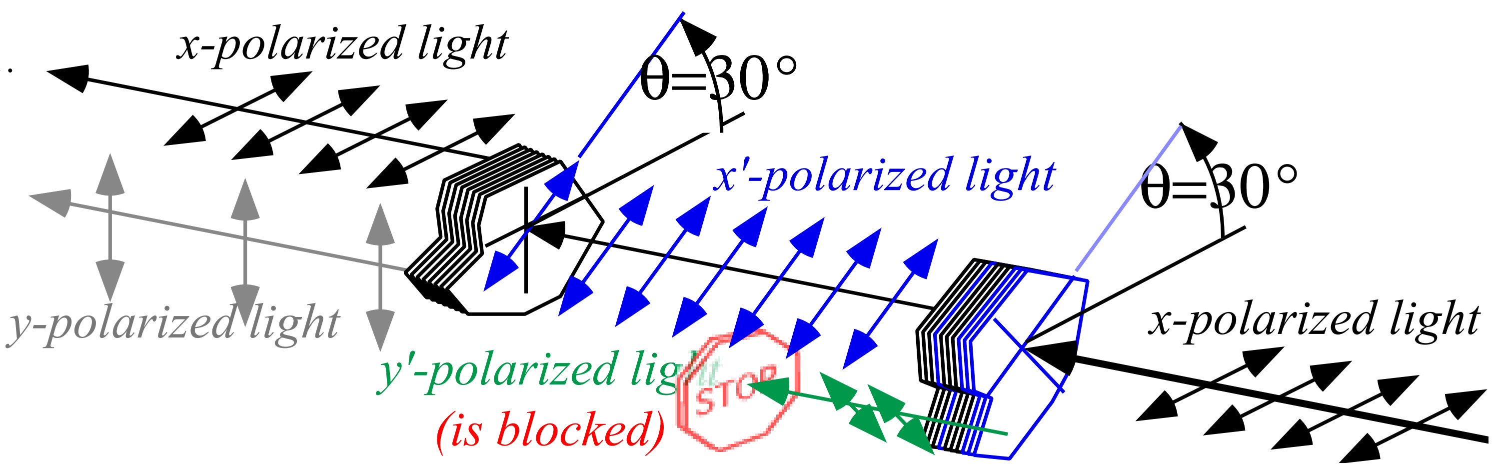
Beam Sorters in Series and Transformation Matrices

Feynman-Dirac Interpretation of

$$\langle m | n' \rangle$$

=Amplitude of state-m after state-n′ is forced to choose from available m-type states



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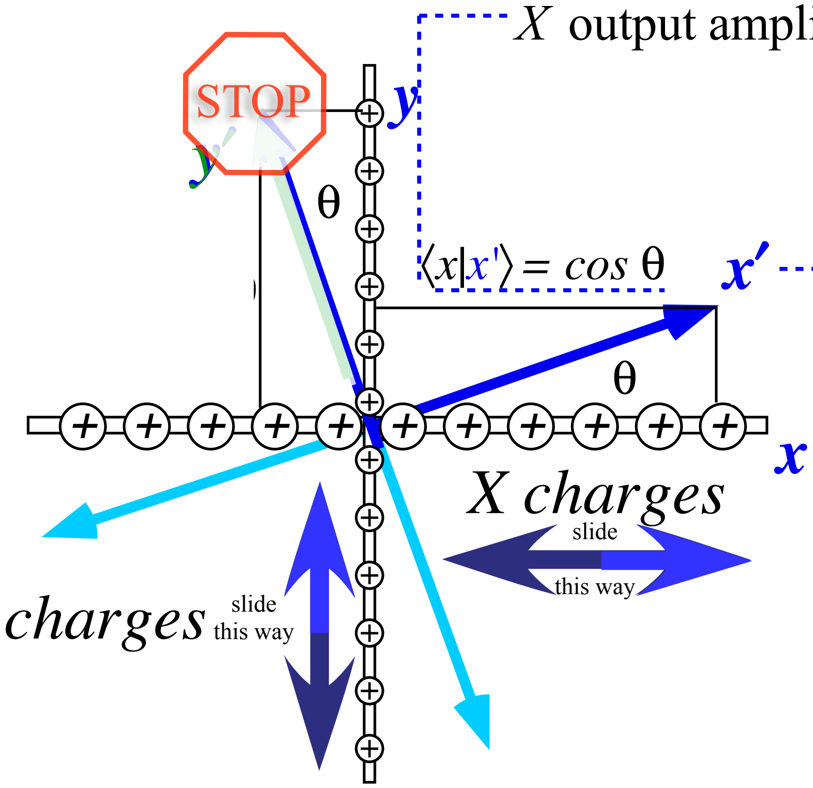
Introducing Dirac bra-ket notation.

If we y′-blocked and let x′ through: *(to be (x,y)-analyzed*

X output amplitude due to x' input $= \langle x | x' \rangle = \cos\theta$



$sin\theta = sin 30° = 1/2 = 0.5$

$cos\theta = cos 30° = \sqrt{3}/2 = 0.866$

Fig. 1.2.2 Geometry of photon beam sorter for input polarizations (x′,y′) tilted by angle θ [relative to (x,y)].

Beam Sorters in Series and Transformation Matrices

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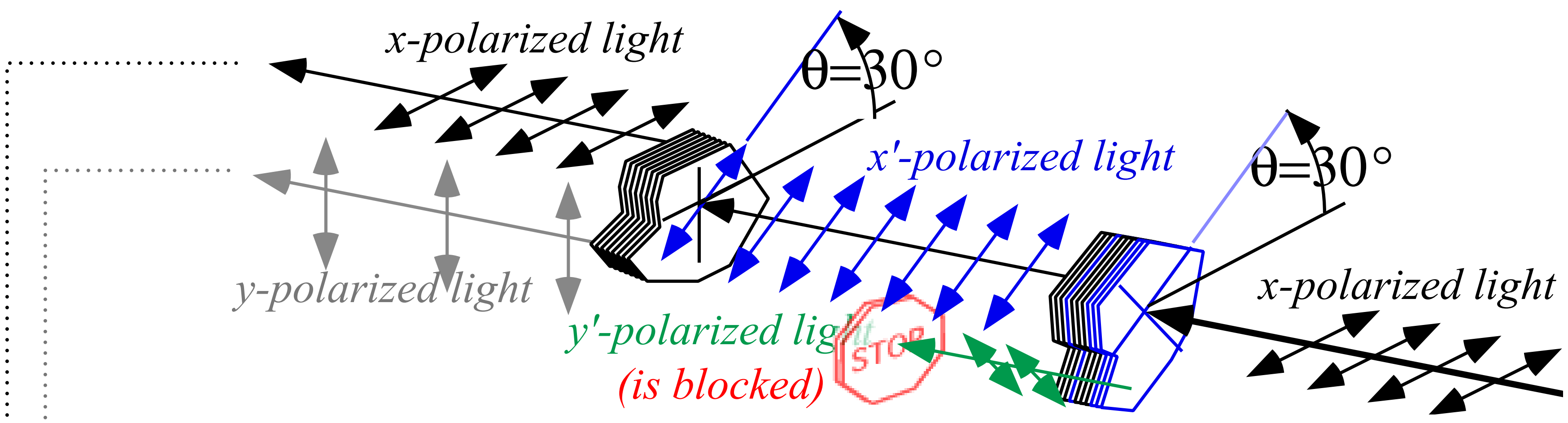

Fig. 1.2.1 Photon beam sorters in series with the first one y-blocked and tilted by angle θ=30°.

Introducing Dirac bra-ket notation.

If we y′-blocked and let x′ through: *(to be (x,y)-analyzed)*

$$X \text{ output amplitude due to } x' \text{ input} = \langle x | x' \rangle = \cos\theta,$$

$$Y \text{ output amplitude due to } x' \text{ input} = \langle y | x' \rangle = \sin\theta$$

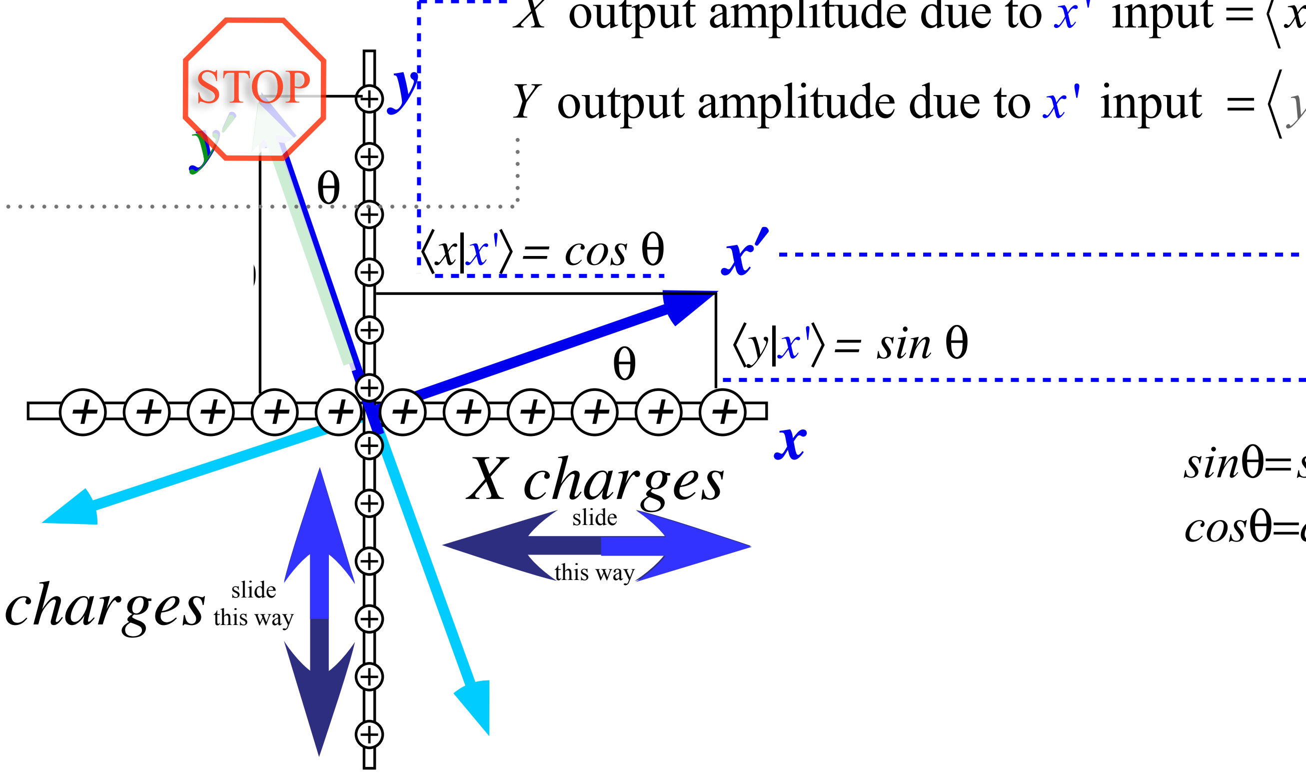

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Fig. 1.2.2 Geometry of photon beam sorter for input polarizations (x′,y′) tilted by angle θ [relative to (x,y)].

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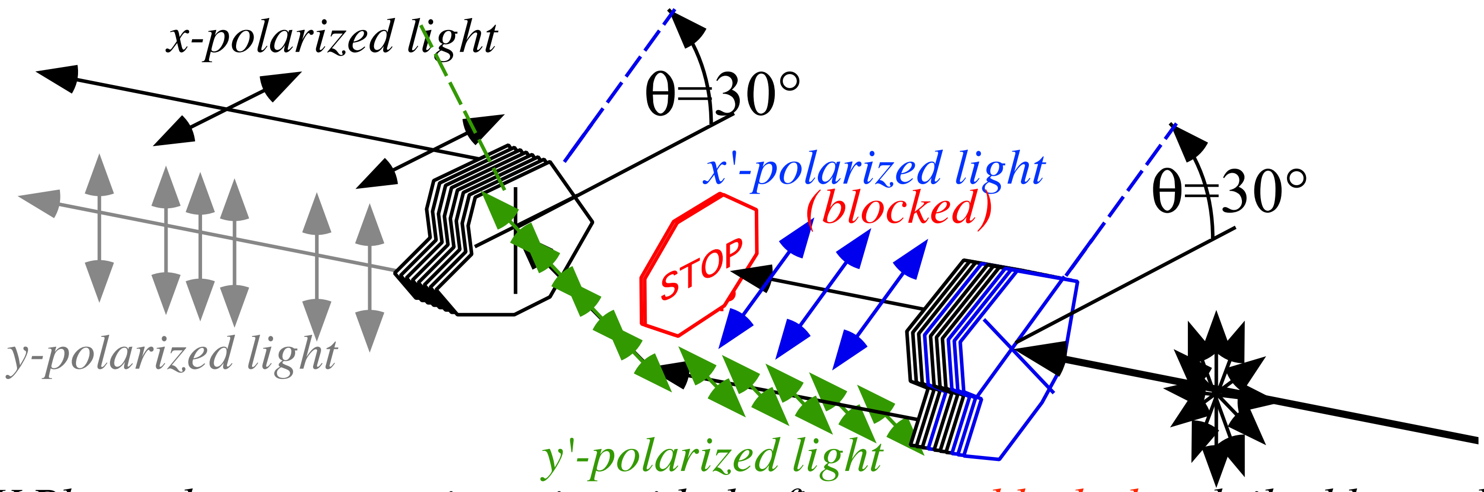

Fig. 1.2.X Photon beam sorters in series with the first one x-blocked and tilted by angle θ=30°.

Introducing Dirac bra-ket notation.

If we x′-blocked and let y′ through instead:

If we y′-blocked and let x′ through:

X output amplitude due to x' input $= \langle x | x' \rangle = \cos\theta$,

Y output amplitude due to x' input $= \langle y | x' \rangle = \sin\theta$

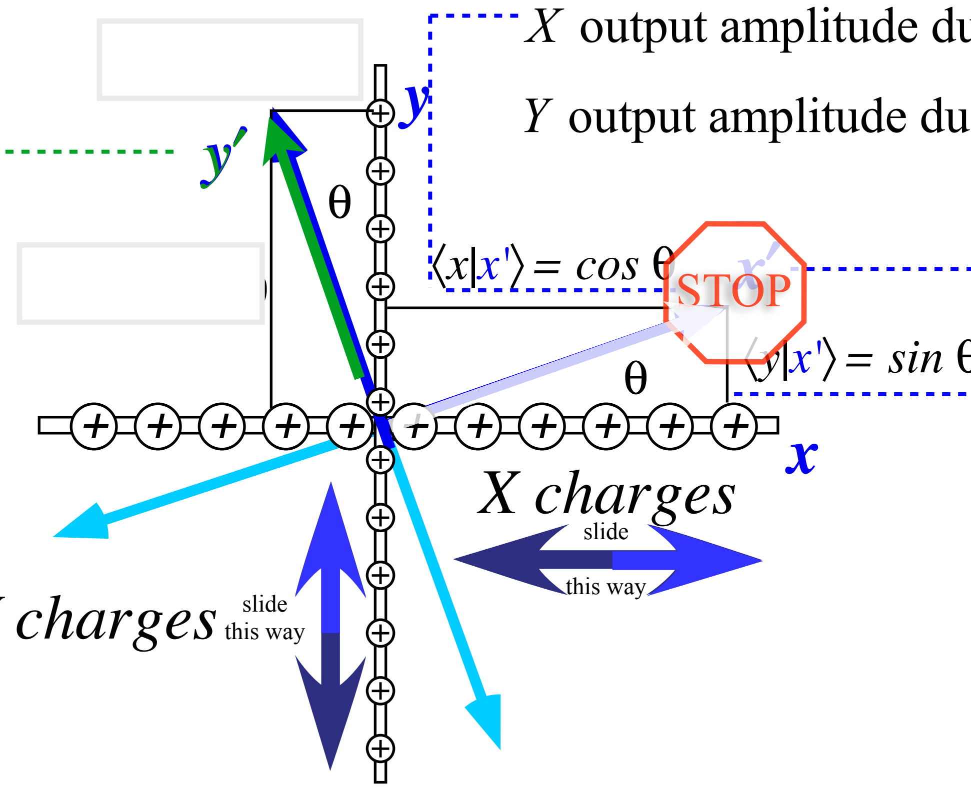

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Fig. 1.2.2 Geometry of photon beam sorter for input polarizations (x′,y′) tilted by angle θ [relative to (x,y)].

Beam Sorters in Series and Transformation Matrices

Feynman-Dirac Interpretation of

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=Amplitude of state-m after state-n′ is forced to choose from available m-type states

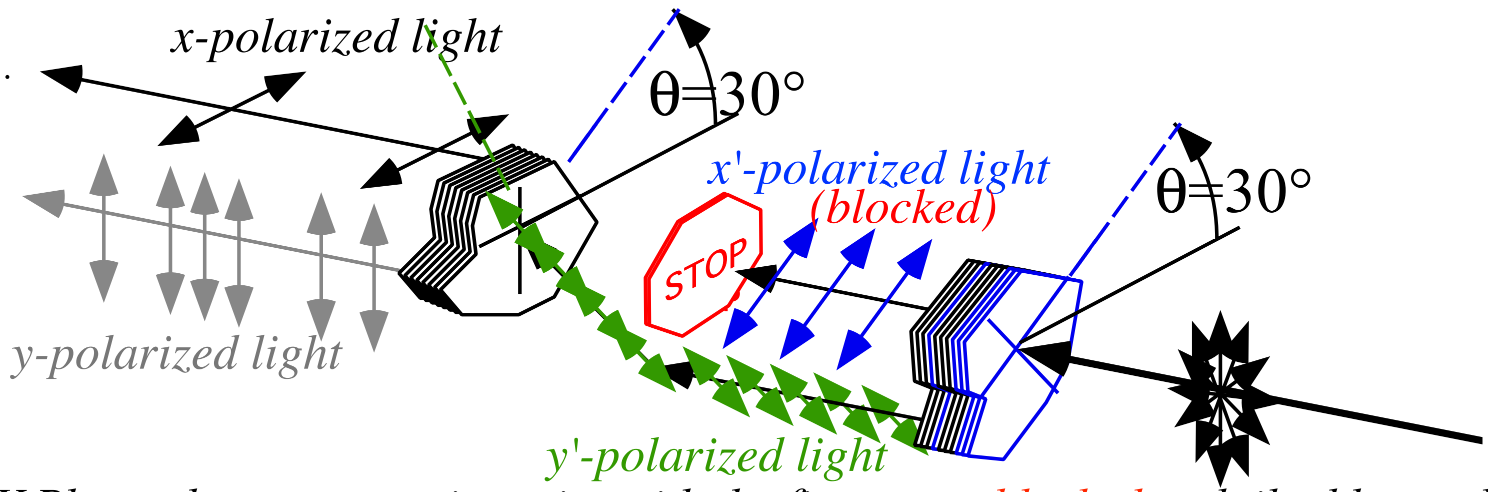

Fig. 1.2.X Photon beam sorters in series with the first one x-blocked and tilted by angle θ=30°.

Introducing Dirac bra-ket notation.

If we x′-blocked and let y′ through instead:

X output amplitude due to y' input $= \langle x | y' \rangle = -\sin\theta$

If we y′-blocked and let x′ through:

X output amplitude due to x' input $= \langle x | x' \rangle = \cos\theta,$

Y output amplitude due to x' input $= \langle y | x' \rangle = \sin\theta$

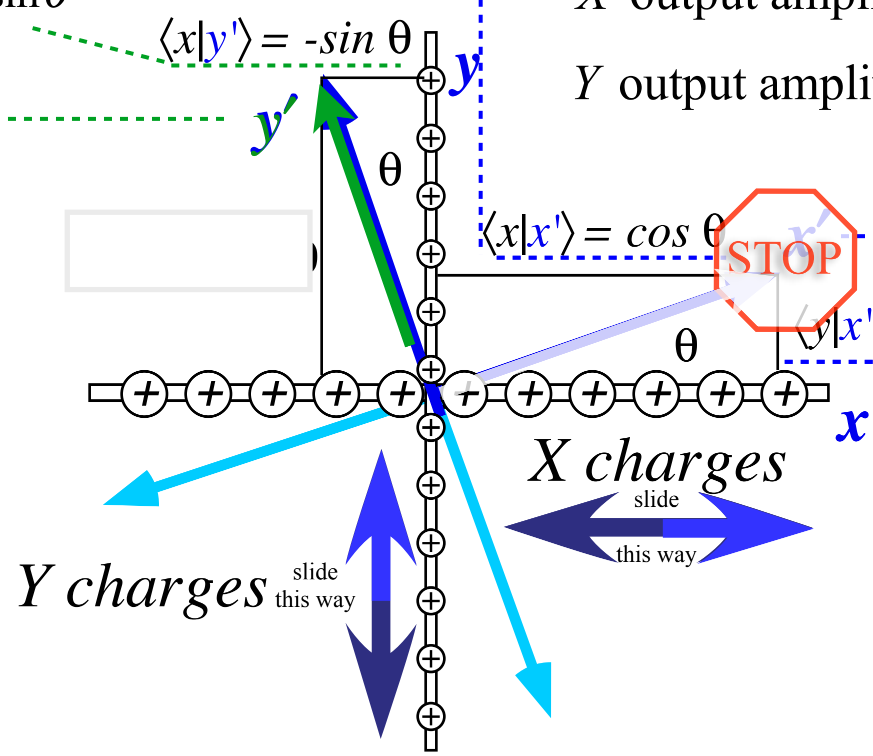

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Fig. 1.2.2 Geometry of photon beam sorter for input polarizations (x′,y′) tilted by angle θ [relative to (x,y)].

Beam Sorters in Series and Transformation Matrices

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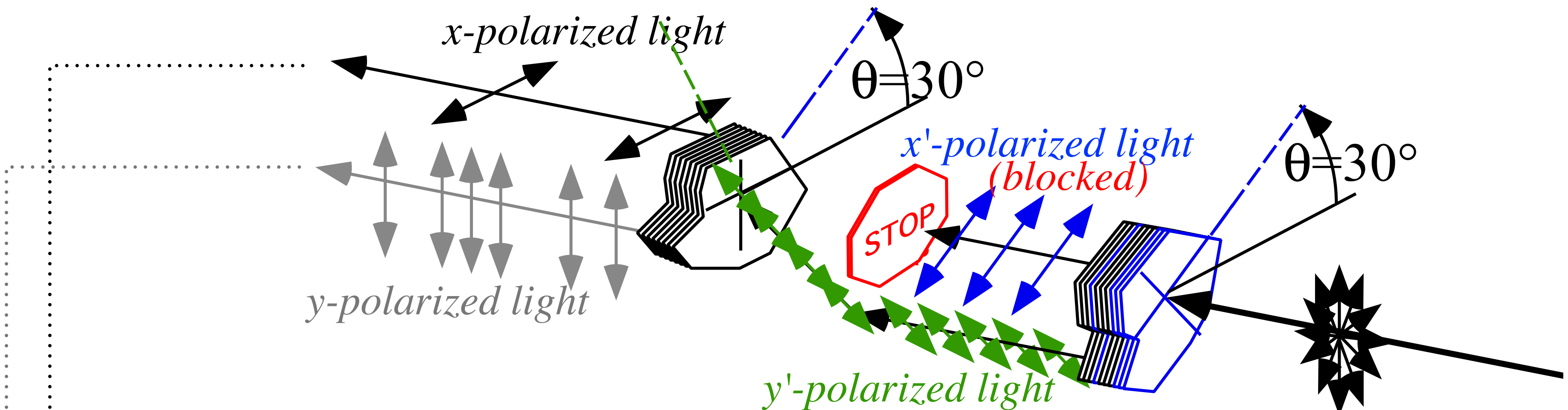

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If we x′-blocked and let y′ through instead:

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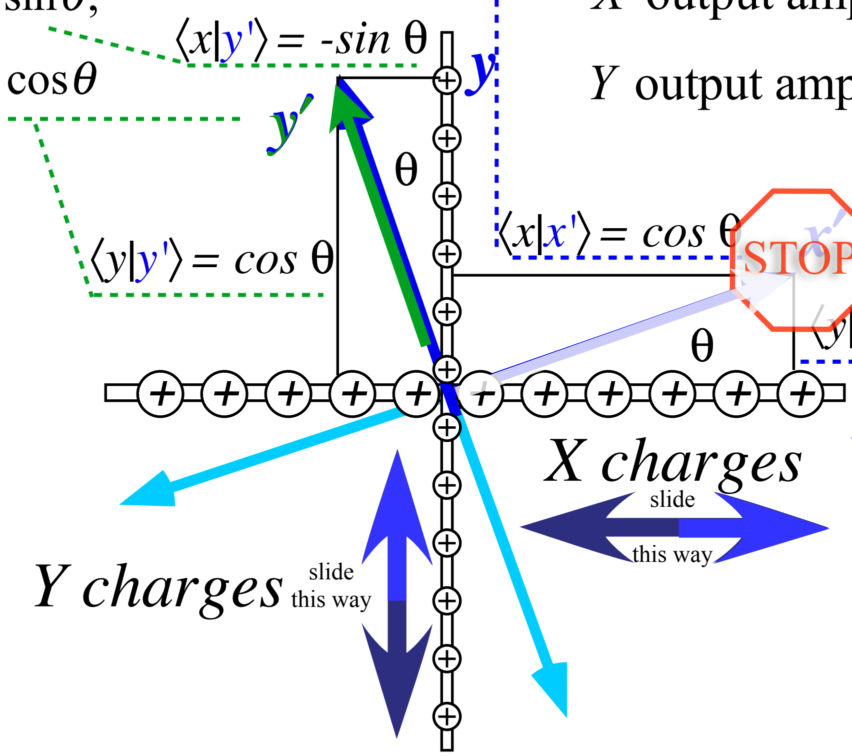

$sin\theta = sin 30° = 1/2 = 0.5$

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Fig. 1.2.2 Geometry of photon beam sorter for input polarizations (x′,y′) tilted by angle θ [relative to (x,y)].

Beam Sorters in Series and Transformation Matrices

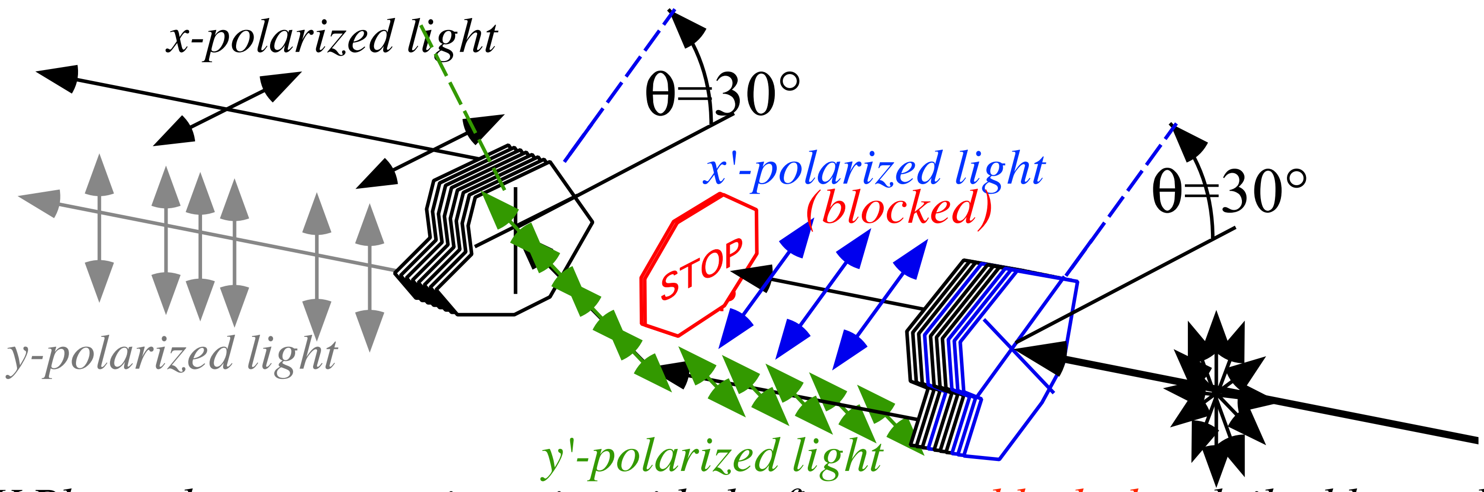

Fig. 1.2.X Photon beam sorters in series with the first one x-blocked and tilted by angle θ=30°.

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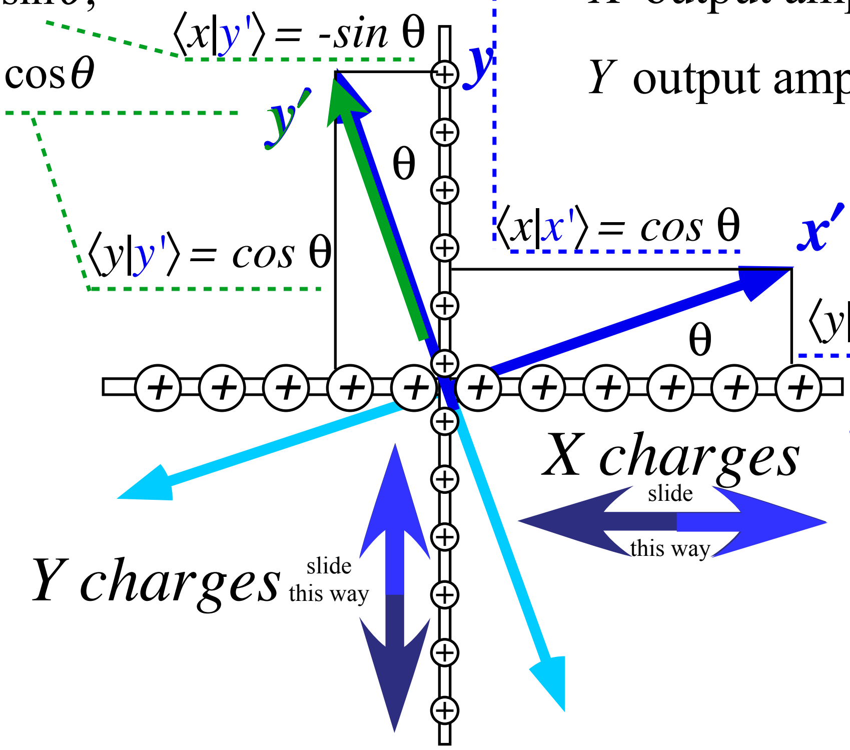

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$$\begin{pmatrix} \langle x | x' \rangle & \langle x | y' \rangle \\ \langle y | x' \rangle & \langle y | y' \rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Introducing bra-ket Transformation Matrix

$T_{m,n'} = \langle m | n' \rangle$

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➡ *"Abstraction" of bra and ket vectors from a Transformation Matrix*

Introducing scalar and matrix products

“Abstraction” of bra and ket vectors from a Transformation Matrix

Ket or column vectors

Bra or row vectors

Given Transformation Matrix $T_{m,n'}$:
$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

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Abstracting ket $|\, n'\rangle$ state vectors
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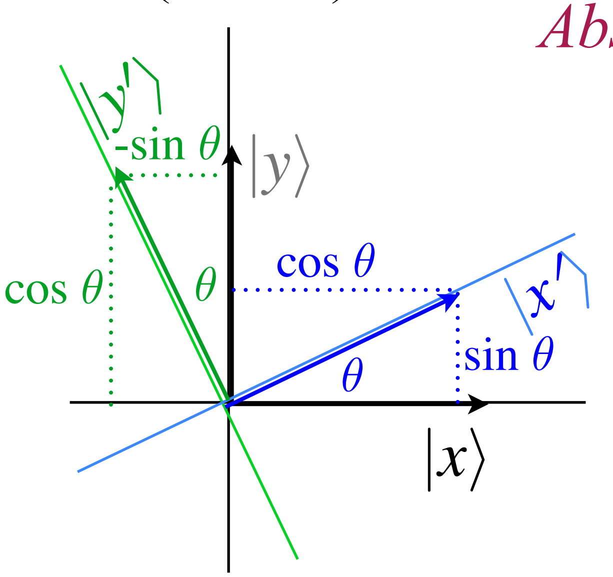

Abstracting ket $|\ n'\rangle$ *state vectors*
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($\theta=+30°$)-Rotated kets $\{|\mathbf{x}'\rangle, |\mathbf{y}'\rangle\}$ or $\{\mathbf{x}', \mathbf{y}'\}$
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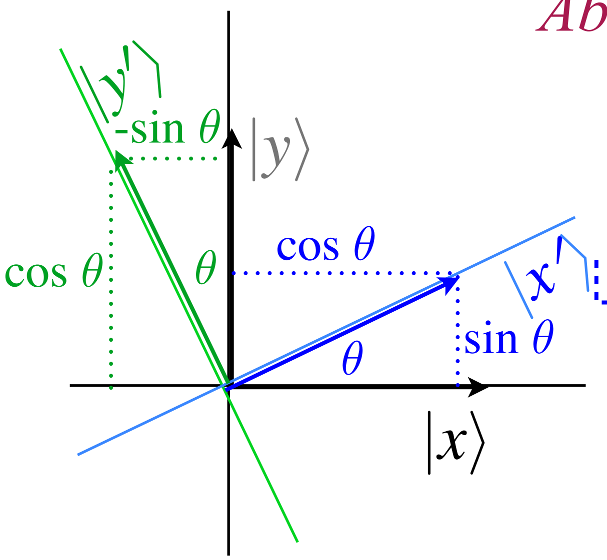

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$$T_{m,n'} = \langle m|n'\rangle$$

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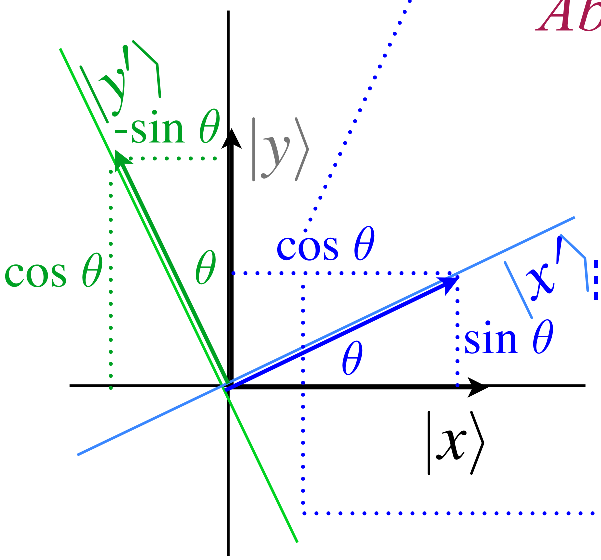

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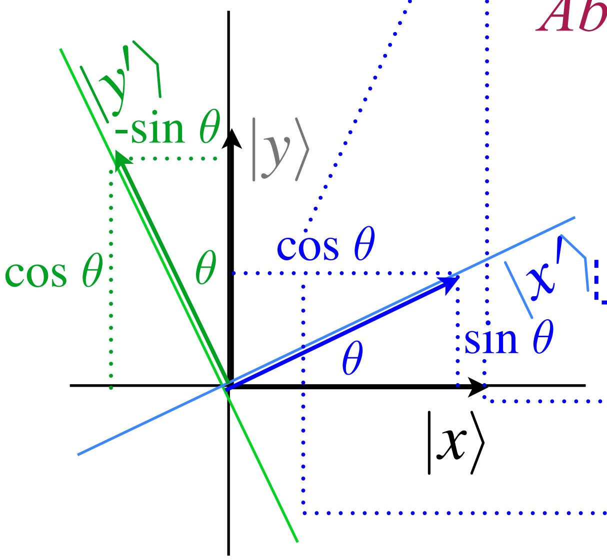

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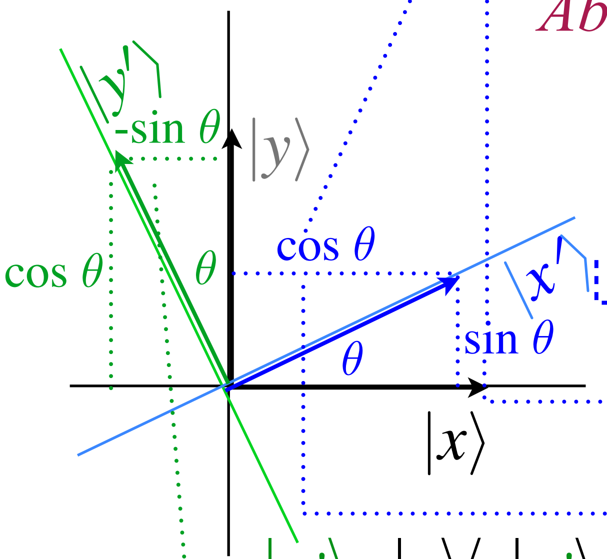
Abstracting ket $|n'\rangle$ state vectors from Transformation Matrix

$$T_{m,n'} = \langle m|n'\rangle$$



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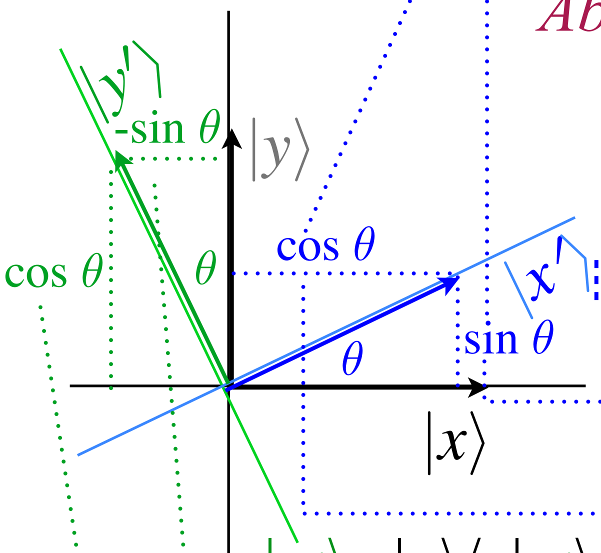

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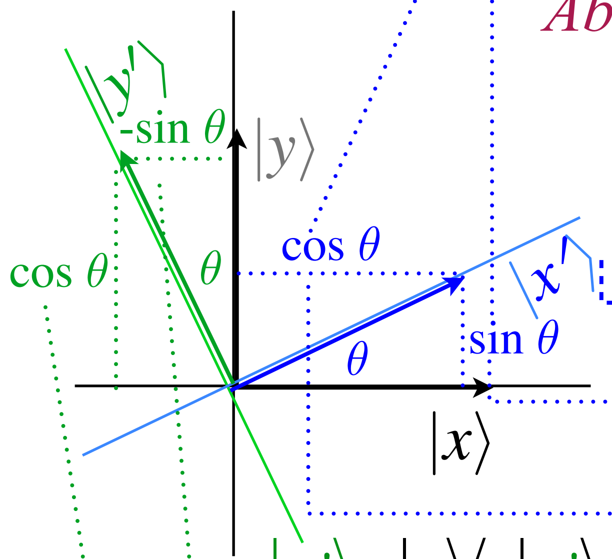

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Beam Sorters

2-State Sorters: spin-1/2 vs. optical polarization

Geometry of optical polarization selection and Brewster's angle

Feynman's lever

Beam Sorters in Series and Transformation Matrices

Introducing Dirac bra-ket notation

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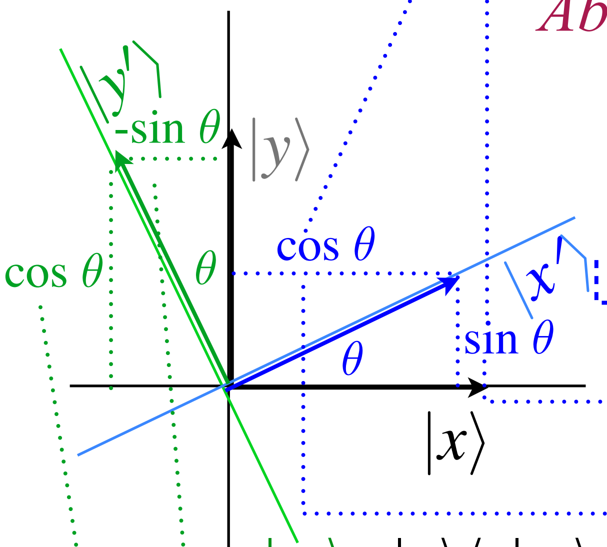

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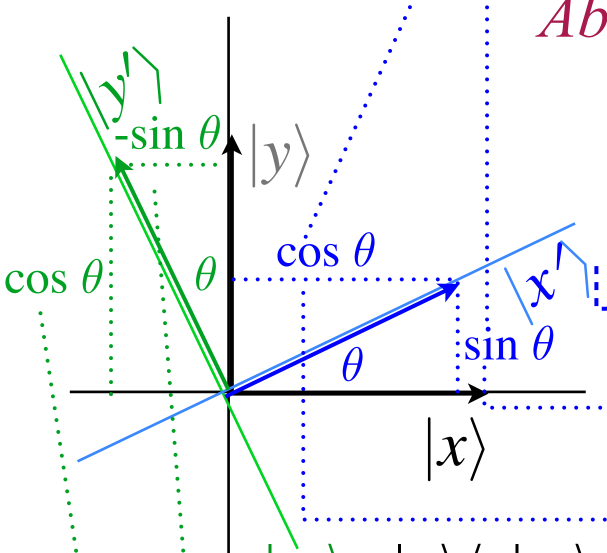

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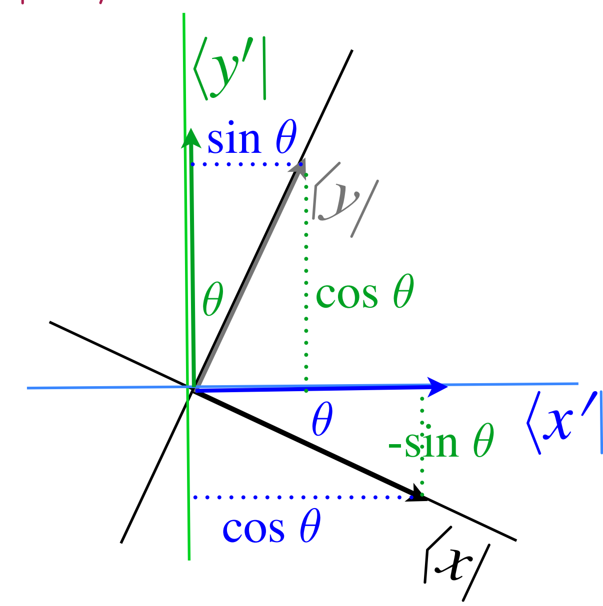

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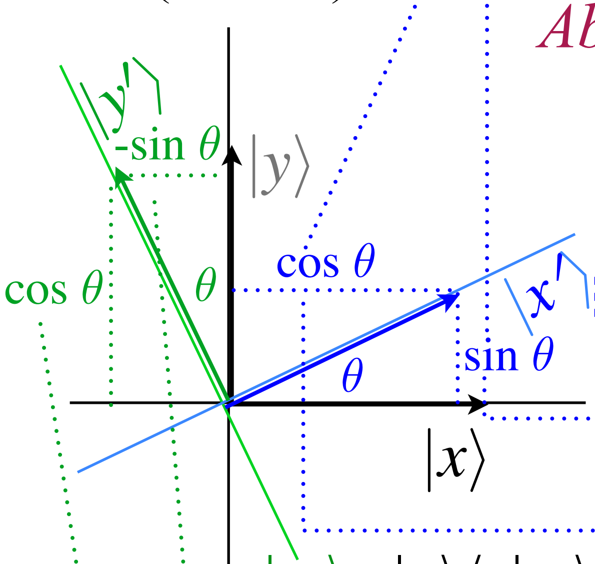

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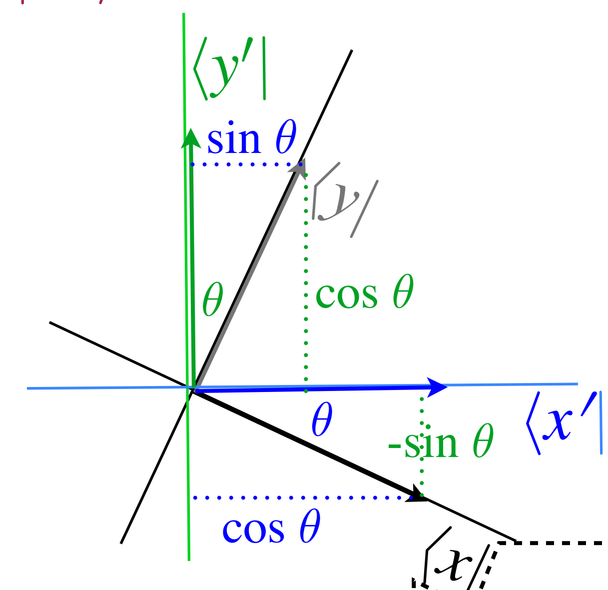

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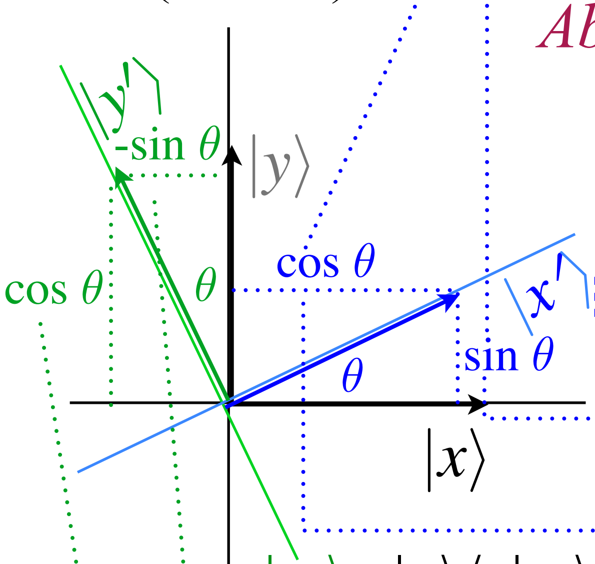

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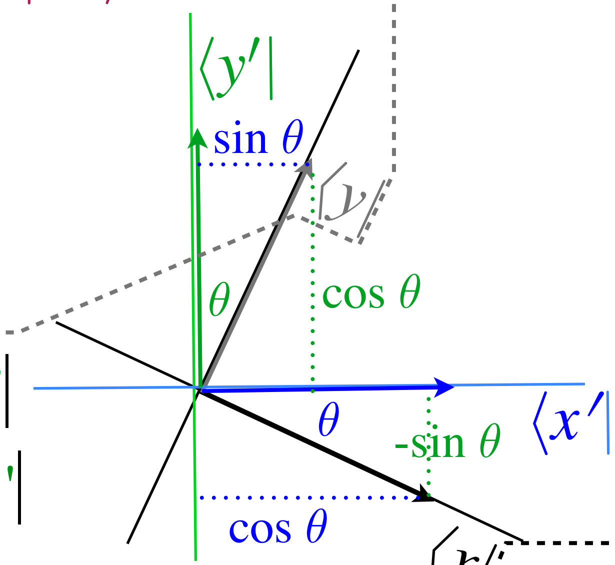

$$\langle y| = \langle y|x'\rangle\langle x'| + \langle y|y'\rangle\langle y'| = (\sin\theta)\langle x'| + (\cos\theta)\langle y'|$$

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($\theta=-30°$)-Rotated bras $\{\langle\mathbf{x}|, \langle\mathbf{y}|\}$ or $\{\mathbf{x}, \mathbf{y}\}$ represented in page-aligned $\{|\mathbf{x}'\rangle, |\mathbf{y}'\rangle\}$ basis.

The same thing in Gibbs vector notation:

$$\mathbf{x}' = \mathbf{x}(\mathbf{x}\bullet\mathbf{x}') + \mathbf{y}(\mathbf{y}\bullet\mathbf{x}'), \quad \mathbf{y}' = \mathbf{x}(\mathbf{x}\bullet\mathbf{y}') + \mathbf{y}(\mathbf{y}\bullet\mathbf{y}'),$$

$$= \mathbf{x}(\cos\theta) + \mathbf{y}(\sin\theta), \qquad = \mathbf{x}(-\sin\theta) + \mathbf{y}(\cos\theta).$$

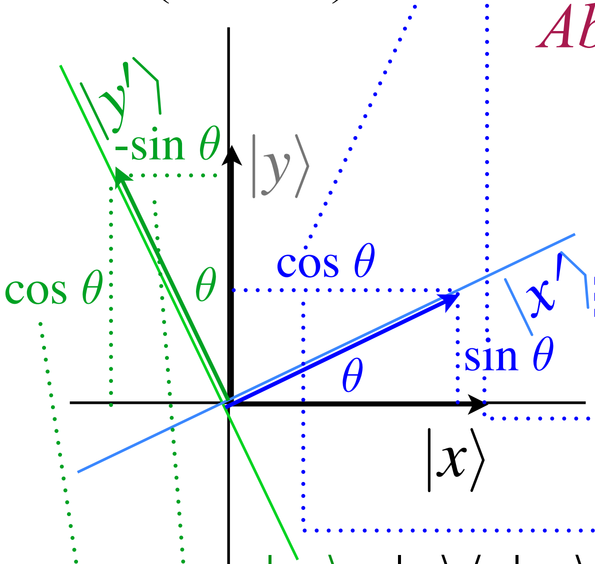
"Abstraction" of bra and ket vectors from a Transformation Matrix

Ket or column vectors

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$|x'\rangle = \begin{pmatrix} \langle x|x'\rangle \\ \langle y|x'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad |y'\rangle = \begin{pmatrix} \langle x|y'\rangle \\ \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

Abstracting ket $|n'\rangle$ state vectors from Transformation Matrix $T_{m,n'} = \langle m|n'\rangle$



$$|x'\rangle = |x\rangle\langle x|x'\rangle + |y\rangle\langle y|x'\rangle = |x\rangle(\cos\theta) + |y\rangle(\sin\theta)$$

$$|y'\rangle = |x\rangle\langle x|y'\rangle + |y\rangle\langle y|y'\rangle = |x\rangle(-\sin\theta) + |y\rangle(\cos\theta)$$

(θ=+30°)-Rotated kets $\{|\mathbf{x}'\rangle, |\mathbf{y}'\rangle\}$ or $\{\mathbf{x}', \mathbf{y}'\}$ represented in page-aligned $\{|\mathbf{x}\rangle, |\mathbf{y}\rangle\}$ basis.

The same thing in Gibbs vector notation:

$$\mathbf{x}' = \mathbf{x}(\mathbf{x}\bullet\mathbf{x}') + \mathbf{y}(\mathbf{y}\bullet\mathbf{x}'), \quad \mathbf{y}' = \mathbf{x}(\mathbf{x}\bullet\mathbf{y}') + \mathbf{y}(\mathbf{y}\bullet\mathbf{y}'),$$

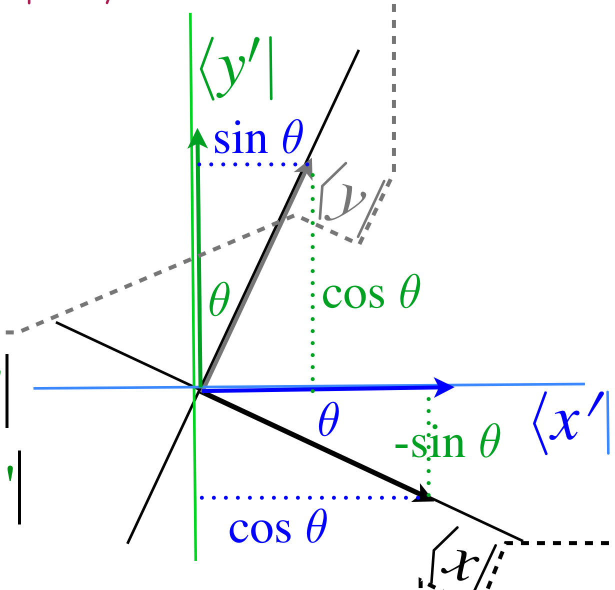
$$= \mathbf{x}(\cos\theta) + \mathbf{y}(\sin\theta), \qquad = \mathbf{x}(-\sin\theta) + \mathbf{y}(\cos\theta).$$

Bra or row vectors

$$\Rightarrow \langle x| = \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \end{pmatrix}$$

$$\Rightarrow \langle y| = \begin{pmatrix} \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \sin\theta & \cos\theta \end{pmatrix}$$

Abstracting bra $\langle m|$ state vectors from Transformation Matrix $T_{m,n'} = \langle m|n'\rangle$



$$\langle y| = \langle y|x'\rangle\langle x'| + \langle y|y'\rangle\langle y'| = (\sin\theta)\langle x'| + (\cos\theta)\langle y'|$$

$$\langle x| = \langle x|x'\rangle\langle x'| + \langle x|y'\rangle\langle y'| = (\cos\theta)\langle x'| + (-\sin\theta)\langle y'|$$

(θ=-30°)-Rotated bras $\{\langle\mathbf{x}|, \langle\mathbf{y}|\}$ or $\{\mathbf{x}, \mathbf{y}\}$ represented in page-aligned $\{|\mathbf{x}'\rangle, |\mathbf{y}'\rangle\}$ basis.

The same thing in Gibbs vector notation:

$$\mathbf{x} = (\mathbf{x}\bullet\mathbf{x}')\mathbf{x}' + (\mathbf{x}\bullet\mathbf{y}')\mathbf{y}', \quad \mathbf{y} = (\mathbf{y}\bullet\mathbf{x}')\mathbf{x}' + (\mathbf{y}\bullet\mathbf{y}')\mathbf{y}',$$

$$\mathbf{x} = (\cos\theta)\mathbf{x}' + (-\sin\theta)\mathbf{y}', \quad \mathbf{y} = (\sin\theta)\mathbf{x}' + (\cos\theta)\mathbf{y}'.$$

"Abstraction" of bra and ket vectors from a Transformation Matrix

Ket or column vectors

Bra or row vectors

Given Transformation Matrix $T_{m,n'}$:

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

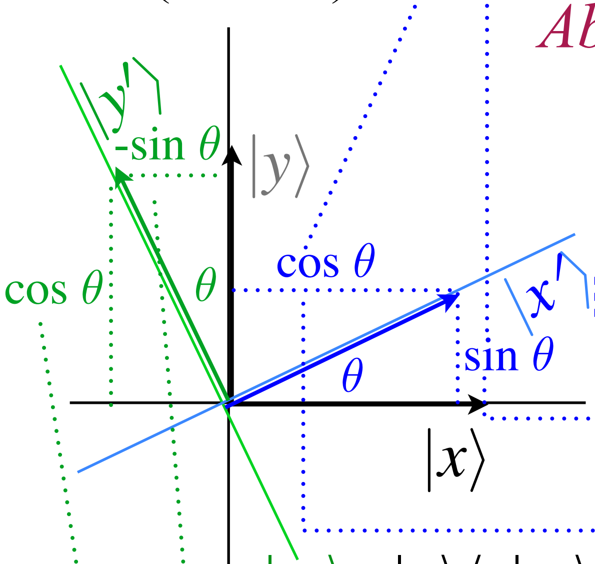
$$\Rightarrow \langle x| = \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \end{pmatrix}$$

$$\Rightarrow \langle y| = \begin{pmatrix} \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \sin\theta & \cos\theta \end{pmatrix}$$

$$|x'\rangle = \begin{pmatrix} \langle x|x'\rangle \\ \langle y|x'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad |y'\rangle = \begin{pmatrix} \langle x|y'\rangle \\ \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

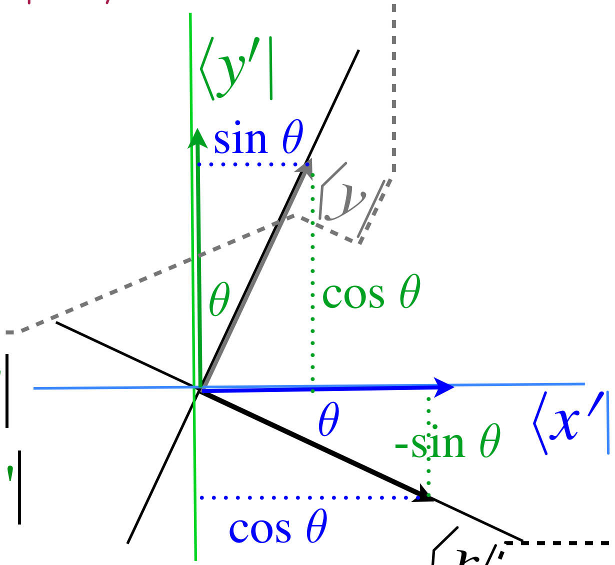
Abstracting ket $|n'\rangle$ state vectors from Transformation Matrix $T_{m,n'} = \langle m|n'\rangle$

Abstracting bra $\langle m|$ state vectors from Transformation Matrix $T_{m,n'} = \langle m|n'\rangle$



$$|x'\rangle = |x\rangle\langle x|x'\rangle + |y\rangle\langle y|x'\rangle = |x\rangle(\cos\theta) + |y\rangle(\sin\theta)$$

$$|y'\rangle = |x\rangle\langle x|y'\rangle + |y\rangle\langle y|y'\rangle = |x\rangle(-\sin\theta) + |y\rangle(\cos\theta)$$



$$\langle y| = \langle y|x'\rangle\langle x'| + \langle y|y'\rangle\langle y'| = (\sin\theta)\langle x'| + (\cos\theta)\langle y'|$$

$$\langle x| = \langle x|x'\rangle\langle x'| + \langle x|y'\rangle\langle y'| = (\cos\theta)\langle x'| + (-\sin\theta)\langle y'|$$

(θ=+30°)-Rotated kets $\{|\mathbf{x'}\rangle, |\mathbf{y'}\rangle\}$ or $\{\mathbf{x'}, \mathbf{y'}\}$ represented in page-aligned $\{|\mathbf{x}\rangle, |\mathbf{y}\rangle\}$ basis.

(θ=-30°)-Rotated bras $\{\langle\mathbf{x}|, \langle\mathbf{y}|\}$ or $\{\mathbf{x}, \mathbf{y}\}$ represented in page-aligned $\{|\mathbf{x'}\rangle, |\mathbf{y'}\rangle\}$ basis.

Ket vector algebra has the order of $T_{m,n'}$ transposed

$$|x'\rangle = |x\rangle\langle x|x'\rangle + |y\rangle\langle y|x'\rangle = |x\rangle(\cos\theta) + |y\rangle(\sin\theta)$$

$$|y'\rangle = |x\rangle\langle x|y'\rangle + |y\rangle\langle y|y'\rangle = |x\rangle(-\sin\theta) + |y\rangle(\cos\theta)$$

Bra vector algebra has the same order as $T_{m,n'}$

$$\langle x| = \langle x|x'\rangle\langle x'| + \langle x|y'\rangle\langle y'| = (\cos\theta)\langle x'| + (-\sin\theta)\langle y'|$$

$$\langle y| = \langle y|x'\rangle\langle x'| + \langle y|y'\rangle\langle y'| = (\sin\theta)\langle x'| + (\cos\theta)\langle y'|$$

Unit vector kets $|\mathbf{x}\rangle$ and $|\mathrm{y}\rangle$ or $\mathbf{x}'$ and $\mathbf{y}'$ are represented (in their own $|\mathrm{x}\rangle$and$|\mathrm{y}\rangle$ basis) as follows.

$$|x\rangle = \begin{pmatrix} \langle x|x\rangle \\ \langle y|x\rangle \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |y\rangle = \begin{pmatrix} \langle x|y\rangle \\ \langle y|y\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Beam Sorters

2-State Sorters: spin-1/2 vs. optical polarization

Geometry of optical polarization selection and Brewster's angle

Feynman's lever

Beam Sorters in Series and Transformation Matrices

Introducing Dirac bra-ket notation

"Abstraction" of bra and ket vectors from a Transformation Matrix

→ *Introducing scalar and matrix products*

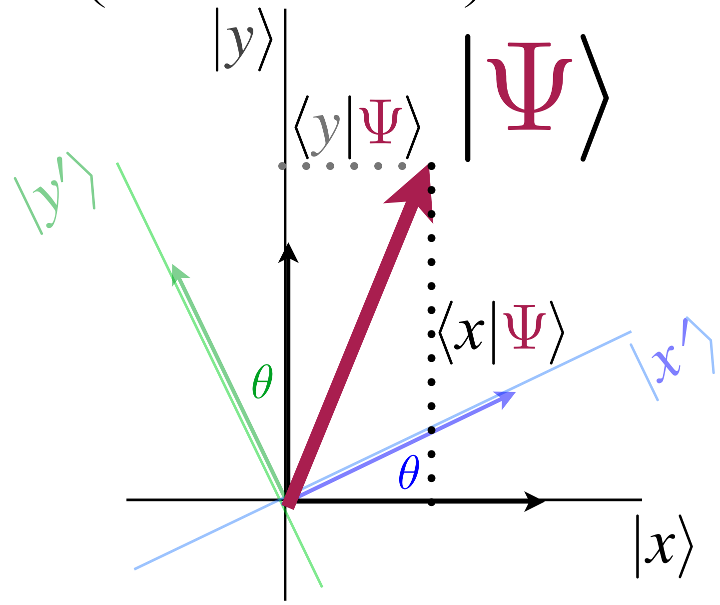
Transformation matrix $T_{m,n'} = \langle m | n' \rangle$ is array of dot or *scalar products* (dot products) of unit vectors or *direction cosines*.

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} (\mathbf{x}\bullet\mathbf{x'}) & (\mathbf{x}\bullet\mathbf{y'}) \\ (\mathbf{y}\bullet\mathbf{x'}) & (\mathbf{y}\bullet\mathbf{y'}) \end{pmatrix}$$

$\{\langle x|, \langle y|\}$ components of $|\Psi\rangle$:

$\langle x|\Psi\rangle = \Psi_x$

$\langle y|\Psi\rangle = \Psi_y$



Any state $|\Psi\rangle$ can be expanded in any basis $\{\langle x|, \langle y|\}$

$$|\Psi\rangle = |x\rangle\langle x|\Psi\rangle + |y\rangle\langle y|\Psi\rangle$$

Transformation matrix $T_{m,n'} = \langle m | n' \rangle$ is array of dot or *scalar products* (dot products) of unit vectors or *direction cosines*.

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} (\mathbf{x}\bullet\mathbf{x'}) & (\mathbf{x}\bullet\mathbf{y'}) \\ (\mathbf{y}\bullet\mathbf{x'}) & (\mathbf{y}\bullet\mathbf{y'}) \end{pmatrix}$$

$\{\langle x|, \langle y|\}$ components of $|\Psi\rangle$:

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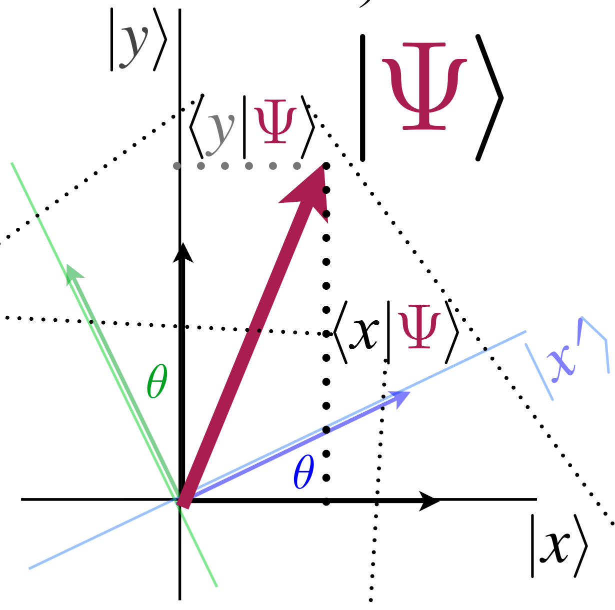
$\langle y|\Psi\rangle = \Psi_y$



Any state $|\Psi\rangle$ can be expanded in any basis $\{\langle x|, \langle y|\}$

$$|\Psi\rangle = |x\rangle\langle x|\Psi\rangle + |y\rangle\langle y|\Psi\rangle$$

Transformation matrix $T_{m,n'}=\langle m|n'\rangle$ is array of dot or *scalar products* (dot products) of unit vectors or *direction cosines*.

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} (\mathbf{x}\bullet\mathbf{x'}) & (\mathbf{x}\bullet\mathbf{y'}) \\ (\mathbf{y}\bullet\mathbf{x'}) & (\mathbf{y}\bullet\mathbf{y'}) \end{pmatrix}$$

$\{\langle x|,\langle y|\}$ components of $|\Psi\rangle$:

$\langle x|\Psi\rangle=\Psi_x$

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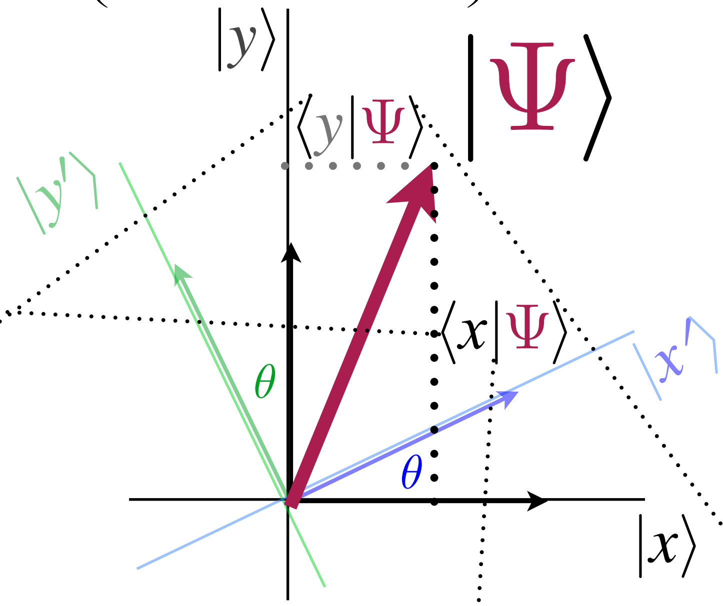
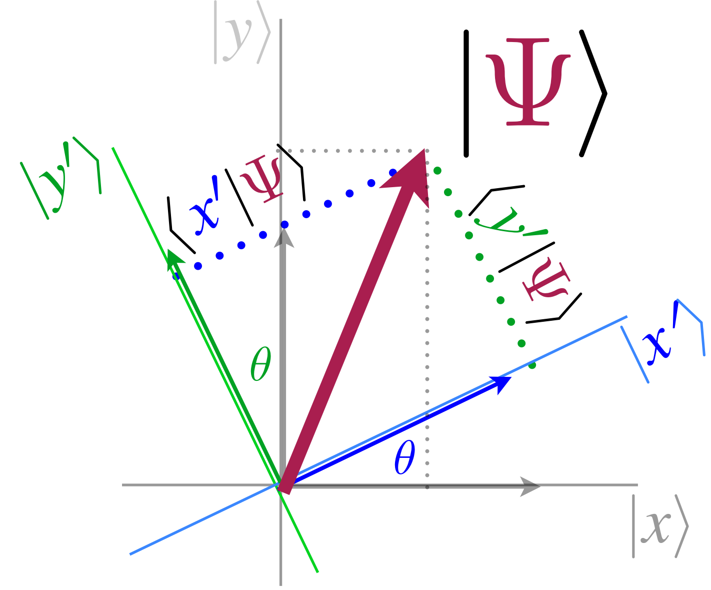




$\{\langle x'|,\langle y'|\}$ components of $|\Psi\rangle$:

$\langle x'|\Psi\rangle=\Psi_{x'}$

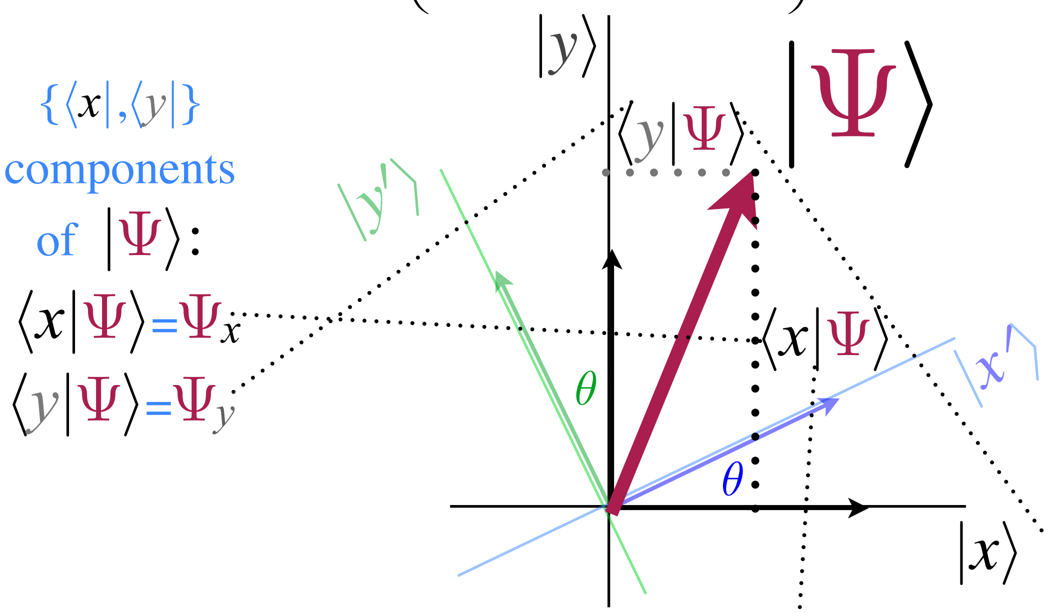
$\langle y'|\Psi\rangle=\Psi_{y'}$

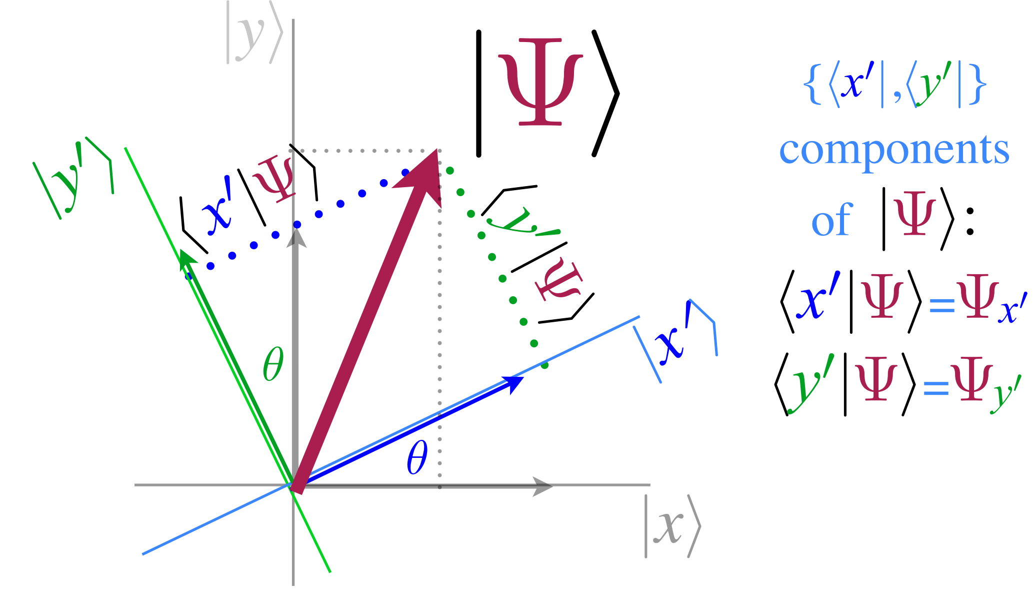
Any state $|\Psi\rangle$ can be expanded in any basis $\{\langle x|,\langle y|\}$, or $\{\langle x'|,\langle y'|\}$, ...*etc*.

$$|\Psi\rangle=|x\rangle\langle x|\Psi\rangle + |y\rangle\langle y|\Psi\rangle = |x'\rangle\langle x'|\Psi\rangle + |y'\rangle\langle y'|\Psi\rangle$$

Transformation matrix $T_{m,n'} = \langle m | n' \rangle$ is array of dot or *scalar products* (dot products) of unit vectors or *direction cosines*.

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} (\mathbf{x}\bullet\mathbf{x}') & (\mathbf{x}\bullet\mathbf{y}') \\ (\mathbf{y}\bullet\mathbf{x}') & (\mathbf{y}\bullet\mathbf{y}') \end{pmatrix}$$





Any state $|\Psi\rangle$ can be expanded in any basis $\{\langle x|, \langle y|\}$, or $\{\langle x'|, \langle y'|\}$, ...*etc*.

$$|\Psi\rangle = |x\rangle\langle x|\Psi\rangle + |y\rangle\langle y|\Psi\rangle = |x'\rangle\langle x'|\Psi\rangle + |y'\rangle\langle y'|\Psi\rangle$$

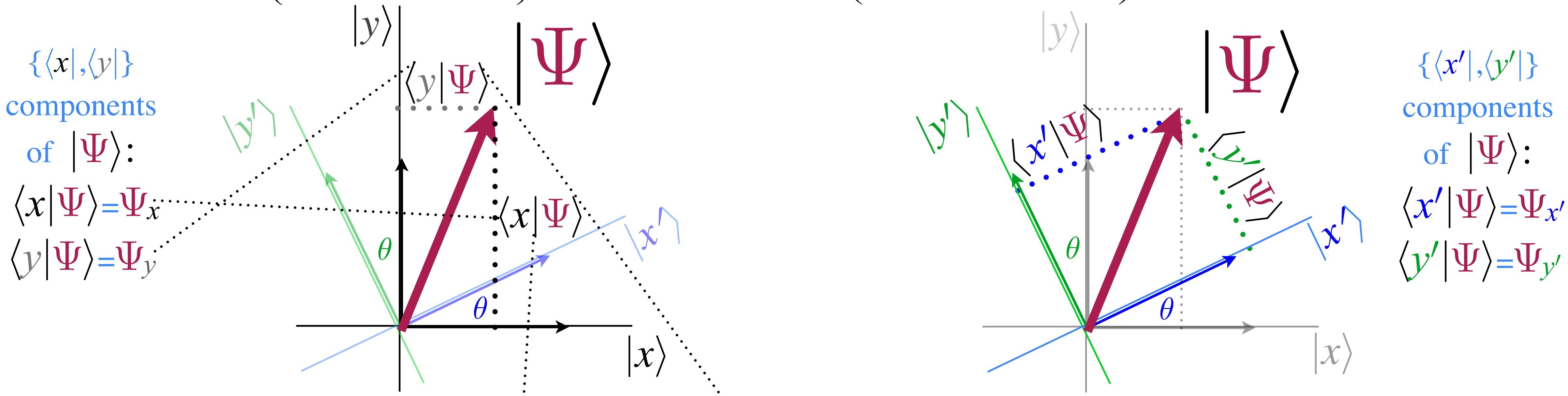
Transformation matrix $T_{m,n'}$ relates $\{\langle x|\Psi\rangle, \langle y|\Psi\rangle\}$ amplitudes to $\{\langle x'|\Psi\rangle, \langle y'|\Psi\rangle\}$.

$$\begin{pmatrix} \langle x|\Psi\rangle \\ \langle y|\Psi\rangle \end{pmatrix} = \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} \begin{pmatrix} \langle x'|\Psi\rangle \\ \langle y'|\Psi\rangle \end{pmatrix} \quad \text{or:} \quad \begin{pmatrix} \Psi_x \\ \Psi_y \end{pmatrix} = \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} \begin{pmatrix} \Psi_{x'} \\ \Psi_{y'} \end{pmatrix}$$

Hybrid
Gibbs-Dirac
notation
(Ug-ly!)

Transformation matrix $T_{m,n'} = \langle m | n' \rangle$ is array of dot or *scalar products* (dot products) of unit vectors or *direction cosines*.

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} (\mathbf{x}\bullet\mathbf{x'}) & (\mathbf{x}\bullet\mathbf{y'}) \\ (\mathbf{y}\bullet\mathbf{x'}) & (\mathbf{y}\bullet\mathbf{y'}) \end{pmatrix}$$



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$$|\Psi\rangle = |x\rangle\langle x|\Psi\rangle + |y\rangle\langle y|\Psi\rangle = |x'\rangle\langle x'|\Psi\rangle + |y'\rangle\langle y'|\Psi\rangle$$

Transformation matrix $T_{m,n'}$ relates $\{\langle x|\Psi\rangle, \langle y|\Psi\rangle\}$ amplitudes to $\{\langle x'|\Psi\rangle, \langle y'|\Psi\rangle\}$.

$$\begin{pmatrix} \langle x|\Psi\rangle \\ \langle y|\Psi\rangle \end{pmatrix} = \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} \begin{pmatrix} \langle x'|\Psi\rangle \\ \langle y'|\Psi\rangle \end{pmatrix} \quad \text{or:} \quad \begin{pmatrix} \Psi_x \\ \Psi_y \end{pmatrix} = \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} \begin{pmatrix} \Psi_{x'} \\ \Psi_{y'} \end{pmatrix}$$

Hybrid Gibbs-Dirac notation *(Ug-ly!)*

Proof: $\langle x| = \langle x|x'\rangle\langle x'| + \langle x|y'\rangle\langle y'|$ implies: $\langle x|\Psi\rangle = \langle x|x'\rangle\langle x'|\Psi\rangle + \langle x|y'\rangle\langle y'|\Psi\rangle$

Transformation matrix $T_{m,n'} = \langle m | n' \rangle$ is array of dot or *scalar products* (dot products) of unit vectors or *direction cosines*.

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} (\mathbf{x}\bullet\mathbf{x}') & (\mathbf{x}\bullet\mathbf{y}') \\ (\mathbf{y}\bullet\mathbf{x}') & (\mathbf{y}\bullet\mathbf{y}') \end{pmatrix}$$

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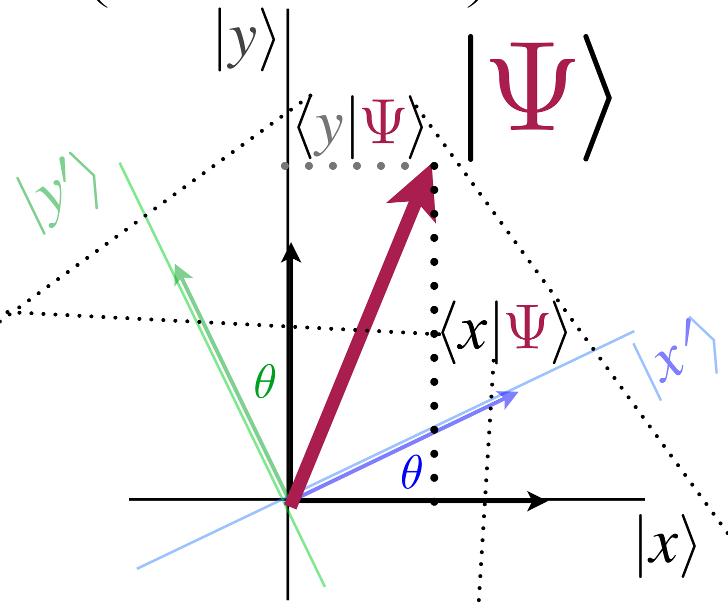

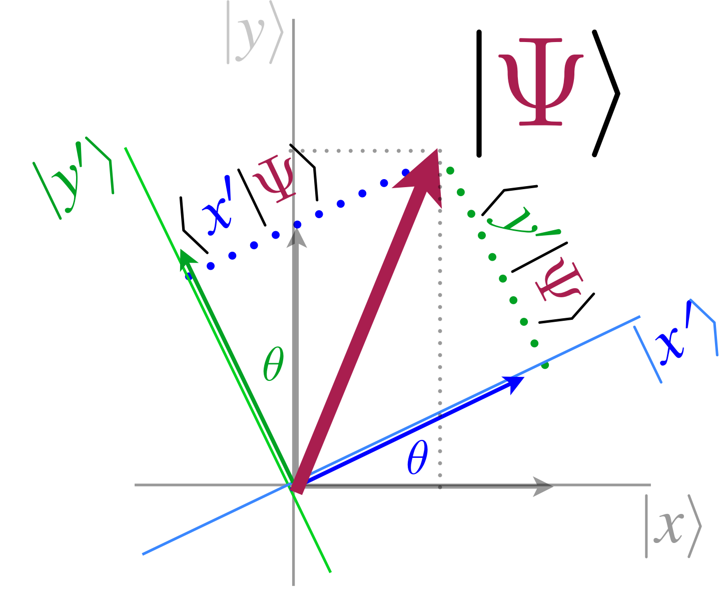

$\{\langle x'|,\langle y'|\}$ components of $|\Psi\rangle$:

$\langle x'|\Psi\rangle = \Psi_{x'}$

$\langle y'|\Psi\rangle = \Psi_{y'}$

Any state $|\Psi\rangle$ can be expanded in any basis $\{\langle x|,\langle y|\}$, or $\{\langle x'|,\langle y'|\}$, ...*etc*.

$$|\Psi\rangle = |x\rangle\langle x|\Psi\rangle + |y\rangle\langle y|\Psi\rangle = |x'\rangle\langle x'|\Psi\rangle + |y'\rangle\langle y'|\Psi\rangle$$

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Hybrid Gibbs-Dirac notation *(Ug-ly!)*

Proof: $\langle x| = \langle x|x'\rangle\langle x'| + \langle x|y'\rangle\langle y'|$ implies: $\langle x|\Psi\rangle = \langle x|x'\rangle\langle x'|\Psi\rangle + \langle x|y'\rangle\langle y'|\Psi\rangle$

$\langle y| = \langle y|x'\rangle\langle x'| + \langle y|y'\rangle\langle y'|$ implies: $\langle y|\Psi\rangle = \langle y|x'\rangle\langle x'|\Psi\rangle + \langle y|y'\rangle\langle y'|\Psi\rangle$

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$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} (\mathbf{x}\bullet\mathbf{x'}) & (\mathbf{x}\bullet\mathbf{y'}) \\ (\mathbf{y}\bullet\mathbf{x'}) & (\mathbf{y}\bullet\mathbf{y'}) \end{pmatrix}$$

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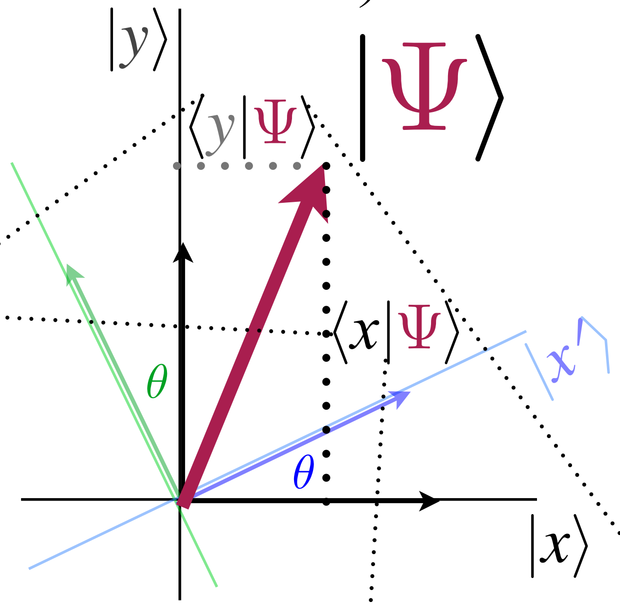

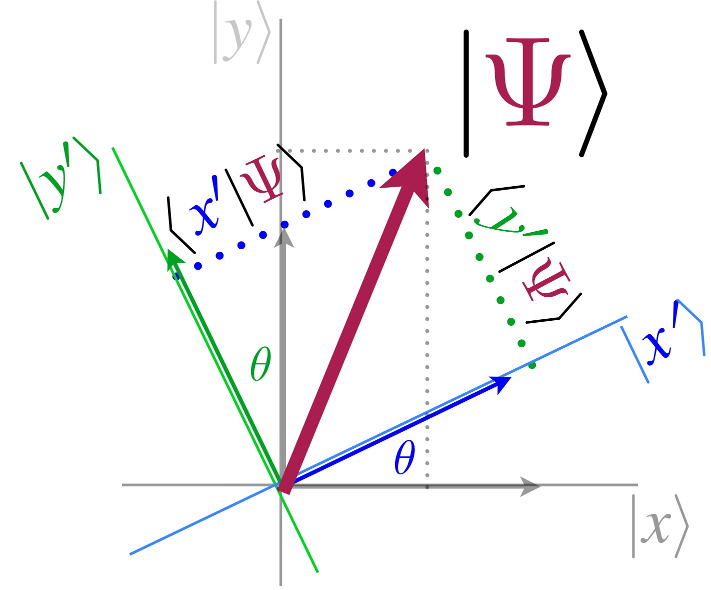

$\{\langle x'|,\langle y'|\}$ components of $|\Psi\rangle$:

$\langle x'|\Psi\rangle = \Psi_{x'}$

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Any state $|\Psi\rangle$ can be expanded in any basis $\{\langle x|,\langle y|\}$, or $\{\langle x'|,\langle y'|\}$, ...*etc*.

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$$\begin{pmatrix} \langle x|\Psi\rangle \\ \langle y|\Psi\rangle \end{pmatrix} = \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} \begin{pmatrix} \langle x'|\Psi\rangle \\ \langle y'|\Psi\rangle \end{pmatrix} \quad \text{or:} \quad \begin{pmatrix} \Psi_x \\ \Psi_y \end{pmatrix} = \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} \begin{pmatrix} \Psi_{x'} \\ \Psi_{y'} \end{pmatrix}$$

Hybrid Gibbs-Dirac notation *(Ug-ly!)*

Inverse (†=T*=-1) matrix $T_{n',m}$ relates $\{\langle x'|\Psi\rangle,\langle y'|\Psi\rangle\}$ amplitudes to $\{\langle x|\Psi\rangle,\langle y|\Psi\rangle\}$.

$$\begin{pmatrix} \langle x'|\Psi\rangle \\ \langle y'|\Psi\rangle \end{pmatrix} = \begin{pmatrix} \langle x'|x\rangle & \langle x'|y\rangle \\ \langle y'|x\rangle & \langle y'|y\rangle \end{pmatrix} \begin{pmatrix} \langle x|\Psi\rangle \\ \langle y|\Psi\rangle \end{pmatrix} \quad \text{or:} \quad \begin{pmatrix} \Psi_{x'} \\ \Psi_{y'} \end{pmatrix} = \begin{pmatrix} \langle x'|x\rangle & \langle x'|y\rangle \\ \langle y'|x\rangle & \langle y'|y\rangle \end{pmatrix} \begin{pmatrix} \Psi_x \\ \Psi_y \end{pmatrix}$$

Hybrid Gibbs-Dirac notation *(Still Ug-ly!)*