Group Theory in Quantum Mechanics

Lecture 19 (4.06.17)

Octahedral-tetrahedral $O \sim T_d$ symmetries

(Int.J.Mol.Sci, 14, 714(2013) p.755-774 , QTCA Unit 5 Ch. 15)

(PSDS - Ch. 4)

Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$
- Octahedral-cubic O symmetry and group operations
- Tetrahedral symmetry becomes Icosahedral
- Octahedral groups $O_h \supset O \sim T_d \supset T$
  - Octahedral O and spin-$O \subset U(2)$

Tetrahedral T class algebra
- Tetrahedral T class minimal equations
- Tetrahedral T class projectors and characters

Octahedral O class algebra
- Octahedral O class minimal equations
- Octahedral O class projectors and characters

Octahedral $O_h \supset O$:Inversion (g&u) parity
- Octahedral $O_h \supset O \supset C_I$ subgroup correlations
- $O_h \supset O \supset D_4$ subgroup correlations
- $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations
- Preview of applications to high resolution spectroscopy

Three groups: O, D_4, and D_3 let you "do"
Most of the other 32 crystal point groups

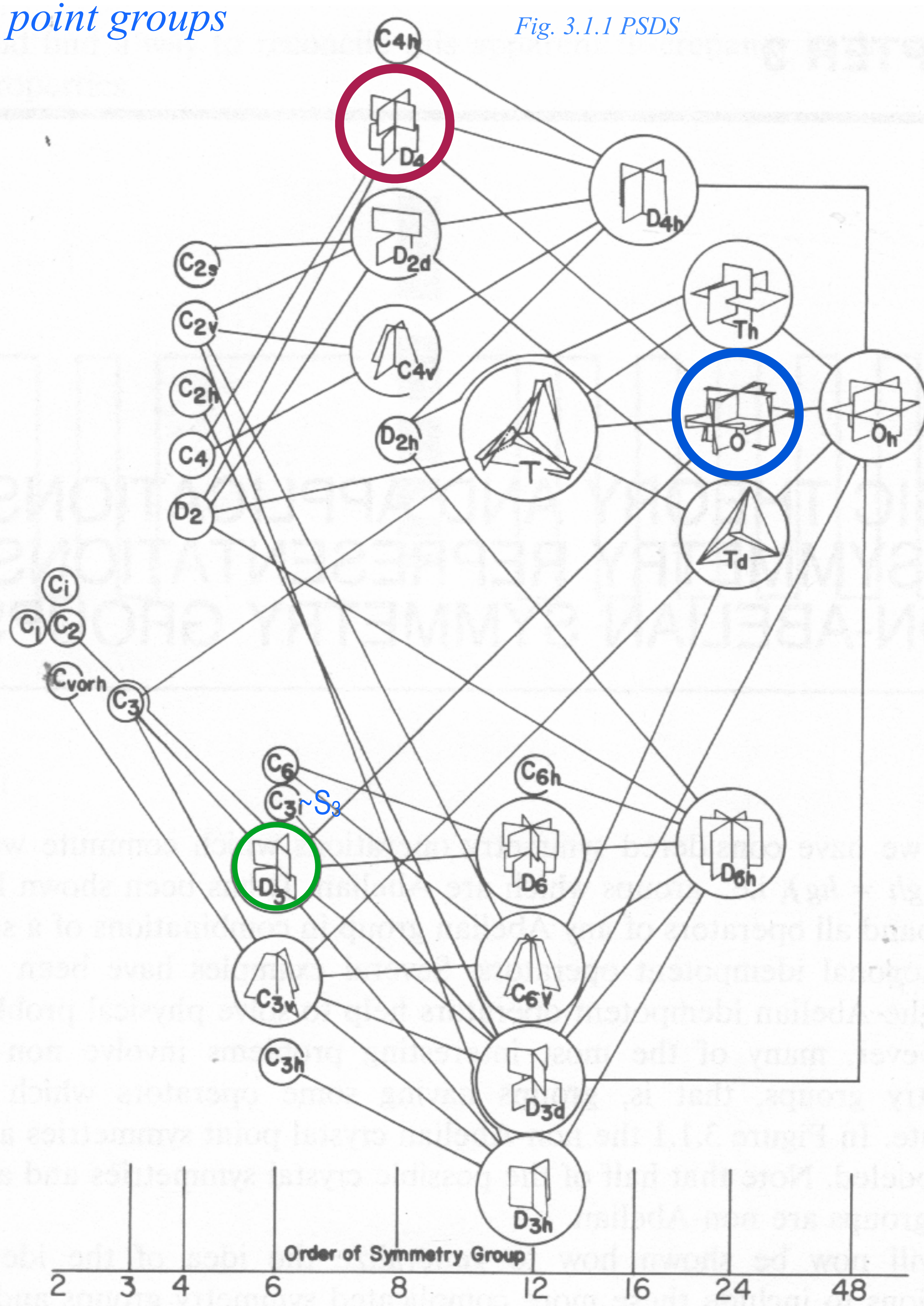
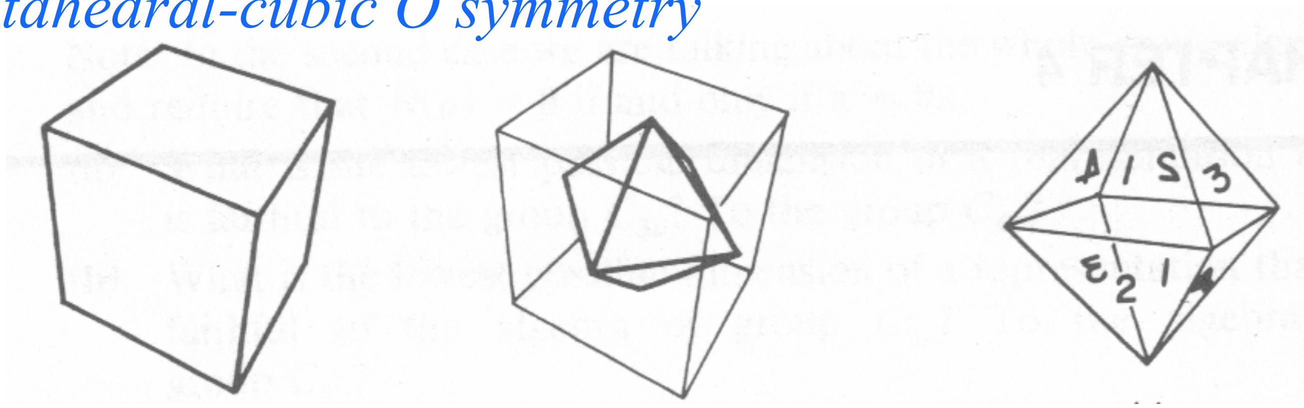


Fig. 3.1.1 PSDS

Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

Octahedral-cubic O symmetry



Order °O=6 hexahedron squares · 4 pts =24
=8 octahedron triangles · 3 pts =24
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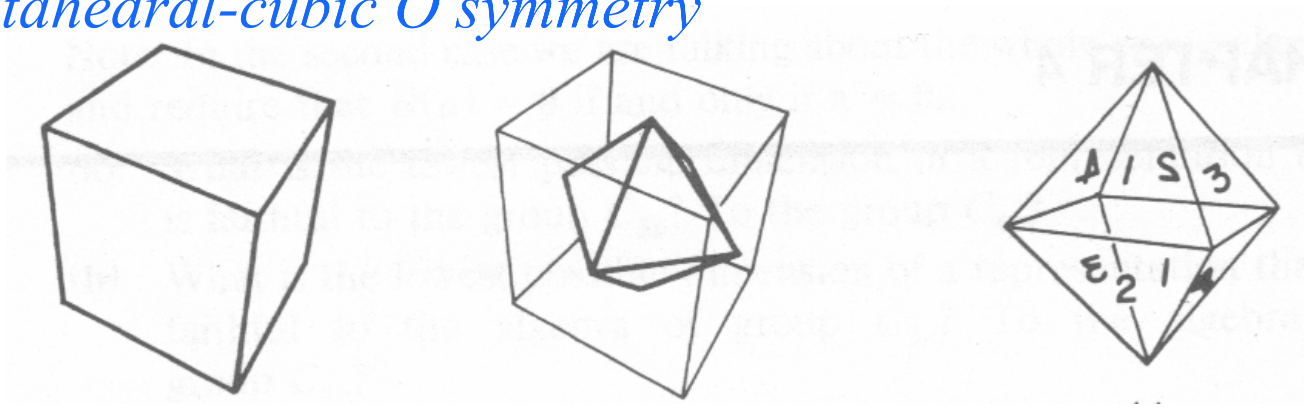

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Octahedral group O operations

Class of 1: **1**

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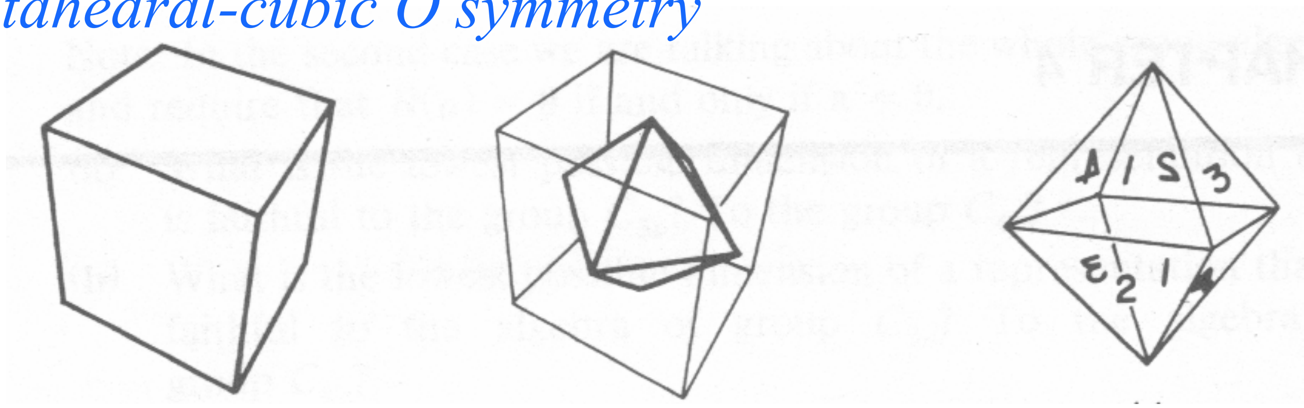
Octahedral-cubic O symmetry



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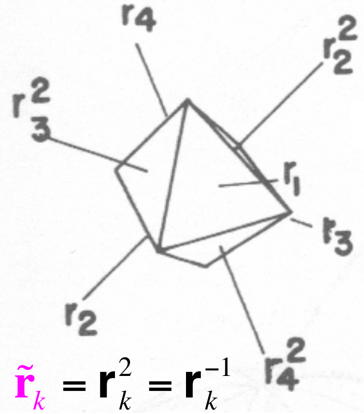
Octahedral group O operations

Class of 1: **1**
$\mathbf{r}_k = \mathbf{r}_k$

Class of 8:
120° rotations
on [111] axes



$\tilde{\mathbf{r}}_k = \mathbf{r}_k^2 = \mathbf{r}_k^{-1}$

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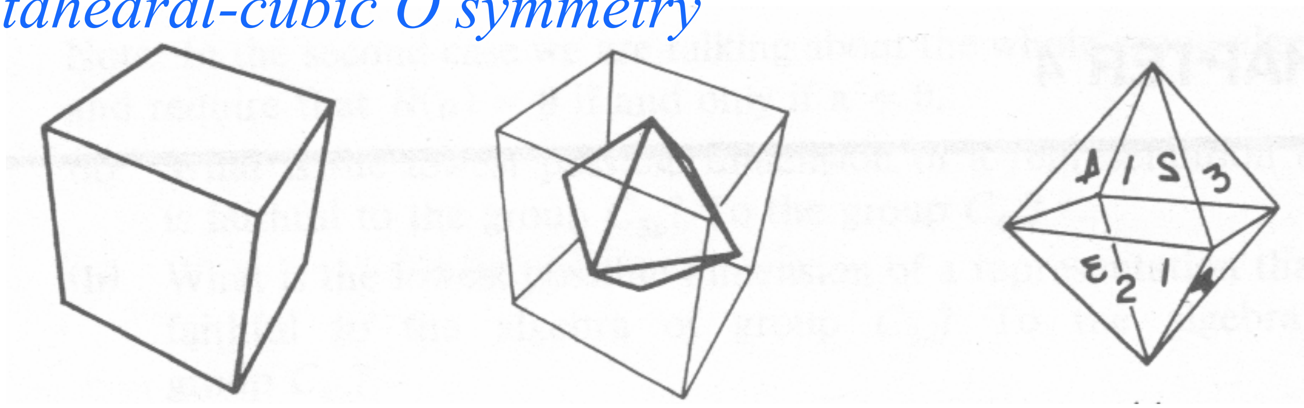

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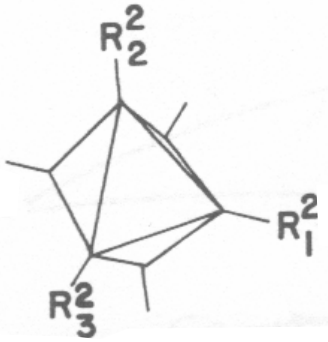

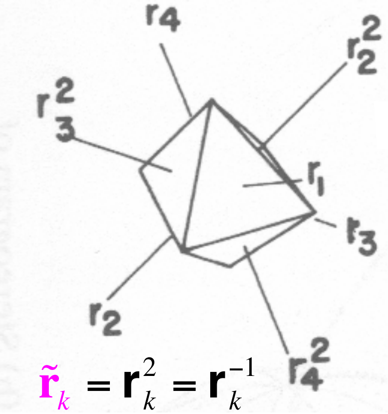

Class of 3:
180° rotations
on [100] axes

$\rho_{x,y,z} = \mathbf{R}^2_{1,2,3}$

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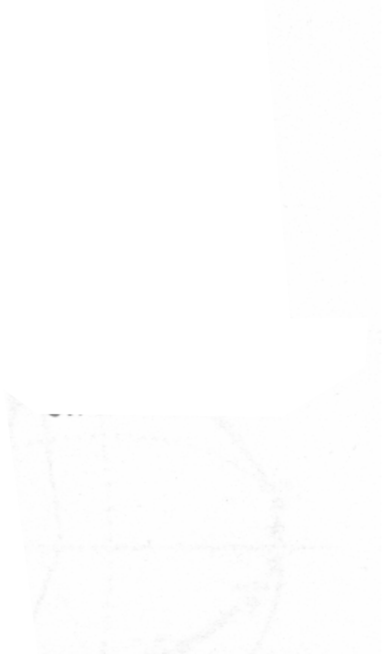

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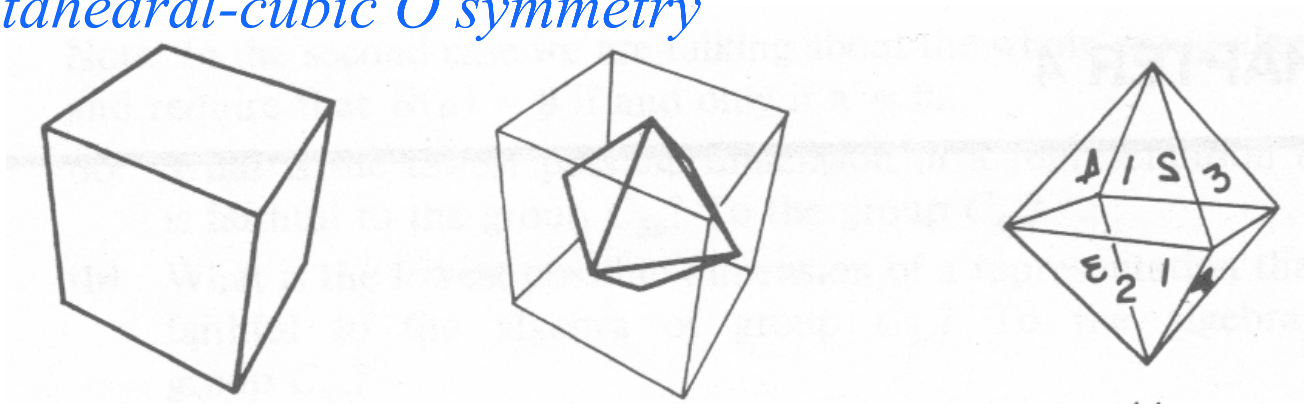
Octahedral-cubic O symmetry



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Octahedral group O operations

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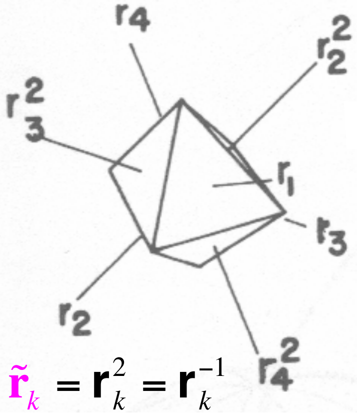

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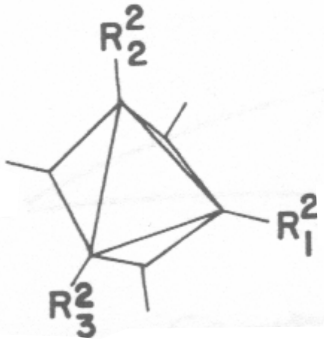

Class of 3:
180° rotations
on [100] *axes*

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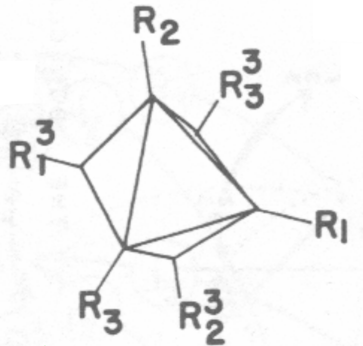
$\mathbf{R}_{x,y,z} = \mathbf{R}_{1,2,3}$

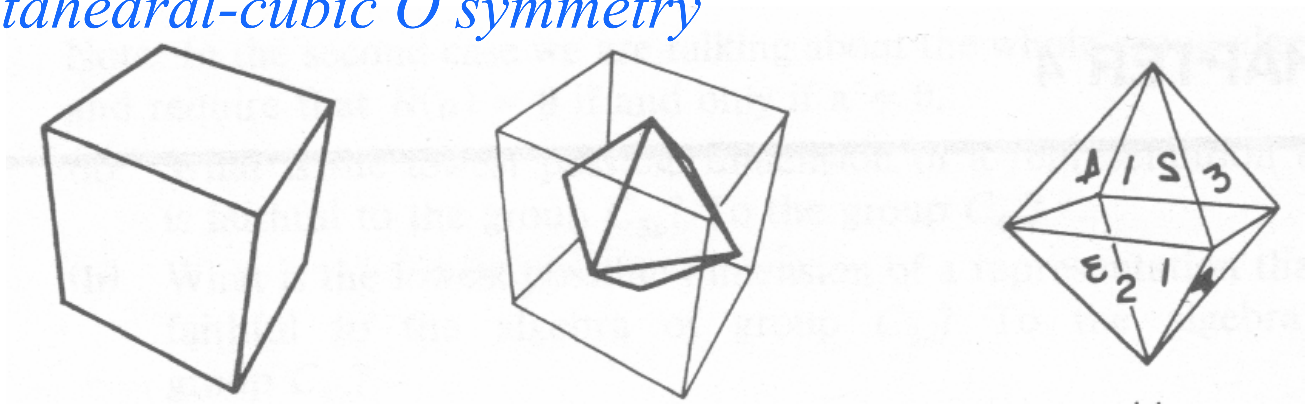
Class of 6:
±90° rotations
on[100] *axes*



$\tilde{\mathbf{R}}_{x,y,z} = \mathbf{R}^3_{1,2,3} = \mathbf{R}^{-1}_{1,2,3}$

Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

Octahedral-cubic O symmetry



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Octahedral group O operations

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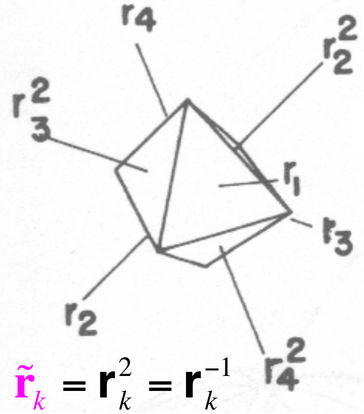

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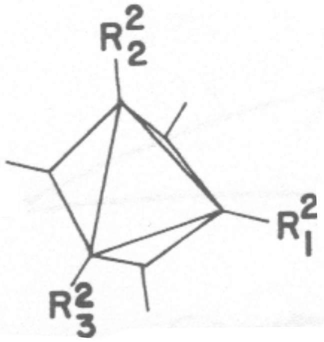

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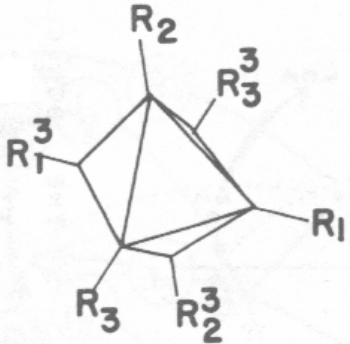

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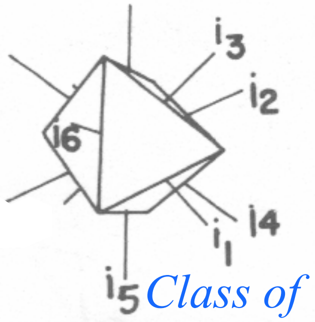

Class of 6:
180° rotations
on [110] diagonals

$\mathbf{i}_k = \mathbf{i}_k$

Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$
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- *Tetrahedral symmetry becomes Icosahedral*
- *Octahedral groups* $O_h \supset O \sim T_d \supset T$
  - *Octahedral O and spin-*$O \subset U(2)$

Tetrahedral T class algebra
- *Tetrahedral T class minimal equations*
- *Tetrahedral T class projectors and characters*

Octahedral O class algebra
- *Octahedral O class minimal equations*
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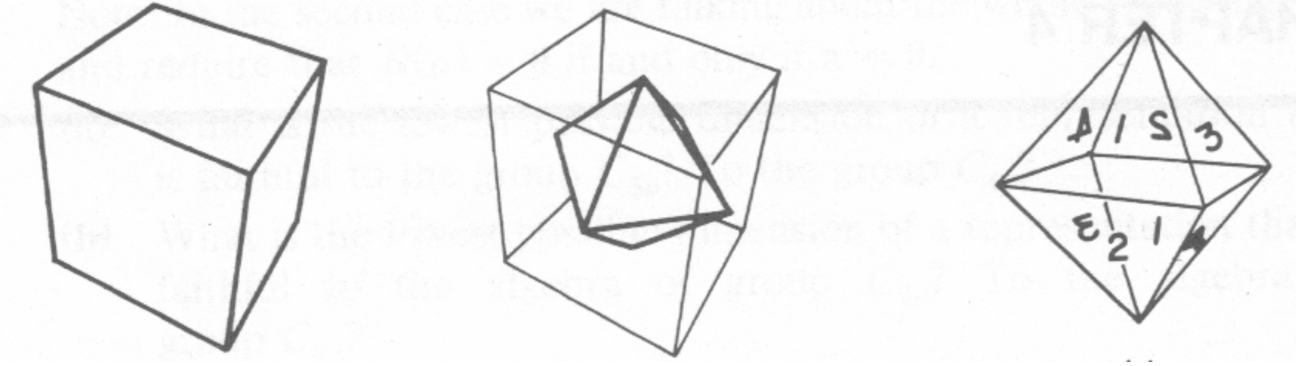
Octahedral $O_h \supset O$ *subgroup correlations*
- *Octahedral* $O_h \supset O$ *subgroup correlations*
- $O_h \supset O \supset D_4$ *subgroup correlations*
- $O_h \supset O \supset D_4 \supset C_4$ *subgroup correlations*
- *Preview of applications to high resolution spectroscopy*

Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

Octahedral-cubic O symmetry



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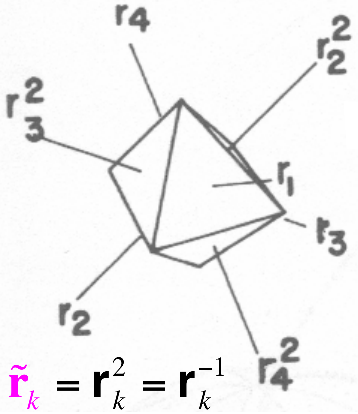

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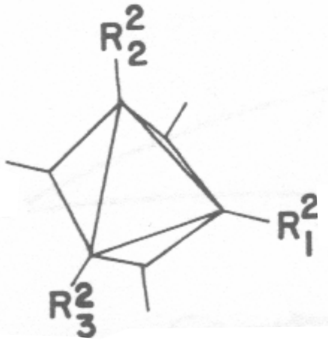

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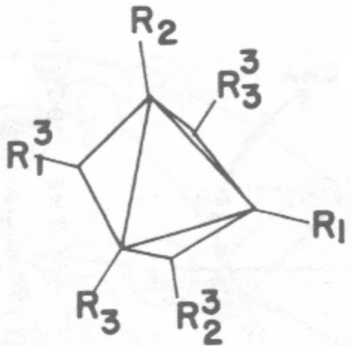

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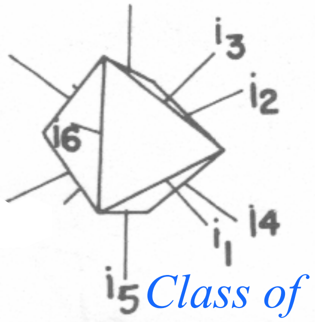

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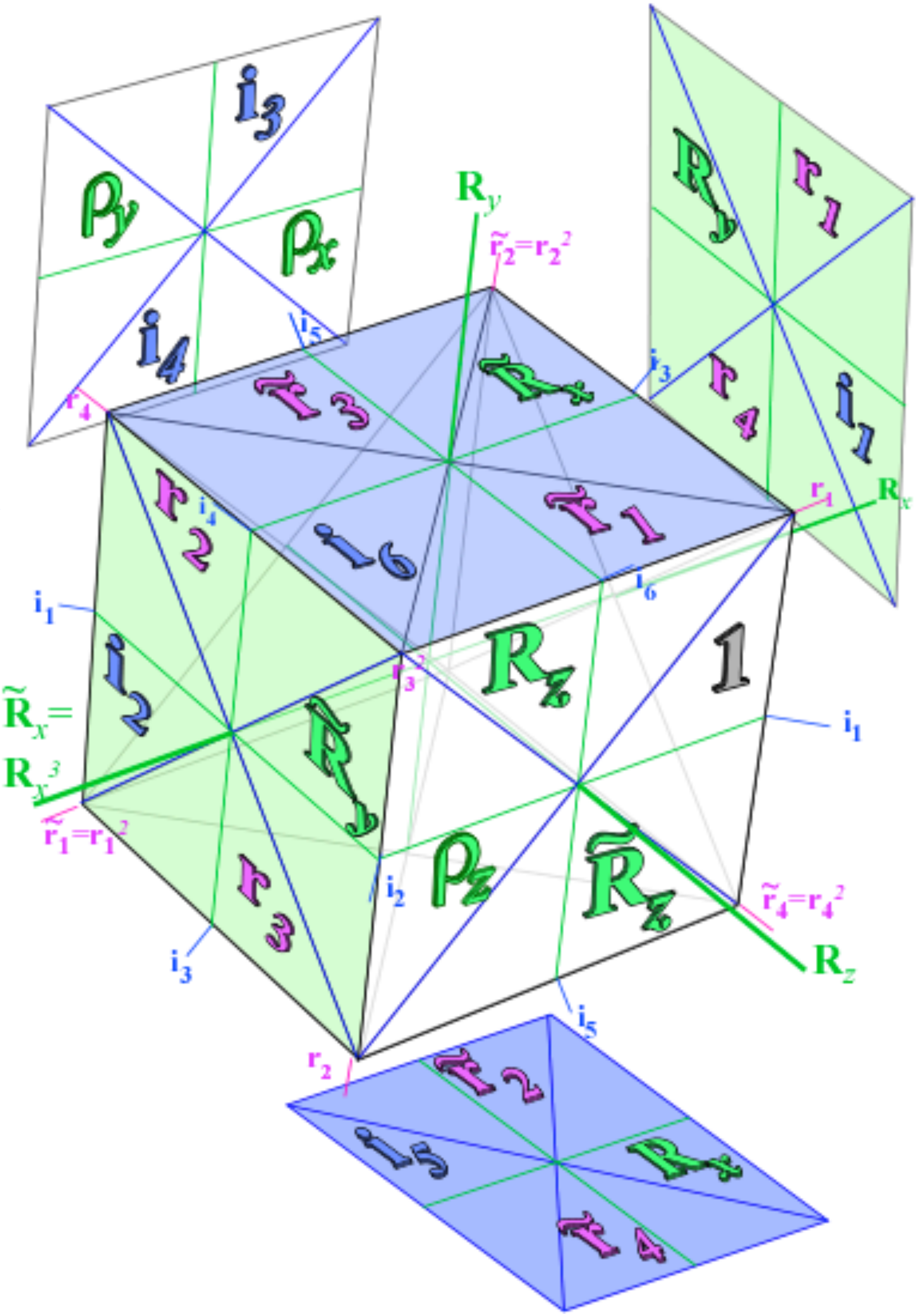

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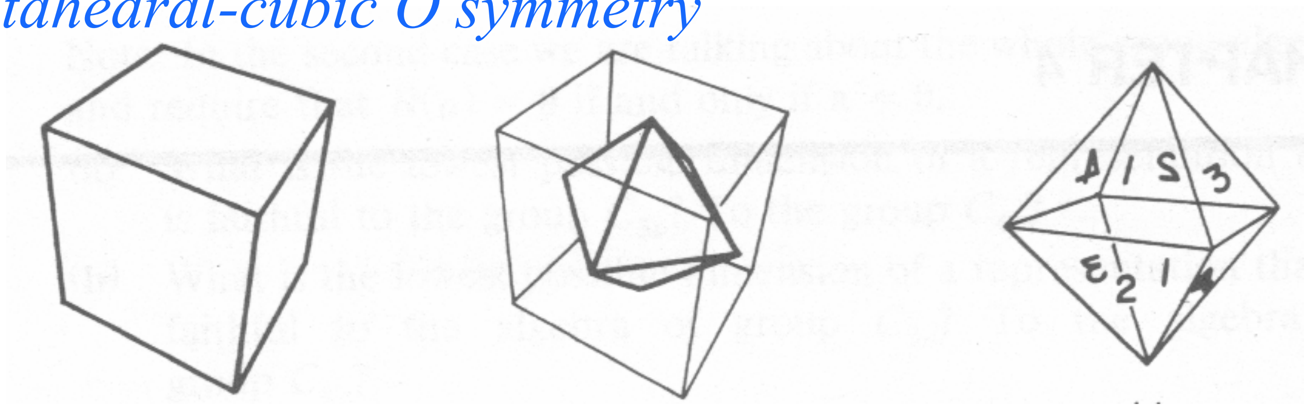

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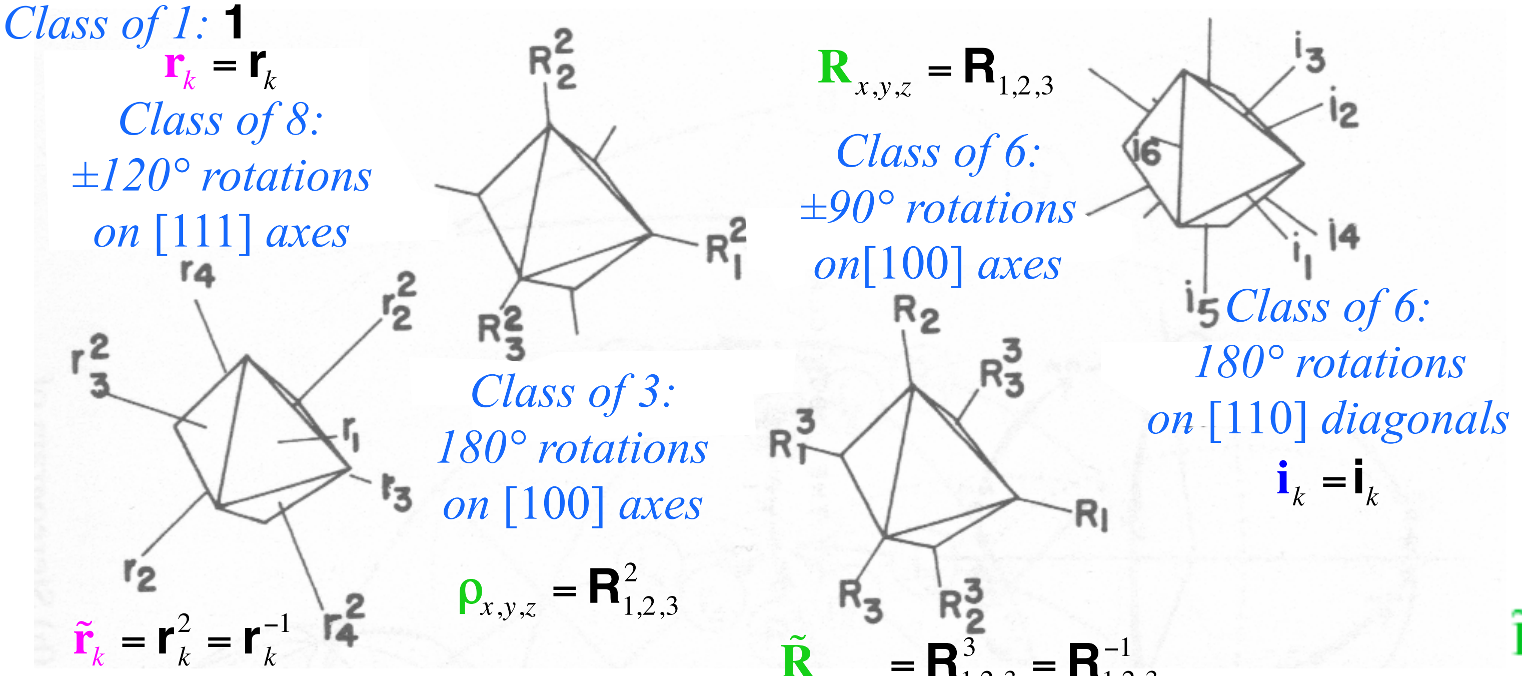
Octahedral group O operations



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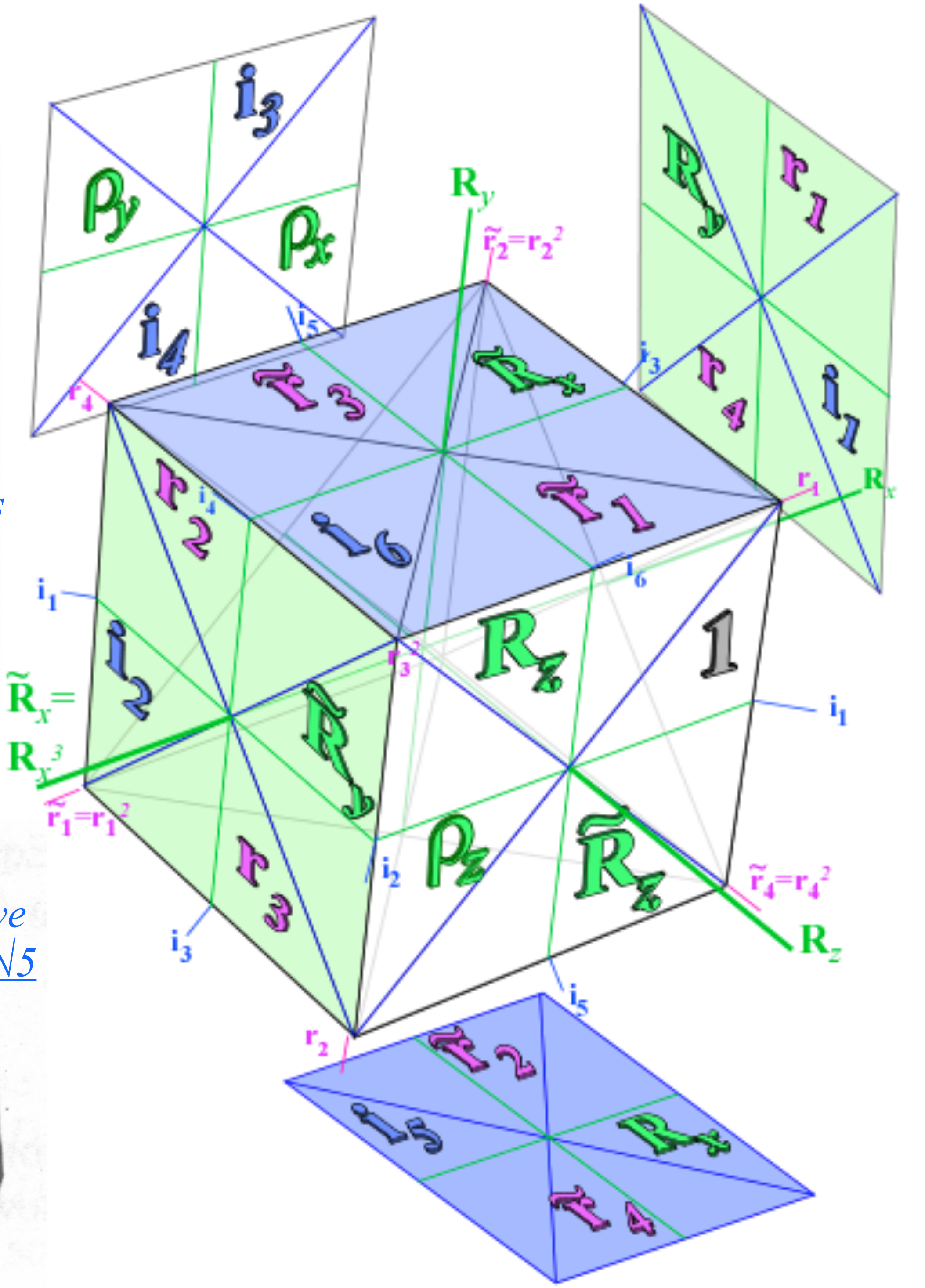
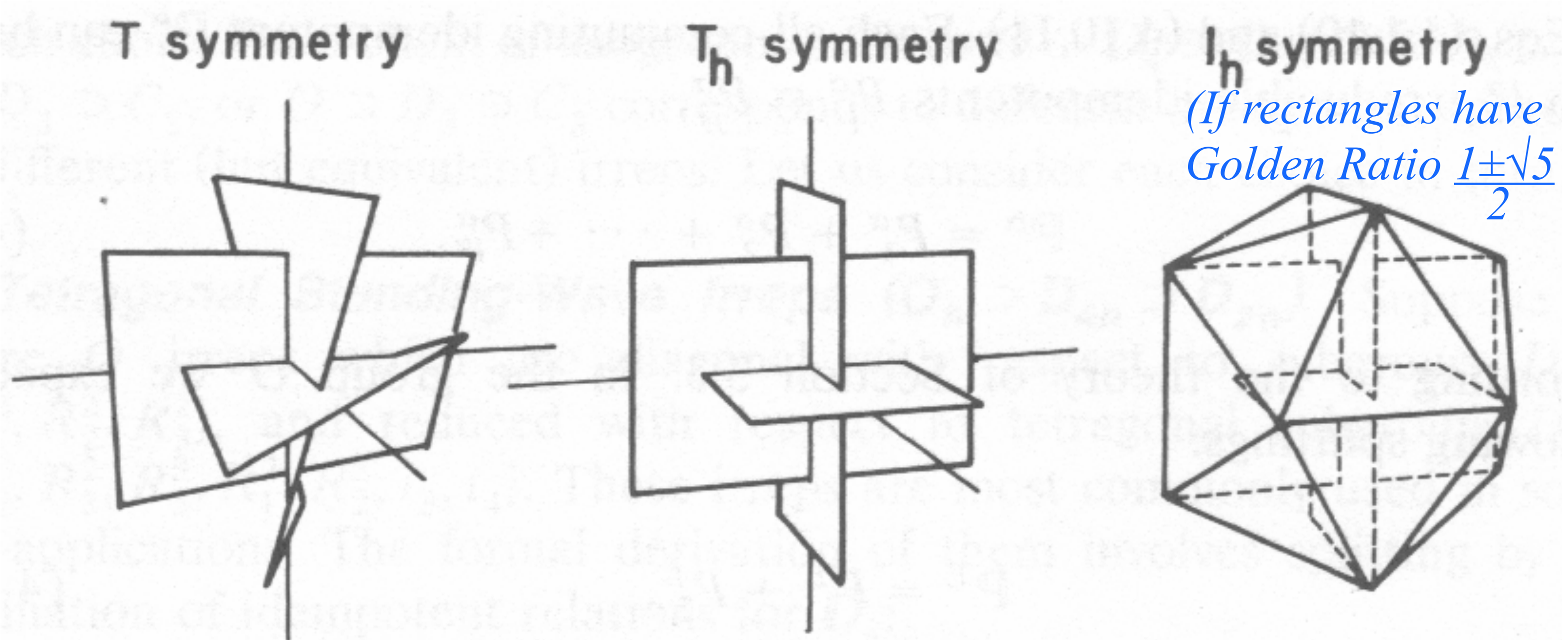


Tetrahedral symmetry becomes Icosahedral



(If rectangles have Golden Ratio $\frac{1 \pm \sqrt{5}}{2}$

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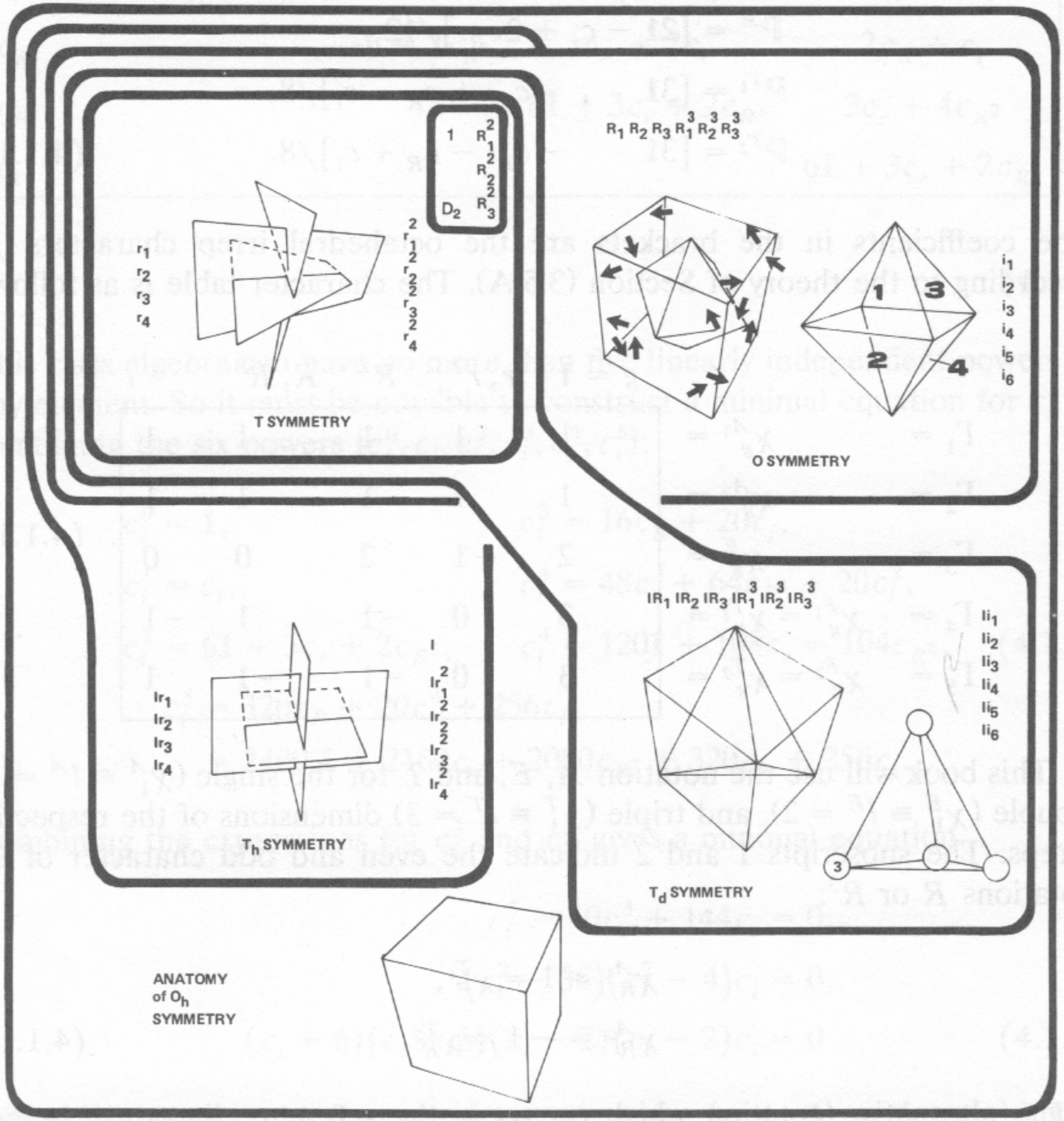

Figure 4.1.5 The full octahedral group (O_h) and four non-Abelian subgroups T, T_h, T_d, and O. The Abelian D_2 subgroup of T is indicated also.

Introduction to octahedral tetrahedral symmetry$O_h\supset O\sim T_d\supset T$

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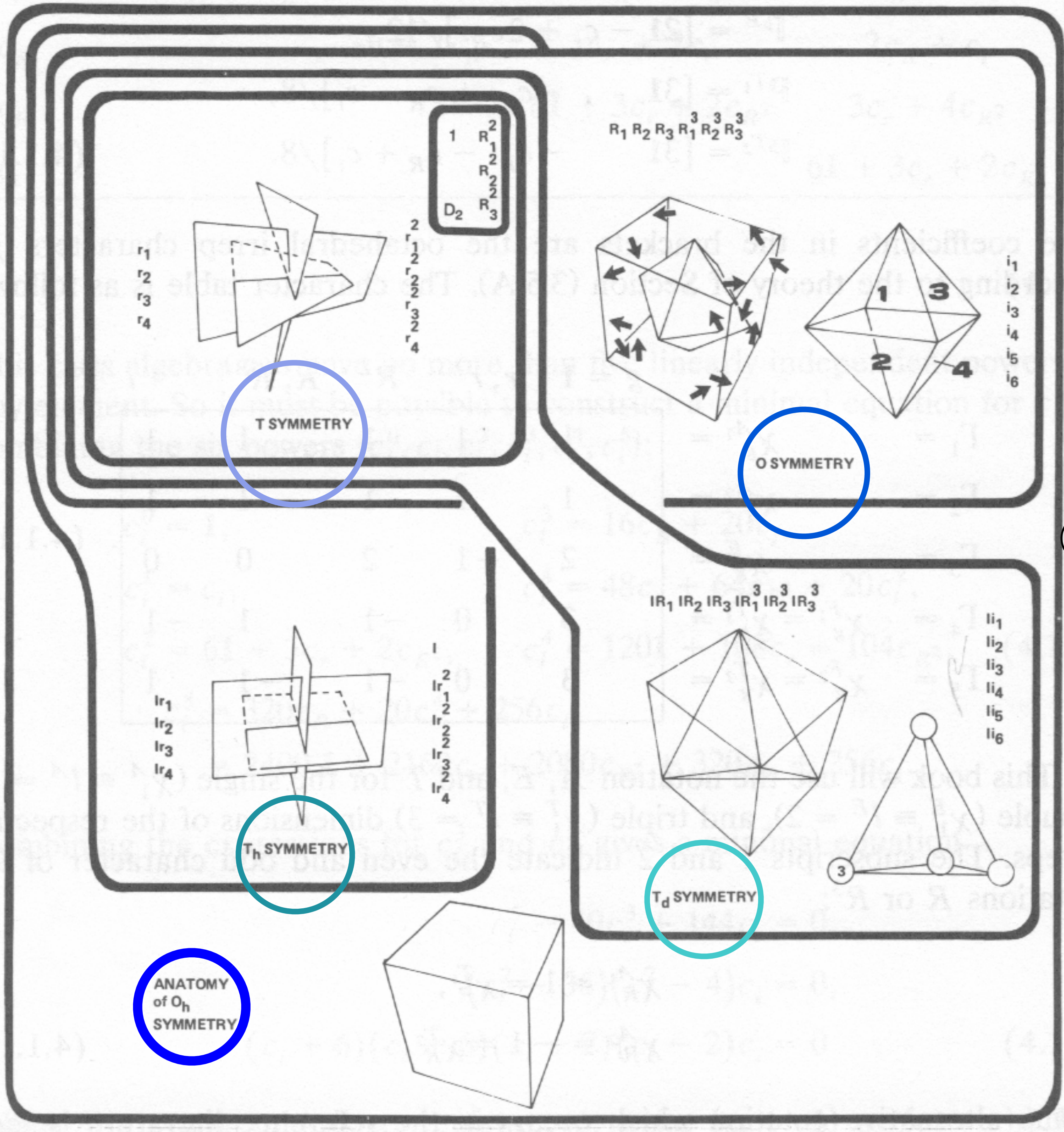

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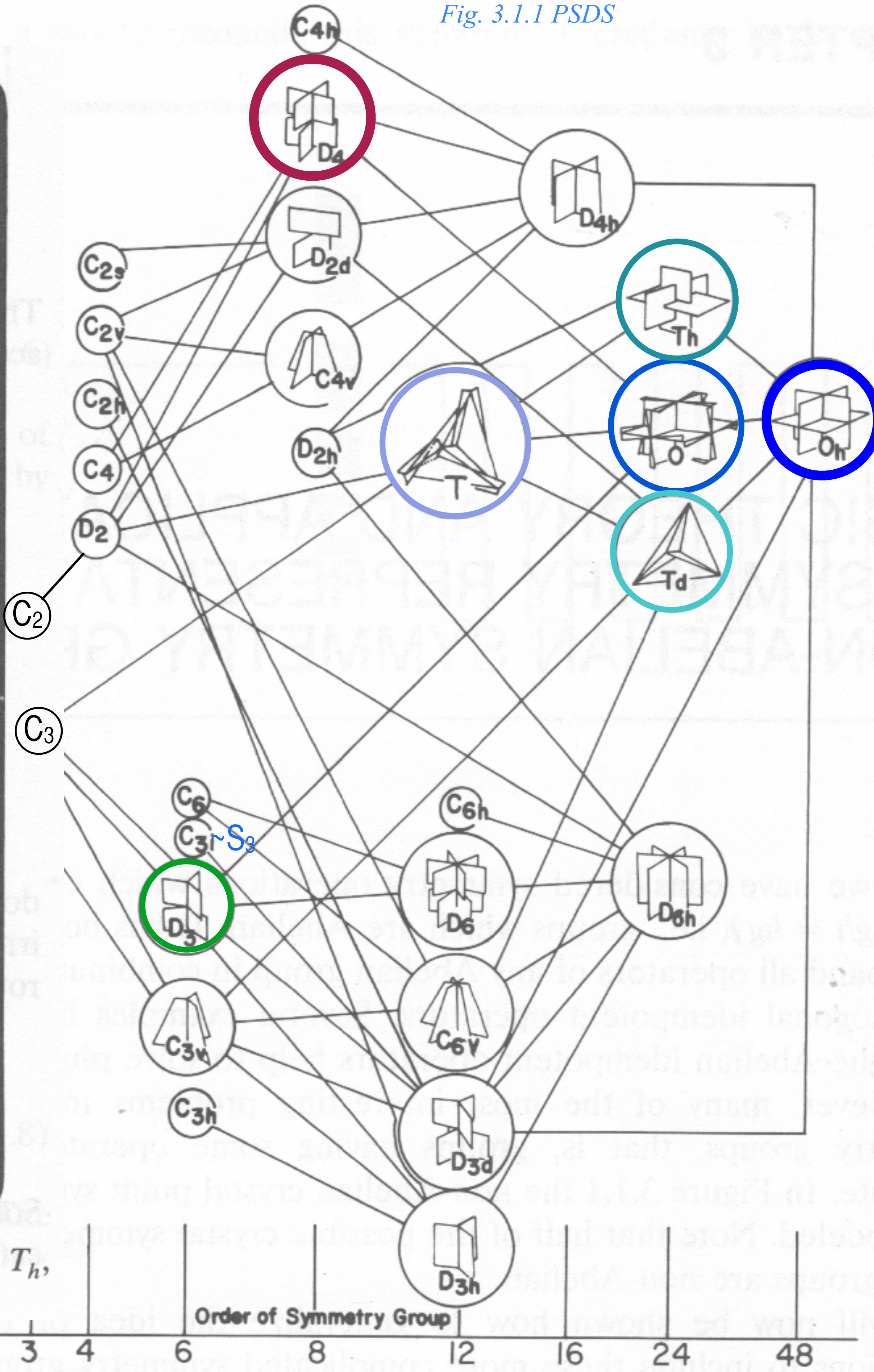
Fig. 3.1.1 PSDS



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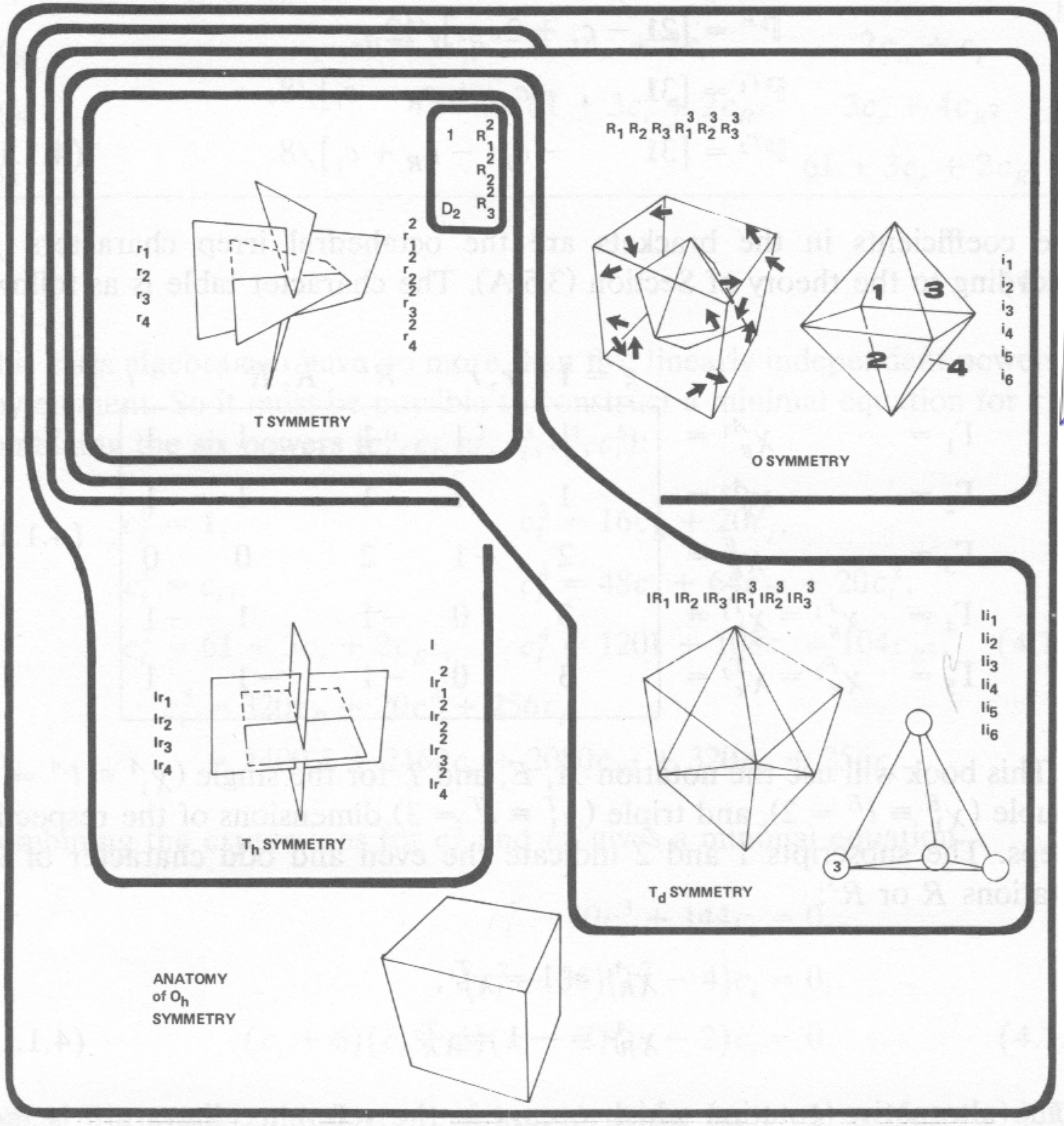

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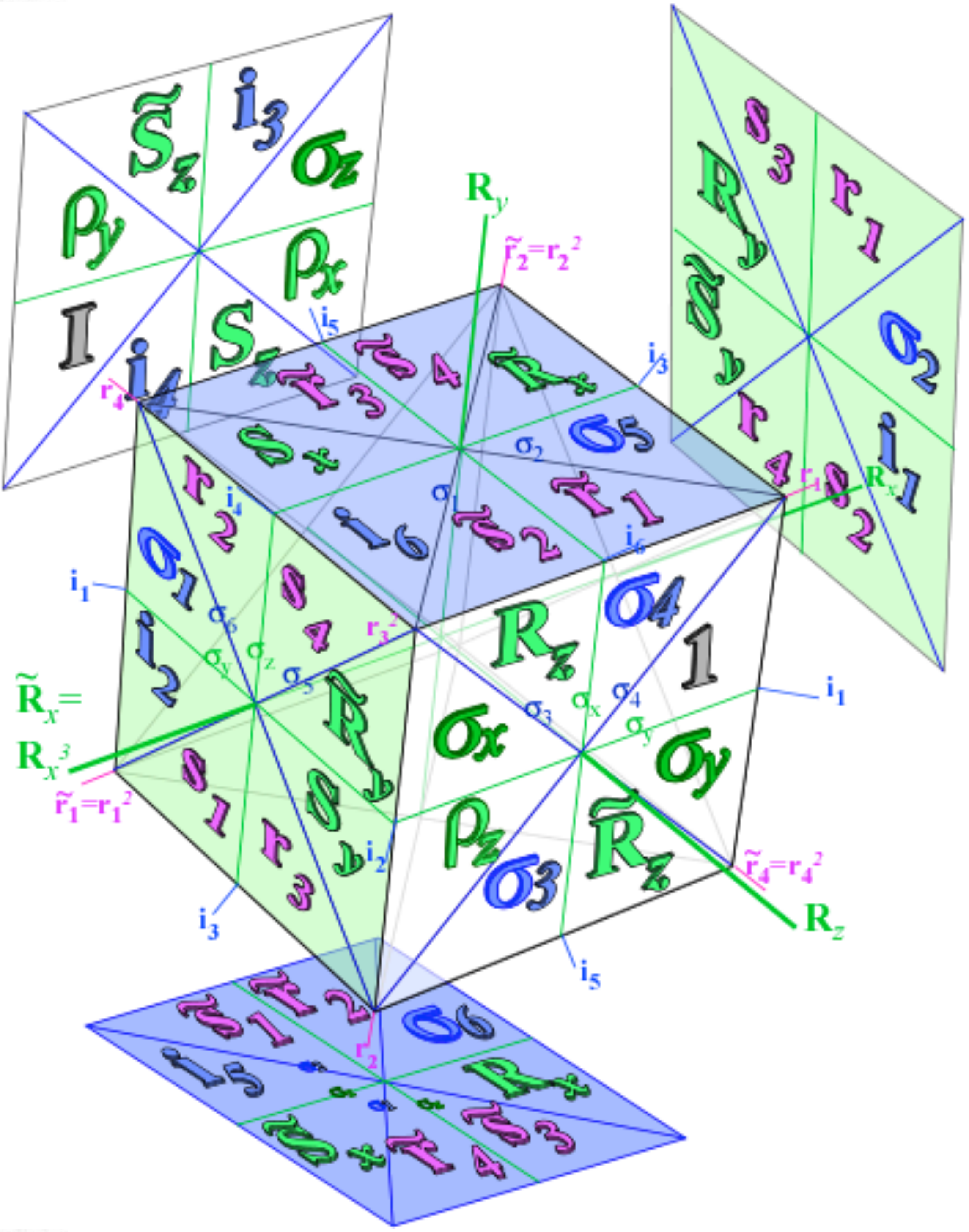

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Octahedral groups $O_h \supset O \sim T_d$ and $O_h \supset T_h \supset T$

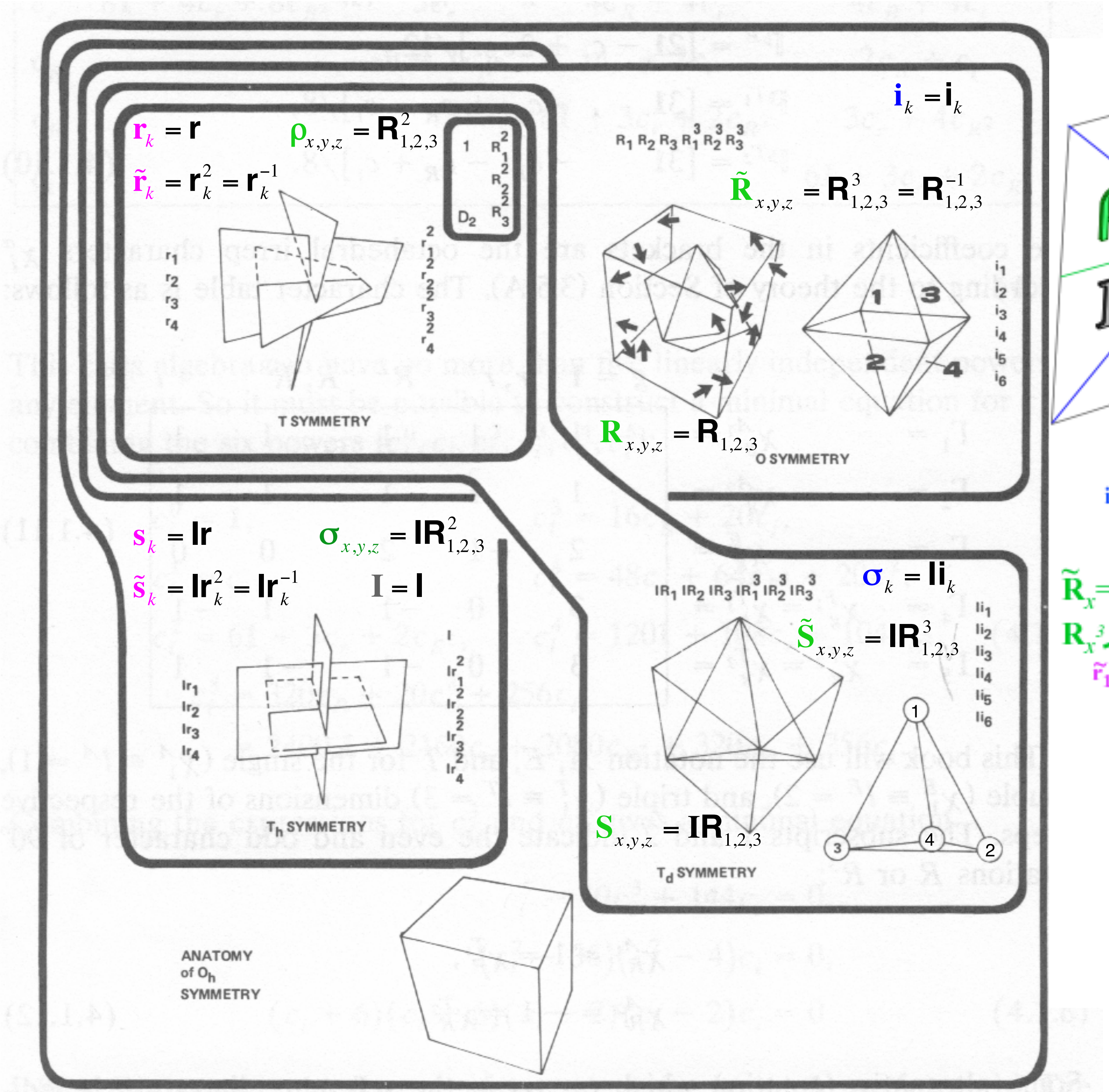

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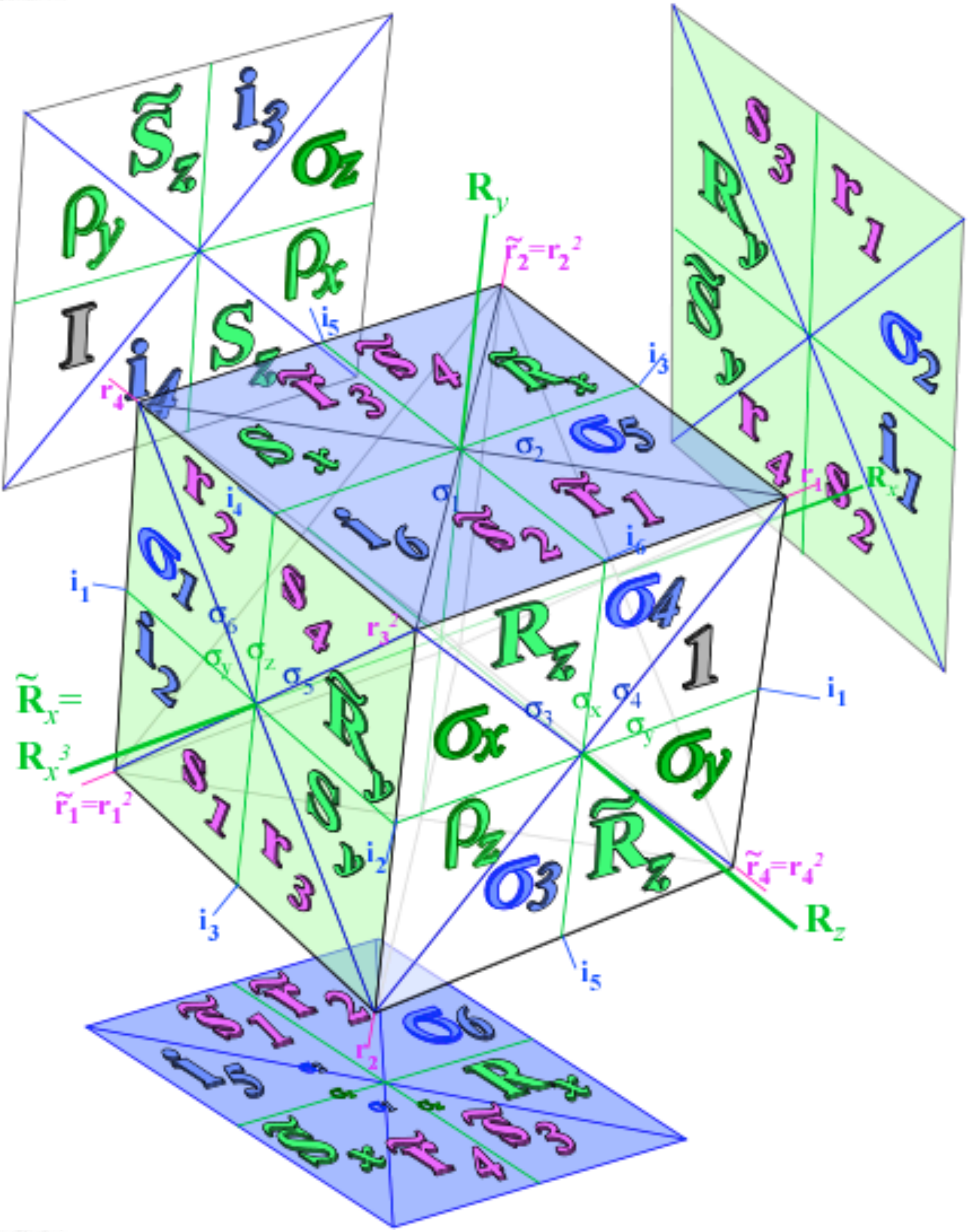
Fig. 4.1.5 from Principles of Symmetry, Dynamics and Spectroscopy



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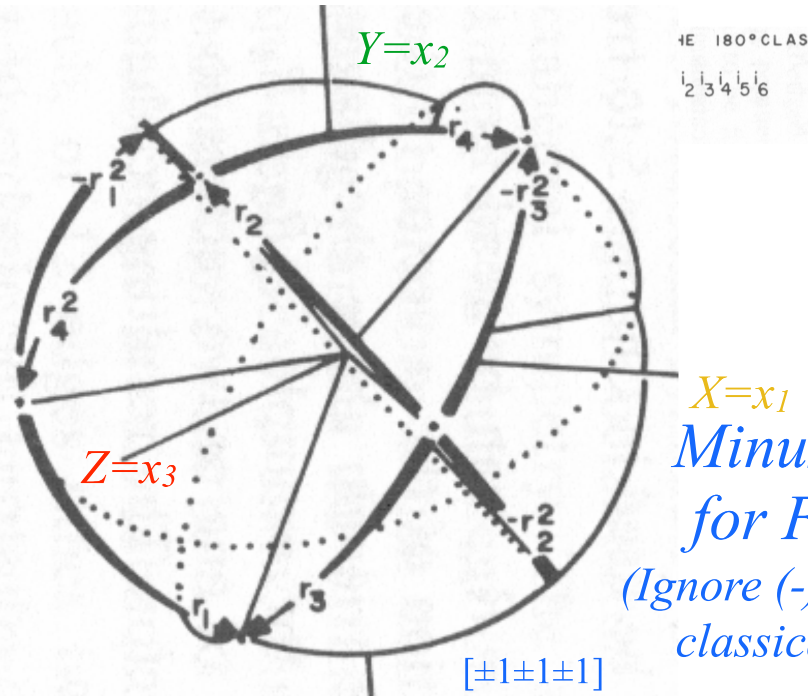
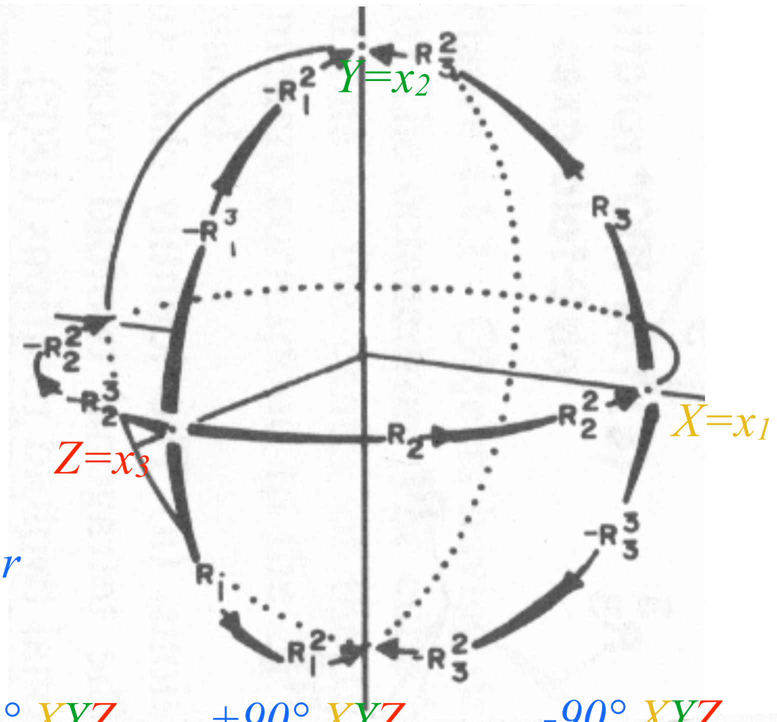
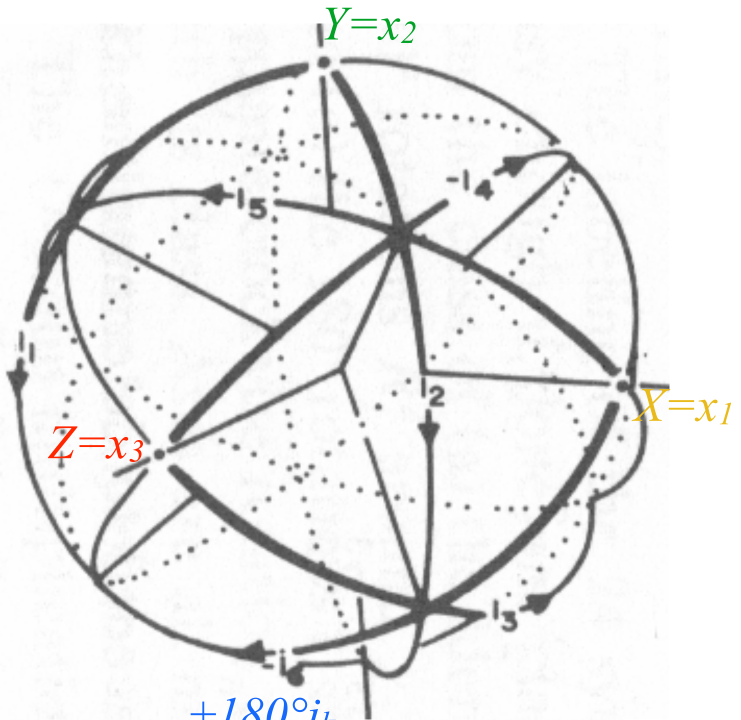


Minus (-)signs for Fermions

(Ignore (-) for Bosons or classical particles)





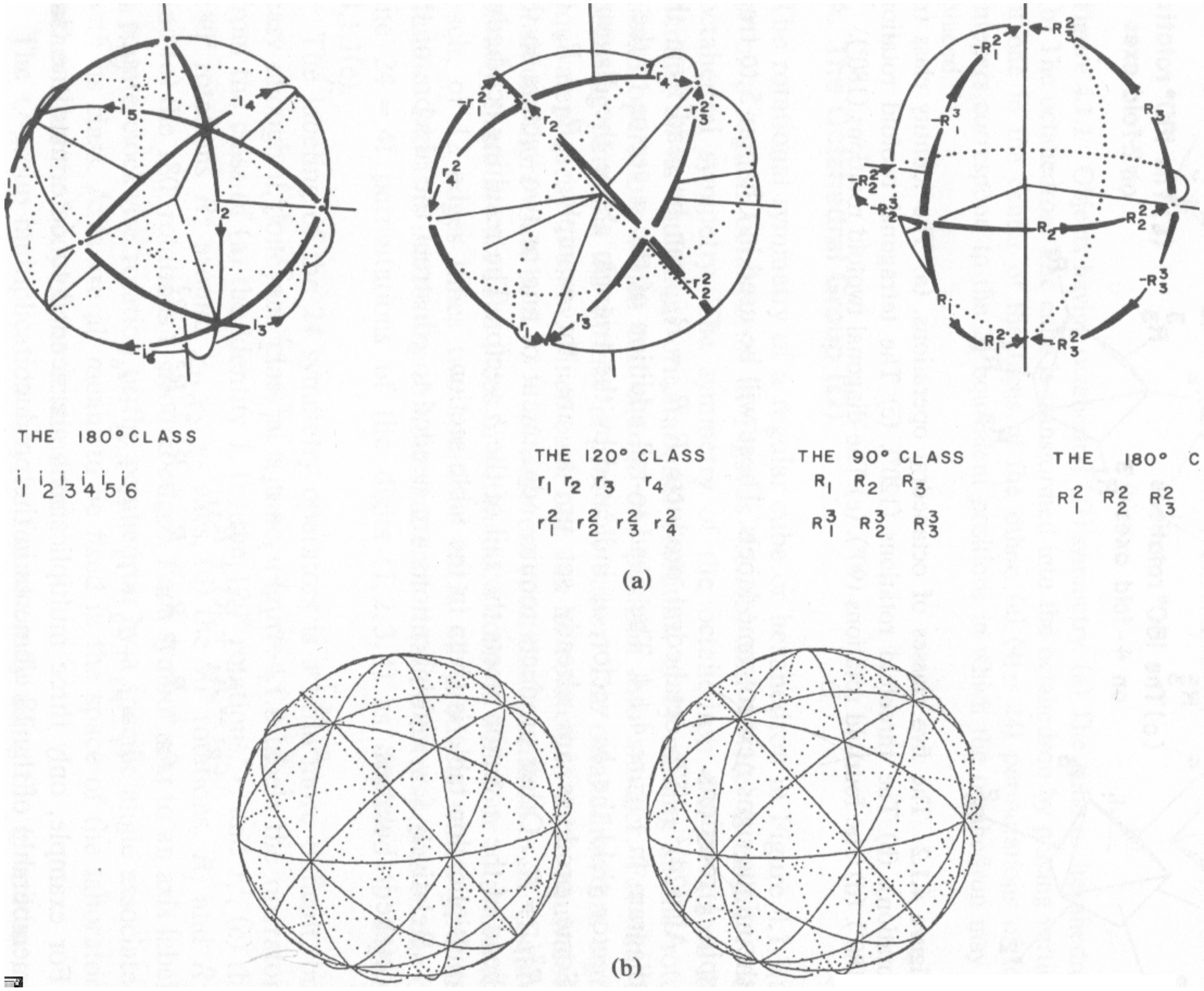
[±1±1±1]

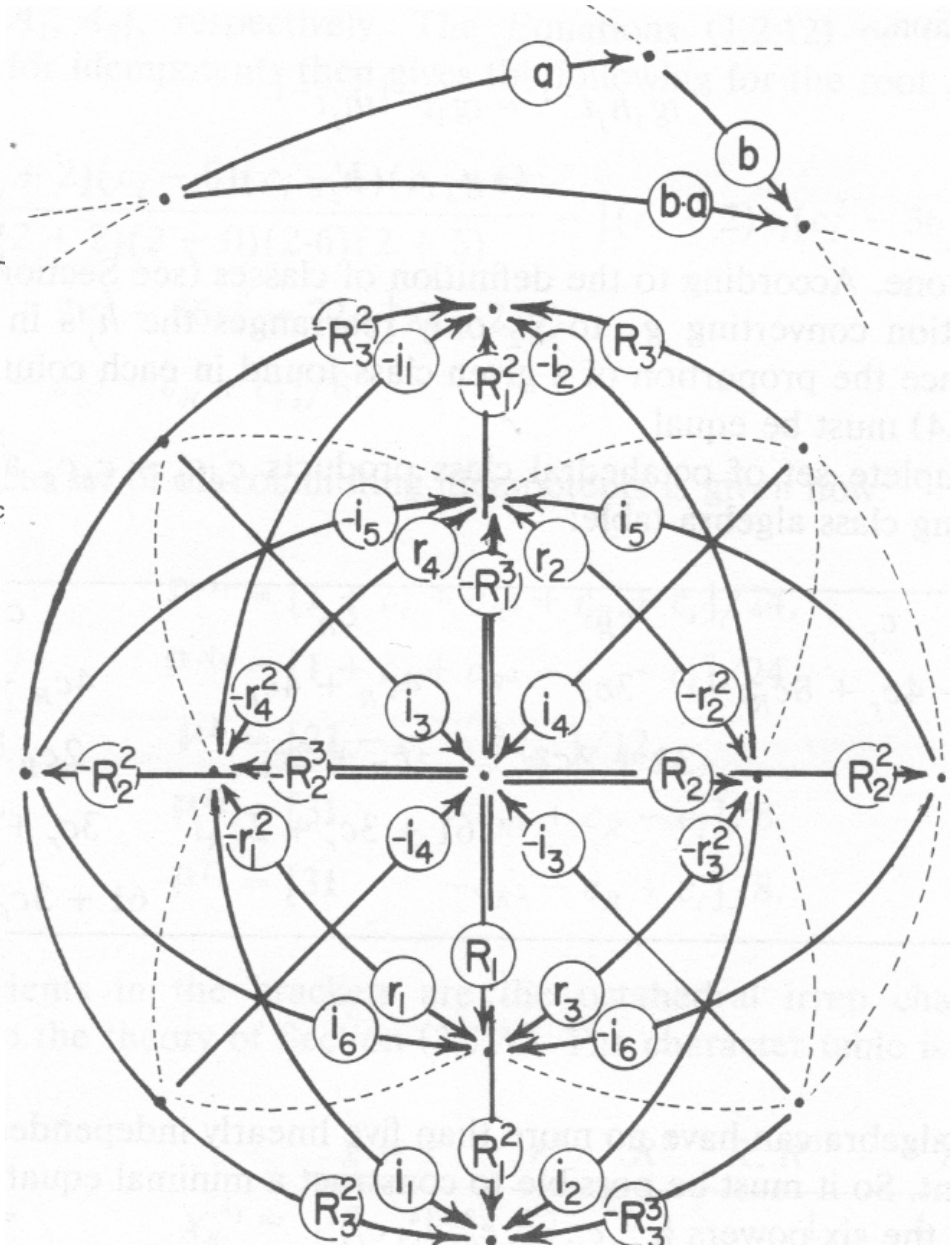
+120° | -120° | ±180° XYZ | +90° XYZ | -90° XYZ | ±180°i_k

	$[1\,1\,1]$	$[\bar{1}\,\bar{1}\,1]$	$[1\,\bar{1}\,\bar{1}]$	$[\bar{1}\,1\,\bar{1}]$	$[\bar{1}\,\bar{1}\,\bar{1}]$	$[1\,1\,\bar{1}]$	$[\bar{1}\,1\,1]$	$[1\,\bar{1}\,1]$	$[1\,0\,0]$	$[0\,1\,0]$	$[0\,0\,1]$	$[1\,0\,0]$	$[0\,1\,0]$	$[0\,0\,1]$	$[\bar{1}\,0\,0]$	$[0\,\bar{1}\,0]$	$[0\,0\,\bar{1}]$	$[1\,0\,1]$	$[1\,0\,\bar{1}]$	$[1\,1\,0]$	$[1\,\bar{1}\,0]$	$[0\,1\,\bar{1}]$	$[0\,1\,1]$
1	r_1	r_2	r_3	r_4	r_1^2	r_2^2	r_3^2	r_4^2	R_1^2	R_2^2	R_3^2	R_1	R_2	R_3	R_1^3	R_2^3	R_3^3	i_1	i_2	i_3	i_4	i_5	i_6
r_1	r_1^2	$-r_4^2$	$-r_2^2$	$-r_3^2$	-1	$-R_2^2$	$-R_3^2$	$-R_1^2$	$-r_2$	$-r_3$	$-r_4$	i_3	i_6	i_1	$-R_3$	$-R_1$	$-R_2$	R_1^3	i_5	R_2^3	i_2	$-i_4$	R_3^3
r_2	$-r_3^2$	r_2^2	$-r_4^2$	$-r_1^2$	R_2^2	-1	R_1^2	$-R_3^2$	r_1	r_4	$-r_3$	R_3	$-R_1^3$	i_2	i_3	$-i_5$	R_2^3	i_6	$-R_1$	R_2	$-i_1$	R_3^3	i_4
r_3	$-r_4^2$	$-r_1^2$	r_3^2	$-r_2^2$	R_3^2	$-R_1^2$	-1	R_2^2	$-r_4$	r_1	r_2	$-i_4$	R_1	$-R_2^3$	R_3^3	i_6	i_2	i_5	$-R_1^3$	i_1	R_2	$-i_3$	R_3
r_4	$-r_2^2$	$-r_3^2$	$-r_1^2$	r_4^2	R_1^2	R_3^2	$-R_2^2$	-1	r_3	$-r_2$	r_1	$-R_3^3$	$-i_5$	R_2	$-i_4$	R_1^3	i_1	R_1	i_6	$-i_2$	R_2^3	R_3	i_3
r_1^2	-1	R_1^2	R_2^2	R_3^2	$-r_1$	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2	R_2^3	R_3^3	R_1^3	$-i_1$	$-i_3$	$-i_6$	$-R_3$	$-i_4$	$-R_1$	i_5	$-i_2$	$-R_2$
r_2^2	$-R_1^2$	-1	R_3^2	$-R_2^2$	r_4	$-r_2$	r_1	r_3	$-r_3^2$	$-r_1^2$	r_4^2	i_2	$-i_3$	$-R_1$	R_2	$-R_3^3$	$-i_5$	i_4	$-R_3$	$-R_1^3$	$-i_6$	R_2^3	$-i_1$
r_3^2	$-R_2^2$	$-R_3^2$	-1	R_1^2	r_2	r_4	$-r_3$	r_1	r_2^2	$-r_4^2$	$-r_1^2$	$-R_2$	$-i_4$	$-i_6$	i_2	R_3	$-R_1^3$	$-i_3$	$-R_3^3$	i_5	R_1	$-i_1$	$-R_2^3$
r_4^2	$-R_3^2$	R_2^2	$-R_1^2$	-1	r_3	r_1	r_2	$-r_4$	$-r_1^2$	r_3^2	$-r_2^2$	$-i_1$	$-R_3$	$-i_5$	$-R_2^3$	$-i_4$	R_1	$-R_3^3$	i_3	$-i_6$	R_1^3	R_2	$-i_2$
R_1^2	$-r_4$	r_3	$-r_2$	r_1	r_2^2	$-r_1^2$	r_4^2	$-r_3^2$	-1	R_3^2	$-R_2^2$	R_1^3	i_1	$-i_4$	$-R_1$	i_2	$-i_3$	$-R_2$	$-R_2^3$	R_3^3	R_3	$-i_6$	i_5
R_2^2	$-r_2$	r_1	r_4	$-r_3$	r_3^2	$-r_4^2$	$-r_1^2$	r_2^2	$-R_3^2$	-1	R_1^2	$-i_5$	R_2^3	i_3	$-i_6$	$-R_2$	$-i_4$	$-i_2$	i_1	$-R_3$	R_3^3	R_1	R_1^3
R_3^2	$-r_3$	$-r_4$	r_1	r_2	r_4^2	r_3^2	$-r_2^2$	$-r_1^2$	R_2^2	$-R_1^2$	-1	i_6	i_2	R_3^3	$-i_5$	$-i_1$	$-R_3$	R_2^3	$-R_2$	i_4	$-i_3$	R_1^3	$-R_1$
R_1	i_1	$-R_2^3$	$-i_2$	R_2	R_3^3	$-i_3$	$-R_3$	i_4	R_1^3	i_6	i_5	R_1^2	r_1	$-r_4^2$	-1	$-r_3$	r_2^2	$-r_4$	r_2	r_1^2	$-r_3^2$	$-R_2^2$	R_3^2
R_2	i_3	R_3	$-R_3^3$	i_4	R_1^3	i_5	$-i_6$	$-R_1$	$-i_2$	R_2^3	i_1	$-r_2^2$	R_2^2	r_1	r_3^2	-1	$-r_4$	R_1^2	R_3^2	$-r_2$	$-r_3$	$-r_4^2$	r_1^2
R_3	i_6	i_5	R_1	$-R_1^3$	R_2^3	$-R_2$	$-i_2$	$-i_1$	i_3	i_4	R_3^3	r_1	$-r_3^2$	R_2^2	$-r_2$	r_4^2	-1	r_1^2	r_2^2	R_2^2	$-R_1^2$	$-r_4$	$-r_3$
R_1^3	$-R_2$	$-i_2$	R_2^3	i_1	$-i_3$	$-R_3^3$	i_4	R_3	$-R_1$	i_5	$-i_6$	-1	$-r_4$	r_3^2	$-R_1^2$	r_2	$-r_1^2$	$-r_1$	r_3	r_2^2	$-r_4^2$	$-R_3^2$	$-R_2^2$
R_2^3	$-R_3$	i_3	i_4	R_3^3	$-i_6$	R_1	$-R_1^3$	i_5	$-i_1$	$-R_2$	$-i_2$	r_4^2	-1	$-r_2$	$-r_1^2$	$-R_2^2$	r_3	$-R_3^2$	R_1^2	$-r_1$	$-r_4$	$-r_2^2$	r_3^2
R_3^3	$-R_1$	R_1^3	i_6	i_5	$-i_1$	$-i_2$	R_2	$-R_2^3$	i_4	$-i_3$	$-R_3$	$-r_3$	r_2^2	-1	r_4	$-r_1^2$	$-R_3^2$	r_4^2	r_3^2	$-R_1^2$	$-R_2^2$	$-r_2$	$-r_1$
i_1	R_3^3	$-i_4$	i_3	R_3	$-R_1$	$-i_6$	$-i_5$	$-R_1^3$	R_2^3	i_2	$-R_2$	r_1^2	R_3^2	$-r_4$	r_4^2	$-R_1^2$	$-r_1$	-1	$-R_2^2$	$-r_3$	r_2	r_3^2	r_2^2
i_2	i_4	R_3^3	R_3	$-i_3$	$-i_5$	R_1^3	R_1	$-i_6$	R_2	$-i_1$	R_2^3	$-r_3^2$	$-R_1^2$	$-r_3$	$-r_2^2$	$-R_3^2$	$-r_2$	R_2^2	-1	r_4	$-r_1$	r_1^2	r_4^2
i_3	R_1^3	R_1	$-i_5$	i_6	$-R_2$	$-R_2^3$	$-i_1$	i_2	$-R_3$	R_3^3	$-i_4$	$-r_2$	r_1^2	R_1^2	$-r_1$	r_2^2	$-R_2^2$	r_3^2	$-r_4^2$	-1	R_3^2	r_3	$-r_4$
i_4	$-i_5$	i_6	$-R_1^3$	$-R_1$	$-i_2$	i_1	$-R_2^3$	$-R_2$	$-R_3^3$	$-R_3$	i_3	r_4	r_4^2	R_2^2	r_3	r_3^2	R_1^2	$-r_2^2$	r_1^2	$-R_3^3$	-1	r_1	$-r_2$
i_5	i_2	$-R_2$	i_1	$-R_2^3$	i_4	$-R_3$	i_3	$-R_3^3$	i_6	$-R_1^3$	$-R_1$	R_3^2	r_2	r_2^2	R_2^2	r_4	r_4^2	$-r_3$	$-r_1$	$-r_3^2$	$-r_1^2$	-1	$-R_1^2$
i_6	R_2^3	i_1	R_2	i_2	$-R_3$	$-i_4$	$-R_3^3$	$-i_3$	$-i_5$	$-R_1$	R_1^3	R_2^2	$-r_3$	r_1^2	$-R_3^2$	$-r_1$	r_3^2	$-r_2$	$-r_4$	r_4^2	r_2^2	R_1^2	-1

Octahedral O and spin-O⊂U(2) rotation product Table F.2.1 from Principles of Symmetry, Dynamics and Spectroscopy

Octahedral O and spin-O⊂U(2) rotation nomogram from Fig. 4.1.3-4 Principles of Symmetry, Dynamics and Spectroscopy





Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$
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- Tetrahedral symmetry becomes Icosahedral
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  - Octahedral O and spin-$O \subset U(2)$

Tetrahedral T class algebra
- Tetrahedral T class minimal equations
- Tetrahedral T class projectors and characters

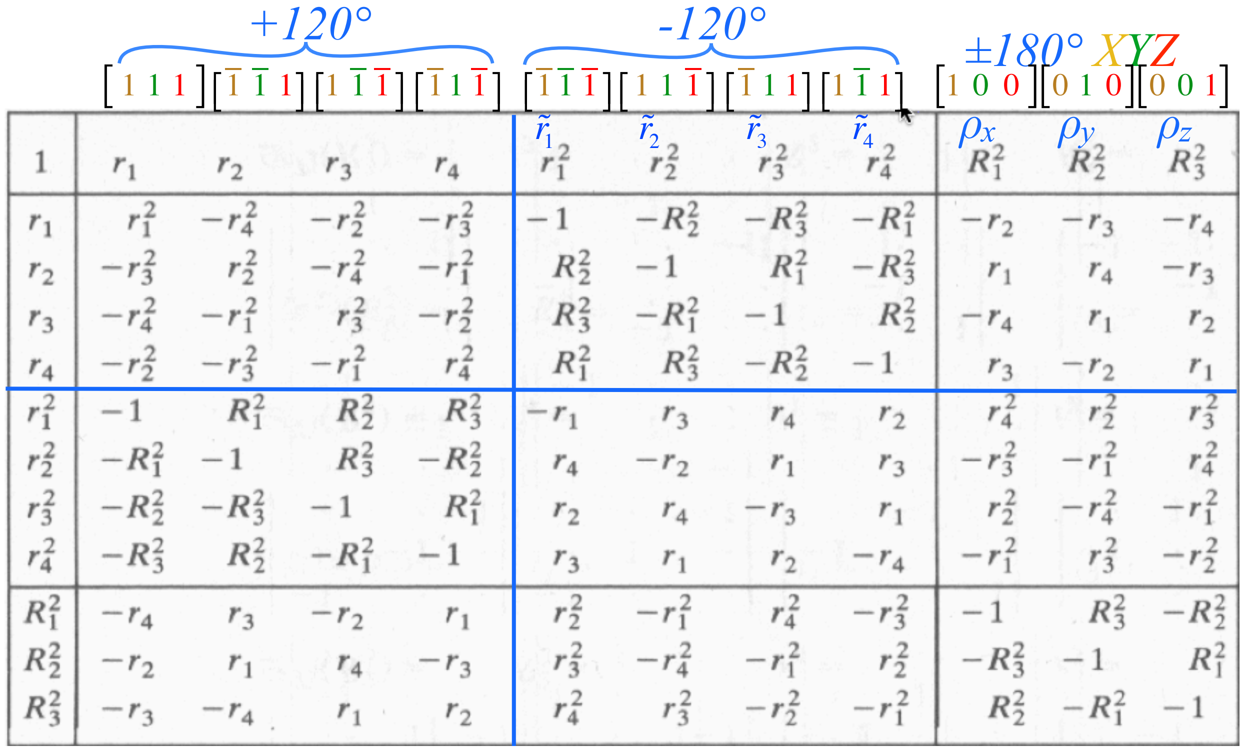
Octahedral O class algebra
- Octahedral O class minimal equations
- Octahedral O class projectors and characters

Octahedral $O_h \supset O$ subgroup correlations
- Octahedral $O_h \supset O$ subgroup correlations
- $O_h \supset O \supset D_4$ subgroup correlations
- $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations
- Preview of applications to high resolution spectroscopy

Tetrahedral T class algebra

$\mathbf{c}_1=\mathbf{1}$, $\mathbf{c}_r=\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4$, $\mathbf{c}^\dagger_r=\mathbf{r}_1{}^2+\mathbf{r}_2{}^2+\mathbf{r}_3{}^2+\mathbf{r}_4{}^2$, $\mathbf{c}_\rho=\mathbf{R}_1{}^2+\mathbf{R}_2{}^2+\mathbf{R}_3{}^2$,

T group products



	+120°				-120°				±180° XYZ		
	$[1\ 1\ 1]$	$[\bar{1}\ \bar{1}\ 1]$	$[1\ \bar{1}\ \bar{1}]$	$[\bar{1}\ 1\ \bar{1}]$	$[\bar{1}\ \bar{1}\ \bar{1}]$	$[1\ 1\ \bar{1}]$	$[\bar{1}\ 1\ 1]$	$[1\ \bar{1}\ 1]$	$[1\ 0\ 0]$	$[0\ 1\ 0]$	$[0\ 0\ 1]$
					$\tilde{r}_1$	$\tilde{r}_2$	$\tilde{r}_3$	$\tilde{r}_4$	ρ_x	ρ_y	ρ_z
1	r_1	r_2	r_3	r_4	r_1^2	r_2^2	r_3^2	r_4^2	R_1^2	R_2^2	R_3^2
r_1	r_1^2	$-r_4^2$	$-r_2^2$	$-r_3^2$	-1	$-R_2^2$	$-R_3^2$	$-R_1^2$	$-r_2$	$-r_3$	$-r_4$
r_2	$-r_3^2$	r_2^2	$-r_4^2$	$-r_1^2$	R_2^2	-1	R_1^2	$-R_3^2$	r_1	r_4	$-r_3$
r_3	$-r_4^2$	$-r_1^2$	r_3^2	$-r_2^2$	R_3^2	$-R_1^2$	-1	R_2^2	$-r_4$	r_1	r_2
r_4	$-r_2^2$	$-r_3^2$	$-r_1^2$	r_4^2	R_1^2	R_3^2	$-R_2^2$	-1	r_3	$-r_2$	r_1
r_1^2	-1	R_1^2	R_2^2	R_3^2	$-r_1$	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2
r_2^2	$-R_1^2$	-1	R_3^2	$-R_2^2$	r_4	$-r_2$	r_1	r_3	$-r_3^2$	$-r_1^2$	r_4^2
r_3^2	$-R_2^2$	$-R_3^2$	-1	R_1^2	r_2	r_4	$-r_3$	r_1	r_2^2	$-r_4^2$	$-r_1^2$
r_4^2	$-R_3^2$	R_2^2	$-R_1^2$	-1	r_3	r_1	r_2	$-r_4$	$-r_1^2$	r_3^2	$-r_2^2$
R_1^2	$-r_4$	r_3	$-r_2$	r_1	r_2^2	$-r_1^2$	r_4^2	$-r_3^2$	-1	R_3^2	$-R_2^2$
R_2^2	$-r_2$	r_1	r_4	$-r_3$	r_3^2	$-r_4^2$	$-r_1^2$	r_2^2	$-R_3^2$	-1	R_1^2
R_3^2	$-r_3$	$-r_4$	r_1	r_2	r_4^2	r_3^2	$-r_2^2$	$-r_1^2$	R_2^2	$-R_1^2$	-1

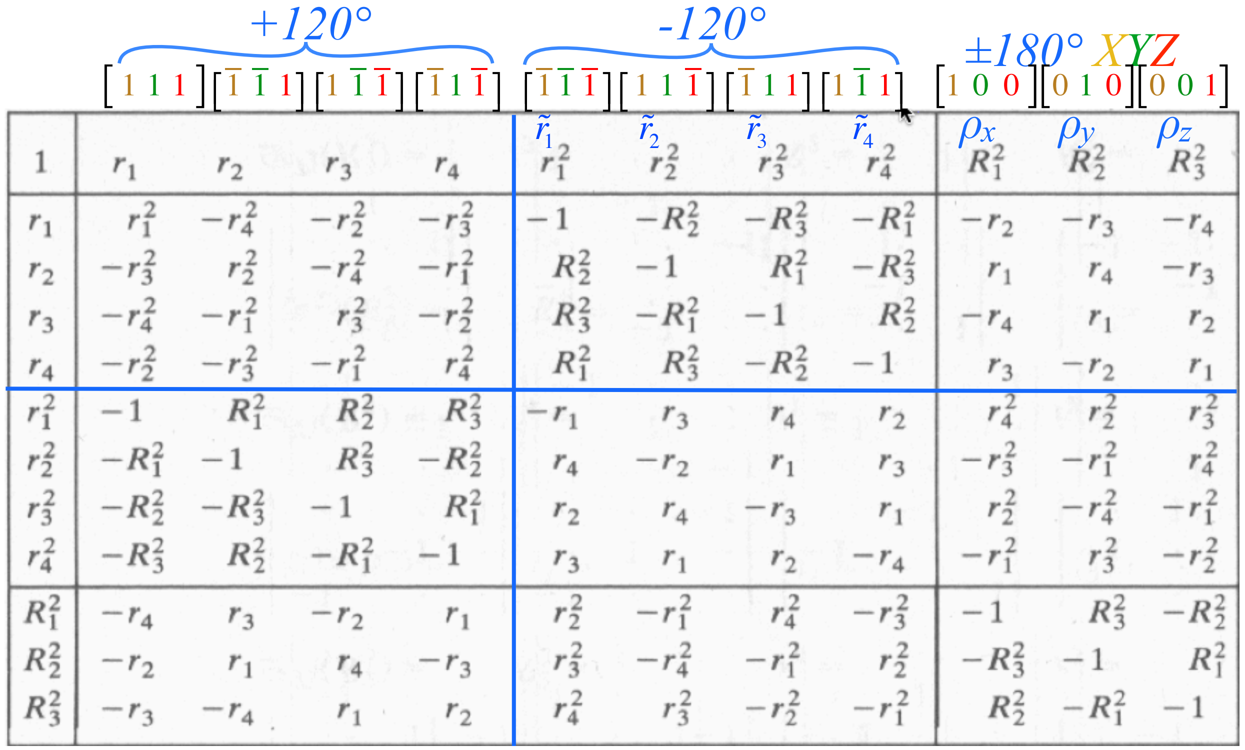
Minus (-)signs for Fermions

(Ignore (-) for Bosons or classical particles)

Tetrahedral T class algebra

$\mathbf{c}_1=\mathbf{1}$, $\mathbf{c}_r=\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4$, $\mathbf{c}^\dagger_r=\mathbf{r}_1{}^2+\mathbf{r}_2{}^2+\mathbf{r}_3{}^2+\mathbf{r}_4{}^2$, $\mathbf{c}_\rho=\mathbf{R}_1{}^2+\mathbf{R}_2{}^2+\mathbf{R}_3{}^2$,

T group products



	+120°				-120°				±180° XYZ		
	$[1\,1\,1]$	$[\bar{1}\,\bar{1}\,1]$	$[1\,\bar{1}\,\bar{1}]$	$[\bar{1}\,1\,\bar{1}]$	$[\bar{1}\,\bar{1}\,\bar{1}]$	$[1\,1\,\bar{1}]$	$[\bar{1}\,1\,1]$	$[1\,\bar{1}\,1]$	$[1\,0\,0]$	$[0\,1\,0]$	$[0\,0\,1]$
					$\tilde{r}_1$	$\tilde{r}_2$	$\tilde{r}_3$	$\tilde{r}_4$	ρ_x	ρ_y	ρ_z
1	r_1	r_2	r_3	r_4	r_1^2	r_2^2	r_3^2	r_4^2	R_1^2	R_2^2	R_3^2
r_1	r_1^2	$-r_4^2$	$-r_2^2$	$-r_3^2$	-1	$-R_2^2$	$-R_3^2$	$-R_1^2$	$-r_2$	$-r_3$	$-r_4$
r_2	$-r_3^2$	r_2^2	$-r_4^2$	$-r_1^2$	R_2^2	-1	R_1^2	$-R_3^2$	r_1	r_4	$-r_3$
r_3	$-r_4^2$	$-r_1^2$	r_3^2	$-r_2^2$	R_3^2	$-R_1^2$	-1	R_2^2	$-r_4$	r_1	r_2
r_4	$-r_2^2$	$-r_3^2$	$-r_1^2$	r_4^2	R_1^2	R_3^2	$-R_2^2$	-1	r_3	$-r_2$	r_1
r_1^2	-1	R_1^2	R_2^2	R_3^2	$-r_1$	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2
r_2^2	$-R_1^2$	-1	R_3^2	$-R_2^2$	r_4	$-r_2$	r_1	r_3	$-r_3^2$	$-r_1^2$	r_4^2
r_3^2	$-R_2^2$	$-R_3^2$	-1	R_1^2	r_2	r_4	$-r_3$	r_1	r_2^2	$-r_4^2$	$-r_1^2$
r_4^2	$-R_3^2$	R_2^2	$-R_1^2$	-1	r_3	r_1	r_2	$-r_4$	$-r_1^2$	r_3^2	$-r_2^2$
R_1^2	$-r_4$	r_3	$-r_2$	r_1	r_2^2	$-r_1^2$	r_4^2	$-r_3^2$	-1	R_3^2	$-R_2^2$
R_2^2	$-r_2$	r_1	r_4	$-r_3$	r_3^2	$-r_4^2$	$-r_1^2$	r_2^2	$-R_3^2$	-1	R_1^2
R_3^2	$-r_3$	$-r_4$	r_1	r_2	r_4^2	r_3^2	$-r_2^2$	$-r_1^2$	R_2^2	$-R_1^2$	-1

Minus (-)signs
for Fermions

(Ignore (-) for Bosons or
classical particles)

T class products

$\mathbf{1}=\mathbf{c}_1$	$\mathbf{c}_r$	$\mathbf{c}^\dagger_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\mathbf{c}^\dagger_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\mathbf{c}^\dagger_r$		$4\mathbf{c}_r$	$3\mathbf{c}^\dagger_r$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

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→ Tetrahedral T class algebra ←

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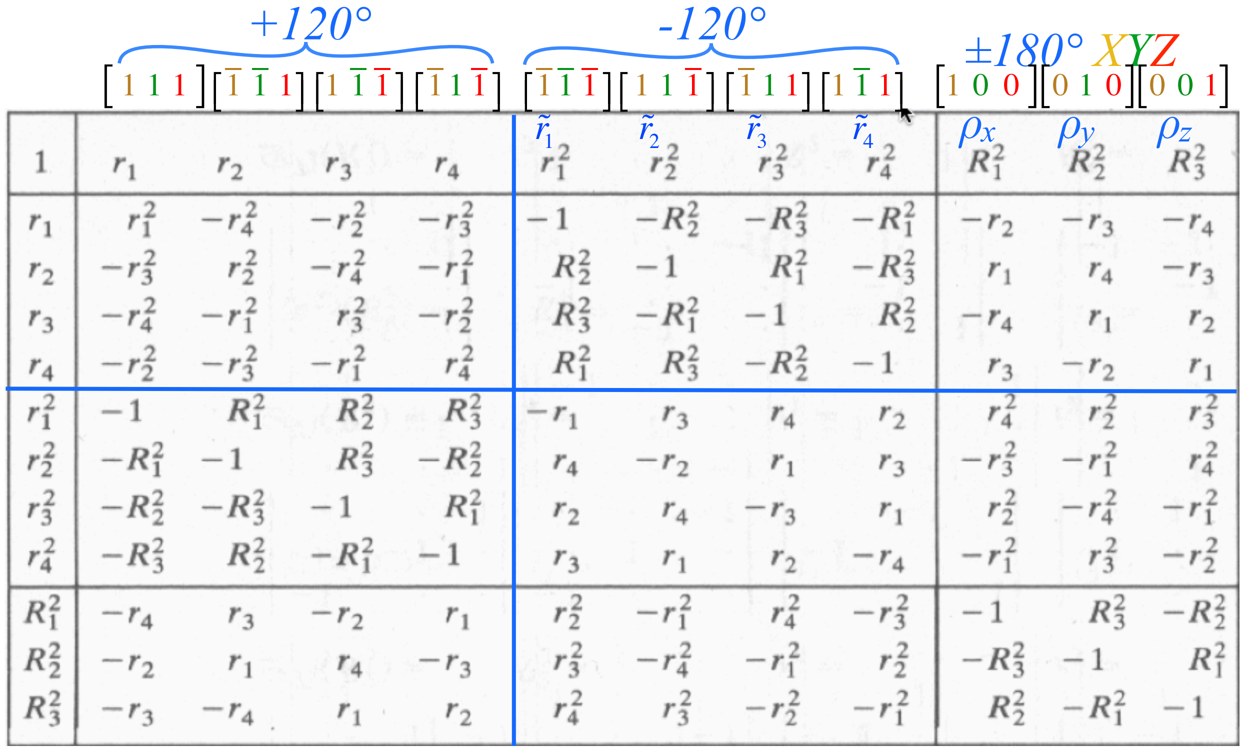
$\mathbf{c}_1=\mathbf{1}$, $\mathbf{c}_r=\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4$, $\mathbf{c}^\dagger_r=\mathbf{r}_1^2+\mathbf{r}_2^2+\mathbf{r}_3^2+\mathbf{r}_4^2$, $\mathbf{c}_\rho=\mathbf{R}_1^2+\mathbf{R}_2^2+\mathbf{R}_3^2$,

T group products



	+120°				-120°				±180° XYZ		
	$[1\ 1\ 1]$	$[\bar{1}\ \bar{1}\ 1]$	$[1\ \bar{1}\ \bar{1}]$	$[\bar{1}\ 1\ \bar{1}]$	$[\bar{1}\ \bar{1}\ \bar{1}]$	$[1\ 1\ \bar{1}]$	$[\bar{1}\ 1\ 1]$	$[1\ \bar{1}\ 1]$	$[1\ 0\ 0]$	$[0\ 1\ 0]$	$[0\ 0\ 1]$
					$\tilde{r}_1$	$\tilde{r}_2$	$\tilde{r}_3$	$\tilde{r}_4$	ρ_x	ρ_y	ρ_z
1	r_1	r_2	r_3	r_4	r_1^2	r_2^2	r_3^2	r_4^2	R_1^2	R_2^2	R_3^2
r_1	r_1^2	$-r_4^2$	$-r_2^2$	$-r_3^2$	-1	$-R_2^2$	$-R_3^2$	$-R_1^2$	$-r_2$	$-r_3$	$-r_4$
r_2	$-r_3^2$	r_2^2	$-r_4^2$	$-r_1^2$	R_2^2	-1	R_1^2	$-R_3^2$	r_1	r_4	$-r_3$
r_3	$-r_4^2$	$-r_1^2$	r_3^2	$-r_2^2$	R_3^2	$-R_1^2$	-1	R_2^2	$-r_4$	r_1	r_2
r_4	$-r_2^2$	$-r_3^2$	$-r_1^2$	r_4^2	R_1^2	R_3^2	$-R_2^2$	-1	r_3	$-r_2$	r_1
r_1^2	-1	R_1^2	R_2^2	R_3^2	$-r_1$	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2
r_2^2	$-R_1^2$	-1	R_3^2	$-R_2^2$	r_4	$-r_2$	r_1	r_3	$-r_3^2$	$-r_1^2$	r_4^2
r_3^2	$-R_2^2$	$-R_3^2$	-1	R_1^2	r_2	r_4	$-r_3$	r_1	r_2^2	$-r_4^2$	$-r_1^2$
r_4^2	$-R_3^2$	R_2^2	$-R_1^2$	-1	r_3	r_1	r_2	$-r_4$	$-r_1^2$	r_3^2	$-r_2^2$
R_1^2	$-r_4$	r_3	$-r_2$	r_1	r_2^2	$-r_1^2$	r_4^2	$-r_3^2$	-1	R_3^2	$-R_2^2$
R_2^2	$-r_2$	r_1	r_4	$-r_3$	r_3^2	$-r_4^2$	$-r_1^2$	r_2^2	$-R_3^2$	-1	R_1^2
R_3^2	$-r_3$	$-r_4$	r_1	r_2	r_4^2	r_3^2	$-r_2^2$	$-r_1^2$	R_2^2	$-R_1^2$	-1

Minus (-)signs
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T class products

$\mathbf{1}=\mathbf{c}_1$	$\mathbf{c}_r$	$\mathbf{c}_r^\dagger$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\mathbf{c}_r^\dagger$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\mathbf{c}_r^\dagger$		$4\mathbf{c}_r$	$3\mathbf{c}_r^\dagger$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_\rho$

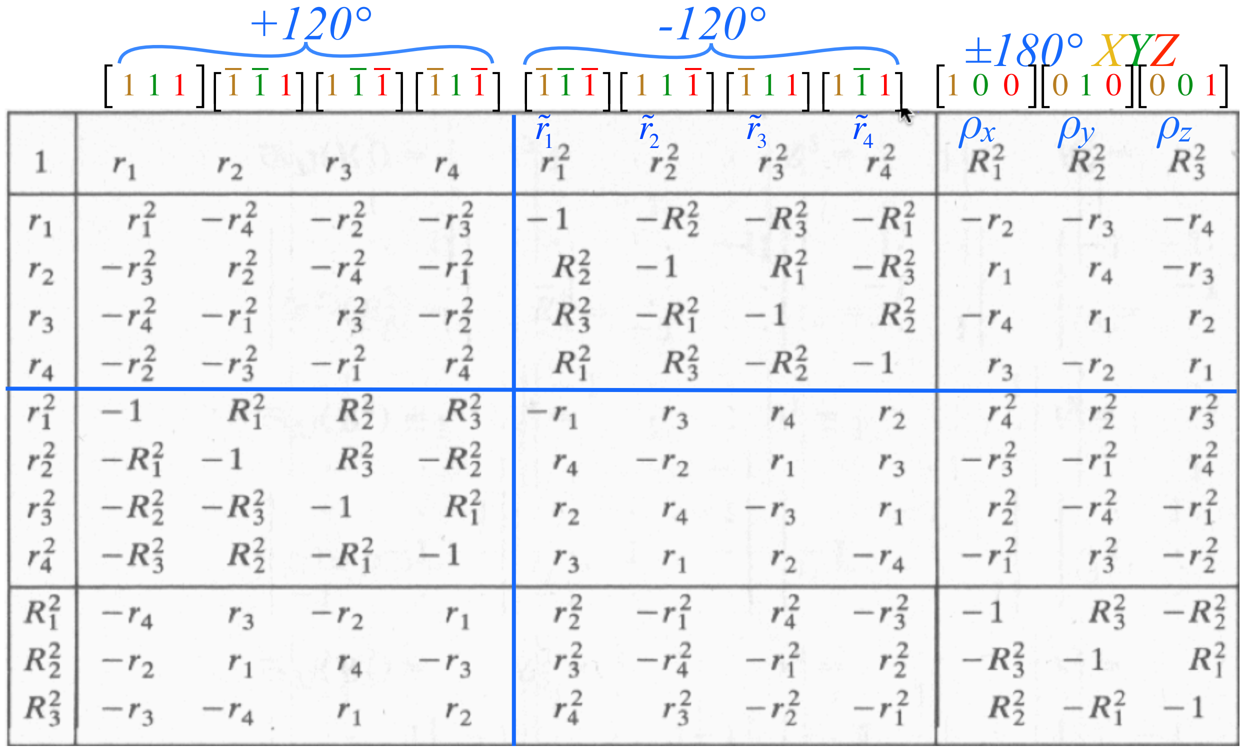
$$\mathbf{c}_\rho^2 = 3\cdot\mathbf{1} + 2\mathbf{c}_\rho$$

$$\mathbf{c}_\rho^2 - 2\mathbf{c}_\rho - 3\cdot\mathbf{1} = \mathbf{0}$$

Tetrahedral T class algebra

$\mathbf{c}_1=\mathbf{1}$, $\mathbf{c}_r=\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4$, $\mathbf{c}^\dagger{}_r=\mathbf{r}_1{}^2+\mathbf{r}_2{}^2+\mathbf{r}_3{}^2+\mathbf{r}_4{}^2$, $\mathbf{c}_\rho=\mathbf{R}_1{}^2+\mathbf{R}_2{}^2+\mathbf{R}_3{}^2$,

T group products



	+120°				-120°				±180° XYZ		
	$[1\ 1\ 1]$	$[\bar{1}\ \bar{1}\ 1]$	$[1\ \bar{1}\ \bar{1}]$	$[\bar{1}\ 1\ \bar{1}]$	$[\bar{1}\ \bar{1}\ \bar{1}]$	$[1\ 1\ \bar{1}]$	$[\bar{1}\ 1\ 1]$	$[1\ \bar{1}\ 1]$	$[1\ 0\ 0]$	$[0\ 1\ 0]$	$[0\ 0\ 1]$
					$\tilde{r}_1$	$\tilde{r}_2$	$\tilde{r}_3$	$\tilde{r}_4$	ρ_x	ρ_y	ρ_z
1	r_1	r_2	r_3	r_4	r_1^2	r_2^2	r_3^2	r_4^2	R_1^2	R_2^2	R_3^2
r_1	r_1^2	$-r_4^2$	$-r_2^2$	$-r_3^2$	-1	$-R_2^2$	$-R_3^2$	$-R_1^2$	$-r_2$	$-r_3$	$-r_4$
r_2	$-r_3^2$	r_2^2	$-r_4^2$	$-r_1^2$	R_2^2	-1	R_1^2	$-R_3^2$	r_1	r_4	$-r_3$
r_3	$-r_4^2$	$-r_1^2$	r_3^2	$-r_2^2$	R_3^2	$-R_1^2$	-1	R_2^2	$-r_4$	r_1	r_2
r_4	$-r_2^2$	$-r_3^2$	$-r_1^2$	r_4^2	R_1^2	R_3^2	$-R_2^2$	-1	r_3	$-r_2$	r_1
r_1^2	-1	R_1^2	R_2^2	R_3^2	$-r_1$	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2
r_2^2	$-R_1^2$	-1	R_3^2	$-R_2^2$	r_4	$-r_2$	r_1	r_3	$-r_3^2$	$-r_1^2$	r_4^2
r_3^2	$-R_2^2$	$-R_3^2$	-1	R_1^2	r_2	r_4	$-r_3$	r_1	r_2^2	$-r_4^2$	$-r_1^2$
r_4^2	$-R_3^2$	R_2^2	$-R_1^2$	-1	r_3	r_1	r_2	$-r_4$	$-r_1^2$	r_3^2	$-r_2^2$
R_1^2	$-r_4$	r_3	$-r_2$	r_1	r_2^2	$-r_1^2$	r_4^2	$-r_3^2$	-1	R_3^2	$-R_2^2$
R_2^2	$-r_2$	r_1	r_4	$-r_3$	r_3^2	$-r_4^2$	$-r_1^2$	r_2^2	$-R_3^2$	-1	R_1^2
R_3^2	$-r_3$	$-r_4$	r_1	r_2	r_4^2	r_3^2	$-r_2^2$	$-r_1^2$	R_2^2	$-R_1^2$	-1

Minus (-)signs
for Fermions

(Ignore (-) for Bosons or
classical particles)

T class products

$\mathbf{1}=\mathbf{c}_1$	$\mathbf{c}_r$	$\mathbf{c}_r^\dagger$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\mathbf{c}_r^\dagger$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\mathbf{c}_r^\dagger$		$4\mathbf{c}_r$	$3\mathbf{c}_r^\dagger$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_\rho$

$$\mathbf{c}_\rho^2 = 3\cdot\mathbf{1} + 2\mathbf{c}_\rho$$

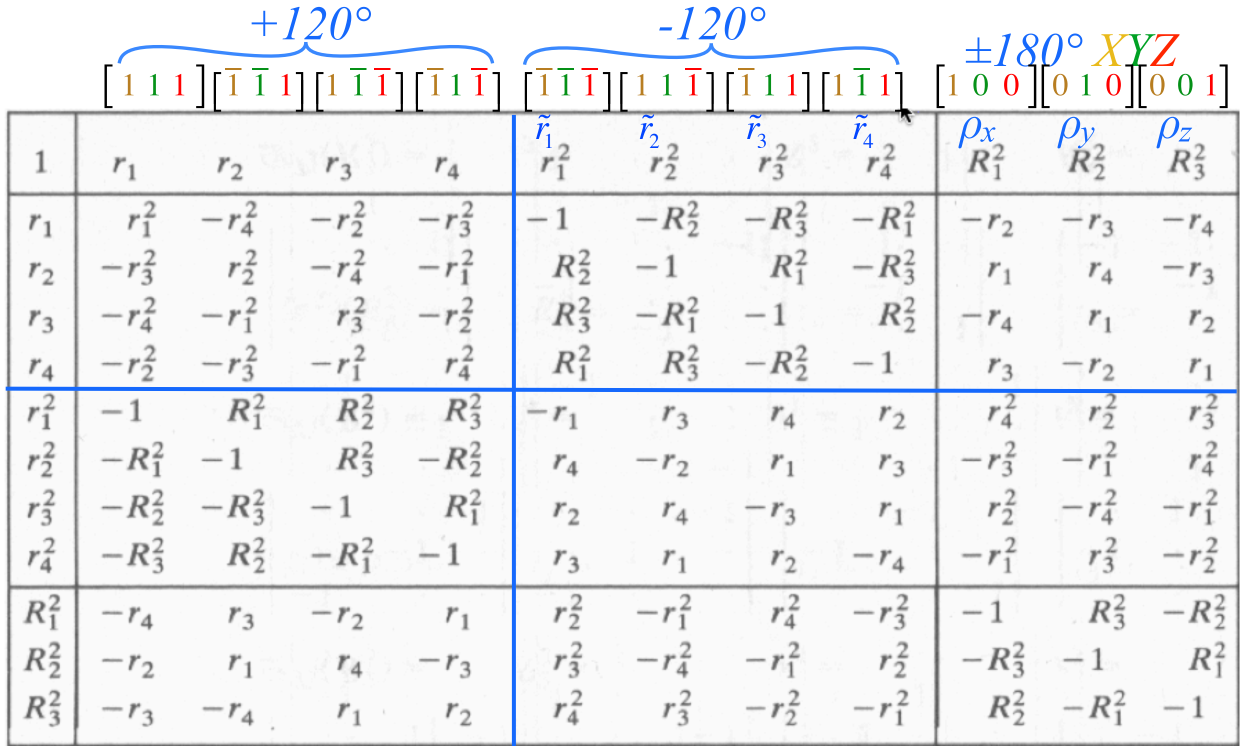
$$\mathbf{c}_\rho^2 - 2\mathbf{c}_\rho - 3\cdot\mathbf{1} = \mathbf{0}$$

$$(\mathbf{c}_\rho - 3\cdot\mathbf{1})(\mathbf{c}_\rho + \mathbf{1}) = \mathbf{0}$$

Tetrahedral T class algebra

$\mathbf{c}_1=\mathbf{1}$, $\mathbf{c}_r=\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4$, $\mathbf{c}^\dagger{}_r=\mathbf{r}_1{}^2+\mathbf{r}_2{}^2+\mathbf{r}_3{}^2+\mathbf{r}_4{}^2$, $\mathbf{c}_\rho=\mathbf{R}_1{}^2+\mathbf{R}_2{}^2+\mathbf{R}_3{}^2$,

T group products



+120° -120° ±180° XYZ

1	$[1\,1\,1]$ r_1	$[\bar1\,\bar1\,1]$ r_2	$[1\,\bar1\,\bar1]$ r_3	$[\bar1\,1\,\bar1]$ r_4	$[\bar1\,\bar1\,\bar1]$ $\tilde r_1$ r_1^2	$[1\,1\,\bar1]$ $\tilde r_2$ r_2^2	$[\bar1\,1\,1]$ $\tilde r_3$ r_3^2	$[1\,\bar1\,1]$ $\tilde r_4$ r_4^2	$[1\,0\,0]$ ρ_x R_1^2	$[0\,1\,0]$ ρ_y R_2^2	$[0\,0\,1]$ ρ_z R_3^2
r_1	r_1^2	$-r_4^2$	$-r_2^2$	$-r_3^2$	-1	$-R_2^2$	$-R_3^2$	$-R_1^2$	$-r_2$	$-r_3$	$-r_4$
r_2	$-r_3^2$	r_2^2	$-r_4^2$	$-r_1^2$	R_2^2	-1	R_1^2	$-R_3^2$	r_1	r_4	$-r_3$
r_3	$-r_4^2$	$-r_1^2$	r_3^2	$-r_2^2$	R_3^2	$-R_1^2$	-1	R_2^2	$-r_4$	r_1	r_2
r_4	$-r_2^2$	$-r_3^2$	$-r_1^2$	r_4^2	R_1^2	R_3^2	$-R_2^2$	-1	r_3	$-r_2$	r_1
r_1^2	-1	R_1^2	R_2^2	R_3^2	$-r_1$	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2
r_2^2	$-R_1^2$	-1	R_3^2	$-R_2^2$	r_4	$-r_2$	r_1	r_3	$-r_3^2$	$-r_1^2$	r_4^2
r_3^2	$-R_2^2$	$-R_3^2$	-1	R_1^2	r_2	r_4	$-r_3$	r_1	r_2^2	$-r_4^2$	$-r_1^2$
r_4^2	$-R_3^2$	R_2^2	$-R_1^2$	-1	r_3	r_1	r_2	$-r_4$	$-r_1^2$	r_3^2	$-r_2^2$
R_1^2	$-r_4$	r_3	$-r_2$	r_1	r_2^2	$-r_1^2$	r_4^2	$-r_3^2$	-1	R_3^2	$-R_2^2$
R_2^2	$-r_2$	r_1	r_4	$-r_3$	r_3^2	$-r_4^2$	$-r_1^2$	r_2^2	$-R_3^2$	-1	R_1^2
R_3^2	$-r_3$	$-r_4$	r_1	r_2	r_4^2	r_3^2	$-r_2^2$	$-r_1^2$	R_2^2	$-R_1^2$	-1

Minus (-)signs for Fermions

(Ignore (-) for Bosons or classical particles)

T class products

$\mathbf{1}=\mathbf{c}_1$	$\mathbf{c}_r$	$\mathbf{c}_r^\dagger$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\mathbf{c}_r^\dagger$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\mathbf{c}_r^\dagger$		$4\mathbf{c}_r$	$3\mathbf{c}_r^\dagger$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_r$

$$\mathbf{c}_r^2=4\tilde{\mathbf{c}}_r$$

Minimal equation for $\mathbf{c}_\rho$

$$\mathbf{c}_\rho^2=3\cdot\mathbf{1}+2\mathbf{c}_\rho$$

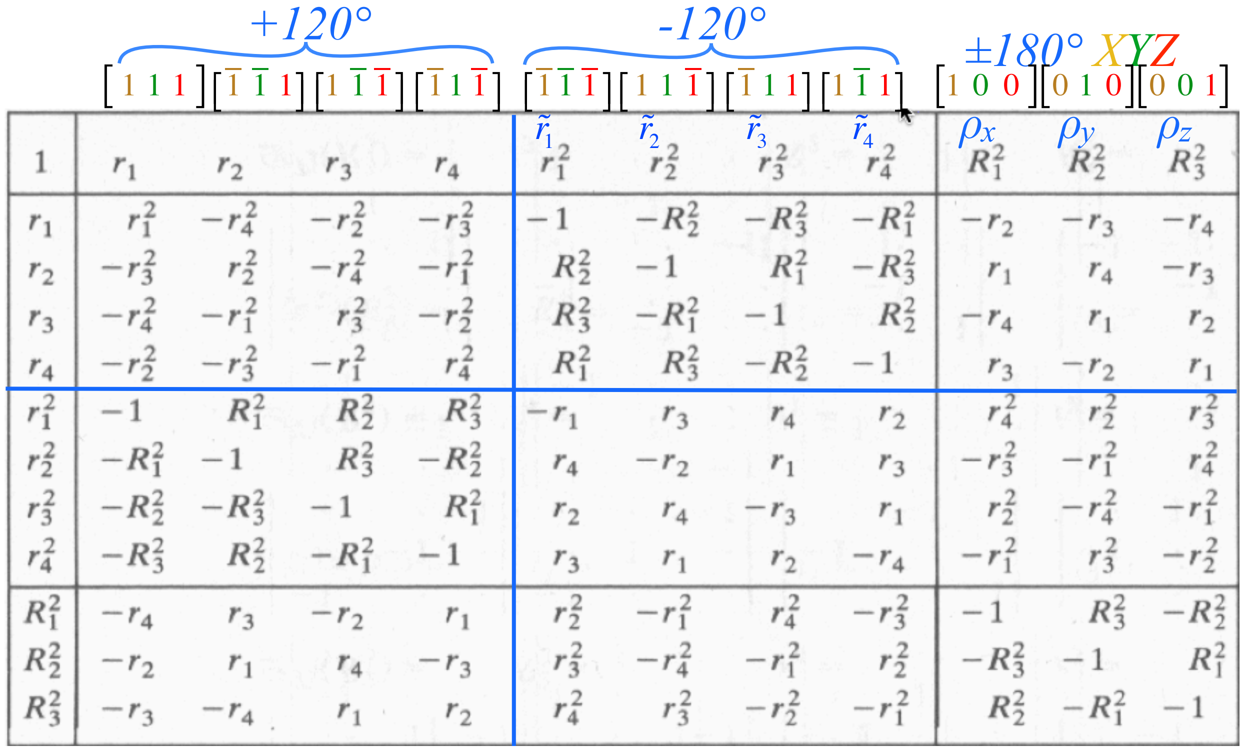
$$\mathbf{c}_\rho^2-2\mathbf{c}_\rho-3\cdot\mathbf{1}=\mathbf{0}$$

$$(\mathbf{c}_\rho-3\cdot\mathbf{1})(\mathbf{c}_\rho+\mathbf{1})=\mathbf{0}$$

Tetrahedral T class algebra

$\mathbf{c}_1=\mathbf{1}$, $\mathbf{c}_r=\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4$, $\mathbf{c}^\dagger{}_r=\mathbf{r}_1{}^2+\mathbf{r}_2{}^2+\mathbf{r}_3{}^2+\mathbf{r}_4{}^2$, $\mathbf{c}_\rho=\mathbf{R}_1{}^2+\mathbf{R}_2{}^2+\mathbf{R}_3{}^2$,

T group products



	+120°				-120°				±180° XYZ		
	$[1\,1\,1]$	$[\bar{1}\,\bar{1}\,1]$	$[1\,\bar{1}\,\bar{1}]$	$[\bar{1}\,1\,\bar{1}]$	$[\bar{1}\,\bar{1}\,\bar{1}]$	$[1\,1\,\bar{1}]$	$[\bar{1}\,1\,1]$	$[1\,\bar{1}\,1]$	$[1\,0\,0]$	$[0\,1\,0]$	$[0\,0\,1]$
					$\tilde{r}_1$	$\tilde{r}_2$	$\tilde{r}_3$	$\tilde{r}_4$	ρ_x	ρ_y	ρ_z
1	r_1	r_2	r_3	r_4	r_1^2	r_2^2	r_3^2	r_4^2	R_1^2	R_2^2	R_3^2
r_1	r_1^2	$-r_4^2$	$-r_2^2$	$-r_3^2$	-1	$-R_2^2$	$-R_3^2$	$-R_1^2$	$-r_2$	$-r_3$	$-r_4$
r_2	$-r_3^2$	r_2^2	$-r_4^2$	$-r_1^2$	R_2^2	-1	R_1^2	$-R_3^2$	r_1	r_4	$-r_3$
r_3	$-r_4^2$	$-r_1^2$	r_3^2	$-r_2^2$	R_3^2	$-R_1^2$	-1	R_2^2	$-r_4$	r_1	r_2
r_4	$-r_2^2$	$-r_3^2$	$-r_1^2$	r_4^2	R_1^2	R_3^2	$-R_2^2$	-1	r_3	$-r_2$	r_1
r_1^2	-1	R_1^2	R_2^2	R_3^2	$-r_1$	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2
r_2^2	$-R_1^2$	-1	R_3^2	$-R_2^2$	r_4	$-r_2$	r_1	r_3	$-r_3^2$	$-r_1^2$	r_4^2
r_3^2	$-R_2^2$	$-R_3^2$	-1	R_1^2	r_2	r_4	$-r_3$	r_1	r_2^2	$-r_4^2$	$-r_1^2$
r_4^2	$-R_3^2$	R_2^2	$-R_1^2$	-1	r_3	r_1	r_2	$-r_4$	$-r_1^2$	r_3^2	$-r_2^2$
R_1^2	$-r_4$	r_3	$-r_2$	r_1	r_2^2	$-r_1^2$	r_4^2	$-r_3^2$	-1	R_3^2	$-R_2^2$
R_2^2	$-r_2$	r_1	r_4	$-r_3$	r_3^2	$-r_4^2$	$-r_1^2$	r_2^2	$-R_3^2$	-1	R_1^2
R_3^2	$-r_3$	$-r_4$	r_1	r_2	r_4^2	r_3^2	$-r_2^2$	$-r_1^2$	R_2^2	$-R_1^2$	-1

Minus (-)signs
for Fermions

(Ignore (-) for Bosons or
classical particles)

T class products

$\mathbf{1}=\mathbf{c}_1$	$\mathbf{c}_r$	$\mathbf{c}_r^\dagger$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\mathbf{c}_r^\dagger$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\mathbf{c}_r^\dagger$		$4\mathbf{c}_r$	$3\mathbf{c}_r^\dagger$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_r$

$$\mathbf{c}_r^2=4\tilde{\mathbf{c}}_r$$

$$\mathbf{c}_r^3=4\tilde{\mathbf{c}}_r\mathbf{c}_r=4(4\cdot\mathbf{1}+4\mathbf{c}_\rho)=16\cdot\mathbf{1}+16\mathbf{c}_\rho$$

Minimal equation for $\mathbf{c}_\rho$

$$\mathbf{c}_\rho^2=3\cdot\mathbf{1}+2\mathbf{c}_\rho$$

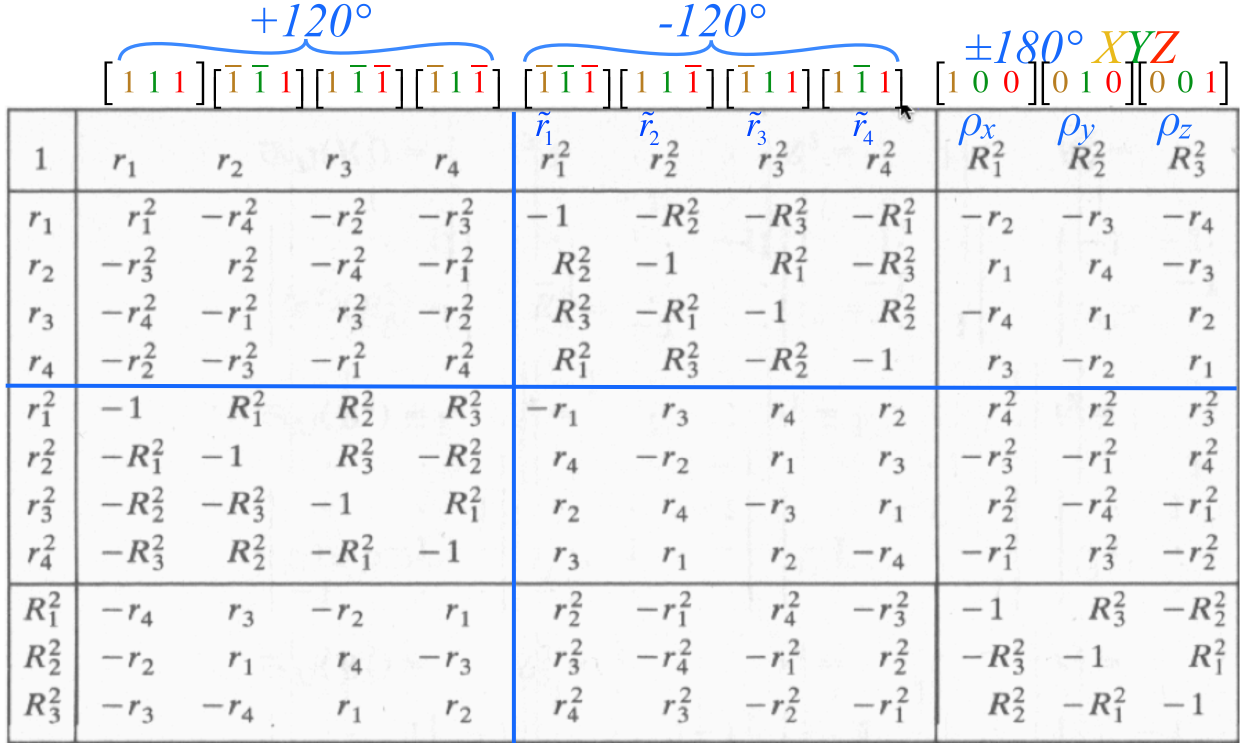
$$\mathbf{c}_\rho^2-2\mathbf{c}_\rho-3\cdot\mathbf{1}=\mathbf{0}$$

$$(\mathbf{c}_\rho-3\cdot\mathbf{1})(\mathbf{c}_\rho+\mathbf{1})=\mathbf{0}$$

Tetrahedral T class algebra

$\mathbf{c}_1=\mathbf{1}$, $\mathbf{c}_r=\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4$, $\mathbf{c}^\dagger_r=\mathbf{r}_1^2+\mathbf{r}_2^2+\mathbf{r}_3^2+\mathbf{r}_4^2$, $\mathbf{c}_\rho=\mathbf{R}_1^2+\mathbf{R}_2^2+\mathbf{R}_3^2$,

T group products



+120° *-120°* *±180° XYZ*

1	$[1\,1\,1]$ r_1	$[\bar1\,\bar1\,1]$ r_2	$[1\,\bar1\,\bar1]$ r_3	$[\bar1\,1\,\bar1]$ r_4	$[\bar1\,\bar1\,\bar1]$ $\tilde r_1$ r_1^2	$[1\,1\,\bar1]$ $\tilde r_2$ r_2^2	$[\bar1\,1\,1]$ $\tilde r_3$ r_3^2	$[1\,\bar1\,1]$ $\tilde r_4$ r_4^2	$[1\,0\,0]$ ρ_x R_1^2	$[0\,1\,0]$ ρ_y R_2^2	$[0\,0\,1]$ ρ_z R_3^2
r_1	r_1^2	$-r_4^2$	$-r_2^2$	$-r_3^2$	-1	$-R_2^2$	$-R_3^2$	$-R_1^2$	$-r_2$	$-r_3$	$-r_4$
r_2	$-r_3^2$	r_2^2	$-r_4^2$	$-r_1^2$	R_2^2	-1	R_1^2	$-R_3^2$	r_1	r_4	$-r_3$
r_3	$-r_4^2$	$-r_1^2$	r_3^2	$-r_2^2$	R_3^2	$-R_1^2$	-1	R_2^2	$-r_4$	r_1	r_2
r_4	$-r_2^2$	$-r_3^2$	$-r_1^2$	r_4^2	R_1^2	R_3^2	$-R_2^2$	-1	r_3	$-r_2$	r_1
r_1^2	-1	R_1^2	R_2^2	R_3^2	$-r_1$	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2
r_2^2	$-R_1^2$	-1	R_3^2	$-R_2^2$	r_4	$-r_2$	r_1	r_3	$-r_3^2$	$-r_1^2$	r_4^2
r_3^2	$-R_2^2$	$-R_3^2$	-1	R_1^2	r_2	r_4	$-r_3$	r_1	r_2^2	$-r_4^2$	$-r_1^2$
r_4^2	$-R_3^2$	R_2^2	$-R_1^2$	-1	r_3	r_1	r_2	$-r_4$	$-r_1^2$	r_3^2	$-r_2^2$
R_1^2	$-r_4$	r_3	$-r_2$	r_1	r_2^2	$-r_1^2$	r_4^2	$-r_3^2$	-1	R_3^2	$-R_2^2$
R_2^2	$-r_2$	r_1	r_4	$-r_3$	r_3^2	$-r_4^2$	$-r_1^2$	r_2^2	$-R_3^2$	-1	R_1^2
R_3^2	$-r_3$	$-r_4$	r_1	r_2	r_4^2	r_3^2	$-r_2^2$	$-r_1^2$	R_2^2	$-R_1^2$	-1

Minus (-)signs for Fermions

(Ignore (-) for Bosons or classical particles)

T class products

$\mathbf{1}=\mathbf{c}_1$	$\mathbf{c}_r$	$\mathbf{c}_r^\dagger$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\mathbf{c}_r^\dagger$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\mathbf{c}_r^\dagger$		$4\mathbf{c}_r$	$3\mathbf{c}_r^\dagger$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_r$

$$\mathbf{c}_r^2 = 4\tilde{\mathbf{c}}_r$$

$$\mathbf{c}_r^3 = 4\tilde{\mathbf{c}}_r\mathbf{c}_r = 4(4\cdot\mathbf{1}+4\mathbf{c}_\rho) = 16\cdot\mathbf{1}+16\mathbf{c}_\rho$$

$$\mathbf{c}_r^4 = 16\cdot\mathbf{1}\mathbf{c}_r+16\mathbf{c}_\rho\mathbf{c}_r = 16\cdot\mathbf{1}\mathbf{c}_r+16(3\mathbf{c}_r)$$

Minimal equation for $\mathbf{c}_\rho$

$$\mathbf{c}_\rho^2 = 3\cdot\mathbf{1}+2\mathbf{c}_\rho$$

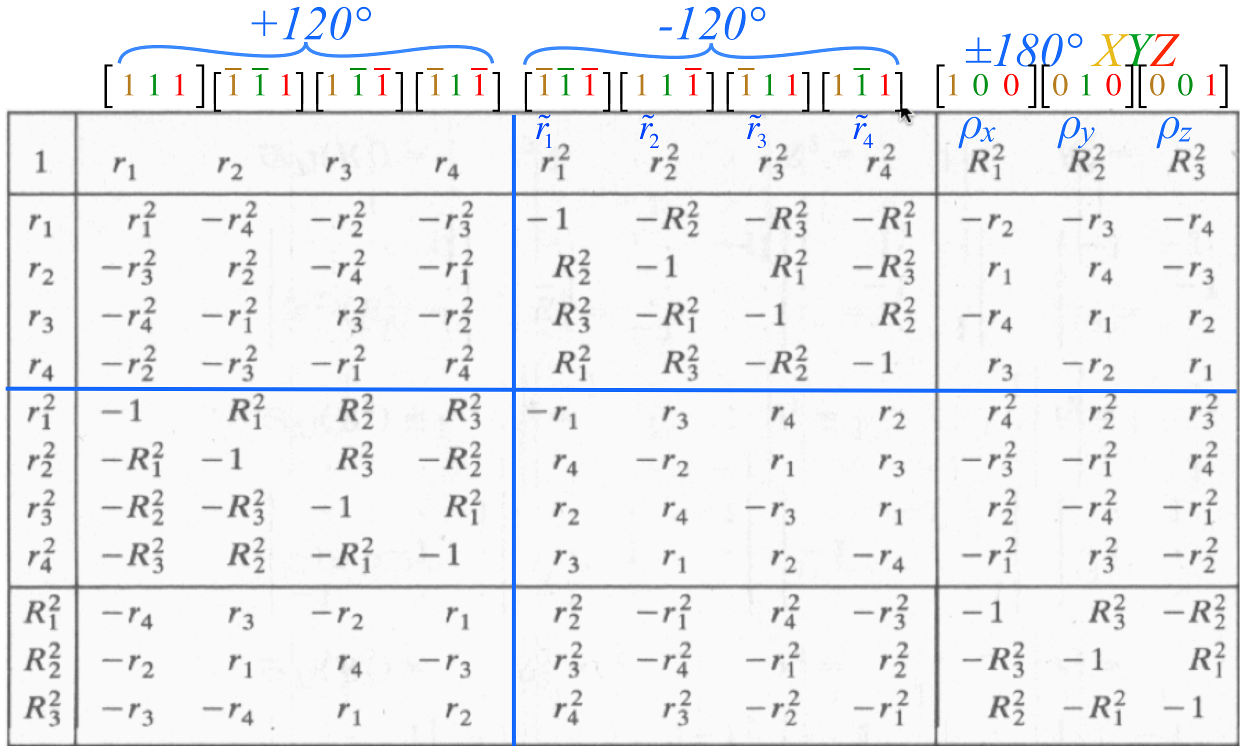
$$\mathbf{c}_\rho^2-2\mathbf{c}_\rho-3\cdot\mathbf{1}=\mathbf{0}$$

$$(\mathbf{c}_\rho-3\cdot\mathbf{1})(\mathbf{c}_\rho+\mathbf{1})=\mathbf{0}$$

Tetrahedral T class algebra

$\mathbf{c}_1=\mathbf{1}$, $\mathbf{c}_r=\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4$, $\mathbf{c}^\dagger{}_r=\mathbf{r}_1{}^2+\mathbf{r}_2{}^2+\mathbf{r}_3{}^2+\mathbf{r}_4{}^2$, $\mathbf{c}_\rho=\mathbf{R}_1{}^2+\mathbf{R}_2{}^2+\mathbf{R}_3{}^2$,

T group products



	+120°				-120°				±180° XYZ		
	$[1\ 1\ 1]$	$[\bar{1}\ \bar{1}\ 1]$	$[1\ \bar{1}\ \bar{1}]$	$[\bar{1}\ 1\ \bar{1}]$	$[\bar{1}\ \bar{1}\ \bar{1}]$	$[1\ 1\ \bar{1}]$	$[\bar{1}\ 1\ 1]$	$[1\ \bar{1}\ 1]$	$[1\ 0\ 0]$	$[0\ 1\ 0]$	$[0\ 0\ 1]$
1	r_1	r_2	r_3	r_4	$\tilde{r}_1$ r_1^2	$\tilde{r}_2$ r_2^2	$\tilde{r}_3$ r_3^2	$\tilde{r}_4$ r_4^2	ρ_x R_1^2	ρ_y R_2^2	ρ_z R_3^2
r_1	r_1^2	$-r_4^2$	$-r_2^2$	$-r_3^2$	-1	$-R_2^2$	$-R_3^2$	$-R_1^2$	$-r_2$	$-r_3$	$-r_4$
r_2	$-r_3^2$	r_2^2	$-r_4^2$	$-r_1^2$	R_2^2	-1	R_1^2	$-R_3^2$	r_1	r_4	$-r_3$
r_3	$-r_4^2$	$-r_1^2$	r_3^2	$-r_2^2$	R_3^2	$-R_1^2$	-1	R_2^2	$-r_4$	r_1	r_2
r_4	$-r_2^2$	$-r_3^2$	$-r_1^2$	r_4^2	R_1^2	R_3^2	$-R_2^2$	-1	r_3	$-r_2$	r_1
r_1^2	-1	R_1^2	R_2^2	R_3^2	$-r_1$	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2
r_2^2	$-R_1^2$	-1	R_3^2	$-R_2^2$	r_4	$-r_2$	r_1	r_3	$-r_3^2$	$-r_1^2$	r_4^2
r_3^2	$-R_2^2$	$-R_3^2$	-1	R_1^2	r_2	r_4	$-r_3$	r_1	r_2^2	$-r_4^2$	$-r_1^2$
r_4^2	$-R_3^2$	R_2^2	$-R_1^2$	-1	r_3	r_1	r_2	$-r_4$	$-r_1^2$	r_3^2	$-r_2^2$
R_1^2	$-r_4$	r_3	$-r_2$	r_1	r_2^2	$-r_1^2$	r_4^2	$-r_3^2$	-1	R_3^2	$-R_2^2$
R_2^2	$-r_2$	r_1	r_4	$-r_3$	r_3^2	$-r_4^2$	$-r_1^2$	r_2^2	$-R_3^2$	-1	R_1^2
R_3^2	$-r_3$	$-r_4$	r_1	r_2	r_4^2	r_3^2	$-r_2^2$	$-r_1^2$	R_2^2	$-R_1^2$	-1

Minus (-)signs
for Fermions

(Ignore (-) for Bosons or classical particles)

T class products

$\mathbf{1}=\mathbf{c}_1$	$\mathbf{c}_r$	$\mathbf{c}_r^\dagger$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\mathbf{c}_r^\dagger$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\mathbf{c}_r^\dagger$		$4\mathbf{c}_r$	$3\mathbf{c}_r^\dagger$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_r$

$$\mathbf{c}_r^2=4\tilde{\mathbf{c}}_r$$

$$\mathbf{c}_r^3=4\tilde{\mathbf{c}}_r\mathbf{c}_r=4(4\cdot\mathbf{1}+4\mathbf{c}_\rho)=16\cdot\mathbf{1}+16\mathbf{c}_\rho$$

$$\mathbf{c}_r^4=16\cdot\mathbf{1}\mathbf{c}_r+16\mathbf{c}_\rho\mathbf{c}_r=16\cdot\mathbf{1}\mathbf{c}_r+16(3\mathbf{c}_r)$$

$$\mathbf{c}_r^4-64\mathbf{c}_r=(\mathbf{c}_r^3-64\cdot\mathbf{1})\mathbf{c}_r=\mathbf{0}$$

Minimal equation for $\mathbf{c}_\rho$

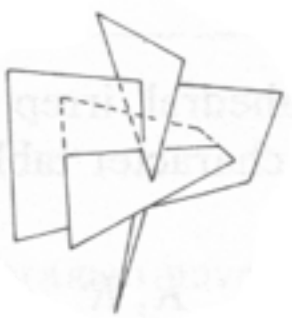
$$\mathbf{c}_\rho^2=3\cdot\mathbf{1}+2\mathbf{c}_\rho$$

$$\mathbf{c}_\rho^2-2\mathbf{c}_\rho-3\cdot\mathbf{1}=\mathbf{0}$$

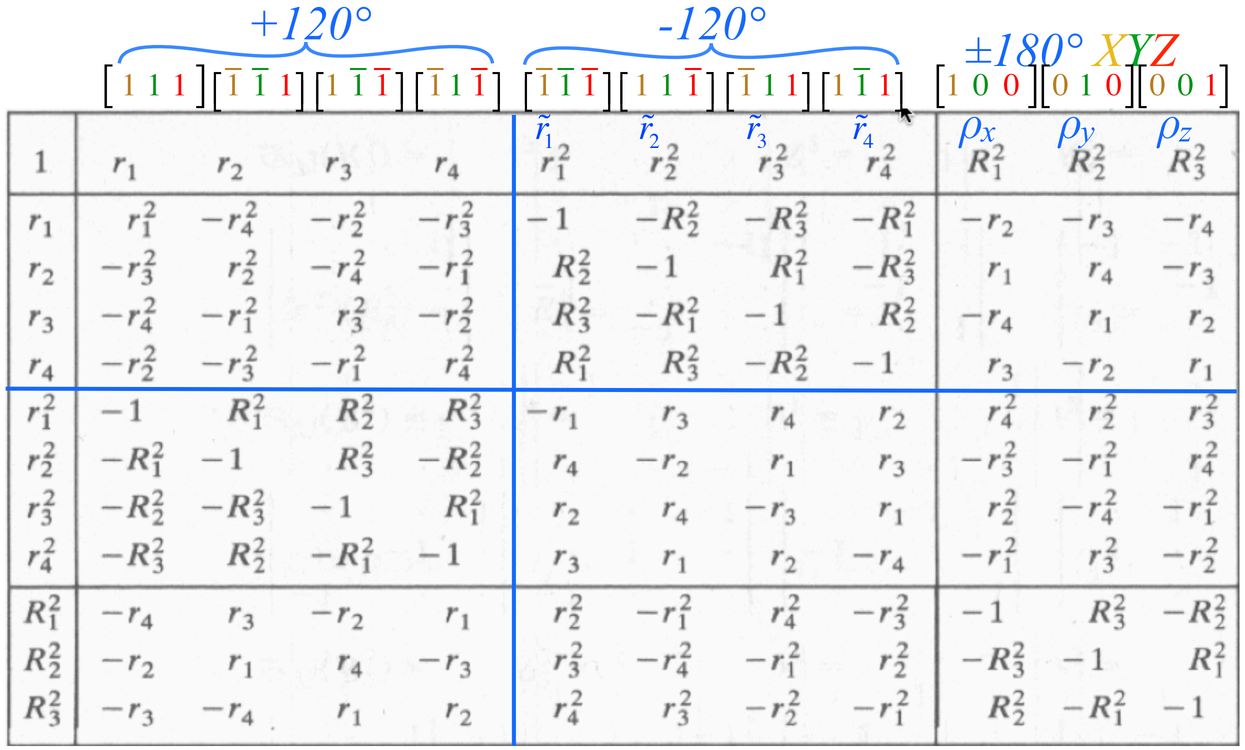
$$(\mathbf{c}_\rho-3\cdot\mathbf{1})(\mathbf{c}_\rho+\mathbf{1})=\mathbf{0}$$

Tetrahedral T class algebra

$\mathbf{c}_1=\mathbf{1}$, $\mathbf{c}_r=\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4$, $\mathbf{c}^\dagger{}_r=\mathbf{r}_1{}^2+\mathbf{r}_2{}^2+\mathbf{r}_3{}^2+\mathbf{r}_4{}^2$, $\mathbf{c}_\rho=\mathbf{R}_1{}^2+\mathbf{R}_2{}^2+\mathbf{R}_3{}^2$,



T group products



+120° -120° ±180° XYZ

	$[1\,1\,1]$	$[\bar1\,\bar1\,1]$	$[1\,\bar1\,\bar1]$	$[\bar1\,1\,\bar1]$	$[\bar1\,\bar1\,\bar1]$	$[1\,1\,\bar1]$	$[\bar1\,1\,1]$	$[1\,\bar1\,1]$	$[1\,0\,0]$	$[0\,1\,0]$	$[0\,0\,1]$
1	r_1	r_2	r_3	r_4	$\tilde r_1$ r_1^2	$\tilde r_2$ r_2^2	$\tilde r_3$ r_3^2	$\tilde r_4$ r_4^2	ρ_x R_1^2	ρ_y R_2^2	ρ_z R_3^2
r_1	r_1^2	$-r_4^2$	$-r_2^2$	$-r_3^2$	-1	$-R_2^2$	$-R_3^2$	$-R_1^2$	$-r_2$	$-r_3$	$-r_4$
r_2	$-r_3^2$	r_2^2	$-r_4^2$	$-r_1^2$	R_2^2	-1	R_1^2	$-R_3^2$	r_1	r_4	$-r_3$
r_3	$-r_4^2$	$-r_1^2$	r_3^2	$-r_2^2$	R_3^2	$-R_1^2$	-1	R_2^2	$-r_4$	r_1	r_2
r_4	$-r_2^2$	$-r_3^2$	$-r_1^2$	r_4^2	R_1^2	R_3^2	$-R_2^2$	-1	r_3	$-r_2$	r_1
r_1^2	-1	R_1^2	R_2^2	R_3^2	$-r_1$	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2
r_2^2	$-R_1^2$	-1	R_3^2	$-R_2^2$	r_4	$-r_2$	r_1	r_3	$-r_3^2$	$-r_1^2$	r_4^2
r_3^2	$-R_2^2$	$-R_3^2$	-1	R_1^2	r_2	r_4	$-r_3$	r_1	r_2^2	$-r_4^2$	$-r_1^2$
r_4^2	$-R_3^2$	R_2^2	$-R_1^2$	-1	r_3	r_1	r_2	$-r_4$	$-r_1^2$	r_3^2	$-r_2^2$
R_1^2	$-r_4$	r_3	$-r_2$	r_1	r_2^2	$-r_1^2$	r_4^2	$-r_3^2$	-1	R_3^2	$-R_2^2$
R_2^2	$-r_2$	r_1	r_4	$-r_3$	r_3^2	$-r_4^2$	$-r_1^2$	r_2^2	$-R_3^2$	-1	R_1^2
R_3^2	$-r_3$	$-r_4$	r_1	r_2	r_4^2	r_3^2	$-r_2^2$	$-r_1^2$	R_2^2	$-R_1^2$	-1

Minus (-)signs for Fermions

(Ignore (-) for Bosons or classical particles)

T class products

$\mathbf{1}=\mathbf{c}_1$	$\mathbf{c}_r$	$\mathbf{c}_r^\dagger$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\mathbf{c}_r^\dagger$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\mathbf{c}_r^\dagger$		$4\mathbf{c}_r$	$3\mathbf{c}_r^\dagger$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_r$

$$\mathbf{c}_r^2 = 4\tilde{\mathbf{c}}_r$$

$$\mathbf{c}_r^3 = 4\tilde{\mathbf{c}}_r\mathbf{c}_r = 4(4\cdot\mathbf{1}+4\mathbf{c}_\rho) = 16\cdot\mathbf{1}+16\mathbf{c}_\rho$$

$$\mathbf{c}_r^4 = 16\cdot\mathbf{1}\mathbf{c}_r + 16\mathbf{c}_\rho\mathbf{c}_r = 16\cdot\mathbf{1}\mathbf{c}_r + 16(3\mathbf{c}_r)$$

$$\mathbf{c}_r^4 - 64\mathbf{c}_r = (\mathbf{c}_r^3 - 64\cdot\mathbf{1})\mathbf{c}_r = \mathbf{0}$$

$$\boxed{(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}}$$

Minimal equation for $\mathbf{c}_\rho$

$$\mathbf{c}_\rho^2 = 3\cdot\mathbf{1} + 2\mathbf{c}_\rho$$

$$\mathbf{c}_\rho^2 - 2\mathbf{c}_\rho - 3\cdot\mathbf{1} = \mathbf{0}$$

$$(\mathbf{c}_\rho - 3\cdot\mathbf{1})(\mathbf{c}_\rho + \mathbf{1}) = \mathbf{0}$$

Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$
- Octahedral-cubic O symmetry and group operations
- Tetrahedral symmetry becomes Icosahedral
- Octahedral groups $O_h \supset O \sim T_d \supset T$
  - Octahedral O and spin-$O \subset U(2)$

Tetrahedral T class algebra
- Tetrahedral T class minimal equations
- Tetrahedral T class projectors and characters

Octahedral O class algebra
- Octahedral O class minimal equations
- Octahedral O class projectors and characters

Octahedral $O_h \supset O$ subgroup correlations
- Octahedral $O_h \supset O$ subgroup correlations
- $O_h \supset O \supset D_4$ subgroup correlations
- $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations
- Preview of applications to high resolution spectroscopy

Tetrahedral T class projectors

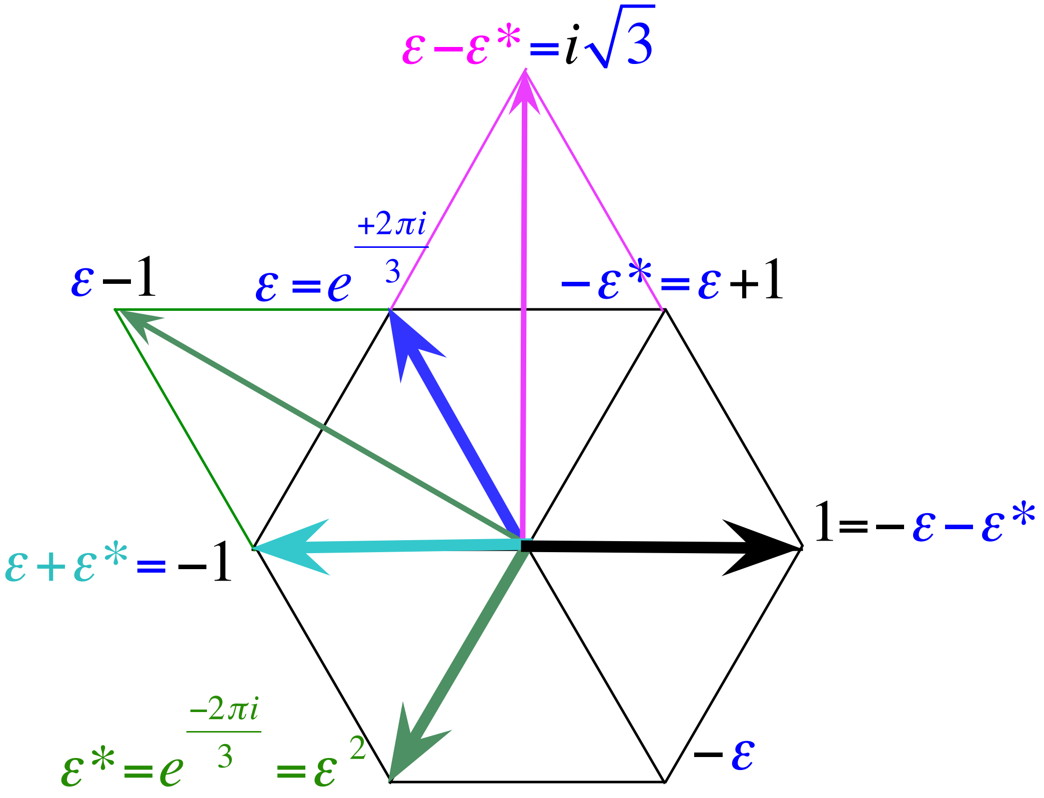
$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_{g}\chi_{g}^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_{g}^{\mu *} \mathbf{c_g}}$$

Minimal equation for $\mathbf{c}_r : (\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4\varepsilon^* \mathbf{1})(\mathbf{c}_r - 4 \mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}$

T class products

$\mathbf{1} = \mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$



Minimal equation for $\mathbf{c}_r$

$$\boxed{(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}}$$

Tetrahedral T class projectors

$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_{g}\chi^{\mu}_{g}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G}\chi^{\mu *}_{g}\mathbf{c_g}}$$

Minimal equation for $\mathbf{c}_r : (\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}$

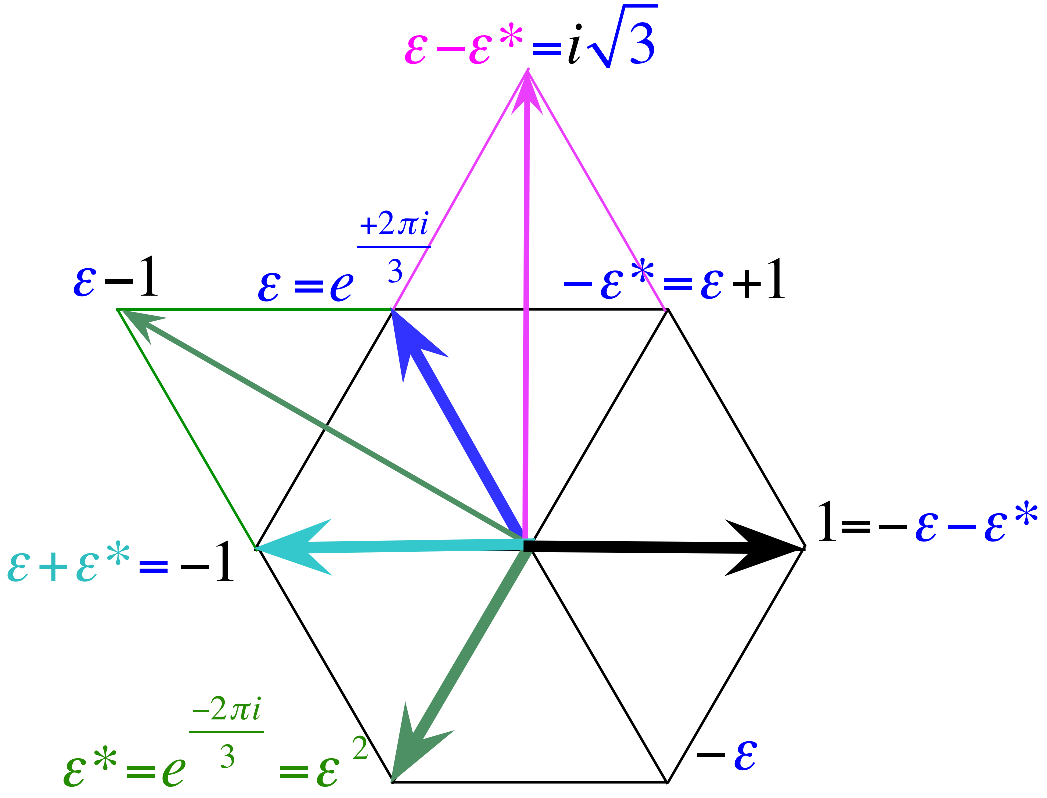
$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)}$$

T class products

$\mathbf{1} = \mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_r$

$$\boxed{(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}}$$



Tetrahedral T class projectors

$$\boxed{\mathbf{c}_\mathbf{g} = \sum_\mu \frac{{}^\circ c_\mathbf{g}\,\chi^\mu_\mathbf{g}}{\ell^\mu}\,\mathbb{P}^\mu}$$

$$\boxed{\mathbb{P}^\mu = \sum_{classes\,c_\mathbf{g}} \frac{\ell^\mu}{{}^\circ G}\chi^{\mu*}_g\,\mathbf{c}_\mathbf{g}}$$

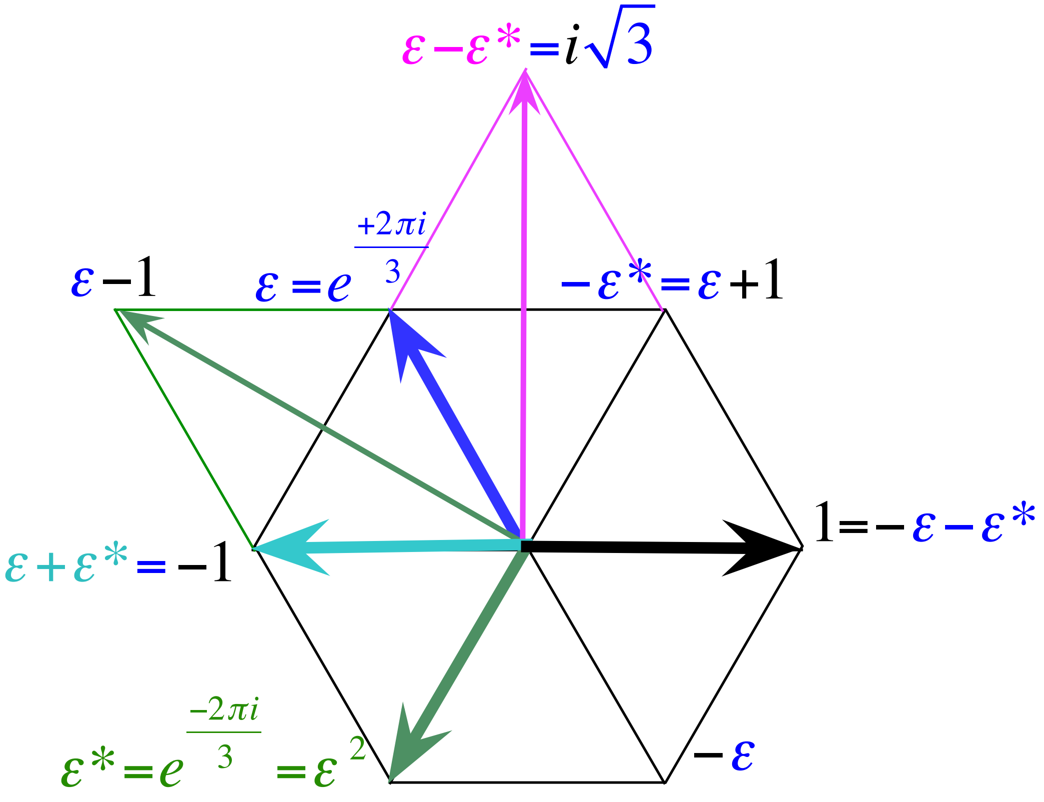
T class products

$\mathbf{1} = \mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_r$: $(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)}$$



Minimal equation for $\mathbf{c}_r$

$$\boxed{(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}}$$

Tetrahedral T class projectors

$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_g^{\mu *} \mathbf{c_g}}$$

T class products

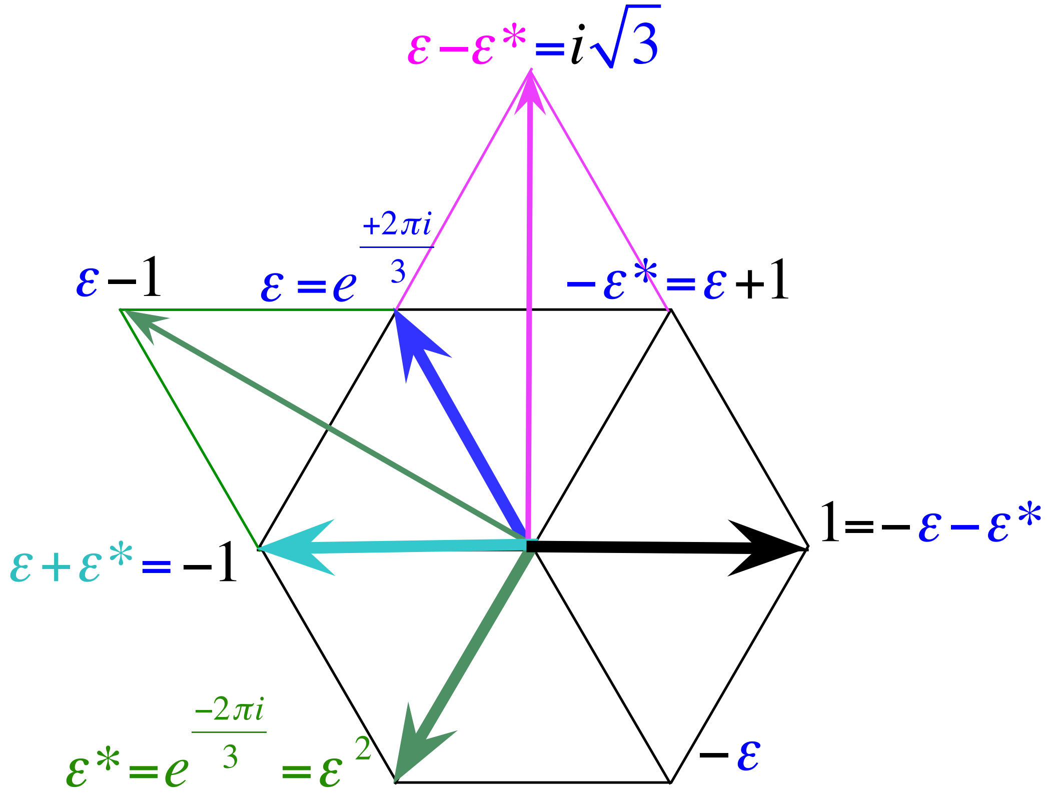
$\mathbf{1} = \mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_r$: $(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)}$$



Minimal equation for $\mathbf{c}_r$

$$\boxed{(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}}$$

Tetrahedral T class projectors

$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^\circ c_g \chi_g^\mu}{\ell^\mu} \mathbb{P}^\mu}$$

$$\boxed{\mathbb{P}^\mu = \sum_{classes\, c_\mathbf{g}} \frac{\ell^\mu}{{}^\circ G} \chi_g^{\mu *} \mathbf{c_g}}$$

T class products

$\mathbf{1} = \mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

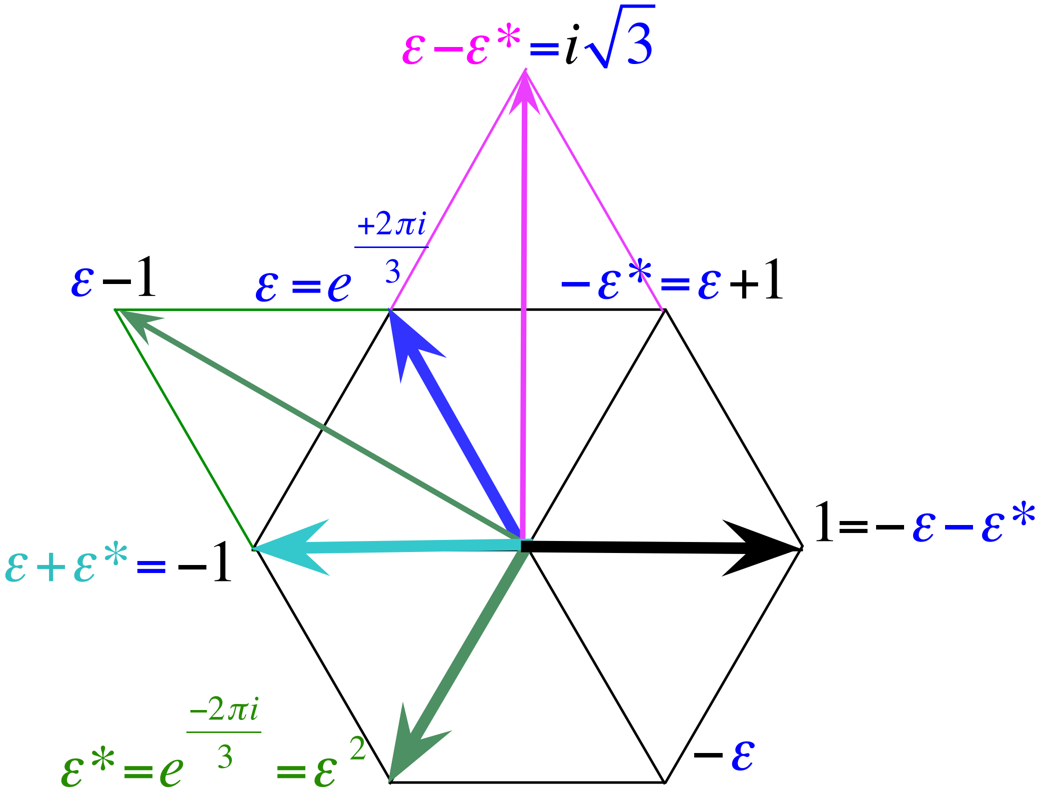
Minimal equation for $\mathbf{c}_r : (\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)}$$



Minimal equation for $\mathbf{c}_r$

$$\boxed{(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}}$$

Tetrahedral T class projectors

$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_g^{\mu *} \mathbf{c_g}}$$

Minimal equation for $\mathbf{c}_r : (\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^* + 1)\mathbf{c}_r - 16\cdot\varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)}$$

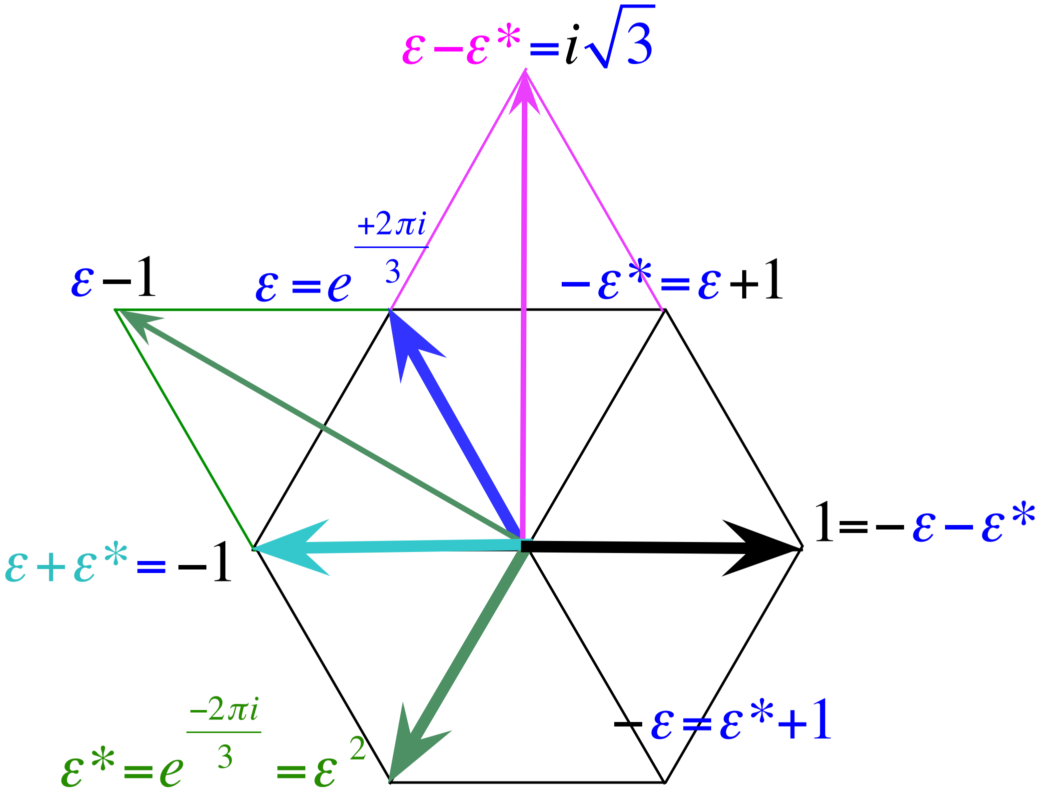
$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)}$$

Minimal equation for $\mathbf{c}_r$

$$\boxed{(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}}$$

T class products

$\mathbf{1} = \mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$



Tetrahedral T class projectors

$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_{g}\chi_{g}^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_{g}^{\mu *} \mathbf{c_g}}$$

T class products

$\mathbf{1} = \mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
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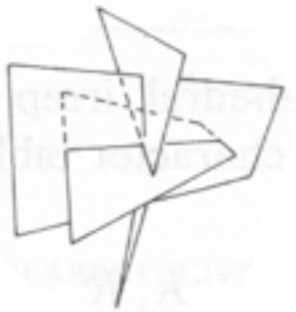


Minimal equation for $\mathbf{c}_r : (\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}$

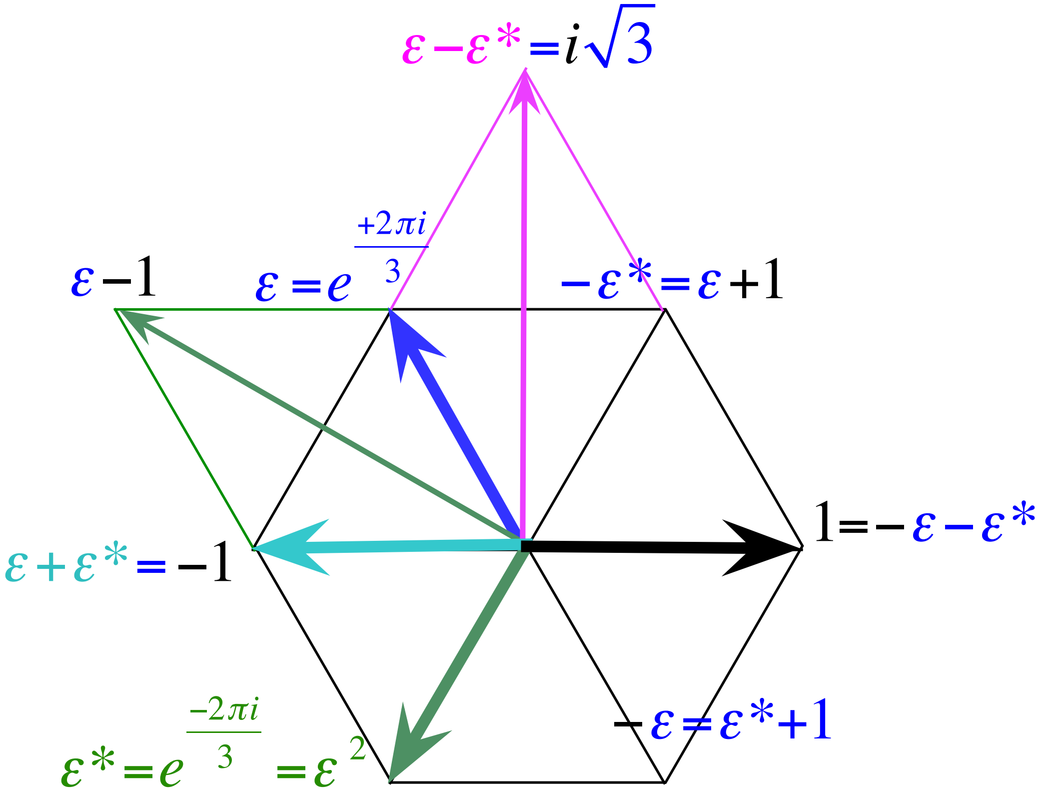
$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^* + 1)\mathbf{c}_r + 16\cdot\varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)}$$



Minimal equation for $\mathbf{c}_r$

$$\boxed{(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}}$$

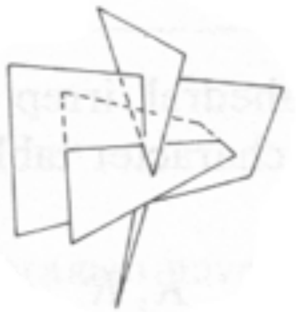
Tetrahedral T class projectors

$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^\circ c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^\circ G} \chi_g^{\mu *} \mathbf{c_g}}$$

T class products

$\mathbf{1} = \mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$



Minimal equation for $\mathbf{c}_r$: $(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^* + 1)\mathbf{c}_r + 16\cdot\varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_\rho) + 16\varepsilon\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})}$$

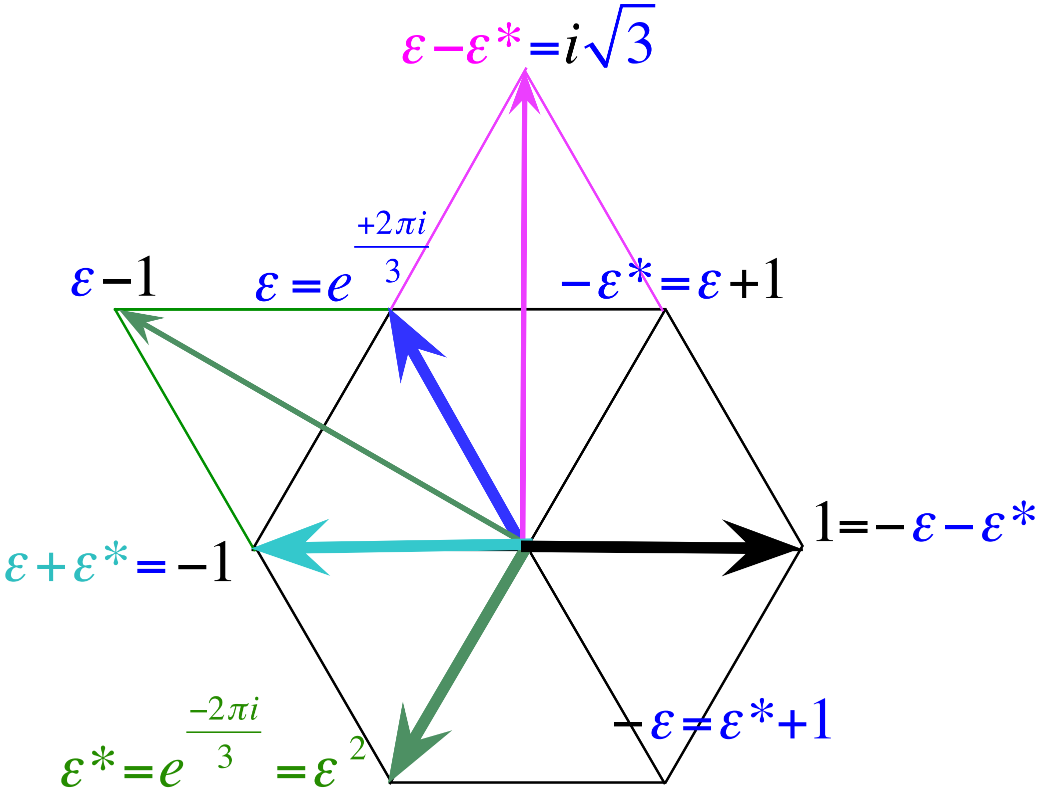
$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)}$$

Minimal equation for $\mathbf{c}_r$

$$\boxed{(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}}$$



Tetrahedral T class projectors

$$\boxed{\mathbf{c}_{\mathbf{g}} = \sum_{\mu} \frac{{}^{\circ}c_{g}\chi^{\mu}_{g}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi^{\mu *}_{g} \mathbf{c}_{\mathbf{g}}}$$

T class products

$\mathbf{1}=\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_r : (\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}$

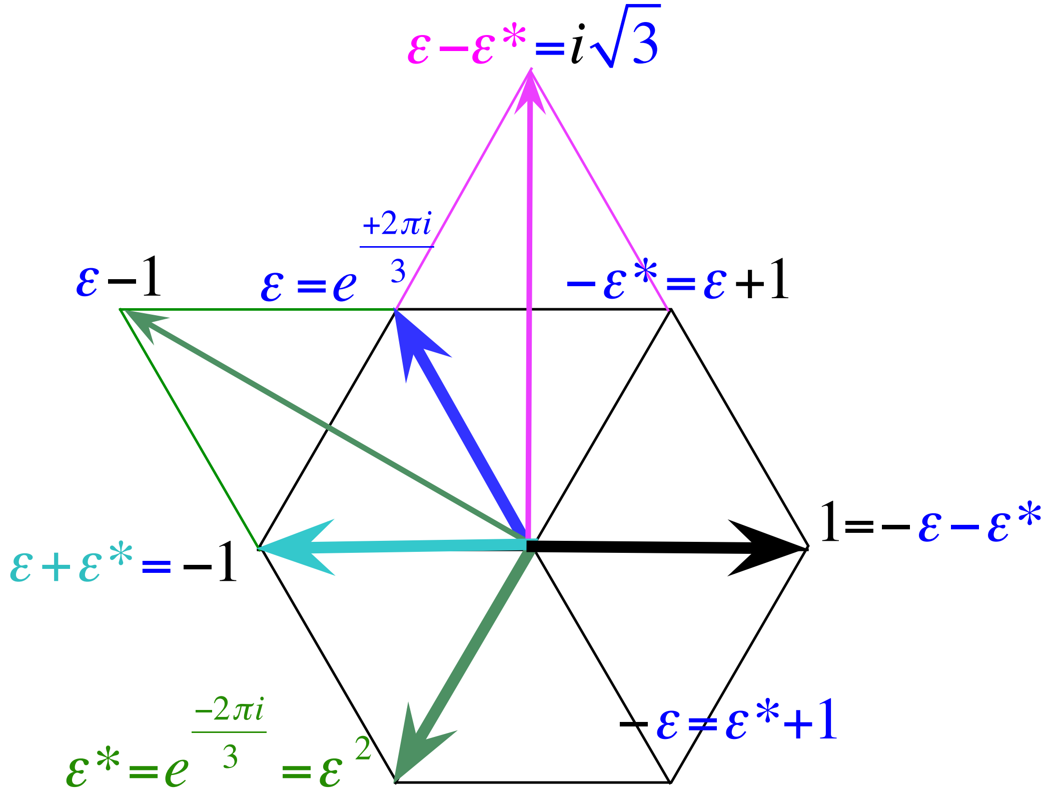
$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^* + 1)\mathbf{c}_r + 16\cdot\varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_\rho) + 16\varepsilon\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})} = \boxed{\frac{\mathbf{1} + \mathbf{c}_\rho + \varepsilon\tilde{\mathbf{c}}_r + \varepsilon^*\mathbf{c}_r}{12}}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)} = \boxed{\frac{\mathbf{1} + \mathbf{c}_\rho + \varepsilon^*\tilde{\mathbf{c}}_r + \varepsilon\mathbf{c}_r}{12}}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)}$$



Minimal equation for $\mathbf{c}_r$

$$\boxed{(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}}$$

Tetrahedral T class projectors

$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_g^{\mu *} \mathbf{c_g}}$$

Minimal equation for $\mathbf{c}_r : (\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^* + 1)\mathbf{c}_r + 16\cdot\varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_\rho) + 16\varepsilon\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})} = \boxed{\frac{\mathbf{1} + \mathbf{c}_\rho + \varepsilon\tilde{\mathbf{c}}_r + \varepsilon^*\mathbf{c}_r}{12}}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)} = \boxed{\frac{\mathbf{1} + \mathbf{c}_\rho + \varepsilon^*\tilde{\mathbf{c}}_r + \varepsilon\mathbf{c}_r}{12}}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)}$$

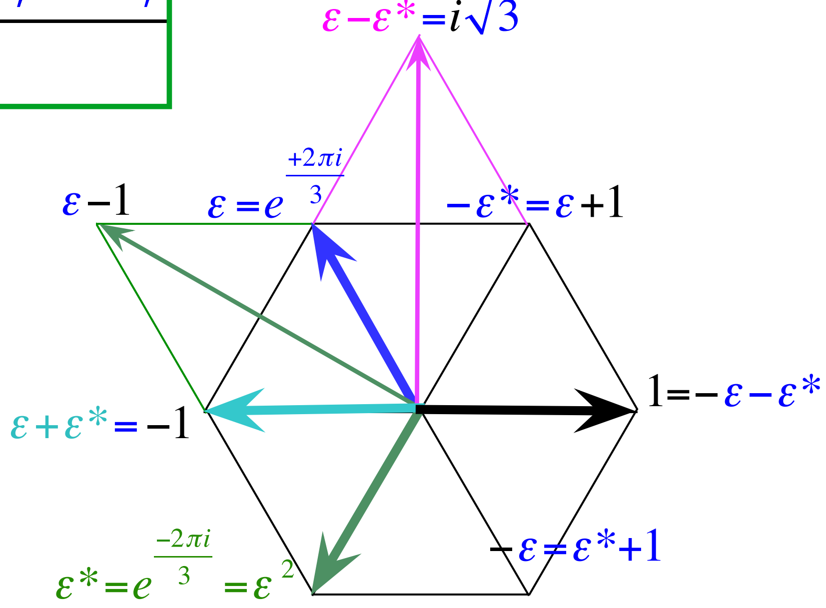
$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)}$$

Minimal equation for $\mathbf{c}_r$

$$\boxed{(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}}$$

T class products

$\mathbf{1} = \mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$



$T : \mathbf{c}_g =$	$\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
.				
$\chi_g^{\varepsilon} =$	1	ε^*	ε	1
$\chi_g^{\varepsilon^*} =$	1	ε	ε^*	1
.				

Tetrahedral T class projectors

$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_g^{\mu *} \mathbf{c_g}}$$

Minimal equation for $\mathbf{c}_r : (\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^* + 1)\mathbf{c}_r + 16\cdot\varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_\rho) + 16\varepsilon\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})} = \boxed{\frac{\mathbf{1} + \mathbf{c}_\rho + \varepsilon\tilde{\mathbf{c}}_r + \varepsilon^*\mathbf{c}_r}{12}}$$

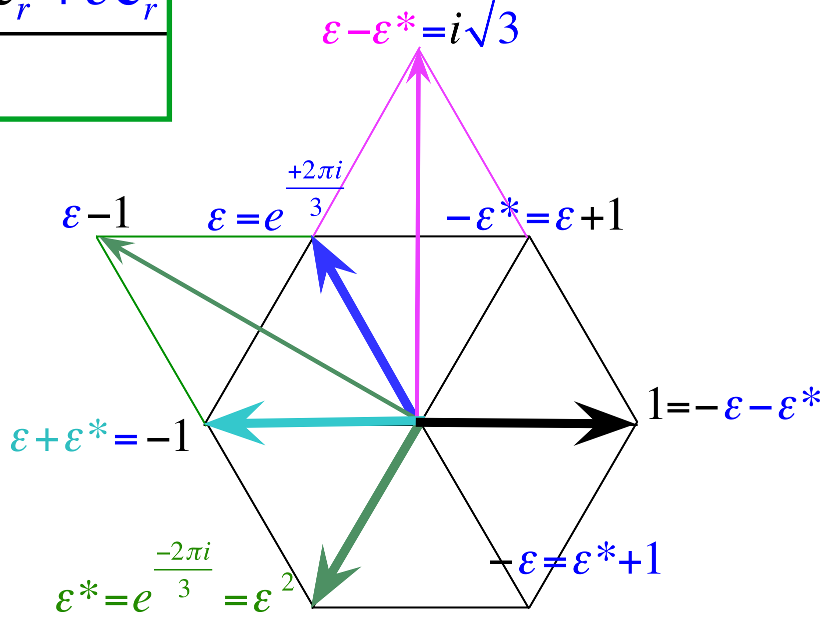
$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)} = \boxed{\frac{\mathbf{1} + \mathbf{c}_\rho + \varepsilon^*\tilde{\mathbf{c}}_r + \varepsilon\mathbf{c}_r}{12}}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\mathbf{1})\mathbf{c}_r}{64(1 - (\varepsilon + \varepsilon^*) + 1)}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)}$$

T class products

$\mathbf{1} = \mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1} + 4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1} + 2\mathbf{c}_\rho$



$T : \mathbf{c}_g =$	$\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\cdot$				
$\chi_g^{\varepsilon} =$	1	ε^*	ε	1
$\chi_g^{\varepsilon^*} =$	1	ε	ε^*	1
$\cdot$				

Minimal equation for $\mathbf{c}_r$

$$\boxed{(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}}$$

Tetrahedral T class projectors

$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_g^{\mu *} \mathbf{c_g}}$$

T class products

$\mathbf{1} = \mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_r : (\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^* + 1)\mathbf{c}_r + 16\cdot\varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

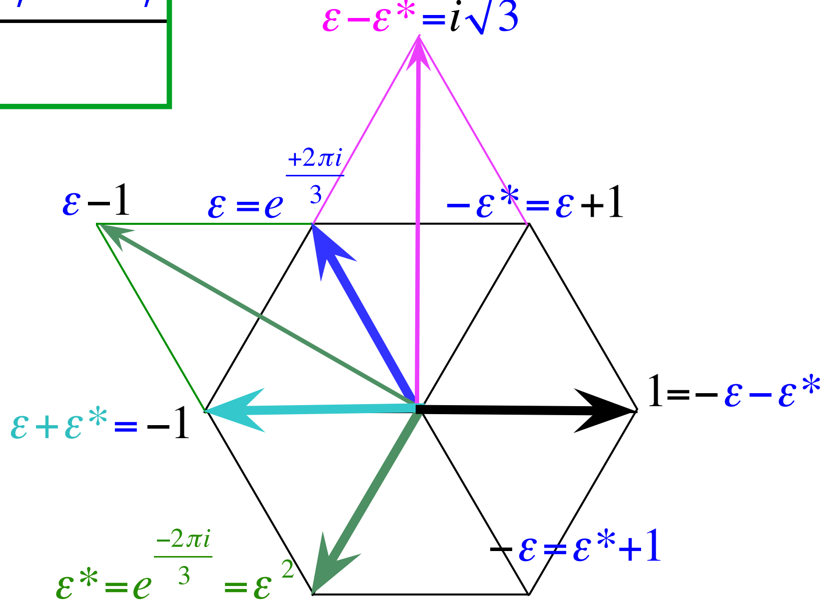
$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_\rho) + 16\varepsilon\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})} = \boxed{\frac{\mathbf{1} + \mathbf{c}_\rho + \varepsilon\tilde{\mathbf{c}}_r + \varepsilon^*\mathbf{c}_r}{12}}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)} = \boxed{\frac{\mathbf{1} + \mathbf{c}_\rho + \varepsilon^*\tilde{\mathbf{c}}_r + \varepsilon\mathbf{c}_r}{12}}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\mathbf{1})\mathbf{c}_r}{64(1 - (\varepsilon + \varepsilon^*) + 1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4\,4\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1 + 1 + 1)}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)}$$



$T : \mathbf{c}_g =$	$\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
.				
$\chi_g^{\varepsilon} =$	1	ε^*	ε	1
$\chi_g^{\varepsilon^*} =$	1	ε	ε^*	1
.				

Minimal equation for $\mathbf{c}_r$

$$\boxed{(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}}$$

Tetrahedral T class projectors

T class products

$\mathbf{1}=\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

$$\boxed{\mathbf{c}_{\mathbf{g}}=\sum_\mu \frac{{}^\circ c_g \chi_g^\mu}{\ell^\mu}\mathbb{P}^\mu}$$

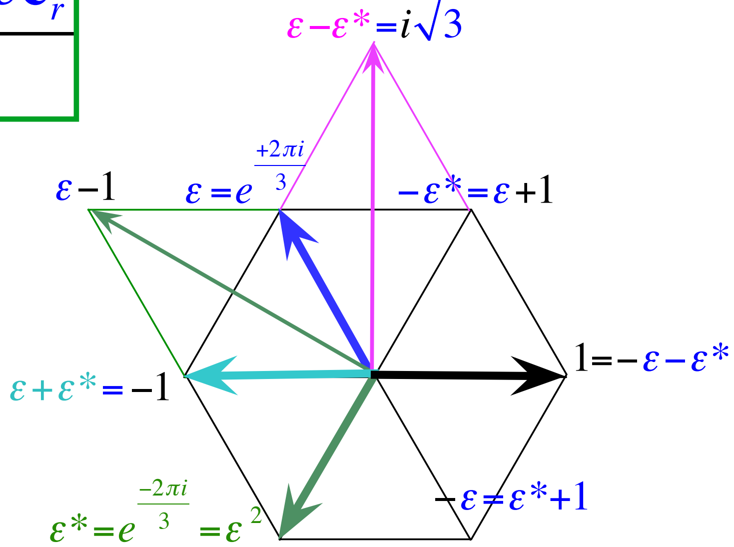
$$\boxed{\mathbb{P}^\mu=\sum_{classes\, c_{\mathbf{g}}} \frac{\ell^\mu}{{}^\circ G}\chi_g^{\mu*}\mathbf{c}_{\mathbf{g}}}$$

Minimal equation for $\mathbf{c}_r$: $(\mathbf{c}_r-4\varepsilon\mathbf{1})(\mathbf{c}_r-4\varepsilon^*\mathbf{1})(\mathbf{c}_r-4\cdot\mathbf{1})(\mathbf{c}_r-0)=\mathbf{0}$

$$\mathbf{P}^{(4\varepsilon)}=\frac{(\mathbf{c}_r-4\varepsilon^*\mathbf{1})(\mathbf{c}_r-4\cdot\mathbf{1})(\mathbf{c}_r-0)}{(4\varepsilon-4\varepsilon^*)(4\varepsilon-4)(4\varepsilon-0)}=\frac{(\mathbf{c}_r^2-4(\varepsilon^*+1)\mathbf{c}_r+16\cdot\varepsilon^*)\mathbf{c}_r}{64(\varepsilon-\varepsilon^*)(\varepsilon-1)\varepsilon}$$

$$=\frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r+4(\varepsilon)4\tilde{\mathbf{c}}_r+16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2-\varepsilon)}=\frac{16(\mathbf{1}+\mathbf{c}_\rho)+16\varepsilon\tilde{\mathbf{c}}_r+16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})}=\boxed{\frac{\mathbf{1}+\mathbf{c}_\rho+\varepsilon\tilde{\mathbf{c}}_r+\varepsilon^*\mathbf{c}_r}{12}}$$

$$\mathbf{P}^{(4\varepsilon^*)}=\frac{(\mathbf{c}_r-4\varepsilon\mathbf{1})(\mathbf{c}_r-4\cdot\mathbf{1})(\mathbf{c}_r-0)}{(4\varepsilon^*-4\varepsilon)(4\varepsilon^*-4)(4\varepsilon^*-0)}=\boxed{\frac{\mathbf{1}+\mathbf{c}_\rho+\varepsilon^*\tilde{\mathbf{c}}_r+\varepsilon\mathbf{c}_r}{12}}$$



$$\mathbf{P}^{(4)}=\frac{(\mathbf{c}_r-4\varepsilon\mathbf{1})(\mathbf{c}_r-4\varepsilon^*\mathbf{1})(\mathbf{c}_r-0)}{(4-4\varepsilon)(4-4\varepsilon^*)(4-0)}=\frac{(\mathbf{c}_r^2-4(\varepsilon+\varepsilon^*)\mathbf{c}_r+16\mathbf{1})\mathbf{c}_r}{64(1-(\varepsilon+\varepsilon^*)+1)}$$

$$=\frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r+4\cdot 4\tilde{\mathbf{c}}_r+16\mathbf{c}_r}{64(1+1+1)}=\frac{16(\mathbf{1}+\mathbf{c}_\rho)+16\tilde{\mathbf{c}}_r+16\mathbf{c}_r}{64(1+1+1)}=\boxed{\frac{\mathbf{1}+\mathbf{c}_\rho+\tilde{\mathbf{c}}_r+\mathbf{c}_r}{12}}$$

$$\mathbf{P}^{(0)}=\frac{(\mathbf{c}_r-4\varepsilon\mathbf{1})(\mathbf{c}_r-4\varepsilon^*\mathbf{1})(\mathbf{c}_r-4\cdot\mathbf{1})}{(0-4\varepsilon)(0-4\varepsilon^*)(0-4)}$$

$T:\mathbf{c}_g=$	$\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\chi_g^A=$	1	1	1	1
$\chi_g^\varepsilon=$	1	ε^*	ε	1
$\chi_g^{\varepsilon^*}=$	1	ε	ε^*	1
.				

Minimal equation for $\mathbf{c}_r$

$$\boxed{(\mathbf{c}_r-4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r-4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r-4\cdot\mathbf{1})(\mathbf{c}_r-0)=\mathbf{0}}$$

Tetrahedral T class projectors

$$\boxed{\mathbf{c}_\mathbf{g} = \sum_\mu \frac{{}^\circ c_g \chi_g^\mu}{\ell^\mu}\mathbb{P}^\mu}\qquad \boxed{\mathbb{P}^\mu = \sum_{classes\, c_\mathbf{g}} \frac{\ell^\mu}{{}^\circ G}\chi_g^{\mu *}\mathbf{c}_\mathbf{g}}$$

T class products

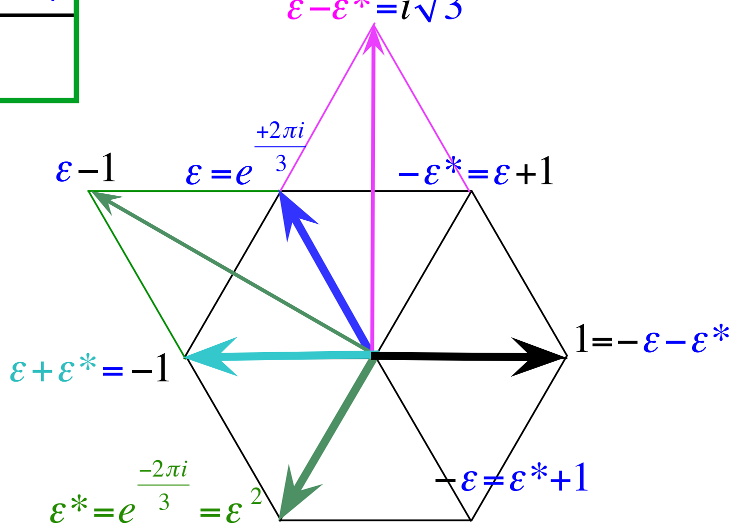
$\mathbf{1}=\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_r$: $(\mathbf{c}_r-4\varepsilon\mathbf{1})(\mathbf{c}_r-4\varepsilon^*\mathbf{1})(\mathbf{c}_r-4\cdot\mathbf{1})(\mathbf{c}_r-0)=\mathbf{0}$

$$\mathbf{P}^{(4\varepsilon)}=\frac{(\mathbf{c}_r-4\varepsilon^*\mathbf{1})(\mathbf{c}_r-4\cdot\mathbf{1})(\mathbf{c}_r-0)}{(4\varepsilon-4\varepsilon^*)(4\varepsilon-4)(4\varepsilon-0)}=\frac{(\mathbf{c}_r^2-4(\varepsilon^*+1)\mathbf{c}_r+16\cdot\varepsilon^*)\mathbf{c}_r}{64(\varepsilon-\varepsilon^*)(\varepsilon-1)\varepsilon}$$

$$=\frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r+4(\varepsilon)4\tilde{\mathbf{c}}_r+16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2-\varepsilon)}=\frac{16(\mathbf{1}+\mathbf{c}_\rho)+16\varepsilon\tilde{\mathbf{c}}_r+16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})}=\boxed{\frac{\mathbf{1}+\mathbf{c}_\rho+\varepsilon\tilde{\mathbf{c}}_r+\varepsilon^*\mathbf{c}_r}{12}}$$

$$\mathbf{P}^{(4\varepsilon^*)}=\frac{(\mathbf{c}_r-4\varepsilon\mathbf{1})(\mathbf{c}_r-4\cdot\mathbf{1})(\mathbf{c}_r-0)}{(4\varepsilon^*-4\varepsilon)(4\varepsilon^*-4)(4\varepsilon^*-0)}=\boxed{\frac{\mathbf{1}+\mathbf{c}_\rho+\varepsilon^*\tilde{\mathbf{c}}_r+\varepsilon\mathbf{c}_r}{12}}$$



$$\mathbf{P}^{(4)}=\frac{(\mathbf{c}_r-4\varepsilon\mathbf{1})(\mathbf{c}_r-4\varepsilon^*\mathbf{1})(\mathbf{c}_r-0)}{(4-4\varepsilon)(4-4\varepsilon^*)(4-0)}=\frac{(\mathbf{c}_r^2-4(\varepsilon+\varepsilon^*)\mathbf{c}_r+16\mathbf{1})\mathbf{c}_r}{64(1-(\varepsilon+\varepsilon^*)+1)}$$

$$=\frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r+4\cdot 4\tilde{\mathbf{c}}_r+16\mathbf{c}_r}{64(1+1+1)}=\frac{16(\mathbf{1}+\mathbf{c}_\rho)+16\tilde{\mathbf{c}}_r+16\mathbf{c}_r}{64(1+1+1)}=\boxed{\frac{\mathbf{1}+\mathbf{c}_\rho+\tilde{\mathbf{c}}_r+\mathbf{c}_r}{12}}$$

$$\mathbf{P}^{(0)}=\frac{(\mathbf{c}_r-4\varepsilon\mathbf{1})(\mathbf{c}_r-4\varepsilon^*\mathbf{1})(\mathbf{c}_r-4\cdot\mathbf{1})}{(0-4\varepsilon)(0-4\varepsilon^*)(0-4)}=\frac{(\mathbf{c}_r^2-4(\varepsilon+\varepsilon^*)\mathbf{c}_r+16\cdot\mathbf{1})(\mathbf{c}_r-4\cdot\mathbf{1})}{64(-\varepsilon)(-\varepsilon^*)(-1)}$$

$T:\mathbf{c}_g=$	$\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\chi_g^A=$	1	1	1	1
$\chi_g^\varepsilon=$	1	ε^*	ε	1
$\chi_g^{\varepsilon^*}=$	1	ε	ε^*	1
.				

Minimal equation for $\mathbf{c}_r$

$$\boxed{(\mathbf{c}_r-4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r-4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r-4\cdot\mathbf{1})(\mathbf{c}_r-0)=\mathbf{0}}$$

Tetrahedral T class projectors

$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_g^{\mu *} \mathbf{c_g}}$$

T class products

$\mathbf{1}=\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_r : (\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^* + 1)\mathbf{c}_r + 16\cdot\varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_\rho) + 16\varepsilon\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})} = \boxed{\frac{\mathbf{1} + \mathbf{c}_\rho + \varepsilon\tilde{\mathbf{c}}_r + \varepsilon^*\mathbf{c}_r}{12}}$$

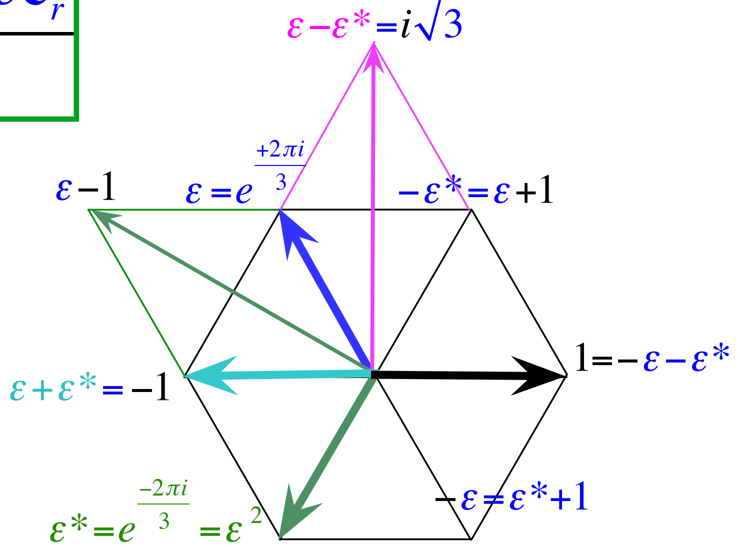
$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)} = \boxed{\frac{\mathbf{1} + \mathbf{c}_\rho + \varepsilon^*\tilde{\mathbf{c}}_r + \varepsilon\mathbf{c}_r}{12}}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\mathbf{1})\mathbf{c}_r}{64(1 - (\varepsilon + \varepsilon^*) + 1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4\cdot 4\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1 + 1 + 1)} = \frac{16(\mathbf{1} + \mathbf{c}_\rho) + 16\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1 + 1 + 1)} = \boxed{\frac{\mathbf{1} + \mathbf{c}_\rho + \tilde{\mathbf{c}}_r + \mathbf{c}_r}{12}}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\cdot\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{64(-\varepsilon)(-\varepsilon^*)(-1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r + 4\mathbf{c}_r + 16\cdot\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{-64}$$



$T : \mathbf{c}_g =$	$\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\chi_g^A =$	1	1	1	1
$\chi_g^\varepsilon =$	1	ε^*	ε	1
$\chi_g^{\varepsilon^*} =$	1	ε	ε^*	1
.				

Minimal equation for $\mathbf{c}_r$

$$\boxed{(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}}$$

Tetrahedral T class projectors

$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_g^{\mu *} \mathbf{c_g}}$$

T class products

$\mathbf{1}=\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_r$: $(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^* + 1)\mathbf{c}_r + 16\cdot\varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_\rho) + 16\varepsilon\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})} = \boxed{\frac{\mathbf{1} + \mathbf{c}_\rho + \varepsilon\tilde{\mathbf{c}}_r + \varepsilon^*\mathbf{c}_r}{12}}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)} = \boxed{\frac{\mathbf{1} + \mathbf{c}_\rho + \varepsilon^*\tilde{\mathbf{c}}_r + \varepsilon\mathbf{c}_r}{12}}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\mathbf{1})\mathbf{c}_r}{64(1 - (\varepsilon + \varepsilon^*) + 1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4\cdot 4\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1+1+1)} = \frac{16(\mathbf{1} + \mathbf{c}_\rho) + 16\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1+1+1)} = \boxed{\frac{\mathbf{1} + \mathbf{c}_\rho + \tilde{\mathbf{c}}_r + \mathbf{c}_r}{12}}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\cdot\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{64(-\varepsilon)(-\varepsilon^*)(-1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r + 4\mathbf{c}_r + 16\cdot\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{-64}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4\mathbf{c}_r^2 + 16\mathbf{c}_r - 16\tilde{\mathbf{c}}_r - 16\mathbf{c}_r - 64\cdot\mathbf{1})}{-64} =$$

$T:\mathbf{c}_g =$	$\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\chi_g^A =$	1	1	1	1
$\chi_g^\varepsilon =$	1	ε^*	ε	1
$\chi_g^{\varepsilon^*} =$	1	ε	ε^*	1
.				

Tetrahedral T class projectors

$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_{g}\chi_{g}^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_{g}^{\mu *} \mathbf{c_g}}$$

T class products

$\mathbf{1}=\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_r : (\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^* + 1)\mathbf{c}_r + 16\cdot\varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_\rho) + 16\varepsilon\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})} = \frac{\mathbf{1} + \mathbf{c}_\rho + \varepsilon\tilde{\mathbf{c}}_r + \varepsilon^*\mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)} = \frac{\mathbf{1} + \mathbf{c}_\rho + \varepsilon^*\tilde{\mathbf{c}}_r + \varepsilon\mathbf{c}_r}{12}$$

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$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4\cdot4\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1+1+1)} = \frac{16(\mathbf{1} + \mathbf{c}_\rho) + 16\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1+1+1)} = \frac{\mathbf{1} + \mathbf{c}_\rho + \tilde{\mathbf{c}}_r + \mathbf{c}_r}{12}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\cdot\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{64(-\varepsilon)(-\varepsilon^*)(-1)}$$

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$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4\mathbf{c}_r^2 + 16\mathbf{c}_r - 16\tilde{\mathbf{c}}_r - 16\mathbf{c}_r - 64\cdot\mathbf{1})}{-64}$$

$$= \frac{4(4\mathbf{1} + 4\mathbf{c}_\rho) + 16\tilde{\mathbf{c}}_r + 16\mathbf{c}_r - 16\tilde{\mathbf{c}}_r - 16\mathbf{c}_r - 64\cdot\mathbf{1})}{-64}$$

$T : \mathbf{c}_g =$	$\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\chi_g^A =$	1	1	1	1
$\chi_g^\varepsilon =$	1	ε^*	ε	1
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.				

Tetrahedral T class characters

$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^\circ c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^\circ G} \chi_g^{\mu *} \mathbf{c_g}}$$

T class products

$\mathbf{1} = \mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_r : (\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^* + 1)\mathbf{c}_r + 16\cdot\varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_\rho) + 16\varepsilon\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})} = \boxed{\frac{\mathbf{1} + \mathbf{c}_\rho + \varepsilon\tilde{\mathbf{c}}_r + \varepsilon^*\mathbf{c}_r}{12}}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)} = \boxed{\frac{\mathbf{1} + \mathbf{c}_\rho + \varepsilon^*\tilde{\mathbf{c}}_r + \varepsilon\mathbf{c}_r}{12}}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\mathbf{1})\mathbf{c}_r}{64(1 - (\varepsilon + \varepsilon^*) + 1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4\cdot 4\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1 + 1 + 1)} = \frac{16(\mathbf{1} + \mathbf{c}_\rho) + 16\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1 + 1 + 1)} = \boxed{\frac{\mathbf{1} + \mathbf{c}_\rho + \tilde{\mathbf{c}}_r + \mathbf{c}_r}{12}}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\cdot\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{64(-\varepsilon)(-\varepsilon^*)(-1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r + 4\mathbf{c}_r + 16\cdot\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{-64}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4\mathbf{c}_r^2 + 16\mathbf{c}_r - 16\tilde{\mathbf{c}}_r - 16\mathbf{c}_r - 64\cdot\mathbf{1})}{-64}$$

$$= \frac{4(4\mathbf{1} + 4\mathbf{c}_\rho) + 16\tilde{\mathbf{c}}_r + 16\mathbf{c}_r - 16\tilde{\mathbf{c}}_r - 16\mathbf{c}_r - 64\cdot\mathbf{1})}{-64} = \frac{-48\cdot\mathbf{1} + 16\mathbf{c}_\rho}{-64} = \boxed{\frac{3}{4}\mathbf{1} - \frac{1}{4}\mathbf{c}_\rho}$$

$T : \mathbf{c}_g =$	$\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\chi_g^A =$	1	1	1	1
$\chi_g^\varepsilon =$	1	ε^*	ε	1
$\chi_g^{\varepsilon^*} =$	1	ε	ε^*	1
$\chi_g^T =$	3	0	0	-1

Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

- Octahedral-cubic O symmetry and group operations
- Tetrahedral symmetry becomes Icosahedral
- Octahedral groups $O_h \supset O \sim T_d \supset T$
  - Octahedral O and spin-$O \subset U(2)$

Tetrahedral T class algebra

- Tetrahedral T class minimal equations
- Tetrahedral T class projectors and characters

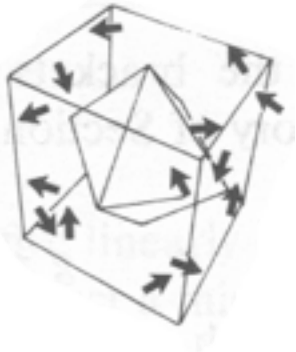
→ Octahedral O class algebra ←

- Octahedral O class minimal equations
- Octahedral O class projectors and characters

Octahedral $O_h \supset O$ subgroup correlations

- Octahedral $O_h \supset O$ subgroup correlations
- $O_h \supset O \supset D_4$ subgroup correlations
- $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations
- Preview of applications to high resolution spectroscopy

Octahedral O class algebra



$\mathbf{c}_1=\mathbf{1}$, $\mathbf{c}_r=\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4+\mathbf{r}_1{}^2+\mathbf{r}_2{}^2+\mathbf{r}_3{}^2+\mathbf{r}_4{}^2$, $\mathbf{c}_\rho=\mathbf{R}_1{}^2+\mathbf{R}_2{}^2+\mathbf{R}_3{}^2$,

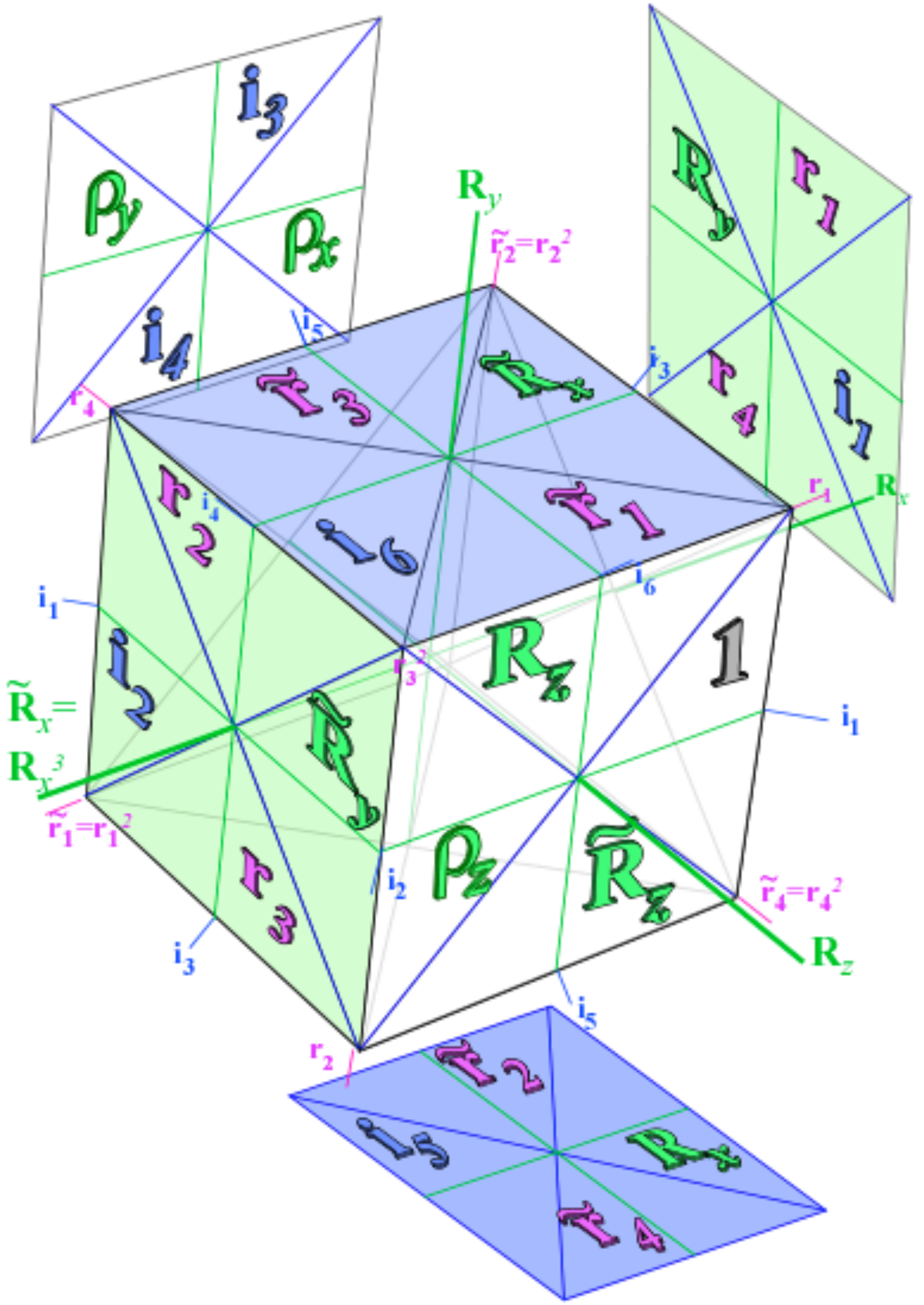
$\mathbf{c}_R=\mathbf{R}_1+\mathbf{R}_2+\mathbf{R}_3+\mathbf{R}_1{}^3+\mathbf{R}_2{}^3+\mathbf{R}_3{}^3$, $\mathbf{c}_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3+\mathbf{i}_4+\mathbf{i}_5+\mathbf{i}_6$

O class products

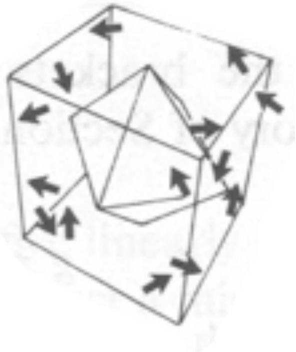
Unnecessary to do $24^2=576$ *products since each row (or column) of* $\mathbf{c}_A\mathbf{c}_B$ *has same class proportion*

For example:

$$\mathbf{c}_\rho\mathbf{c}_i = \mathbf{R}_1{}^2\mathbf{i}_1+\ldots.= \mathbf{R}_2+\ldots.$$
$$+\mathbf{R}_2{}^2\mathbf{i}_1+\ldots. \quad +\mathbf{i}_2+\ldots.$$
$$+\mathbf{R}_3{}^2\mathbf{i}_1+\ldots. \quad +\mathbf{R}_2{}^3+$$



Octahedral O class algebra



$\mathbf{c}_1=\mathbf{1}$, $\mathbf{c}_r=\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4+\mathbf{r}_1^2+\mathbf{r}_2^2+\mathbf{r}_3^2+\mathbf{r}_4^2$, $\mathbf{c}_\rho=\mathbf{R}_1^2+\mathbf{R}_2^2+\mathbf{R}_3^2$,

$\mathbf{c}_R=\mathbf{R}_1+\mathbf{R}_2+\mathbf{R}_3+\mathbf{R}_1^3+\mathbf{R}_2^3+\mathbf{R}_3^3$, $\mathbf{c}_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3+\mathbf{i}_4+\mathbf{i}_5+\mathbf{i}_6$

O class products

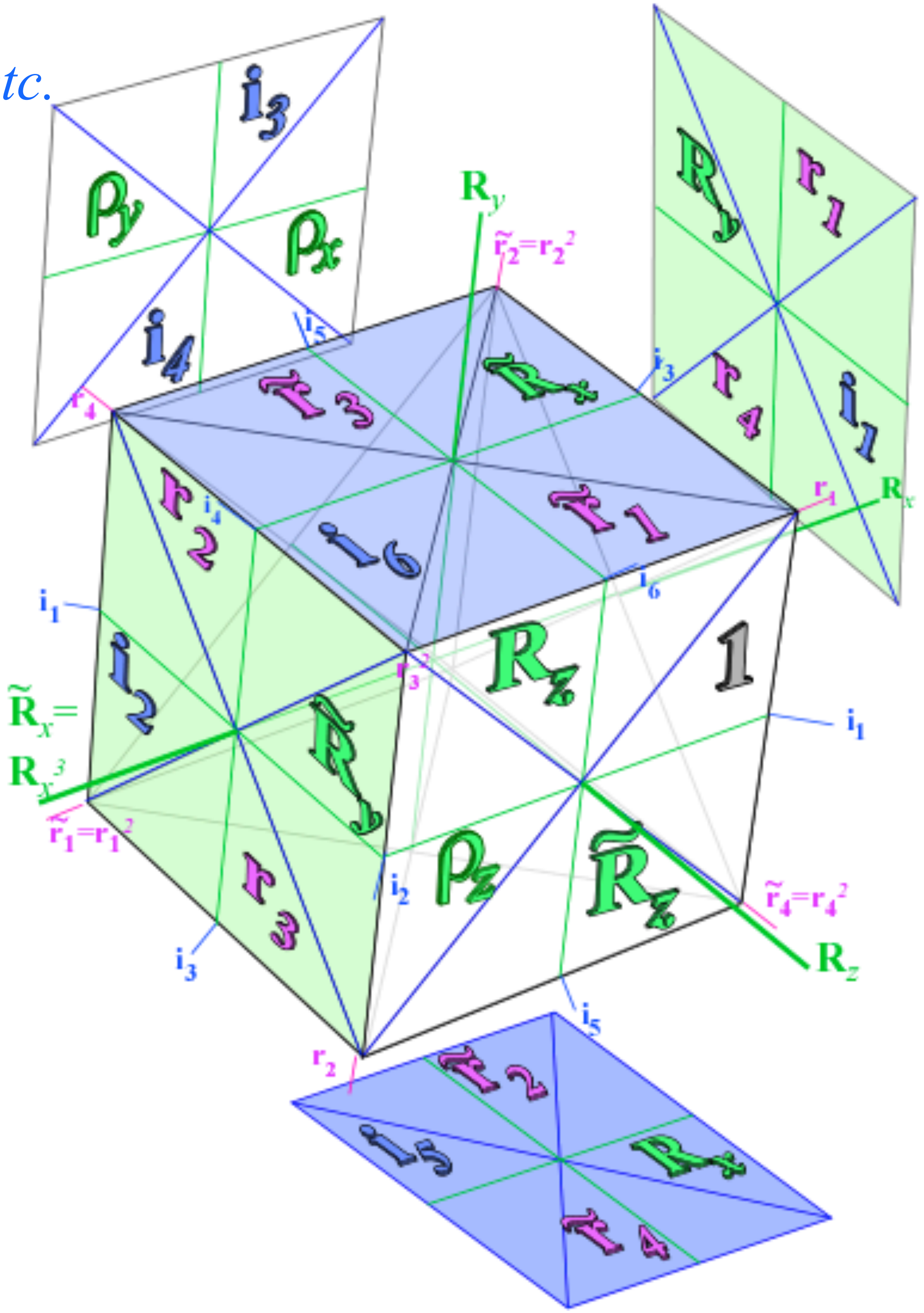
Unnecessary to do $24^2=576$ *products since each row (or column) of* $\mathbf{c}_A\mathbf{c}_B$ *has same class proportion*

For example:

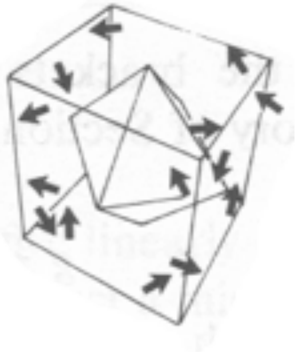
$$\begin{aligned}\mathbf{c}_{R^2}\mathbf{c}_i = \ &\mathbf{R}_1^2\mathbf{i}_1+\ldots.= \mathbf{R}_2+\ldots.\\ &+\mathbf{R}_2^2\mathbf{i}_1+\ldots. \quad +\mathbf{i}_2+\ldots.\\ &+\mathbf{R}_3^2\mathbf{i}_1+\ldots. \quad +\mathbf{R}_2^3+\end{aligned}$$

So there are $2\mathbf{c}_R$ *for each* $\mathbf{c}_i$ *:*

$\mathbf{c}_\rho\mathbf{c}_i=2\mathbf{c}_R+\mathbf{c}_i$ *or:* $4\mathbf{c}_R+2\mathbf{c}_i$ *etc.*



Octahedral O class algebra



$\mathbf{c}_1=1$, $\mathbf{c}_r=\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4+\mathbf{r}_1^2+\mathbf{r}_2^2+\mathbf{r}_3^2+\mathbf{r}_4^2$, $\mathbf{c}_\rho=\mathbf{R}_1^2+\mathbf{R}_2^2+\mathbf{R}_3^2$,

$\mathbf{c}_R=\mathbf{R}_1+\mathbf{R}_2+\mathbf{R}_3+\mathbf{R}_1^3+\mathbf{R}_2^3+\mathbf{R}_3^3$, $\mathbf{c}_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3+\mathbf{i}_4+\mathbf{i}_5+\mathbf{i}_6$

O class products

Unnecessary to do $24^2=576$ products since each row (or column) of $\mathbf{c}_A\mathbf{c}_B$ has same class proportion

For example: $\mathbf{c}_\rho\mathbf{c}_i=?$

$$\begin{aligned}\mathbf{c}_{R^2}\mathbf{c}_i = \ & \mathbf{R}_1^2\mathbf{i}_1+\dots = \mathbf{R}_2+\dots\\ & +\mathbf{R}_2^2\mathbf{i}_1+\dots \quad +\mathbf{i}_2+\dots\\ & +\mathbf{R}_3^2\mathbf{i}_1+\dots \quad +\mathbf{R}_2^3+\end{aligned}$$

So there are $2\mathbf{c}_R$ for each $\mathbf{c}_i$ ....in $({}^\circ c_\rho)\cdot({}^\circ c_i)=(3)\cdot(6)=18$ terms

$\mathbf{c}_\rho\mathbf{c}_i=2\mathbf{c}_R+\mathbf{c}_i$ *or:* $4\mathbf{c}_R+2\mathbf{c}_i$ *etc.*

So: $2({}^\circ c_R)+({}^\circ c_i)=2\cdot 6+6=18$

Proof that class proportion cannot vary:

$$\begin{aligned}\mathbf{c}_g\mathbf{c}_h &= \mathbf{g}_1\mathbf{h}_1+\mathbf{g}_2\mathbf{h}_1+\dots = \mathbf{g}_1\mathbf{h}_1+\mathbf{t}\mathbf{g}_1\mathbf{h}_1\mathbf{t}^{-1}+\dots = \mathbf{g}_1\mathbf{h}_1+\mathbf{t}\mathbf{g}_1\mathbf{t}^{-1}\mathbf{t}\mathbf{h}_1\mathbf{t}^{-1}+\dots = \mathbf{g}_1\mathbf{h}_1+\mathbf{g}_2\ \mathbf{t}\mathbf{h}_1\mathbf{t}^{-1}+\dots\\ &\quad +\mathbf{g}_1\mathbf{h}_2+\mathbf{g}_2\mathbf{h}_2+\dots \quad +\mathbf{g}_1\mathbf{h}_2+\mathbf{t}\mathbf{g}_1\mathbf{h}_2\mathbf{t}^{-1}+\dots \quad +\mathbf{g}_1\mathbf{h}_2+\mathbf{t}\mathbf{g}_1\mathbf{t}^{-1}\mathbf{t}\mathbf{h}_2\mathbf{t}^{-1}+\dots \quad +\mathbf{g}_1\mathbf{h}_2+\mathbf{g}_2\ \mathbf{t}\mathbf{h}_2\mathbf{t}^{-1}+\dots\\ &= \mathbf{g}_1\mathbf{h}_3+\mathbf{g}_2\mathbf{h}_3+\dots \quad +\mathbf{g}_1\mathbf{h}_2+\mathbf{t}\mathbf{g}_1\mathbf{h}_3\mathbf{t}^{-1}+\dots \quad +\mathbf{g}_1\mathbf{h}_2+\mathbf{t}\mathbf{g}_1\mathbf{t}^{-1}\mathbf{t}\mathbf{h}_3\mathbf{t}^{-1}+\dots \quad +\mathbf{g}_1\mathbf{h}_2+\mathbf{g}_2\ \mathbf{t}\mathbf{h}_3\mathbf{t}^{-1}+\dots\end{aligned}$$

O class product table

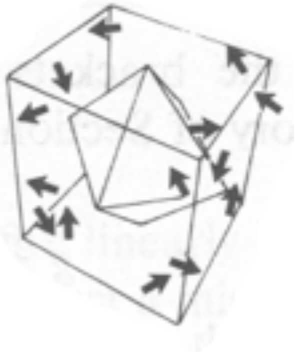
$\mathbf{1}=c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_\rho$

$\mathbf{c}_\rho^2=3\mathbf{1}+2\mathbf{c}_\rho$

$\mathbf{c}_\rho^2-2\mathbf{c}_\rho-3\mathbf{1}=\mathbf{0}$

$(\mathbf{c}_\rho-3\mathbf{1})(\mathbf{c}_\rho+\mathbf{1})=\mathbf{0}$



Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

- *Octahedral-cubic O symmetry and group operations*
- *Tetrahedral symmetry becomes Icosahedral*
- *Octahedral groups $O_h \supset O \sim T_d \supset T$*
  - *Octahedral O and spin-$O \subset U(2)$*

Tetrahedral T class algebra

- *Tetrahedral T class minimal equations*
- *Tetrahedral T class projectors and characters*

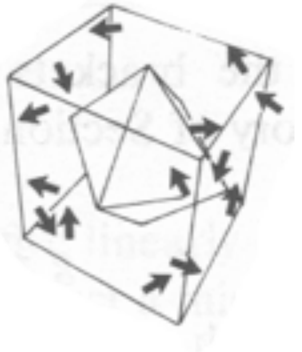
➔ *Octahedral O class algebra* 🡐

- *Octahedral O class minimal equations*
- *Octahedral O class projectors and characters*

Octahedral $O_h \supset O$ subgroup correlations

- *Octahedral $O_h \supset O$ subgroup correlations*
- *$O_h \supset O \supset D_4$ subgroup correlations*
- *$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations*
- *Preview of applications to high resolution spectroscopy*

Octahedral O class minimal equations



$$\boxed{\mathbf{c}_{\mathbf{g}} = \sum_{\mu} \frac{{}^{\circ}c_{g}\chi_{g}^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_{g}^{\mu *} \mathbf{c}_{\mathbf{g}}}$$

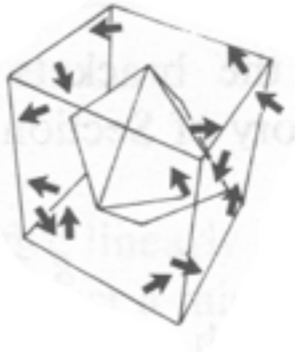
O class product table

$\mathbf{1} = c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_i$

$$\mathbf{c}_i^2 = 6 \cdot \mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$$

Octahedral O class minimal equations



$$\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_g^{\mu *} \mathbf{c_g}$$

O class product table

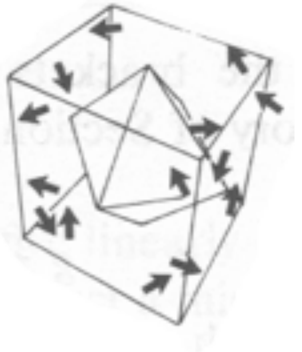
$\mathbf{1} = c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_i$

$$\mathbf{c}_i^2 = 6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = 6\cdot\mathbf{1}\mathbf{c}_i+3\mathbf{c}_r\mathbf{c}_i+2\mathbf{c}_\rho\mathbf{c}_i$$

Octahedral O class minimal equations



$$\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_{g}\chi^{\mu}_{g}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi^{\mu *}_{g} \mathbf{c_g}$$

O class product table

$\mathbf{1} = c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_i$

$$\mathbf{c}_i^2 = 6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = 6\cdot\mathbf{1}\mathbf{c}_i+3\mathbf{c}_r\mathbf{c}_i+2\mathbf{c}_\rho\mathbf{c}_i$$

$$= 6\mathbf{c}_i+3(4\mathbf{c}_R+4\mathbf{c}_i)+2(2\mathbf{c}_R+\mathbf{c}_i)$$

Octahedral O class minimal equations

$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_g \chi^{\mu}_g}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi^{\mu *}_g \mathbf{c_g}}$$

O class product table

$\mathbf{1} = c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_i$

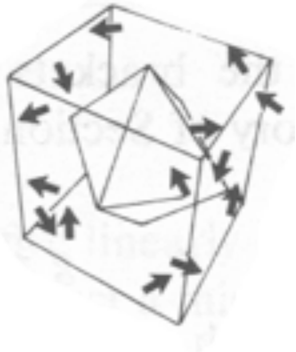
$$\mathbf{c}_i^2 = 6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = 6\cdot\mathbf{1}\mathbf{c}_i+3\mathbf{c}_r\mathbf{c}_i+2\mathbf{c}_\rho\mathbf{c}_i$$

$$=6\mathbf{c}_i+3(4\mathbf{c}_R+4\mathbf{c}_i)+2(2\mathbf{c}_R+\mathbf{c}_i)$$

$$\mathbf{c}_i^3 = 16\mathbf{c}_R+20\mathbf{c}_i$$

Octahedral O class minimal equations



$$\mathbf{c}_{\mathbf{g}} = \sum_{\mu} \frac{{}^{\circ}c_{g}\chi_{g}^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_{g}^{\mu *} \mathbf{c}_{\mathbf{g}}$$

O class product table

$\mathbf{1} = c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_i$

$$\mathbf{c}_i^2 = 6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$$

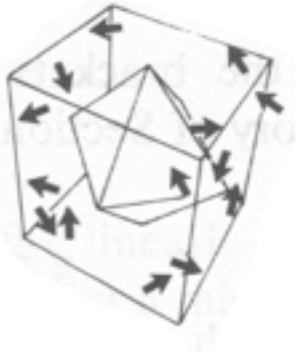
$$\mathbf{c}_i^3 = 6\cdot\mathbf{1}\mathbf{c}_i+3\mathbf{c}_r\mathbf{c}_i+2\mathbf{c}_\rho\mathbf{c}_i$$

$$= 6\mathbf{c}_i+3(4\mathbf{c}_R+4\mathbf{c}_i)+2(2\mathbf{c}_R+\mathbf{c}_i)$$

$$\mathbf{c}_i^3 = 16\mathbf{c}_R+20\mathbf{c}_i$$

$$\mathbf{c}_i^4 = \quad 16\mathbf{c}_R\mathbf{c}_i \quad +20\mathbf{c}_i\mathbf{c}_i$$

Octahedral O class minimal equations



$$\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{classes\, c_\mathbf{g}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_g^{\mu *} \mathbf{c_g}$$

O class product table

$\mathbf{1} = c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_i$

$$\mathbf{c}_i^2 = 6\cdot\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = 6\cdot\mathbf{1}\mathbf{c}_i + 3\mathbf{c}_r\mathbf{c}_i + 2\mathbf{c}_\rho\mathbf{c}_i$$

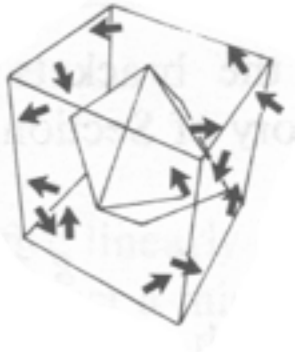
$$= 6\mathbf{c}_i + 3(4\mathbf{c}_R + 4\mathbf{c}_i) + 2(2\mathbf{c}_R + \mathbf{c}_i)$$

$$\mathbf{c}_i^3 = 16\mathbf{c}_R + 20\mathbf{c}_i$$

$$\mathbf{c}_i^4 = \quad 16\mathbf{c}_R\mathbf{c}_i \quad + 20\mathbf{c}_i\mathbf{c}_i$$

$$= 16(3\mathbf{c}_r + 4\mathbf{c}_\rho) + 20(6\cdot\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho)$$

Octahedral O class minimal equations



$$\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_{g}\chi^{\mu}_{g}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi^{\mu *}_{g} \mathbf{c_g}$$

O class product table

$\mathbf{1} = c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_i$

$$\mathbf{c}_i^2 = 6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = 6\cdot\mathbf{1}\mathbf{c}_i+3\mathbf{c}_r\mathbf{c}_i+2\mathbf{c}_\rho\mathbf{c}_i$$

$$=6\mathbf{c}_i+3(4\mathbf{c}_R+4\mathbf{c}_i)+2(2\mathbf{c}_R+\mathbf{c}_i)$$

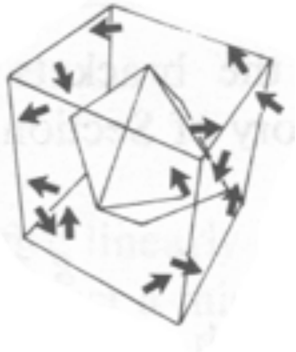
$$\mathbf{c}_i^3 = 16\mathbf{c}_R+20\mathbf{c}_i$$

$$\mathbf{c}_i^4 = \quad 16\mathbf{c}_R\mathbf{c}_i \quad +20\mathbf{c}_i\mathbf{c}_i$$

$$=16(3\mathbf{c}_r+4\mathbf{c}_\rho)+20(6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho)$$

$$=48\mathbf{c}_r+64\mathbf{c}_\rho+120\cdot\mathbf{1}+60\mathbf{c}_r+40\mathbf{c}_\rho)$$

Octahedral O class minimal equations



$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_g^{\mu *} \mathbf{c_g}}$$

O class product table

$\mathbf{1} = c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_i$

$$\mathbf{c}_i^2 = 6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = 6\cdot\mathbf{1}\mathbf{c}_i+3\mathbf{c}_r\mathbf{c}_i+2\mathbf{c}_\rho\mathbf{c}_i$$

$$=6\mathbf{c}_i+3(4\mathbf{c}_R+4\mathbf{c}_i)+2(2\mathbf{c}_R+\mathbf{c}_i)$$

$$\mathbf{c}_i^3 = 16\mathbf{c}_R+20\mathbf{c}_i$$

$$\mathbf{c}_i^4 = \quad 16\mathbf{c}_R\mathbf{c}_i \quad +20\mathbf{c}_i\mathbf{c}_i$$

$$=16(3\mathbf{c}_r+4\mathbf{c}_\rho)+20(6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho)$$

$$=48\mathbf{c}_r+64\mathbf{c}_\rho+120\cdot\mathbf{1}+60\mathbf{c}_r+40\mathbf{c}_\rho)$$

$$=120\cdot\mathbf{1}+108\mathbf{c}_r+104\mathbf{c}_\rho$$

Octahedral O class minimal equations

$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_{g}\chi_{g}^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_{g}^{\mu *} \mathbf{c_g}}$$

O class product table

$\mathbf{1} = c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_i$

$$\mathbf{c}_i{}^2 = 6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$$

$$\mathbf{c}_i{}^3 = 6\cdot\mathbf{1}\mathbf{c}_i+3\mathbf{c}_r\mathbf{c}_i+2\mathbf{c}_\rho\mathbf{c}_i$$

$$=6\mathbf{c}_i+3(4\mathbf{c}_R+4\mathbf{c}_i)+2(2\mathbf{c}_R+\mathbf{c}_i)$$

$$\mathbf{c}_i{}^3 = 16\mathbf{c}_R+20\mathbf{c}_i$$

$$\mathbf{c}_i{}^4 = \quad 16\mathbf{c}_R\mathbf{c}_i \quad +20\mathbf{c}_i\mathbf{c}_i$$

$$=16(3\mathbf{c}_r+4\mathbf{c}_\rho)+20(6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho)$$

$$=48\mathbf{c}_r+64\mathbf{c}_\rho+120\cdot\mathbf{1}+60\mathbf{c}_r+40\mathbf{c}_\rho)$$

$$=120\cdot\mathbf{1}+108\mathbf{c}_r+104\mathbf{c}_\rho$$

$$\mathbf{c}_i{}^5 = 120\mathbf{c}_i+108\mathbf{c}_r\mathbf{c}_i \qquad +104\mathbf{c}_\rho\mathbf{c}_i$$

$$=120\mathbf{c}_i+108(4\mathbf{c}_R+4\mathbf{c}_i)+104(2\mathbf{c}_R+\mathbf{c}_i)$$

$$=640\mathbf{c}_R+656\mathbf{c}_i$$

Octahedral O class minimal equations

$$\mathbf{c_g} = \sum_{\mu} \frac{{}^\circ c_g \chi_g^\mu}{\ell^\mu} \mathbb{P}^\mu$$

$$\mathbb{P}^\mu = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^\mu}{{}^\circ G} \chi_g^{\mu *} \mathbf{c_g}$$

O class product table

$\mathbf{1} = c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_i$

$$\mathbf{c}_i^2 = 6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = 6\cdot\mathbf{1}\mathbf{c}_i+3\mathbf{c}_r\mathbf{c}_i+2\mathbf{c}_\rho\mathbf{c}_i$$

$$=6\mathbf{c}_i+3(4\mathbf{c}_R+4\mathbf{c}_i)+2(2\mathbf{c}_R+\mathbf{c}_i)$$

$$\mathbf{c}_i^3 = 16\mathbf{c}_R+20\mathbf{c}_i$$

$$\mathbf{c}_i^4 = \quad 16\mathbf{c}_R\mathbf{c}_i \quad +20\mathbf{c}_i\mathbf{c}_i$$

$$=16(3\mathbf{c}_r+4\mathbf{c}_\rho)+20(6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho)$$

$$=48\mathbf{c}_r+64\mathbf{c}_\rho+120\cdot\mathbf{1}+60\mathbf{c}_r+40\mathbf{c}_\rho)$$

$$=120\cdot\mathbf{1}+108\mathbf{c}_r+104\mathbf{c}_\rho$$

$$\mathbf{c}_i^5 = 120\mathbf{c}_i+108\mathbf{c}_r\mathbf{c}_i \qquad +104\mathbf{c}_\rho\mathbf{c}_i$$

$$=120\mathbf{c}_i+108(4\mathbf{c}_R+4\mathbf{c}_i)+104(2\mathbf{c}_R+\mathbf{c}_i)$$

$$=640\mathbf{c}_R+656\mathbf{c}_i$$

$$40\mathbf{c}_i^3 = 640\mathbf{c}_R+800\mathbf{c}_i$$

Octahedral O class minimal equations

$$\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_g \chi^{\mu}_g}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi^{\mu *}_g \mathbf{c_g}$$

O class product table

$\mathbf{1} = c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_i$

$$\mathbf{c}_i^2 = 6\cdot\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = 6\cdot\mathbf{1}\mathbf{c}_i + 3\mathbf{c}_r\mathbf{c}_i + 2\mathbf{c}_\rho\mathbf{c}_i$$

$$= 6\mathbf{c}_i + 3(4\mathbf{c}_R + 4\mathbf{c}_i) + 2(2\mathbf{c}_R + \mathbf{c}_i)$$

$$\mathbf{c}_i^3 = 16\mathbf{c}_R + 20\mathbf{c}_i$$

$$\mathbf{c}_i^4 = \quad 16\mathbf{c}_R\mathbf{c}_i \quad + 20\mathbf{c}_i\mathbf{c}_i$$

$$= 16(3\mathbf{c}_r + 4\mathbf{c}_\rho) + 20(6\cdot\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho)$$

$$= 48\mathbf{c}_r + 64\mathbf{c}_\rho + 120\cdot\mathbf{1} + 60\mathbf{c}_r + 40\mathbf{c}_\rho)$$

$$= 120\cdot\mathbf{1} + 108\mathbf{c}_r + 104\mathbf{c}_\rho$$

$$\mathbf{c}_i^5 = 120\mathbf{c}_i + 108\mathbf{c}_r\mathbf{c}_i \qquad + 104\mathbf{c}_\rho\mathbf{c}_i$$

$$= 120\mathbf{c}_i + 108(4\mathbf{c}_R + 4\mathbf{c}_i) + 104(2\mathbf{c}_R + \mathbf{c}_i)$$

$$= 640\mathbf{c}_R + 656\mathbf{c}_i$$

$$40\mathbf{c}_i^3 = 640\mathbf{c}_R + 800\mathbf{c}_i$$

$$\begin{array}{r} 800 \\ -\underline{656} \\ 144 \end{array}$$

$$\mathbf{c}_i^5 - 40\mathbf{c}_i^3 + 144\mathbf{c}_i = \mathbf{0}$$

Octahedral O class minimal equations

$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_g^{\mu *} \mathbf{c_g}}$$

O class product table

$\mathbf{1} = c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_i$

$$\mathbf{c}_i^2 = 6\cdot\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = 6\cdot\mathbf{1}\mathbf{c}_i + 3\mathbf{c}_r\mathbf{c}_i + 2\mathbf{c}_\rho\mathbf{c}_i$$

$$= 6\mathbf{c}_i + 3(4\mathbf{c}_R + 4\mathbf{c}_i) + 2(2\mathbf{c}_R + \mathbf{c}_i)$$

$$\mathbf{c}_i^3 = 16\mathbf{c}_R + 20\mathbf{c}_i$$

$$\mathbf{c}_i^4 = 16\mathbf{c}_R\mathbf{c}_i + 20\mathbf{c}_i\mathbf{c}_i$$

$$= 16(3\mathbf{c}_r + 4\mathbf{c}_\rho) + 20(6\cdot\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho)$$

$$= 48\mathbf{c}_r + 64\mathbf{c}_\rho + 120\cdot\mathbf{1} + 60\mathbf{c}_r + 40\mathbf{c}_\rho)$$

$$= 120\cdot\mathbf{1} + 108\mathbf{c}_r + 104\mathbf{c}_\rho$$

$$\mathbf{c}_i^5 = 120\mathbf{c}_i + 108\mathbf{c}_r\mathbf{c}_i + 104\mathbf{c}_\rho\mathbf{c}_i$$

$$= 120\mathbf{c}_i + 108(4\mathbf{c}_R + 4\mathbf{c}_i) + 104(2\mathbf{c}_R + \mathbf{c}_i)$$

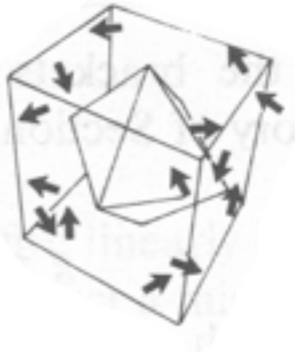
$$= 640\mathbf{c}_R + 656\mathbf{c}_i$$

$$40\mathbf{c}_i^3 = 640\mathbf{c}_R + 800\mathbf{c}_i$$

$$\mathbf{c}_i^5 - 40\mathbf{c}_i^3 + 144\mathbf{c}_i = \mathbf{0} = (\mathbf{c}_i^2 - 36\cdot\mathbf{1})(\mathbf{c}_i^2 - 4\cdot\mathbf{1})(\mathbf{c}_i - 0\cdot\mathbf{1})$$

$$\begin{array}{r} 800 \\ \underline{-656} \\ 144 \end{array}$$

Octahedral O class minimal equations



$$\boxed{\mathbf{c}_{\mathbf{g}} = \sum_{\mu} \frac{{}^{\circ}c_{g}\chi^{\mu}_{g}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G}\chi^{\mu *}_{g}\mathbf{c}_{\mathbf{g}}}$$

O class product table

$\mathbf{1} = c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_i$

$$\mathbf{c}_i^2=6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3=6\cdot\mathbf{1}\mathbf{c}_i+3\mathbf{c}_r\mathbf{c}_i+2\mathbf{c}_\rho\mathbf{c}_i$$

$$=6\mathbf{c}_i+3(4\mathbf{c}_R+4\mathbf{c}_i)+2(2\mathbf{c}_R+\mathbf{c}_i)$$

$$\mathbf{c}_i^3=16\mathbf{c}_R+20\mathbf{c}_i$$

$$\mathbf{c}_i^4= \quad 16\mathbf{c}_R\mathbf{c}_i \quad +20\mathbf{c}_i\mathbf{c}_i$$

$$=16(3\mathbf{c}_r+4\mathbf{c}_\rho)+20(6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho)$$

$$=48\mathbf{c}_r+64\mathbf{c}_\rho+120\cdot\mathbf{1}+60\mathbf{c}_r+40\mathbf{c}_\rho)$$

$$=120\cdot\mathbf{1}+108\mathbf{c}_r+104\mathbf{c}_\rho$$

$$\mathbf{c}_i^5=120\mathbf{c}_i+108\mathbf{c}_r\mathbf{c}_i \qquad +104\mathbf{c}_\rho\mathbf{c}_i$$

$$=120\mathbf{c}_i+108(4\mathbf{c}_R+4\mathbf{c}_i)+104(2\mathbf{c}_R+\mathbf{c}_i)$$

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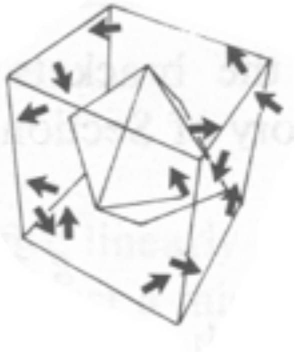
$$40\mathbf{c}_i^3=640\mathbf{c}_R+800\mathbf{c}_i$$

$$\mathbf{c}_i^5-40\mathbf{c}_i^3+144\mathbf{c}_i=\mathbf{0}=(\mathbf{c}_i^2-36\cdot\mathbf{1})(\mathbf{c}_i^2-4\cdot\mathbf{1})(\mathbf{c}_i-0\cdot\mathbf{1})$$

$$\begin{array}{r} 800 \\ \underline{-656} \\ 144 \end{array}$$

Minimal equation for $\mathbf{c}_i$

$$\boxed{\mathbf{0}=(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i-2\cdot\mathbf{1})(\mathbf{c}_i-0\cdot\mathbf{1})}$$



Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$
- *Octahedral-cubic O symmetry and group operations*
- *Tetrahedral symmetry becomes Icosahedral*
- *Octahedral groups $O_h \supset O \sim T_d \supset T$*
  - *Octahedral O and spin-$O \subset U(2)$*

Tetrahedral T class algebra
- *Tetrahedral T class minimal equations*
- *Tetrahedral T class projectors and characters*

Octahedral O class algebra
- *Octahedral O class minimal equations*
- *Octahedral O class projectors and characters*

Octahedral $O_h \supset O$ subgroup correlations
- *Octahedral $O_h \supset O$ subgroup correlations*
- *$O_h \supset O \supset D_4$ subgroup correlations*
- *$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations*
- *Preview of applications to high resolution spectroscopy*

Octahedral O projector algebra

Begin with minimal equation: $\mathbf{0}= (\mathbf{c}_i+2\cdot\mathbf{1})\cancel{(\mathbf{c}_i-2\cdot\mathbf{1})}(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i-0\cdot\mathbf{1})$

$$\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_g \chi^{\mu}_{g}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi^{\mu *}_{g} \mathbf{c_g}$$

$$\mathbf{P}^{(2)} = \frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-0)}{(2+2)(2-6)(2+6)(2-0)}$$

O class product table

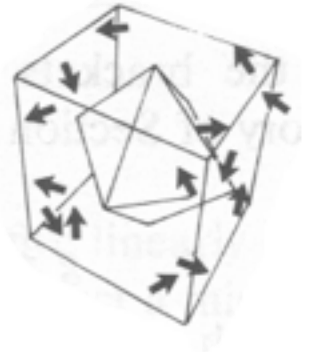
$\mathbf{1}=c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Octahedral O projector algebra

Begin with minimal equation: $\boxed{\mathbf{0}= (\mathbf{c}_i+2\cdot\mathbf{1})(\cancel{\mathbf{c}_i-2\cdot\mathbf{1}})(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i-0\cdot\mathbf{1})}$

$$\boxed{\mathbf{c}_\mathbf{g} = \sum_\mu \frac{{}^\circ c_g \chi_g^\mu}{\ell^\mu}\mathbb{P}^\mu}$$

$$\boxed{\mathbb{P}^\mu = \sum_{classes\, c_\mathbf{g}} \frac{\ell^\mu}{{}^\circ G}\chi_g^{\mu *}\mathbf{c}_\mathbf{g}}$$



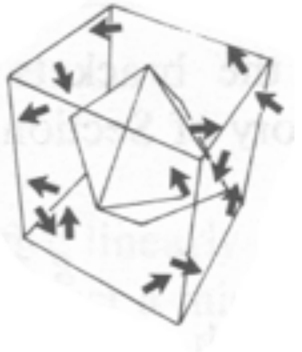
$$\mathbf{P}^{(2)} = \frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-0)}{(2+2)(2-6)(2+6)(2-0)} = \frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i^2-36\cdot\mathbf{1})\mathbf{c}_i}{-256}$$

O class product table

$\mathbf{1}=c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Octahedral O projector algebra

Begin with minimal equation: $\mathbf{0}= (\mathbf{c}_i +2\cdot\mathbf{1})\cancel{(\mathbf{c}_i -2\cdot\mathbf{1})}(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i-0\cdot\mathbf{1})$



$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

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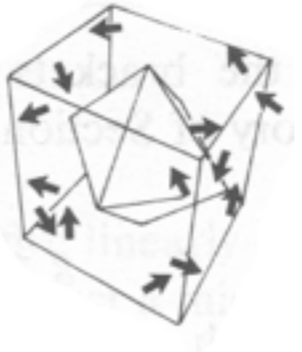
$$\mathbf{P}^{(2)} = \frac{(\mathbf{c}_i + 2\cdot\mathbf{1})(\mathbf{c}_i - 6\cdot\mathbf{1})(\mathbf{c}_i + 6\cdot\mathbf{1})(\mathbf{c}_i - 0)}{(2+2)(2-6)(2+6)(2-0)} = \frac{(\mathbf{c}_i + 2\cdot\mathbf{1})(\mathbf{c}_i^2 - 36\cdot\mathbf{1})\mathbf{c}_i}{-256} = \frac{\mathbf{c}_i^4 + 2\cdot\mathbf{c}_i^3 - 36\,\mathbf{c}_i^2 - 72\,\mathbf{c}_i}{-256}$$

O class product table

$\mathbf{1} = c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Octahedral O projector algebra

Begin with minimal equation: $\mathbf{0}= (\mathbf{c}_i +2\cdot\mathbf{1})\cancel{(\mathbf{c}_i -2\cdot\mathbf{1})}(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i-0\cdot\mathbf{1})$



$$\mathbf{c}_\mathbf{g} = \sum_\mu \frac{{}^\circ c_g \chi_g^\mu}{\ell^\mu}\mathbb{P}^\mu$$

$$\mathbb{P}^\mu = \sum_{classes\, c_\mathbf{g}} \frac{\ell^\mu}{{}^\circ G}\chi_g^{\mu *}\mathbf{c}_\mathbf{g}$$

$$\mathbf{P}^{(2)} = \frac{(\mathbf{c}_i + 2\cdot\mathbf{1})(\mathbf{c}_i - 6\cdot\mathbf{1})(\mathbf{c}_i + 6\cdot\mathbf{1})(\mathbf{c}_i - 0)}{(2+2)(2-6)(2+6)(2-0)} = \frac{(\mathbf{c}_i + 2\cdot\mathbf{1})(\mathbf{c}^2{}_i - 36\cdot\mathbf{1})\mathbf{c}_i}{-256} = \frac{\mathbf{c}^4{}_i + 2\cdot\mathbf{c}^3{}_i - 36\,\mathbf{c}^2{}_i - 72\,\mathbf{c}_{\ i}}{-256}$$

Expanding $\mathbf{P}^{(2)}$

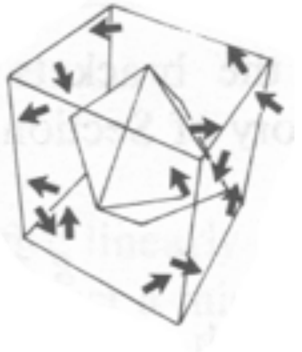
$$\mathbf{c}_i = \qquad\qquad + \qquad \mathbf{c}_i$$

O class product table

$\mathbf{1} = c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Octahedral O projector algebra

Begin with minimal equation: $\boxed{\mathbf{0}=(\mathbf{c}_i+2\cdot\mathbf{1})\cancel{(\mathbf{c}_i-2\cdot\mathbf{1})}(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i-0\cdot\mathbf{1})}$



$$\boxed{\mathbf{c}_\mathbf{g}=\sum_{\mu}\frac{{}^{\circ}c_g\chi_g^{\mu}}{\ell^{\mu}}\mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu}=\sum_{classes\, c_\mathbf{g}}\frac{\ell^{\mu}}{{}^{\circ}G}\chi_g^{\mu*}\mathbf{c}_\mathbf{g}}$$

$$\mathbf{P}^{(2)}=\frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-0)}{(2+2)(2-6)(2+6)(2-0)}=\frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i^2-36\cdot\mathbf{1})\mathbf{c}_i}{-256}=\frac{\mathbf{c}_i^4+2\cdot\mathbf{c}_i^3-36\,\mathbf{c}_i^2-72\,\mathbf{c}_i}{-256}$$

Expanding $\mathbf{P}^{(2)}$

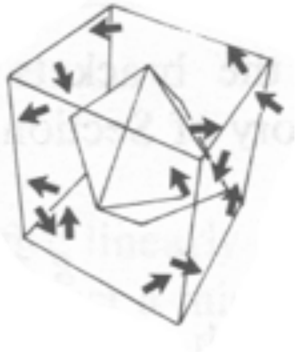
$\mathbf{c}_i^2=\quad 6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

$\mathbf{c}_i\ =\qquad\qquad\qquad\qquad +\qquad \mathbf{c}_i$

O class product table

$\mathbf{1}=c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Octahedral O projector algebra



Begin with minimal equation: $\mathbf{0} = (\mathbf{c}_i + 2\cdot\mathbf{1})(\cancel{\mathbf{c}_i - 2\cdot\mathbf{1}})(\mathbf{c}_i + 6\cdot\mathbf{1})(\mathbf{c}_i - 6\cdot\mathbf{1})(\mathbf{c}_i - 0\cdot\mathbf{1})$

$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^\circ c_g \chi_g^\mu}{\ell^\mu} \mathbb{P}^\mu}$$

$$\boxed{\mathbb{P}^\mu = \sum_{classes\, c_\mathbf{g}} \frac{\ell^\mu}{{}^\circ G} \chi_g^{\mu *} \mathbf{c_g}}$$

$$\mathbf{P}^{(2)} = \frac{(\mathbf{c}_i + 2\cdot\mathbf{1})(\mathbf{c}_i - 6\cdot\mathbf{1})(\mathbf{c}_i + 6\cdot\mathbf{1})(\mathbf{c}_i - 0)}{(2+2)(2-6)(2+6)(2-0)} = \frac{(\mathbf{c}_i + 2\cdot\mathbf{1})(\mathbf{c}_i^2 - 36\cdot\mathbf{1})\mathbf{c}_i}{-256} = \frac{\mathbf{c}_i^4 + 2\cdot\mathbf{c}_i^3 - 36\,\mathbf{c}_i^2 - 72\,\mathbf{c}_i}{-256}$$

Expanding $\mathbf{P}^{(2)}$

$\mathbf{c}_i^3 =$ $\qquad\qquad\qquad$ $+16\mathbf{c}_R + 20\mathbf{c}_i$

$\mathbf{c}_i^2 =$ $\quad 6\cdot\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$

$\mathbf{c}_i =$ $\qquad\qquad\qquad$ $+ \qquad \mathbf{c}_i$

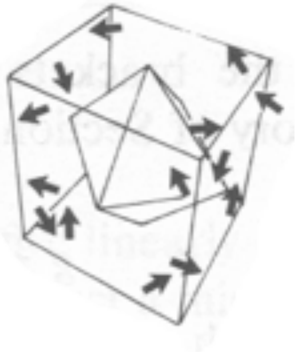
O class product table

$\mathbf{1} = c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Octahedral O projector algebra

Begin with minimal equation:

$$\mathbf{0}= (\mathbf{c}_i +2\cdot\mathbf{1})\cancel{(\mathbf{c}_i -2\cdot\mathbf{1})}(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i-0\cdot\mathbf{1})$$



$$\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu}= \sum_{classes\, c_\mathbf{g}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_g^{\mu *} \mathbf{c_g}$$

$$\mathbf{P}^{(2)} = \frac{(\mathbf{c}_i + 2\cdot\mathbf{1})(\mathbf{c}_i - 6\cdot\mathbf{1})(\mathbf{c}_i + 6\cdot\mathbf{1})(\mathbf{c}_i - 0)}{(2+2)(2-6)(2+6)(2-0)} = \frac{(\mathbf{c}_i + 2\cdot\mathbf{1})(\mathbf{c}_i^2 - 36\cdot\mathbf{1})\mathbf{c}_i}{-256} = \frac{\mathbf{c}_i^4 + 2\cdot\mathbf{c}_i^3 - 36\,\mathbf{c}_i^2 - 72\,\mathbf{c}_i}{-256}$$

Expanding $\mathbf{P}^{(2)}$

$\mathbf{c}_i^4 = 120\cdot\mathbf{1} + 108\mathbf{c}_r + 104\mathbf{c}_\rho$

$\mathbf{c}_i^3 = \qquad + 16\mathbf{c}_R + 20\mathbf{c}_i$

$\mathbf{c}_i^2 = \quad 6\cdot\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$

$\mathbf{c}_i \ = \qquad + \quad \mathbf{c}_i$

O class product table

$\mathbf{1} = c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
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$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
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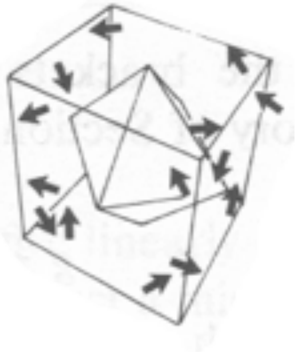

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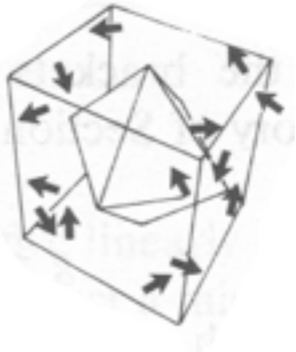

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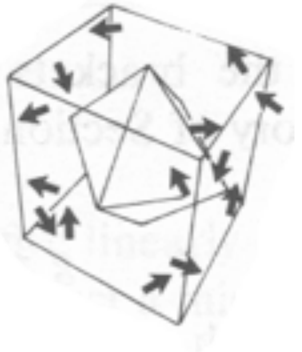

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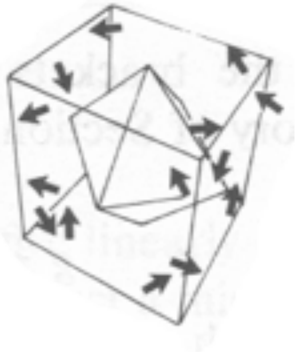

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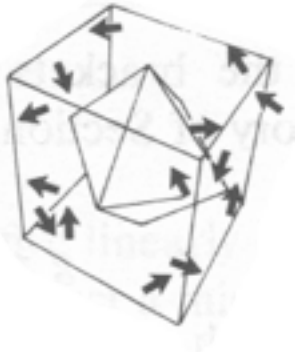

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Expanding $\mathbf{P}^{(2)}$

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\hline
-256\mathbf{P}^{(2)} &= -96\cdot\mathbf{1}+0\,\mathbf{c}_r+32\mathbf{c}_\rho+32\mathbf{c}_R-\;32\mathbf{c}_i
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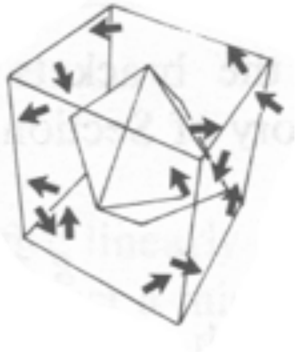

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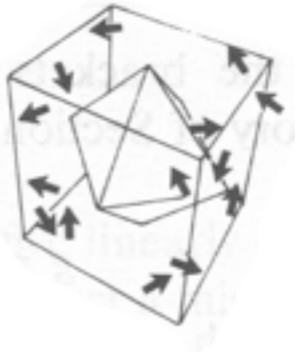

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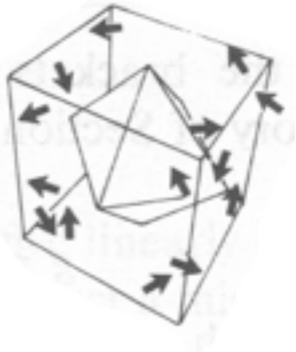
Applying the conventional label T_2 for (2)

χ^{μ}_{g}	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1\ldots4}$	$\boldsymbol{\rho}_{xyz}$	$\mathbf{R}_{xyz}$	$\mathbf{i}_{1\ldots6}$
·					
·					
·					
·					
χ^{T_2}	3	0	−1	−1	1

Octahedral O projector algebra

Begin with minimal equation: $\mathbf{0}=\cancel{(\mathbf{c}_i+2\cdot\mathbf{1})}(\mathbf{c}_i-2\cdot\mathbf{1})(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i-0\cdot\mathbf{1})$



$$\boxed{\mathbf{c}_\mathbf{g}=\sum_{\mu}\frac{{}^\circ c_g\chi_g^\mu}{\ell^\mu}\mathbb{P}^\mu}\qquad\boxed{\mathbb{P}^\mu=\sum_{classes\,c_\mathbf{g}}\frac{\ell^\mu}{{}^\circ G}\chi_g^{\mu*}\mathbf{c}_\mathbf{g}}$$

$$\mathbf{P}^{(2)}=\frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-0)}{(2+2)(2-6)(2+6)(2-0)}=\frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i^2-36\cdot\mathbf{1})\mathbf{c}_i}{-256}=\frac{\mathbf{c}_i^4+2\cdot\mathbf{c}_i^3-36\,\mathbf{c}_i^2-72\,\mathbf{c}_i}{-256}$$

Expanding $\mathbf{P}^{(2)}$

$$\begin{aligned}\mathbf{c}_i^4&=120\cdot\mathbf{1}+108\mathbf{c}_r+104\mathbf{c}_\rho\\ \mathbf{c}_i^3&=\qquad\qquad\qquad+16\mathbf{c}_R+20\mathbf{c}_i\\ \mathbf{c}_i^2&=\quad 6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho\\ \mathbf{c}_i&=\qquad\qquad\qquad\qquad+\quad\mathbf{c}_i\end{aligned}$$

$$\begin{aligned}\mathbf{c}_i^4&=120\cdot\mathbf{1}+108\mathbf{c}_r+104\mathbf{c}_\rho\\ +2\mathbf{c}_i^3&=\qquad\qquad\qquad\qquad\qquad+32\mathbf{c}_R+40\mathbf{c}_i\\ -36\mathbf{c}_i^2&=-216\cdot\mathbf{1}-108\mathbf{c}_r\ \ -72\mathbf{c}_\rho\\ -72\mathbf{c}_i&=\qquad\qquad\qquad\qquad\qquad\qquad\qquad-72\mathbf{c}_i\\ \hline -256\mathbf{P}^{(2)}&=-96\cdot\mathbf{1}+0\,\mathbf{c}_r+32\mathbf{c}_\rho+32\mathbf{c}_R-\ 32\mathbf{c}_i\end{aligned}$$

$$\mathbf{P}^{(2)}=\boxed{\frac{3}{8}\mathbf{1}+\frac{0}{8}\mathbf{c}_r-\frac{1}{8}\mathbf{c}_\rho-\frac{1}{8}\mathbf{c}_R+\frac{1}{8}\mathbf{c}_i}$$

Expansion of $\mathbf{P}^{(-2)}$ *has (-)sign on last 2 terms...*

$$\mathbf{P}^{(-2)}=\boxed{\frac{3}{8}\mathbf{1}+\frac{0}{8}\mathbf{c}_r-\frac{1}{8}\mathbf{c}_\rho+\frac{1}{8}\mathbf{c}_R-\frac{1}{8}\mathbf{c}_i}$$

O class product table

$\mathbf{1}=c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

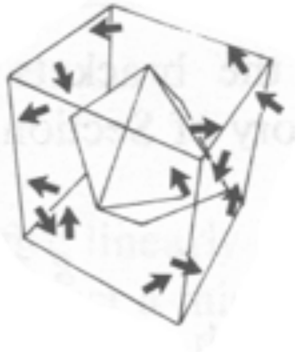
Octahedral O characters

χ_g^μ	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1\ldots4}$	$\boldsymbol{\rho}_{xyz}$	$\mathbf{R}_{xyz}$	$\mathbf{i}_{1\ldots6}$
·					
·					
·					
χ^{T_1}	3	0	−1	1	−1
χ^{T_2}	3	0	−1	−1	1

Applying the conventional label T_2 for (2) and T_1 for (-2)

Octahedral O projector algebra

Begin with minimal equation: $\mathbf{0}= (\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i-2\cdot\mathbf{1})(\mathbf{c}_i+6\cdot\mathbf{1})\cancel{(\mathbf{c}_i-6\cdot\mathbf{1})}(\mathbf{c}_i-0\cdot\mathbf{1})$



$$\boxed{\mathbf{c}_\mathbf{g}=\sum_\mu \frac{{}^\circ c_g \chi_g^\mu}{\ell^\mu}\mathbb{P}^\mu}$$

$$\boxed{\mathbb{P}^\mu=\sum_{classes\, c_\mathbf{g}} \frac{\ell^\mu}{{}^\circ G}\chi_g^{\mu *}\mathbf{c}_\mathbf{g}}$$

$$\mathbf{P}^{(2)}=\frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-0)}{(2+2)(2-6)(2+6)(2-0)}=\frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i^2-36\cdot\mathbf{1})\mathbf{c}_i}{-256}=\frac{\mathbf{c}_i^4+2\cdot\mathbf{c}_i^3-36\,\mathbf{c}_i^2-72\,\mathbf{c}_i}{-256}$$

$$\begin{aligned}
\mathbf{c}_i^4&=120\cdot\mathbf{1}+108\mathbf{c}_r+104\mathbf{c}_\rho\\
\mathbf{c}_i^3&=\qquad\qquad\qquad\qquad +16\mathbf{c}_R+20\mathbf{c}_i\\
\mathbf{c}_i^2&=\quad 6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho\\
\mathbf{c}_i\ &=\qquad\qquad\qquad\qquad\qquad +\quad \mathbf{c}_i
\end{aligned}$$

$$\begin{aligned}
\mathbf{c}_i^4 &=120\cdot\mathbf{1}+108\mathbf{c}_r+104\mathbf{c}_\rho\\
+2\mathbf{c}_i^3 &= \qquad\qquad\qquad\qquad\qquad +32\mathbf{c}_R+40\mathbf{c}_i\\
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-72\mathbf{c}_i &= \qquad\qquad\qquad\qquad\qquad\qquad\qquad -72\mathbf{c}_i\\
\hline
-256\mathbf{P}^{(2)}&=-96\cdot\mathbf{1}+0\,\mathbf{c}_r+32\mathbf{c}_\rho+32\mathbf{c}_R-\ 32\mathbf{c}_i
\end{aligned}$$

$$\mathbf{P}^{(2)}=\boxed{\frac{3}{8}\mathbf{1}+\frac{0}{8}\mathbf{c}_r-\frac{1}{8}\mathbf{c}_\rho-\frac{1}{8}\mathbf{c}_R+\frac{1}{8}\mathbf{c}_i}$$

Expansion of $\mathbf{P}^{(-2)}$ has (-)sign on last 2 terms...

$$\mathbf{P}^{(-2)}=\boxed{\frac{3}{8}\mathbf{1}+\frac{0}{8}\mathbf{c}_r-\frac{1}{8}\mathbf{c}_\rho+\frac{1}{8}\mathbf{c}_R-\frac{1}{8}\mathbf{c}_i}$$

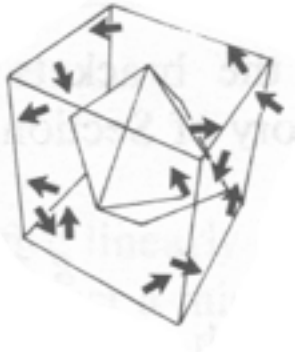
O class product table

$\mathbf{1}=c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Octahedral O characters

(Remaining character derivations left as an exercise)

χ_g^μ	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1\ldots4}$	$\boldsymbol{\rho}_{xyz}$	$\mathbf{R}_{xyz}$	$\mathbf{i}_{1\ldots6}$
χ^{A_1}	1	1	1	1	1
χ^{A_2}	1	1	1	−1	−1
χ^{E}	2	−1	2	0	0
χ^{T_1}	3	0	−1	1	−1
χ^{T_2}	3	0	−1	−1	1



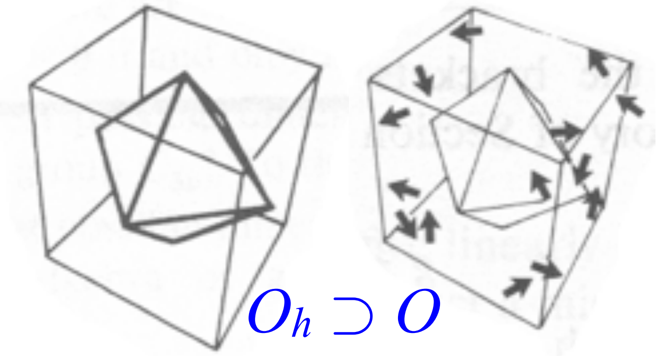
Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$
  Octahedral-cubic O symmetry and group operations
  Tetrahedral symmetry becomes Icosahedral
  Octahedral groups $O_h \supset O \sim T_d \supset T$
    Octahedral O and spin-$O \subset U(2)$

Tetrahedral T class algebra
  Tetrahedral T class minimal equations
  Tetrahedral T class projectors and characters

Octahedral O class algebra
  Octahedral O class minimal equations
  Octahedral O class projectors and characters

Octahedral $O_h \supset O$:Inversion (g&u) parity
  Octahedral $O_h \supset O \supset C_I$ subgroup correlations
  $O_h \supset O \supset D_4$ subgroup correlations
  $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations
  Preview of applications to high resolution spectroscopy

Octahedral $O_h = O\times\{\mathbf{1},\mathbf{I}\}$ characters of $O\times C_{\mathbf{I}}\supset O$



$O_h \supset O$ symmetry

EVEN parity *(gerade)*

	χ^μ_g	$\mathbf{g=1}$	$\mathbf{r}_{1...4}$	$\boldsymbol{\rho}_{xyz}$	$\mathbf{R}_{xyz}$	$\mathbf{i}_{1...6}$
A_{1g}	$\chi^{A_{1g}}$	1	1	1	1	1
A_{2g}	$\chi^{A_{2g}}$	1	1	1	−1	−1
E_g	χ^{E_g}	2	−1	2	0	0
T_{1g}	$\chi^{T_{1g}}$	3	0	−1	1	−1
T_{2g}	$\chi^{T_{2g}}$	3	0	−1	−1	1

3D – Inversion

$$\langle\mathbf{I}\rangle = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

$C_{\mathbf{I}}$-symmetry

1	$\mathbf{I}$
$\mathbf{I}$	1

$C_{\mathbf{I}}$-characters

$C_{\mathbf{I}}$	1	$\mathbf{I}$	± Parity P
g	*1*	*1*	*(gerade)*
u	*1*	*-1*	*(ungerade)*

O class product table

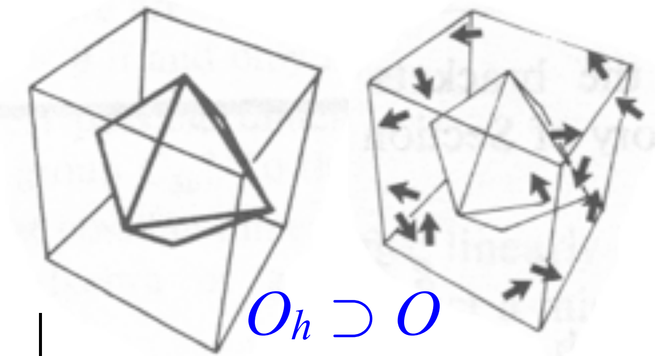
$\mathbf{1}=c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

(Remaining character derivations left as an exercise)

Octahedral O characters

χ^μ_g	$\mathbf{g=1}$	$\mathbf{r}_{1...4}$	$\boldsymbol{\rho}_{xyz}$	$\mathbf{R}_{xyz}$	$\mathbf{i}_{1...6}$
χ^{A_1}	1	1	1	1	1
χ^{A_2}	1	1	1	−1	−1
χ^{E}	2	−1	2	0	0
χ^{T_1}	3	0	−1	1	−1
χ^{T_2}	3	0	−1	−1	1

Octahedral $O_h = O\times\{\mathbf{1},\mathbf{I}\}$ characters of $O\times C_{\mathbf{I}}\supset O$



$O_h \supset O$ symmetry

	χ^{μ}_{g}	$\mathbf{g=1}$	$\mathbf{r}_{1\ldots4}$	$\boldsymbol{\rho}_{xyz}$	$\mathbf{R}_{xyz}$	$\mathbf{i}_{1\ldots6}$	$\mathbf{g=I}$	$\mathbf{Ir}_{1\ldots4}$	$\mathbf{I}\boldsymbol{\rho}_{xyz}$	$\mathbf{IR}_{xyz}$	$\mathbf{Ii}_{1\ldots6}$
A_{1g}	$\chi^{A_{1g}}$	1	1	1	1	1	1	1	1	1	1
A_{2g}	$\chi^{A_{2g}}$	1	1	1	−1	−1	1	1	1	−1	−1
E_{g}	$\chi^{E_{g}}$	2	−1	2	0	0	2	−1	2	0	0
T_{1g}	$\chi^{T_{1g}}$	3	0	−1	1	−1	3	0	−1	1	−1
T_{2g}	$\chi^{T_{2g}}$	3	0	−1	−1	1	3	0	−1	−1	1

EVEN
parity
(gerade)

3D – Inversion

$$\langle\mathbf{I}\rangle = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

$C_{\mathbf{I}}$-symmetry

1	**I**
I	1

$C_{\mathbf{I}}$-characters

$C_{\mathbf{I}}$	1	**I**
g	*1*	*1*
u	*1*	*-1*

±
Parity P
(gerade)
(ungerade)

O class product table

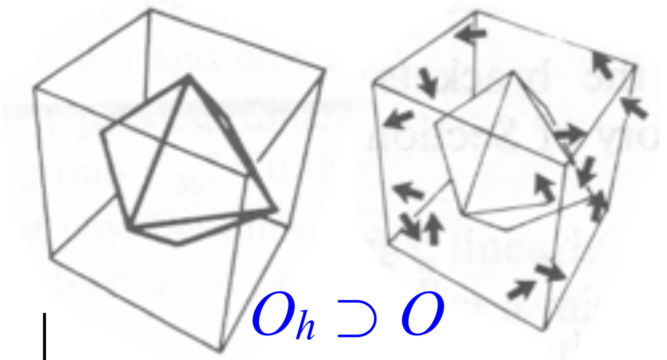
$\mathbf{1}=c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Octahedral O characters

(Remaining character derivations left as an exercise)

χ^{μ}_{g}	$\mathbf{g=1}$	$\mathbf{r}_{1\ldots4}$	$\boldsymbol{\rho}_{xyz}$	$\mathbf{R}_{xyz}$	$\mathbf{i}_{1\ldots6}$
χ^{A_1}	1	1	1	1	1
χ^{A_2}	1	1	1	−1	−1
χ^{E}	2	−1	2	0	0
χ^{T_1}	3	0	−1	1	−1
χ^{T_2}	3	0	−1	−1	1

Octahedral $O_h = O \times \{\mathbf{1},\mathbf{I}\}$ *characters of* $O \times C_{\mathbf{I}} \supset O$



$O_h \supset O$ symmetry

		χ^{μ}_{g}	$\mathbf{g=1}$	$\mathbf{r}_{1\ldots4}$	$\boldsymbol{\rho}_{xyz}$	$\mathbf{R}_{xyz}$	$\mathbf{i}_{1\ldots6}$	$\mathbf{g=I}$	$\mathbf{Ir}_{1\ldots4}$	$\mathbf{I\rho}_{xyz}$	$\mathbf{IR}_{xyz}$	$\mathbf{Ii}_{1\ldots6}$
EVEN parity *(gerade)*	A_{1g}	$\chi^{A_{1g}}$	1	1	1	1	1	1	1	1	1	1
	A_{2g}	$\chi^{A_{2g}}$	1	1	1	−1	−1	1	1	1	−1	−1
	E_g	χ^{E_g}	2	−1	2	0	0	2	−1	2	0	0
	T_{1g}	$\chi^{T_{1g}}$	3	0	−1	1	−1	3	0	−1	1	−1
	T_{2g}	$\chi^{T_{2g}}$	3	0	−1	−1	1	3	0	−1	−1	1
ODD parity *(ungerade)*	A_{1u}	$\chi^{A_{1u}}$	1	1	1	1	1					
	A_{2u}	$\chi^{A_{2u}}$	1	1	1	−1	−1					
	E_u	χ^{E_u}	2	−1	2	0	0					
	T_{1u}	$\chi^{T_{1u}}$	3	0	−1	1	−1					
	T_{2u}	$\chi^{T_{2u}}$	3	0	−1	−1	1					

3D – Inversion

$$\langle \mathbf{I} \rangle = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

$C_{\mathbf{I}}$-symmetry

1	**I**
I	1

$C_{\mathbf{I}}$-characters

$C_{\mathbf{I}}$	1	**I**	± Parity P
g	*1*	*1*	*(gerade)*
u	*1*	*-1*	*(ungerade)*

O class product table

$\mathbf{1} = c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Octahedral O characters

(Remaining character derivations left as an exercise)

χ^{μ}_{g}	$\mathbf{g=1}$	$\mathbf{r}_{1\ldots4}$	$\boldsymbol{\rho}_{xyz}$	$\mathbf{R}_{xyz}$	$\mathbf{i}_{1\ldots6}$
χ^{A_1}	1	1	1	1	1
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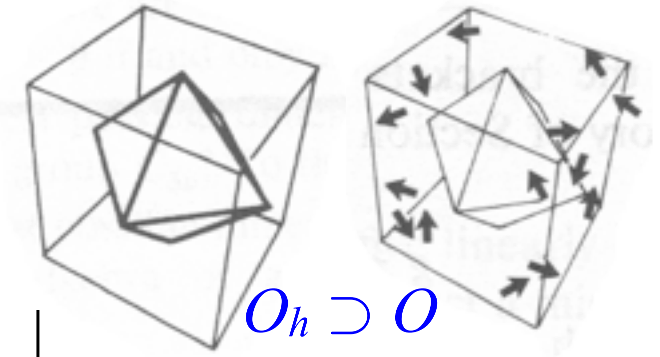
χ^{E}	2	−1	2	0	0
χ^{T_1}	3	0	−1	1	−1
χ^{T_2}	3	0	−1	−1	1

Octahedral $O_h = O\times\{\mathbf{1},\mathbf{I}\}$ characters of $O\times C_{\mathbf{I}}\supset O$

O_h easily derived from those of O and $C_{\mathbf{I}}$!



$O_h \supset O$ symmetry

		χ^{μ}_{g}	$\mathbf{g=1}$	$\mathbf{r}_{1\ldots4}$	$\boldsymbol{\rho}_{xyz}$	$\mathbf{R}_{xyz}$	$\mathbf{i}_{1\ldots6}$	$\mathbf{g=I}$	$\mathbf{Ir}_{1\ldots4}$	$\mathbf{I}\boldsymbol{\rho}_{xyz}$	$\mathbf{IR}_{xyz}$	$\mathbf{Ii}_{1\ldots6}$
EVEN parity *(gerade)*	A_{1g}	$\chi^{A_{1g}}$	1	1	1	1	1	1	1	1	1	1
	A_{2g}	$\chi^{A_{2g}}$	1	1	1	−1	−1	1	1	1	−1	−1
	E_g	χ^{E_g}	2	−1	2	0	0	2	−1	2	0	0
	T_{1g}	$\chi^{T_{1g}}$	3	0	−1	1	−1	3	0	−1	1	−1
	T_{2g}	$\chi^{T_{2g}}$	3	0	−1	−1	1	3	0	−1	−1	1
ODD parity *(ungerade)*	A_{1u}	$\chi^{A_{1u}}$	1	1	1	1	1	−1	−1	−1	−1	−1
	A_{2u}	$\chi^{A_{2u}}$	1	1	1	−1	−1	−1	−1	−1	+1	+1
	E_u	χ^{E_u}	2	−1	2	0	0	−2	+1	−2	0	0
	T_{1u}	$\chi^{T_{1u}}$	3	0	−1	1	−1	−3	0	+1	−1	+1
	T_{2u}	$\chi^{T_{2u}}$	3	0	−1	−1	1	−3	0	+1	+1	−1

3D – Inversion

$$\langle\mathbf{I}\rangle = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

$C_{\mathbf{I}}$-symmetry

1	$\mathbf{I}$
$\mathbf{I}$	1

$C_{\mathbf{I}}$-characters

$C_{\mathbf{I}}$	1	$\mathbf{I}$	± Parity P
g	*1*	*1*	*(gerade)*
u	*1*	*-1*	*(ungerade)*

O class product table

$\mathbf{1} = c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
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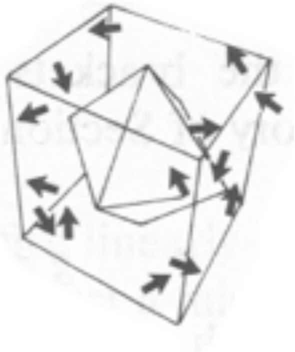
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Octahedral O characters

χ^{μ}_{g}	$\mathbf{g=1}$	$\mathbf{r}_{1\ldots4}$	$\boldsymbol{\rho}_{xyz}$	$\mathbf{R}_{xyz}$	$\mathbf{i}_{1\ldots6}$
χ^{A_1}	1	1	1	1	1
χ^{A_2}	1	1	1	−1	−1
χ^{E}	2	−1	2	0	0
χ^{T_1}	3	0	−1	1	−1
χ^{T_2}	3	0	−1	−1	1

(Remaining character derivations left as an exercise)



- *Introduction to octahedral/ tetrahedral symmetry* $O_h \supset O \sim T_d \supset T$
    - *Octahedral-cubic O symmetry and group operations*
    - *Tetrahedral symmetry becomes Icosahedral*
    - *Octahedral groups* $O_h \supset O \sim T_d \supset T$
        - *Octahedral O and spin-*$O \subset U(2)$

- *Tetrahedral T class algebra*
    - *Tetrahedral T class minimal equations*
    - *Tetrahedral T class projectors and characters*

- *Octahedral O class algebra*
    - *Octahedral O class minimal equations*
    - *Octahedral O class projectors and characters*

- *Octahedral* $O_h \supset O$*:Inversion (g&u) parity*
    - *Octahedral* $O_h \supset O \supset C_1$ *subgroup correlations*
    - $O_h \supset O \supset D_4$ *subgroup correlations*
    - $O_h \supset O \supset D_4 \supset C_4$ *subgroup correlations*
    - *Preview of applications to high resolution spectroscopy*

Octahedral $O_h \supset O$ subgroup correlations

χ_g^μ	**g = 1**	$\mathbf{r}_{1...4}$	$\boldsymbol{\rho}_{xyz}$	$\mathbf{R}_{xyz}$	$\mathbf{i}_{1...6}$
χ^{A_1}	1	1	1	1	1
χ^{A_2}	1	1	1	−1	−1
χ^{E}	2	−1	2	0	0
χ^{T_1}	3	0	−1	1	−1
χ^{T_2}	3	0	−1	−1	1

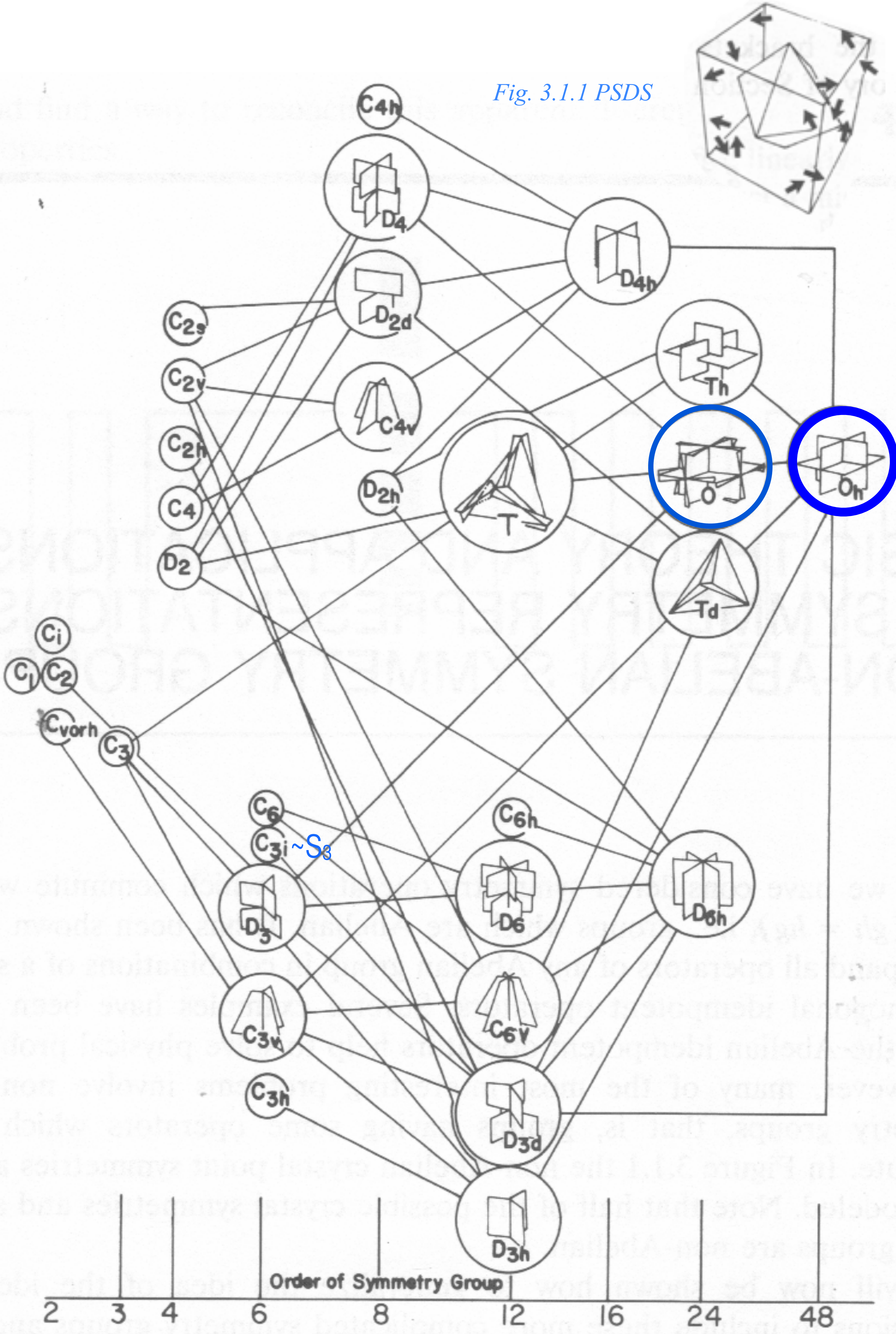


Fig. 3.1.1 PSDS

Octahedral $O_h \supset O$ subgroup correlations

$\chi_g^{\mu_p}$	**1**	$\mathbf{r}_{1...4}$	$\boldsymbol{\rho}_{xyz}$	$\mathbf{R}_{xyz}$	$\mathbf{i}_{1...6}$	**I**	**Ir=** $\mathbf{s}_{1...4}$	**Iρ=** $\boldsymbol{\sigma}_{xyz}$	**IR=** $\mathbf{S}_{xyz}$	**Ii=** $\boldsymbol{\sigma}_{1...6}$
$\chi^{A_{1g}}$	1	1	1	1	1	1	1	1	1	1
$\chi^{A_{2g}}$	1	1	1	−1	−1	1	1	1	−1	−1
χ^{E_g}	2	−1	2	0	0	2	−1	2	0	0
$\chi^{T_{1g}}$	3	0	−1	1	−1	3	0	−1	1	−1
$\chi^{T_{2g}}$	3	0	−1	−1	1	3	0	−1	−1	1
$\chi^{A_{1u}}$	1	1	1	1	1	−1	−1	−1	−1	−1
$\chi^{A_{2u}}$	1	1	1	−1	−1	−1	−1	−1	1	1
χ^{E_u}	2	−1	2	0	0	−2	1	−2	0	0
$\chi^{T_{1u}}$	3	0	−1	1	−1	−3	0	1	−1	1
$\chi^{T_{2u}}$	3	0	−1	−1	1	−3	0	1	1	−1

$O_h \supset O$	A_1	A_2	E	T_1	T_2
A_{1g}	1	·	·	·	·
A_{2g}	·	1	·	·	·
E_g	·	·	1	·	·
T_{1g}	·	·	·	1	·
T_{2g}	·	·	·	·	1
A_{1u}	1	·	·	·	·
A_{2u}	·	1	·	·	·
E_u	·	·	1	·	·
T_{1u}	·	·	·	1	·
T_{2u}	·	·	·	·	1

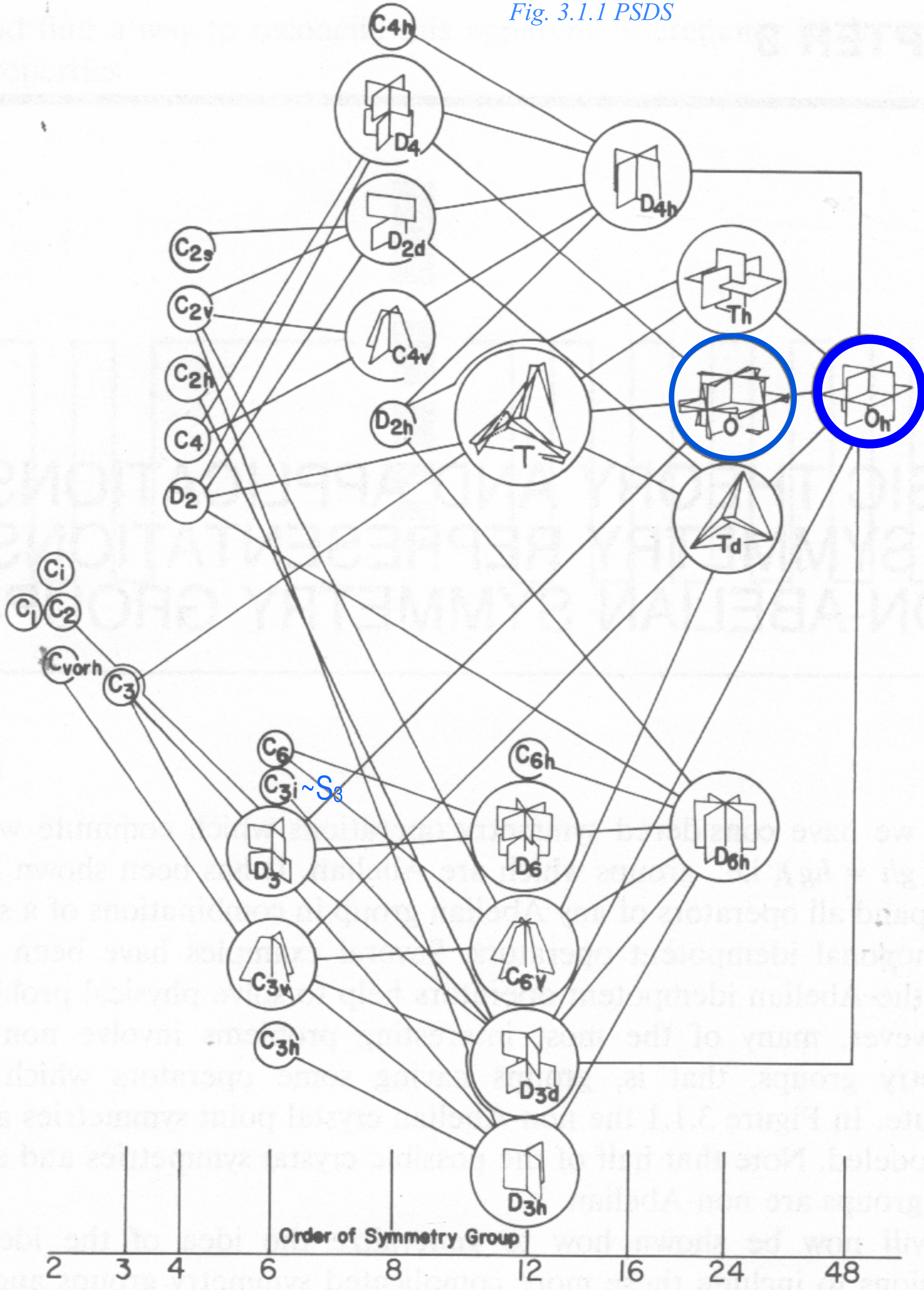


Fig. 3.1.1 PSDS

Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$
- *Octahedral-cubic O symmetry and group operations*
- *Tetrahedral symmetry becomes Icosahedral*
- *Octahedral groups $O_h \supset O \sim T_d \supset T$*
  - *Octahedral O and spin-$O \subset U(2)$*

Tetrahedral T class algebra
- *Tetrahedral T class minimal equations*
- *Tetrahedral T class projectors and characters*

Octahedral O class algebra
- *Octahedral O class minimal equations*
- *Octahedral O class projectors and characters*

Octahedral $O_h \supset O$ subgroup correlations (Parity)
- *Octahedral $O_h \supset O$ subgroup correlations*
- ➔ *$O_h \supset O \supset D_4$ subgroup correlations* ⬅
- *$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations*
- *Preview of applications to high resolution spectroscopy*

Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

D_4: $\mathbf{1}, \rho_{z180°}, \mathbf{R}_{z\pm 90°}, \rho_{z180°}, \mathbf{i}_{3,4}$

$\chi_g^\mu(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1\ldots4}$	180° ρ_{xyz}	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

$\chi_g^\mu(D_4)$	$\mathbf{g}=\mathbf{1}$	$\rho_{z180°}$	$\mathbf{R}_{z\pm 90°}$	$\rho_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0

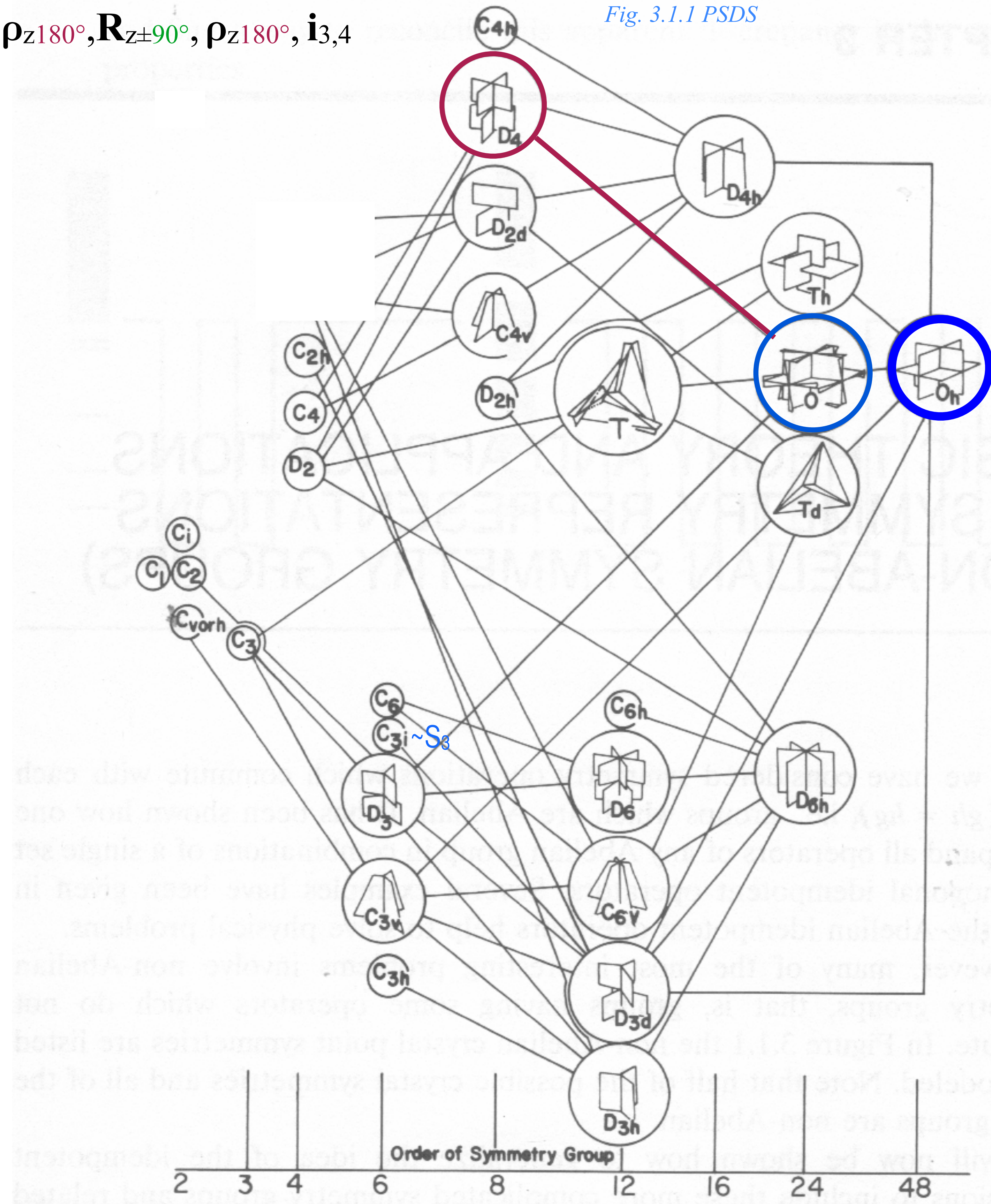


Fig. 3.1.1 PSDS

Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

D_4: **1**, $\boldsymbol{\rho}_{z180°}$, $\mathbf{R}_{z\pm90°}$, $\boldsymbol{\rho}_{x,y180°}$, $\mathbf{i}_{3,4}$

$\chi^{\mu}_{g}(O)$	$\mathbf{g=1}$	$\mathbf{r}_{1\ldots4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	1	1	1	1
A_2	1	1	1	−1	−1
E	2	−1	2	0	0
T_1	3	0	−1	1	−1
T_2	3	0	−1	−1	1

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g=1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g=1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i

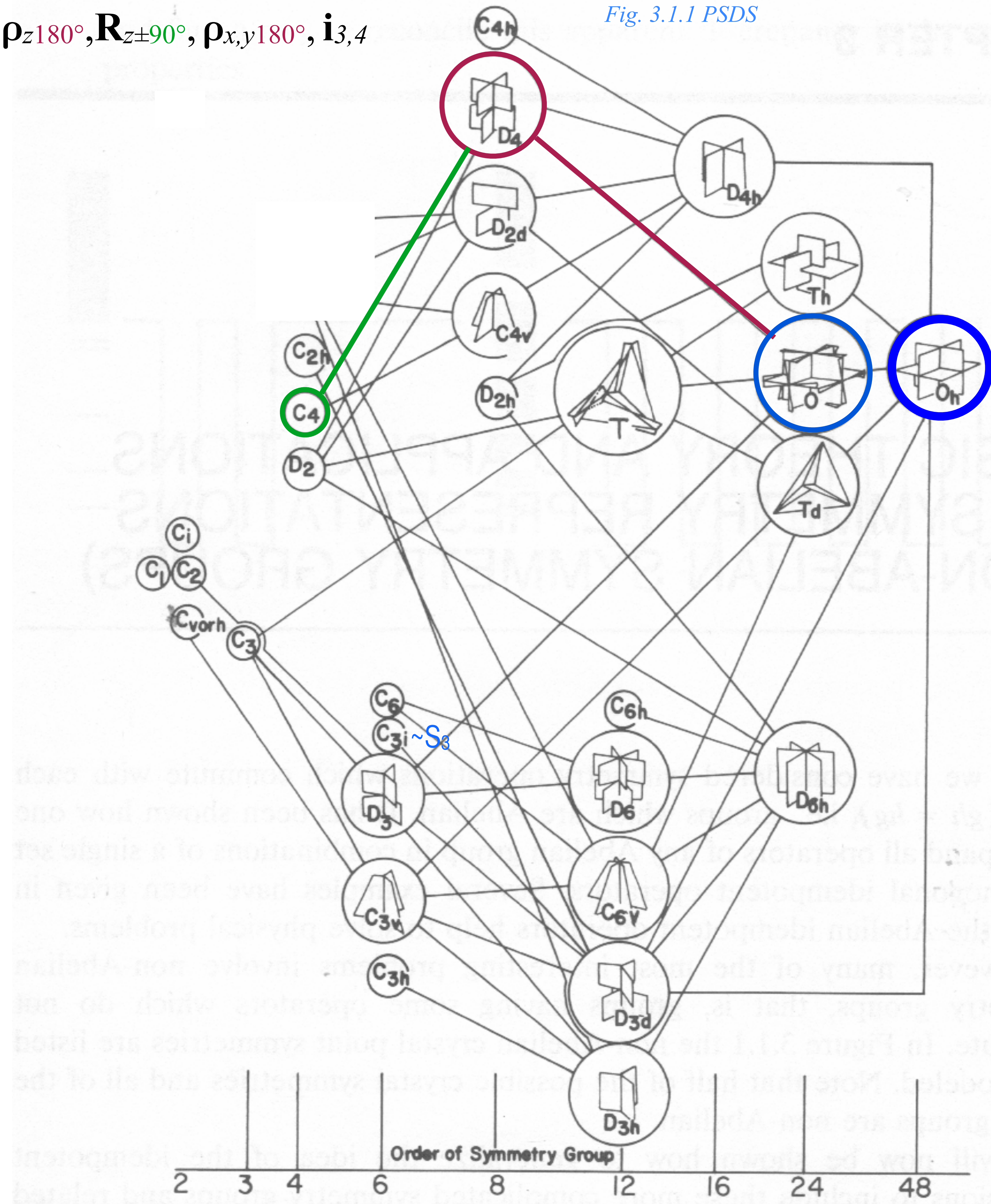


Fig. 3.1.1 PSDS

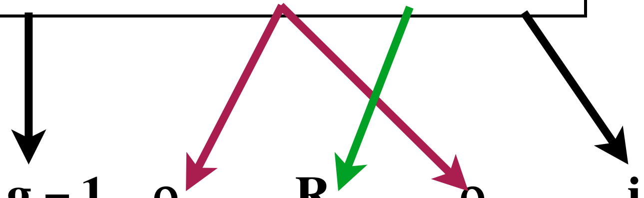
Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O\downarrow D_4$ subduction

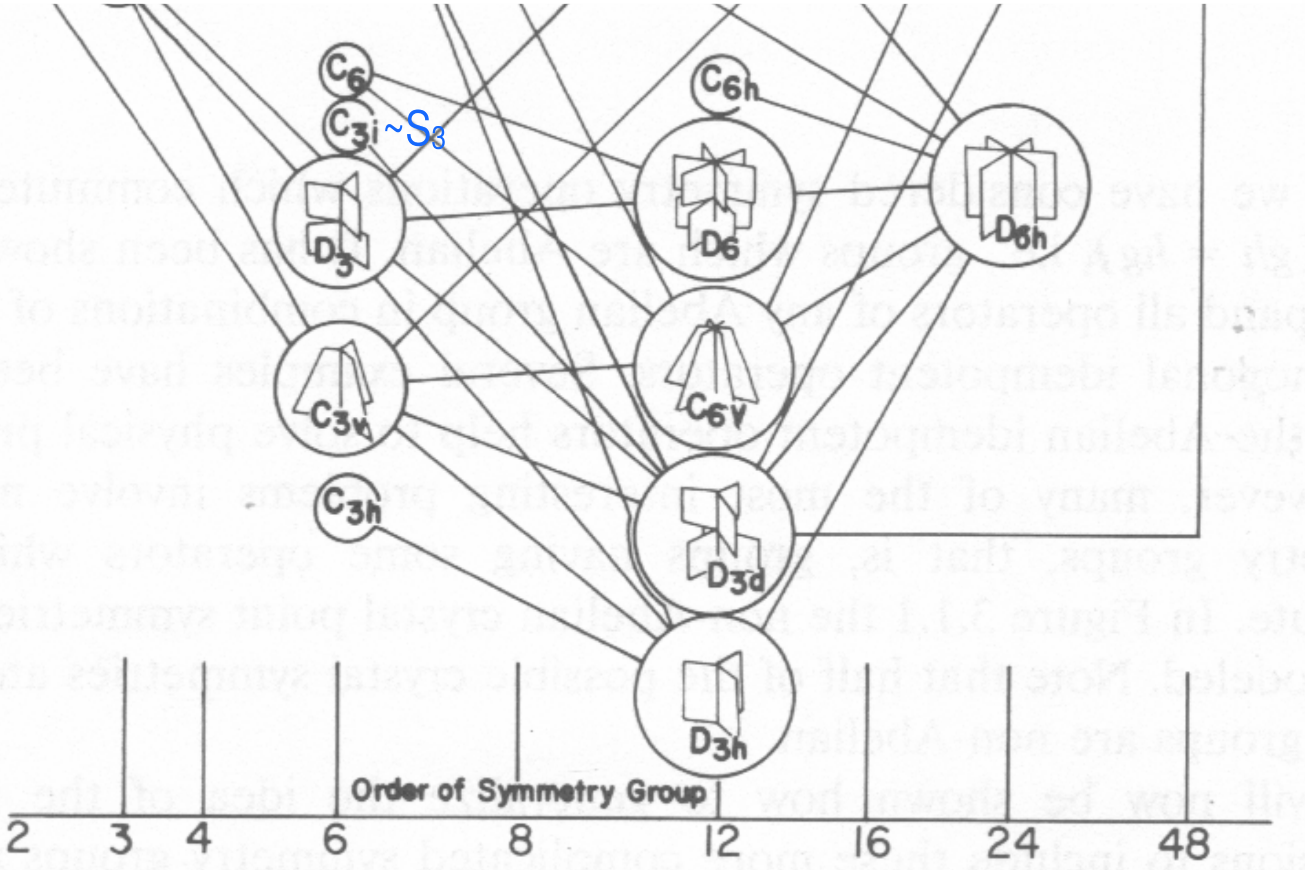
D_4: $\mathbf{1}, \rho_{z180°}, \mathbf{R}_{z\pm 90°}, \rho_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1, \quad 1, \quad 1, \quad 1, \quad 1 .$

$\chi_g^\mu(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1\ldots4}$	180° ρ_{xyz}	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1



$\chi_g^\mu(D_4)$	$\mathbf{g}=\mathbf{1}$	$\rho_{z180°}$	$\mathbf{R}_{z\pm 90°}$	$\rho_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

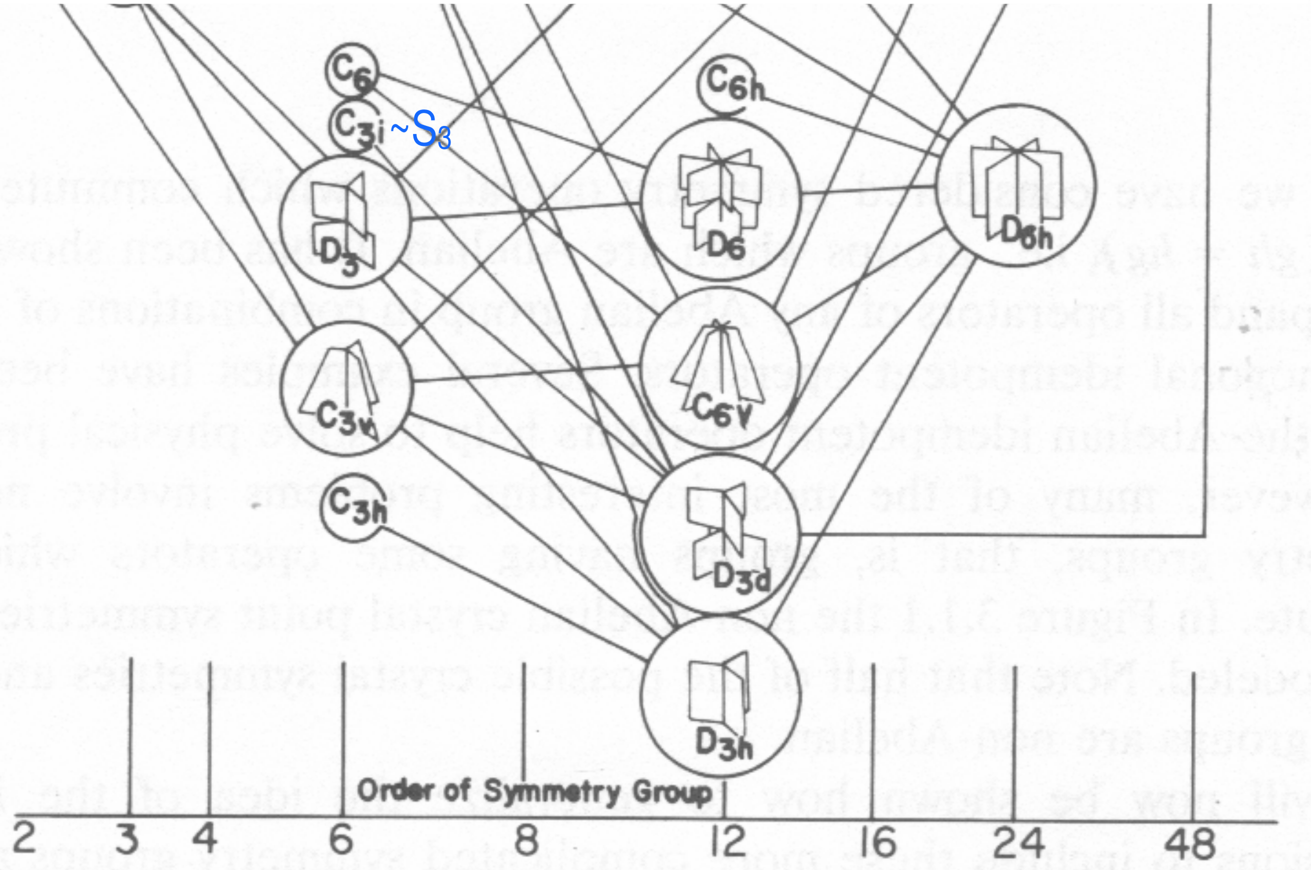
$O \downarrow D_4$ subduction

D_4: $\mathbf{1}, \rho_{z180°}, \mathbf{R}_{z\pm 90°}, \rho_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1, \quad 1, \quad 1, \quad 1, \quad 1.$ $\quad A_1(O)\downarrow D_4 = A_1(D_4)$

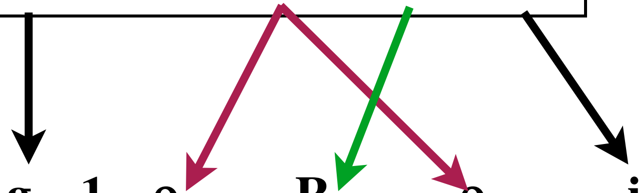
$\chi_g^\mu(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1\ldots4}$	180° ρ_{xyz}	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	1	1	1	1
A_2	1	1	1	−1	−1
E	2	−1	2	0	0
T_1	3	0	−1	1	−1
T_2	3	0	−1	−1	1

$\chi_g^\mu(D_4)$	$\mathbf{g}=\mathbf{1}$	$\rho_{z180°}$	$\mathbf{R}_{z\pm 90°}$	$\rho_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

			180°	90°	180°
$\chi_g^\mu(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1\ldots4}$	$\boldsymbol{\rho}_{xyz}$	$\mathbf{R}_{xyz}$	$\mathbf{i}_{1\ldots6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1



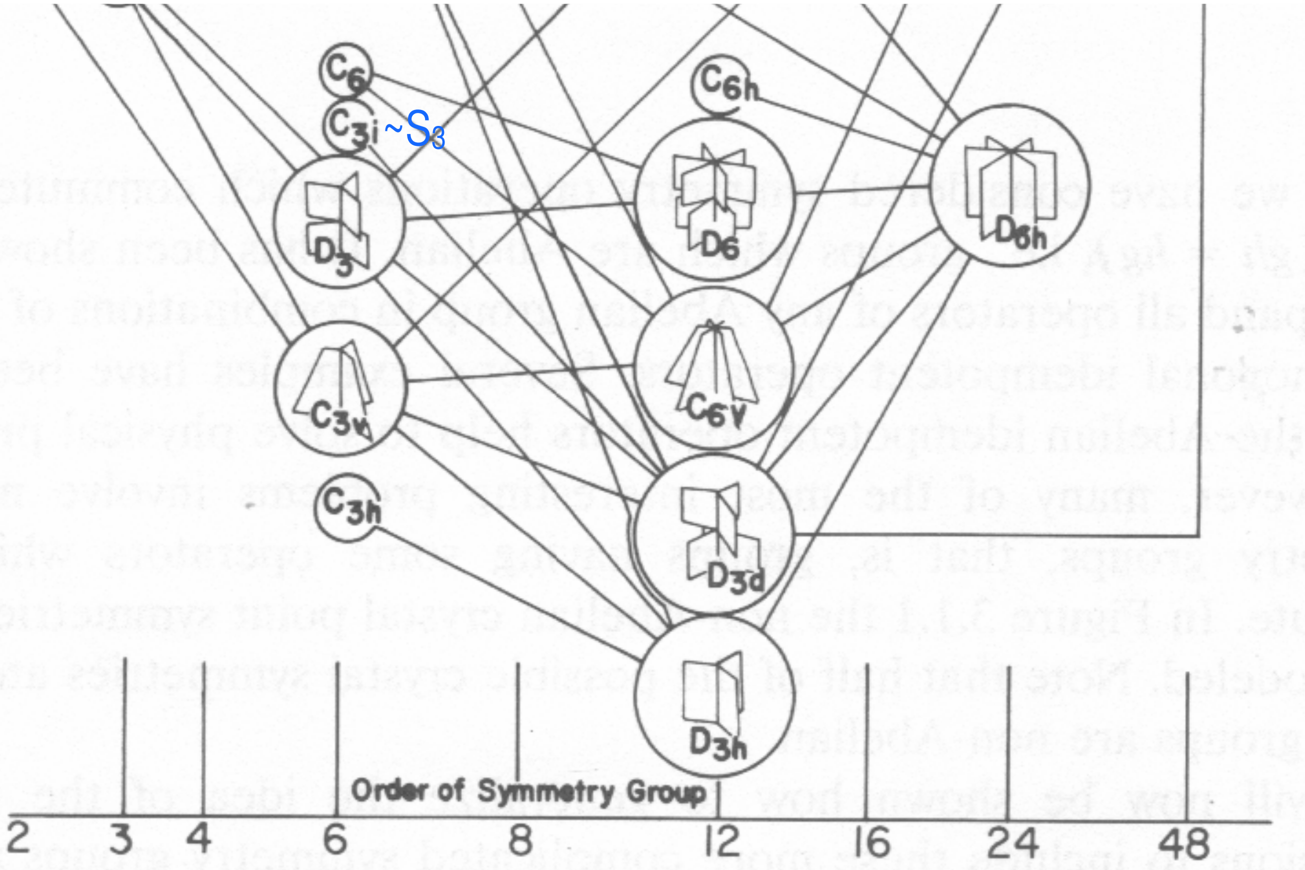
$\chi_g^\mu(D_4)$	$\mathbf{g}=\mathbf{1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0

$O{\downarrow}D_4$ subduction

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

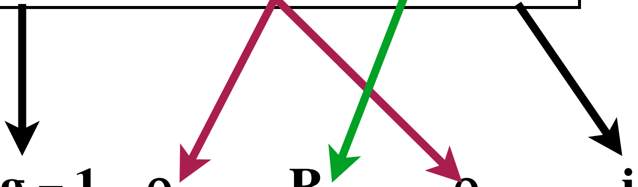
$A_1(O){\downarrow}D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$ $A_1(O){\downarrow}D_4 = A_1(D_4)$

$A_2(O){\downarrow}D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi^{\mu}_{g}(O)$	$\mathbf{g=1}$	~~$\mathbf{r}_{1\ldots4}$~~	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1



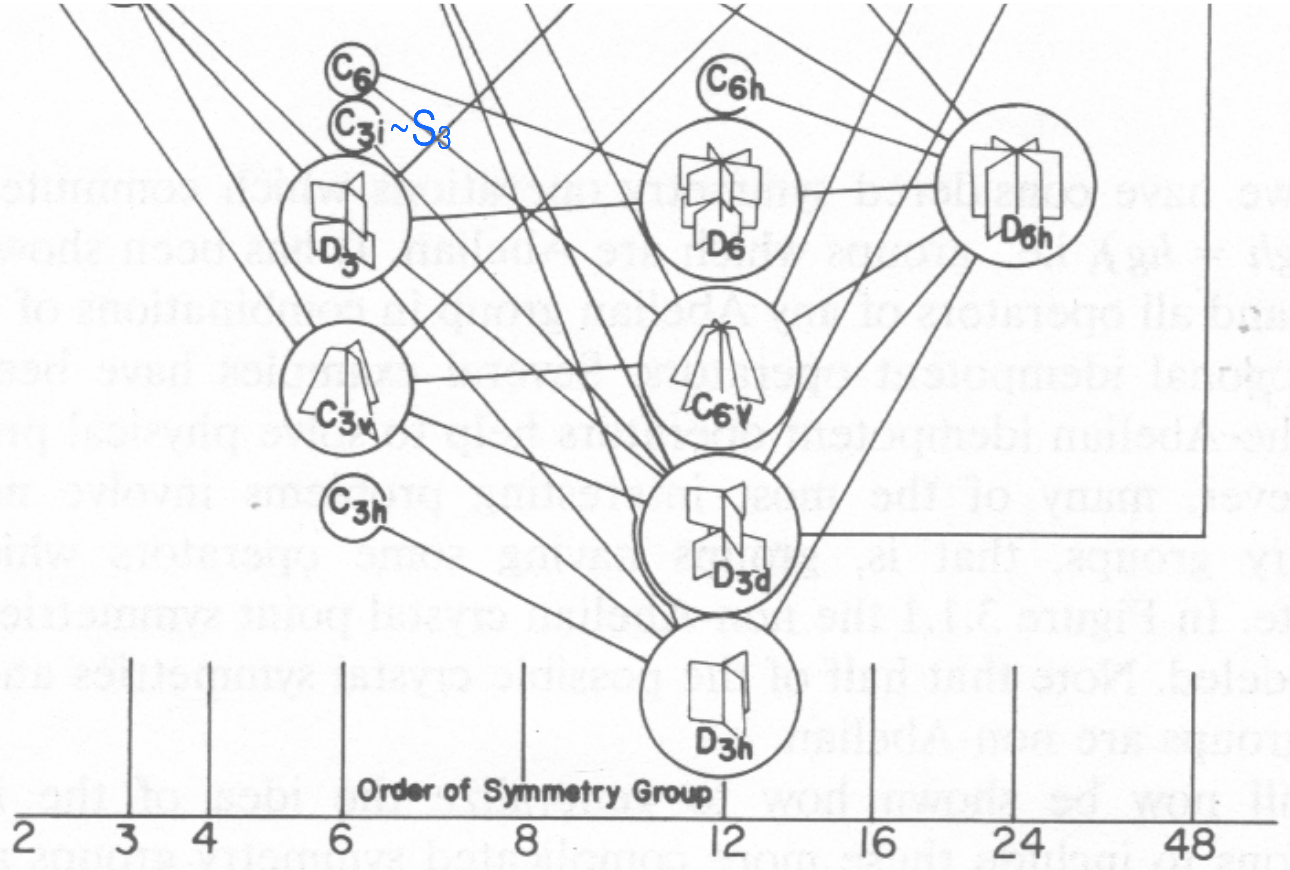
$\chi^{\mu}_{g}(D_4)$	$\mathbf{g=1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0

$O\downarrow D_4$ subduction

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$ $A_1(O)\downarrow D_4 = A_1(D_4)$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$ $A_2(O)\downarrow D_4 = B_1(D_4)$



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm 90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

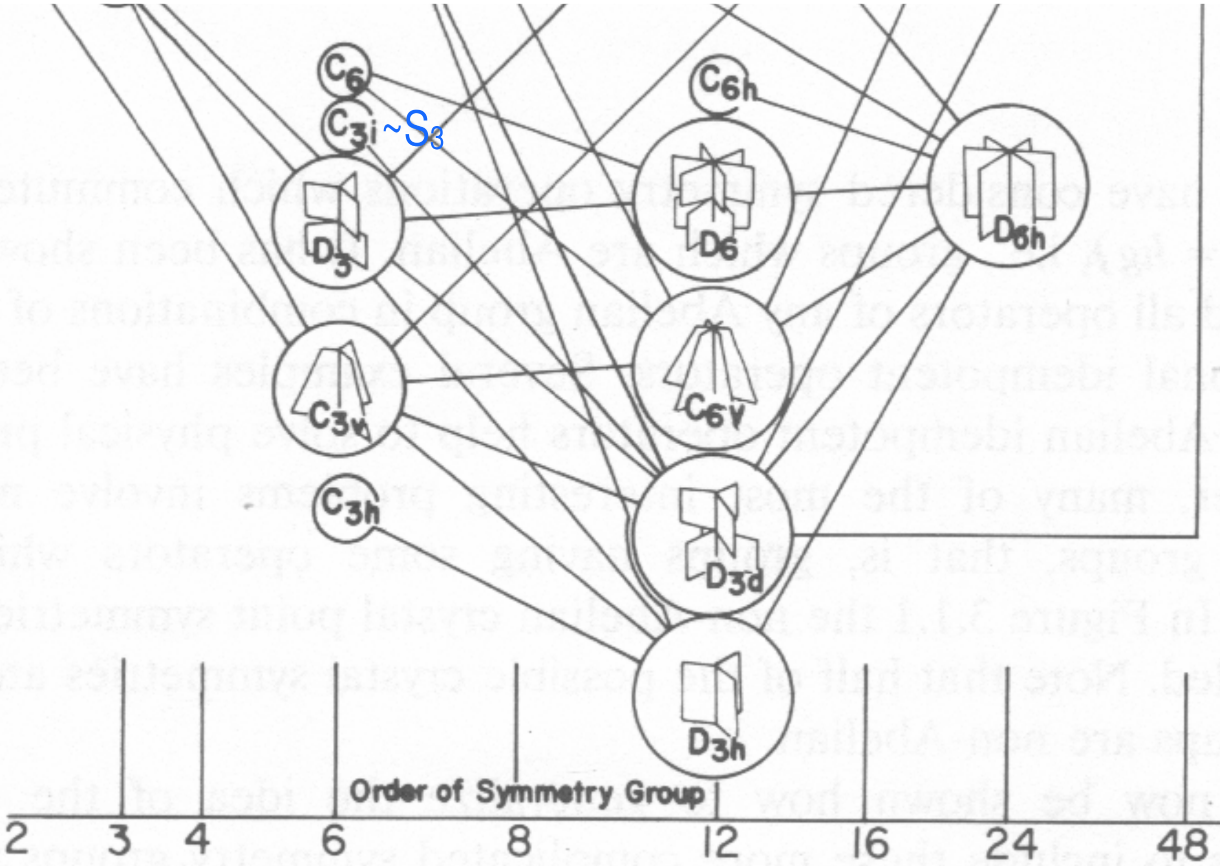
$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$ $A_1(O)\downarrow D_4 = A_1(D_4)$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$ $A_2(O)\downarrow D_4 = B_1(D_4)$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

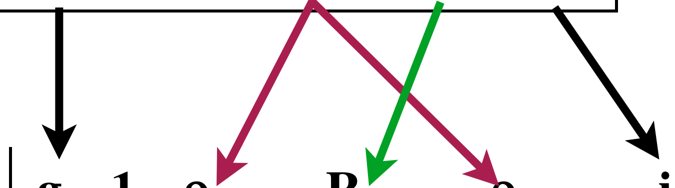
$\chi_g^\mu(O)$	$\mathbf{g=1}$	$\mathbf{r}_{1\ldots4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

$\chi_g^\mu(D_4)$	$\mathbf{g=1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi^{\mu}_{g}(O)$	$\mathbf{g=1}$	$\mathbf{r}_{1\ldots4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1



$\chi^{\mu}_{g}(D_4)$	$\mathbf{g=1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0

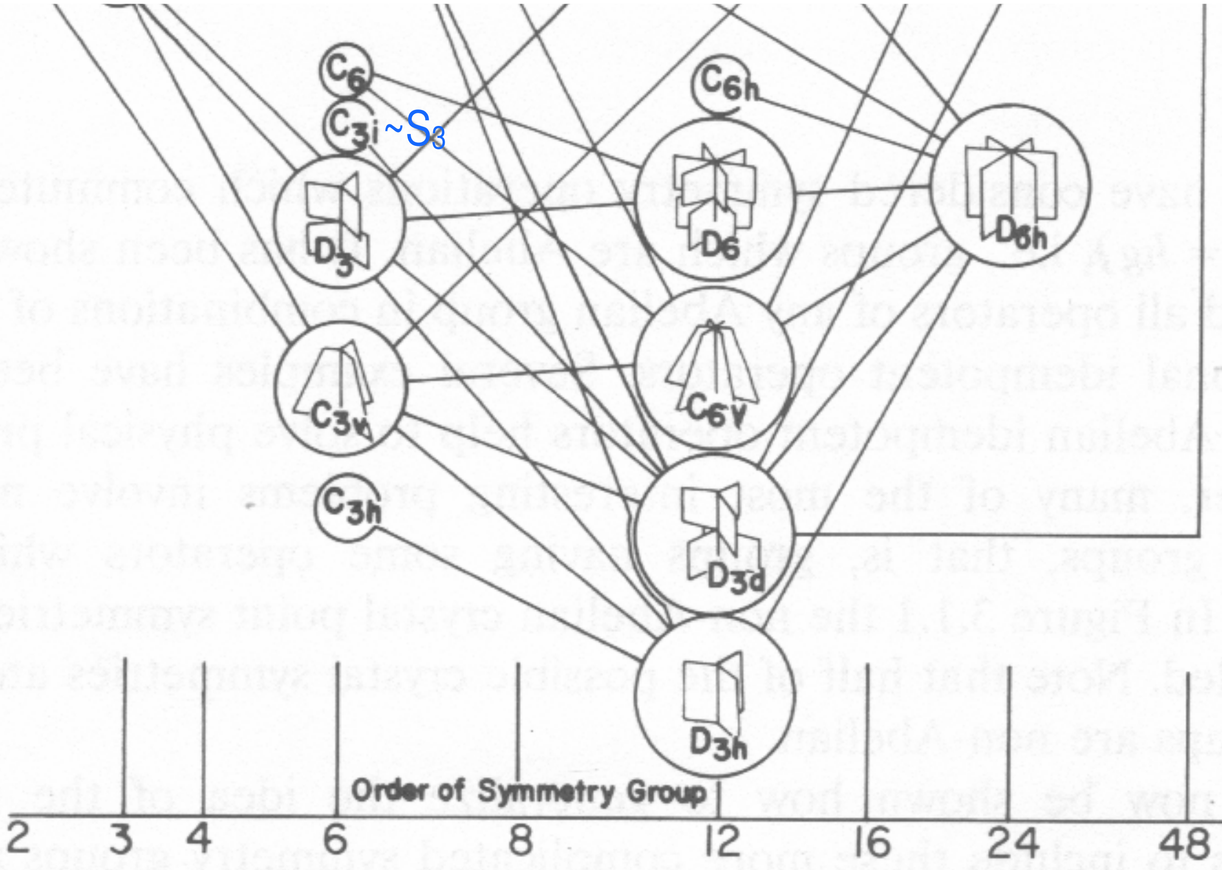
$O{\downarrow}D_4$ subduction

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O){\downarrow}D_4 = 1, \quad 1, \quad 1, \quad 1, \quad 1\,.$ $\quad A_1(O){\downarrow}D_4 = A_1(D_4)$

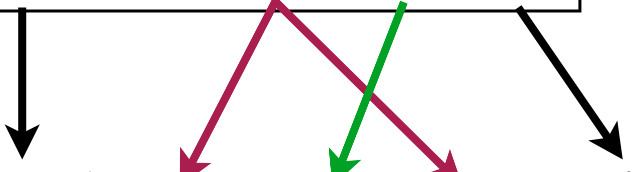
$A_2(O){\downarrow}D_4 = 1, \quad 1, \quad -1, \quad 1, \quad -1\,.$ $\quad A_2(O){\downarrow}D_4 = B_1(D_4)$

$E(O){\downarrow}D_4 = 2, \quad 2, \quad 0, \quad 2, \quad 0\,.$ $\quad E(O){\downarrow}D_4 = A_1 \oplus B_1(D_4)$



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi^{\mu}_{g}(O)$	$\mathbf{g=1}$	$\mathbf{r}_{1\ldots4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1



$\chi^{\mu}_{g}(D_4)$	$\mathbf{g=1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0

$O\downarrow D_4$ subduction

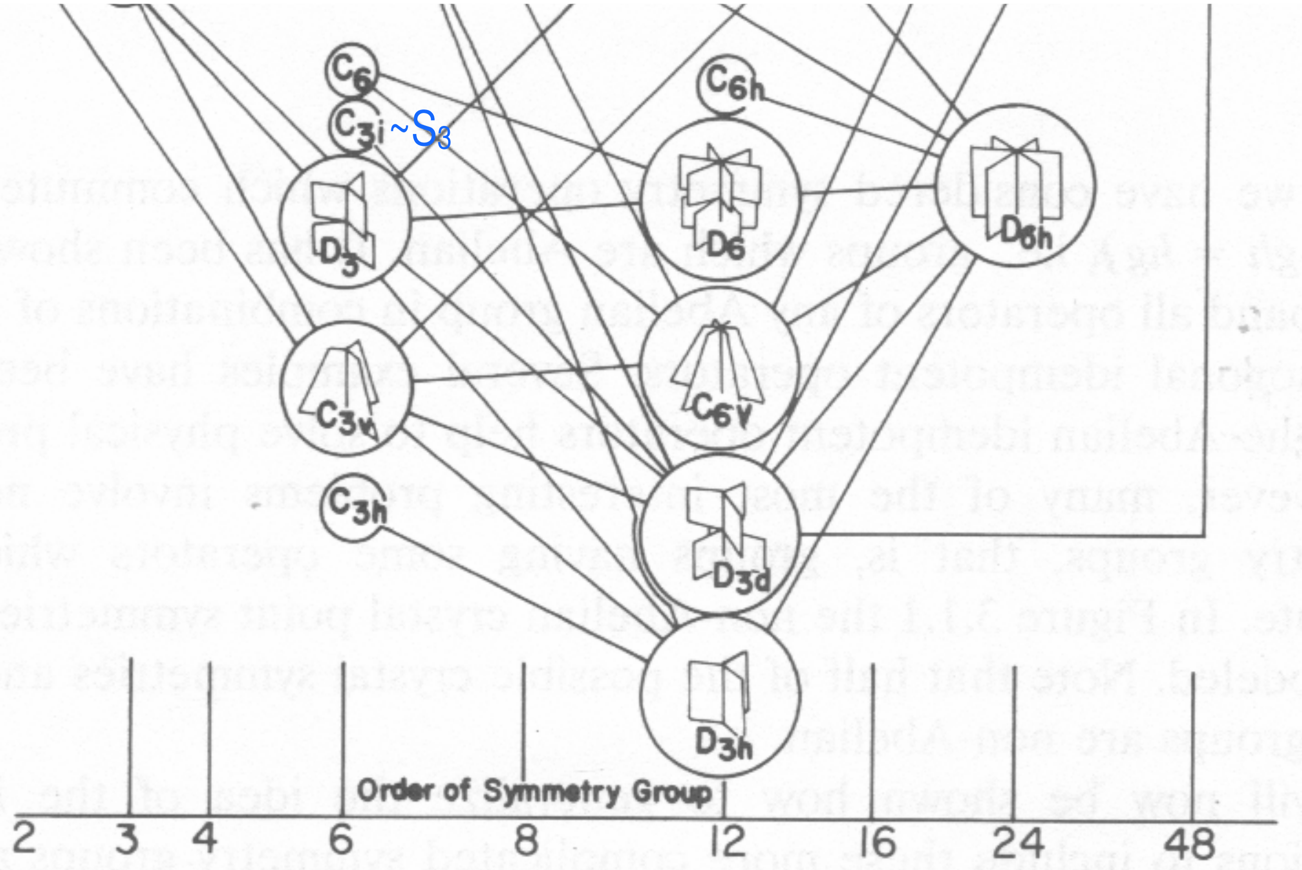
D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$ $A_1(O)\downarrow D_4 = A_1(D_4)$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$ $A_2(O)\downarrow D_4 = B_1(D_4)$

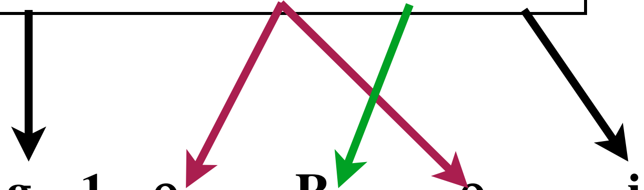
$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$ $E(O)\downarrow D_4 = A_1 \oplus B_1(D_4)$

$T_1(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi_g^\mu(O)$	$\mathbf{g=1}$	$\mathbf{r}_{1\ldots4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1



$\chi_g^\mu(D_4)$	$\mathbf{g=1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0

$O{\downarrow}D_4$ subduction

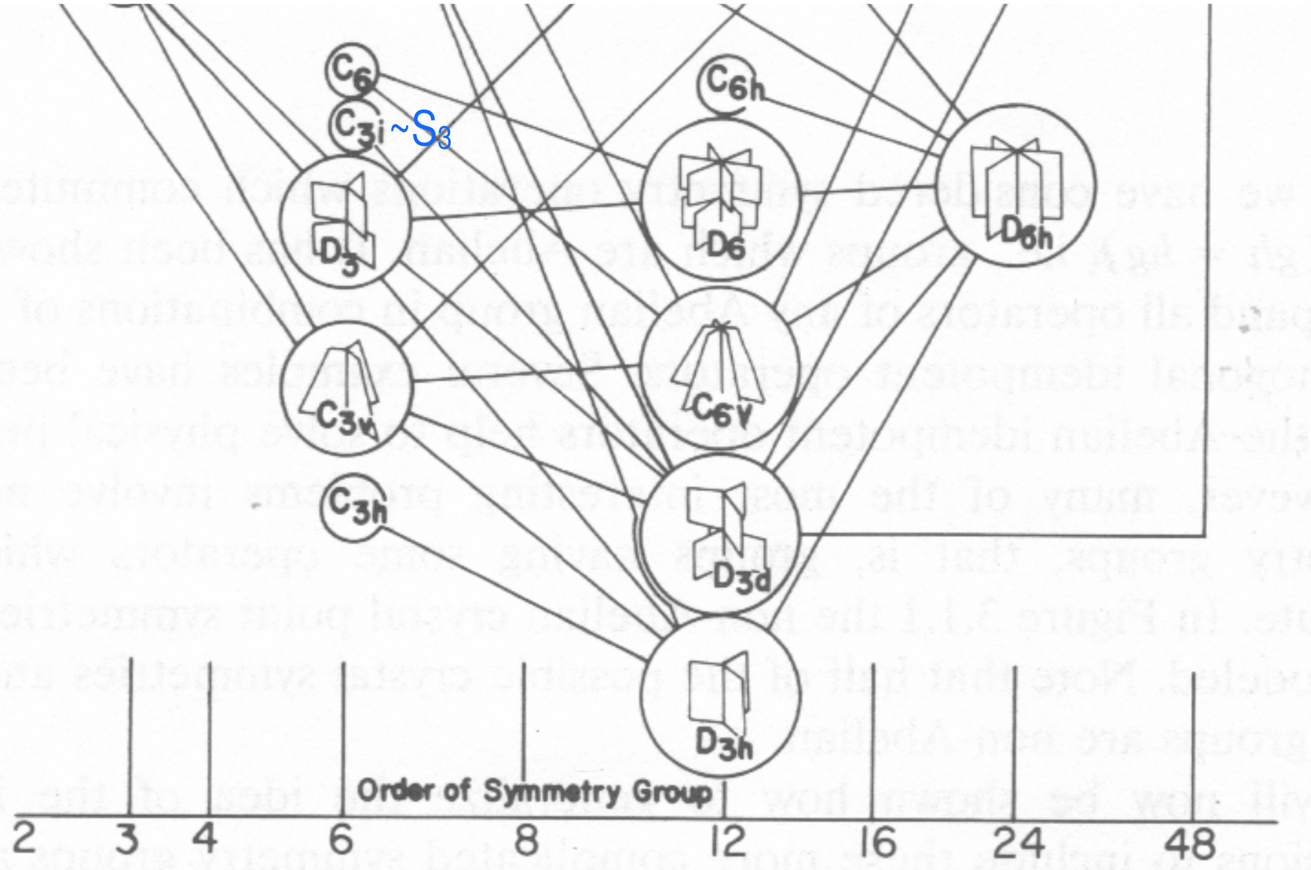
D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O){\downarrow}D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$ $A_1(O){\downarrow}D_4 = A_1(D_4)$

$A_2(O){\downarrow}D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$ $A_2(O){\downarrow}D_4 = B_1(D_4)$

$E(O){\downarrow}D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$ $E(O){\downarrow}D_4 = A_1 \oplus B_1(D_4)$

$T_1(O){\downarrow}D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$ $T_1(O){\downarrow}D_4 = E \oplus A_2(D_4)$



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi_g^\mu(O)$	$\mathbf{g=1}$	~~$\mathbf{r}_{1\ldots4}$~~	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

$O{\downarrow}D_4$ subduction

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O){\downarrow}D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$ $\quad A_1(O){\downarrow}D_4 = A_1(D_4)$

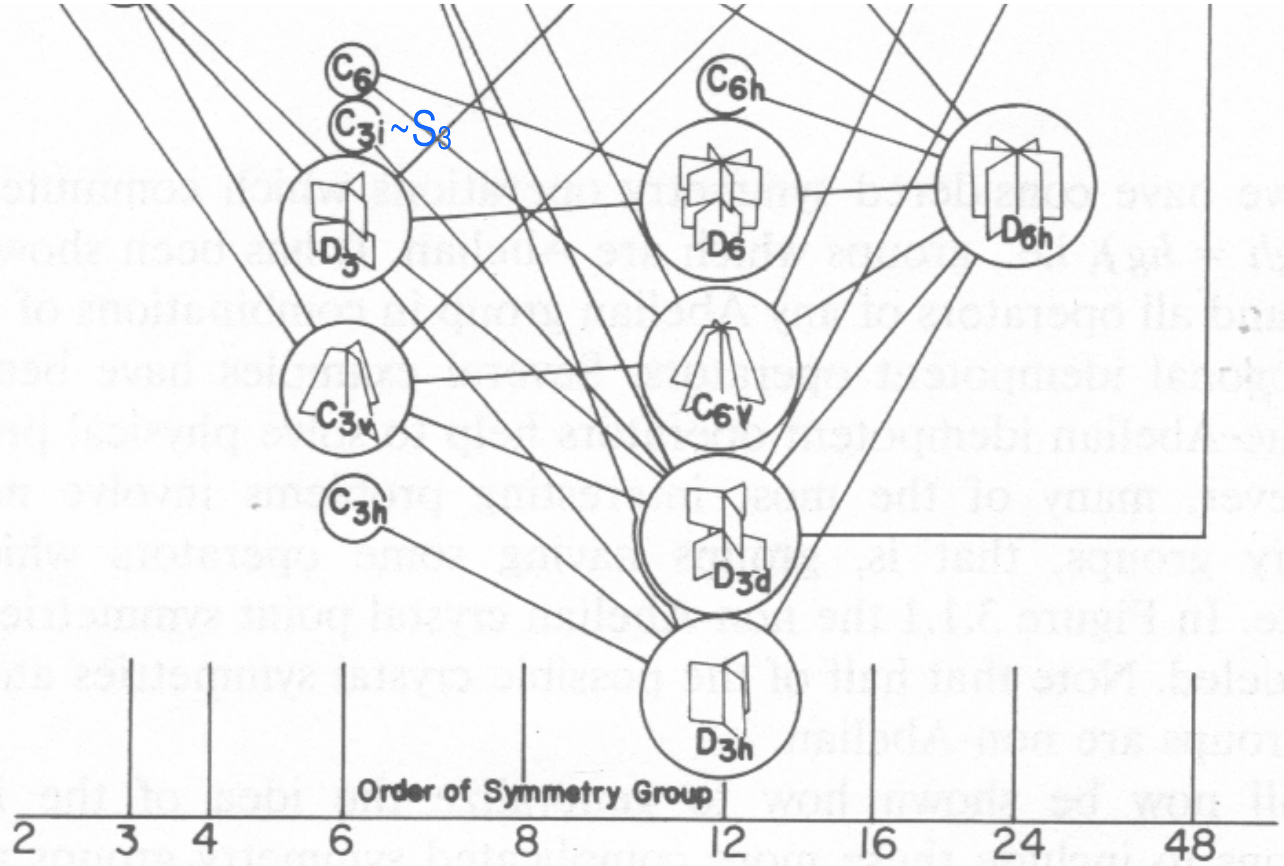
$A_2(O){\downarrow}D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$ $\quad A_2(O){\downarrow}D_4 = B_1(D_4)$

$E(O){\downarrow}D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$ $\quad E(O){\downarrow}D_4 = A_1 \oplus B_1(D_4)$

$T_1(O){\downarrow}D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$ $\quad T_1(O){\downarrow}D_4 = E \oplus A_2(D_4)$

$T_2(O){\downarrow}D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

$\chi_g^\mu(D_4)$	$\mathbf{g=1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi^{\mu}_{g}(O)$	$\mathbf{g=1}$	$\mathbf{r}_{1\ldots4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

$O{\downarrow}D_4$ subduction

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O){\downarrow}D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$ $A_1(O){\downarrow}D_4 = A_1(D_4)$

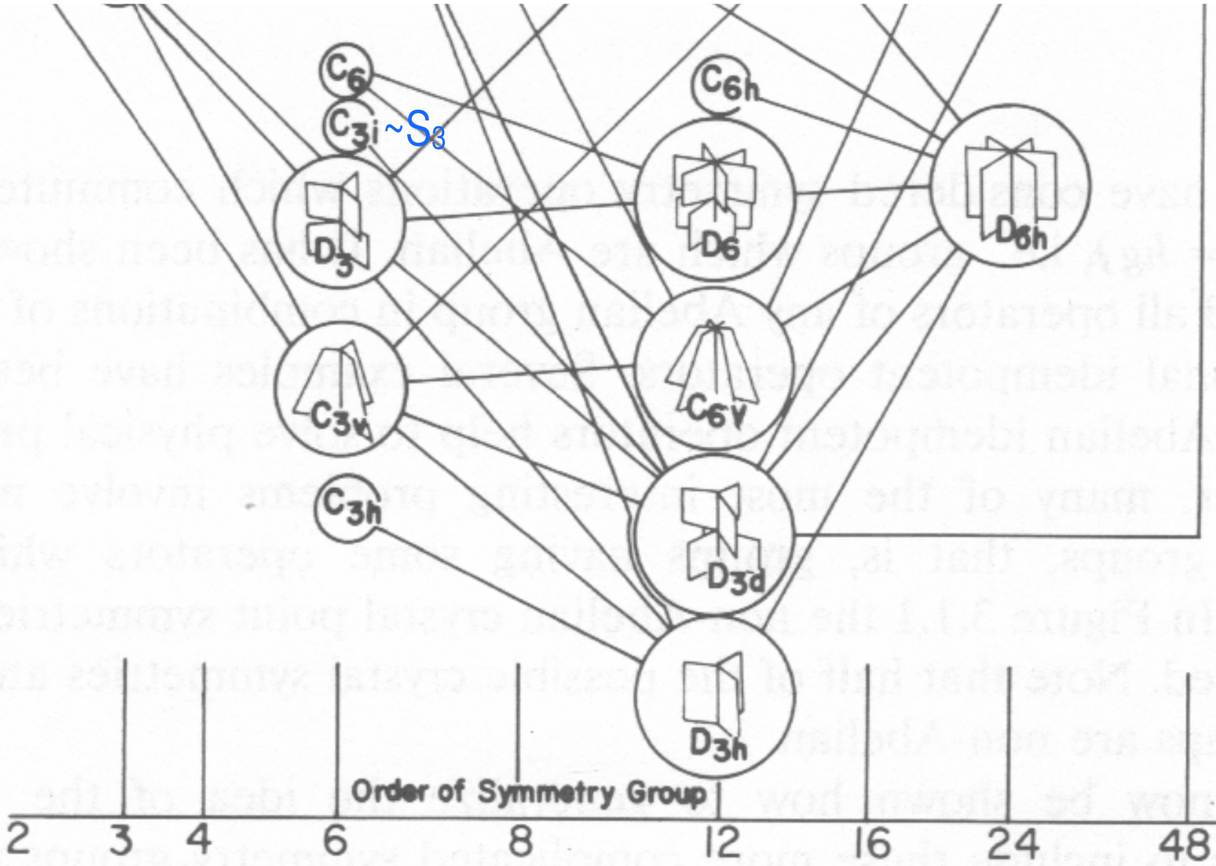
$A_2(O){\downarrow}D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$ $A_2(O){\downarrow}D_4 = B_1(D_4)$

$E(O){\downarrow}D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$ $E(O){\downarrow}D_4 = A_1 \oplus B_1(D_4)$

$T_1(O){\downarrow}D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$ $T_1(O){\downarrow}D_4 = E \oplus A_2(D_4)$

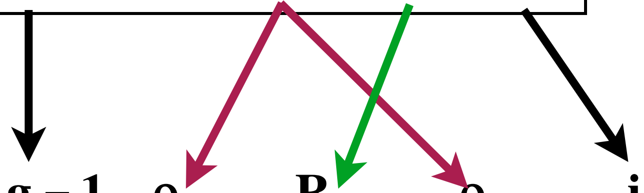
$T_2(O){\downarrow}D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$ $T_2(O){\downarrow}D_4 = E \oplus B_2(D_4)$

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g=1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi_g^\mu(O)$	$\mathbf{g=1}$	$\mathbf{r}_{1\ldots4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1



$\chi_g^\mu(D_4)$	$\mathbf{g=1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0

$O\downarrow D_4$ subduction

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\,.$ $\quad A_1(O)\downarrow D_4 = A_1(D_4)$

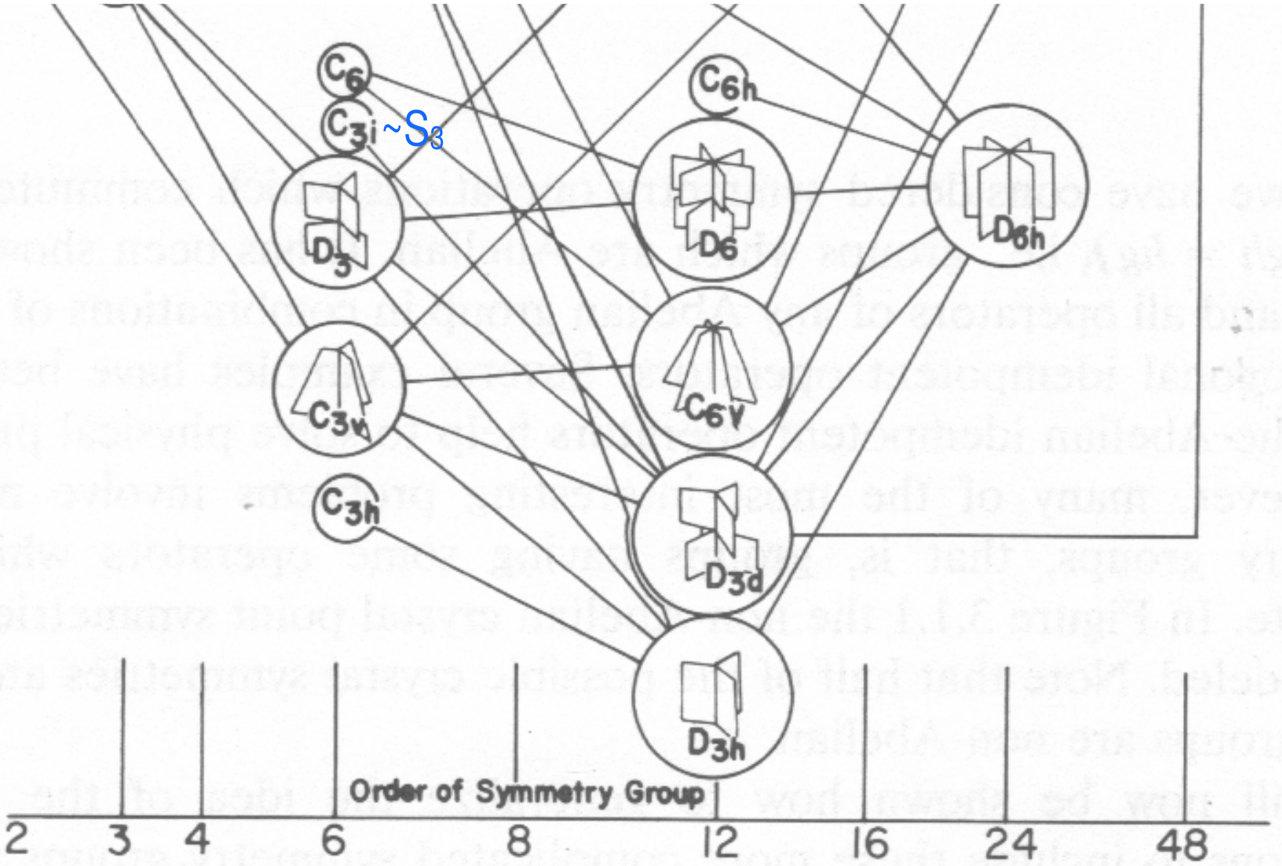
$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\,.$ $\quad A_2(O)\downarrow D_4 = B_1(D_4)$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\,.$ $\quad E(O)\downarrow D_4 = A_1 \oplus B_1(D_4)$

$T_1(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\,.$ $\quad T_1(O)\downarrow D_4 = E \oplus A_2(D_4)$

$T_2(O)\downarrow D_4 = 3,\ -1,\ -1,\ -1,\ 1\,.$ $\quad T_2(O)\downarrow D_4 = E \oplus B_2(D_4)$

$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi^{\mu}_{g}(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1...4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1...6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

$O\downarrow D_4$ subduction

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm 90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

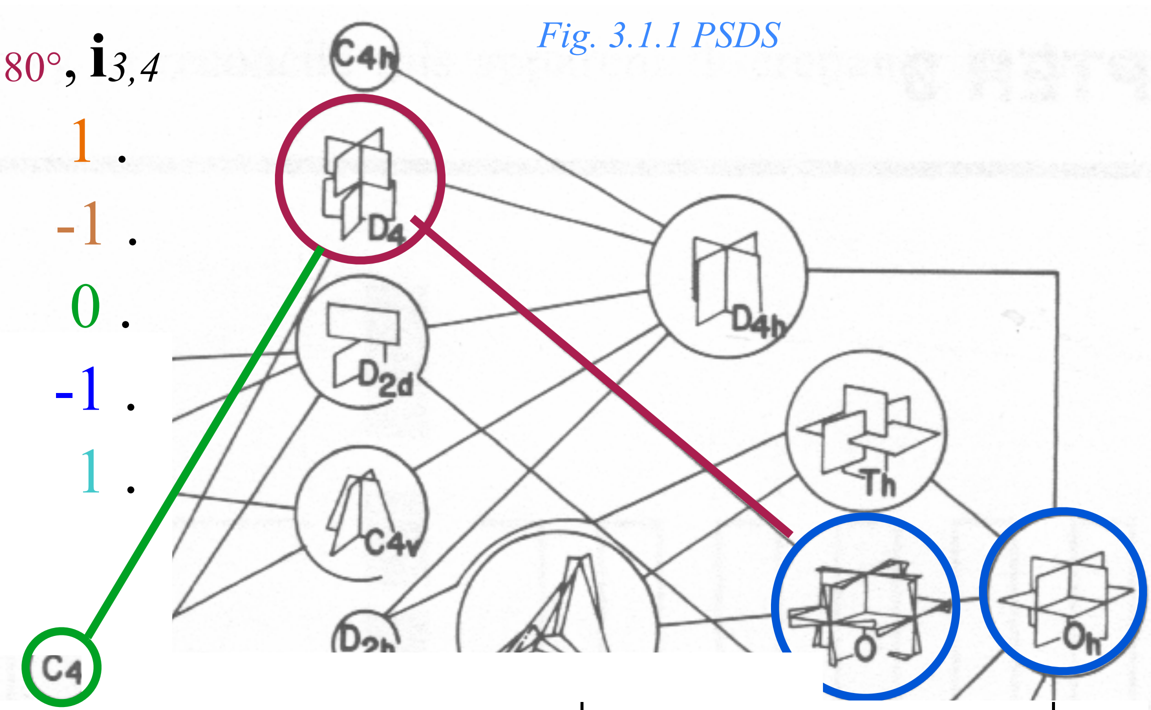
$T_2(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

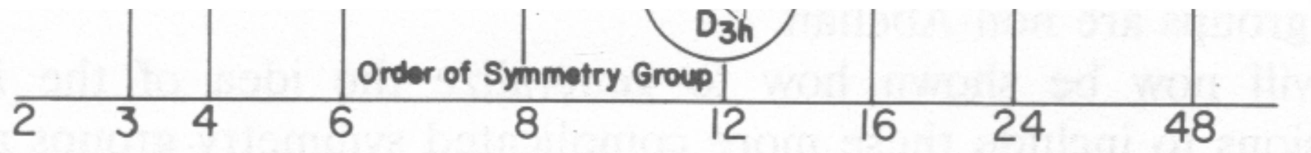
$\chi^{\mu}_{g}(D_4)$	$\mathbf{g}=\mathbf{1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm 90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	~~1~~	~~1~~
B_1	1	1	−1	~~1~~	~~−1~~
A_2	1	1	1	~~−1~~	~~−1~~
B_2	1	1	−1	~~−1~~	~~1~~
E	2	−2	0	~~0~~	~~0~~

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i

Fig. 3.1.1 PSDS



$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1



- *Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$*
  - *Octahedral-cubic O symmetry and group operations*
  - *Tetrahedral symmetry becomes Icosahedral*
  - *Octahedral groups $O_h \supset O \sim T_d \supset T$*
    - *Octahedral O and spin-$O \subset U(2)$*

- *Tetrahedral T class algebra*
  - *Tetrahedral T class minimal equations*
  - *Tetrahedral T class projectors and characters*

- *Octahedral O class algebra*
  - *Octahedral O class minimal equations*
  - *Octahedral O class projectors and characters*

- *Octahedral $O_h \supset O$:Inversion (g&u) parity*
  - *Octahedral $O_h \supset O \supset C_{\mathbf{I}}$ subgroup correlations*
  - *$O_h \supset O \supset D_4$ subgroup correlations*
  - *$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations*
  - *Preview of applications to high resolution spectroscopy*

Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi^{\mu}_{g}(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1...4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1...6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

$O \downarrow D_4$ subduction

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm 90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

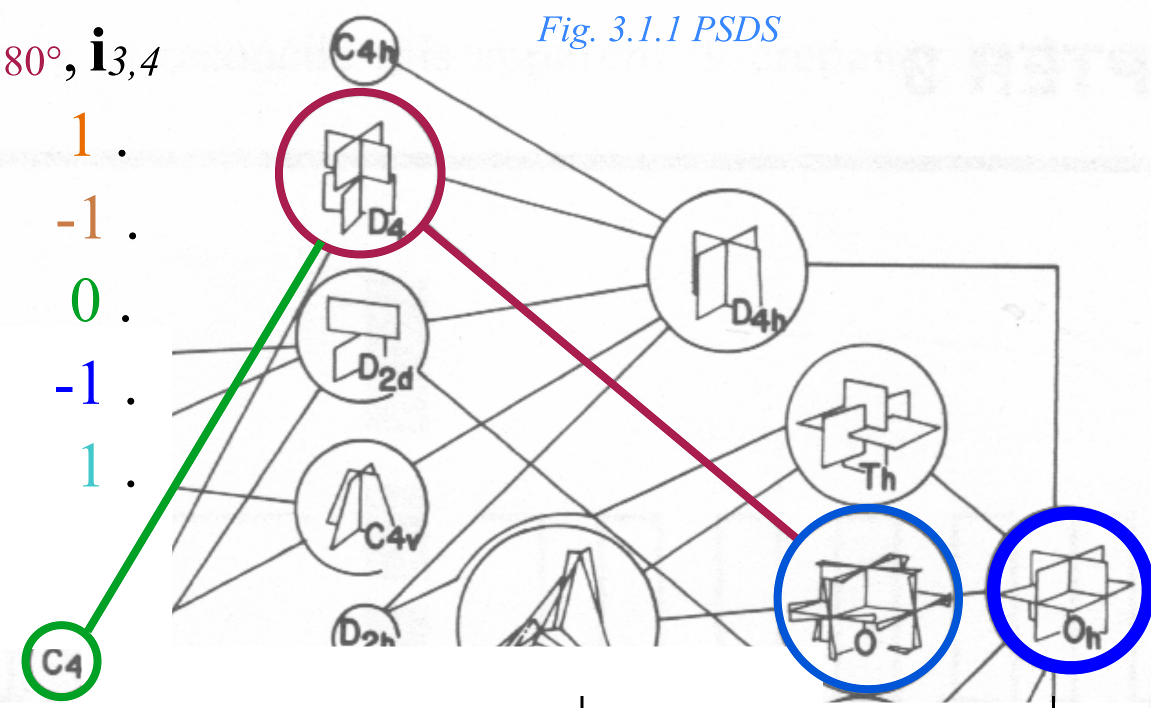
$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

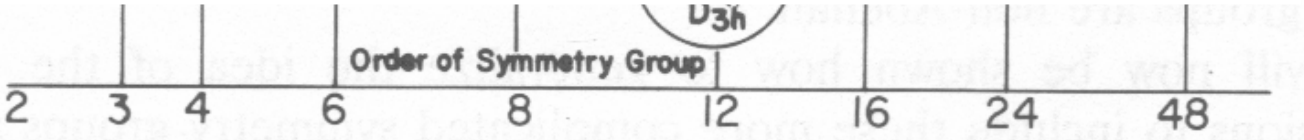
$T_2(O)\downarrow D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$



$\chi^{\mu}_{g}(D_4)$	$\mathbf{g}=\mathbf{1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm 90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	~~1~~	~~1~~
B_1	1	1	−1	~~1~~	~~−1~~
A_2	1	1	1	~~−1~~	~~−1~~
B_2	1	1	−1	~~−1~~	~~1~~
E	2	−2	0	~~0~~	~~0~~

$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm 90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

$\chi_g^\mu(O)$	$\mathbf{g=1}$	$\mathbf{r}_{1\ldots4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

$\chi_g^\mu(D_4)$	$\mathbf{g=1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	~~1~~	~~1~~
B_1	1	1	−1	~~1~~	~~−1~~
A_2	1	1	1	~~−1~~	~~−1~~
B_2	1	1	−1	~~−1~~	~~1~~
E	2	−2	0	~~0~~	~~0~~

$D_4 \downarrow C_4$ subduction

C_4 : $\mathbf{1}, \mathbf{R}_{z+90°}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z-90°}$

$A_1(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1.$

$\chi_g^\mu(C_4)$	$\mathbf{g=1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i

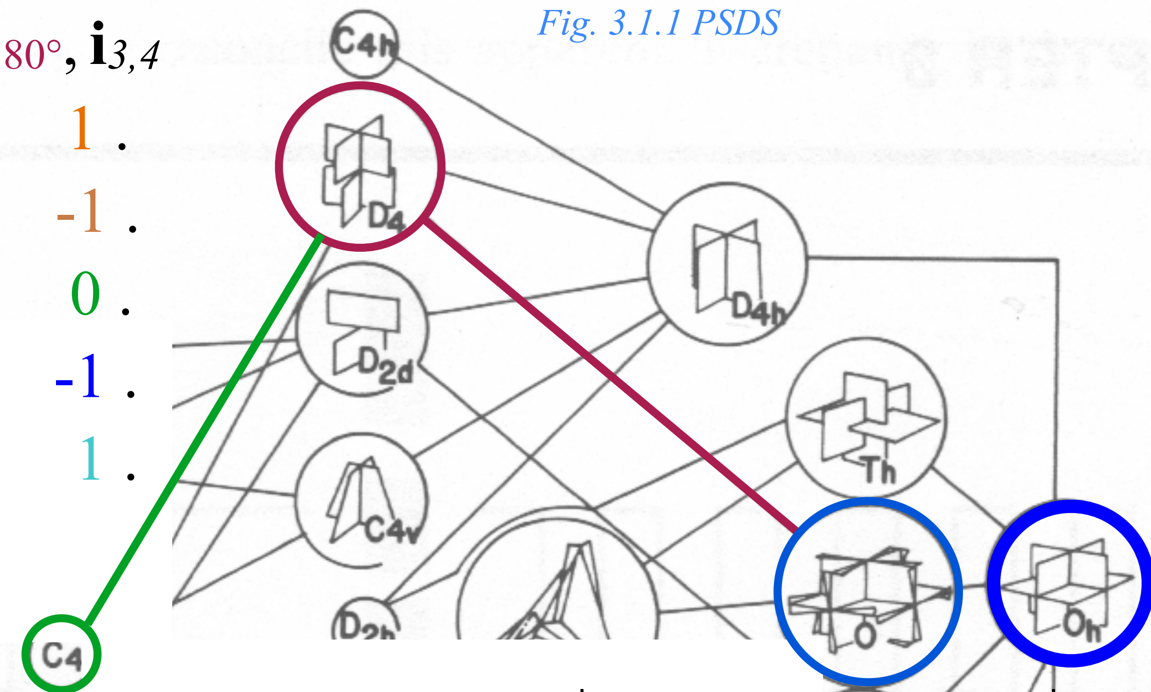
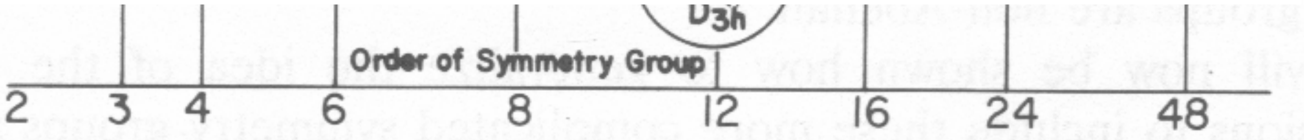


Fig. 3.1.1 PSDS

$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi^{\mu}_{g}(O)$	$\mathbf{g=1}$	$\mathbf{r}_{1...4}$	ρ_{xyz} 180°	$\mathbf{R}_{xyz}$ 90°	$\mathbf{i}_{1...6}$ 180°
A_1	1	1	1	1	1
A_2	1	1	1	−1	−1
E	2	−1	2	0	0
T_1	3	0	−1	1	−1
T_2	3	0	−1	−1	1

O↓D₄ subduction

D_4: **1**, $\boldsymbol{\rho}_{z180°}$, $\mathbf{R}_{z\pm90°}$, $\boldsymbol{\rho}_{x,y180°}$, $\mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

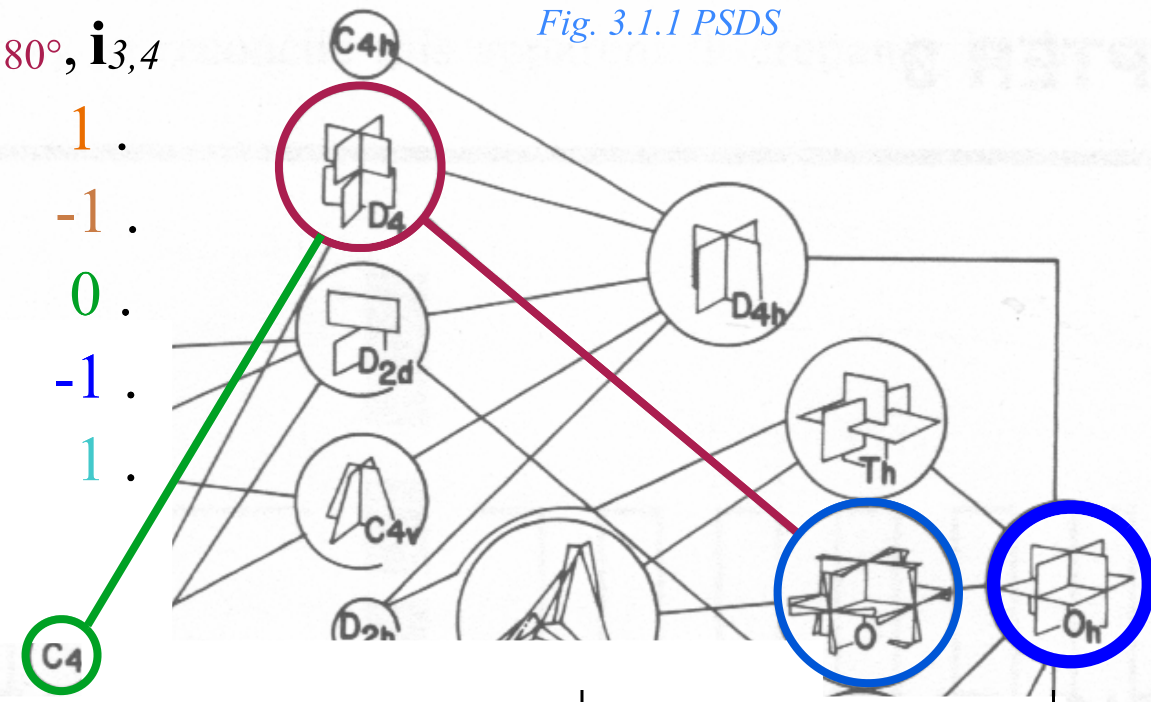


Fig. 3.1.1 PSDS

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g=1}$	$\rho_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\rho_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0

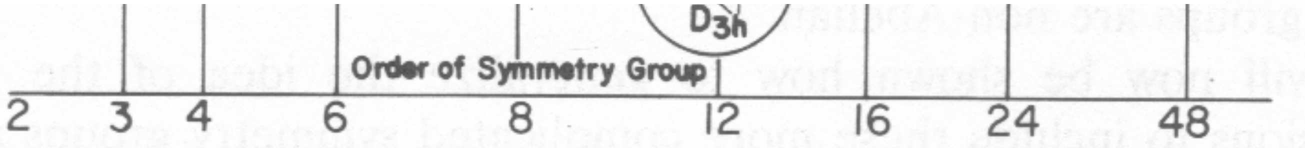
D₄↓C₄ subduction

C_4 : **1**, $\mathbf{R}_{z+90°}$, $\boldsymbol{\rho}_{z180°}$, $\mathbf{R}_{z-90°}$

$A_1(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g=1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i

$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi^{\mu}_{g}(O)$	$\mathbf{g=1}$	$\mathbf{r}_{1\ldots4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

O↓D₄ subduction

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1\ .$

$A_2(O)\downarrow D_4 = 1,\ \ 1,\ \ -1,\ \ 1,\ \ -1\ .$

$E(O)\downarrow D_4 = 2,\ \ 2,\ \ 0,\ \ 2,\ \ 0\ .$

$T_2(O)\downarrow D_4 = 3,\ \ -1,\ \ 1,\ \ -1,\ \ -1\ .$

$T_2(O)\downarrow D_4 = 3,\ \ -1,\ \ -1,\ \ -1,\ \ 1\ .$

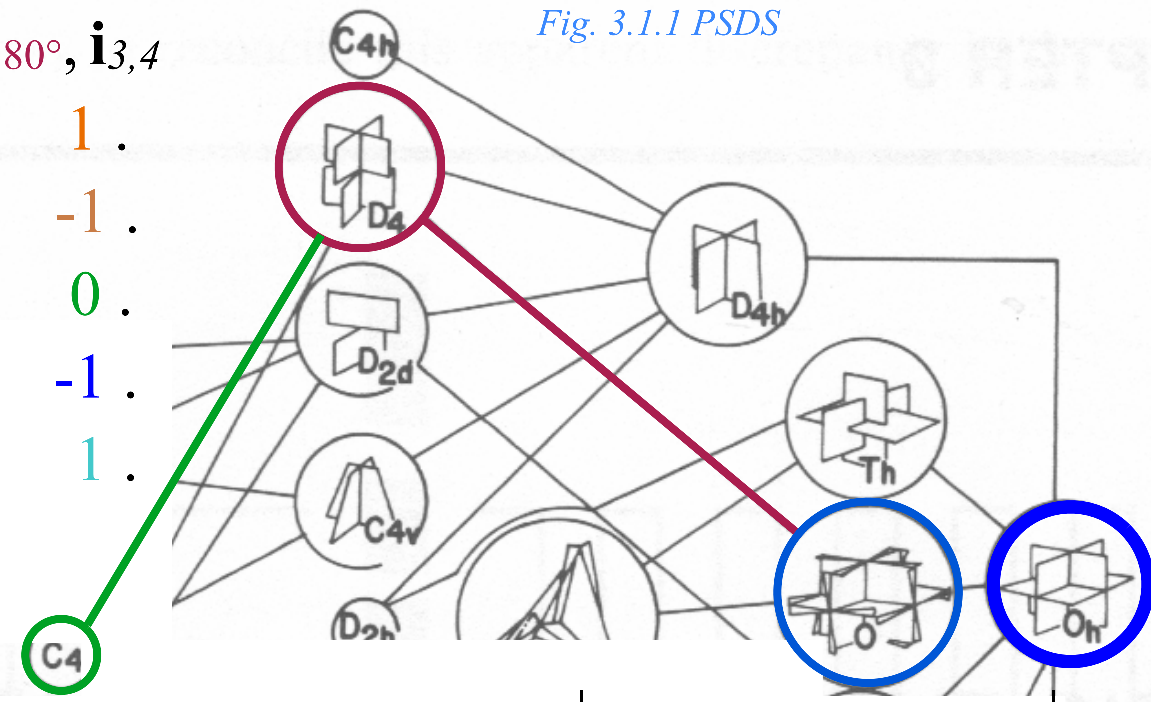


Fig. 3.1.1 PSDS

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g=1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	~~1~~	~~1~~
B_1	1	1	−1	~~1~~	~~−1~~
A_2	1	1	1	~~−1~~	~~−1~~
B_2	1	1	−1	~~−1~~	~~1~~
E	2	−2	0	~~0~~	~~0~~

D₄↓C₄ subduction

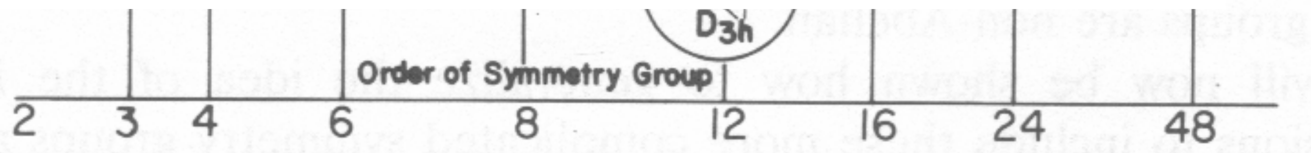
C_4 : $\mathbf{1}, \mathbf{R}_{z+90°}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z-90°}$

$A_1(D_4)\downarrow C_4 = 1,\ \ 1,\ \ 1,\ \ 1. \quad =(0)_4$

$B_1(D_4)\downarrow C_4 = 1,\ \ -1,\ \ 1,\ \ -1.$

$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g=1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi^{\mu}_{g}(O)$	$\mathbf{g=1}$	$\mathbf{r}_{1...4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1...6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

O↓D₄ subduction

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm 90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

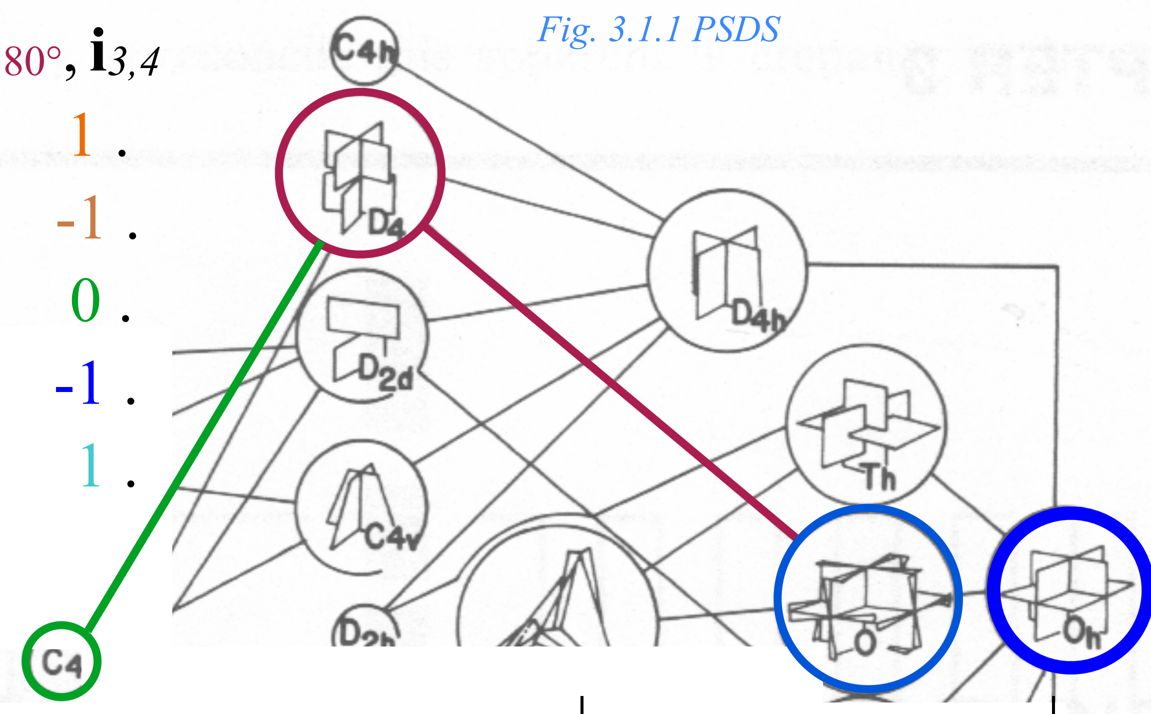


Fig. 3.1.1 PSDS

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g=1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm 90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	~~1~~	~~1~~
B_1	1	1	−1	~~1~~	~~−1~~
A_2	1	1	1	~~−1~~	~~−1~~
B_2	1	1	−1	~~−1~~	~~1~~
E	2	−2	0	~~0~~	~~0~~

D₄↓C₄ subduction

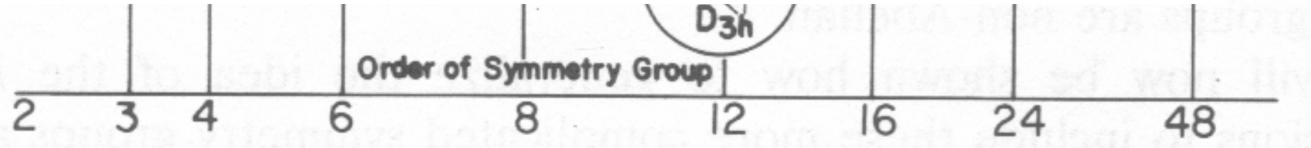
C_4: $\mathbf{1}, \mathbf{R}_{z+90°}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z-90°}$

$A_1(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$B_1(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$

$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g=1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi^{\mu}_{g}(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1...4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1...6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

$O{\downarrow}D_4$ subduction

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm 90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O){\downarrow}D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O){\downarrow}D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O){\downarrow}D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

$T_2(O){\downarrow}D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

$T_2(O){\downarrow}D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

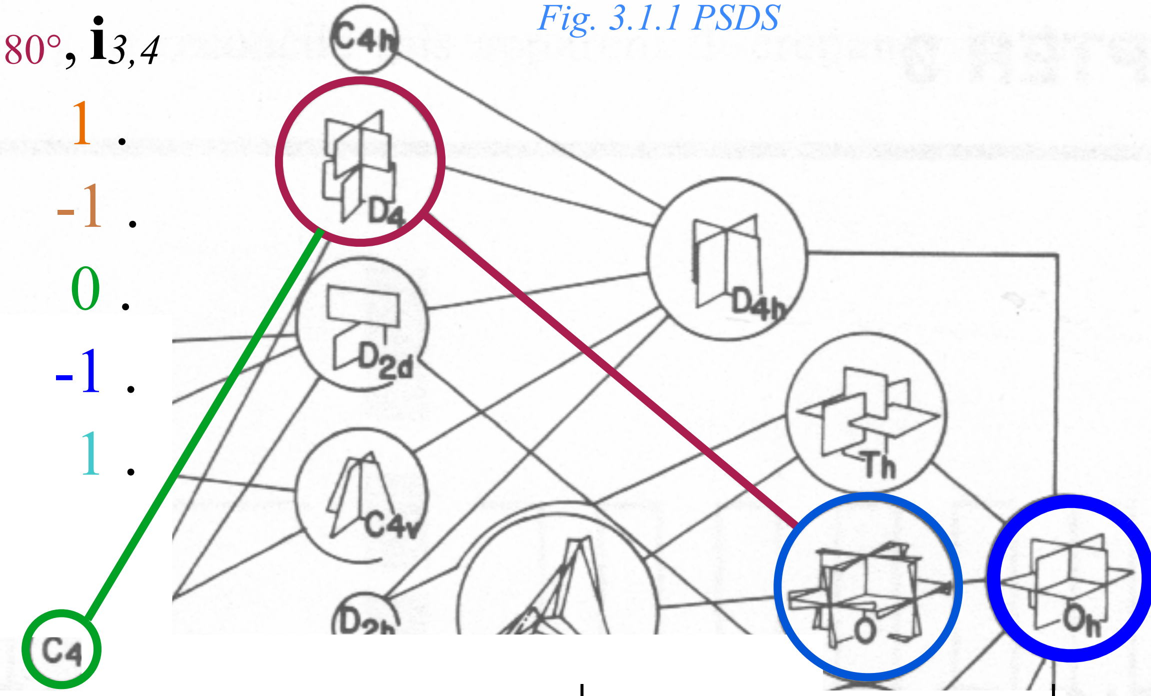


Fig. 3.1.1 PSDS

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g}=\mathbf{1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm 90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	~~1~~	~~1~~
B_1	1	1	−1	~~1~~	~~−1~~
A_2	1	1	1	~~−1~~	~~−1~~
B_2	1	1	−1	~~−1~~	~~1~~
E	2	−2	0	~~0~~	~~0~~

$D_4{\downarrow}C_4$ subduction

C_4: $\mathbf{1}, \mathbf{R}_{z+90°}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z-90°}$

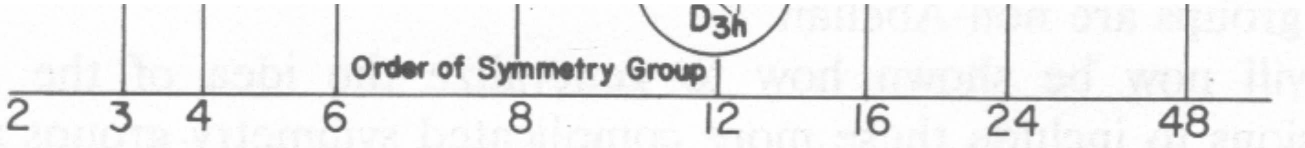
$A_1(D_4){\downarrow}C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$B_1(D_4){\downarrow}C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$

$A_2(D_4){\downarrow}C_4 = 1,\ 1,\ 1,\ 1.$

$O{\downarrow}D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

			180°	90°	180°
$\chi^{\mu}_{g}(O)$	$\mathbf{g=1}$	$\mathbf{r}_{1...4}$	$\boldsymbol{\rho}_{xyz}$	$\mathbf{R}_{xyz}$	$\mathbf{i}_{1...6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

$O{\downarrow}D_4$ subduction

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O){\downarrow}D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O){\downarrow}D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O){\downarrow}D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

$T_2(O){\downarrow}D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

$T_2(O){\downarrow}D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

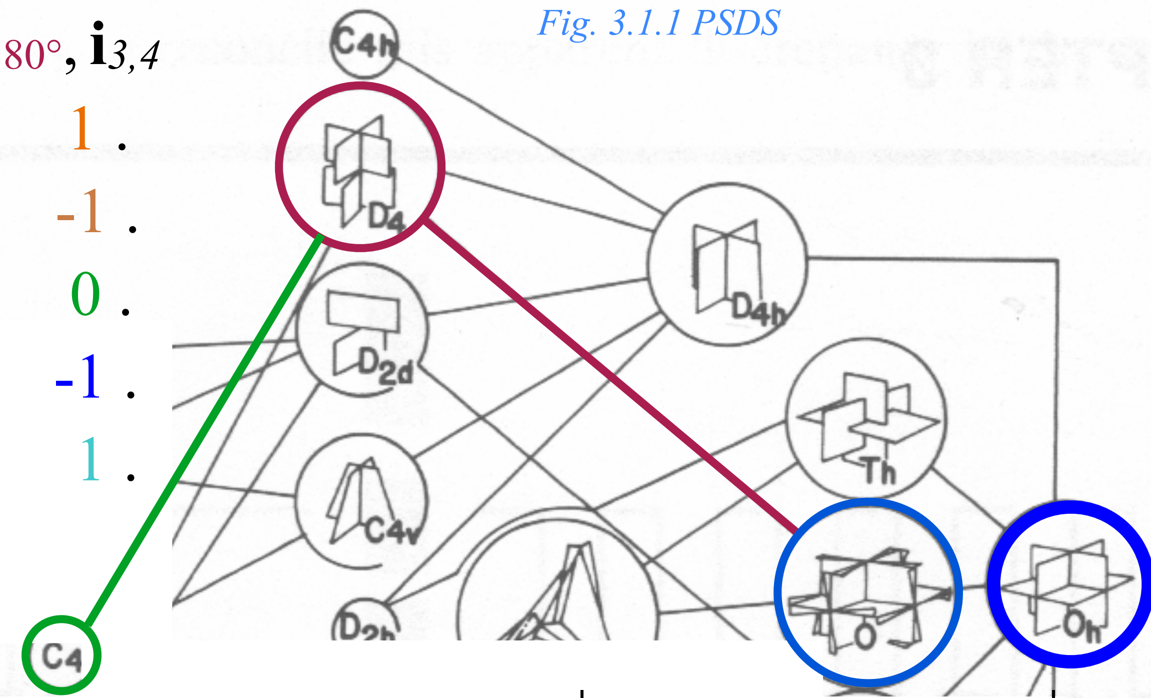


Fig. 3.1.1 PSDS

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g=1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	~~1~~	~~1~~
B_1	1	1	−1	~~1~~	~~−1~~
A_2	1	1	1	~~−1~~	~~−1~~
B_2	1	1	−1	~~−1~~	~~1~~
E	2	−2	0	~~0~~	~~0~~

$D_4{\downarrow}C_4$ subduction

C_4: $\mathbf{1}, \mathbf{R}_{z+90°}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z-90°}$

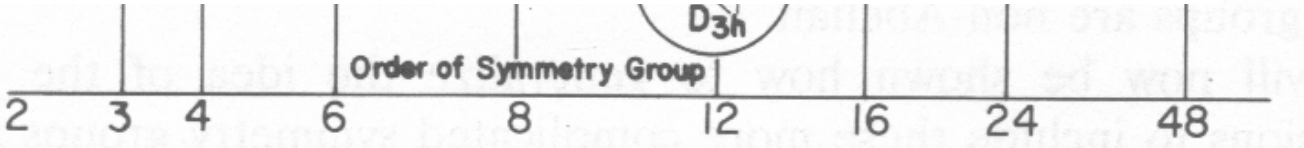
$A_1(D_4){\downarrow}C_4 = 1,\ 1,\ 1,\ 1.\quad =(0)_4$

$B_1(D_4){\downarrow}C_4 = 1,\ -1,\ 1,\ -1.\quad =(2)_4$

$A_2(D_4){\downarrow}C_4 = 1,\ 1,\ 1,\ 1.\quad =(0)_4$

$O{\downarrow}D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g=1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

O↓D_4 subduction

$\chi^{\mu}_{g}(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1\ldots4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	–1	–1
E	2	~~–1~~	2	0	0
T_1	3	~~0~~	–1	1	–1
T_2	3	~~0~~	–1	–1	1

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1, \;1, \;1, \;1, \;1\,.$

$A_2(O)\downarrow D_4 = 1, \;1, \;-1, \;1, \;-1\,.$

$E(O)\downarrow D_4 = 2, \;2, \;0, \;2, \;0\,.$

$T_2(O)\downarrow D_4 = 3, \;-1, \;1, \;-1, \;-1\,.$

$T_2(O)\downarrow D_4 = 3, \;-1, \;-1, \;-1, \;1\,.$

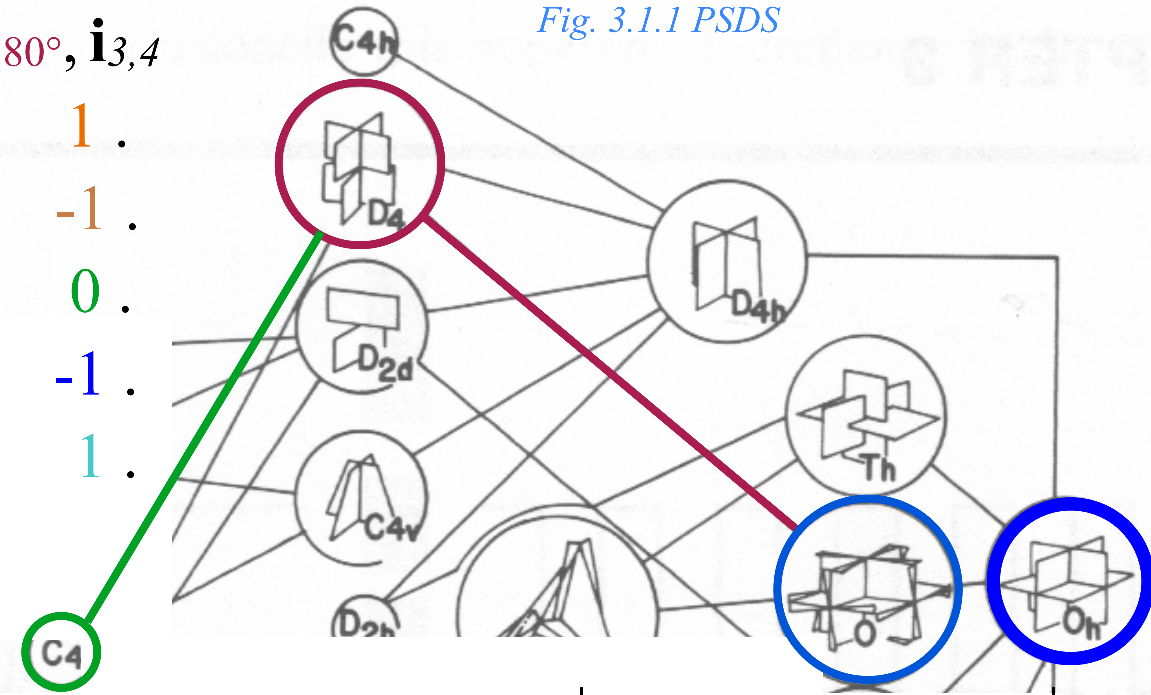


Fig. 3.1.1 PSDS

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g}=\mathbf{1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	~~1~~	~~1~~
B_1	1	1	–1	~~1~~	~~–1~~
A_2	1	1	1	~~–1~~	~~–1~~
B_2	1	1	–1	~~–1~~	~~1~~
E	2	–2	0	~~0~~	~~0~~

D_4↓C_4 subduction

C_4 : $\mathbf{1}, \mathbf{R}_{z+90°}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z-90°}$

$A_1(D_4)\downarrow C_4 = 1, \;1, \;1, \;1. \quad =(0)_4$

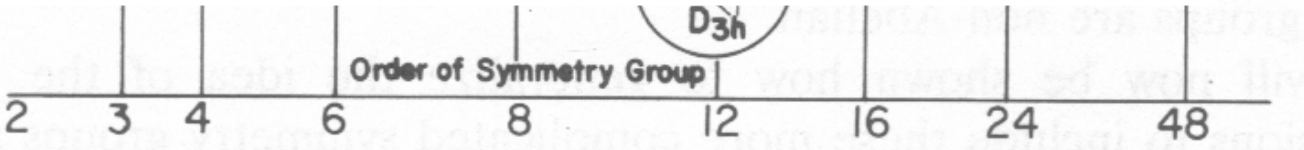
$B_1(D_4)\downarrow C_4 = 1, \;-1, \;1, \;-1. \quad =(2)_4$

$A_2(D_4)\downarrow C_4 = 1, \;1, \;1, \;1. \quad =(0)_4$

$B_2(D_4)\downarrow C_4 = 1, \;-1, \;1, \;-1.$

$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	–1	$-i$
$(2)_4$	1	–1	1	–1
$(3)_4$	1	$-i$	–1	i



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi^{\mu}_{g}(O)$	$\mathbf{g=1}$	$\mathbf{r}_{1...4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1...6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

$O\downarrow D_4$ subduction

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm 90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

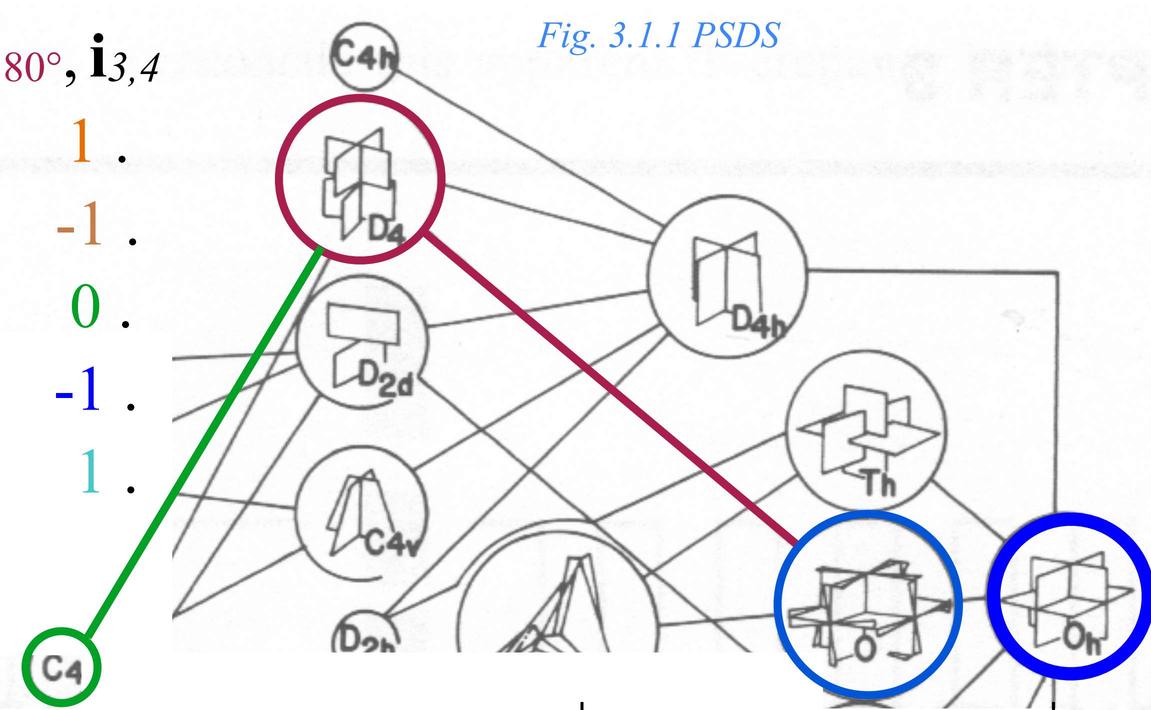


Fig. 3.1.1 PSDS

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g=1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm 90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	~~1~~	~~1~~
B_1	1	1	−1	~~1~~	~~−1~~
A_2	1	1	1	~~−1~~	~~−1~~
B_2	1	1	−1	~~−1~~	~~1~~
E	2	−2	0	~~0~~	~~0~~

$D_4\downarrow C_4$ subduction

C_4: $\mathbf{1}, \mathbf{R}_{z+90°}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z-90°}$

$A_1(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1.\quad =(0)_4$

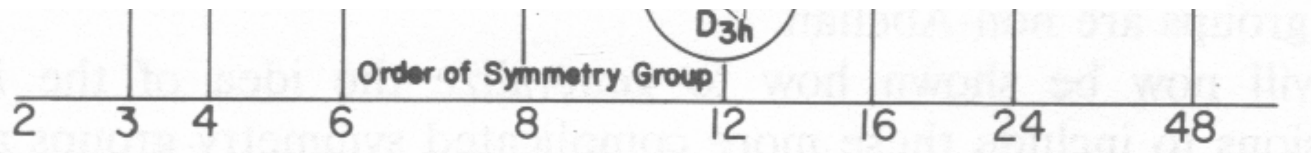
$B_1(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1.\quad =(2)_4$

$A_2(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1.\quad =(0)_4$

$B_2(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1.\quad =(2)_4$

$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g=1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi^{\mu}_{g}(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1\ldots4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

$O\downarrow D_4$ subduction

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

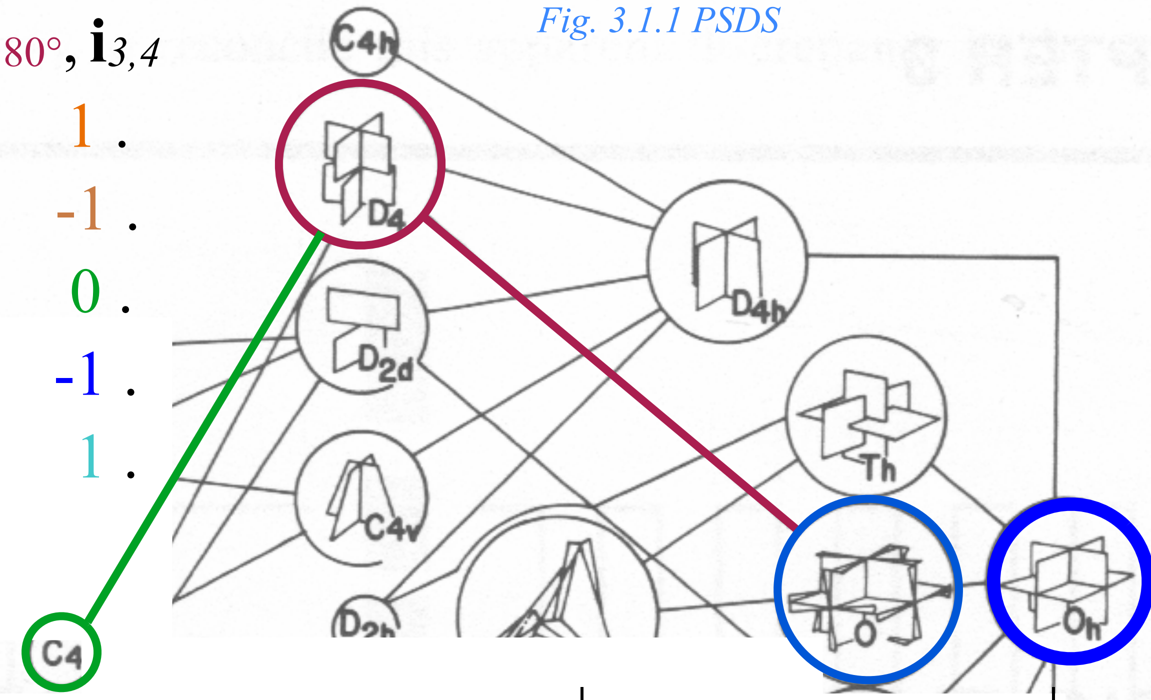


Fig. 3.1.1 PSDS

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g}=\mathbf{1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	~~1~~	~~1~~
B_1	1	1	−1	~~1~~	~~−1~~
A_2	1	1	1	~~−1~~	~~−1~~
B_2	1	1	−1	~~−1~~	~~1~~
E	2	−2	0	~~0~~	~~0~~

$D_4\downarrow C_4$ subduction

C_4: $\mathbf{1}, \mathbf{R}_{z+90°}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z-90°}$

$A_1(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$B_1(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$

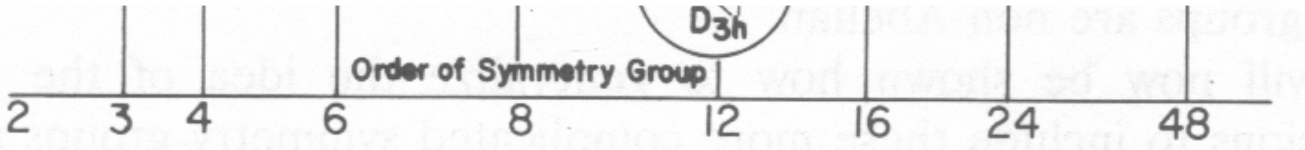
$A_2(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$B_2(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$

$E(D_4)\downarrow C_4 = 2,\ 0,\ -2,\ 0\ .$

$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O{\downarrow}D_4$ subduction

			180°	90°	180°
$\chi^{\mu}_{g}(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1...4}$	$\boldsymbol{\rho}_{xyz}$	$\mathbf{R}_{xyz}$	$\mathbf{i}_{1...6}$
A_1	1	$\cancel{1}$	1	1	1
A_2	1	$\cancel{1}$	1	−1	−1
E	2	$\cancel{-1}$	2	0	0
T_1	3	$\cancel{0}$	−1	1	−1
T_2	3	$\cancel{0}$	−1	−1	1

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm 90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O){\downarrow}D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O){\downarrow}D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O){\downarrow}D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

$T_2(O){\downarrow}D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

$T_2(O){\downarrow}D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

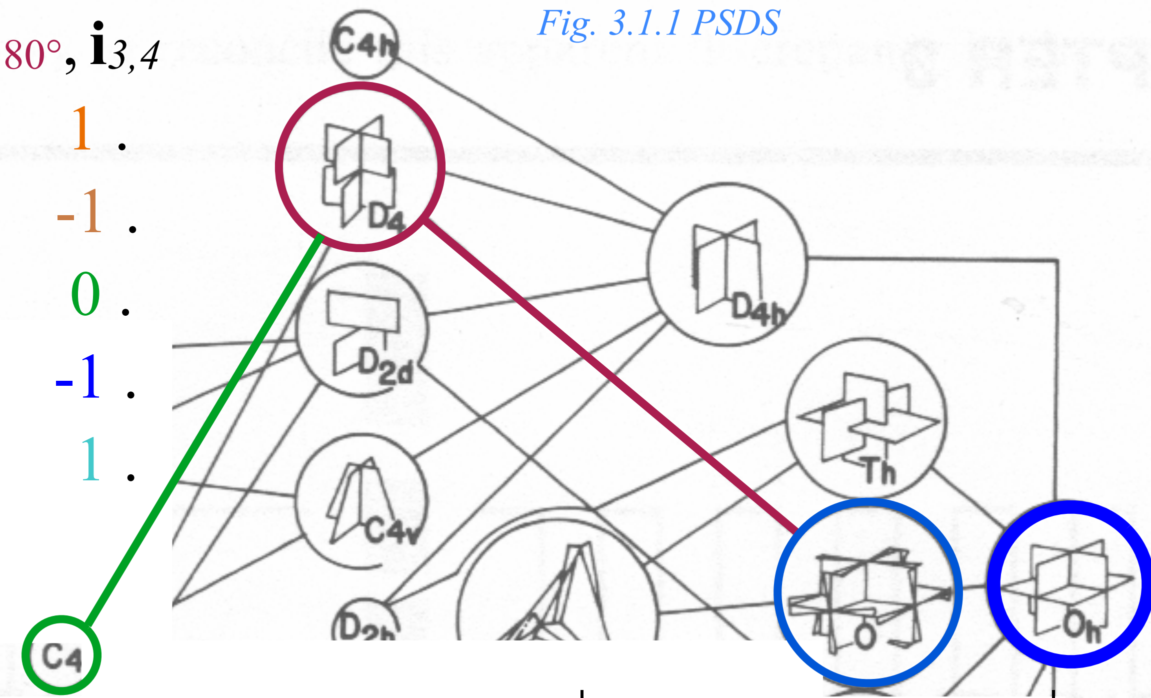


Fig. 3.1.1 PSDS

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g}=\mathbf{1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm 90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	$\cancel{1}$	$\cancel{1}$
B_1	1	1	−1	$\cancel{1}$	$\cancel{-1}$
A_2	1	1	1	$\cancel{-1}$	$\cancel{-1}$
B_2	1	1	−1	$\cancel{-1}$	$\cancel{1}$
E	2	−2	0	$\cancel{0}$	$\cancel{0}$

$D_4{\downarrow}C_4$ subduction

C_4: $\mathbf{1}, \mathbf{R}_{z+90°}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z-90°}$

$A_1(D_4){\downarrow}C_4 = 1,\ 1,\ 1,\ 1.\ =(0)_4$

$B_1(D_4){\downarrow}C_4 = 1,\ -1,\ 1,\ -1.\ =(2)_4$

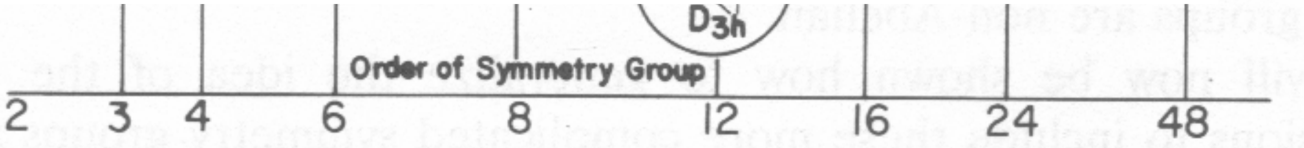
$A_2(D_4){\downarrow}C_4 = 1,\ 1,\ 1,\ 1.\ =(0)_4$

$B_2(D_4){\downarrow}C_4 = 1,\ -1,\ 1,\ -1.\ =(2)_4$

$E(D_4){\downarrow}C_4 = 2,\ 0,\ -2,\ 0\ .\ =(1)_4 \oplus (3)_4$

$O{\downarrow}D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi^{\mu}_{g}(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1\ldots4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	1	1	1	1
A_2	1	1	1	−1	−1
E	2	−1	2	0	0
T_1	3	0	−1	1	−1
T_2	3	0	−1	−1	1

$O\downarrow D_4$ subduction

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

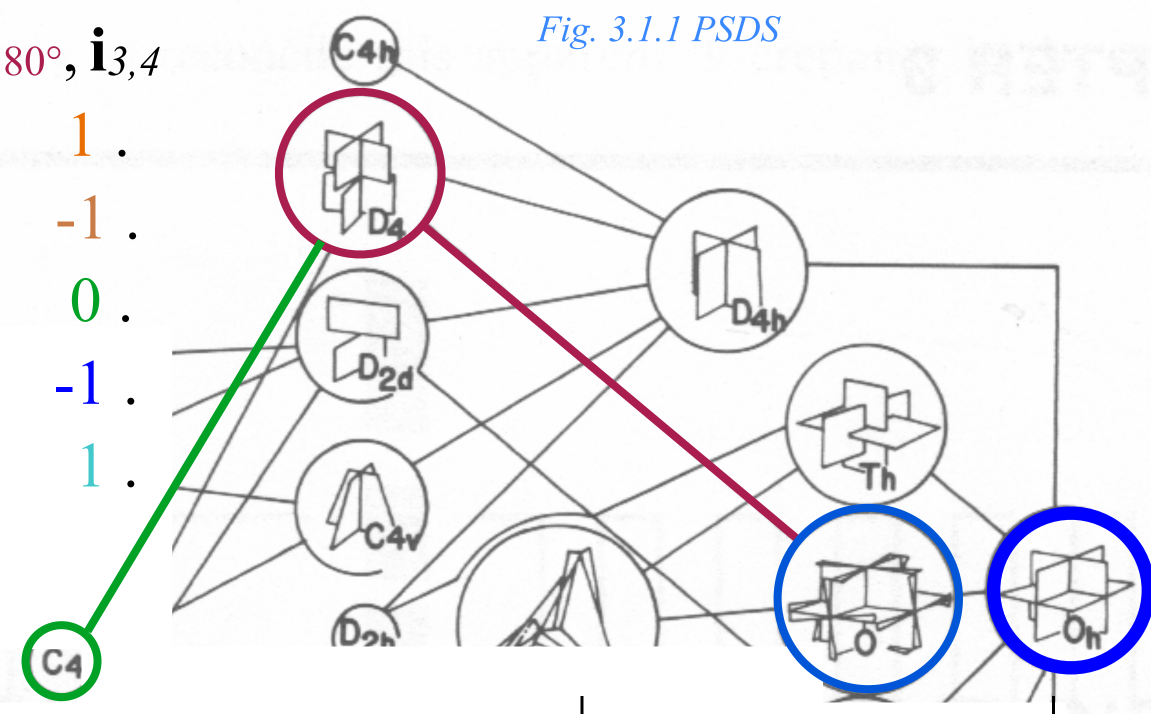


Fig. 3.1.1 PSDS

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g}=\mathbf{1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0

$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$D_4\downarrow C_4$ subduction

C_4: $\mathbf{1}, \mathbf{R}_{z+90°}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z-90°}$

$A_1(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. = (0)_4$

$B_1(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1. = (2)_4$

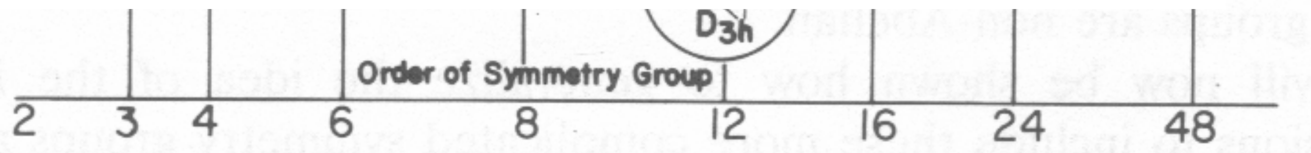
$A_2(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. = (0)_4$

$B_2(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1. = (2)_4$

$E(D_4)\downarrow C_4 = 2,\ 0,\ -2,\ 0 = (1)_4 \oplus (3)_4$

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i

$D_4\downarrow C_4$	0_4	1_4	2_4	$3_4 = \bar{1}_4$
A_1	1	·	·	·
B_1	·	·	1	·
A_2	1	·	·	·
B_2	·	·	1	·
E	·	1	·	1



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi_g^\mu(O)$	$\mathbf{g=1}$	$\mathbf{r}_{1...4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1...6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

$O \downarrow D_4$ subduction

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm 90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

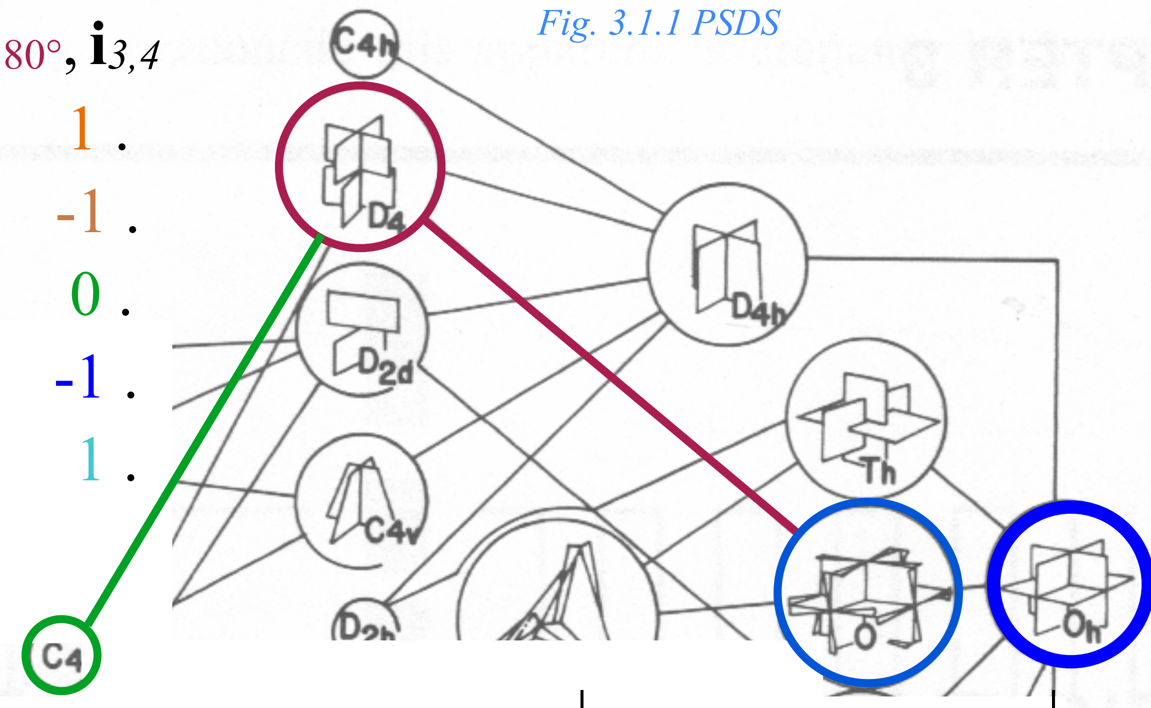


Fig. 3.1.1 PSDS

$\chi_g^\mu(D_4)$	$\mathbf{g=1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm 90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	~~1~~	~~1~~
B_1	1	1	−1	~~1~~	~~−1~~
A_2	1	1	1	~~−1~~	~~−1~~
B_2	1	1	−1	~~−1~~	~~1~~
E	2	−2	0	~~0~~	~~0~~

$D_4 \downarrow C_4$ subduction

C_4: $\mathbf{1}, \mathbf{R}_{z+90°}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z-90°}$

$A_1(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$B_1(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$

$A_2(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$B_2(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$

$E(D_4)\downarrow C_4 = 2,\ 0,\ -2,\ 0\ . \quad =(1)_4 \oplus (3)_4$

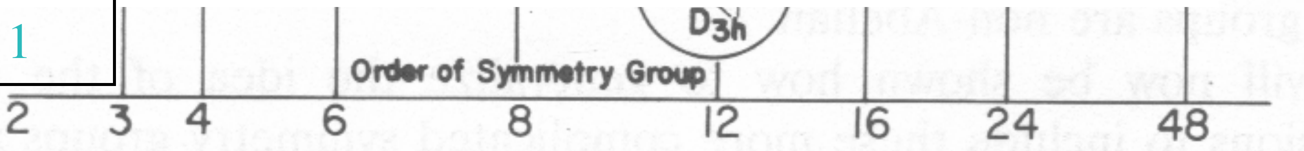
$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi_g^\mu(C_4)$	$\mathbf{g=1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i

$O \downarrow C_4$ subduction

$O\downarrow C_4$	0_4	1_4	2_4	$3_4 = \bar{1}_4$
A_1	1	·	·	·
A_2	·	·	1	·
E	1	·	1	·
T_1	1	1	·	1
T_2	·	1	1	1

$D_4\downarrow C_4$	0_4	1_4	2_4	$3_4 = \bar{1}_4$
A_1	1	·	·	·
B_1	·	·	1	·
A_2	1	·	·	·
B_2	·	·	1	·
E	·	1	·	1



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi^{\mu}_{g}(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1...4}$	180° ρ_{xyz}	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1...6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

O↓D4 subduction

D_4: $\mathbf{1}, \rho_{z180°}, \mathbf{R}_{z\pm 90°}, \rho_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1.$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1.$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0.$

$T_2(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1.$

$T_2(O)\downarrow D_4 = 3,\ -1,\ -1,\ -1,\ 1.$

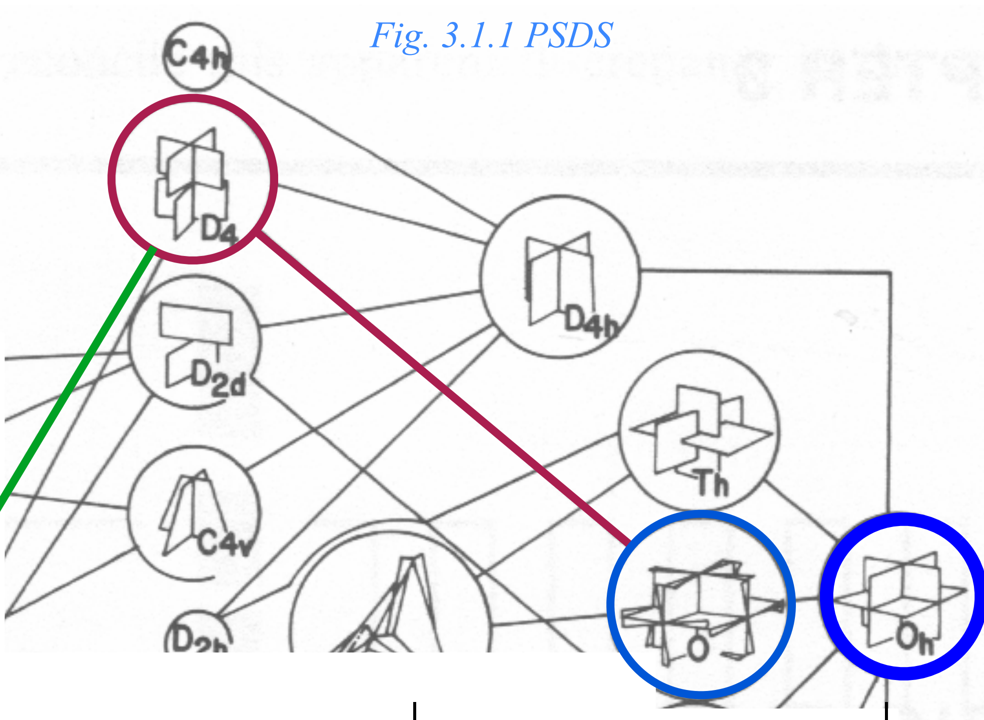


Fig. 3.1.1 PSDS

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g}=\mathbf{1}$	$\rho_{z180°}$	$\mathbf{R}_{z\pm 90°}$	$\rho_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	~~1~~	~~1~~
B_1	1	1	−1	~~1~~	~~−1~~
A_2	1	1	1	~~−1~~	~~−1~~
B_2	1	1	−1	~~−1~~	~~1~~
E	2	−2	0	~~0~~	~~0~~

D4↓C4 subduction

C_4: $\mathbf{1}, \mathbf{R}_{z+90°}, \rho_{z180°}, \mathbf{R}_{z-90°}$

$A_1(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$B_1(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$

$A_2(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$B_2(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$

$E(D_4)\downarrow C_4 = 2,\ 0,\ -2,\ 0. \quad =(1)_4\oplus(3)_4$

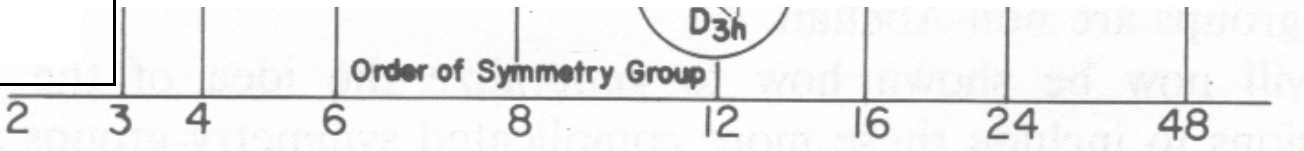
$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i

O↓C4 subduction

$O\downarrow C_4$	0_4	1_4	2_4	$3_4=\bar{1}_4$
A_1	→1	·	·	·
A_2				
E				
T_1				
T_2				

$D_4\downarrow C_4$	0_4	1_4	2_4	$3_4=\bar{1}_4$
A_1	→1	·	·	·
B_1	·	·	1	·
A_2	1	·	·	·
B_2	·	·	1	·
E	·	1	·	1



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi^{\mu}_{g}(O)$	$\mathbf{g=1}$	$\mathbf{r}_{1\ldots4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

$O{\downarrow}D_4$ subduction

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O){\downarrow}D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O){\downarrow}D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O){\downarrow}D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

$T_2(O){\downarrow}D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

$T_2(O){\downarrow}D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

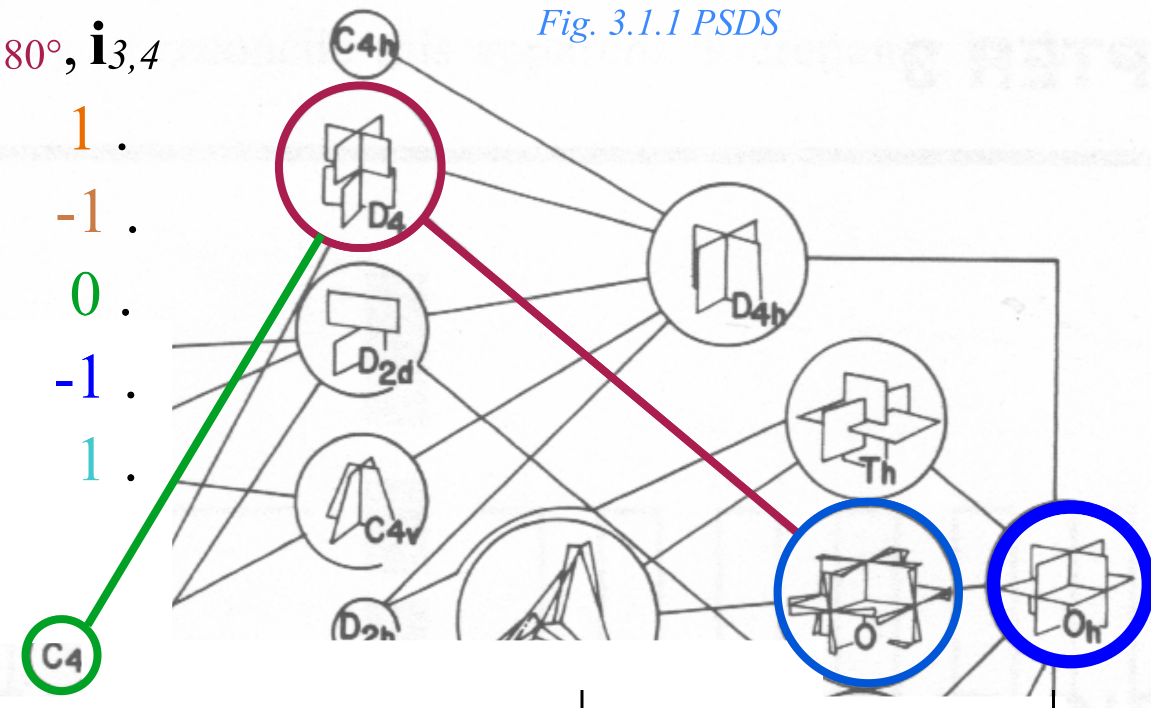


Fig. 3.1.1 PSDS

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g=1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	~~1~~	~~1~~
B_1	1	1	−1	~~1~~	~~−1~~
A_2	1	1	1	~~−1~~	~~−1~~
B_2	1	1	−1	~~−1~~	~~1~~
E	2	−2	0	~~0~~	~~0~~

$D_4{\downarrow}C_4$ subduction

C_4: $\mathbf{1}, \mathbf{R}_{z+90°}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z-90°}$

$A_1(D_4){\downarrow}C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$B_1(D_4){\downarrow}C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$

$A_2(D_4){\downarrow}C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$B_2(D_4){\downarrow}C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$

$E(D_4){\downarrow}C_4 = 2,\ 0,\ -2,\ 0\ . \quad =(1)_4 \oplus (3)_4$

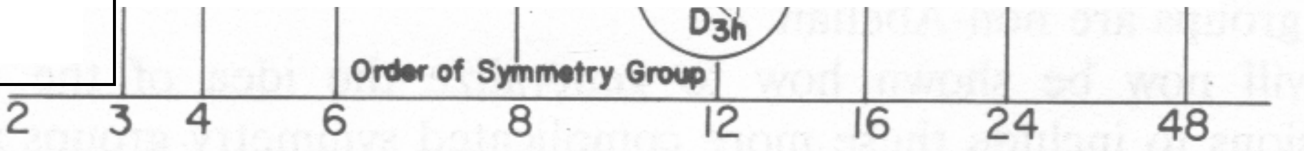
$O{\downarrow}D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g=1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i

$O{\downarrow}C_4$ subduction

$O{\downarrow}C_4$	0_4	1_4	2_4	$3_4 = \bar{1}_4$
A_1	1	·	·	·
A_2	·	·	→1	·
E				
T_1				
T_2				

$D_4{\downarrow}C_4$	0_4	1_4	2_4	$3_4 = \bar{1}_4$
A_1	1	·	·	·
B_1	·	·	→1	·
A_2	1	·	·	·
B_2	·	·	1	·
E	·	1	·	1



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi^{\mu}_{g}(O)$	$\mathbf{g=1}$	$\mathbf{r}_{1...4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1...6}$
A_1	1	$\cancel{1}$	1	1	1
A_2	1	$\cancel{1}$	1	–1	–1
E	2	$\cancel{-1}$	2	0	0
T_1	3	$\cancel{0}$	–1	1	–1
T_2	3	$\cancel{0}$	–1	–1	1

$O\downarrow D_4$ subduction

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

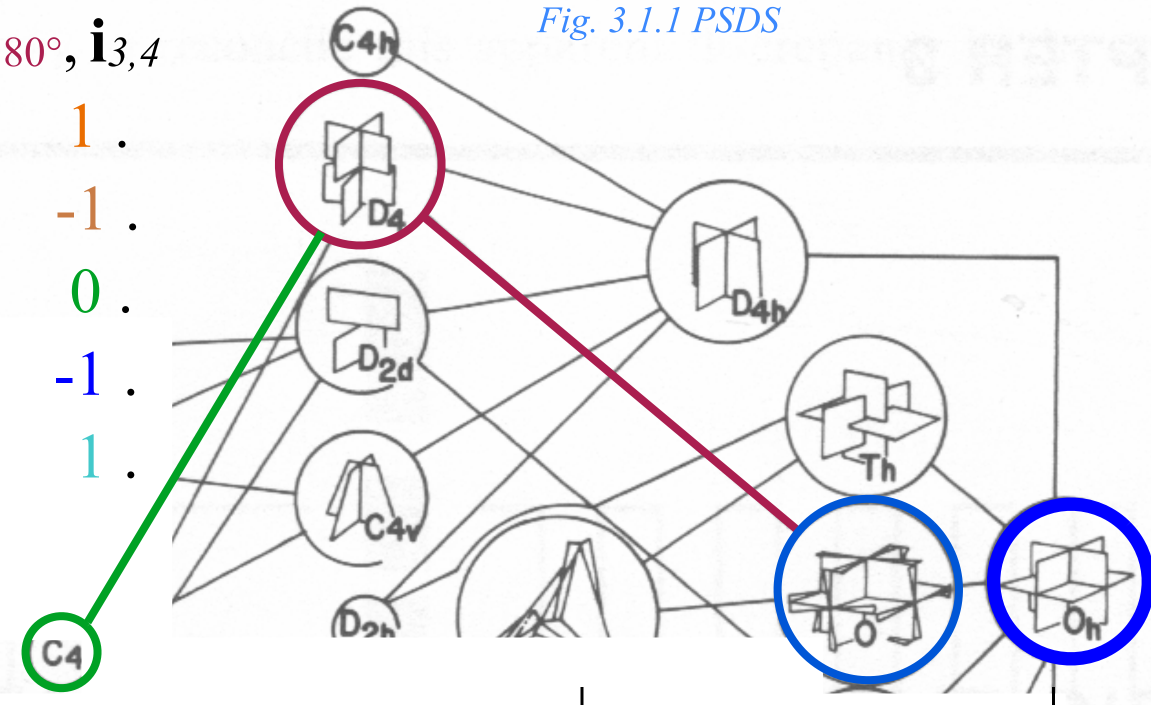


Fig. 3.1.1 PSDS

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g=1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	$\cancel{1}$	$\cancel{1}$
B_1	1	1	–1	$\cancel{1}$	$\cancel{-1}$
A_2	1	1	1	$\cancel{-1}$	$\cancel{-1}$
B_2	1	1	–1	$\cancel{-1}$	$\cancel{1}$
E	2	–2	0	$\cancel{0}$	$\cancel{0}$

$D_4\downarrow C_4$ subduction

C_4 : $\mathbf{1}, \mathbf{R}_{z+90°}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z-90°}$

$A_1(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1.\ =(0)_4$

$B_1(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1.\ =(2)_4$

$A_2(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1.\ =(0)_4$

$B_2(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1.\ =(2)_4$

$E(D_4)\downarrow C_4 = 2,\ 0,\ -2,\ 0\ .\ =(1)_4 \oplus (3)_4$

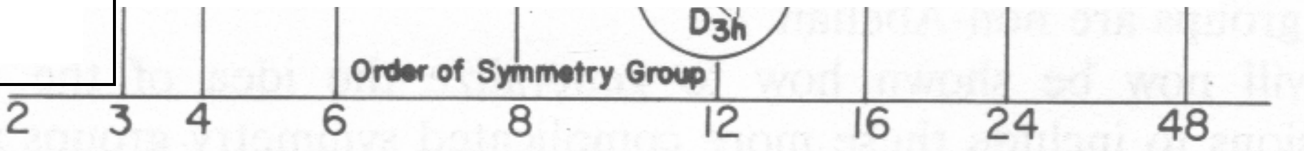
$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g=1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	–1	$-i$
$(2)_4$	1	–1	1	–1
$(3)_4$	1	$-i$	–1	i

$O\downarrow C_4$ subduction

$O\downarrow C_4$	0_4	1_4	2_4	$3_4 = \bar{1}_4$
A_1	1	·	·	·
A_2	·	·	1	·
E	→1	·	→1	·
T_1				
T_2				

$D_4\downarrow C_4$	0_4	1_4	2_4	$3_4 = \bar{1}_4$
A_1	→1	·	·	·
B_1	·	·	→1	·
A_2	1	·	·	·
B_2	·	·	1	·
E	·	1	·	1



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi^{\mu}_{g}(O)$	$\mathbf{g=1}$	$\mathbf{r}_{1...4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1...6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

O↓D₄ subduction

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm 90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

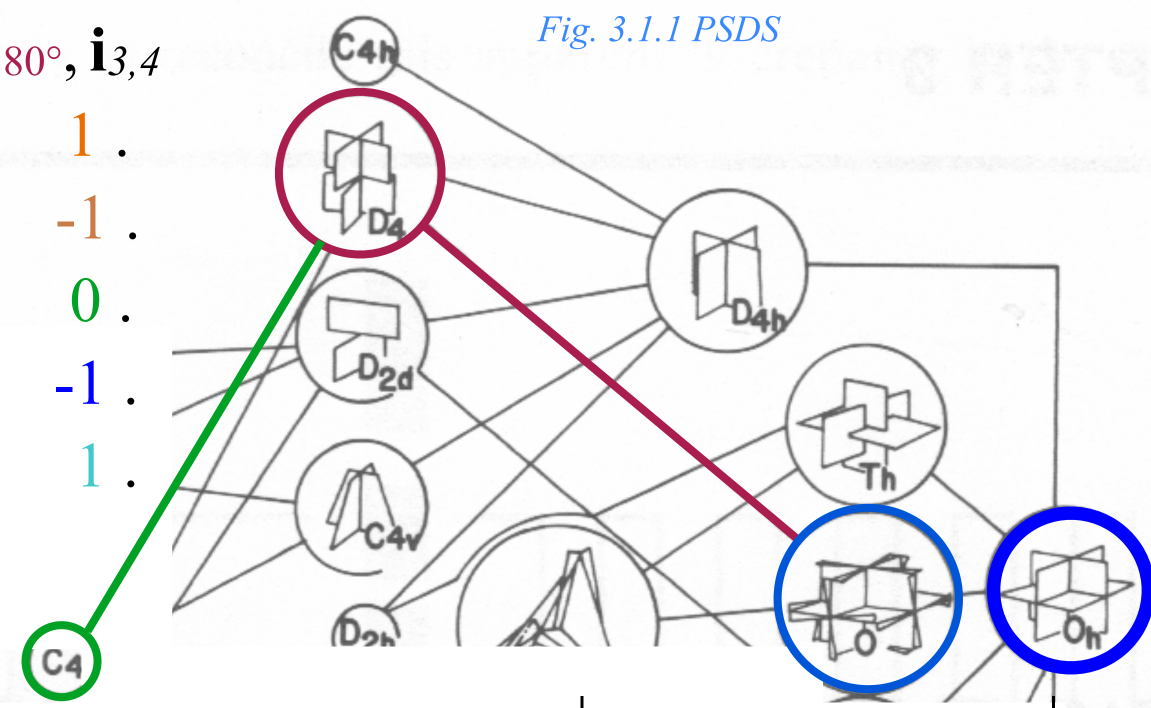


Fig. 3.1.1 PSDS

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g=1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm 90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	~~1~~	~~1~~
B_1	1	1	−1	~~1~~	~~−1~~
A_2	1	1	1	~~−1~~	~~−1~~
B_2	1	1	−1	~~−1~~	~~1~~
E	2	−2	0	~~0~~	~~0~~

D₄↓C₄ subduction

C_4 : $\mathbf{1}, \mathbf{R}_{z+90°}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z-90°}$

$A_1(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$B_1(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$

$A_2(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$B_2(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$

$E(D_4)\downarrow C_4 = 2,\ 0,\ -2,\ 0\ . \quad =(1)_4 \oplus (3)_4$

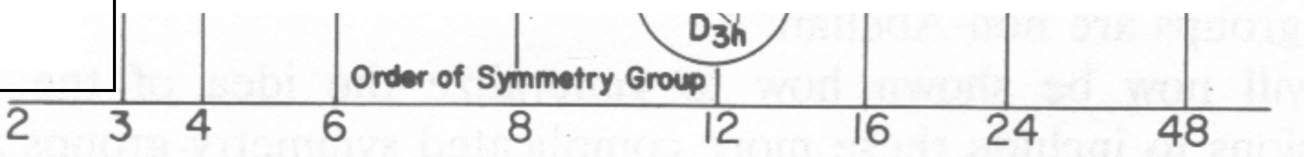
$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g=1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i

O↓C₄ subduction

$O\downarrow C_4$	0_4	1_4	2_4	$3_4 = \bar{1}_4$
A_1	1	·	·	·
A_2	·	·	1	·
E	1	·	1	·
T_1	→1	→1	·	→1
T_2				

$D_4\downarrow C_4$	0_4	1_4	2_4	$3_4 = \bar{1}_4$
A_1	1	·	·	·
B_1	·	·	1	·
A_2	→1	·	·	·
B_2	·	·	1	·
E	·	→1	·	→1



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi^{\mu}_{g}(O)$	$\mathbf{g=1}$	$\mathbf{r}_{1\ldots4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

O↓D₄ subduction

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$$
$$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$$
$$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$$
$$T_2(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$$
$$T_2(O)\downarrow D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$$

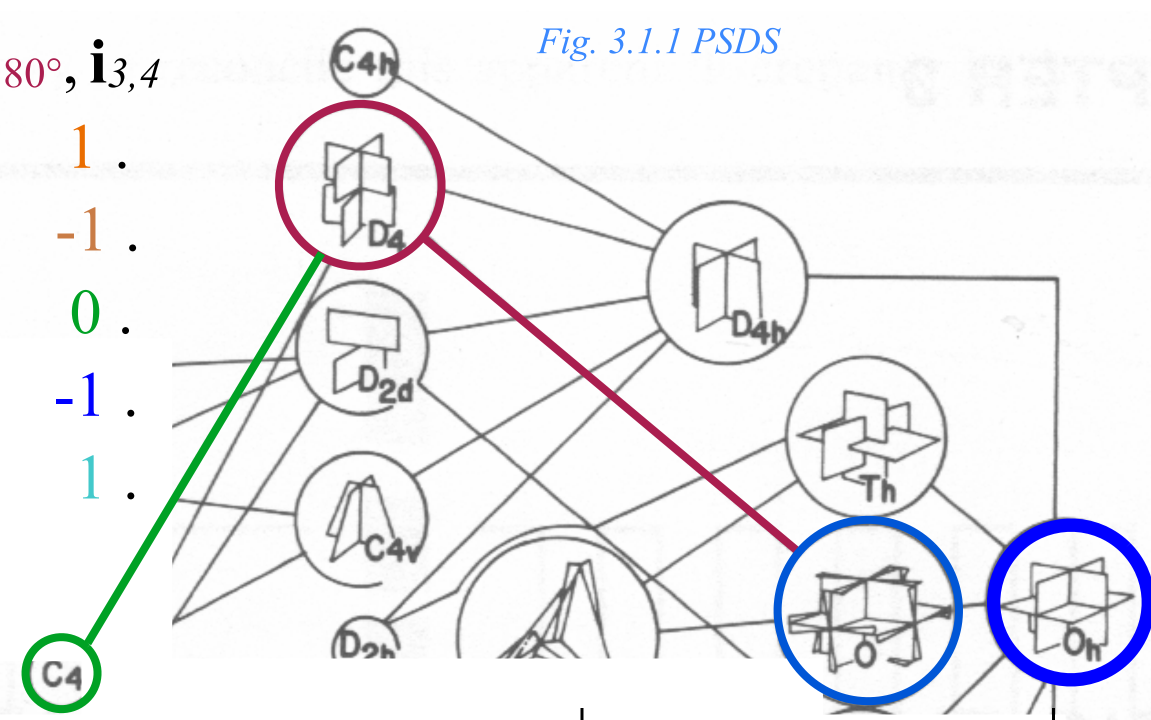


Fig. 3.1.1 PSDS

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g=1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	~~1~~	~~1~~
B_1	1	1	−1	~~1~~	~~−1~~
A_2	1	1	1	~~−1~~	~~−1~~
B_2	1	1	−1	~~−1~~	~~1~~
E	2	−2	0	~~0~~	~~0~~

D₄↓C₄ subduction

C_4: $\mathbf{1}, \mathbf{R}_{z+90°}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z-90°}$

$$A_1(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$$
$$B_1(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$$
$$A_2(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$$
$$B_2(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$$
$$E(D_4)\downarrow C_4 = 2,\ 0,\ -2,\ 0\ . \quad =(1)_4 \oplus (3)_4$$

$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g=1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i

O↓C₄ subduction

$O\downarrow C_4$	0_4	1_4	2_4	$3_4=\bar{1}_4$
A_1	1	·	·	·
A_2	·	·	1	·
E	1	·	1	·
T_1	1	1	·	1
T_2	·	→1	→1	→1

$D_4\downarrow C_4$	0_4	1_4	2_4	$3_4=\bar{1}_4$
A_1	1	·	·	·
B_1	·	·	1	·
A_2	1	·	·	·
B_2	·	·	→1	·
E	·	→1	·	→1

Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O{\downarrow}D_4$ subduction

$$D_4: \quad \mathbf{1}, \rho_{z180°}, \mathbf{R}_{z\pm 90°}, \rho_{x,y180°}, \mathbf{i}_{3,4}$$

$$A_1(O){\downarrow}D_4 = 1, \quad 1, \quad 1, \quad 1, \quad 1\ .$$

$$A_2(O){\downarrow}D_4 = 1, \quad 1, \quad -1, \quad 1, \quad -1\ .$$

$$E(O){\downarrow}D_4 = 2, \quad 2, \quad 0, \quad 2, \quad 0\ .$$

$$T_2(O){\downarrow}D_4 = 3, \quad -1, \quad 1, \quad -1, \quad -1\ .$$

$$T_2(O){\downarrow}D_4 = 3, \quad -1, \quad -1, \quad -1, \quad 1\ .$$

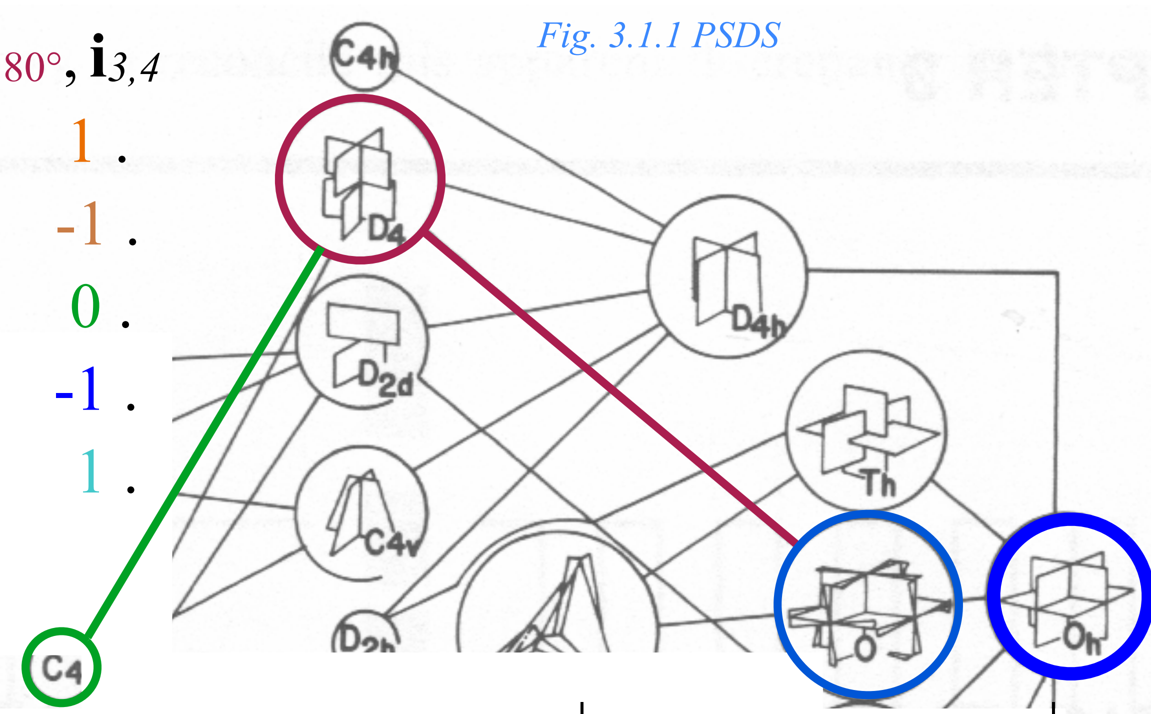
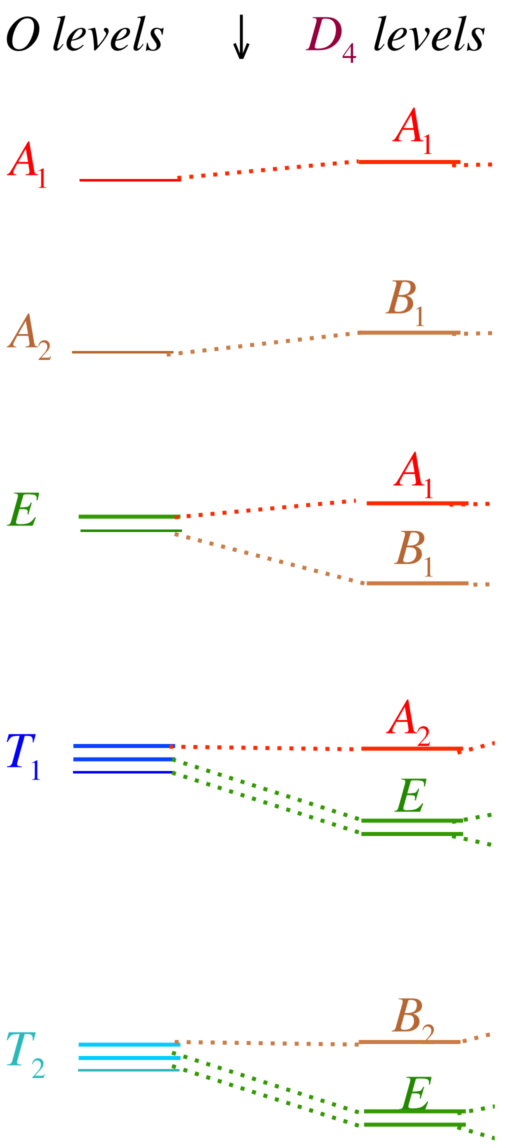


Fig. 3.1.1 PSDS

$O \supset D_4 \supset C_4$ subgroup and level-splitting/relabeling correlations



$D_4{\downarrow}C_4$ subduction

$$C_4: \quad \mathbf{1}, \mathbf{R}_{z+90°}, \rho_{z180°}, \mathbf{R}_{z-90°}$$

$$A_1(D_4){\downarrow}C_4 = 1, \quad 1, \quad 1, \quad 1. \quad =(0)_4$$

$$B_1(D_4){\downarrow}C_4 = 1, \quad -1, \quad 1, \quad -1. \quad =(2)_4$$

$$A_2(D_4){\downarrow}C_4 = 1, \quad 1, \quad 1, \quad 1. \quad =(0)_4$$

$$B_2(D_4){\downarrow}C_4 = 1, \quad -1, \quad 1, \quad -1. \quad =(2)_4$$

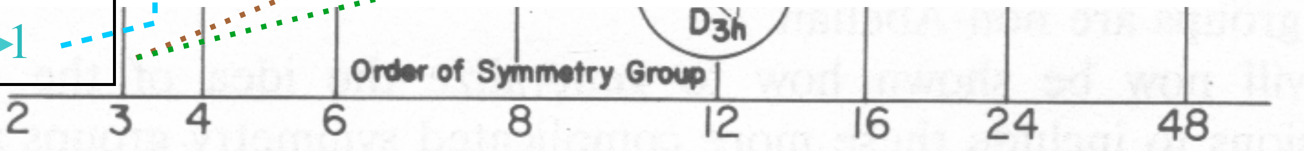
$$E(D_4){\downarrow}C_4 = 2, \quad 0, \quad -2, \quad 0\ . \quad =(1)_4 \oplus (3)_4$$

$O{\downarrow}D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$O{\downarrow}C_4$ subduction

$O{\downarrow}C_4$	0_4	1_4	2_4	$3_4 = \bar{1}_4$
A_1	1	·	·	·
A_2	·	·	1	·
E	1	·	1	·
T_1	1	1	·	1
T_2	·	→1	→1	→1

$D_4{\downarrow}C_4$	0_4	1_4	2_4	$3_4 = \bar{1}_4$
A_1	1	·	·	·
B_1	·	·	1	·
A_2	1	·	·	·
B_2	·	·	→1	·
E	·	→1	·	→1



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

D_4: $\mathbf{1}, \rho_{z180°}, \mathbf{R}_{z\pm 90°}, \rho_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O) \downarrow D_4 = 1, \quad 1, \quad 1, \quad 1, \quad 1\ .$

$A_2(O) \downarrow D_4 = 1, \quad 1, \quad -1, \quad 1, \quad -1\ .$

$E(O) \downarrow D_4 = 2, \quad 2, \quad 0, \quad 2, \quad 0\ .$

$T_2(O) \downarrow D_4 = 3, \quad -1, \quad 1, \quad -1, \quad -1\ .$

$T_2(O) \downarrow D_4 = 3, \quad -1, \quad -1, \quad -1, \quad 1\ .$

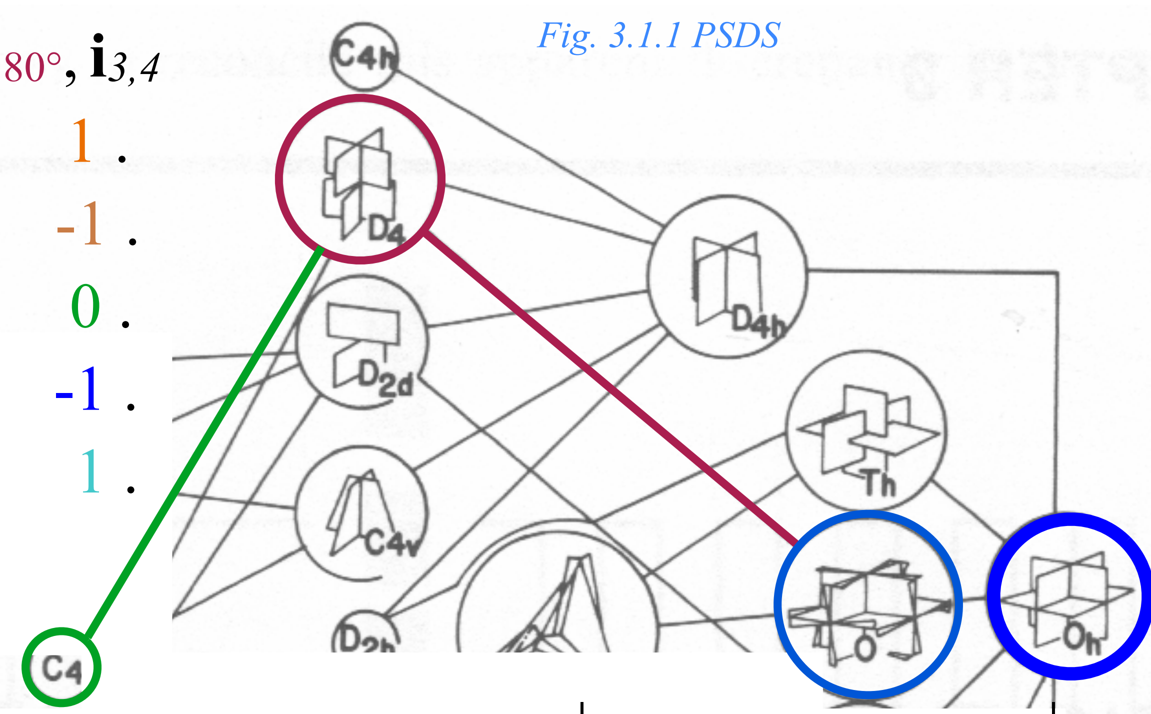
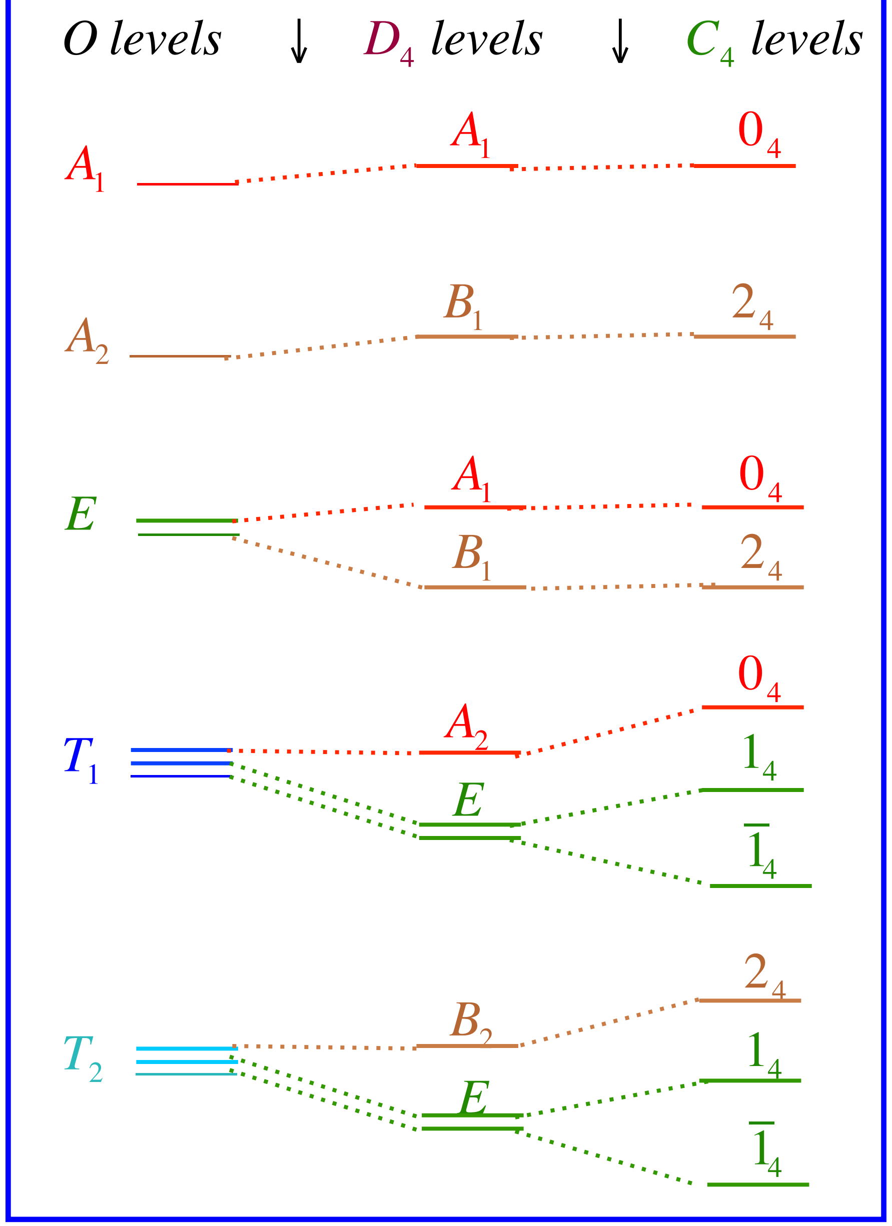


Fig. 3.1.1 PSDS

$O \supset D_4 \supset C_4$ subgroup and level-splitting/relabeling correlations



$D_4 \downarrow C_4$ subduction

C_4: $\mathbf{1}, \mathbf{R}_{z+90°}, \rho_{z180°}, \mathbf{R}_{z-90°}$

$A_1(D_4) \downarrow C_4 = 1, \quad 1, \quad 1, \quad 1. \quad =(0)_4$

$B_1(D_4) \downarrow C_4 = 1, \quad -1, \quad 1, \quad -1. \quad =(2)_4$

$A_2(D_4) \downarrow C_4 = 1, \quad 1, \quad 1, \quad 1. \quad =(0)_4$

$B_2(D_4) \downarrow C_4 = 1, \quad -1, \quad 1, \quad -1. \quad =(2)_4$

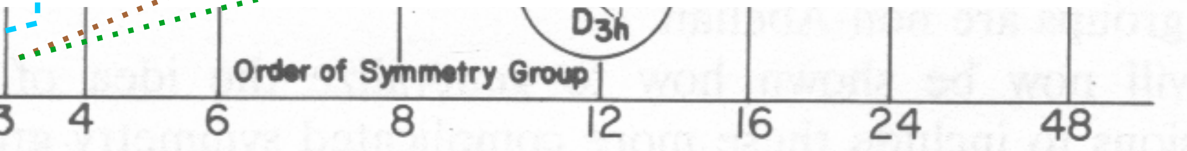
$E(D_4) \downarrow C_4 = 2, \quad 0, \quad -2, \quad 0\ . \quad =(1)_4 \oplus (3)_4$

$O \downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$D_4 \downarrow C_4$	0_4	1_4	2_4	$3_4 = \bar{1}_4$
A_1	1	·	·	·
B_1	·	·	1	·
A_2	1	·	·	·
B_2	·	·	→1	·
E	·	→1	·	→1

$O \downarrow C_4$ subduction

$O \downarrow C_4$	0_4	1_4	2_4	$3_4 = \bar{1}_4$
A_1	1	·	·	·
A_2	·	·	1	·
E	1	·	1	·
T_1	1	1	·	1
T_2	·	→1	→1	→1



Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$
  - *Octahedral-cubic O symmetry and group operations*
  - *Tetrahedral symmetry becomes Icosahedral*
  - *Octahedral groups $O_h \supset O \sim T_d \supset T$*
    - *Octahedral O and spin-$O \subset U(2)$*

Tetrahedral T class algebra
  - *Tetrahedral T class minimal equations*
  - *Tetrahedral T class projectors and characters*

Octahedral O class algebra
  - *Octahedral O class minimal equations*
  - *Octahedral O class projectors and characters*

Octahedral $O_h \supset O$:Inversion (g&u) parity
  - *Octahedral $O_h \supset O \supset C_{\mathbf{I}}$ subgroup correlations*
  - *$O_h \supset O \supset D_4$ subgroup correlations*
  - *$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations*
  - *Preview of applications to high resolution spectroscopy*

Summary of some Octahedral symmetry results:

$\ell^{A_1} = 1$

$\ell^{A_2} = 1$

$\ell^{E} = 2$

$\ell^{T_1} = 3$

$\ell^{T_2} = 3$

Cubic-Octahedral Group *O*

Centrum: $\kappa(O) = \Sigma_{(\alpha)} (\ell^{\alpha})^{0} = 1^{0}+1^{0}+2^{0}+3^{0}+3^{0}=5$

Rank: $\rho(O) = \Sigma_{(\alpha)} (\ell^{\alpha})^{1} = 1^{1}+1^{1}+2^{1}+3^{1}+3^{1}=10$

Order: ${}^{o}(O) = \Sigma_{(\alpha)} (\ell^{\alpha})^{0} = 1^{2}+1^{2}+2^{2}+3^{2}+3^{2}=24$

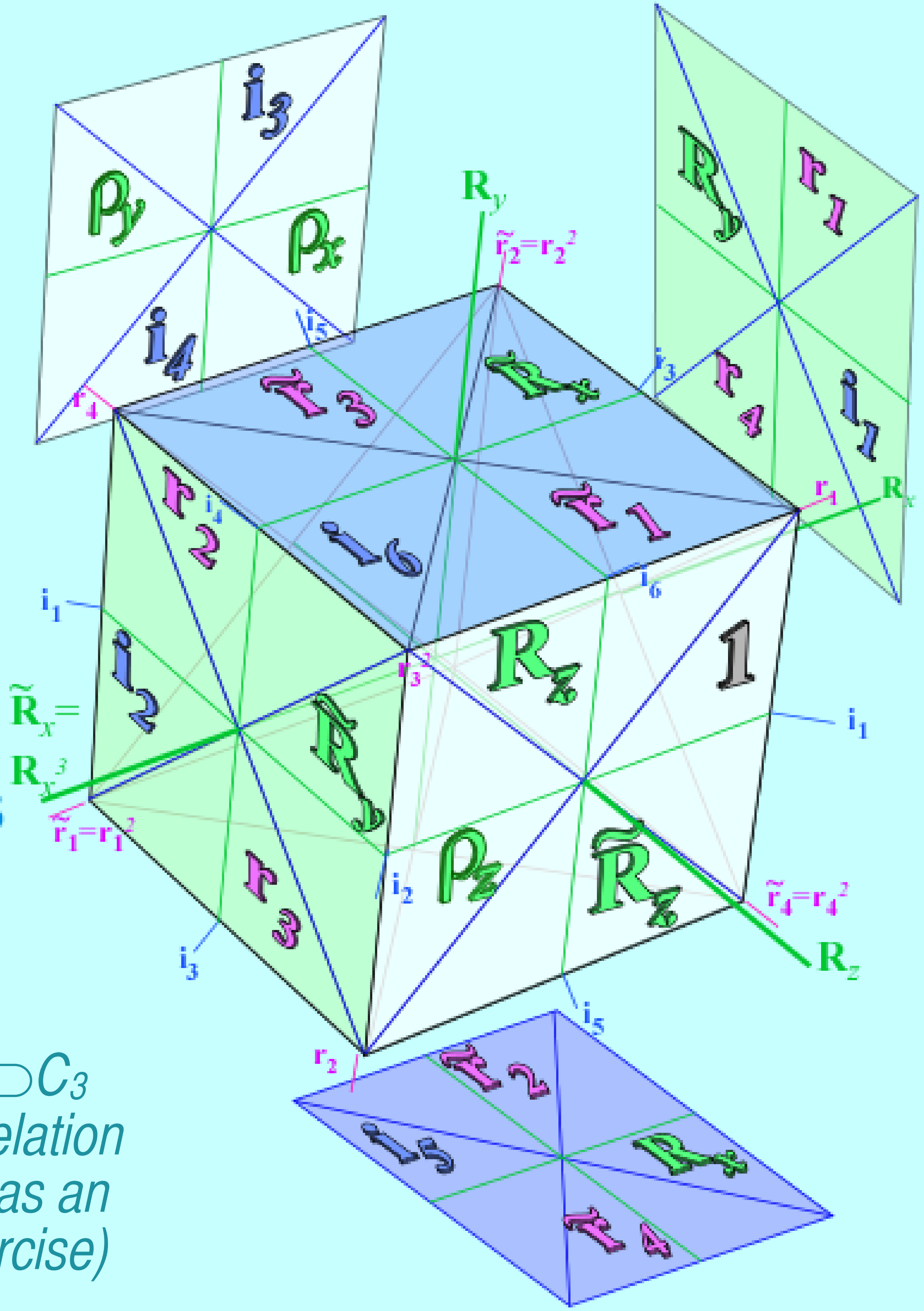
O group $\chi^{\alpha}_{\kappa_g}$	$g=1$	r_{1-4} $\tilde{r}_{1-4}$	ρ_{xyz}	R_{xyz} $\tilde{R}_{xyz}$	i_{1-6}
$\alpha = A_1$	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

s-orbital r^2 → A_1

d-orbitals $\{x^2+y^2-2z^2, x^2-y^2\}$ → E

p-orbitals $\{x, y, z\}$ → T_1

$\{xz, yz, xy\}$ d-orbitals → T_2



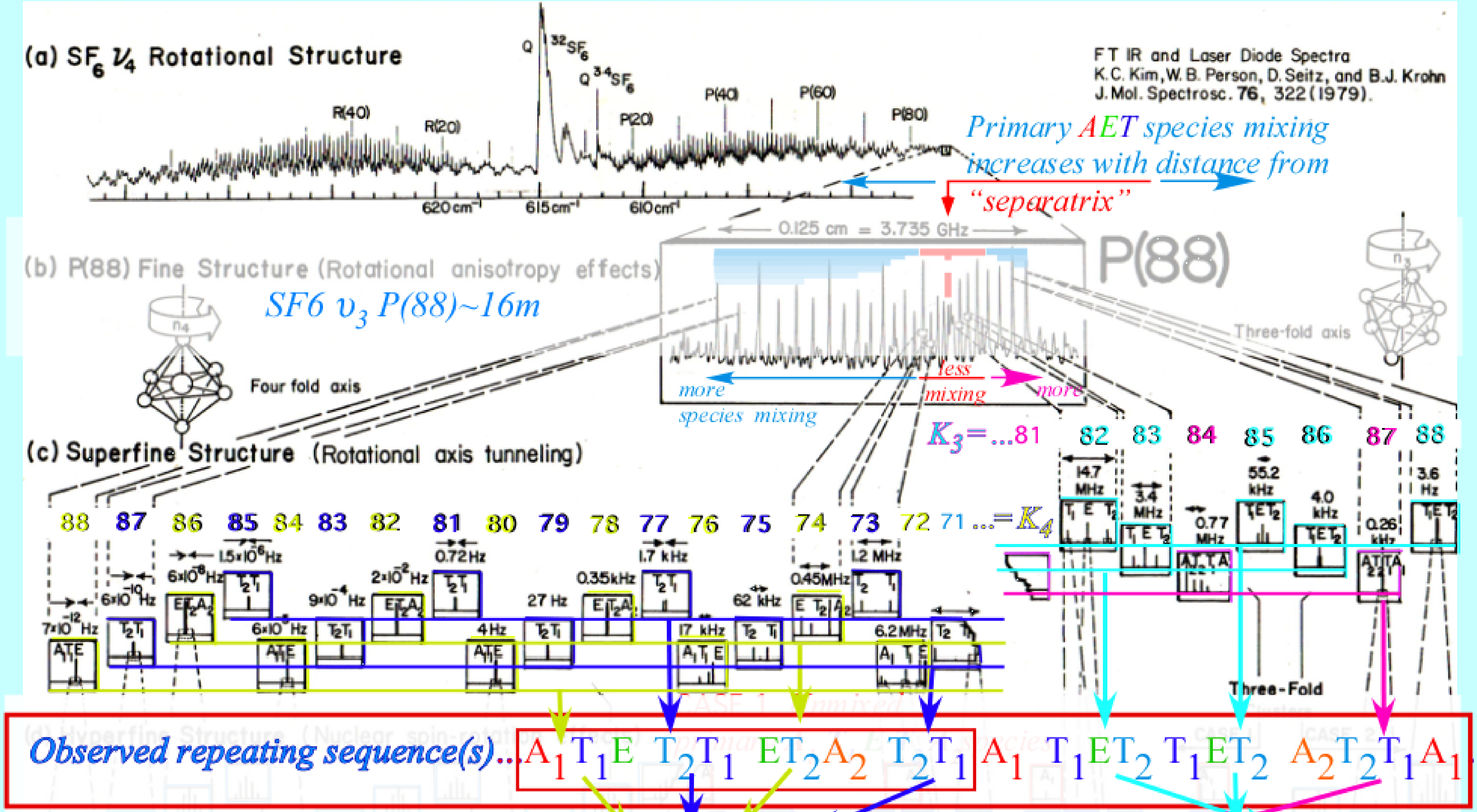
O⊃C₄

	$(0)_4$	$(1)_4$	$(2)_4$	$(3)_4=(-1)_4$
A_1	1	•	•	•
A_2	•	•	1	•
E	1	•	1	•
T_1	1	1	•	1
T_2	•	1	1	1

O⊃C₃

	$(0)_3$	$(1)_3$	$(2)_3=(-1)_3$
A_1	1	•	•
A_2	1	•	•
E	•	1	1
T_1	1	1	1
T_2	1	1	1

(O⊃C₃ correlation left as an exercise)



Observed repeating sequence(s)... $A_1\ T_1\ E\ \ T_2\ T_1\ \ E\ T_2\ A_2\ \ T_2\ T_1\ \ A_1\ \ T_1\ E\ T_2\ \ T_1\ E\ T_2\ \ A_2\ T_2\ T_1\ A_1$...

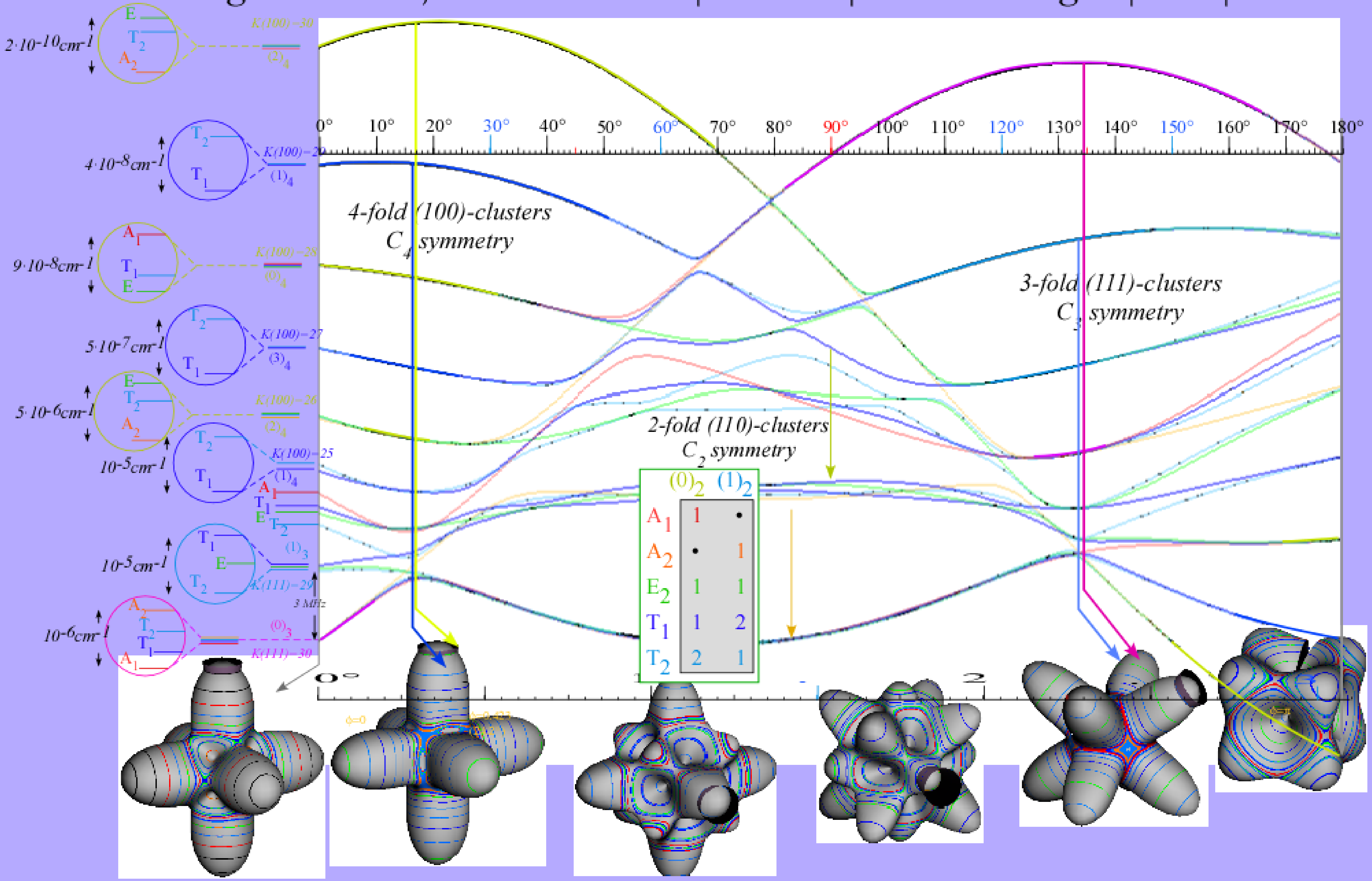
Local correlations explain clustering...
... but what about spacing and ordering?...

...and physical consequences?

$O\supset C_4$	$(0)_4$	$(1)_4$	$(2)_4$	$(3)_4=(-1)_4$
A_1	1	•	•	•
A_2	•	•	1	•
E	1	•	1	•
T_1	1	1	•	1
T_2	•	1	1	1

$O\supset C_3$	$(0)_3$	$(1)_3$	$(2)_3=(-1)_3$
A_1	1	•	•
A_2	1	•	•
E	•	1	1
T_1	1	1	1
T_2	1	1	1

Eigenvalues of $\mathbf{H}=B\mathbf{J}^2+\cos\phi\,\mathbf{T}^{[4]}+\sin\phi\,\mathbf{T}^{[6]}$ *vs. mix angle* ϕ: $0<\phi<\pi$



	$(0)_2$	$(1)_2$
A_1	1	•
A_2	•	1
E_2	1	1
T_1	1	2
T_2	2	1