Group Theory in Quantum Mechanics

Lecture 15 (3.09.17)

Smallest non-Abelian group $D_3$ (and isomorphic $C_{3v} \sim D_3$)

(Small non-Abelian group $D_3$ and isomorphic $C_{3v} \sim D_3$)

(3-Dihedral-axes group $D_3$ vs. 3-Vertical-mirror-plane group $C_{3v}$

$D_3$ and $C_{3v}$ are isomorphic ($D_3 \sim C_{3v}$ share product table)

Deriving $D_3 \sim C_{3v}$ products:

By group definition $|g\rangle = g|1\rangle$ of position ket $|g\rangle$ 

By nomograms based on $U(2)$ Hamilton-turns

Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes

Non-commutative symmetry expansion and Global-Local solution

Global vs Local symmetry and Mock-Mach principle

Global vs Local matrix duality for $D_3$

Global vs Local symmetry expansion of $D_3$ Hamiltonian

1st-Stage spectral decomposition of global/local $D_3$ Hamiltonian

Group theory of equivalence transformations and classes

Lagrange theorems

All-commuting operators and $D_3$-invariant class algebra (center)

All-commuting projectors and $D_3$-invariant characters

Group invariant numbers: Centrum, Rank, and Order

(Fig. 15.2.1 QTCA)
3-Dihedral-axes group $D_3$ vs. 3-Vertical-mirror-plane group $C_{3v}$
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  - Global vs Local symmetry expansion of $D_3$ Hamiltonian

1st-Stage spectral decomposition of global/local $D_3$ Hamiltonian
- Group theory of equivalence transformations and classes
  - Lagrange theorems
- All-commuting operators and $D_3$-invariant class algebra
- All-commuting projectors and $D_3$-invariant characters
- Group invariant numbers: Centrum, Rank, and Order

Crystal-field splitting: $O(3) \supset D_3$ symmetry reduction and $D_3 \downarrow D_3$ splitting
Figure 3.1.1 Crystal point symmetry groups. Models are sketched in circles for the 16 non-Abelian groups. (See also Figure 2.11.1.)
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3-Dihedral-axes group \( D_3 \) vs. 3-Vertical-mirror-plane group \( C_{3v} \)

\( D_3 \) and \( C_{3v} \) are isomorphic (\( D_3 \sim C_{3v} \) share product table)

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Fig. 3.1.3 PSDS

3-Dihedral-axes group $D_3$ vs. 3-Vertical-mirror-plane group $C_{3v}$

*isomorphic means mathematically the same abstract group even if physically different action.*

Showing that $D_3$ and $C_{3v}$ are isomorphic* ($D_3 \sim C_{3v}$ share product table)
3-Dihedral-axes group $D_3$ vs. 3-Vertical-mirror-plane group $C_{3v}$

**Figure 3.1.3** Pictorial comparison of $D_3$ and $C_{3v}$ symmetry. A propeller having $D_3$ symmetry is shown next to a three-plane paddle having $C_{3v}$ symmetry. The group operations are labeled by arrows, which indicate the effect they have. For example, $\rho_3$ is a $180^\circ$ rotation around the y axis, while $I\rho_3 = \sigma_3$ is a reflection through the xz plane. (Here axes are fixed and the objects rotate.)

$180^\circ D_3$-Y-axis-rotation: $\rho_3 = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & +1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$ maps to: XZ-mirror-plane reflection: $\sigma_3 = \begin{pmatrix} +1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & +1 \end{pmatrix}$

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3-Dihedral-axes group $D_3$ vs. 3-Vertical-mirror-plane group $C_{3v}$

$\rho_3 (180^\circ)$

$\rho_1 (180^\circ)$

$r (120^\circ)$

$\rho_2 (180^\circ)$

$\sigma_3 \rho_3$

$\sigma_1 \rho_3$

$\sigma_2 \rho_3$

$\rho_3 I = I \rho_3$

$180^\circ$ \perp \text{-axial-rotation-inversion} \quad \sigma = R \cdot I = I \cdot R$

$\sigma_3 = \begin{pmatrix} +1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & +1 \end{pmatrix}$

$\sigma_3 = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & +1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$

$\sigma_3 = \begin{pmatrix} +1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & +1 \end{pmatrix}$

180° $D_3$-Y-axis-rotation: $\rho_3 = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & +1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$ maps to: XZ-mirror-plane reflection: $\sigma_3 = \begin{pmatrix} -1 & \cdot & +1 \\ \cdot & -1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$

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Figure 3.1.3 PSDS

Mirror-plane-reflection $\sigma$ equals
$180^\circ$ ⊥-axial-rotation-inversion
$\sigma = R \cdot I = I \cdot R$

$\sigma_3 = \begin{pmatrix} +1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & +1 \end{pmatrix}$

$= \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & +1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix} = \rho_3 \cdot I = I \cdot \rho_3$

$180^\circ D_3$-Y-axis-rotation: $\rho_3 = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & +1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$

maps to: XZ-mirror-plane reflection: $\sigma_3 = \begin{pmatrix} +1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & +1 \end{pmatrix}$

Inversion $I = -1$ commutes with all $R$

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**Fig. 3.1.3 PSDS**

**Mirror-plane-reflection $\sigma$ equals**

$180^\circ \perp$-axial-rotation-inversion

$\sigma = R \cdot I = I \cdot R$

$\sigma_3 = \begin{pmatrix} +1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & +1 \end{pmatrix} = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & +1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$

$= \rho_3 \cdot I = I \cdot \rho_3$

$180^\circ D_3$-Y-axis-rotation: $\rho_3 = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & +1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$ maps to: XZ-mirror-plane reflection: $\sigma_3 = \begin{pmatrix} +1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & +1 \end{pmatrix}$

$180^\circ D_3$-$\rho_2$-axis-rotation: $\rho_2$ maps to: $\perp \rho_2$-mirror-plane reflection: $\sigma_2 = \rho_2 \cdot I = I \cdot \rho_2$

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**Mirror-plane-reflection $\sigma$**

$180^\circ \perp$-axial-rotation-inversion

$\sigma = R \cdot I = I \cdot R$

$\sigma_3 = \begin{pmatrix} +1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & +1 \end{pmatrix}$

$= \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & +1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$

$= \rho_3 \cdot I = I \cdot \rho_3$

**Inversion**

$I = -I$ commutes with all $R$

*isomorphic means mathematically the same abstract group even if physically different action.*

**Showing that $D_3$ and $C_{3v}$ are isomorphic** ($D_3 \sim C_{3v}$ share product table)

**Fig. 3.1.3 PSDS**

**Figure 3.1.3** Pictorial comparison of $D_3$ and $C_{3v}$ symmetry. A propeller having $D_3$ symmetry is shown next to a three-plane paddle having $C_{3v}$ symmetry. The group operations are labeled by arrows, which indicate the effect they have. For example, $\rho_3$ is a $180^\circ$ rotation around the $y$ axis, while $I \rho_3 = \sigma_3$ is a reflection through the $xz$ plane. (Here axes are fixed and the objects rotate.)

$180^\circ D_3$-$Y$-axis-rotation: $\rho_3 = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & +1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$ maps to: XZ-mirror-plane reflection: $\sigma_3 = \begin{pmatrix} +1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & +1 \end{pmatrix}$

$180^\circ D_3$-$D_2$-axis-rotation: $\rho_2$ maps to: $\perp \rho_2$-mirror-plane reflection: $\sigma_2 = \rho_2 \cdot I = I \cdot \rho_2$

$180^\circ D_3$-$D_1$-axis-rotation: $\rho_1$ maps to: $\perp \rho_1$-mirror-plane reflection: $\sigma_1 = \rho_1 \cdot I = I \cdot \rho_1$
3-Dihedral-axes group $D_3$ vs. 3-Vertical-mirror-plane group $C_{3v}$

Fig. 3.1.3 PSDS

$180^\circ D_3$ - Y-axis-rotation: $\rho_3 = \begin{pmatrix} -1 & \cdots & \cdots \\ \cdots & +1 & \cdots \\ \cdots & \cdots & -1 \end{pmatrix}$ maps to: XZ-mirror-plane reflection: $\sigma_3 = \begin{pmatrix} +1 & \cdots & \cdots \\ \cdots & -1 & \cdots \\ \cdots & \cdots & +1 \end{pmatrix}$

$180^\circ D_3$ - $\rho_2$-axis-rotation: $\rho_2$

maps to: $\bot \rho_2$ - mirror-plane reflection: $\sigma_2 = \rho_2 \cdot I = I \rho_2$

$180^\circ D_3$ - $\rho_1$-axis-rotation: $\rho_1$

maps to: $\bot \rho_1$ - mirror-plane reflection: $\sigma_1 = \rho_1 \cdot I = I \rho_1$

$D_3$-product: $\rho_1 \rho_2$

maps to: $C_{3v}$-product: $\sigma_1 \sigma_2 = \rho_1 \cdot I \rho_2 = \rho_1 \rho_2$

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180°$D_3$-Y-axis-rotation: $\rho_3 = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & +1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$ maps to: XZ-mirror-plane reflection: $\sigma_3 = \begin{pmatrix} +1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & +1 \end{pmatrix}$

180°$D_3$-$\rho_2$-axis-rotation: $\rho_2$ maps to: $\bot\rho_2$-mirror-plane reflection: $\sigma_2 = \rho_2\I = \I\rho_2$

180°$D_3$-$\rho_1$-axis-rotation: $\rho_1$ maps to: $\bot\rho_1$-mirror-plane reflection: $\sigma_1 = \rho_1\I = \I\rho_1$

$D_3$-product: $\rho_1\rho_2$ maps to: $C_{3v}$-product: $\sigma_1\sigma_2 = \rho_1\I\rho_2 = \rho_1\rho_2$

$D_3$-product: $\rho_1r^p$ maps to: $C_{3v}$-product: $\sigma_1r^p = \rho_1\I r^p = \rho_1 r^p \I = \I \rho_1 r^p$

Mirror-plane-reflection $\sigma$ equals 180° $\bot$-axial-rotation-inversion $\sigma = R\I = \I R$

Inversion $\I = -\I$ commutes with all $R$

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$D_3$ and $C_{3v}$ are isomorphic ($D_3 \sim C_{3v}$ share product table)

**Deriving $D_3 \sim C_{3v}$ products:**

- **By group definition** $|g\rangle = g |I\rangle$ of position ket $|g\rangle$
- **By nomograms based on $U(2)$ Hamilton-turns**

**Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes**

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Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle = g |1\rangle$ of position ket $|g\rangle$

$\mathbf{r}^1 |1\rangle = |\mathbf{r}^1\rangle$

$\sigma_1 |1\rangle = |\sigma_1\rangle$

$\sigma_2 |1\rangle = |\sigma_2\rangle$
Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle = g|1\rangle$ of position ket $|g\rangle$

Building $C_{3v}$ Group “slide-rule”
3-Dihedral-axes group $D_3$ vs. 3-Vertical-mirror-plane group $C_{3v}$

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Example: Find $C_{3v}$ product $\sigma_1 r^1|1\rangle = \sigma_1|r^1\rangle$

Using $C_{3v}$ Group "slide-rule"
Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle = g|1\rangle$ of position ket $|g\rangle$

Example: Find $C_{3v}$ product $\sigma_1 r^1 |1\rangle = \sigma_1 |r^1\rangle$

Using $C_{3v}$ Group “slide-rule”

Result: $\sigma_1 r^1 = \sigma_2$

Factor $r^1$ on right acts first

Left is last (like Hebrew)
Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle = g|1\rangle$ of position ket $|g\rangle$

Example: Find $C_{3v}$ product $\sigma_1 r^1 |1\rangle = \sigma_1 |r^1\rangle$

Using $C_{3v}$ Group "slide-rule" result: $\sigma_1 r^1 = \sigma_2$

Other $\sigma_1$ results from graph:

$\sigma_1 \{ 1, \ r^1, \ r^2, \ \sigma_1, \ \sigma_2, \ \sigma_3 \} = \{ \sigma_1, \sigma_2, \sigma_3, \ 1, \ r^1, \ r^2 \}$
Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle = g|I\rangle$ of position ket $|g\rangle$

Example: Find $C_{3v}$ product $\sigma_1 r^1 |I\rangle = \sigma_1 |r^1\rangle$

Using $C_{3v}$ Group “slide-rule”

Other $\sigma_1$ results from graph:

$\sigma_1 \{1, r^1, r^2, \sigma_1, \sigma_2, \sigma_3\} = \{\sigma_1, \sigma_2, \sigma_3, 1, r^1, r^2\}$

....whole $C_{3v}$ group table:
Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle=g|I\rangle$ of position ket $|g\rangle$

$D_3$ and $C_{3v}$ clearly are isomorphic $D_3\sim C_{3v}$ share group table

...except for notation $\rho_k \leftrightarrow \sigma_k$
Figure 3.1.1 Crystal point symmetry groups. Models are sketched in circles for the 16 non-Abelian groups. (See also Figure 2.11.1.)
Figure 3.1.1 Crystal point symmetry groups. Models are sketched in circles for the 16 non-Abelian groups. (See also Figure 2.11.1.)

Total number $N_g$ of distinct groups

Number $N_A$ are Abelian

$D_4$ and $C_{4v}$ are related similarly to $D_3$ and $C_{3v}$.
3-Dihedral-axes group $D_3$ vs. 3-Vertical-mirror-plane group $C_{3v}$

$D_3$ and $C_{3v}$ are isomorphic ($D_3 \sim C_{3v}$ share product table)

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Deriving $D_3 \sim C_{3v}$ products by nomograms based on $U(2)$ Hamilton-turns

(Fig. 3.1.5 PSDS)

(Fig. 3.1.6 PSDS)

(From Lect. 8 p. 65-78)
Deriving $D_3 \sim C_{3v}$ products by nomograms based on $U(2)$ Hamilton-turns

(Fig. 3.1.5 PSDS)

Rotation vector $\Theta$
Rotation angle $= \Theta$

1st Mirror plane
2nd Mirror plane

Hamilton Turn
$N_1 \rightarrow N_2$
($\Theta/2$ Arc)

(Fig. 10.A.7 QTCA)

(From Lect. 8 p. 63-78)

(Fig. 3.1.6 PSDS)

(Rotation $R[\Theta'] R[\omega] = R[\omega'']$

(Fig. 10.A.8 QTCA)

(Product $R[\Theta''] = R[\Theta'] R[\Theta]

(From Lect. 8 p. 63-78)
Deriving $D_3 \sim C_{3v}$ products by nomograms based on $U(2)$ Hamilton-turns

**Figure 3.1.7** Geometrical definition of symmetry group $D_3$. (a) Hamilton arc vectors are drawn for rotations $r$, $i_1$, and $i_3$. (b) Group nomogram is obtained by projecting (a) onto the $xy$ plane.

*Note* $h^2 = r^1$ and $h^4 = r^2$ for $D_6$ notation

<table>
<thead>
<tr>
<th></th>
<th>$h^2$</th>
<th>$h^4$</th>
<th>$\rho_1$</th>
<th>$\rho_2$</th>
<th>$\rho_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$h^4$</td>
<td>$-1$</td>
<td>$-\rho_2$</td>
<td>$-\rho_3$</td>
<td>$\rho_1$</td>
</tr>
<tr>
<td></td>
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<td>$h^4$</td>
<td>$-\rho_3$</td>
<td>$\rho_1$</td>
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<td>$h^4$</td>
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<tr>
<td>$\rho_3$</td>
<td>$-\rho_1$</td>
<td>$-\rho_2$</td>
<td>$h^2$</td>
<td>$h^4$</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

**$U(2)$ result:**

$-\rho_1 r^1 = \rho_2$

**$R(3)$ result:**

$\rho_1 r^1 = \rho_2$
Deriving $D_3 \sim C_{3v}$ products by nomograms based on $U(2)$ Hamilton-turns

**Figure 5.7.4** Hamilton arcs and vector nomogram for $D_2$

\[
\begin{array}{cccc}
1 & R_x & R_y & R_z \\
R_x & -1 & R_z & -R_y \\
R_y & -R_z & -1 & R_x \\
R_z & R_y & -R_x & -1 \\
\end{array}
\]

\[
-i\sigma_B
\]

\[
-i\sigma_C
\]

\[
-i\sigma_A
\]

\[
\mathcal{H}^E(R_x) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \mathcal{H}^E(R_y) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{H}^E(R_z) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.
\]
Note $h^2 = r^1$ and $h^4 = r^2$ for $D_6 \supset D_3$ notation

Later we show: $D_6 = D_3 \times C_2$
Later we show:

\[ D_6 \supset D_3 \]

Note \( h_2 = r_1 \) and \( h_4 = r_2 \)

for \( D_6 \supset D_3 \) notation

Later we show:

\[ D_6 = D_3 \times C_2 \]
3-Dihedral-axes group $D_3$ vs. 3-Vertical-mirror-plane group $C_{3v}$

$D_3$ and $C_{3v}$ are isomorphic ($D_3 \sim C_{3v}$ share product table)

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Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes

Figure 3.2.1 Showing class equivalence using Hamilton’s vectors. Operation $R$ is equivalent to $R' = TRT^{-1}$.

Product $R[\Theta''] = R[\Theta'] \cdot R[\Theta]$

Product $R[\Theta'''] = R[\Theta] \cdot R[\Theta']$

Product $R[\Theta] \cdot R^{-1}[\Theta]$

(Product $R[\Theta'] \cdot R^{-1}[\Theta]$)
Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes

Transforming $D_3$ operators using $D_3$ operators

Example 1: Rotating $\rho_3$ axis crank using $r^1$ puts it down onto $\rho_1$

<table>
<thead>
<tr>
<th>$D_3 \gg_f$†</th>
<th>1</th>
<th>$r^2$</th>
<th>$r^1$</th>
<th>$\rho_1$</th>
<th>$\rho_2$</th>
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Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes

Transforming $D_3$ operators using $D_3$ operators

Example 1: Rotating $\rho_3$ axis crank using $r^1$ puts it down onto $\rho_1$

Seems to imply: $r^1 \rho_3 (r^1)^{-1} = r^1 \rho_3 r^2 = \rho_1$

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Need to check that with table:

$$r^1 \rho_3 r^2 = \rho_2 r^2$$
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Need to check that with table:

$r^1 \rho_3 r^2 = \rho_2 r^2 = \rho_1$

Checks out!
**Deriving \( D_3 \sim C_{3v} \) equivalence transformations and classes**

**Transforming \( D_3 \) operators using \( D_3 \) operators**

**Example 2:** Rotating \( \rho_3 \) axis crank using \( \rho_1 \) puts it down onto \( \rho_2 \)

Seems to imply:

\[
\rho_1 \rho_3 (\rho_1)^{-1} = \rho_1 \rho_3 \rho_1 = \rho_2
\]

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Need to check that with table:

\[
\rho_1 \rho_3 \rho_1 = r^2 \rho_1 = \rho_2
\]

Checks out!
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All-commuting projectors and $D_3$-invariant characters
Group invariant numbers: Centrum, Rank, and Order
What has been done so far:

**Abelian (Commutative) $C_2$, $C_3$, ..., $C_6$ ...**

$H$ diagonalized by $r^p$ symmetry operators that **COMMUTE** with $H$ \( (r^p H = H r^p) \),
*and* with each other \( (r^p r^q = r^{p+q} = r^q r^p) \).
What has been done so far:

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and with each other ($r^p r^q = r^{p+q} = r^q r^p$).

What we need to learn now:

**Non-Abelian** (do not commute) $D_3, O_h, \ldots$

While all $H$ symmetry operations **COMMUTE** with $H$ \quad ($U H = H U$)

most do **not** with each other ($U V \neq V U$).
What has been done so far:

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While all $H$ symmetry operations **COMMUTE** with $H$ \( (U H = H U) \)

most do **not** with each other \( (U V \neq V U) \).

**Q:** So how do we write $H$ in terms of non-commutative $U$?
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“Give me a place to stand...
and I will move the Earth”
Archimedes 287-212 B.C.E

Ideas of duality/relativity go way back (...VanVleck, Casimir..., Mach, Newton, Archimedes...)

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$\mathbf{R}$ commutes with all $\overline{\mathbf{R}}$
(because they’re independent or “unentangled”)

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Mock-Mach relativity principle
\[ R|1\rangle = \bar{R}^{-1}|1\rangle \]
...for one state \( |1\rangle \) only!
**Global vs Local symmetry and Mock-Mach principle**

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**Mock-Mach relativity principle**
\[ R |1\rangle = \bar{R}^{-1} |1\rangle \]
...for one state \( |1\rangle \) only!

...But *how* do you actually *make* the \( R \) and \( \bar{R} \) operations?
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Example of GLOBAL vs LOCAL symmetry algebra for $D_3 \sim C_{3v}$
Example of GLOBAL vs LOCAL symmetry algebra for $D_3 \sim C_{3v}$

$D_3$-defined local-wave bases

$D_3$ Group “slide-rule”

Lab-fixed (Extrinsic-Global) operations and rotation axes
Example of RELATIVITY-DUALITY for $D_3 \sim C_{3v}$

To represent external \{..T,U,V,...\} switch $g \leftrightarrow g^\dagger$ on top of group table

$$
R^G(1) = \begin{pmatrix}
1 & \cdots & \cdots & \cdots & 1 \\
. & 1 & \cdots & \cdots & . \\
. & . & 1 & \cdots & . \\
. & . & . & 1 & . \\
. & . & . & . & 1 \\
\end{pmatrix},
R^G(r) = \begin{pmatrix}
1 & \cdots & \cdots & \cdots & 1 \\
1 & 1 & \cdots & \cdots & . \\
. & 1 & \cdots & \cdots & . \\
. & . & 1 & \cdots & . \\
. & . & . & 1 & . \\
\end{pmatrix},
R^G(r^2) = \begin{pmatrix}
1 & \cdots & \cdots & \cdots & 1 \\
. & 1 & \cdots & \cdots & . \\
. & . & 1 & \cdots & . \\
. & . & . & 1 & . \\
. & . & . & . & 1 \\
\end{pmatrix},
R^G(i_1) = \begin{pmatrix}
1 & \cdots & \cdots & \cdots & 1 \\
. & 1 & \cdots & \cdots & . \\
. & . & 1 & \cdots & . \\
. & . & . & 1 & . \\
. & . & . & . & 1 \\
\end{pmatrix},
R^G(i_2) = \begin{pmatrix}
1 & \cdots & \cdots & \cdots & 1 \\
. & 1 & \cdots & \cdots & . \\
. & . & 1 & \cdots & . \\
. & . & . & 1 & . \\
. & . & . & . & 1 \\
\end{pmatrix},
R^G(i_3) = \begin{pmatrix}
1 & \cdots & \cdots & \cdots & 1 \\
. & 1 & \cdots & \cdots & . \\
. & . & 1 & \cdots & . \\
. & . & . & 1 & . \\
. & . & . & . & 1 \\
\end{pmatrix},
$$
Example of RELATIVITY-DUALITY for $D_3 \sim C_{3v}$

To represent external $\{..T,U,V,..\}$ switch $g \leftrightarrow g^\dagger$ on top of group table

$$
R^g(1) = R^g(r) = R^g(r^2) = R^g(i_1) = R^g(i_2) = R^g(i_3) =

\begin{array}{ccccccc}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array}
$$

$D_3$ global
$gg^\dagger$-table

To represent internal $\{..\bar{T},\bar{U},\bar{V},..\}$ switch $g \leftrightarrow g^\dagger$ on side of group table

$$
R^g(\bar{1}) = R^g(\bar{r}) = R^g(\bar{r}^2) = R^g(\bar{i}_1) = R^g(\bar{i}_2) = R^g(\bar{i}_3) =

\begin{array}{ccccccc}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array}
$$

$D_3$ local
$g^\dagger g$-table
Example of RELATIVITY-DUALITY for $D_3\sim C_{3v}$

To represent *external* \{..$T, U, V, ...$\} switch $g \leftrightarrow g^\dagger$ on **top** of group table

\[
R^G(1) = \begin{pmatrix} 1 & \vdots & \vdots & \vdots \\ \vdots & 1 & \vdots & \vdots \\ \vdots & \vdots & 1 & \vdots \\ \vdots & \vdots & \vdots & 1 \end{pmatrix}, \quad R^G(r) = \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{pmatrix}, \quad R^G(r^2) = \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{pmatrix}, \quad R^G(i) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad R^G(i_1^2) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad R^G(i_2^2) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad R^G(i_3^2) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}
\]

**RESULT:**
Any $R(T)$ **commute** (Even if $T$ and $U$ do not...) with any $R(U)$...

...and $T \cdot U = V$ if & only if $\overline{T} \cdot \overline{U} = \overline{V}$.

To represent *internal* \{..$\overline{T}, \overline{U}, \overline{V}, ...$\} switch $g \leftrightarrow g^\dagger$ on **side** of group table
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Group invariant numbers: Centrum, Rank, and Order
**Example of RELATIVITY-DUALITY for \(D_5-C_{3v}\)**

To represent *external* \(\{..T,U,V,...\}\) switch \(g \leftrightarrow g^\dagger\) on *top* of group table

\[
\begin{align*}
R^G(\mathbf{1}) &= R^G(\mathbf{r}) = R^G(\mathbf{r}^2) = R^G(\mathbf{i}_1) = R^G(\mathbf{i}_2) = R^G(\mathbf{i}_3) = \\
\begin{pmatrix}
1 & . & . & . & . & . \\
. & 1 & . & . & . & . \\
. & . & 1 & . & . & . \\
. & . & . & 1 & . & . \\
. & . & . & . & 1 & . \\
. & . & . & . & . & 1
\end{pmatrix}
\end{align*}
\]

**RESULT:**

Any \(R(T)\) commute (Even if \(T\) and \(U\) do not...)

with any \(R(U)\)... 

...and \(T \cdot U = V\) if & only if \(\overline{T} \cdot \overline{U} = \overline{V}\).

So an \(H\)-matrix having *Global* symmetry \(D_5\)

\[
H = H_1^0 + r_1 \mathbf{r}_1 + r_2 \mathbf{r}_2^2 + i_1 \mathbf{i}_1 + i_2 \mathbf{i}_2 + i_3 \mathbf{i}_3
\]

is made from *Local* symmetry matrices

To represent *internal* \(\{..\overline{T},\overline{U},\overline{V},...\}\) switch \(g \leftrightarrow g^\dagger\) on *side* of group table

\[
\begin{align*}
R^G(\overline{\mathbf{1}}) &= R^G(\overline{\mathbf{r}}) = R^G(\overline{\mathbf{r}^2}) = R^G(\overline{\mathbf{i}_1}) = R^G(\overline{\mathbf{i}_2}) = R^G(\overline{\mathbf{i}_3}) = \\
\begin{pmatrix}
1 & . & . & . & . & . \\
. & 1 & . & . & . & . \\
. & . & 1 & . & . & . \\
. & . & . & 1 & . & . \\
. & . & . & . & 1 & . \\
. & . & . & . & . & 1
\end{pmatrix}
\end{align*}
\]
Example of RELATIVITY-DUALITY for $D_3 \sim C_{3\nu}$

To represent external $\{..\text{T}, \text{U}, \text{V},...\}$ switch $g \leftrightarrow g^\dagger$ on top of group table

$$\begin{align*}
R^G(\text{1}) &= R^G(\text{r}) = R^G(\text{r}^2) = R^G(\text{i}_1) = R^G(\text{i}_2) = R^G(\text{i}_3) \\
\begin{pmatrix}
1 & \ldots & 1 & \ldots & 1 \\
1 & \ldots & 1 & \ldots & 1 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
1 & \ldots & 1 & \ldots & 1 \\
\end{pmatrix}
\end{align*}$$

RESULT: Any $R(\text{T})$ commute (Even if $\text{T}$ and $\text{U}$ do not) with any $R(\bar{\text{U}})$...

...and $\text{T} \cdot \text{U} = \text{V}$ if & only if $\bar{\text{T}} \cdot \bar{\text{U}} = \bar{\text{V}}$.

So an $\mathbb{H}$-matrix having Global symmetry $D_3$

$$\mathbb{H} = H I^0 r_1 i_1 r_1 + r_2 i_2 r_2 + i_1 i_1 i_1 + i_2 i_2 i_2 + i_3 i_3 i_3$$

is made from Local symmetry matrices

To represent internal $\{..\bar{\text{T}}, \bar{\text{U}}, \bar{\text{V}},...\}$ switch $g \leftrightarrow g^\dagger$ on side of group table

$$\begin{align*}
R^G(\bar{\text{1}}) &= R^G(\bar{\text{r}}) = R^G(\bar{\text{r}}^2) = R^G(\bar{\text{i}}_1) = R^G(\bar{\text{i}}_2) = R^G(\bar{\text{i}}_3) \\
\begin{pmatrix}
1 & \ldots & 1 & \ldots & 1 \\
1 & \ldots & 1 & \ldots & 1 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
1 & \ldots & 1 & \ldots & 1 \\
\end{pmatrix}
\end{align*}$$

Local $\mathbb{H}$ matrix parametrized by $g$'s
Example of RELATIVITY-DUALITY for $D_3\sim C_{3v}$

To represent external $\{..T,U,V,...\}$ switch $g \leftrightarrow g^\dagger$ on top of group table

$$R^G(\mathbf{1}) = R^G(r) = R^G(r^2) = R^G(i_1) = R^G(i_2) = R^G(i_3) =$$

$$\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{bmatrix}$$

RESULT: Any $R(T)$ commute (Even if $T$ and $U$ do not) with any $R(U)$...

...and $T\cdot U = V$ if & only if $\bar{T}\cdot \bar{U} = \bar{V}$.

To represent internal $\{..\bar{T},\bar{U},\bar{V},...\}$ switch $g \leftrightarrow g^\dagger$ on side of group table

$$R^G(\bar{\mathbf{1}}) = R^G(\bar{r}) = R^G(\bar{r}^2) = R^G(\bar{i}_1) = R^G(\bar{i}_2) = R^G(\bar{i}_3) =$$

$$\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{bmatrix}$$

So an $H$-matrix having Global symmetry $D_3$

$$H = H\mathbf{1} + r_1 \bar{r}_1 + r_2 \bar{r}_2^* + i_1 \bar{i}_1 + i_2 \bar{i}_2 + i_3 \bar{i}_3$$

is made from Local symmetry matrices

All the global $g$ commute with general local $H$ matrix.
**Example of RELATIVITY-DUALITY for D**

To represent *external* \{..T,U,V,...\}...

\[ R^G(T) = R^G(U) = R^G(V) = R^G(i) = \]

\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{pmatrix}
\]

**RESULT:**

Any \( R(T) \) commute (Even if \( T \) and \( U \) do not) with any \( R(U) \)...

...and \( T \cdot U = V \) if & only if \( T \cdot \bar{U} = \bar{V} \).

To represent *internal* \{..\bar{T},\bar{U},\bar{V},...\}....

\[ R^G(\bar{T}) = R^G(\bar{U}) = R^G(\bar{V}) = R^G(\bar{i}) = \]

\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{pmatrix}
\]

\[
H = \langle 1 | H | 1 \rangle = H^* \\
r_1 = \langle r | H | 1 \rangle = r_2^* \\
r_2 = \langle r^2 | H | 1 \rangle = r_1^* \\
i_1 = \langle i_1 | H | 1 \rangle = i_1^* \\
i_2 = \langle i_2 | H | 1 \rangle = i_2^* \\
i_3 = \langle i_3 | H | 1 \rangle = i_3^* \\
\]

So an \( H \)-matrix having *Global* symmetry\( D_3 \)

\[
H = H1 + r_1 \bar{r}_1 + r_2 \bar{r}_2 + i_1 \bar{i}_1 + i_2 \bar{i}_2 + i_3 \bar{i}_3
\]

is made from *Local* symmetry matrices

\[
\begin{pmatrix}
H & r_1 & r_2 & i_1 & i_2 & i_3 \\
r_1 & H & r_1 & i_2 & i_3 & i_1 \\
r_2 & r_1 & H & i_3 & i_1 & i_2 \\
i_1 & i_2 & i_3 & H & r_1 & r_2 \\
i_2 & i_3 & i_1 & r_2 & H & r_1 \\
i_3 & i_1 & i_2 & r_1 & r_2 & H \\
\end{pmatrix}
\]
3-Dihedral-axes group $D_3$ vs. 3-Vertical-mirror-plane group $C_{3v}$

$D_3$ and $C_{3v}$ are isomorphic ($D_3 \sim C_{3v}$ share product table)

Deriving $D_3 \sim C_{3v}$ products:
- By group definition $|g\rangle = g|1\rangle$ of position ket $|g\rangle$
- By nomograms based on $U(2)$ Hamilton-turns

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- Global vs Local symmetry and Mock-Mach principle
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- Lagrange theorems
- All-commuting operators and $D_3$-invariant class algebra
- All-commuting projectors and $D_3$-invariant characters
- Group invariant numbers: Centrum, Rank, and Order
Review: Spectral resolution of $D_3$ Center (Class algebra)

Class-sum $\kappa_k$ commutes with all $g_i$

Class-sum $\kappa_k$ invariance: $g_i \kappa_k = \kappa_k g_i$
Review: Spectral resolution of $D_3$ Center (Class algebra)

Class-sum $\kappa_k$ commutes with all $g_i$

Class-sum $\kappa_k$ invariance: $g_i \kappa_k = \kappa_k g_i$

° $G = \text{order of group}$: $(°D_3 = 6)$

° $\kappa_k = \text{order of class } \kappa_k$: $(°\kappa_1 = 1, °\kappa_r = 2, °\kappa_i = 3)$
Review: Spectral resolution of $D_3$ Center (Class algebra)

**Class-sum $\kappa_k$ commutes with all $g_i$**

Class-sum $\kappa_k$ invariance:

$$g_i \kappa_k = \kappa_k g_i$$

$^\circ G = $ order of group:

$$^\circ D_3 = 6$$

$^\circ \kappa_k = $ order of class $\kappa_k$:

$$^\circ \kappa_1 = 1, \ ^\circ \kappa_r = 2, \ ^\circ \kappa_i = 3$$

$$g_i \kappa_k g_i^{-1} = \kappa_k$$

where:

$$\kappa_k = \sum_{j=1}^{\kappa_j} g_j$$

**D₃ Algebra**

$D_3$ class algebra

$D_3$ Center (All-commuting operators)

$$\kappa_i = i_1 + i_2 + i_3$$

$$\kappa_r = r^2 + r$$

A Maximal Set of Commuting Operators

$P_{A1}$

$P_{A2}$

$P_{E1}$

$r^2$

$r$

$P_{E12}$

$P_{E_{21}}$

$P_{E_{11}}$

$P_{E_{22}}$

$i_1$

$i_2$

$i_3$
Review: Spectral resolution of $D_3$ Center (Class algebra)

Class-sum $\kappa_k$ commutes with all $g_i$

Class-sum $\kappa_k$ invariance: $g_i \kappa_k = \kappa_k g_i$

- $G$ = order of group: $(^G D_3 = 6)$
- $\kappa_k$ = order of class $\kappa_k$: $(^{\kappa_1 = 1, \kappa_r = 2, \kappa_i = 3})$

$g_i \kappa_k g_i^{-1} = \kappa_k$ where: $\kappa_k = \sum_{j=1}^{^{G \kappa_k}} g_j = \frac{1}{^{S_k}} \sum_{l=1}^{^G} g_l g_k g_l^{-1}$

- $s_k$ = order of $g_k$-self-symmetry: $(^{s_1 = 6, s_r = 3, s_i = 2})$

$\kappa_i = i_1 + i_2 + i_3$

$D_3$ Algebra

$\kappa_j = r^2 + r$

$D_3$ class algebra

$P_{A_1}$

$P_{A_2}$

$P_{E_1}$

$P_{E_2}$

$P_{E_3}$

$D_3$ Center

(All-commuting operators)

A Maximal Set of Commuting Operators

$r$

$r^2$

$P E_{11}$

$P E_{22}$

$P E_{12}$

$P E_{21}$

$P E_{xx}$

$P E_{yy}$

$P E_{xy}$

$P E_{yx}$
Review: Spectral resolution of $D_3$ Center (Class algebra)

**$D_3$ Algebra**

<table>
<thead>
<tr>
<th>1</th>
<th>$r^2$</th>
<th>$r$</th>
<th>$i_1$</th>
<th>$i_2$</th>
<th>$i_3$</th>
</tr>
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<tbody>
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<td>$i_2$</td>
<td>$i_3$</td>
<td>$i_1$</td>
</tr>
<tr>
<td>$i_1$</td>
<td>$i_3$</td>
<td>$i_2$</td>
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<td>$r$</td>
<td>$r^2$</td>
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<tr>
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<td>$i_1$</td>
<td>$i_3$</td>
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<td>$i_1$</td>
<td>$r$</td>
<td>$r^2$</td>
<td>1</td>
</tr>
</tbody>
</table>

Class-sum $\kappa_k$ commutes with all $g_i$

Class-sum $\kappa_k$ invariance:

$g_i \kappa_k = \kappa_k g_i$

$\circ G = \text{order of group: } (\circ D_3 = 6)$

$\circ \kappa_k = \text{order of class } \kappa_k : (\circ \kappa_1 = 1, \circ \kappa_r = 2, \circ \kappa_i = 3)$

$g_r \kappa_k g_r^{-1} = \kappa_k \quad \text{where: } \kappa_k = \sum_{j=1}^{\circ \kappa_k} g_j = \frac{1}{\circ s_k} \sum_{t=1}^{\circ G} g_t \kappa_k g_t^{-1}$

$\circ s_k = \text{order of } g_k$-self-symmetry: ($\circ s_1 = 6, \circ s_r = 3, \circ s_i = 2$)

$\circ s_k = \frac{\circ G}{\circ \kappa_k} \quad \circ s_k$ is an integer count of $D_3$ operators $g_s$ that commute with $g_k$. 

$D_3$ class algebra

$\kappa_i = i_1 + i_2 + i_3$

$\kappa_r = r^2 + r$

A Maximal Set of Commuting Operators

$P_{A_1}$

$P_{A_2}$

$P_{E_1}$

$P_{E_{11}}$

$P_{E_{21}}$

$P_{E_{22}}$
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All-commuting projectors and $D_3$-invariant characters
Group invariant numbers: Center, Rank, and Order
**Review:** Spectral resolution of \( D_3 \) Center (Class algebra)

<table>
<thead>
<tr>
<th>( i )</th>
<th>( \text{r}^2 )</th>
<th>( \text{r} )</th>
<th>( i_1 )</th>
<th>( i_2 )</th>
<th>( i_3 )</th>
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<tbody>
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<td>( \text{r}^2 )</td>
<td>( i_3 )</td>
<td>( i_1 )</td>
<td>( i_2 )</td>
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</tr>
<tr>
<td>( \text{r}^2 )</td>
<td>( \text{r} )</td>
<td>( i_2 )</td>
<td>( i_3 )</td>
<td>( i_1 )</td>
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<td>( \text{r}^2 )</td>
<td></td>
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<tr>
<td>( i_2 )</td>
<td>( i_1 )</td>
<td>( i_3 )</td>
<td>( \text{r}^2 )</td>
<td>( \text{r} )</td>
<td></td>
</tr>
<tr>
<td>( i_3 )</td>
<td>( i_2 )</td>
<td>( i_1 )</td>
<td>( \text{r} )</td>
<td>( \text{r}^2 )</td>
<td></td>
</tr>
</tbody>
</table>

\[
\kappa_j = 1, \quad \kappa_r = \text{r} + \text{r}^2, \quad \kappa_i = i_1 + i_2 + i_3
\]

[D3 Algebra Diagram]

Class-sum \( \kappa_k \) commutes with all \( g_i \)

Class-sum \( \kappa_k \) invariance:

\[
g_i \kappa_k = \kappa_k g_i
\]

\( ^{\circ}G \) = order of group:

\( ^{\circ}D_3 = 6 \)

\( ^{\circ}\kappa_k \) = order of class \( \kappa_k \):

\( ^{\circ} \kappa_1 = 1, \quad ^{\circ} \kappa_r = 2, \quad ^{\circ} \kappa_i = 3 \)

\[
g_i \kappa_k g_i^{-1} = \kappa_k \quad \text{where:} \quad \kappa_k = \sum_{j=1}^{j=\kappa_k} g_j = \frac{1}{s_k} \sum_{i=1}^{i=\kappa_i} g_i g_k g_i^{-1}
\]

\( s_k \) = order of \( g_k \)-self-symmetry:

\( ^{\circ}s_1 = 6, \quad ^{\circ}s_r = 3, \quad ^{\circ}s_i = 2 \)

\( ^{\circ}s_k = ^{\circ}G / ^{\circ}\kappa_k \quad ^{\circ}s_k \text{ is an integer count of } D_3 \text{ operators } g_s \text{ that commute with } g_k.\)
Review: Spectral resolution of $D_3$ Center (Class algebra)

$D_3$ Algebra

Class-sum $\kappa_k$ commutes with all $g_i$

Class-sum $\kappa_k$ invariance: $g_i \kappa_k = \kappa_k g_i$

$G$ = order of group: ($G = 6$)

$\kappa_k$ = order of class $\kappa_k$: ($\kappa_1 = 1$, $\kappa_r = 2$, $\kappa_i = 3$)

$g_i \kappa_k g_i^{-1} = \kappa_k$ where: $\kappa_k = \sum_{j=1}^{\kappa_k} g_j = \frac{1}{n} \sum_{t=1}^{G} g_t \kappa_k g_t^{-1}$

$\kappa_k$ = order of $g_k$-self-symmetry: ($\kappa_1 = 6$, $\kappa_r = 3$, $\kappa_i = 2$)

$\kappa_k = G / \kappa_k$ $\kappa_k$ is an integer count of $D_3$ operators $g_s$ that commute with $g_k$.

These operators $g_s$ form the $g_k$-self-symmetry group $s_k$. Each $g_s$ transforms $g_k$ into itself: $g_s g_k g_s^{-1} = g_k$
**Review:** Spectral resolution of $D_3$ Center (Class algebra)

### $D_3$ Algebra

#### Class-sum $\kappa_k$ commutes with all $g_i$

- **Class-sum $\kappa_k$ invariance:** $g_i \kappa_k = \kappa_k g_i$
- **$G$ = order of group:** $|D_3| = 6$
- **$\kappa_k$ = order of class $\kappa_k$:** $|\kappa_1| = 1$, $|\kappa_r| = 2$, $|\kappa_i| = 3$

$$g_i \kappa_k g_i^{-1} = \kappa_k$$

where: $\kappa_k = \sum_{j=1}^{g_i} g_j = \frac{1}{|s_k|} \sum_{t=1}^{G} g_i g_k g_i^{-1}$

- **$s_k$ = order of $g_k$-self-symmetry:** $|s_1| = 6$, $|s_r| = 3$, $|s_i| = 2$
- **$s_k = |G|/|\kappa_k|$. $s_k$ is an integer count of $D_3$ operators $g_s$ that commute with $g_k$.**

These operators $g_s$ form the $g_k$-self-symmetry group $s_k$. Each $g_s$ transforms $g_k$ into itself: $g_s g_k g_s^{-1} = g_k$

If an operator $g_t$ transforms $g_k$ into a different element $g'_k$ of its class: $g_t g_k g_t^{-1} = g'_k$, then so does $g_s g_t$. That is: $g_s g_t (g_s g_k g_s^{-1}) = g_s g_t g_s^{-1} g_t^{-1} = g_s g_s^{-1} g_t^{-1} = g'_k$.

<table>
<thead>
<tr>
<th>$D_3$ class algebra</th>
<th>$D_3$ Center</th>
</tr>
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<tbody>
<tr>
<td>$\kappa_1$</td>
<td>$\kappa_r = \kappa_1$</td>
</tr>
<tr>
<td>$\kappa_r$</td>
<td>$\kappa_i = \kappa_1 + \kappa_2 + \kappa_3$</td>
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</table>

<table>
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<th>$\kappa_1$</th>
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<th>$\kappa_i$</th>
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<table>
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</tr>
<tr>
<td>$s_3$</td>
<td>$x_3$</td>
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<td>$z_3$</td>
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Another Maximal Set of Commuting Operators

$D_3$ Center:

- $PA_1$
- $PA_2$
- $PE_1$
- $PE_2$
- $PE_3$

A Maximal Set of Commuting Operators:

- $PE_{11} = r$
- $PE_{22} = r^2$
- $PE_{12} = r^3$
- $PE_{21} = r^4$
- $PE_{11} = r^5$

<table>
<thead>
<tr>
<th>$PE_{11}$</th>
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**Review:** Spectral resolution of **$D_3$ Center** (Class algebra)

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<td>$i_1$</td>
<td>$r^2$</td>
<td>$r$</td>
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</tbody>
</table>

Class-sum $\kappa_k$ commutes with all $g_r$

Class-sum $\kappa_k$ invariance: 
\[ g_r \kappa_k = \kappa_k g_r \]

$G$ = order of group: 
\[ (D_3 = 6) \]

$\kappa_k$ = order of class $\kappa_k$: 
\[ (\kappa_1 = 1, \kappa_r = 2, \kappa_i = 3) \]

\[ g_r \kappa_k g_r^{-1} = \kappa_k \]  
where: 
\[ \kappa_k = \sum_{j=1}^{r_j} g_j = \sum_{l=1}^{r_i} g_l g_s g_i^{-1} \]

$s_k$ = order of $g_k$-self-symmetry: 
\[ (s_r = 6, s_r = 3, s_i = 2) \]

$s_k = \frac{G}{\kappa_k}$  
$s_k$ is an integer count of $D_3$ operators $g_s$ that commute with $g_k$.

These operators $g_s$ form the $g_k$-self-symmetry group $s_k$. Each $g_s$ transforms $g_k$ into itself: 
\[ g_s g_k g_s^{-1} = g_k \]

If an operator $g_l$ transforms $g_k$ into a different element $g'_k$ of its class: 
\[ g_l g_k g_l^{-1} = g'_k, \]  
then so does $g_l g_s$.  
that is:

**Subgroup** $s_k = \{ g_0=1, g_1=g_k, g_2, \ldots \}$ has $l=(\kappa_k-1)$ **Left Cosets** (one coset for each member of class $\kappa_k$).  

$g_1 s_k = g_l \{ g_0=1, g_1=g_k, g_2, \ldots \}$,
Review: \textbf{Spectral resolution of $D_3$ Center (Class algebra)}

\begin{center}
\begin{tabular}{|c|c|c|c|}
\hline
1 & $r^2$ & $r$ & \(i_1i_2i_3\) \\
\hline
$r$ & $1$ & $r^2$ & $i_1i_2i_3$ \\
\hline
$r^2$ & $1$ & $r$ & $i_1i_2i_3$ \\
\hline
\end{tabular}
\end{center}

Class-sum $\kappa_k$ commutes with all $g_i$

Class-sum $\kappa_k$ invariance:

\[ g_i \kappa_k = \kappa_k g_i \]

\begin{itemize}
  \item \( G \) = order of group: \( ( \kappa D_3 = 6 ) \)
  \item \( \kappa_k \) = order of class $\kappa_k$:
    \( ( \kappa_1 = 1, \kappa_r = 2, \kappa_i = 3 ) \)
\end{itemize}

\[ g_i \kappa_k g_j^{-1} = \kappa_k \] where: \( \kappa_k = \sum_{j=1}^{r} g_j = \frac{1}{\kappa_k} \sum_{j=1}^{G} g_i \kappa_k g_j^{-1} \)

\( \kappa_k \) = order of $g_i$ self-symmetry: \( ( \kappa_i = 6, \kappa_r = 3, \kappa_i = 2 ) \)

\( \kappa_k \) = \( \text{order of } \kappa_k \) = \( \text{order of } G / \kappa_k \)

\( \kappa_k \) is an integer count of $D_3$ operators $g_s$ that commute with $g_k$.

These operators $g_s$ form the $g_k$-self-symmetry group $s_k$. Each $g_s$ transforms $g_k$ into itself: $g_s g_k g_s^{-1} = g_k$

If an operator $g_t$ transforms $g_k$ into a different element $g'_k$ of its class: $g_t g_k g_t^{-1} = g'_k$, then so does $g_s g_t$.

Subgroup $s_k = \{ g_0 = 1, g_1 = g_k, g_2, \ldots \}$ has \( 1 = ( \kappa_k - 1 ) \) \textbf{Left Cosets} (one coset for each member of class $\kappa_k$).

\begin{align*}
  g_1 s_k &= g_1 \{ g_0 = 1, g_1 = g_k, g_2, \ldots \}, \\
  g_2 s_k &= g_2 \{ g_0 = 1, g_1 = g_k, g_2, \ldots \}, \\
  \vdots \\
  g_l s_k &= g_l \{ g_0 = 1, g_1 = g_k, g_2, \ldots \}
\end{align*}
Review: Spectral resolution of $D_3$ Center (Class algebra)

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>$r^2$</th>
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</tbody>
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Class-sum $\kappa_k$ commutes with all $g_i$

Class-sum $\kappa_k$ invariance: $g_i \kappa_k = \kappa_k g_i$

$G = \text{order of group}$: $(^G D_3 = 6)$

$\kappa_k = \text{order of class} \kappa_k$: $(^\kappa_1 = 1, ^\kappa_r = 2, ^\kappa_i = 3)$

$g_k \kappa_k g_i^{-1} = \kappa_k$ where: $\kappa_k = \sum_{j=1}^{j=\kappa_k} g_j = \frac{1}{\kappa_k} \sum_{i=1}^{i=\kappa_k} g_i g_k g_i^{-1}$

$\kappa_j = \text{order of } g_k$-self-symmetry: $(^\kappa_1 = 6, ^\kappa_r = 3, ^\kappa_i = 2)$

$\kappa_s = ^G / ^\kappa_k$ $\kappa_s$ is an integer count of $D_3$ operators $g_s$ that commute with $g_k$.

These operators $g_s$ form the $g_k$-self-symmetry group $s_k$. Each $g_s$ transforms $g_k$ into itself: $g_s g_k g_s^{-1} = g_k$

If an operator $g_t$ transforms $g_k$ into a different element $g'_k$ of its class: $g_t g_k g_t^{-1} = g'_k$, then so does $g_s g_t$.

Subgroup $s_k = \{g_0=1, \ g_1=g_k, \ g_2, \ldots\}$ has $l = (^{\kappa_k-1})$ Left Cosets (one coset for each member of class $\kappa_k$).

$g_1 s_k = g_1 \{g_0=1, \ g_1=g_k, \ g_2, \ldots\}$,

$g_2 s_k = g_2 \{g_0=1, \ g_1=g_k, \ g_2, \ldots\}$, ...

They will divide the group of order $^G D_3 = ^\kappa_k \cdot ^{\kappa_s}$ evenly into $^\kappa_k$ subsets each of order $^\kappa_s$. 
Review: Spectral resolution of **D₃ Center** (Class algebra)

**Class-sum** $\kappa_k$ commutes with all $g_i$

Class-sum $\kappa_k$ invariance: $g_i \kappa_k = \kappa_k g_i$

$^\circ G$ = order of group: $(^\circ D_3 = 6)$

$^\circ \kappa_k$ = order of class $\kappa_k$: $(^\circ \kappa_1 = 1, ^\circ \kappa_r = 2, ^\circ \kappa_i = 3)$

\[
g_i g_k g_l^{-1} = g_k \quad \text{where:} \quad \kappa_k = \sum_{j=1}^{^\circ \kappa_k} g_j = \frac{1}{^\circ \kappa_k} \sum_{i=1}^{^\circ G} g_i g_k g_l^{-1}
\]

$^\circ s_k$ = order of $g_k$-self-symmetry: $(^\circ s_1 = 6, ^\circ s_r = 3, ^\circ s_i = 2)$

$^\circ s_k = ^\circ G / ^\circ \kappa_k$

$^\circ s_k$ is an integer count of $D_3$ operators $g_s$ that commute with $g_k$.

These operators $g_s$ form the $g_k$-self-symmetry group $s_k$. Each $g_s$ transforms $g_k$ into itself: $g_s g_k g_l^{-1} = g_k$

If an operator $g_l$ transforms $g_k$ into a different element $g'_k$ of its class: $g_s g_l g_l^{-1} = g'_k$, then so does $g_l g_s$.

That is:

Subgroup $s_k = \{g_0=1, g_1=g_k, g_2,\ldots\}$ has $l=(^\circ \kappa_k-1)$ **Left Cosets** (one coset for each member of class $\kappa_k$).

$g_1 s_k = g_1 \{g_0=1, g_1=g_k, g_2,\ldots\}$,

$g_2 s_k = g_2 \{g_0=1, g_1=g_k, g_2,\ldots\}$,

These results are known as **Lagrange’s Coset Theorem(s)**

They will divide the group of order $^\circ D_3 = ^\circ \kappa_k \cdot ^\circ s_k$ evenly into $^\circ \kappa_k$ subsets each of order $^\circ s_k$. 

---

**D₃ Algebra**

### D₃ class algebra

- **All-commuting operators**
  - $\kappa_{i} = i_{1} + i_{2} + i_{3}$
  - $\kappa_{r} = r^{2} + r$

### A Maximal Set of Commuting Operators

- $P_{A_{1}}$
- $P_{A_{2}}$
- $P_{E_{1}}$

### Center

- $\kappa_{r} = r^{2} + r$
3-Dihedral-axes group $D_3$ vs. 3-Vertical-mirror-plane group $C_{3v}$

$D_3$ and $C_{3v}$ are isomorphic ($D_3 \sim C_{3v}$ share product table)

Deriving $D_3 \sim C_{3v}$ products:

By group definition $|g\rangle = g|I\rangle$ of position ket $|g\rangle$

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All-commuting operators and $D_3$-invariant class algebra

All-commuting projectors and $D_3$-invariant characters

Group invariant numbers: Centrum, Rank, and Order
Spectral analysis of non-commutative “Group-table Hamiltonian”

1st Step: Spectral resolution of $D_3$-Center (Class algebra of $D_3$)

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Each class-sum $\kappa_k$ commutes with all of $D_3$.

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$\kappa_g$'s are mutually commuting with respect to themselves and all-commuting with respect to the whole group.

$$r \ \kappa_i \ r^{-1} = i_2 + i_3 + i_1 = \kappa_i \quad \text{or:} \quad r \ \kappa_i = \kappa_i \ r$$

$$\sum_{h=1}^{G} h g h^{-1} = v_g \ \kappa_g \ , \quad \text{where:} \quad v_g = \frac{G}{\kappa_g} = \text{integer}$$

$^G \kappa_g$ is order of class $\kappa_g$ and must evenly divide group order $^G G$. 


Spectral analysis of non-commutative “Group-table Hamiltonian”

1st Step: Spectral resolution of $D_3$-Center (Class algebra of $D_3$)

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Note also:

$\kappa_2^2 - \kappa_2 - 2 \cdot 1 = 0$

$\kappa_3^2 = 3\cdot \kappa_2 + 3 \cdot 1$
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Class products give spectral polynomial and all-commuting projectors $P^{(\alpha)}$

\[
0 = \kappa_3^3 - 9\kappa_3 = (\kappa_3 - 3 \cdot 1)(\kappa_3 + 3 \cdot 1)(\kappa_3 - 0 \cdot 1) \\
\kappa_3 = 3 \cdot \kappa_2 + 3 \cdot 1
\]

Note also:

\[
\kappa_2^2 - \kappa_2 - 2 \cdot 1 = 0 \\
0 = (\kappa_2 - 2 \cdot 1)(\kappa_2 + 1)
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Class products give spectral polynomial and all-commuting projectors $P^{(\alpha)} = P^{A_1}$, $P^{A_2}$, and $P^E$

Note also:

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\[
\begin{array}{|c|c|c|}
\hline
\kappa_1 & \kappa_2 & \kappa_3 \\
\hline
1 & r^1 + r^2 & i_1 + i_2 + i_3 \\
\hline
\kappa_2 & 2\kappa_1 + \kappa_2 & 2\kappa_3 \\
\hline
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0 &= \kappa_3^3 - 9\kappa_3 = (\kappa_3 - 3 \cdot 1)(\kappa_3 + 3 \cdot 1)(\kappa_3 - 0 \cdot 1) \\
0 &= (\kappa_2 - 2 \cdot 1)(\kappa_2 + 1) \\
0 &= (\kappa_3 - 3 \cdot 1)P^{A_1} \\
\kappa_3 P^{A_1} &= +3 \cdot P^{A_1} \\
\end{align*}
\]

\[
P^{A_1} = \frac{(\kappa_3 + 3 \cdot 1)(\kappa_3 - 0 \cdot 1)}{(+3 + 3)(+3 - 0)}
\]
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<td>$i_3$</td>
<td>$i_1$</td>
<td>$i_2$</td>
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</tr>
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<td>$i_3$</td>
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Each class-sum $\kappa_k$ commutes with all of $D_3$.

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Class products give spectral polynomial and
all-commuting projectors $P^{(\alpha)} = P^{A_1}$, $P^{A_2}$, and $P^E$

Note also:

$\kappa_2 - \kappa_2 - 2\cdot 1 = 0$

$0 = \kappa_3^2 - 9\kappa_3 = (\kappa_3 - 3 \cdot 1)(\kappa_3 + 3 \cdot 1)(\kappa_3 - 0 \cdot 1)$

$0 = (\kappa_2 - 2 \cdot 1)(\kappa_2 + 1)$

$\kappa_3 P^{A_1} = +3 \cdot P^{A_1}$

$0 = (\kappa_3 + 3 \cdot 1)P^{A_2}$

$\kappa_3 P^{A_2} = -3 \cdot P^{A_2}$

$P^{A_1} = \frac{(\kappa_3 + 3 \cdot 1)(\kappa_3 - 0 \cdot 1)}{(+3 + 3)(+3 - 0)}$

$P^{A_2} = \frac{(\kappa_3 - 3 \cdot 1)(\kappa_3 - 0 \cdot 1)}{(-3 - 3)(-3 - 0)}$
Spectral analysis of non-commutative “Group-table Hamiltonian”

1st Step: Spectral resolution of $D_3$-Center (Class algebra of $D_3$)

<table>
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<tr>
<th>$1$</th>
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</tr>
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</table>

Note also:

$\kappa_2^2 - \kappa_2 - 2 \cdot 1 = 0 \quad \Rightarrow \quad \kappa_2 = r^1 + r^2$

$\kappa_3 = i_1 + i_2 + i_3$

Each class-sum $\kappa_k$ commutes with all of $D_3$.

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Class products give spectral polynomial and all-commuting projectors $P^{(\alpha)} = P^{A_1}$, $P^{A_2}$, and $P^E$

$0 = (\kappa_3 - 3 \cdot 1)P^{A_1}$

$\kappa_3 P^{A_1} = +3 \cdot P^{A_1}$

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$0 = (\kappa_3 - 0 \cdot 1)P^E$

$\kappa_3 P^E = +0 \cdot P^E$

$P^{A_1} = \frac{(\kappa_3 + 3 \cdot 1)(\kappa_3 - 0 \cdot 1)}{(+3 + 3)(+3 - 0)}$

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$P^E = \frac{(\kappa_3 - 3 \cdot 1)(\kappa_3 + 3 \cdot 1)}{(+0 - 3)(+0 + 3)}$
Spectral analysis of non-commutative “Group-table Hamiltonian”

1st Step: Spectral resolution of $D_3$-Center (Class algebra of $D_3$)

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Note also:

$\kappa^2 - \kappa - 2 \cdot 1 = 0 \quad 0 = \kappa^3 - 9 \kappa_3 = (\kappa_3 - 3 \cdot 1)(\kappa_3 + 3 \cdot 1)(\kappa_3 - 0 \cdot 1)$

Each class-sum $\kappa_k$ commutes with all of $D_3$.

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<td>$r^1 + r^2$</td>
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</tr>
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Class products give spectral polynomial and all-commuting projectors $P^{(\alpha)} = P^{A_1}$, $P^{A_2}$, and $P^E$

Class resolution into sum of eigenvalue $\cdot$ Projector

$\kappa_1 = 1 \cdot P^{A_1} + 1 \cdot P^{A_2} + 1 \cdot P^E$

$\kappa_r = 2 \cdot P^{A_1} + 2 \cdot P^{A_2} - 1 \cdot P^E$

$\kappa_i = 3 \cdot P^{A_1} - 3 \cdot P^{A_2} + 0 \cdot P^E$

Note also:

$\kappa^2 - \kappa - 2 \cdot 1 = 0 \quad 0 = (\kappa_2 - 2 \cdot 1)(\kappa_2 + 1)$
Spectral analysis of non-commutative “Group-table Hamiltonian”

1st Step: Spectral resolution of $D_3$-Center (Class algebra of $D_3$)

Each class-sum $\kappa_k$ commutes with all of $D_3$.

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$0 = (\kappa_3 - 3 \cdot 1)P^{A_1}$

$\kappa_3P^{A_1} = +3 \cdot P^{A_1}$

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$\kappa_3P^{A_2} = -3 \cdot P^{A_2}$

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$\kappa_r = 2 \cdot P^{A_1} + 2 \cdot P^{A_2} - 1 \cdot P^E \quad \kappa^2_r = \kappa_r + 2 \cdot 1 \Rightarrow (\kappa_r - 2 \cdot 1)(\kappa_r + 1) = 0$

$\kappa_i = 3 \cdot P^{A_1} - 3 \cdot P^{A_2} + 0 \cdot P^E$

Inverse resolution gives $D_3$ Character Table

$P^{A_1} = (\kappa_1 + \kappa_2 + \kappa_3)/6 = (1 + r + r^2 + i_1 + i_2 + i_3)/6$

$P^{A_2} = (\kappa_1 + \kappa_2 + \kappa_3)/6 = (1 + r + r^2 - i_1 - i_2 - i_3)/6$

$P^E = (2\kappa_1 - \kappa_2 + 0)/3 = (21 - r - r^2)/3$
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$\kappa_r = 2 \cdot P^{A_1} + 2 \cdot P^{A_2} - 1 \cdot P^E \quad \iff \quad \kappa_r^2 = \kappa_r + 2 \cdot 1 \Rightarrow (\kappa_r - 2 \cdot 1)(\kappa_r + 1) = 0$

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Spectral analysis of non-commutative “Group-table Hamiltonian”

1st Step: Spectral resolution of $D_3$-Center (Class algebra of $D_3$)

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Irreducible characters are traces $\chi^\alpha_r = Tr D^{(\alpha)}(r_k)$ of irreducible representations $D^{(\alpha)}(r_k)$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\chi_1^\alpha$</th>
<th>$\chi_2^\alpha$</th>
<th>$\chi_3^\alpha$</th>
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<tr>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$A_2$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$E$</td>
<td>2</td>
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</table>
3-Dihedral-axes group $D_3$ vs. 3-Vertical-mirror-plane group $C_{3v}$
$D_3$ and $C_{3v}$ are isomorphic ($D_3 \sim C_{3v}$ share product table)
Deriving $D_3 \sim C_{3v}$ products:
By group definition $|g\rangle = g|1\rangle$ of position ket $|g\rangle$
By nomograms based on $U(2)$ Hamilton-turns
Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes

Non-commutative symmetry expansion and Global-Local solution
Global vs Local symmetry and Mock-Mach principle
Global vs Local matrix duality for $D_3$
Global vs Local symmetry expansion of $D_3$ Hamiltonian

1st-Step in spectral analysis of $D_3$ “group-table” Hamiltonian: Algebra of $D_3$ Center(Classes)
All-commuting operators and $D_3$-invariant class algebra
All-commuting projectors and $D_3$-invariant characters
Group invariant numbers: Centrum, Rank, and Order
Important invariant numbers or “characters”

\[ \ell^\alpha = \text{Irreducible representation (irrep) dimension or level degeneracy} \]

For symmetry group or algebra \( G \)

**Centrum:** \( \kappa(G) = \sum_{\text{irrep}(\alpha)} (\ell^\alpha)^0 = \text{Number of classes, invariants, irrep types, all-commuting ops} \)

**Rank:** \( \rho(G) = \sum_{\text{irrep}(\alpha)} (\ell^\alpha)^1 = \text{Number of irrep idempotents } P_{n,n}^{(\alpha)}, \text{mutually-commuting ops} \)

**Order:** \( o(G) = \sum_{\text{irrep}(\alpha)} (\ell^\alpha)^2 = \text{Total number of irrep projectors } P_{m,n}^{(\alpha)} \text{ or symmetry ops} \)
Important invariant numbers or “characters”

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For symmetry group or algebra \( G \)

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**Order:** \( \circ(G) = \sum_{\text{irrep}(\alpha)} (\ell^\alpha)^2 \) = **Total** number of irrep projectors \( P_{m,n}^{(\alpha)} \) or symmetry ops

\[
\kappa(D_3) = (1)^0 + (1)^0 + (2)^0 = 3 \\
\rho(D_3) = (1)^1 + (1)^1 + (2)^1 = 4 \\
\circ(D_3) = (1)^2 + (1)^2 + (2)^2 = 6
\]