Group Theory in Quantum Mechanics

Lecture 14.5 *(3.07.17)*

C_N symmetry systems coupled, uncoupled, and re-coupled

(Quantum Theory for the Computer Age - Unit 3-5)
(Principles of Symmetry, Dynamics, and Spectroscopy - Sec. 1-12 of Ch. 2)

- *Breaking C_N cyclic coupling into linear chains*
  - *Review of 1D-Bohr-ring related to infinite square well*
  - *Breaking C_{2N+2} to approximate linear N-chain (Examples $C_2 \rightleftarrows C_6 \rightleftarrows C_{14}$)*
    - *Band-It simulation: Intro to scattering approach to quantum symmetry*
      - *How Band-It works: Match each Ψ and DΨ, Let L=0 at Right end*

- *Breaking C_{2N} cyclic coupling down to C_N symmetry*
  - *Acoustical modes vs. Optical modes*
  - *Intro to other examples of band theory*
  - *Avoided crossing view of band-gaps*

- *Finally! Symmetry groups that are not just C_N*
  - *The "4-Group(s)" D_2 and C_{2v}*
  - *Spectral decomposition of D_2*
    - *Some D_2 modes*
  - *Outer product properties and the Crystal-Point Symmetry Group Zoo*
  - *Polygonal geometry of $U(2) \supset C_N$ character spectral function.*
    - *Algebra*
      - *Geometry*

*The **CPT** subgroup of Lorentz Group*

$$\chi^j(2\pi/n) = \frac{\sin\left(\pi(2j+1)/n\right)}{\sin\left(\pi/n\right)}$$

Breaking C_N cyclic coupling into linear chains
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    Band-It simulation: Intro to scattering approach to quantum symmetry
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   Algebra
    Geometry

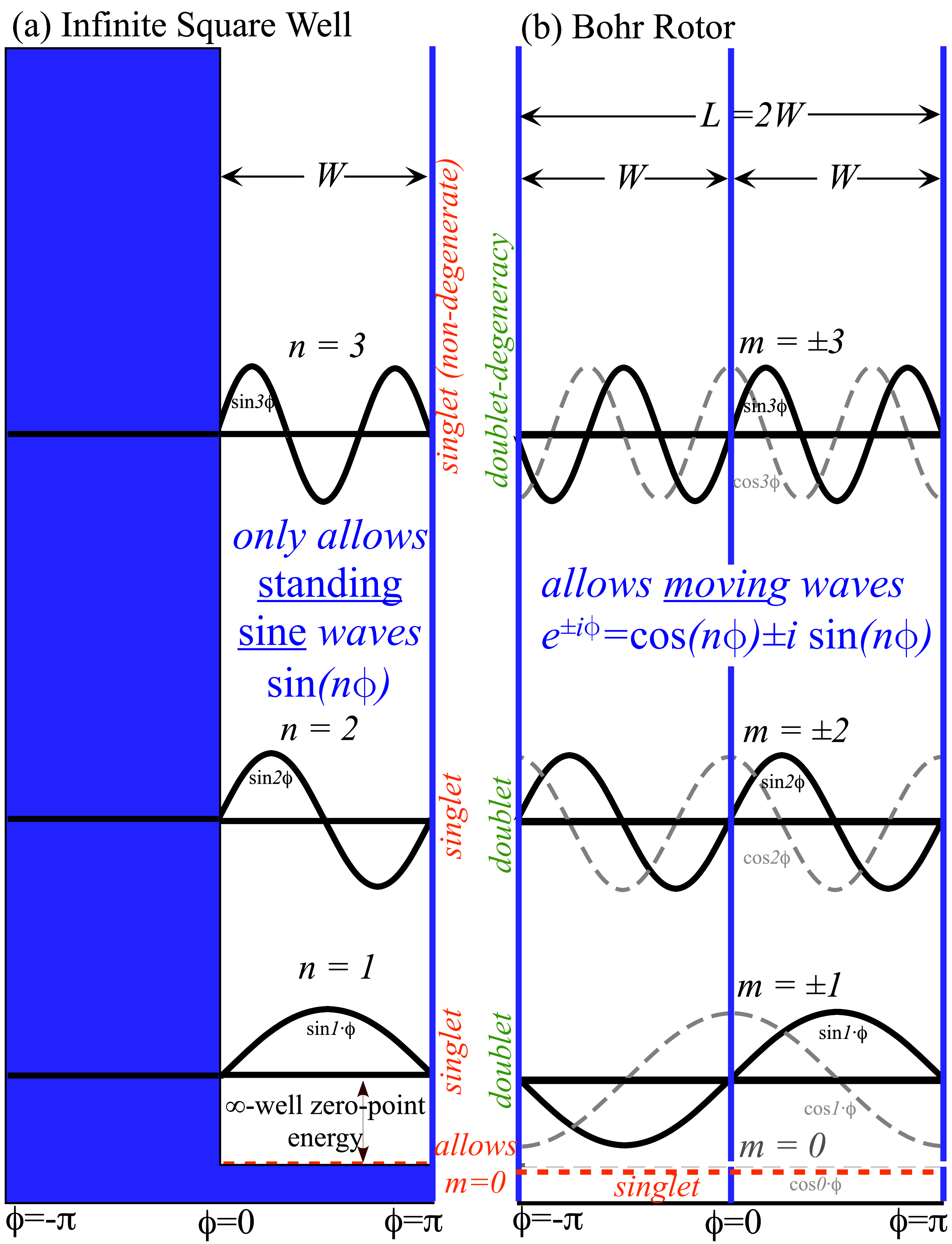
∞-Square well PE versus Bohr rotor

Fig. 12.2.6 Comparison of eigensolutions for (a) Infinite square well, and (b) Bohr rotor.

From QTCA Unit 5 Ch. 12

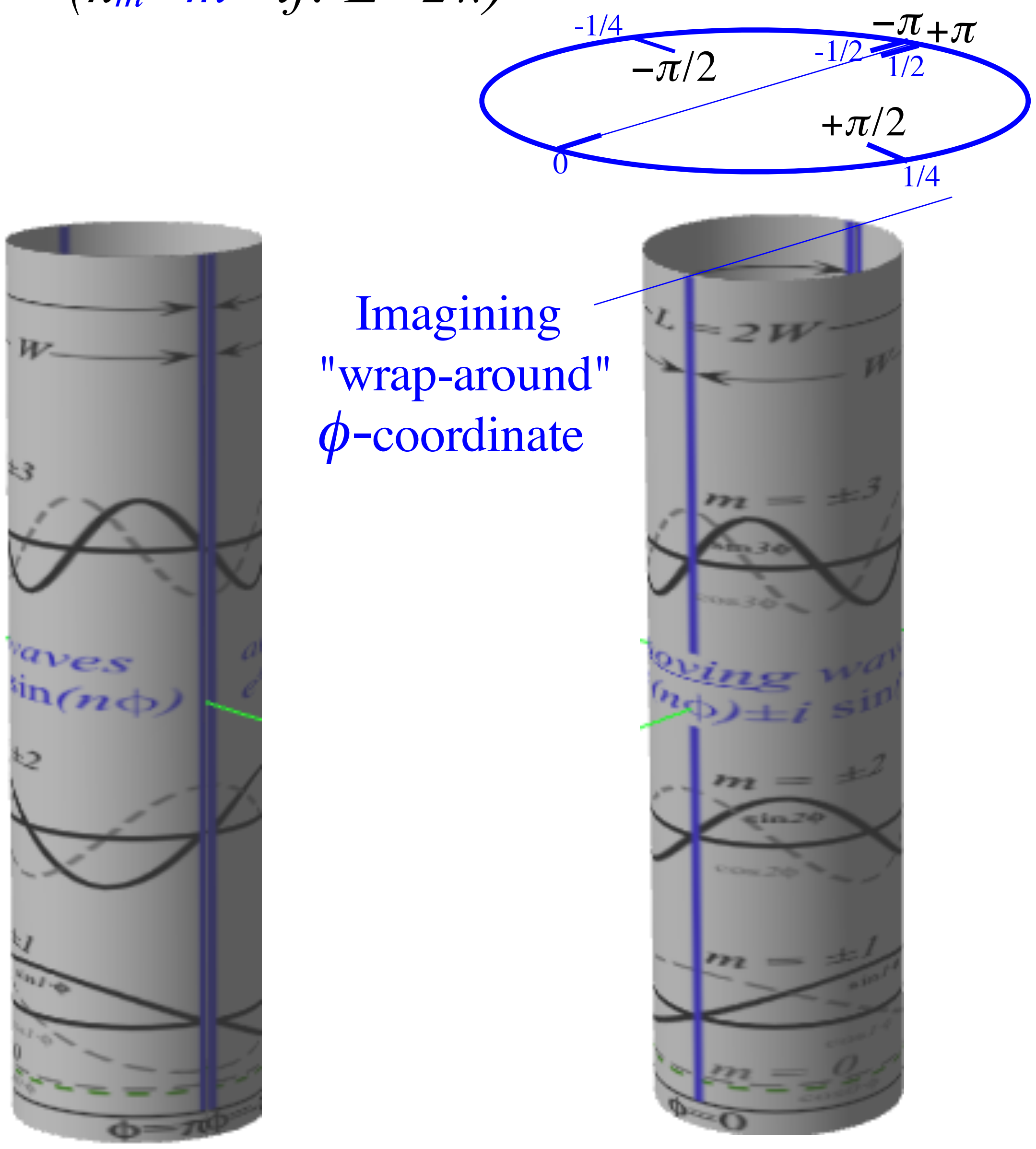
$m=0, \pm 1, \pm 2, \pm 3$,...are momentum quanta in wavevector formula: $k_m=2\pi m/L$

($k_m=m$ if: $L=2\pi$)



only allows standing sine waves $\sin(n\phi)$

allows moving waves $e^{\pm i\phi}=\cos(n\phi)\pm i\sin(n\phi)$



Imagining "wrap-around" ϕ-coordinate

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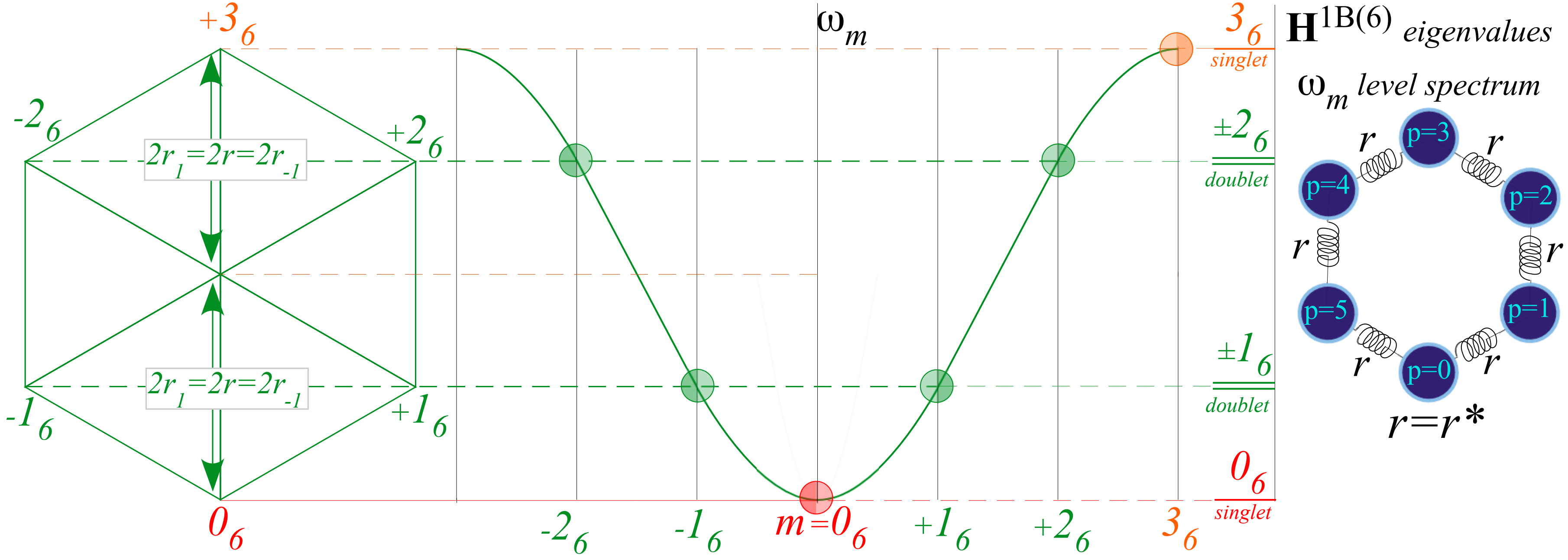
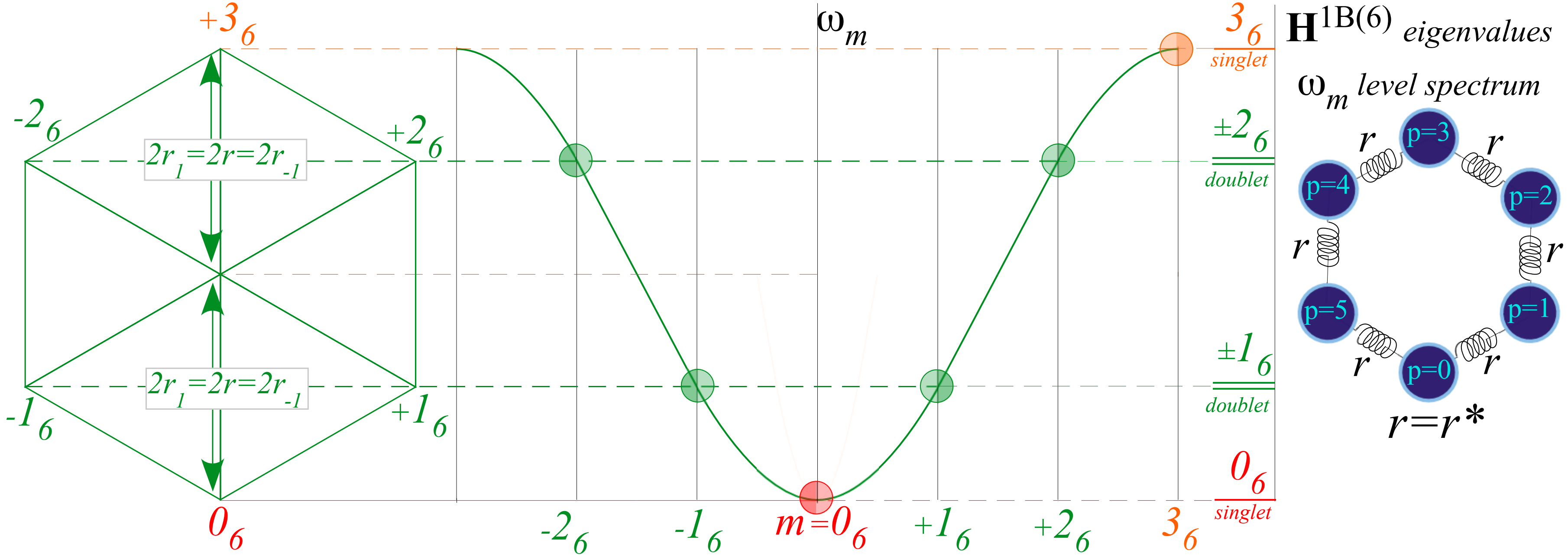
$$\chi^j(2\pi/n) = \frac{\sin(\pi(2j+1)/n)}{\sin(\pi/n)}$$

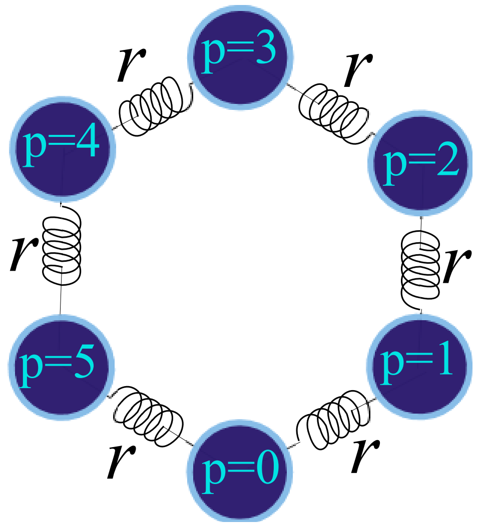
C_6 symmetry: Elementary Bloch Hamiltonian $\mathbf{H}^{1B(6)}$ (1st neighbor coupling)



C_6 symmetry: Elementary Bloch Hamiltonian $\mathbf{H}^{1B(6)}$ *(1st neighbor coupling)*



$$\mathbf{H}^{1B(6)}\begin{pmatrix}\psi^{m}{}_{0}\\ \psi^{m}{}_{1}\\ \psi^{m}{}_{2}\\ \psi^{m}{}_{3}\\ \psi^{m}{}_{4}\\ \psi^{m}{}_{5}\end{pmatrix} = \begin{array}{cccccc} p=0 & 1 & 2 & 3 & 4 & 5\\ 2r & -r & \cdot & \cdot & \cdot & -r\\ -r & 2r & -r & \cdot & \cdot & \cdot\\ \cdot & -r & 2r & -r & \cdot & \cdot\\ \cdot & \cdot & -r & 2r & -r & \cdot\\ \cdot & \cdot & \cdot & -r & 2r & -r\\ -r & \cdot & \cdot & \cdot & -r & 2r \end{array}\begin{pmatrix}\psi^{m}{}_{0}\\ \psi^{m}{}_{1}\\ \psi^{m}{}_{2}\\ \psi^{m}{}_{3}\\ \psi^{m}{}_{4}\\ \psi^{m}{}_{5}\end{pmatrix} = 2r\left(1-\cos\frac{2\pi m}{6}\right)\begin{pmatrix}\psi^{m}{}_{0}\\ \psi^{m}{}_{1}\\ \psi^{m}{}_{2}\\ \psi^{m}{}_{3}\\ \psi^{m}{}_{4}\\ \psi^{m}{}_{5}\end{pmatrix}$$



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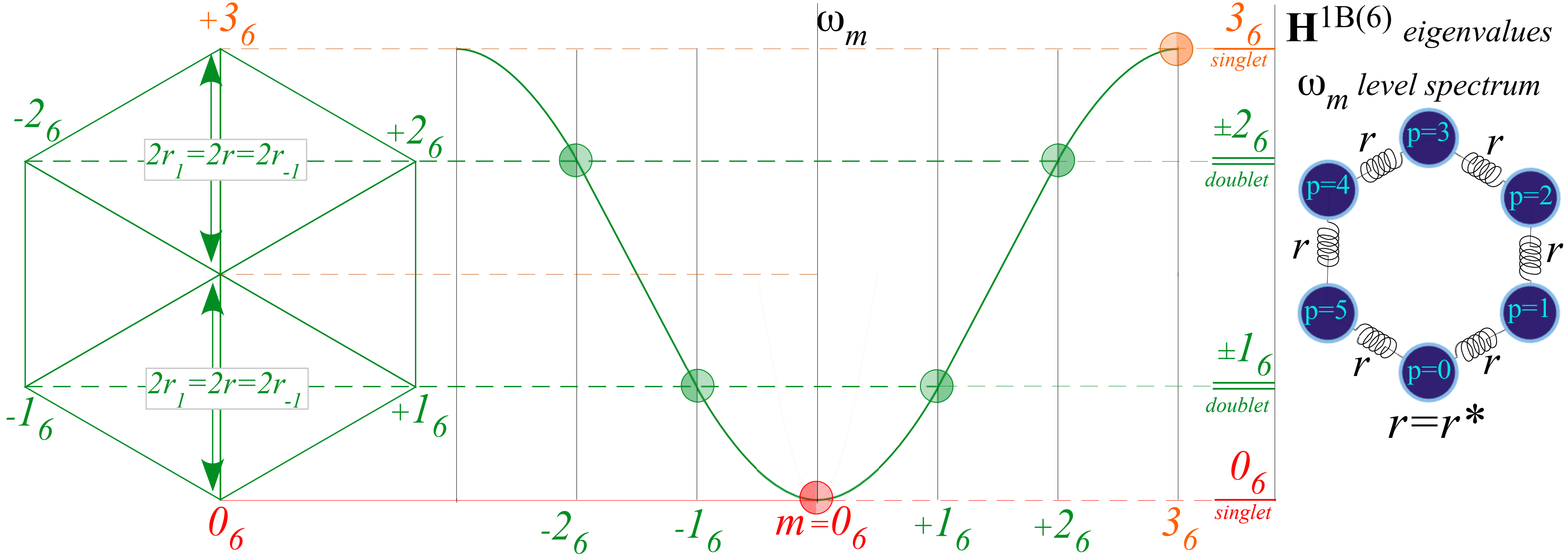
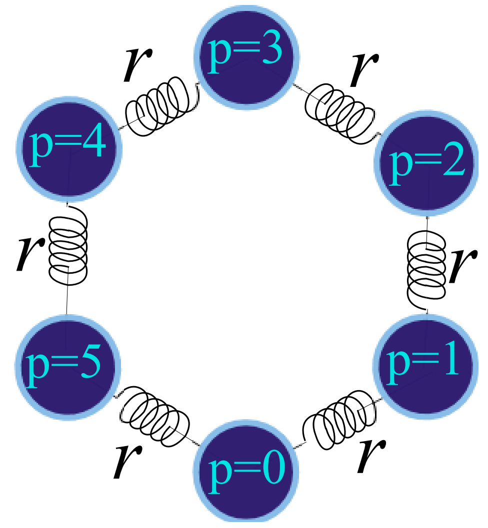


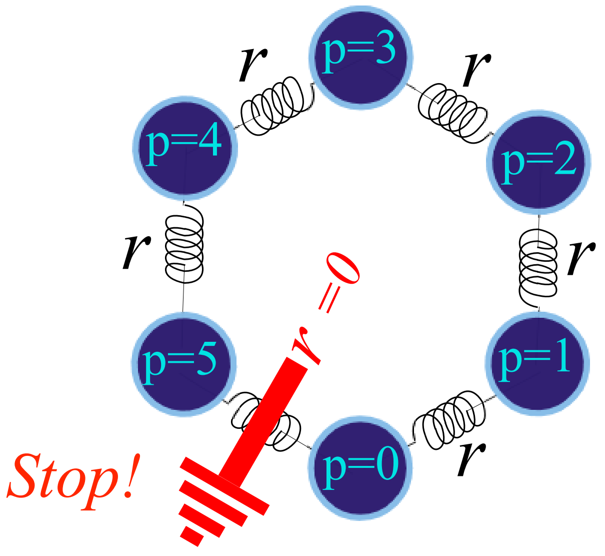
$$\mathbf{H}^{1B(6)}\begin{pmatrix}\psi^m{}_0\\ \psi^m{}_1\\ \psi^m{}_2\\ \psi^m{}_3\\ \psi^m{}_4\\ \psi^m{}_5\end{pmatrix}=\left(\begin{array}{cccccc} p=0 & 1 & 2 & 3 & 4 & 5\\ 2r & -r & \cdot & \cdot & \cdot & -r\\ -r & 2r & -r & \cdot & \cdot & \cdot\\ \cdot & -r & 2r & -r & \cdot & \cdot\\ \cdot & \cdot & -r & 2r & -r & \cdot\\ \cdot & \cdot & \cdot & -r & 2r & -r\\ -r & \cdot & \cdot & \cdot & -r & 2r\end{array}\right)\begin{pmatrix}\psi^m{}_0\\ \psi^m{}_1\\ \psi^m{}_2\\ \psi^m{}_3\\ \psi^m{}_4\\ \psi^m{}_5\end{pmatrix}=2r(1-\cos\frac{2\pi m}{6})\begin{pmatrix}\psi^m{}_0\\ \psi^m{}_1\\ \psi^m{}_2\\ \psi^m{}_3\\ \psi^m{}_4\\ \psi^m{}_5\end{pmatrix}$$



$\mathbf{H}^{1B(6)}$ eigensolutions are very sensitive to zeroing or constraining a coupling!

$$\mathbf{H}^{1B(6)}\begin{pmatrix}\psi^m{}_0\\ \psi^m{}_1\\ \psi^m{}_2\\ \psi^m{}_3\\ \psi^m{}_4\\ \psi^m{}_5\end{pmatrix}=\left(\begin{array}{cccccc} p=0 & 1 & 2 & 3 & 4 & 5\\ 2r & -r & \cdot & \cdot & \cdot & 0\\ -r & 2r & -r & \cdot & \cdot & \cdot\\ \cdot & -r & 2r & -r & \cdot & \cdot\\ \cdot & \cdot & -r & 2r & -r & \cdot\\ \cdot & \cdot & \cdot & -r & 2r & -r\\ 0 & \cdot & \cdot & \cdot & -r & 2r\end{array}\right)\begin{pmatrix}\psi^m{}_0\\ \psi^m{}_1\\ \psi^m{}_2\\ \psi^m{}_3\\ \psi^m{}_4\\ \psi^m{}_5\end{pmatrix}=?\begin{pmatrix}?\\ ?\\ ?\\ ?\\ ?\\ ?\end{pmatrix}$$

(Not eigenvectors)



Consider sine and cosine eigenvectors of a *14-by-14* elementary Bloch matrix $\mathbf{H}^{\text{EB}(14)}$

$$\langle cos^m| = \left(c_0^m=1 \;\vdots\; c_1^m \;\; c_2^m \;\; c_3^m \;\; c_4^m \;\; c_5^m \;\; c_6^m \;\vdots\; c_7^m=1 \;\vdots\; c_{-6}^m \;\; c_{-5}^m \;\; c_{-4}^m \;\; c_{-3}^m \;\; c_{-2}^m \;\; c_{-1}^m \right) \qquad c_p^m = \cos\left(m\cdot p\frac{\pi}{7}\right) = c_{-p}^m$$

$$\langle sin^m| = \left(s_0^m=0 \;\vdots\; s_1^m \;\; s_2^m \;\; s_3^m \;\; s_4^m \;\; s_5^m \;\; s_6^m \;\vdots\; s_7^m=0 \;\vdots\; s_{-6}^m \;\; s_{-5}^m \;\; s_{-4}^m \;\; s_{-3}^m \;\; s_{-2}^m \;\; s_{-1}^m \right) \qquad s_p^m = \sin\left(m\cdot p\frac{\pi}{7}\right) = -s_{-p}^m$$

$$\mathbf{H}^{\text{EB}(14)}|sin^m\rangle \quad = \omega^{m(14)}|sin^m\rangle$$

$$\begin{array}{c|c:cccccc:c:cccccc|}
p \diagup p' & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & -6 & -5 & -4 & -3 & -2 & -1 \\
\hline
0 & 2r & -r & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -r \\
\hdashline
1 & -r & 2r & -r & \cdot & \cdot & \cdot & \cdot & & & & & & & \\
2 & \cdot & -r & 2r & -r & \cdot & \cdot & \cdot & & & & & & & \\
3 & \cdot & \cdot & -r & 2r & -r & \cdot & \cdot & & & & & & & \\
4 & \cdot & \cdot & \cdot & -r & 2r & -r & \cdot & & & & & & & \\
5 & \cdot & \cdot & \cdot & \cdot & -r & 2r & -r & & & & & & & \\
6 & \cdot & \cdot & \cdot & \cdot & \cdot & -r & 2r & -r & & & & & & \\
\hdashline
7 & \cdot & & & & & & -r & 2r & -r & & & & & \\
\hdashline
-6 & \cdot & & & & & & & -r & 2r & -r & \cdot & \cdot & \cdot & \cdot \\
-5 & \cdot & & & & & & & & -r & 2r & -r & \cdot & \cdot & \cdot \\
-4 & \cdot & & & & & & & & \cdot & -r & 2r & -r & \cdot & \cdot \\
-3 & \cdot & & & & & & & & \cdot & \cdot & -r & 2r & -r & \cdot \\
-2 & \cdot & & & & & & & & \cdot & \cdot & \cdot & -r & 2r & -r \\
-1 & -r & & & & & & & & \cdot & \cdot & \cdot & \cdot & -r & 2r
\end{array}
\left\|\begin{array}{c} \vdots \\ \hline 0 \\ \hdashline s_1^m \\ s_2^m \\ s_3^m \\ s_4^m \\ s_5^m \\ s_6^m \\ \hdashline 0 \\ \hdashline s_{-6}^m \\ s_{-5}^m \\ s_{-4}^m \\ s_{-3}^m \\ s_{-2}^m \\ s_{-1}^m \end{array}\right\|
= \omega^{m(14)}
\left\|\begin{array}{c} \vdots \\ \hline 0 \\ \hdashline s_1^m \\ s_2^m \\ s_3^m \\ s_4^m \\ s_5^m \\ s_6^m \\ \hdashline 0 \\ \hdashline s_{-6}^m \\ s_{-5}^m \\ s_{-4}^m \\ s_{-3}^m \\ s_{-2}^m \\ s_{-1}^m \end{array}\right\|$$

where :

$$\omega^{m(14)} = 2r\left(1-\cos\frac{2\pi m}{14}\right)$$

Consider sine and cosine eigenvectors of a *14-by-14* elementary Bloch matrix $\mathbf{H}^{\text{EB}(14)}$

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$$\left\langle sin^{m}\right| = \left(s_0^m = 0 \;\middle|\; s_1^m \;\; s_2^m \;\; s_3^m \;\; s_4^m \;\; s_5^m \;\; s_6^m \;\middle|\; s_7^m = 0 \;\middle|\; s_{-6}^m \;\; s_{-5}^m \;\; s_{-4}^m \;\; s_{-3}^m \;\; s_{-2}^m \;\; s_{-1}^m \right)$$

$$c_p^m = \cos\left(m \cdot p\frac{\pi}{7}\right) = c_{-p}^m$$

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$$\mathbf{H}^{\text{EB}(14)}\left|sin^m\right\rangle \quad = \omega^{m(14)}\left|sin^m\right\rangle$$

$\mathbf{H}^{\text{EB}(14)}$ gives eigensolution of a *6-by-6* constrained Bloch matrix $\mathbf{H}^{\text{CM}(6)}$

$$\begin{array}{c|c|cccccc|c|cccccc}
p \diagup p' & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & -6 & -5 & -4 & -3 & -2 & -1 \\
\hline
0 & 2r & -r & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -r \\
\hline
1 & -r & 2r & -r & \cdot & \cdot & \cdot & 0 & & & & & & & \\
2 & \cdot & -r & 2r & -r & \cdot & \cdot & \cdot & & & & & & & \\
3 & \cdot & \cdot & -r & 2r & -r & \cdot & \cdot & & & & & & & \\
4 & \cdot & \cdot & \cdot & -r & 2r & -r & \cdot & & & & & & & \\
5 & \cdot & \cdot & \cdot & \cdot & -r & 2r & -r & & & & & & & \\
6 & \cdot & 0 & \cdot & \cdot & \cdot & -r & 2r & -r & & & & & & \\
\hline
7 & \cdot & & & & & & -r & 2r & -r & & & & & \\
\hline
-6 & \cdot & & & & & & & -r & 2r & -r & \cdot & \cdot & \cdot & \cdot \\
-5 & \cdot & & & & & & & & -r & 2r & -r & \cdot & \cdot & \cdot \\
-4 & \cdot & & & & & & & & \cdot & -r & 2r & -r & \cdot & \cdot \\
-3 & \cdot & & & & & & & & \cdot & \cdot & -r & 2r & -r & \cdot \\
-2 & \cdot & & & & & & & & \cdot & \cdot & \cdot & -r & 2r & -r \\
-1 & -r & & & & & & & & \cdot & \cdot & \cdot & \cdot & -r & 2r
\end{array}
\begin{pmatrix} \cdot \\ \cdot \\ \hline 0 \\ \hline s_1^m \\ s_2^m \\ s_3^m \\ s_4^m \\ s_5^m \\ s_6^m \\ \hline 0 \\ \hline s_{-6}^m \\ s_{-5}^m \\ s_{-4}^m \\ s_{-3}^m \\ s_{-2}^m \\ s_{-1}^m \end{pmatrix}
= \omega^{m(14)}
\begin{pmatrix} \cdot \\ \cdot \\ \hline 0 \\ \hline s_1^m \\ s_2^m \\ s_3^m \\ s_4^m \\ s_5^m \\ s_6^m \\ \hline 0 \\ \hline s_{-6}^m \\ s_{-5}^m \\ s_{-4}^m \\ s_{-3}^m \\ s_{-2}^m \\ s_{-1}^m \end{pmatrix}$$

where :

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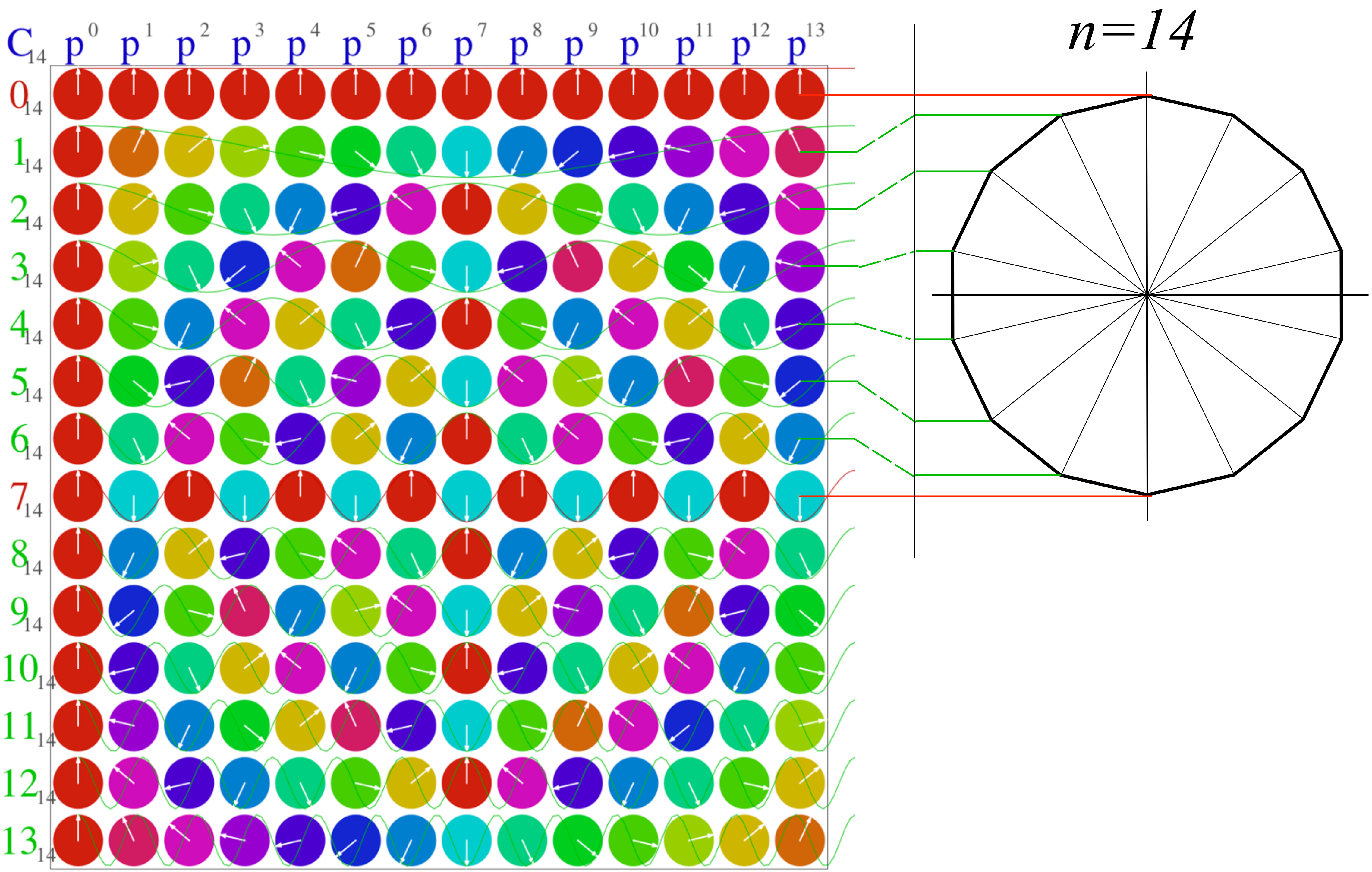
$\mathbf{H}^{EB(14)}$ gives eigensolution of a *6-by-6* constrained Bloch matrix $\mathbf{H}^{CM(6)}$ using its sine-waves only

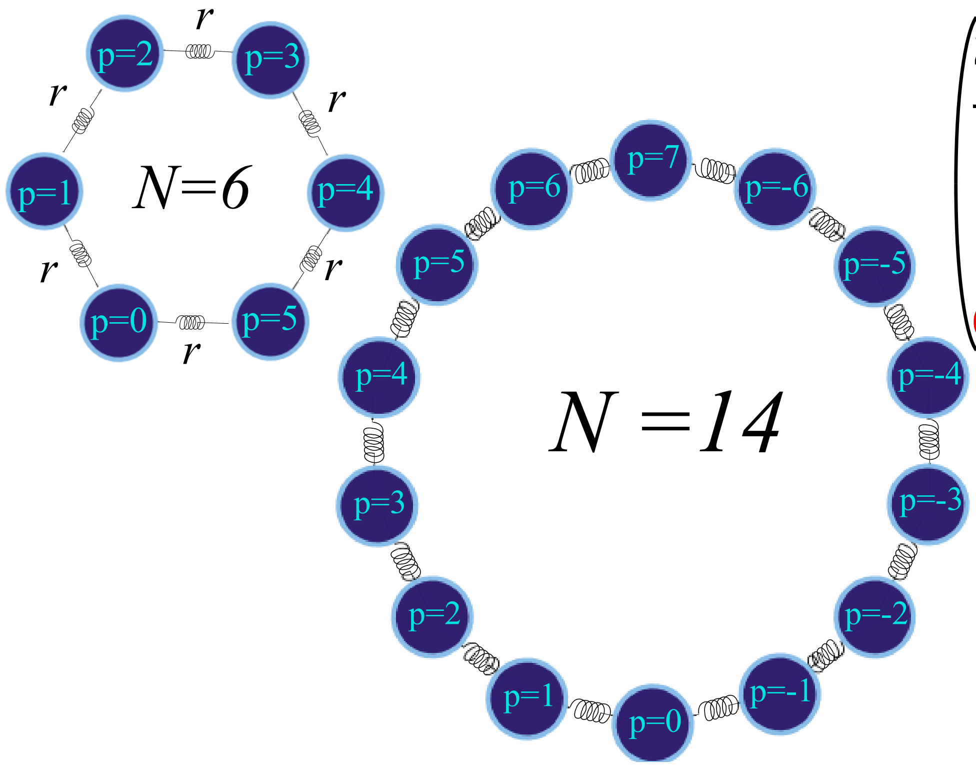


WaveIt Web Simulation

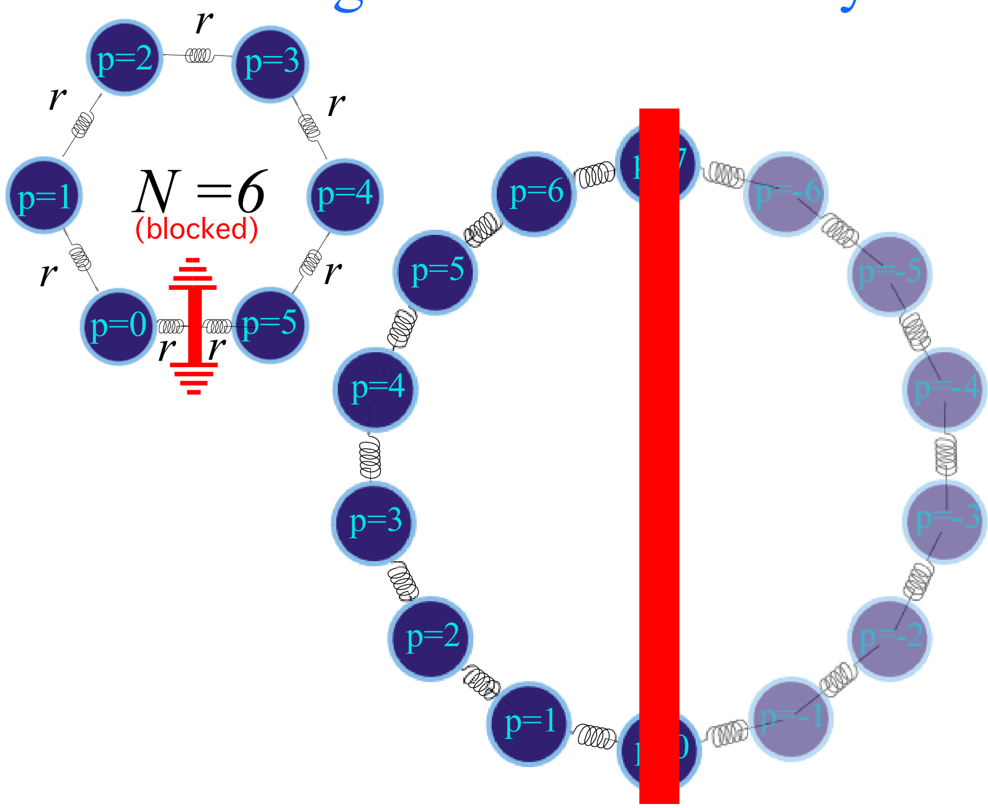
C_{14} Character Phasors

$\mathbf{H}^{EB(14)}$ gives eigensolution of a *6-by-6* constrained Bloch matrix $\mathbf{H}^{CM(6)}$ using its sine-waves only

$$\begin{pmatrix} 2r & -r & \cdot & \cdot & \cdot & -r \\ -r & 2r & -r & \cdot & \cdot & \cdot \\ \cdot & -r & 2r & -r & \cdot & \cdot \\ \cdot & \cdot & -r & 2r & -r & \cdot \\ \cdot & \cdot & \cdot & -r & 2r & -r \\ -r & \cdot & \cdot & \cdot & -r & 2r \end{pmatrix}$$

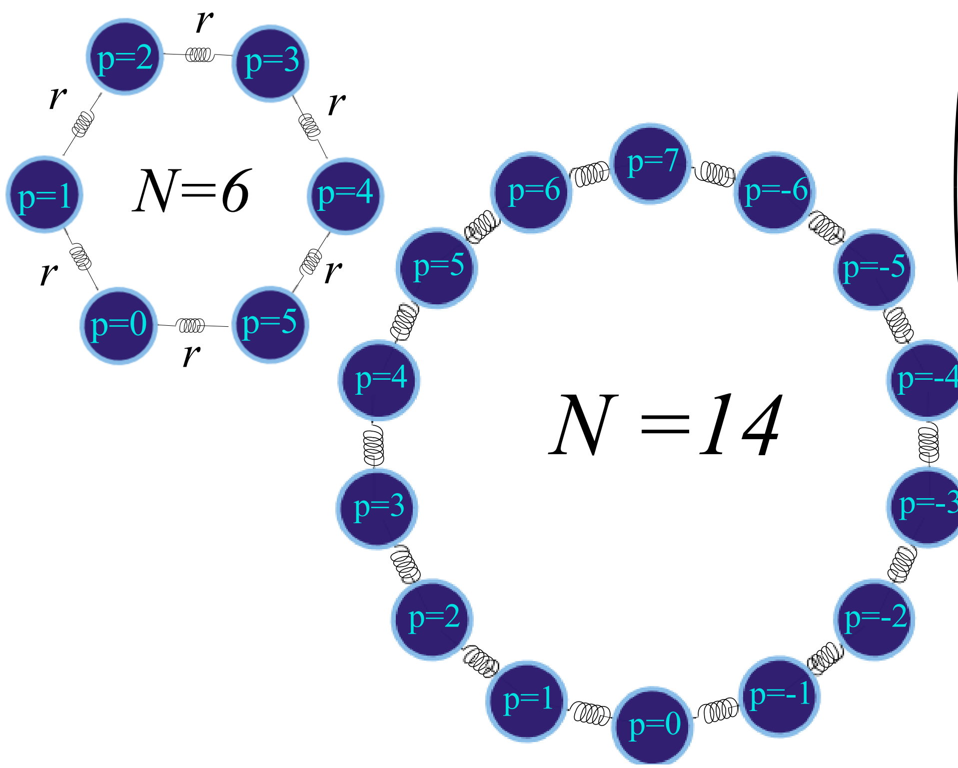


$$\begin{pmatrix} 2r & -r & \cdot & \cdot & \cdot & 0 \\ -r & 2r & -r & \cdot & \cdot & \cdot \\ \cdot & -r & 2r & -r & \cdot & \cdot \\ \cdot & \cdot & -r & 2r & -r & \cdot \\ \cdot & \cdot & \cdot & -r & 2r & -r \\ 0 & \cdot & \cdot & \cdot & -r & 2r \end{pmatrix}$$

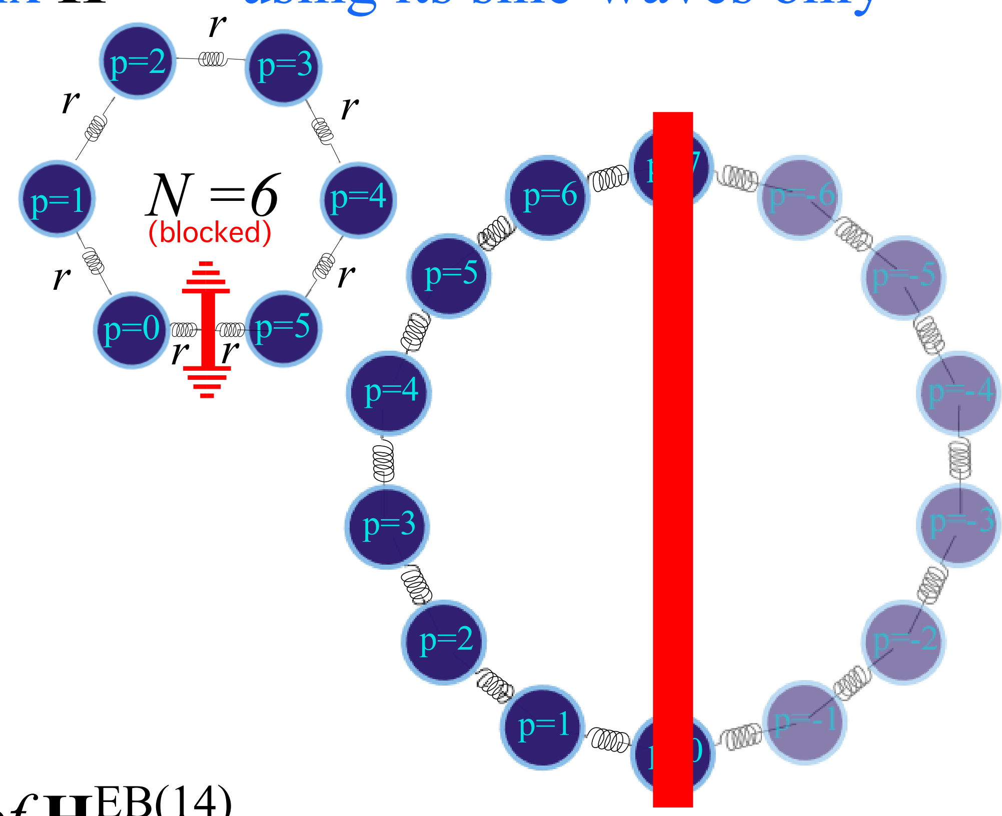


$\mathbf{H}^{EB(14)}$ gives eigensolution of a *6-by-6* constrained Bloch matrix $\mathbf{H}^{CM(6)}$ using its sine-waves only

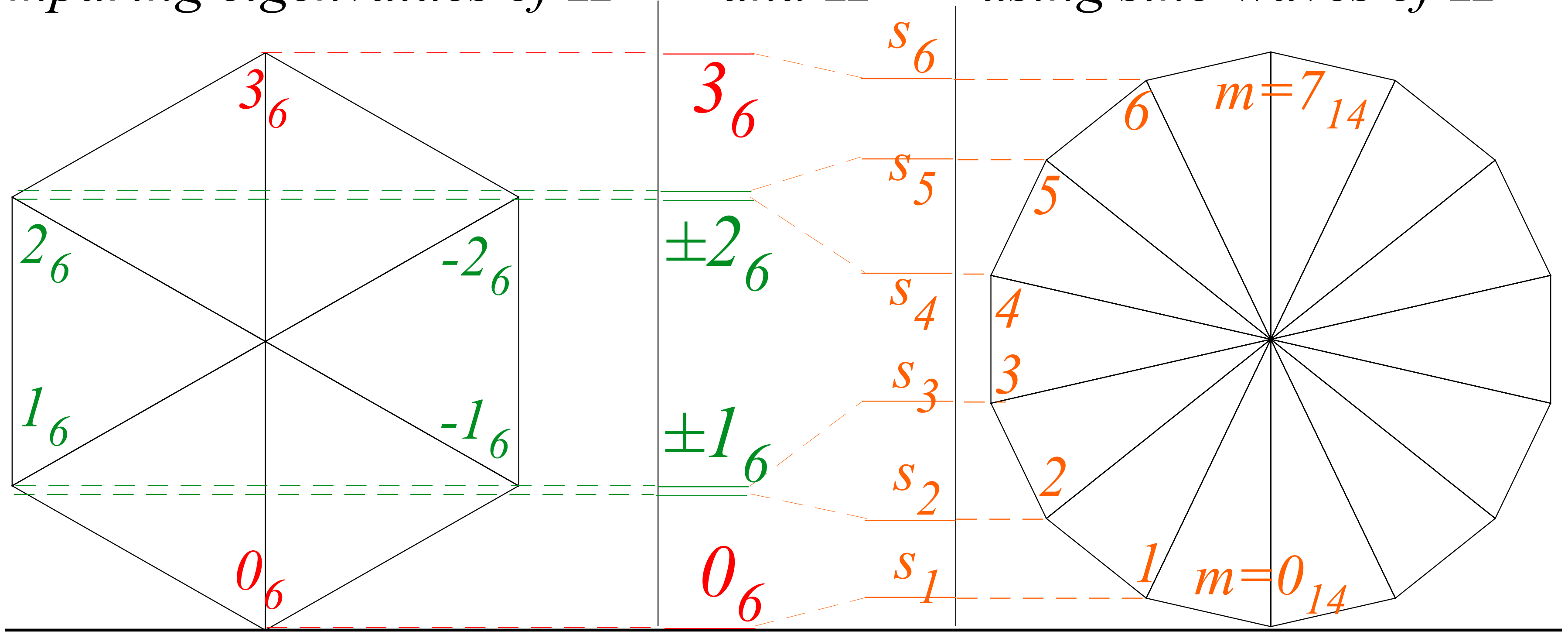
$$\begin{pmatrix} 2r & -r & \cdot & \cdot & \cdot & -r \\ -r & 2r & -r & \cdot & \cdot & \cdot \\ \cdot & -r & 2r & -r & \cdot & \cdot \\ \cdot & \cdot & -r & 2r & -r & \cdot \\ \cdot & \cdot & \cdot & -r & 2r & -r \\ -r & \cdot & \cdot & \cdot & -r & 2r \end{pmatrix}$$



$$\begin{pmatrix} 2r & -r & \cdot & \cdot & \cdot & 0 \\ -r & 2r & -r & \cdot & \cdot & \cdot \\ \cdot & -r & 2r & -r & \cdot & \cdot \\ \cdot & \cdot & -r & 2r & -r & \cdot \\ \cdot & \cdot & \cdot & -r & 2r & -r \\ 0 & \cdot & \cdot & \cdot & -r & 2r \end{pmatrix}$$

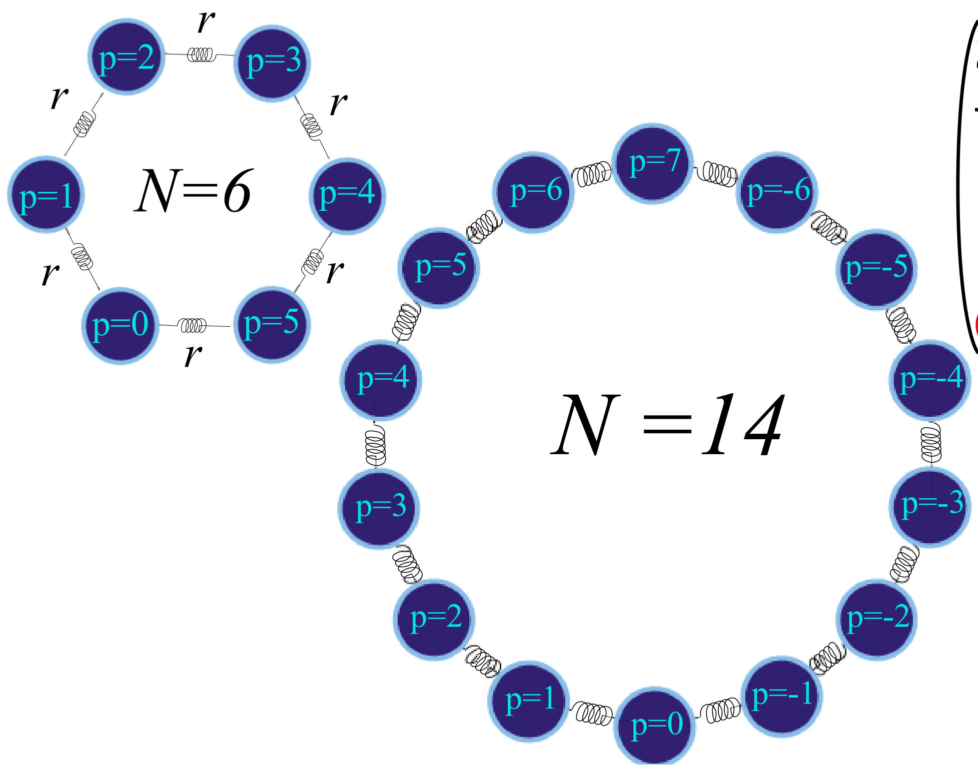


Comparing eigenvalues of $\mathbf{H}^{EB(6)}$ *and* $\mathbf{H}^{CM(6)}$ *using sine-waves of* $\mathbf{H}^{EB(14)}$

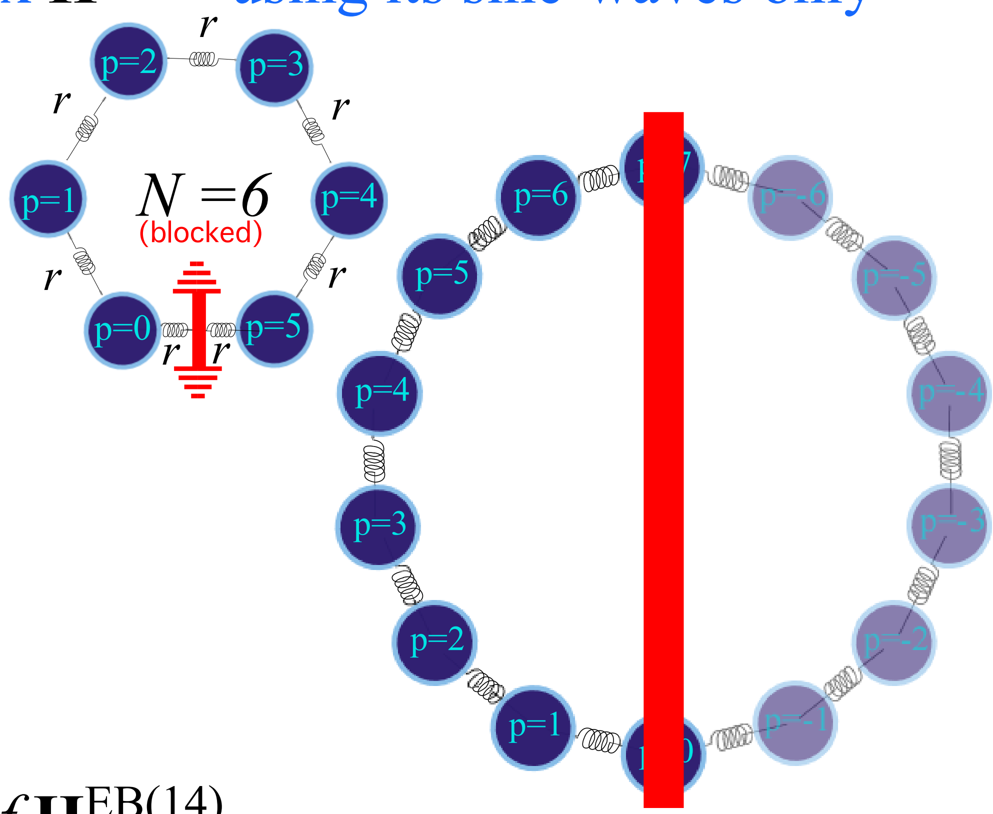


$\mathbf{H}^{EB(14)}$ gives eigensolution of a *6-by-6* constrained Bloch matrix $\mathbf{H}^{CM(6)}$ using its sine-waves only

$$\begin{pmatrix} 2r & -r & \cdot & \cdot & \cdot & -r \\ -r & 2r & -r & \cdot & \cdot & \cdot \\ \cdot & -r & 2r & -r & \cdot & \cdot \\ \cdot & \cdot & -r & 2r & -r & \cdot \\ \cdot & \cdot & \cdot & -r & 2r & -r \\ -r & \cdot & \cdot & \cdot & -r & 2r \end{pmatrix}$$



$$\begin{pmatrix} 2r & -r & \cdot & \cdot & \cdot & 0 \\ -r & 2r & -r & \cdot & \cdot & \cdot \\ \cdot & -r & 2r & -r & \cdot & \cdot \\ \cdot & \cdot & -r & 2r & -r & \cdot \\ \cdot & \cdot & \cdot & -r & 2r & -r \\ 0 & \cdot & \cdot & \cdot & -r & 2r \end{pmatrix}$$



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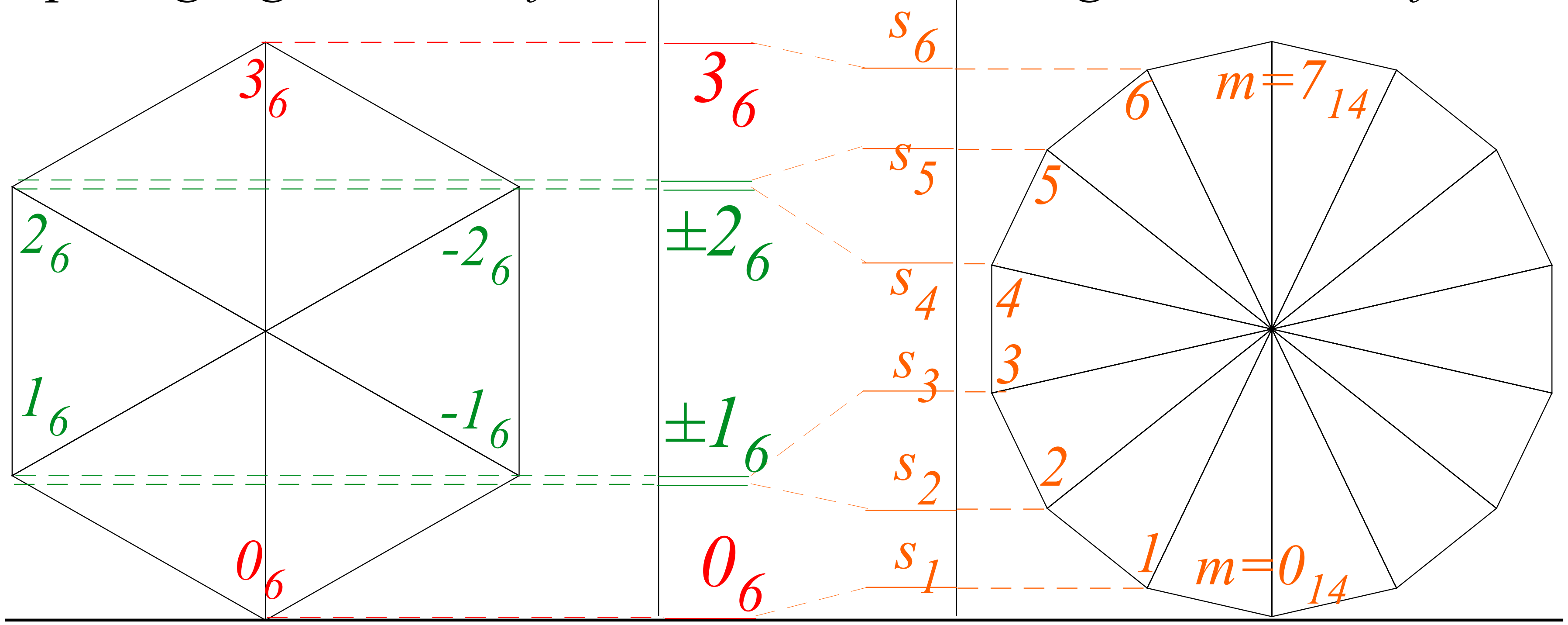

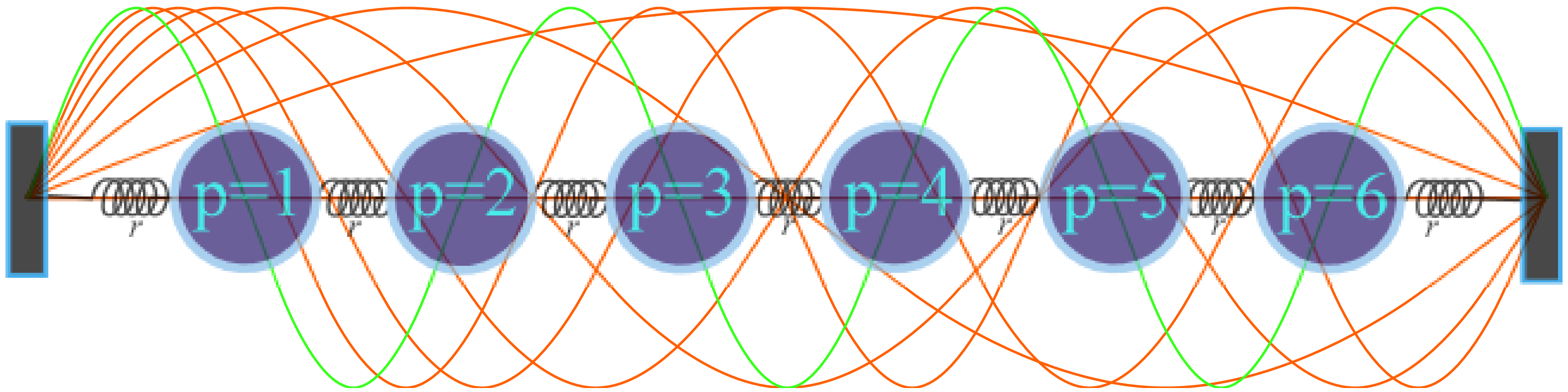

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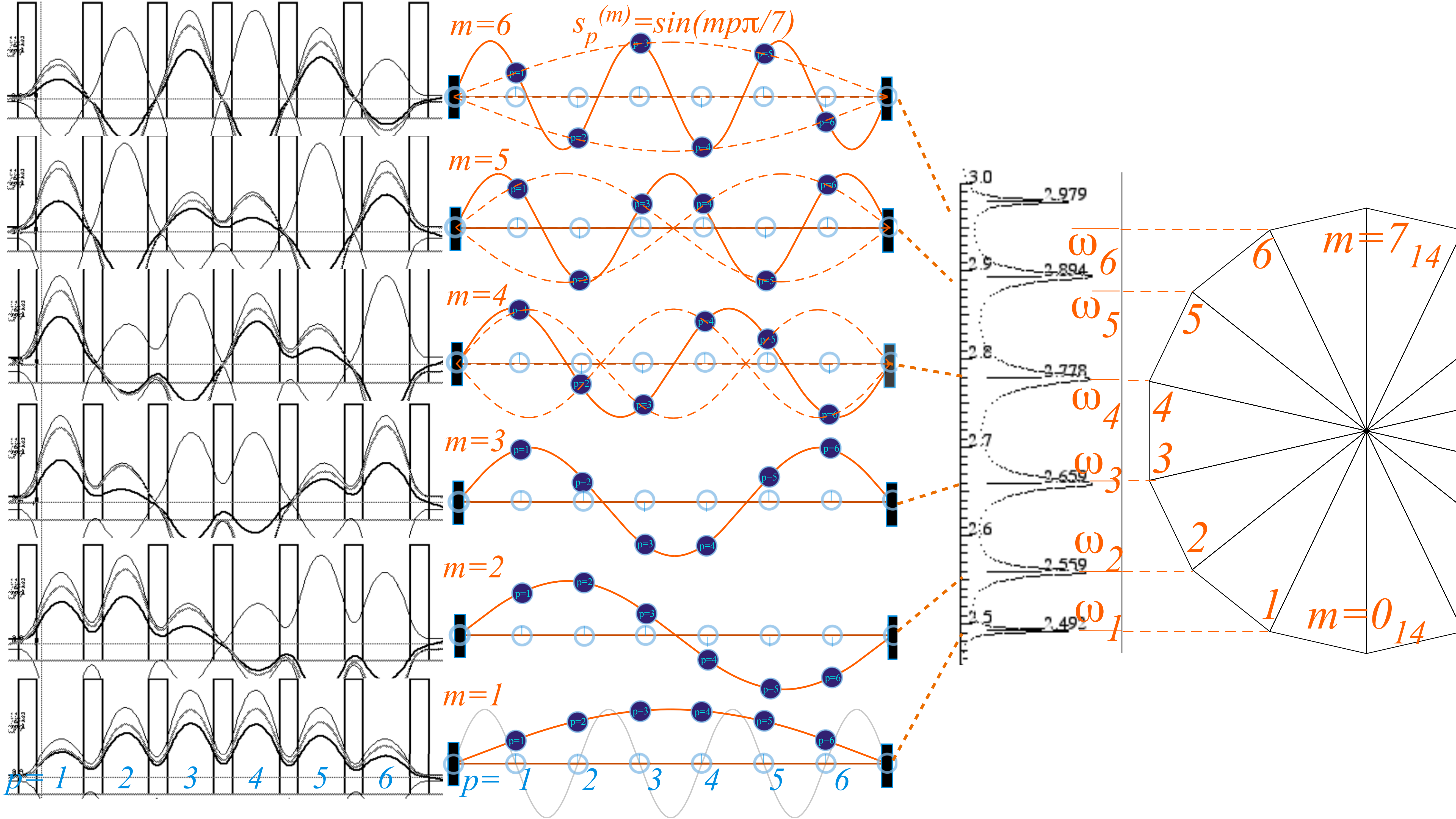
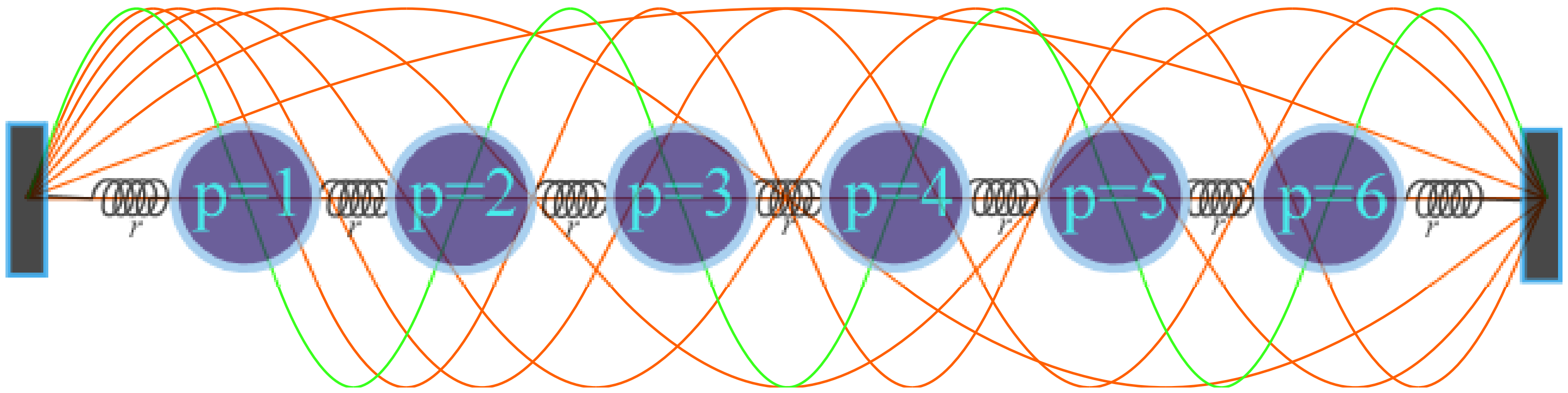


Band-It simulation is Mac OS 9 application not yet converted to web



Breaking C_N cyclic coupling into linear chains

- *Review of 1D-Bohr-ring related to infinite square well*
- *Breaking C_{2N+2} to approximate linear N-chain (Examples $C_2 \rightleftarrows C_6 \rightleftarrows C_{14}$)*
  - *Band-It simulation: Intro to scattering approach to quantum symmetry*
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- *Acoustical modes vs. Optical modes*
- *Intro to other examples of band theory*
- *Avoided crossing view of band-gaps*

Finally! Symmetry groups that are not just C_N

- *The "4-Group(s)" D_2 and C_{2v}*
- *Spectral decomposition of D_2*
  - *Some D_2 modes*
- *Outer product properties and the Crystal-Point Symmetry Group Zoo*
- *Polygonal geometry of $U(2) \supset C_N$ character spectral function*
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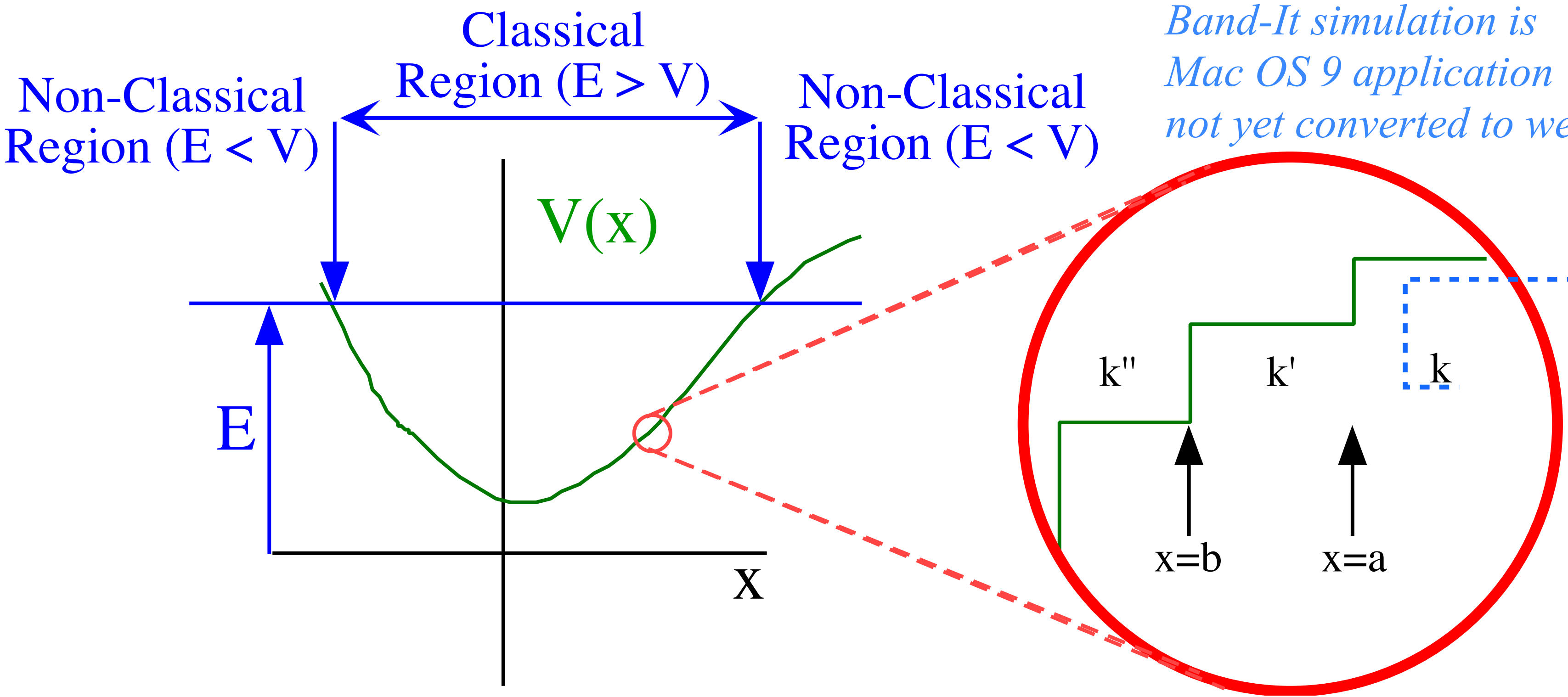
    - *Geometry*

$$\chi^j(2\pi/n) = \frac{\sin\left(\pi(2j+1)/n\right)}{\sin\left(\pi/n\right)}$$

How Band-It simulation works (from QTfCA Unit 4 Chapter 13)



Band-It simulation is Mac OS 9 application not yet converted to web

Fig. 13.1.1 Non-constant potential V(x) approximated by a series of small constant-V steps.

Between each step potential, kinetic energy, and k are assumed constant.

$$\Psi_E(x,0) = Re^{ikx} + Le^{-ikx}$$

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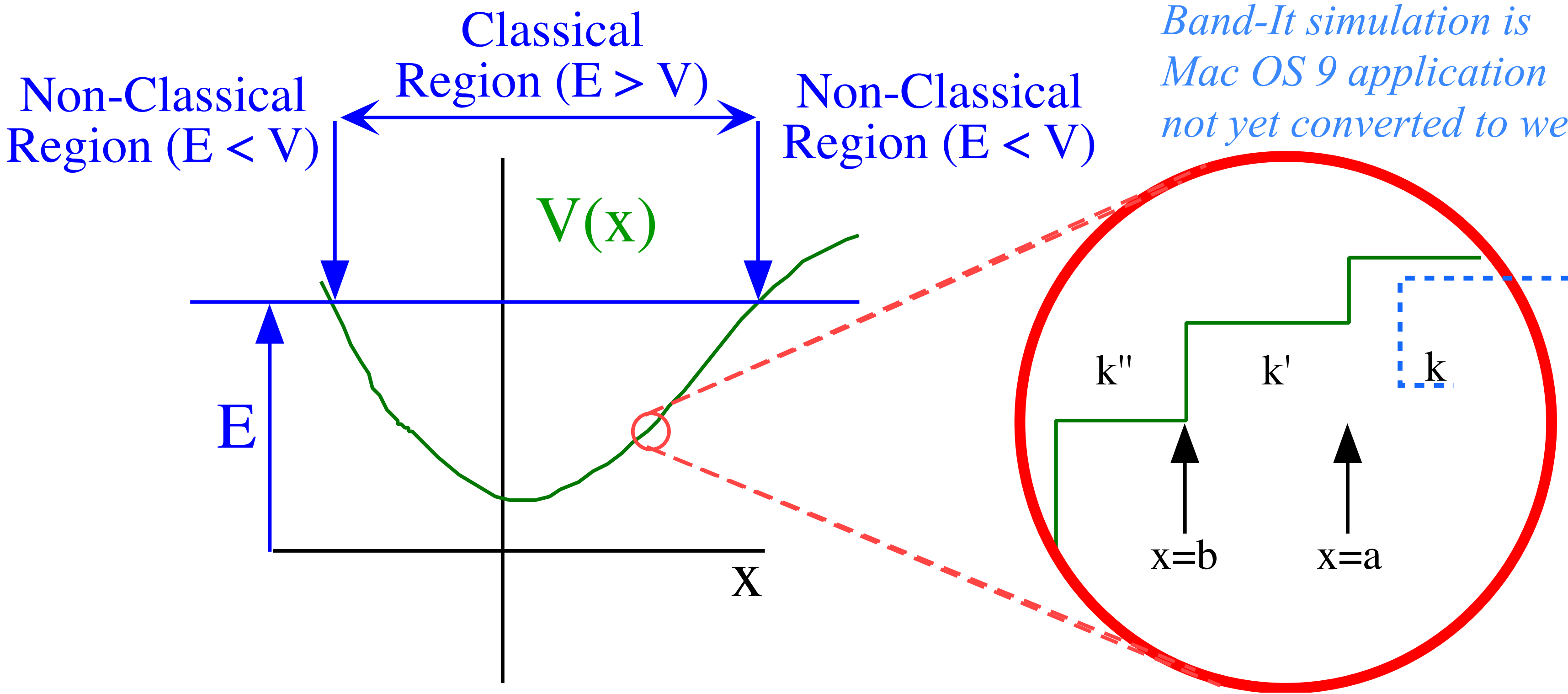

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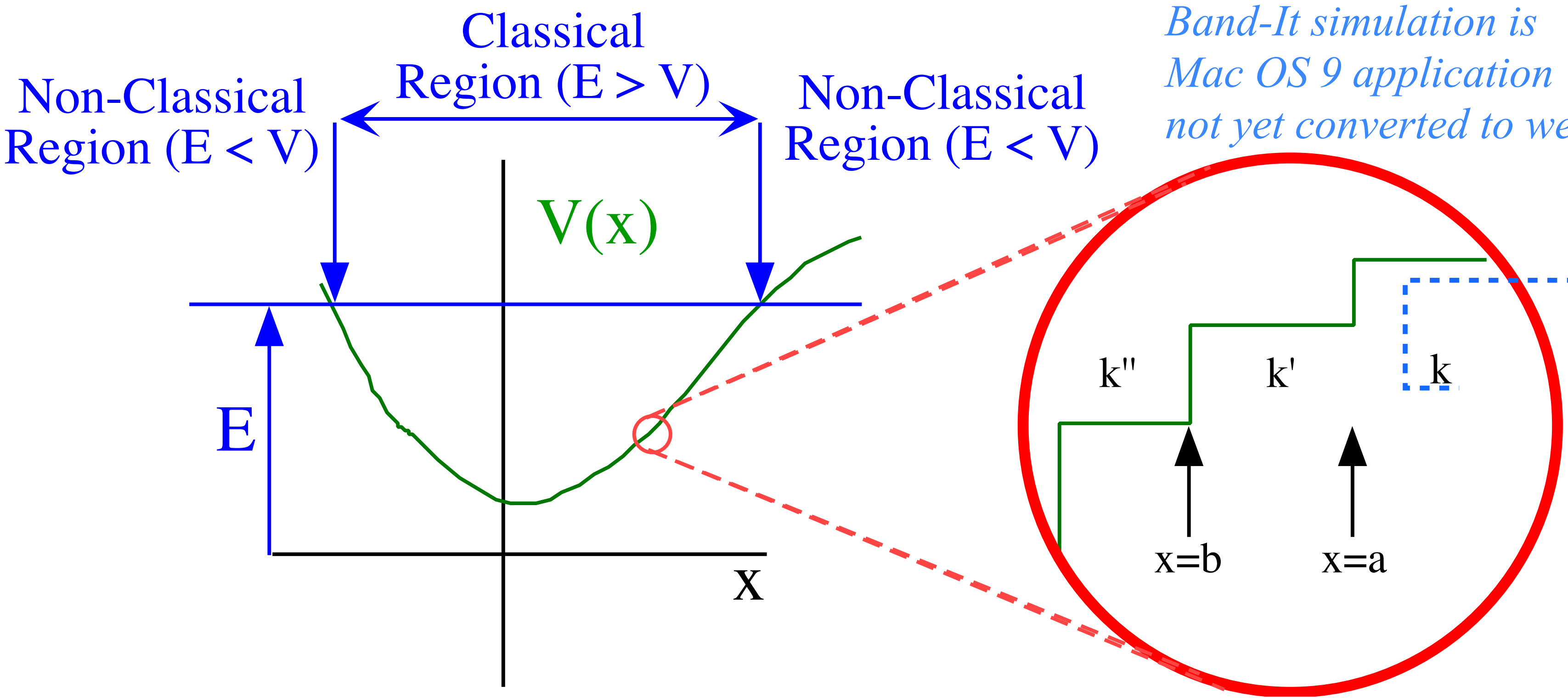

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Relations between the pair (Ψ, $D\Psi$) and amplitudes (R, L) just above $x=a$.

$$\begin{pmatrix} \Psi \\ D\Psi \end{pmatrix} = \begin{pmatrix} e^{ikx} & e^{-ikx} \\ ike^{ikx} & -ike^{-ikx} \end{pmatrix} \begin{pmatrix} R \\ L \end{pmatrix}$$

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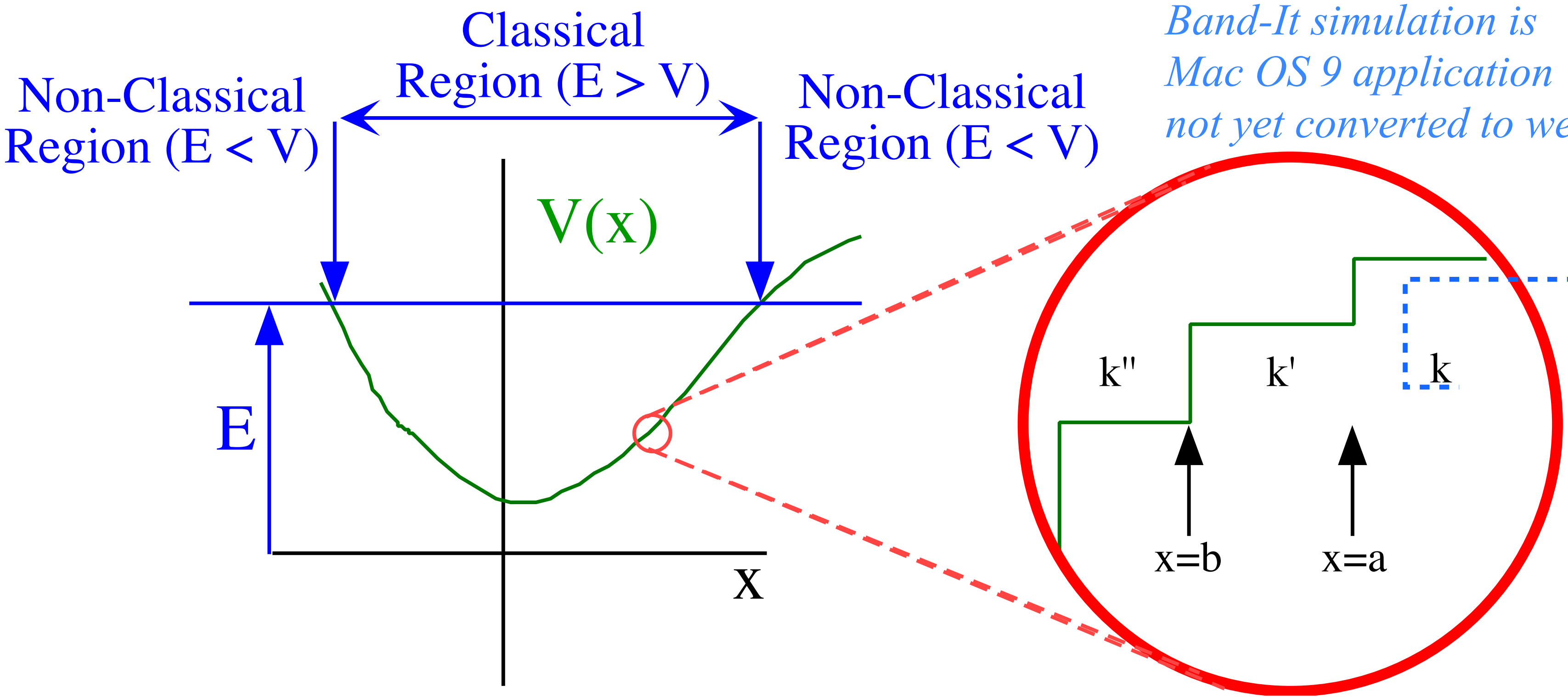

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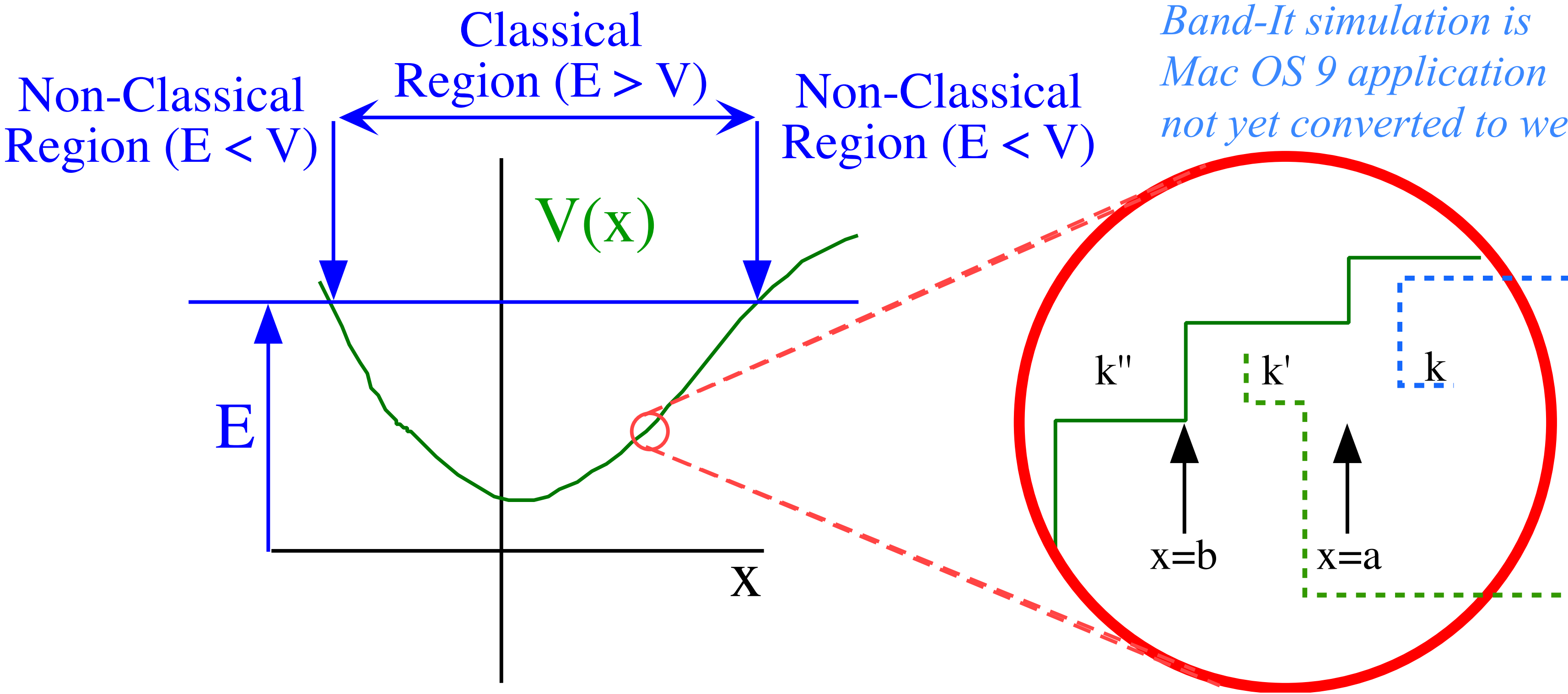
How Band-It simulation works (from QTfCA Unit 4 Chapter 13)



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Relations on the other side of the step boundary just below $x=a$. *(Inverted)*

$$\begin{pmatrix} \Psi' \\ D\Psi' \end{pmatrix} = \begin{pmatrix} e^{ik'x} & e^{-ik'x} \\ ik'e^{ik'x} & -ik'e^{-ik'x} \end{pmatrix} \begin{pmatrix} R' \\ L' \end{pmatrix}, \qquad \begin{pmatrix} R' \\ L' \end{pmatrix} = \frac{i}{2k'} \begin{pmatrix} -ik'e^{-ik'x} & -e^{-ik'x} \\ -ik'e^{ik'x} & e^{ik'x} \end{pmatrix} \begin{pmatrix} \Psi' \\ D\Psi' \end{pmatrix}$$

Breaking C_N cyclic coupling into linear chains

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Algebra

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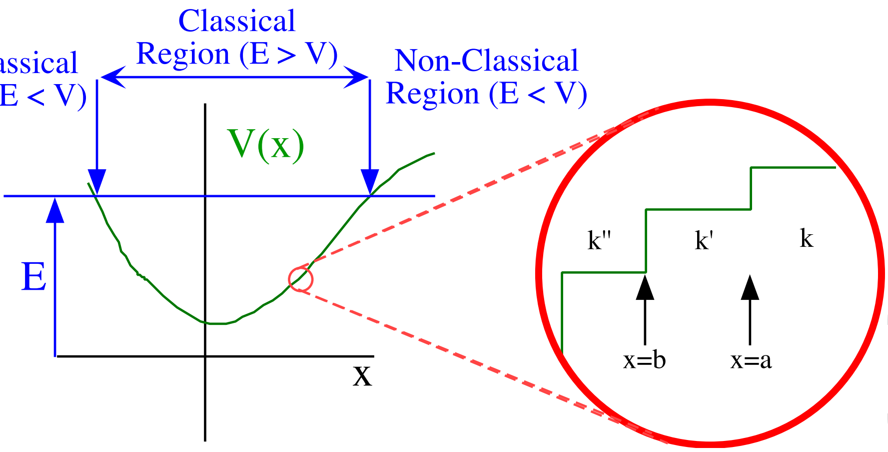
How Band-It simulation works (from QTfCA Unit 4 Chapter 13)



Wave function and derivative at $x=a-\varepsilon$ equals that at $x=a+\varepsilon$.

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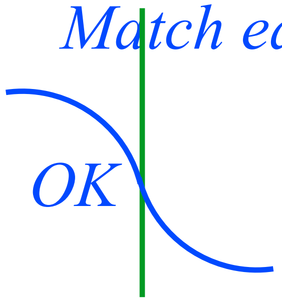
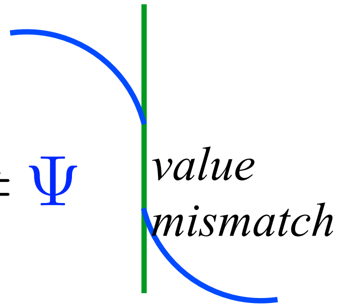
$$\begin{pmatrix}R'\\ L'\end{pmatrix}=\frac{i}{2k'}\begin{pmatrix}-ik'e^{-ik'a} & -e^{-ik'a}\\ -ik'e^{ik'a} & e^{ik'a}\end{pmatrix}\begin{pmatrix}\Psi\\ D\Psi\end{pmatrix}_{x=a}$$

Match each Ψ *and* $D\Psi$

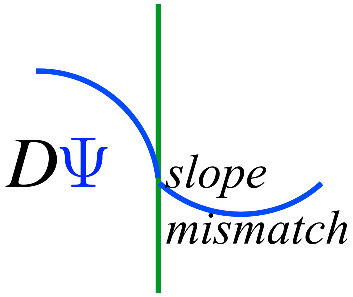


OK

NOT OK: $\Psi'\neq\Psi$



NOT OK: $D\Psi'\neq D\Psi$



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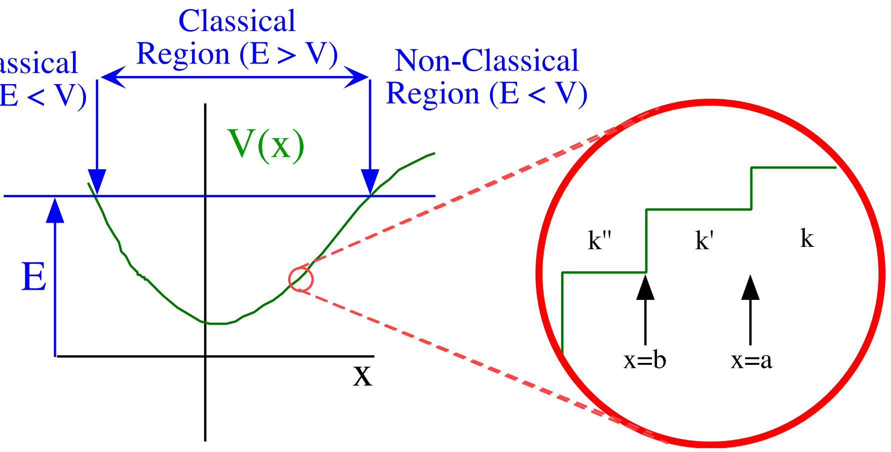
How Band-It simulation works (from QTfCA Unit 4 Chapter 13)



Wave function and derivative at $x=a\text{-}\varepsilon$ equals that at $x=a+\varepsilon$.

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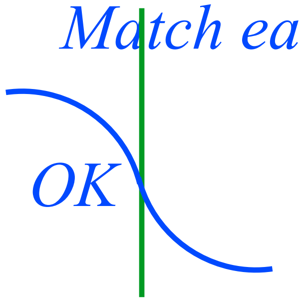
Match each Ψ and DΨ



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How Band-It simulation works (from QTfCA Unit 4 Chapter 13)

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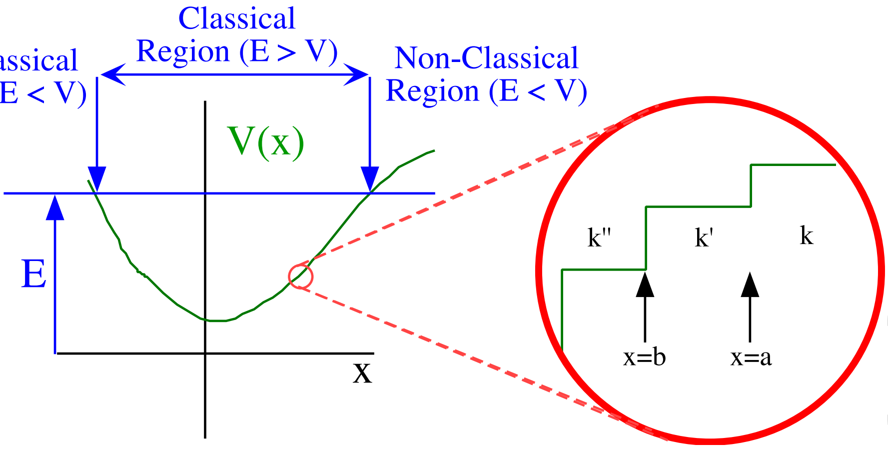
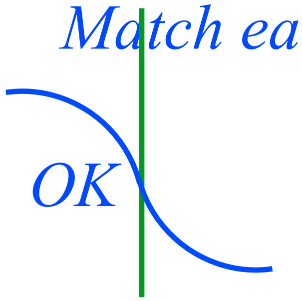


Match each Ψ *and* DΨ



$$\begin{pmatrix}\Psi' \\ D\Psi'\end{pmatrix}_{x=a-\varepsilon} = \begin{pmatrix}\Psi \\ D\Psi\end{pmatrix}_{x=a+\varepsilon}$$

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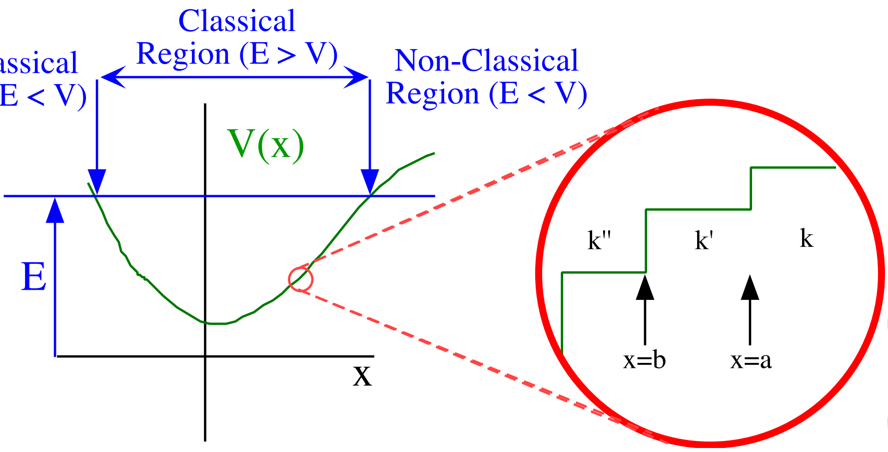

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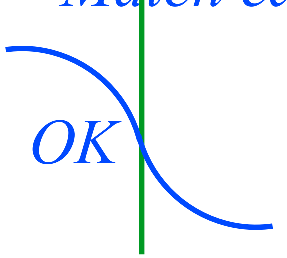
Match each Ψ *and* DΨ



$$\begin{pmatrix}R'\\ L'\end{pmatrix}=\frac{i}{2k'}\begin{pmatrix}-ik'e^{-ik'a} & -e^{-ik'a}\\ -ik'e^{ik'a} & e^{ik'a}\end{pmatrix}\begin{pmatrix}e^{ika} & e^{-ika}\\ ike^{ika} & -ike^{-ika}\end{pmatrix}\begin{pmatrix}R\\ L\end{pmatrix}$$

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A special case: *single input conditions* with no sources or reflectors on one side (say, right hand side) so no incoming waves exist there (say, *L=0* but *R=Outgoing≠ 0*.)

How Band-It simulation works (from QTfCA Unit 4 Chapter 13)

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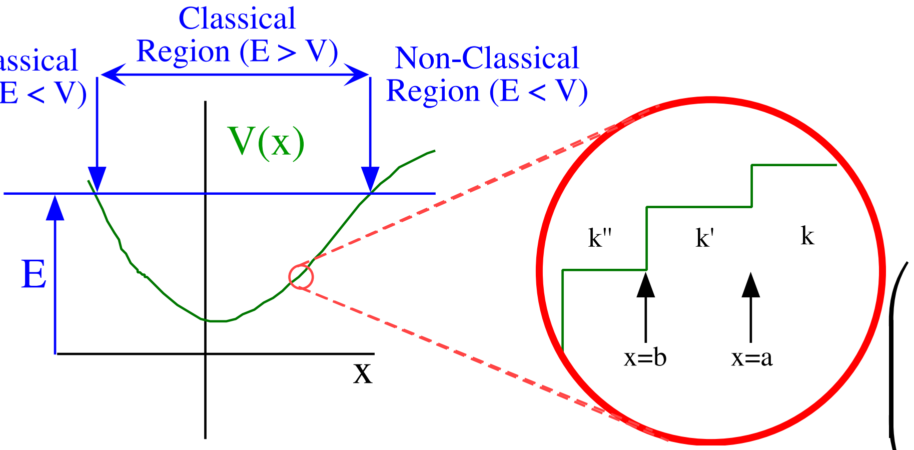

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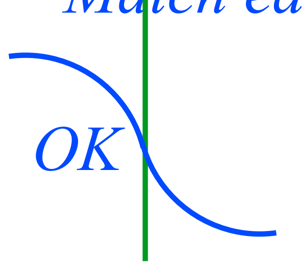
Match each Ψ *and* DΨ



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How Band-It simulation works (from QTfCA Unit 4 Chapter 13)

A special case: *single input conditions* with no sources or reflectors on one side (say, right hand side) so no incoming waves exist there (say, $L=0$ but $R=Outgoing\neq 0$.)

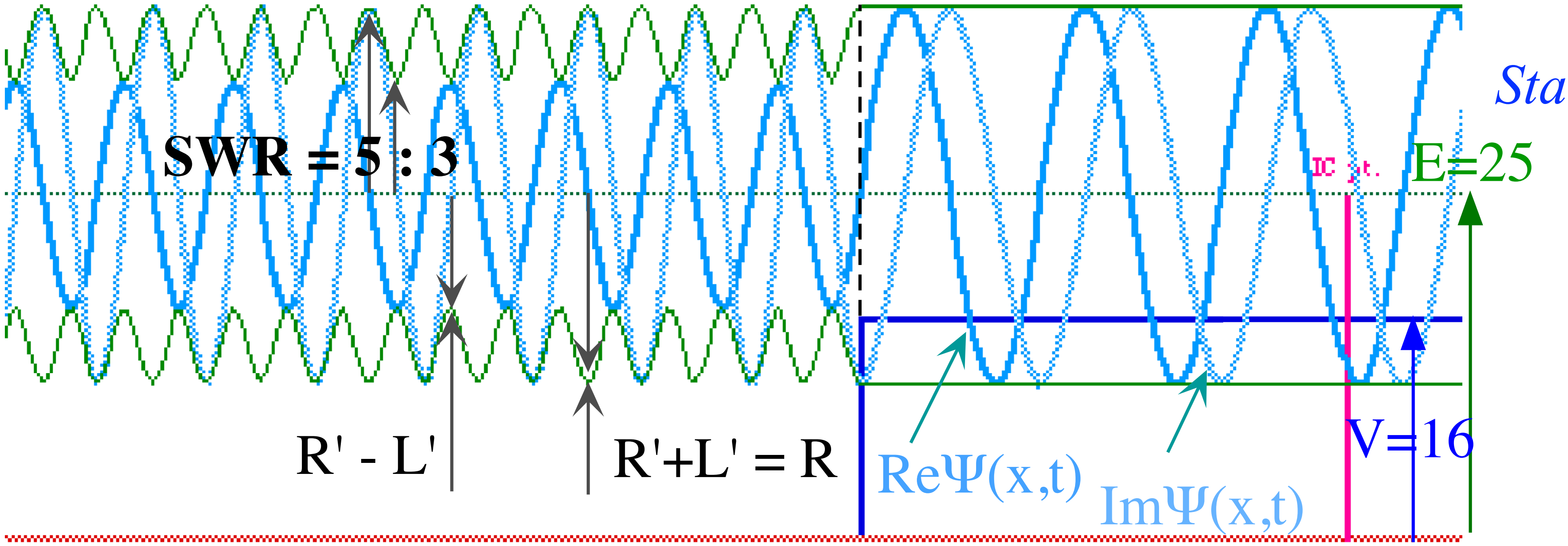
$$\begin{pmatrix} R' \\ L' \end{pmatrix} = \begin{pmatrix} \left(1+\frac{k}{k'}\right)\frac{e^{i(k-k')a}}{2} & \left(1-\frac{k}{k'}\right)\frac{e^{-i(k+k')a}}{2} \\ \left(1-\frac{k}{k'}\right)\frac{e^{i(k+k')a}}{2} & \left(1+\frac{k}{k'}\right)\frac{e^{i(k'-k)a}}{2} \end{pmatrix} \begin{pmatrix} R \\ 0 \end{pmatrix} = \begin{pmatrix} R\left(1+\frac{k}{k'}\right)\frac{e^{i(k-k')a}}{2} \\ R\left(1-\frac{k}{k'}\right)\frac{e^{i(k+k')a}}{2} \end{pmatrix}$$

This gives *transmitted or output amplitude R* and *reflected amplitude L'* given an *input amplitude R'*.

$$R = \frac{2k'}{\left(k+k'\right)} R' e^{i\left(k'-k\right)a} \quad , \qquad L' = \frac{\left(k'-k\right)}{\left(k+k'\right)} R' e^{2ik'a}$$

The *transmission coefficient $T_{transmit}$* and *reflection coefficient $T_{reflect}$* (for $a=0$)

$$T_{transmit} = \frac{\left|R\right|^2}{\left|R'\right|^2} = \frac{4\left|k'\right|^2}{\left|k+k'\right|^2} \ , \qquad T_{reflect} = \frac{\left|L'\right|^2}{\left|R'\right|^2} = \frac{\left|k'-k\right|^2}{\left|k'+k\right|^2} \qquad SWR = \frac{L'-R'}{L'+R'} = \frac{\frac{2kR'}{k+k'}}{\frac{2k'R'}{k+k'}} = \frac{k}{k'} = \frac{\sqrt{E-V}}{\sqrt{E}}$$



Standing Wave Ratio (SWR)

Breaking C_N cyclic coupling into linear chains

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→ *Breaking C_{2N} cyclic coupling down to C_N symmetry* ←

Acoustical modes vs. Optical modes

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Avoided crossing view of band-gaps

Finally! Symmetry groups that are not just C_N

The "4-Group(s)" D_2 and C_{2v}

Spectral decomposition of D_2

Some D_2 modes

Outer product properties and the Crystal-Point Symmetry Group Zoo

Polygonal geometry of $U(2) \supset C_N$ character spectral function $\chi^j(2\pi/n) = \frac{\sin(\pi(2j+1)/n)}{\sin(\pi/n)}$

Algebra

Geometry

C_{24} lattice reduced to C_{12} symmetry

Fig. 2.7.6 PSDS

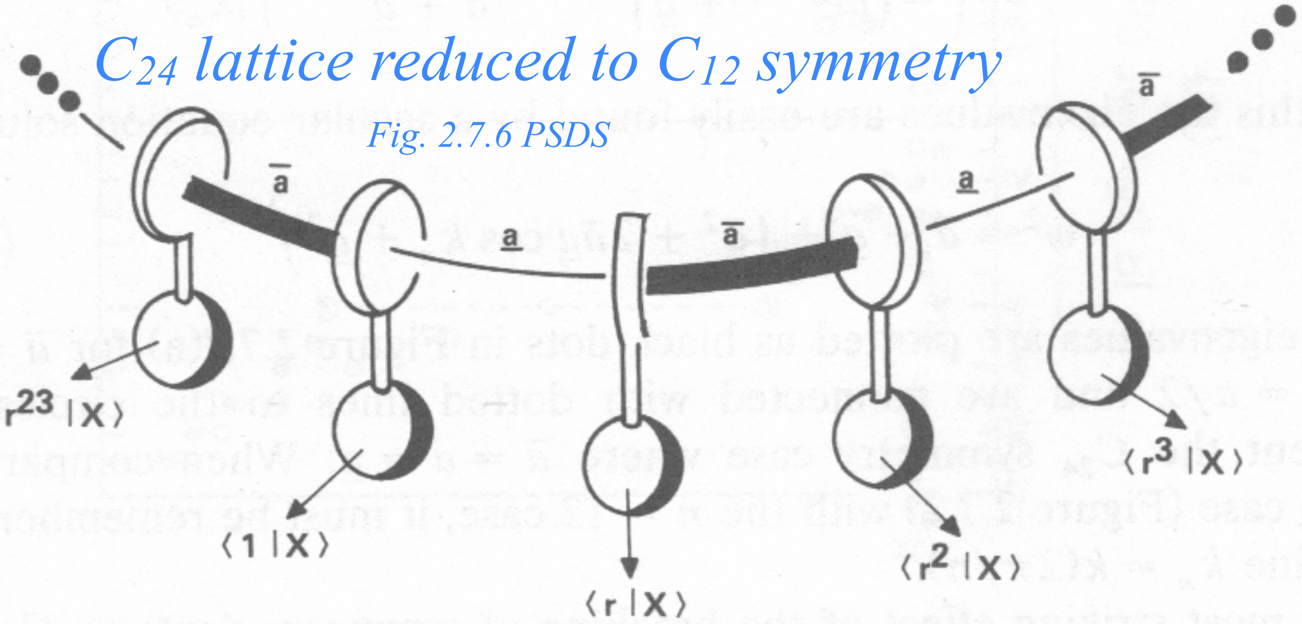


Fig. 2.7.6 PrinciplesSymmetryDynamics&Spectroscopy

C₂₄ lattice reduced to C₁₂ symmetry

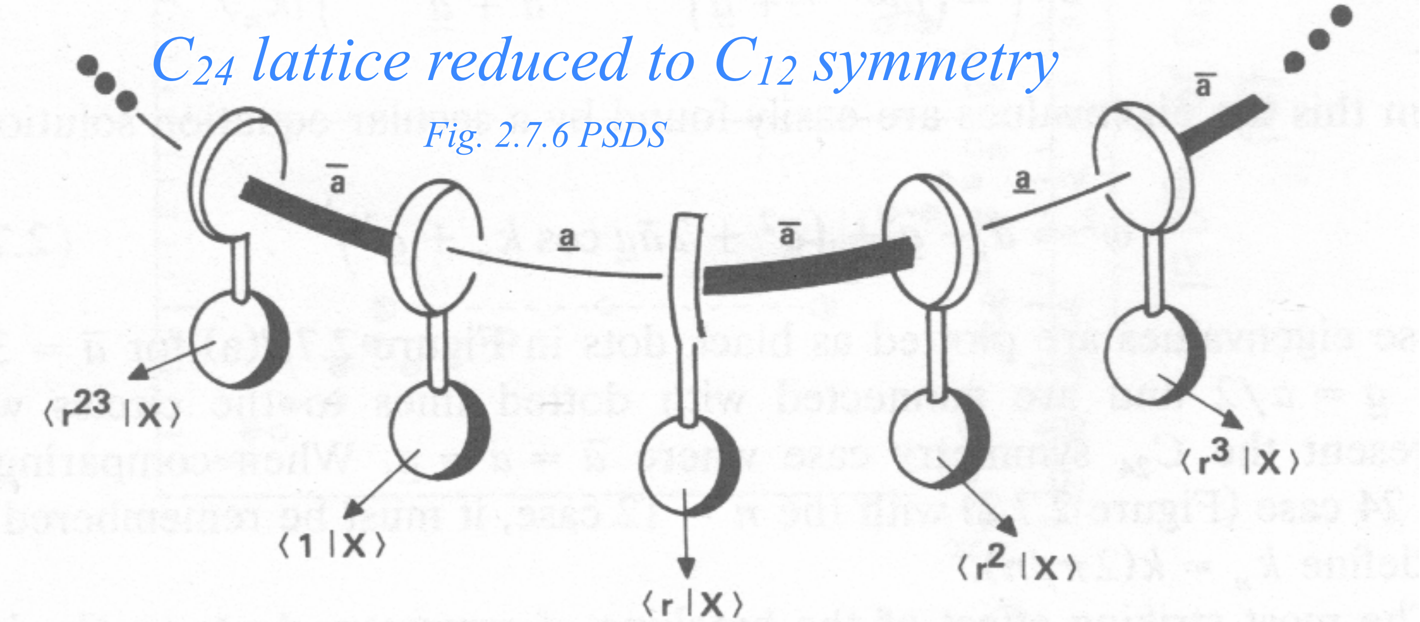


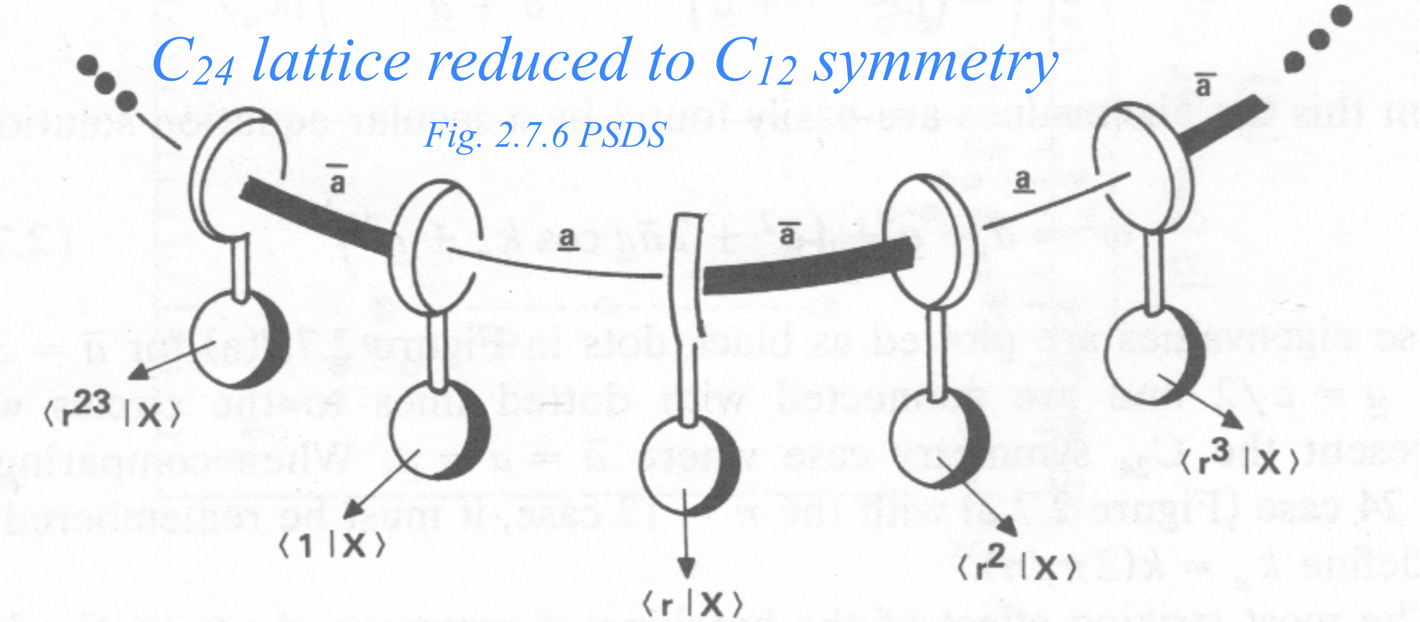
Fig. 2.7.6 PSDS

$$\begin{pmatrix} \langle r^0|\mathbf{K}|r^0\rangle & \langle r^0|\mathbf{K}|r^1\rangle & \langle r^0|\mathbf{K}|r^2\rangle & \ldots & \langle r^0|\mathbf{K}|r^{-1}\rangle \\ \langle r^1|\mathbf{K}|r^0\rangle & \langle r^1|\mathbf{K}|r^1\rangle & \langle r^1|\mathbf{K}|r^2\rangle & \ldots & \langle r^1|\mathbf{K}|r^{-1}\rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$= \begin{pmatrix} \underline{a}+\bar{a} & -\underline{a} & 0 & \ldots & -\bar{a} \\ -\underline{a} & \underline{a}+\bar{a} & -\bar{a} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Fig. 2.7.6 PrinciplesSymmetryDynamics&Spectroscopy

C_{24} lattice reduced to C_{12} symmetry



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Only C_{12} symmetry projectors commute with **K**-matrix if $\underline{a} \neq \bar{a}$. Then C_{24}-symmetry is broken !

C_{24} lattice reduced to C_{12} symmetry

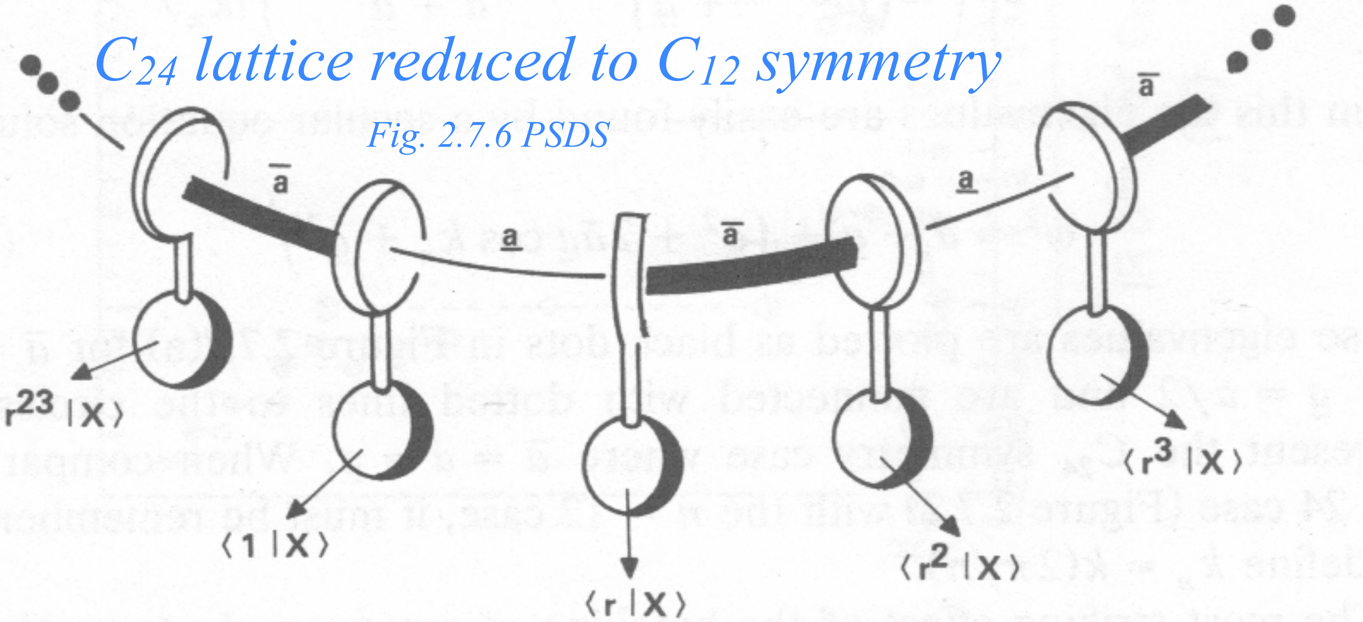

Fig. 2.7.6 PSDS

$$\begin{pmatrix} \langle r^0|\mathbf{K}|r^0\rangle & \langle r^0|\mathbf{K}|r^1\rangle & \langle r^0|\mathbf{K}|r^2\rangle & \dots & \langle r^0|\mathbf{K}|r^{-1}\rangle \\ \langle r^1|\mathbf{K}|r^0\rangle & \langle r^1|\mathbf{K}|r^1\rangle & \langle r^1|\mathbf{K}|r^2\rangle & \dots & \langle r^1|\mathbf{K}|r^{-1}\rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

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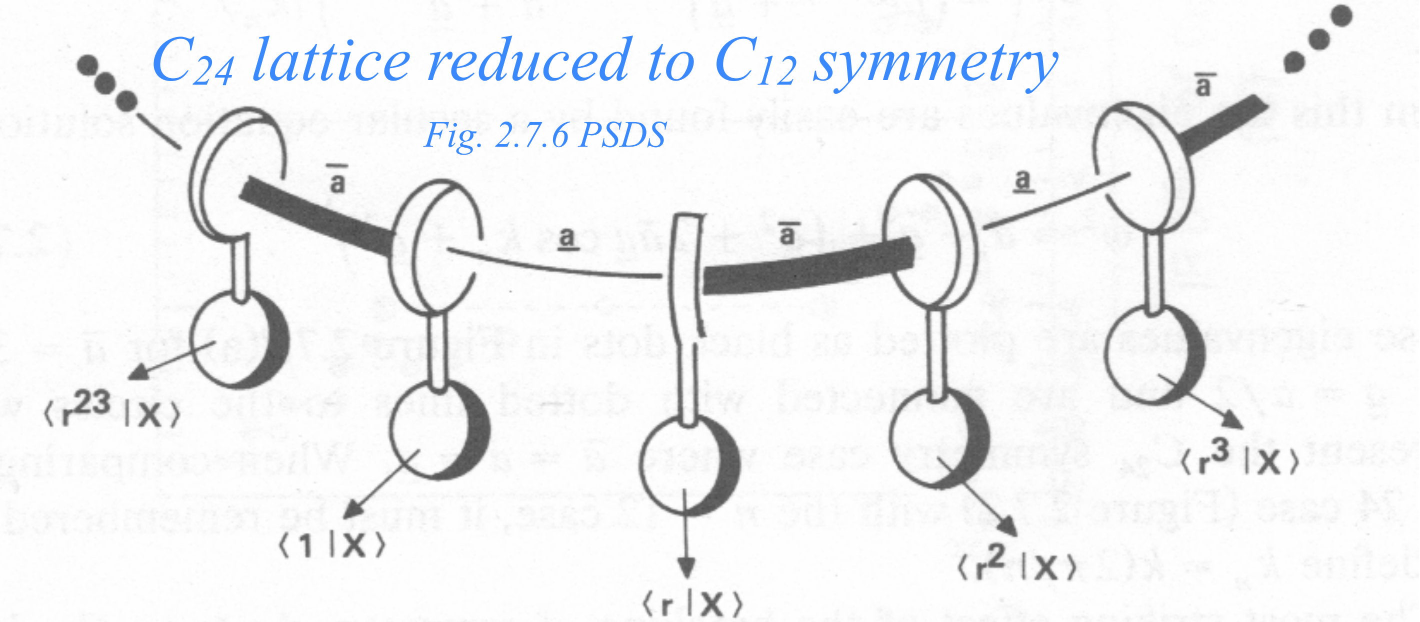
Only C_{12} symmetry projectors commute with **K***-matrix if* $\underline{a} \neq \bar{a}$. *Then C_{24}-symmetry is broken !*

$$\mathbf{P}^{(m)} = \frac{1}{12}\left(\mathbf{1} + e^{-ik_m}\mathbf{r}^{2} + e^{-2ik_m}\mathbf{r}^{4} + e^{-3ik_m}\mathbf{r}^{6} + \dots + e^{+2ik_m}\mathbf{r}^{-4} + e^{+ik_m}\mathbf{r}^{-2}\right) \quad \text{where:} \quad k_m = \frac{2\pi m}{12}$$

C₂₄ lattice reduced to C₁₂ symmetry



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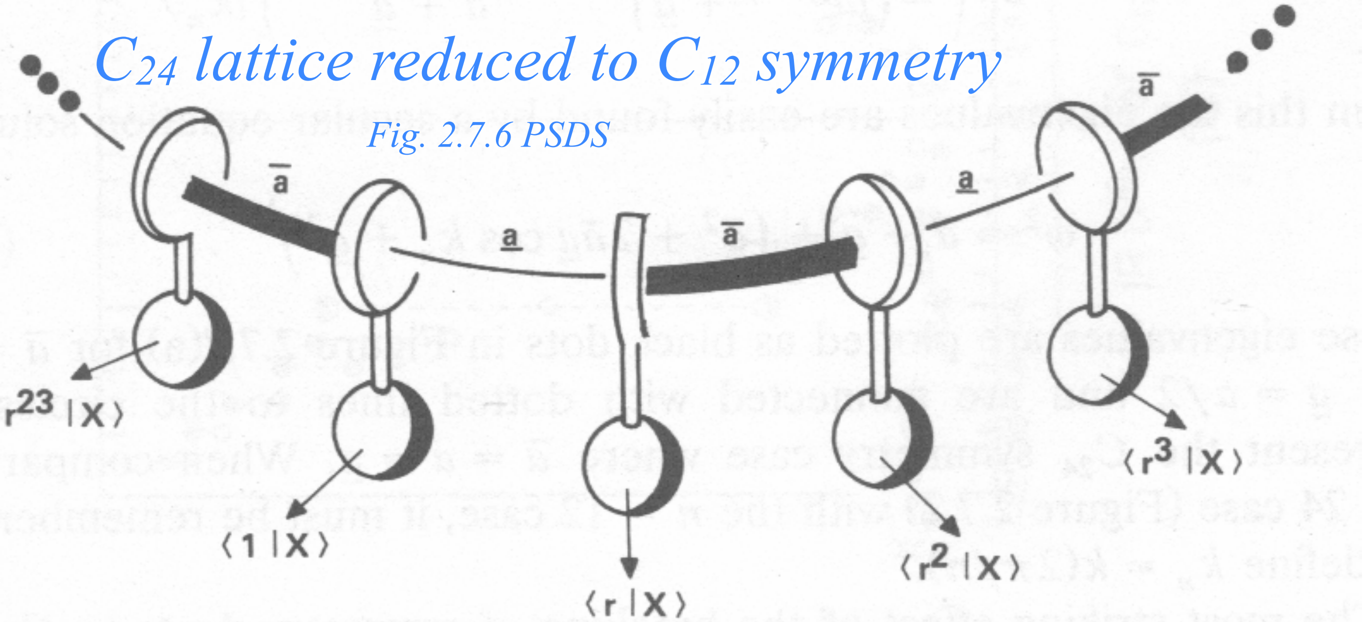
Two kinds of C₁₂ symmetry m-states are coupled by **K**-*matrix.*

C_{24} lattice reduced to C_{12} symmetry



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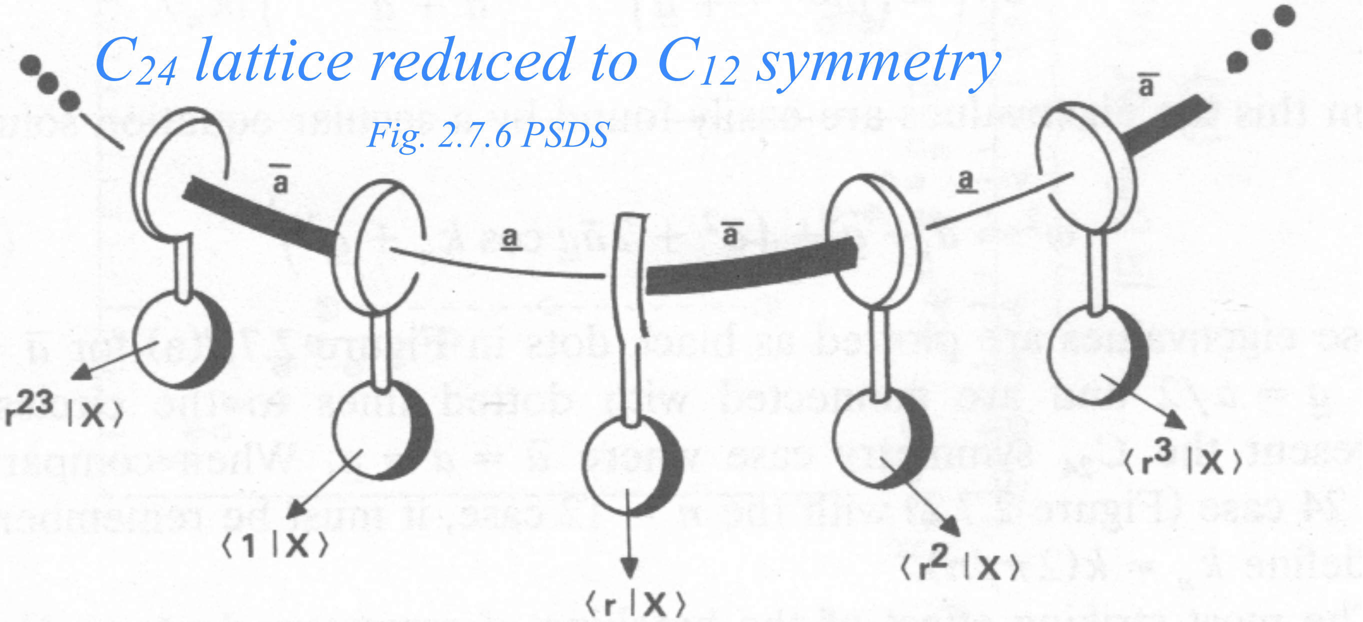
C_{24} lattice reduced to C_{12} symmetry



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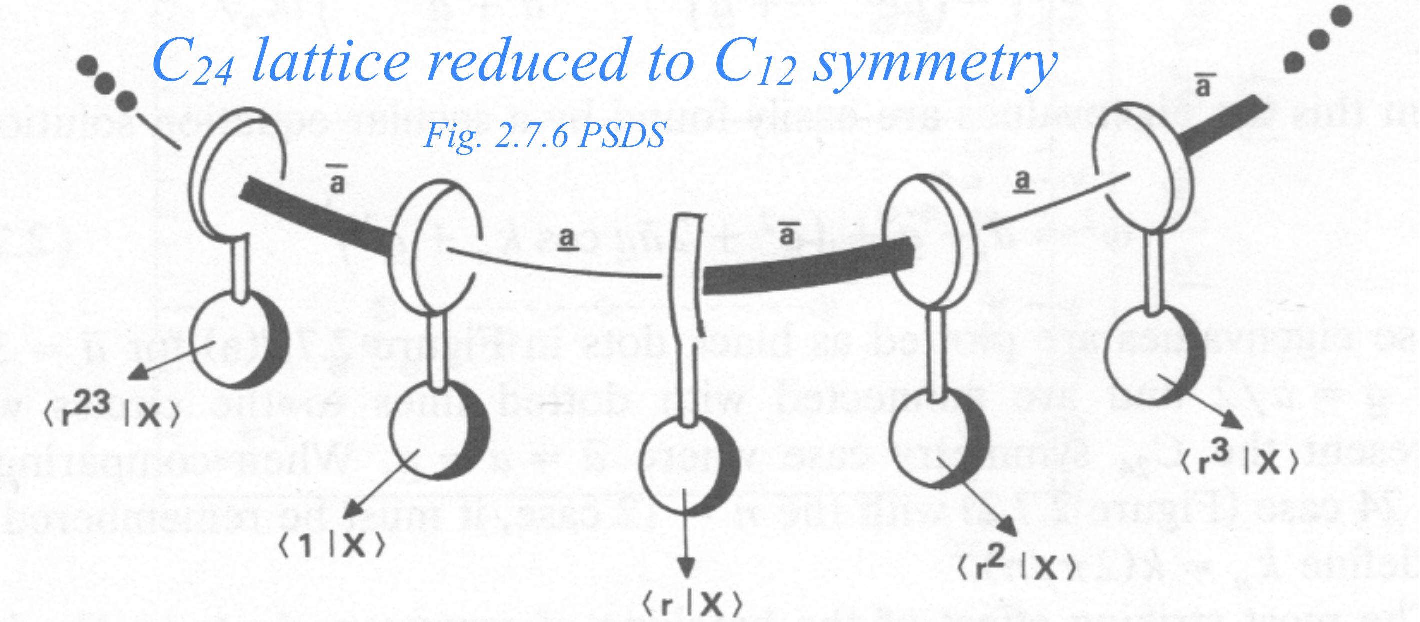
$$\begin{aligned} \langle k_m|\mathbf{K}|k_m\rangle &= \langle r^0|\mathbf{P}^{(m)}\mathbf{K}\mathbf{P}^{(m)}|r^0\rangle\cdot 12 = \langle r^0|\mathbf{K}\mathbf{P}^{(m)}|r^0\rangle\cdot 12 \\ &= \langle r^0|\mathbf{K}|r^0\rangle + e^{-ik_m}\langle r^0|\mathbf{K}|r^2\rangle + e^{-2ik_m}\langle r^0|\mathbf{K}|r^4\rangle + \dots \\ &= \underline{a}+\bar{a} \quad + \quad 0 \quad + \quad 0 \quad + \dots \end{aligned}$$

C_{24} lattice reduced to C_{12} symmetry



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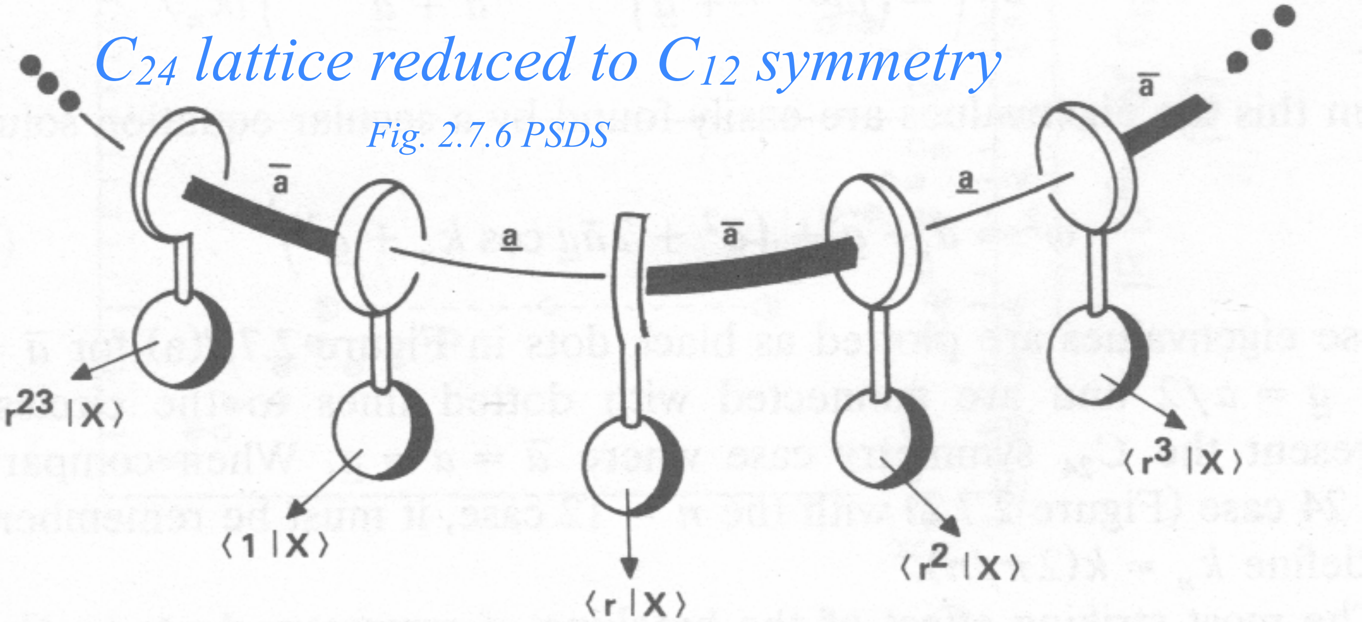
C_{24} lattice reduced to C_{12} symmetry

Fig. 2.7.6 PSDS



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$$= \begin{pmatrix} \underline{a}+\bar{a} & -(\underline{a}+e^{+ik_m}\bar{a}) \\ -(\underline{a}+e^{-ik_m}\bar{a}) & \underline{a}+\bar{a} \end{pmatrix}$$

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Breaking C_N cyclic coupling into linear chains
- *Review of 1D-Bohr-ring related to infinite square well*
- *Breaking C_{2N+2} to approximate linear N-chain (Examples $C_2 \rightleftarrows C_6 \rightleftarrows C_{14}$)*
  - *Band-It simulation: Intro to scattering approach to quantum symmetry*
    - *How Band-It works: Match each Ψ and DΨ, Let L=0 at Right end*

Breaking C_{2N} cyclic coupling down to C_N symmetry
- *Acoustical modes vs. Optical modes*
- *Intro to other examples of band theory*
- *Avoided crossing view of band-gaps*

Finally! Symmetry groups that are not just C_N
- *The "4-Group(s)" D_2 and C_{2v}*
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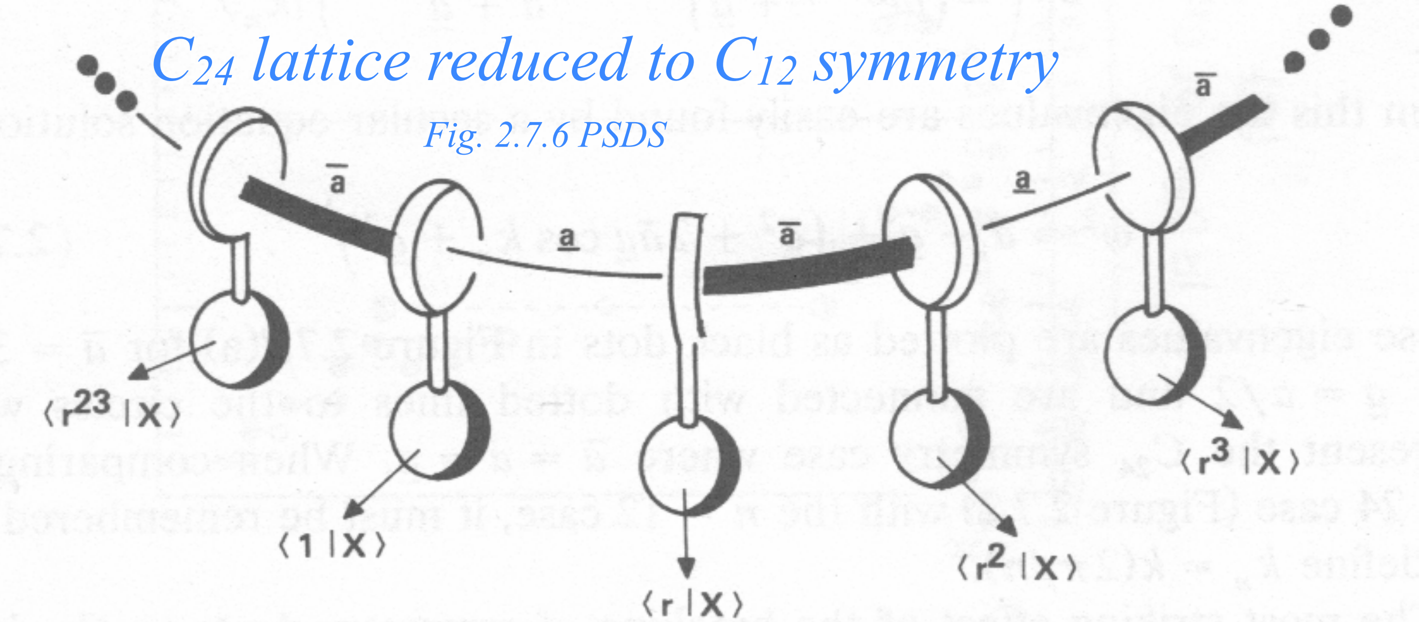
  - *Algebra*
    - *Geometry*

C_{24} lattice reduced to C_{12} symmetry



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Two kinds of C_{12} symmetry m-states are coupled by **K**-*matrix: Even $|r^{even}\rangle$ and odd $|r^{odd}\rangle$ p-points.*

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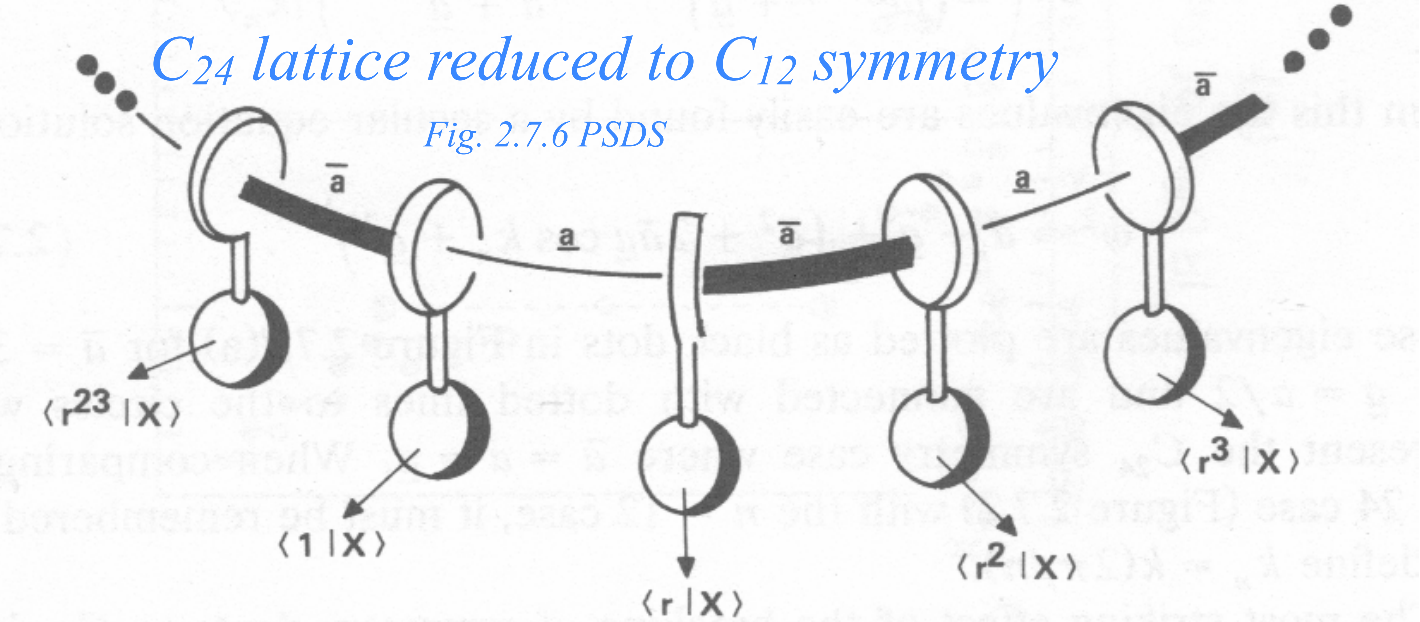

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$$\mathbf{P}^{(m)} = \frac{1}{12}\left(\mathbf{1} + e^{-ik_m}\mathbf{r}^{2} + e^{-2ik_m}\mathbf{r}^{4} + e^{-3ik_m}\mathbf{r}^{6} + \dots + e^{+2ik_m}\mathbf{r}^{-4} + e^{+ik_m}\mathbf{r}^{-2}\right) \quad \text{where:} \quad k_m = \frac{2\pi m}{12}$$

Two kinds of C_{12} symmetry m-states are coupled by **K***-matrix: Even $|r^{even}\rangle$ and odd $|r^{odd}\rangle$ p-points.*

$$|k_m\rangle = \mathbf{P}^{(m)}|r^0\rangle\cdot\sqrt{12} = \left(|r^0\rangle + e^{-ik_m}|r^2\rangle + e^{-2ik_m}|r^4\rangle + \dots\right)/\sqrt{12} \qquad |k'_m\rangle = \mathbf{P}^{(m)}|r^1\rangle\cdot\sqrt{12} = \left(|r^1\rangle + e^{-ik_m}|r^3\rangle + e^{-2ik_m}|r^5\rangle + \dots\right)/\sqrt{12}$$

$$\langle\mathbf{K}\rangle^{k_m} = \begin{pmatrix} \langle k_m|\mathbf{K}|k_m\rangle & \langle k_m|\mathbf{K}|k'_m\rangle \\ \langle k'_m|\mathbf{K}|k_m\rangle & \langle k'_m|\mathbf{K}|k'_m\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \underline{a}+\bar{a} & -(\underline{a}+e^{+ik_m}\bar{a}) \\ -(\underline{a}+e^{-ik_m}\bar{a}) & \underline{a}+\bar{a} \end{pmatrix}$$

Secular Eq.:

$$0 = \kappa^2 - \ Tr\langle\mathbf{K}\rangle^{k_m} + \ Det\langle\mathbf{K}\rangle^{k_m}$$

$$0 = \kappa^2 - 2(\underline{a}+\bar{a})\kappa + (\underline{a}+\bar{a})^2 - (\underline{a}+e^{+ik_m}\bar{a})(\underline{a}+e^{-ik_m}\bar{a})$$

$$0 = \kappa^2 - 2(\underline{a}+\bar{a})\kappa + (\underline{a}+\bar{a})^2 - \underline{a}^2 - \bar{a}^2 - 2\bar{a}\underline{a}\cos k_m$$

$$0 = \kappa^2 - 2(\underline{a}+\bar{a})\kappa + 2\bar{a}\underline{a}(1-\cos k_m)$$

C_{24} lattice reduced to C_{12} symmetry

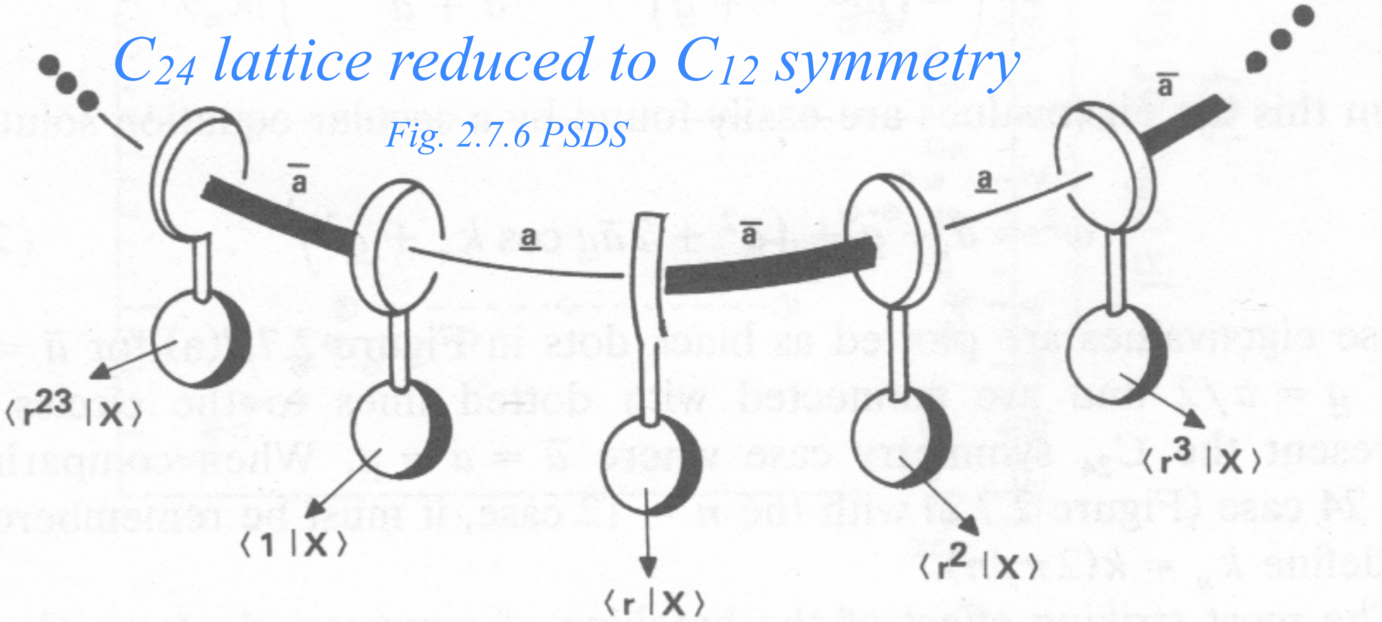

Fig. 2.7.6 PSDS

$$\begin{pmatrix} \langle r^0|\mathbf{K}|r^0\rangle & \langle r^0|\mathbf{K}|r^1\rangle & \langle r^0|\mathbf{K}|r^2\rangle & \dots & \langle r^0|\mathbf{K}|r^{-1}\rangle \\ \langle r^1|\mathbf{K}|r^0\rangle & \langle r^1|\mathbf{K}|r^1\rangle & \langle r^1|\mathbf{K}|r^2\rangle & \dots & \langle r^1|\mathbf{K}|r^{-1}\rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$= \begin{pmatrix} \underline{a}+\bar{a} & -\underline{a} & 0 & \dots & -\bar{a} \\ -\underline{a} & \underline{a}+\bar{a} & -\bar{a} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Only C_{12} symmetry projectors commute with **K***-matrix if* $\underline{a} \neq \bar{a}$. *Then C_{24}-symmetry is broken !*

$$\mathbf{P}^{(m)} = \frac{1}{12}\left(\mathbf{1} + e^{-ik_m}\mathbf{r}^{2} + e^{-2ik_m}\mathbf{r}^{4} + e^{-3ik_m}\mathbf{r}^{6} + \dots + e^{+2ik_m}\mathbf{r}^{-4} + e^{+ik_m}\mathbf{r}^{-2}\right) \quad \text{where:} \quad k_m = \frac{2\pi m}{12}$$

Two kinds of C_{12} symmetry ***m****-states are coupled by* **K***-matrix: Even $|r^{even}\rangle$ and odd $|r^{odd}\rangle$ p-points.*

$$|k_m\rangle = \mathbf{P}^{(m)}|r^0\rangle\cdot\sqrt{12} = \left(|r^0\rangle + e^{-ik_m}|r^2\rangle + e^{-2ik_m}|r^4\rangle + \dots\right)/\sqrt{12}$$

$$|k'_m\rangle = \mathbf{P}^{(m)}|r^1\rangle\cdot\sqrt{12} = \left(|r^1\rangle + e^{-ik_m}|r^3\rangle + e^{-2ik_m}|r^5\rangle + \dots\right)/\sqrt{12}$$

$$\langle\mathbf{K}\rangle^{k_m} = \begin{pmatrix} \langle k_m|\mathbf{K}|k_m\rangle & \langle k_m|\mathbf{K}|k'_m\rangle \\ \langle k'_m|\mathbf{K}|k_m\rangle & \langle k'_m|\mathbf{K}|k'_m\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \underline{a}+\bar{a} & -(\underline{a}+e^{+ik_m}\bar{a}) \\ -(\underline{a}+e^{-ik_m}\bar{a}) & \underline{a}+\bar{a} \end{pmatrix}$$

Secular Eq.:

$$0 = \kappa^2 - \;\; Tr\langle\mathbf{K}\rangle^{k_m} + \qquad Det\langle\mathbf{K}\rangle^{k_m}$$

$$0 = \kappa^2 - 2(\underline{a}+\bar{a})\kappa + (\underline{a}+\bar{a})^2 - (\underline{a}+e^{+ik_m}\bar{a})(\underline{a}+e^{-ik_m}\bar{a})$$

$$0 = \kappa^2 - 2(\underline{a}+\bar{a})\kappa + (\underline{a}+\bar{a})^2 - \underline{a}^2 - \bar{a}^2 - 2\bar{a}\underline{a}\cos k_m$$

$$0 = \kappa^2 - 2(\underline{a}+\bar{a})\kappa + 2\bar{a}\underline{a}(1-\cos k_m)$$

Eigenvalues:

$$\kappa = \omega_{k_m}^2 = \underline{a}+\bar{a} \pm \sqrt{\underline{a}^2 + 2\bar{a}\underline{a}\cos k_m + \bar{a}^2}$$

C_{24} lattice reduced to C_{12} symmetry

Fig. 2.7.7 PSDS

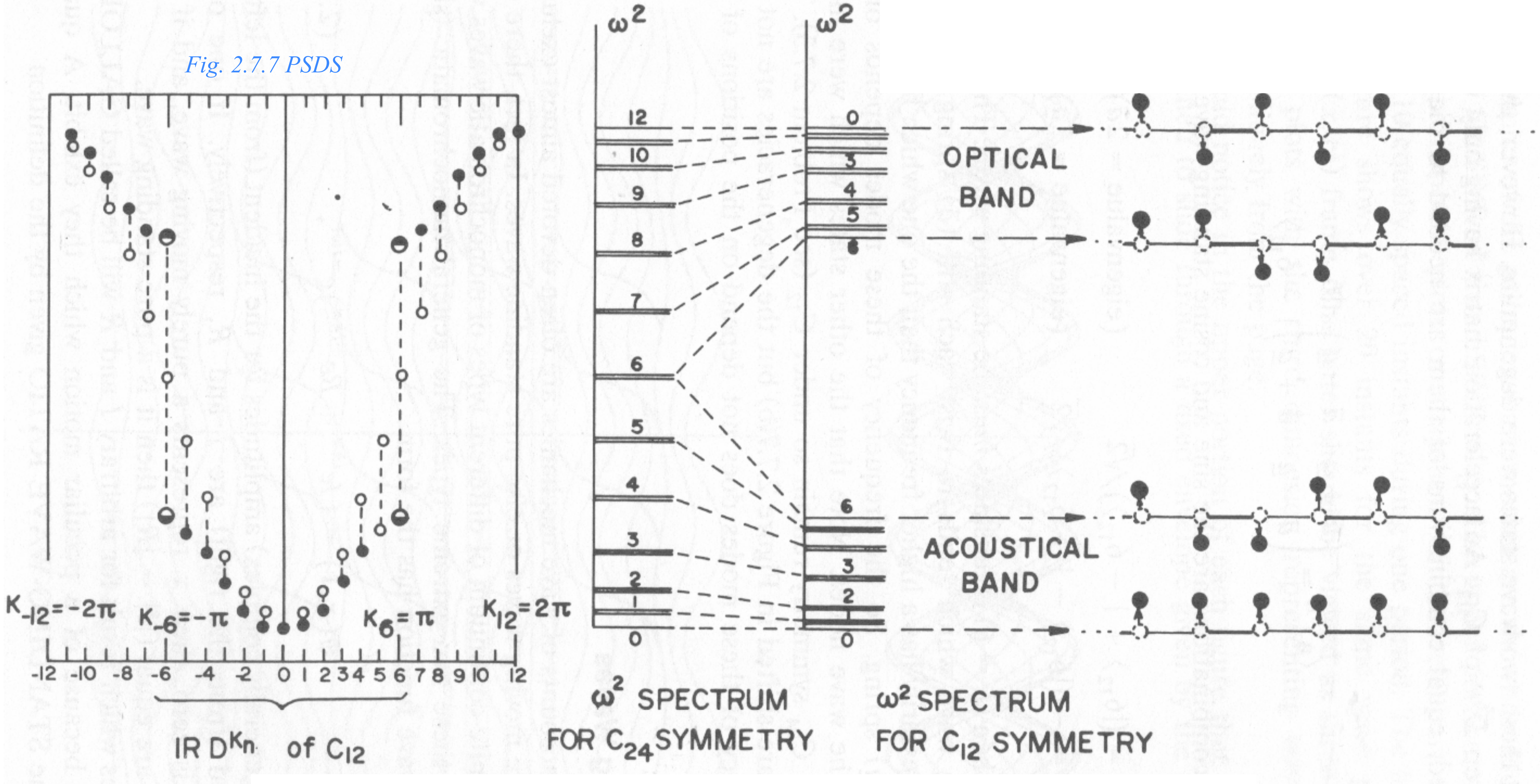

Figure 2.7.7 Band splitting due to C_{24}–C_{12} symmetry breaking.

Eigenvalues:

$$\kappa = \omega^2_{k_m} = \underline{a} + \bar{a} \pm \sqrt{\underline{a}^2 + 2\bar{a}\underline{a}\cos k_m + \bar{a}^2}$$

$$\langle \mathbf{K}\rangle^{k_m} = \begin{pmatrix} \langle k_m|\mathbf{K}|k_m\rangle & \langle k_m|\mathbf{K}|k'_m\rangle \\ \langle k'_m|\mathbf{K}|k_m\rangle & \langle k'_m|\mathbf{K}|k'_m\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \underline{a}+\bar{a} & -(\underline{a}+e^{+ik_m}\bar{a}) \\ -(\underline{a}+e^{-ik_m}\bar{a}) & \underline{a}+\bar{a} \end{pmatrix}$$

C_{24} lattice reduced to C_{12} symmetry

Fig. 2.7.7 PSDS

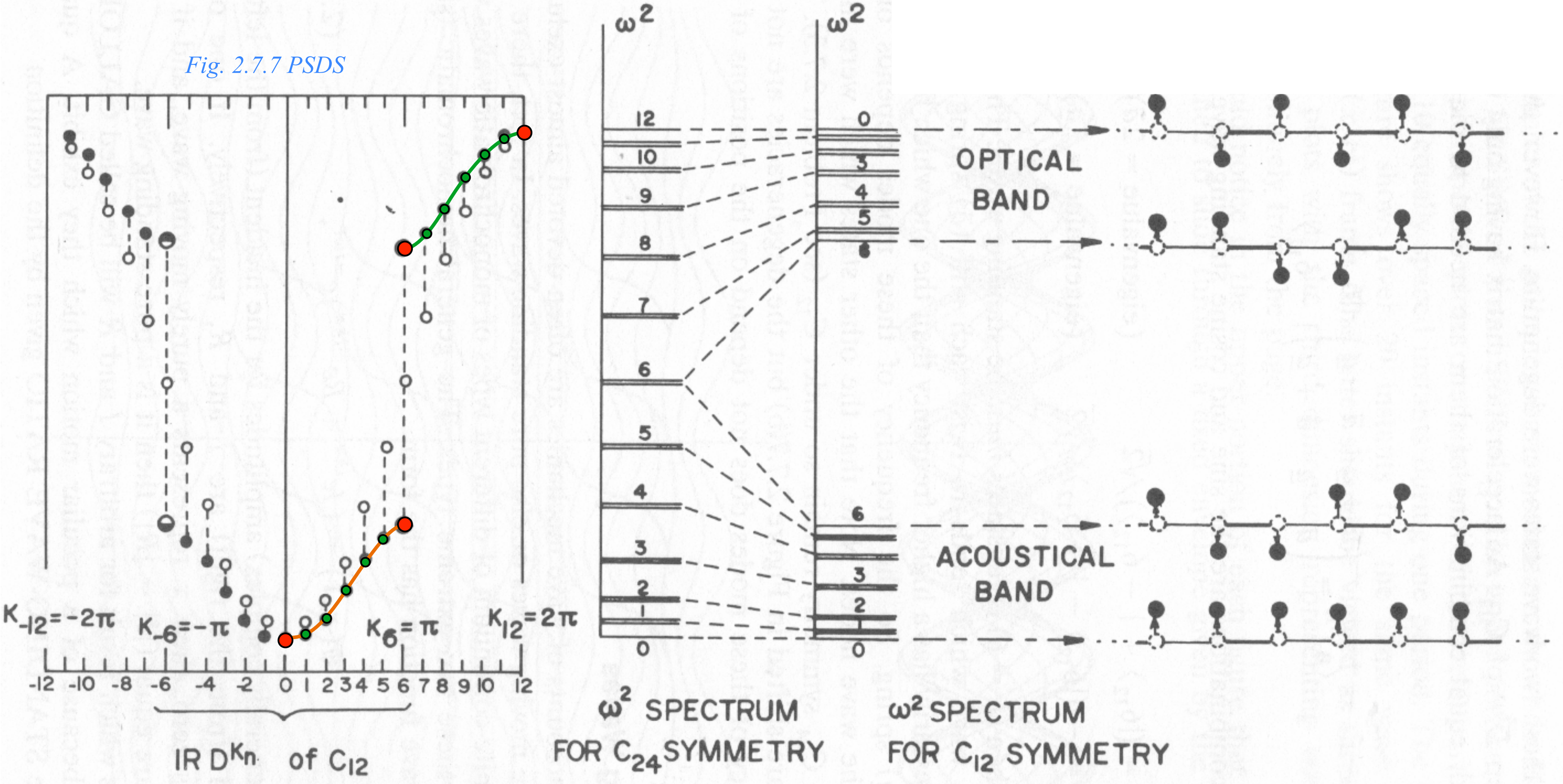

Figure 2.7.7 Band splitting due to C_{24}–C_{12} symmetry breaking.

Eigenvalues:

$$\kappa = \omega_{k_m}^2 = \underline{a} + \bar{a} \pm \sqrt{\underline{a}^2 + 2\bar{a}\underline{a}\cos k_m + \bar{a}^2}$$

$$\langle \mathbf{K} \rangle^{k_m} = \begin{pmatrix} \langle k_m | \mathbf{K} | k_m \rangle & \langle k_m | \mathbf{K} | k'_m \rangle \\ \langle k'_m | \mathbf{K} | k_m \rangle & \langle k'_m | \mathbf{K} | k'_m \rangle \end{pmatrix}$$

$$= \begin{pmatrix} \underline{a} + \bar{a} & -(\underline{a} + e^{+ik_m}\bar{a}) \\ -(\underline{a} + e^{-ik_m}\bar{a}) & \underline{a} + \bar{a} \end{pmatrix}$$

C_{24} lattice reduced to C_{12} symmetry

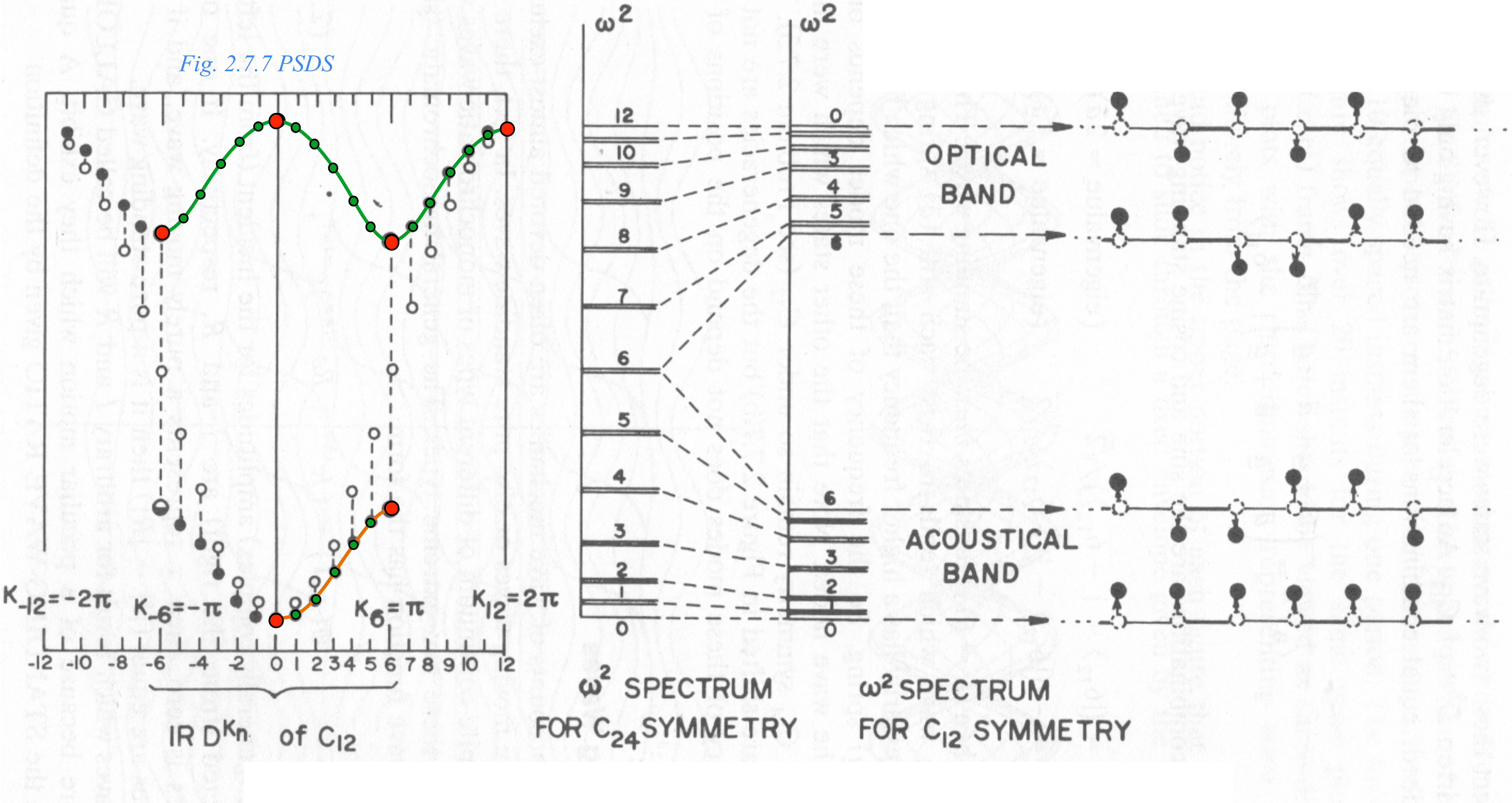

Figure 2.7.7 Band splitting due to C_{24}–C_{12} symmetry breaking.

Eigenvalues:

$$\kappa = \omega^2_{k_m} = \underline{a} + \bar{a} \pm \sqrt{\underline{a}^2 + 2\bar{a}\underline{a}\cos k_m + \bar{a}^2}$$

$$\langle \mathbf{K} \rangle^{k_m} = \begin{pmatrix} \langle k_m|\mathbf{K}|k_m\rangle & \langle k_m|\mathbf{K}|k'_m\rangle \\ \langle k'_m|\mathbf{K}|k_m\rangle & \langle k'_m|\mathbf{K}|k'_m\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \underline{a}+\bar{a} & -(\underline{a}+e^{+ik_m}\bar{a}) \\ -(\underline{a}+e^{-ik_m}\bar{a}) & \underline{a}+\bar{a} \end{pmatrix}$$

Breaking C_N cyclic coupling into linear chains
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→ *Intro to other examples of band theory* ←
Avoided crossing view of band-gaps

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Algebra
Geometry

Intro to other examples of band theory

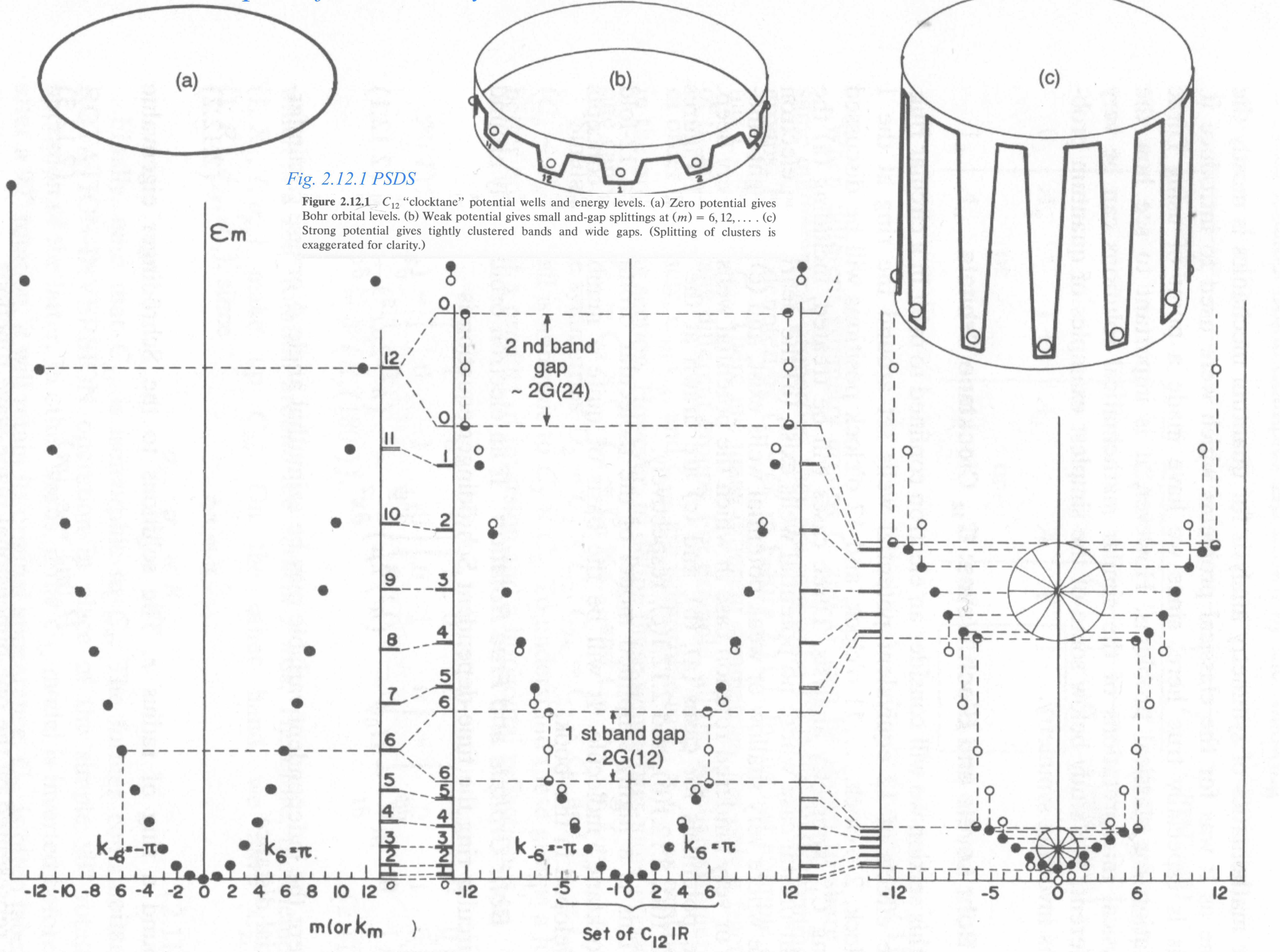


Fig. 2.12.1 PSDS

Figure 2.12.1 C_{12} "clocktane" potential wells and energy levels. (a) Zero potential gives Bohr orbital levels. (b) Weak potential gives small and-gap splittings at $(m) = 6, 12, \ldots$. (c) Strong potential gives tightly clustered bands and wide gaps. (Splitting of clusters is exaggerated for clarity.)

Intro to other examples of band theory

Crossing equations for a pair of humps

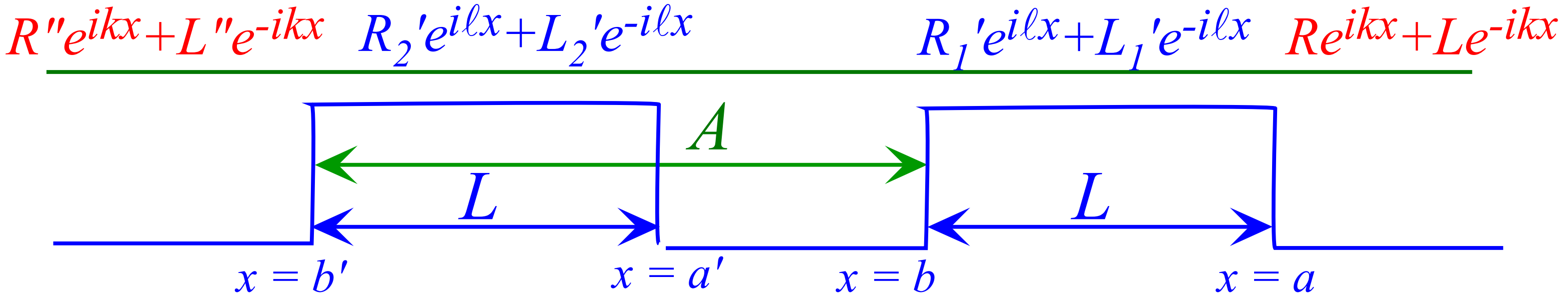


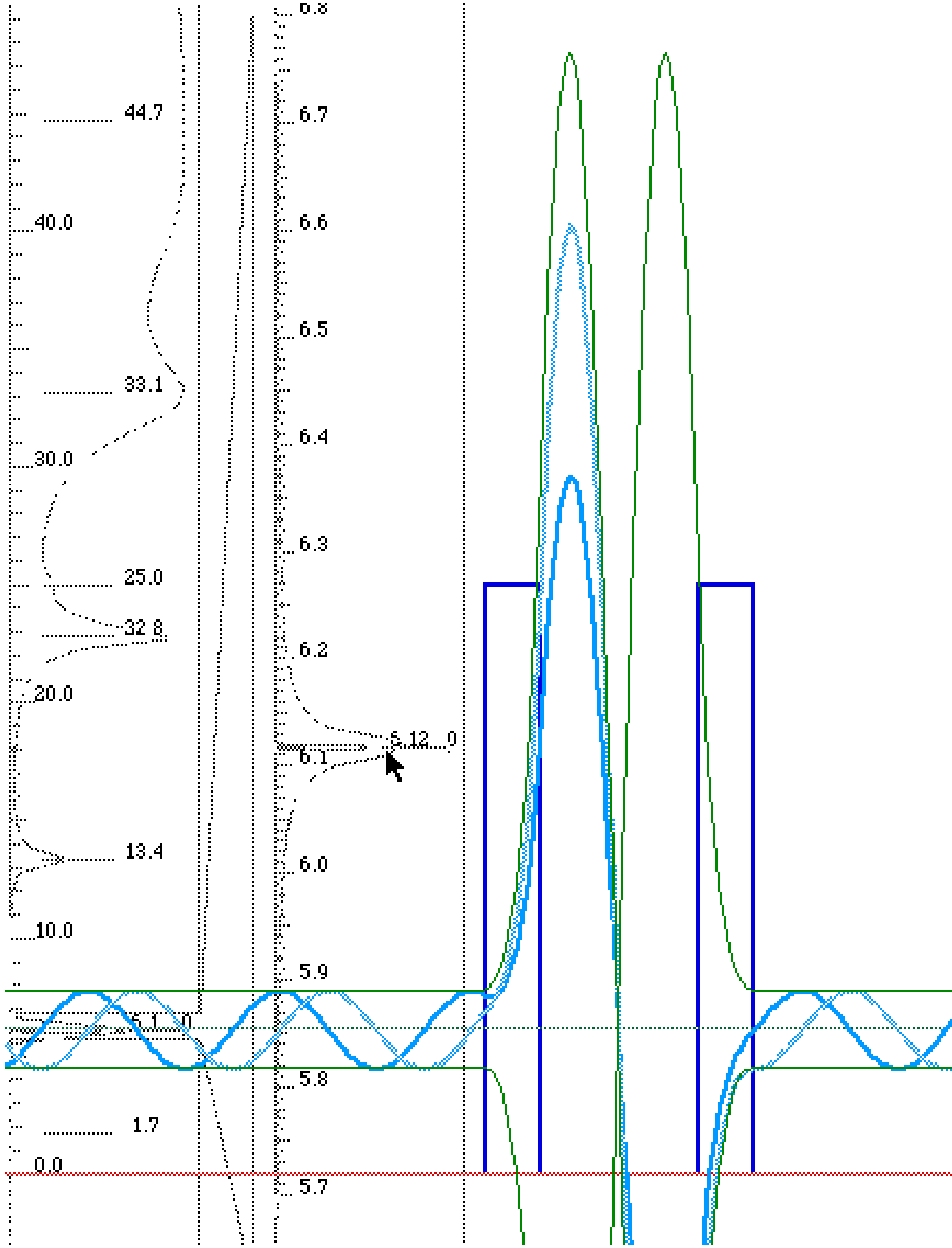
Fig. 14.1.5 C$_2$-symmetric double barrier .

$$\begin{pmatrix} R'' \\ L'' \end{pmatrix} = \begin{pmatrix} e^{i2kL}\chi^{*2} + e^{-i2kA}\xi^2 & -i\xi\left(e^{-i2kb}\chi^* + e^{-i2ka'}\chi\right) \\ i\xi\left(e^{i2kb}\chi + e^{i2ka'}\chi^*\right) & e^{-i2kL}\chi^2 + e^{i2kA}\xi^2 \end{pmatrix}\begin{pmatrix} R \\ L \end{pmatrix}$$

$\chi = \cosh \kappa L - i \sinh 2\beta \; \sinh \kappa L$, and: $\xi = \cosh 2\beta \; \sinh \kappa L$

$$\cosh 2\beta = \frac{1}{2}\left(\frac{\kappa}{k} + \frac{k}{\kappa}\right) = \frac{\kappa^2 + k^2}{2k\kappa}, \qquad \sinh 2\beta = \frac{1}{2}\left(\frac{\kappa}{k} - \frac{k}{\kappa}\right) = \frac{\kappa^2 - k^2}{2k\kappa}$$

Fig. 14.1.7 Second (E= 6.117) resonance in L=0.5 well between two width=0.5 barriers(V=25) .



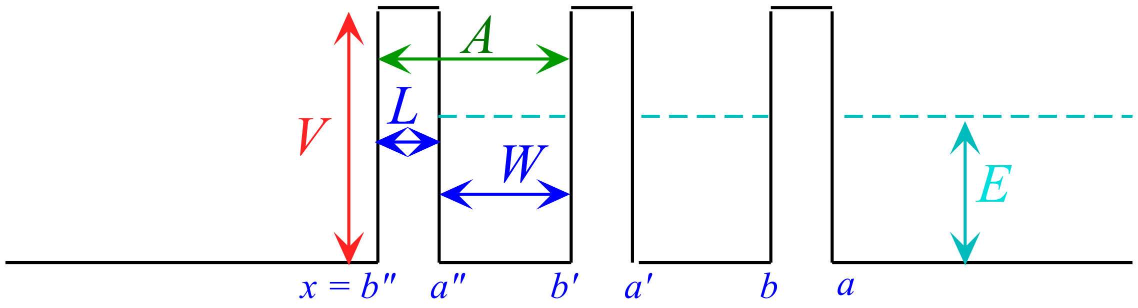
Intro to other examples of band theory

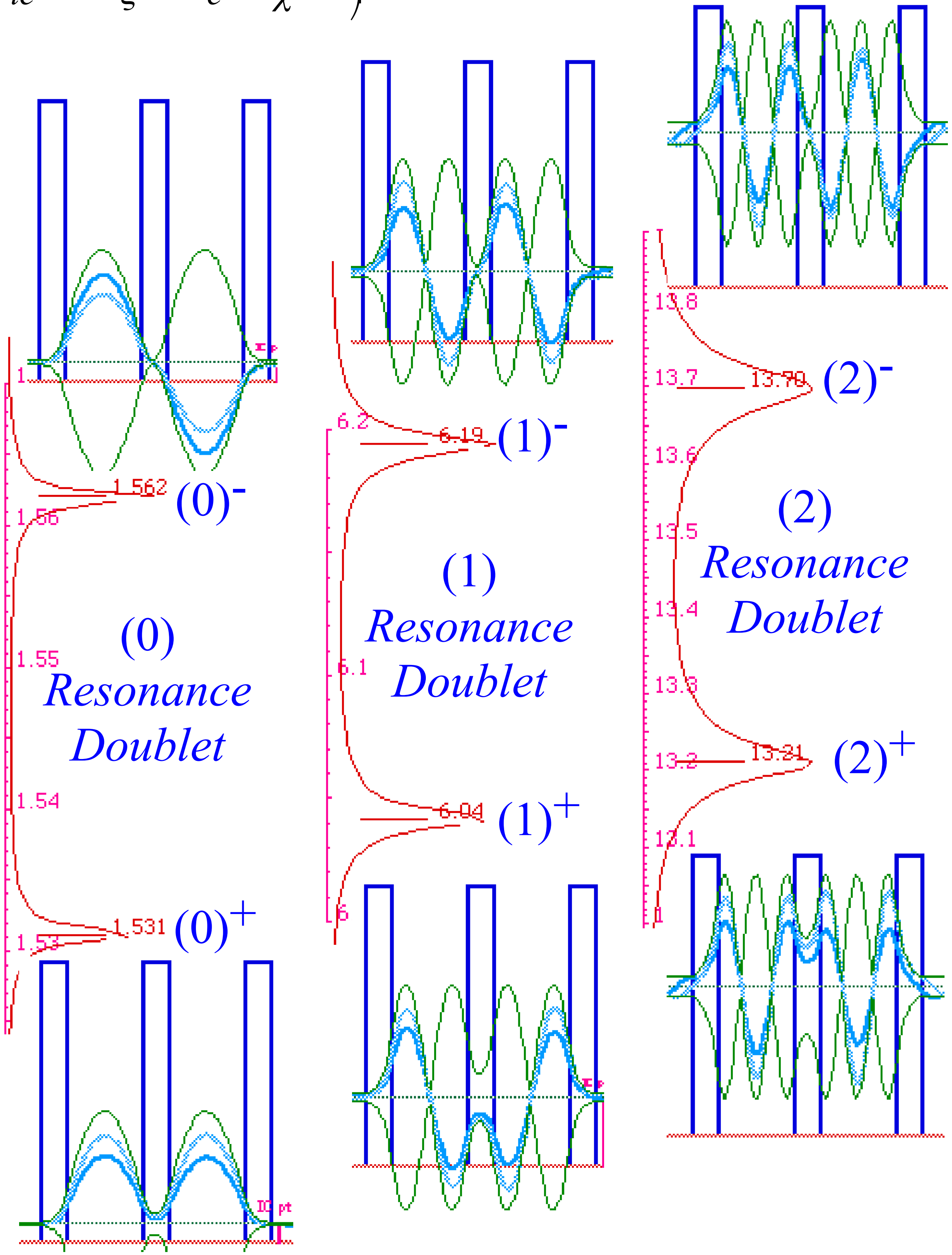
$$C^{3-barrier} = C'' \cdot C' \cdot C$$

$$= \begin{pmatrix} e^{ikL}\chi^* & -ie^{-ik(a''+b'')}\xi \\ ie^{ik(a''+b'')}\xi & e^{-ikL}\chi \end{pmatrix} \cdot \begin{pmatrix} e^{ikL}\chi^* & -ie^{-ik(a'+b')}\xi \\ ie^{ik(a'+b')}\xi & e^{-ikL}\chi \end{pmatrix} \cdot \begin{pmatrix} e^{ikL}\chi^* & -ie^{-ik(a+b)}\xi \\ ie^{ik(a+b)}\xi & e^{-ikL}\chi \end{pmatrix}$$

Crossing equations for three humps

Fig. 14.1.10 Triple-barrier double-well potential





Breaking C_N cyclic coupling into linear chains
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$$\chi^j(2\pi/n) = \frac{\sin(\pi(2j+1)/n)}{\sin(\pi/n)}$$

Algebra
Geometry

Intro to other examples of band theory

Bohr-It simulations assume ring-periodic-boundary conditions

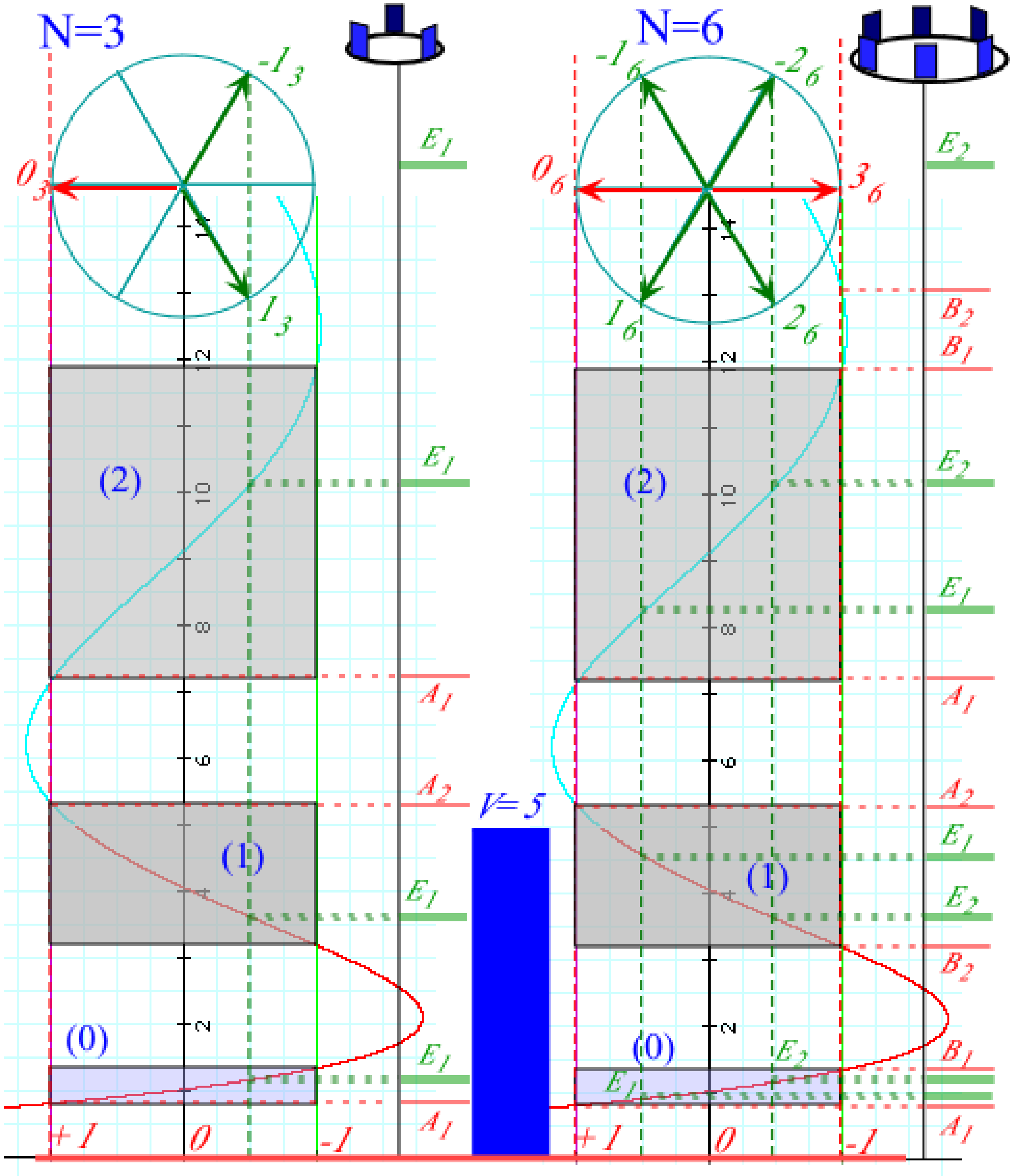

Fig. 14.2.8 Multiplets for V=5.
(W=15nm well ,L=5nm barrier) for (N=3)-ring and (N=6)-ring.

Intro to other examples of band theory

Bohr-It simulations assume ring-periodic-boundary conditions

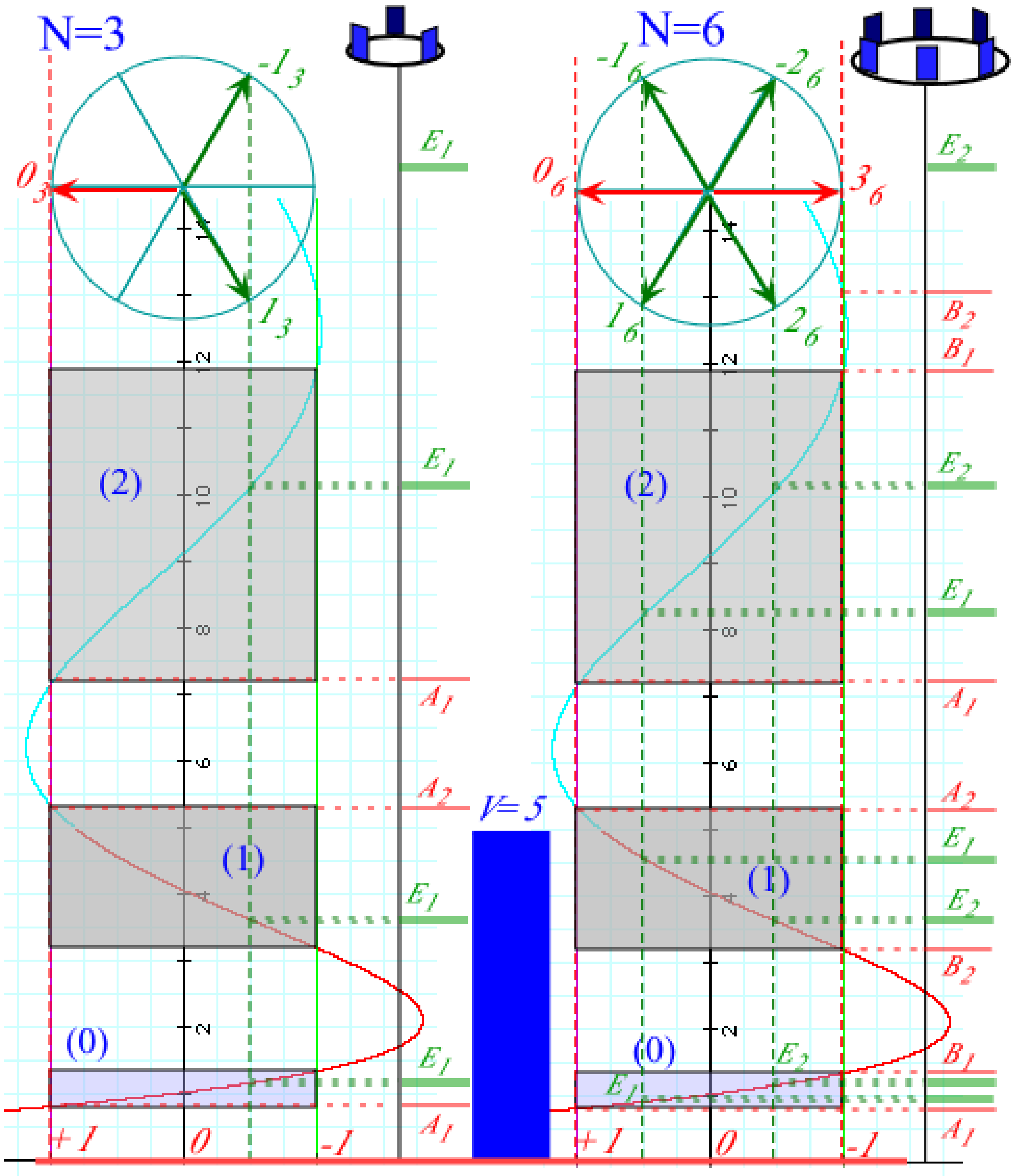

Fig. 14.2.8 Multiplets for V=5.
(W=15nm well ,L=5nm barrier) for (N=3)-ring and (N=6)-ring.

Band-It simulations line-non-periodic scattering conditions

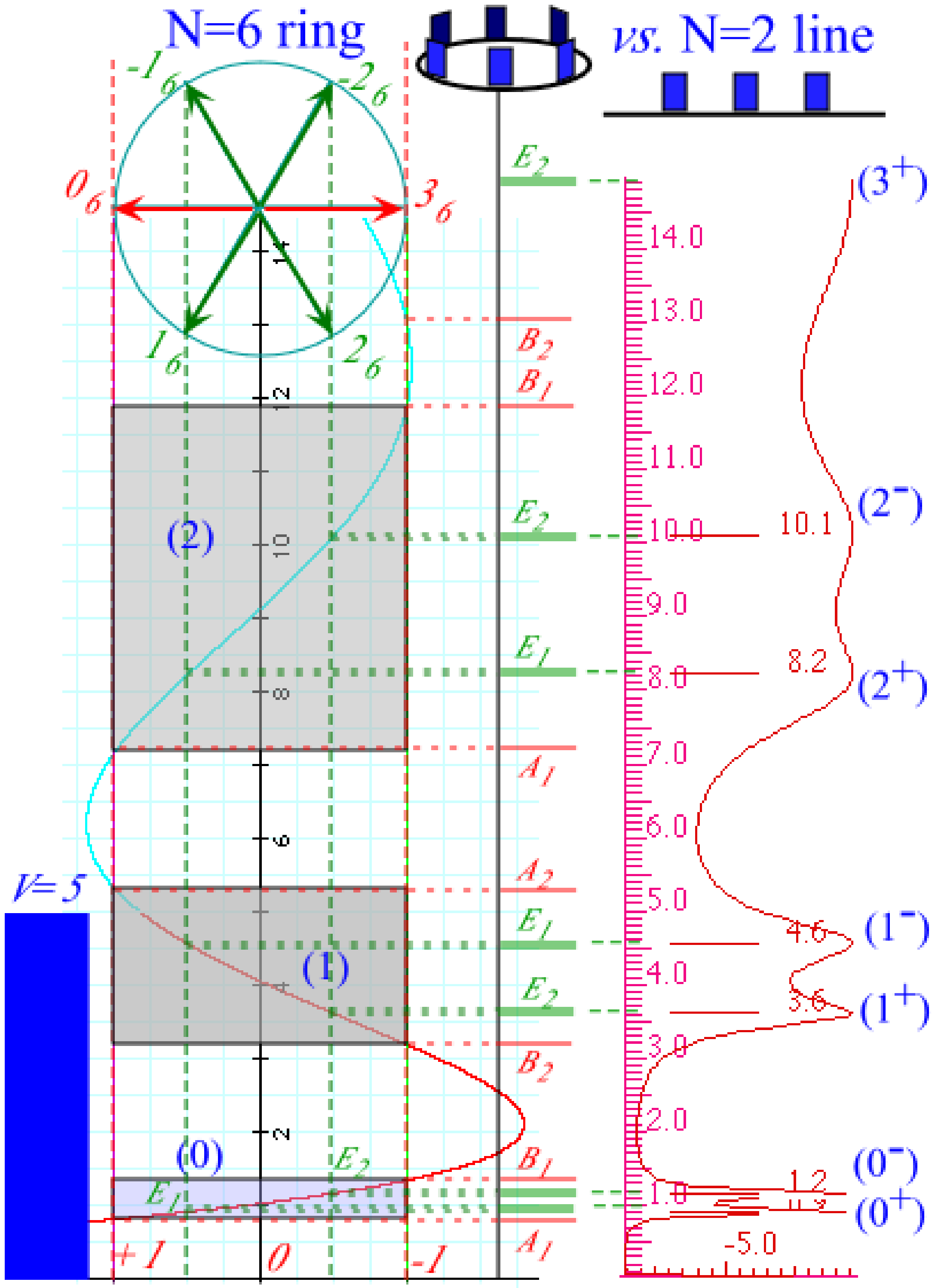

Fig. 14.2.9 (N=6)-ring and (N=2)-line potential.
(V=5, W=15nm well ,L=5nm barrier)

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Algebra
Geometry

Pure Type-B Hamiltonian

NH₃ (Ammonia)

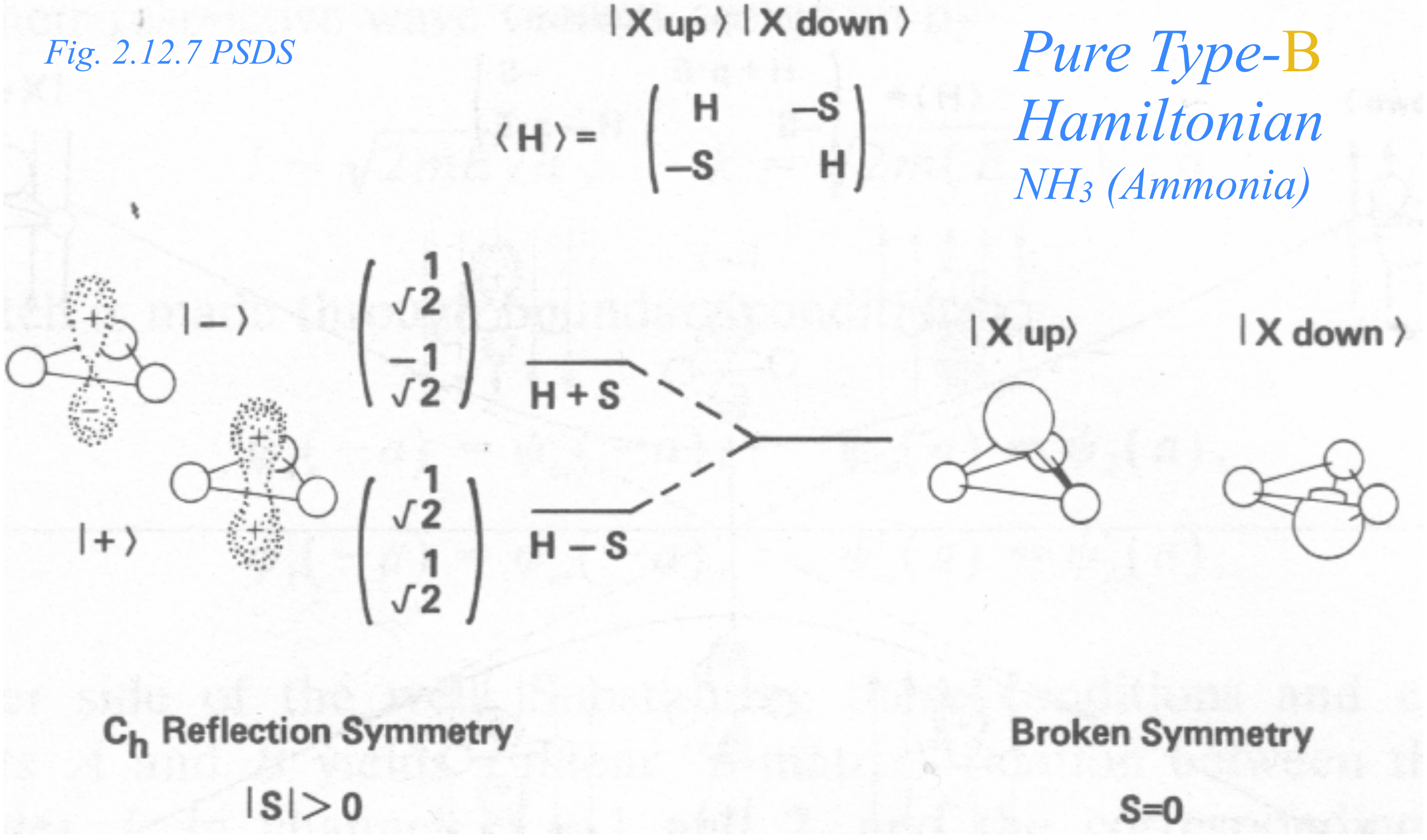
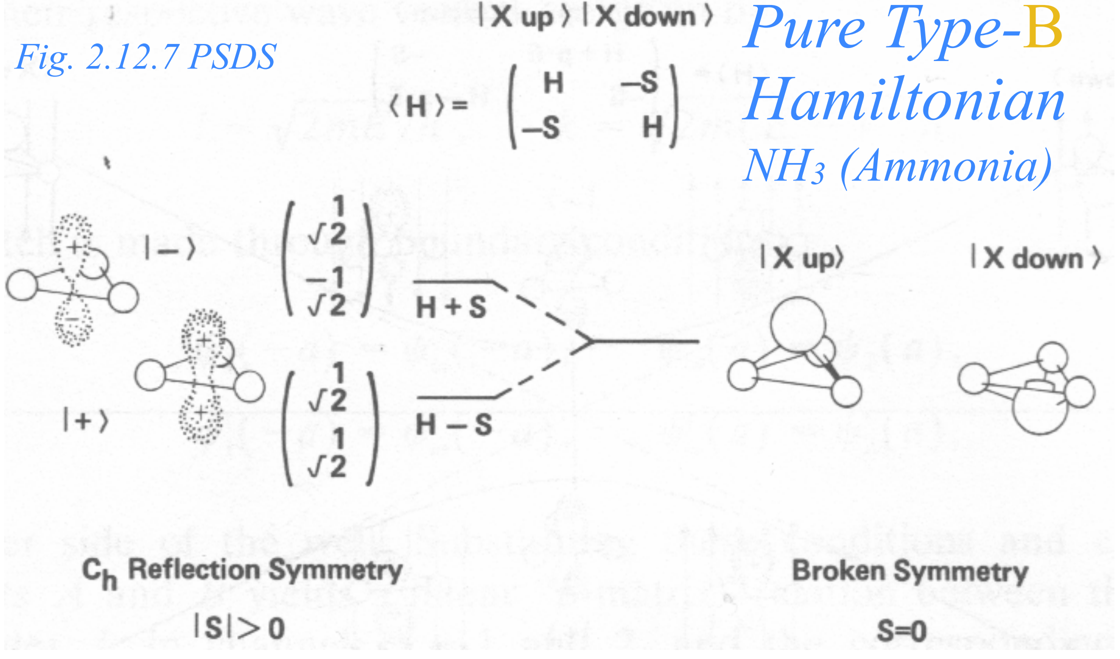


Fig. 2.12.7 PSDS

Fig. 2.12.7 PSDS

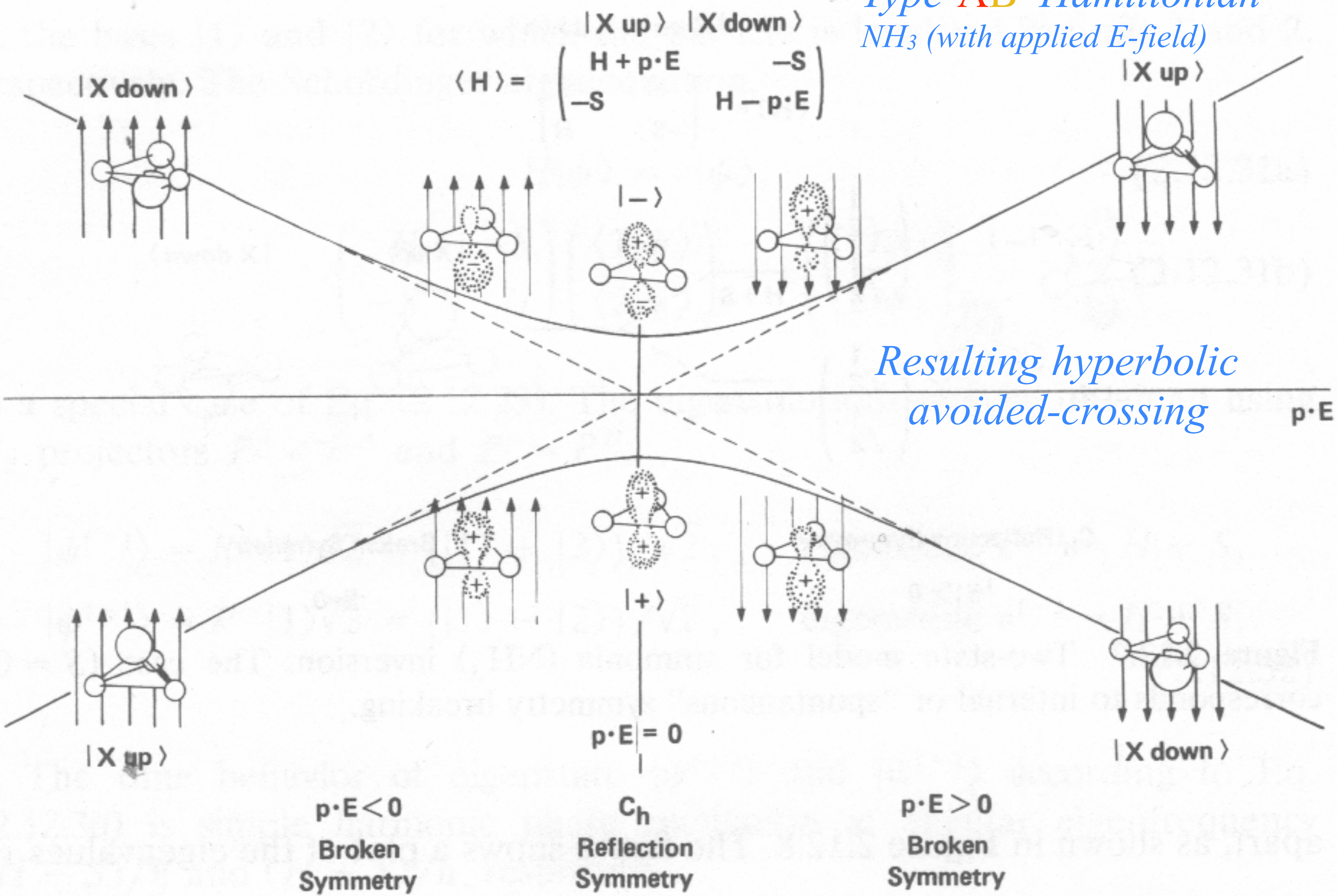


Pure Type-B Hamiltonian

NH_3 (Ammonia)

Type-AB Hamiltonian

NH_3 (with applied E-field)



Resulting hyperbolic avoided-crossing

Fig. 2.12.8 PSDS

Transform **H**(*A – basis*) *into* **H**(*B – basis*)

$$\frac{\begin{pmatrix}1&1\\1&-1\end{pmatrix}}{\sqrt{2}}\begin{pmatrix}+A&B\\B&-A\end{pmatrix}\frac{\begin{pmatrix}1&1\\1&-1\end{pmatrix}}{\sqrt{2}}$$

$$=\frac{1}{2}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}+A&B\\B&-A\end{pmatrix}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix}+A+B&B-A\\+A-B&B+A\end{pmatrix}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix}2B&2A\\2A&-2B\end{pmatrix}$$

$$=\begin{pmatrix}+B&A\\A&-B\end{pmatrix}$$

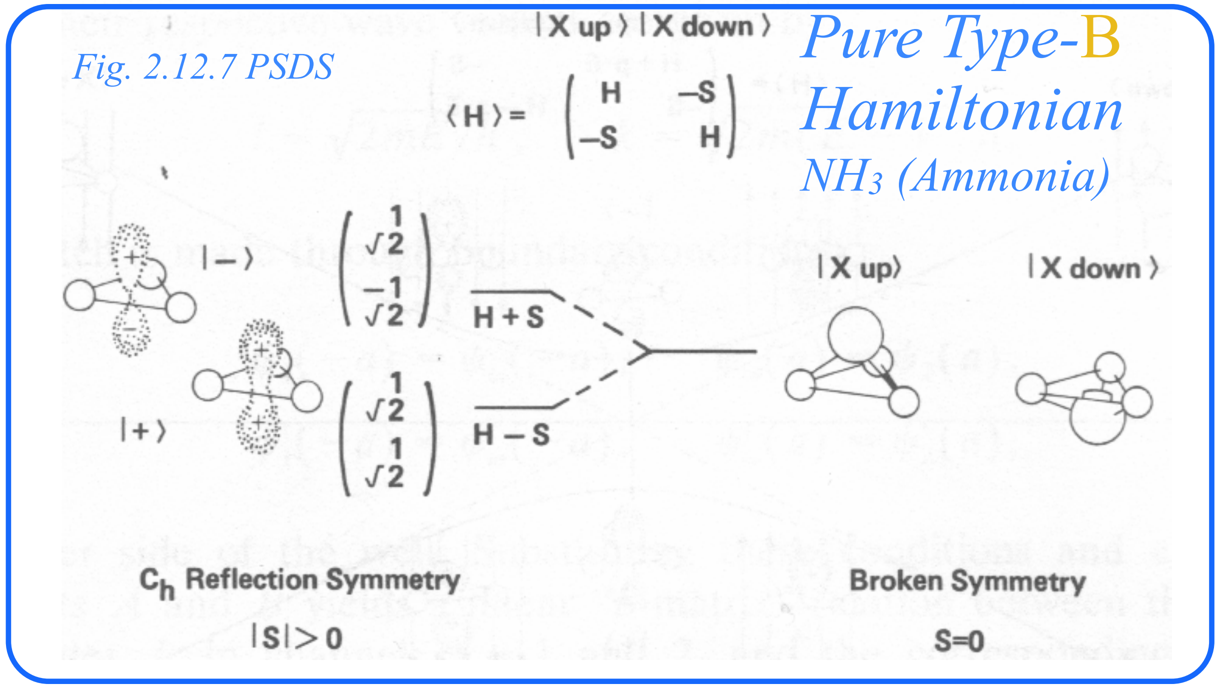


Fig. 2.12.7 PSDS

Pure Type-B Hamiltonian

NH3 (Ammonia)

Review of
Lecture 10
p. 65 to 73

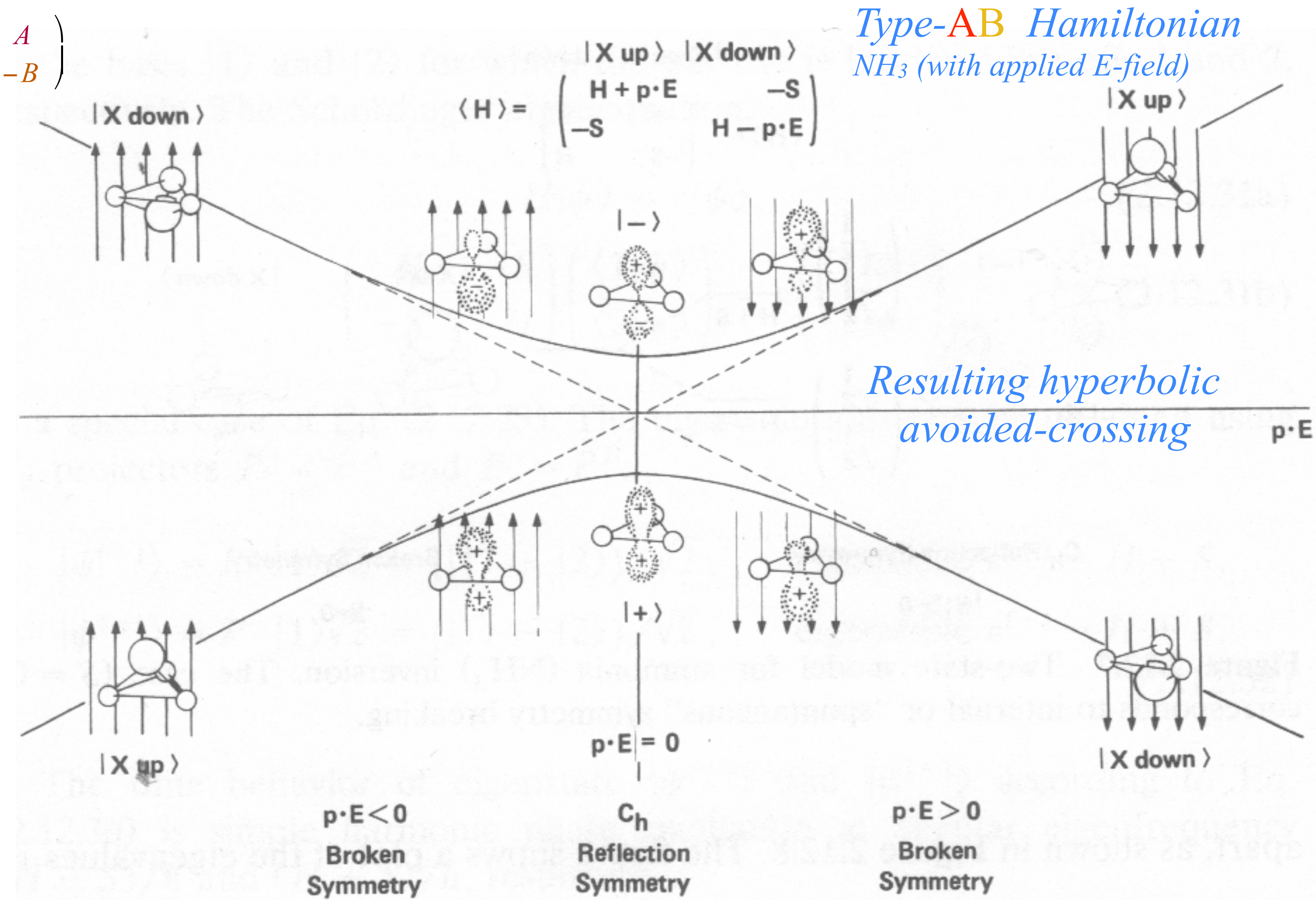


Fig. 2.12.8 PSDS

Type-AB Hamiltonian

NH3 (with applied E-field)

Resulting hyperbolic avoided-crossing

A to B to A Symmetry breaking described by hyperbolic eigenvalues of $A\boldsymbol{\sigma}_A+B\boldsymbol{\sigma}_B=\mathbf{H}=\begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$

$\mathbf{H}=\begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$ Secular equation: $\varepsilon^2-0\cdot\varepsilon-(A^2+B^2)$ gives *hyperbolic* energy levels: $\varepsilon=\pm\sqrt{A^2+B^2}$

$\mathbf{H}(B-basis)$ $\qquad\qquad$ $\mathbf{H}(A-basis)$

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}=\frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}}\begin{pmatrix} +A & B \\ B & -A \end{pmatrix}\frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}}$$

$$=\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} +A & B \\ B & -A \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix} +A+B & B-A \\ +A-B & B+A \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix} 2B & 2A \\ 2A & -2B \end{pmatrix}$$

$$=\begin{pmatrix} +B & A \\ A & -B \end{pmatrix}$$

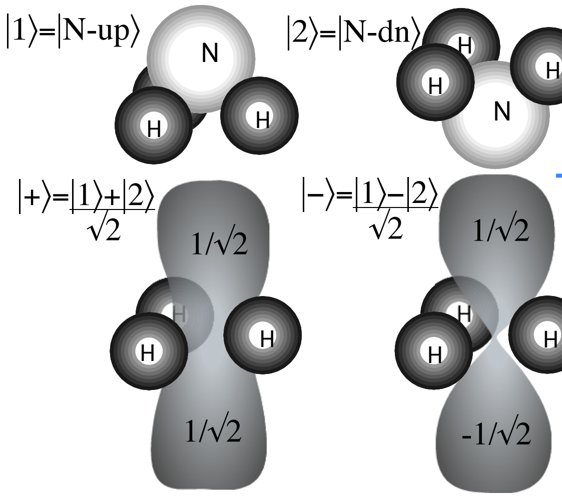
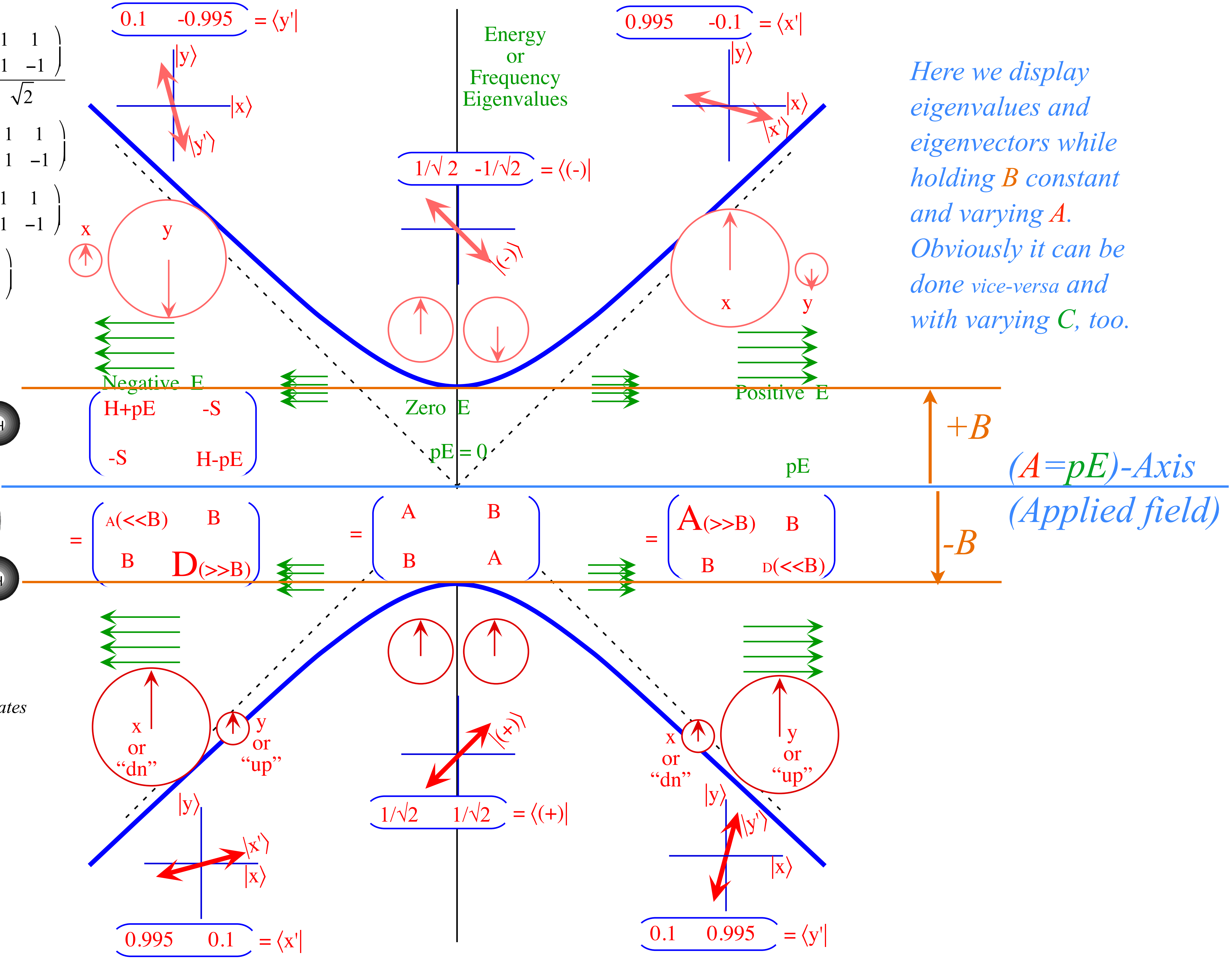


Fig. 10.3.2 Ammonia (NH$_3$) inversion states
(a) Base states (b) C$_2$-Eigenstates



Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa and with varying C, too.

Fig. 10.3.1 (b) Wigner avoided level crossing. (Fixed tunneling B=-S and variable A-D=pE field.)

Review of
Lecture 10
p. 73

Breaking C_N cyclic coupling into linear chains
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Algebra
Geometry

Finally! Symmetry groups that are not just C_N (And some that are)

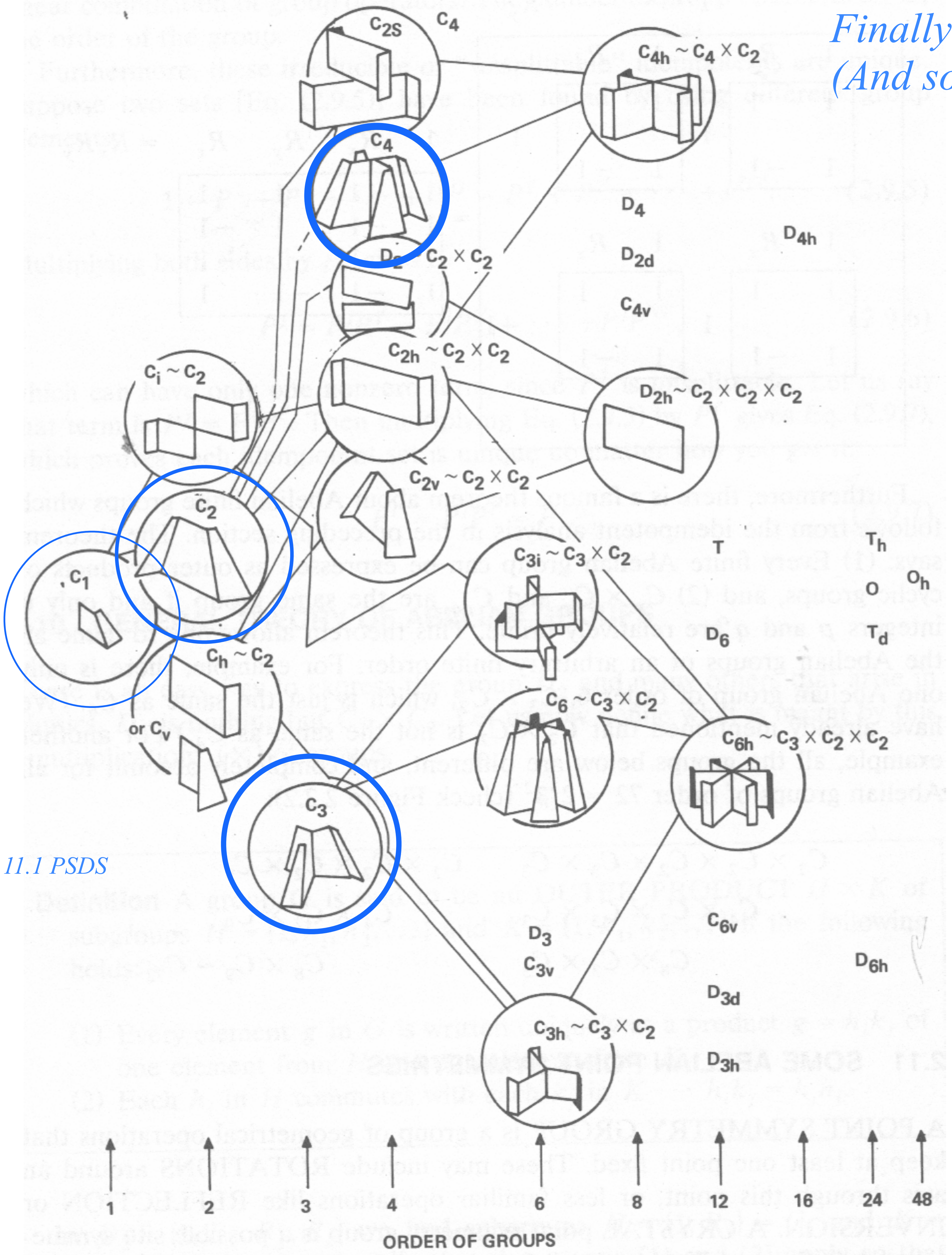

Fig. 2.11.1 PSDS

Figure 2.11.1 Abelian crystal point groups. Sixteen of the 32 crystal point groups are Abelian and are illustrated by models drawn in circles.

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Algebra
Geometry

Finally! Symmetry groups that are not just C_N
(And some that are)
Starting with D_2

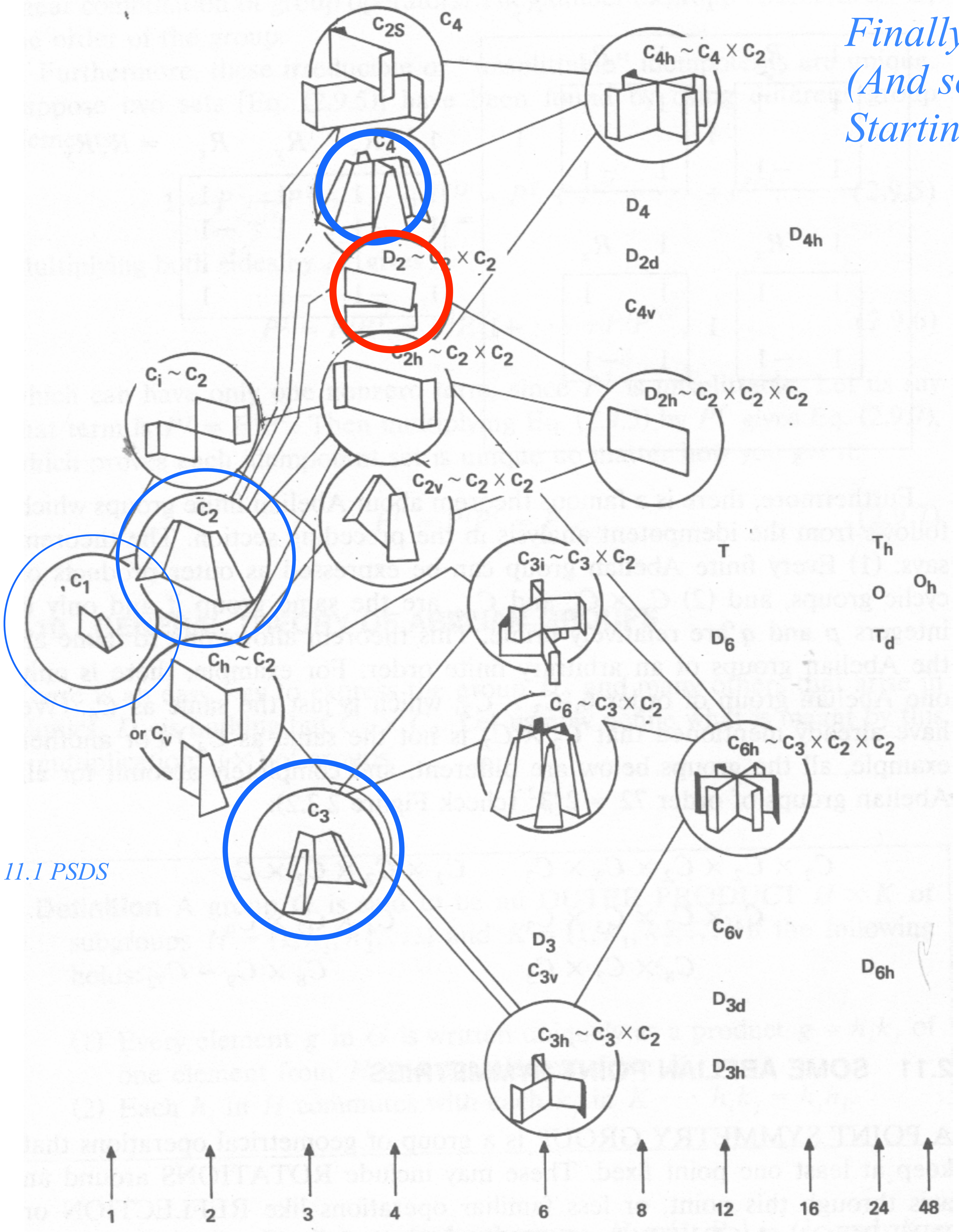


Fig. 2.11.1 PSDS

Figure 2.11.1 Abelian crystal point groups. Sixteen of the 32 crystal point groups are Abelian and are illustrated by models drawn in circles.

Finally! Symmetry groups that are not just C_N

(And some that are)

Starting with D_2 and C_{2h} and C_{2v}

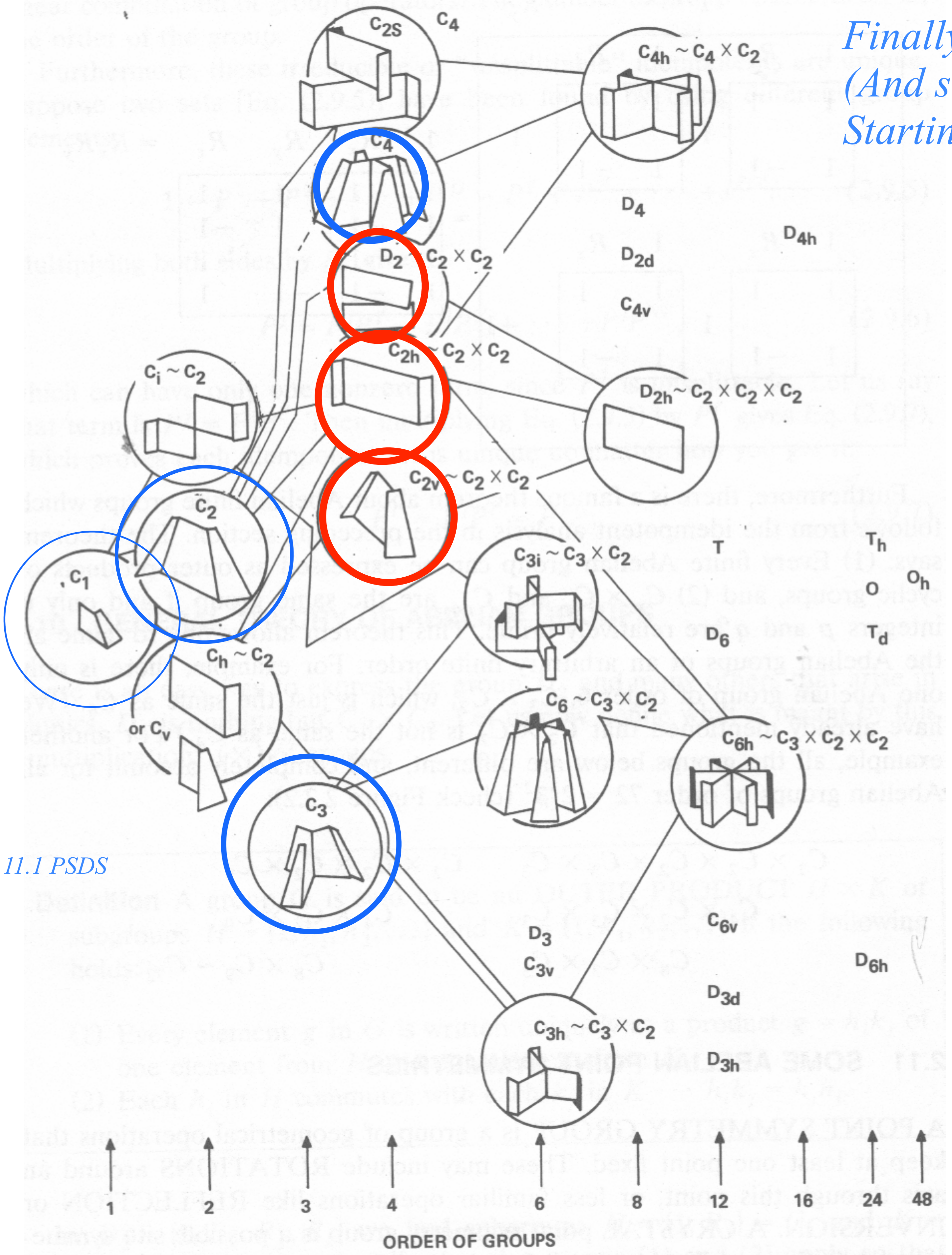


Fig. 2.11.1 PSDS

Figure 2.11.1 Abelian crystal point groups. Sixteen of the 32 crystal point groups are Abelian and are illustrated by models drawn in circles.

Breaking C_N cyclic coupling into linear chains

Review of 1D-Bohr-ring related to infinite square well

Breaking C_{2N+2} to approximate linear N-chain (Examples $C_2 \rightleftarrows C_6 \leftrightarrows C_{14}$)

Band-It simulation: Intro to scattering approach to quantum symmetry

How Band-It works: Match each Ψ and DΨ, Let L=0 at Right end

Breaking C_{2N} cyclic coupling down to C_N symmetry

Acoustical modes vs. Optical modes

Intro to other examples of band theory

Type-AB Avoided crossing view of band-gaps

Finally! Symmetry groups that are not just C_N

The "4-Group(s)" D_2 and C_{2v}

Spectral decomposition of D_2

Some D_2 modes

Outer product properties and the Crystal-Point Symmetry Group Zoo

Polygonal geometry of $U(2) \supset C_N$ character spectral function

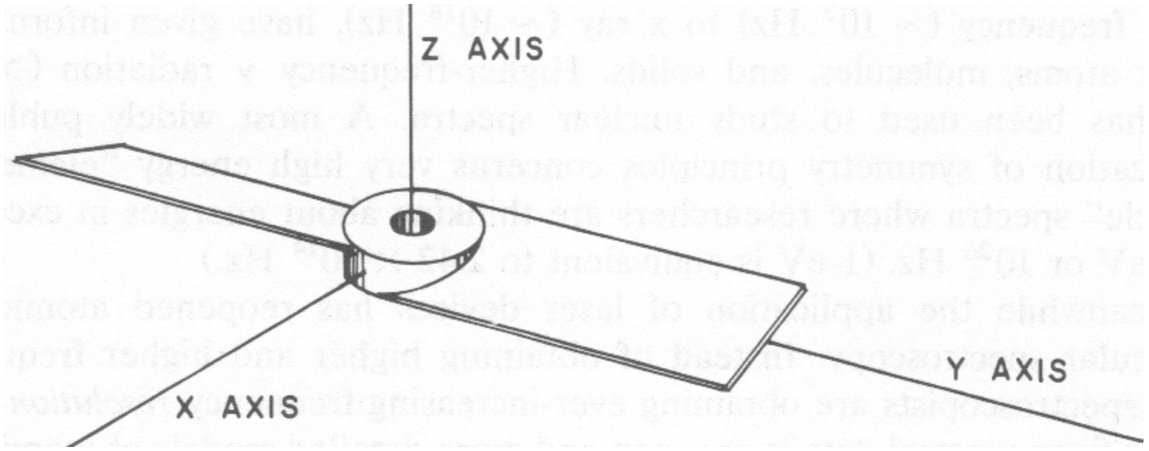
$$\chi^j(2\pi/n) = \frac{\sin(\pi(2j+1)/n)}{\sin(\pi/n)}$$

Algebra

Geometry

*The **CPT** subgroup of Lorentz Group*

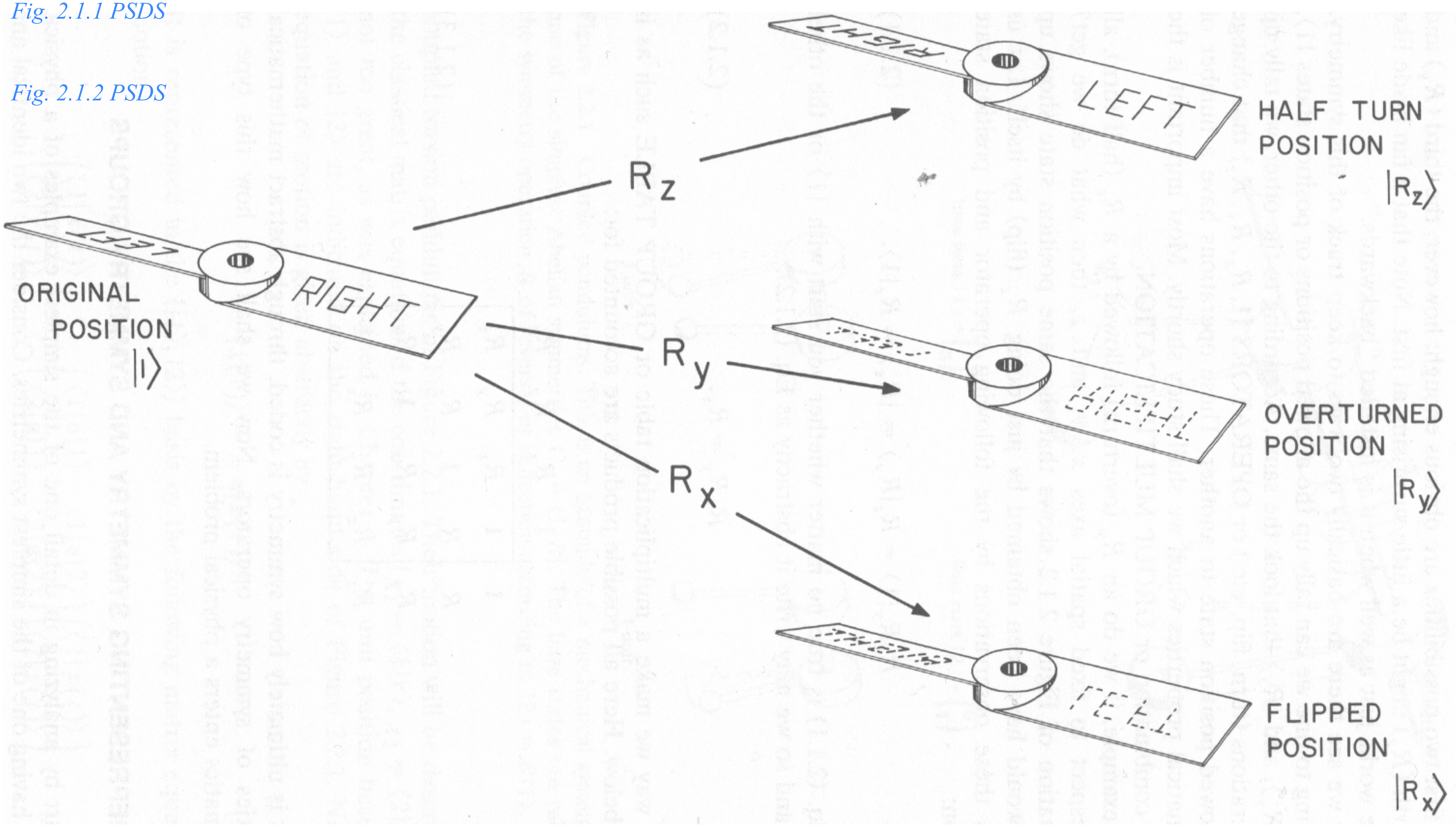
D_2 Symmetry (The 4-Group)



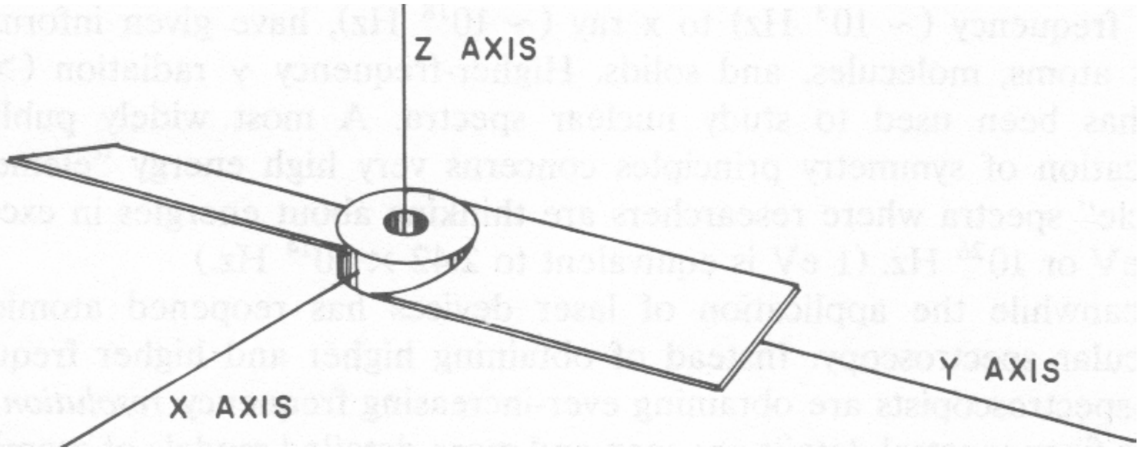
1 :	THE ORIGINAL POSITION	Don't touch the fan blade.
R_z:	THE HALF-TURN POSITION	Rotate it by 180° around its axle or the z axis.
R_y:	THE OVERTURNED POSITION	Overturn it 180° around the y axis.
R_x:	THE FLIPPED POSITION	Flip it 180° around the x axis.

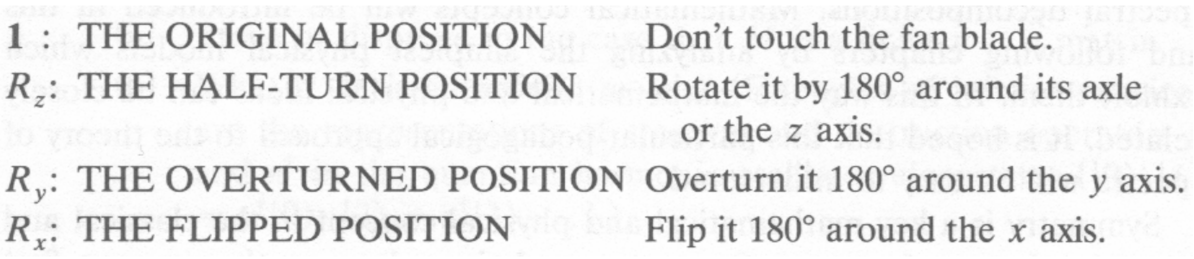
Fig. 2.1.1 PSDS

Fig. 2.1.2 PSDS



D_2 Symmetry (The 4-Group)





1 :	THE ORIGINAL POSITION	Don't touch the fan blade.
R_z:	THE HALF-TURN POSITION	Rotate it by 180° around its axle or the z axis.
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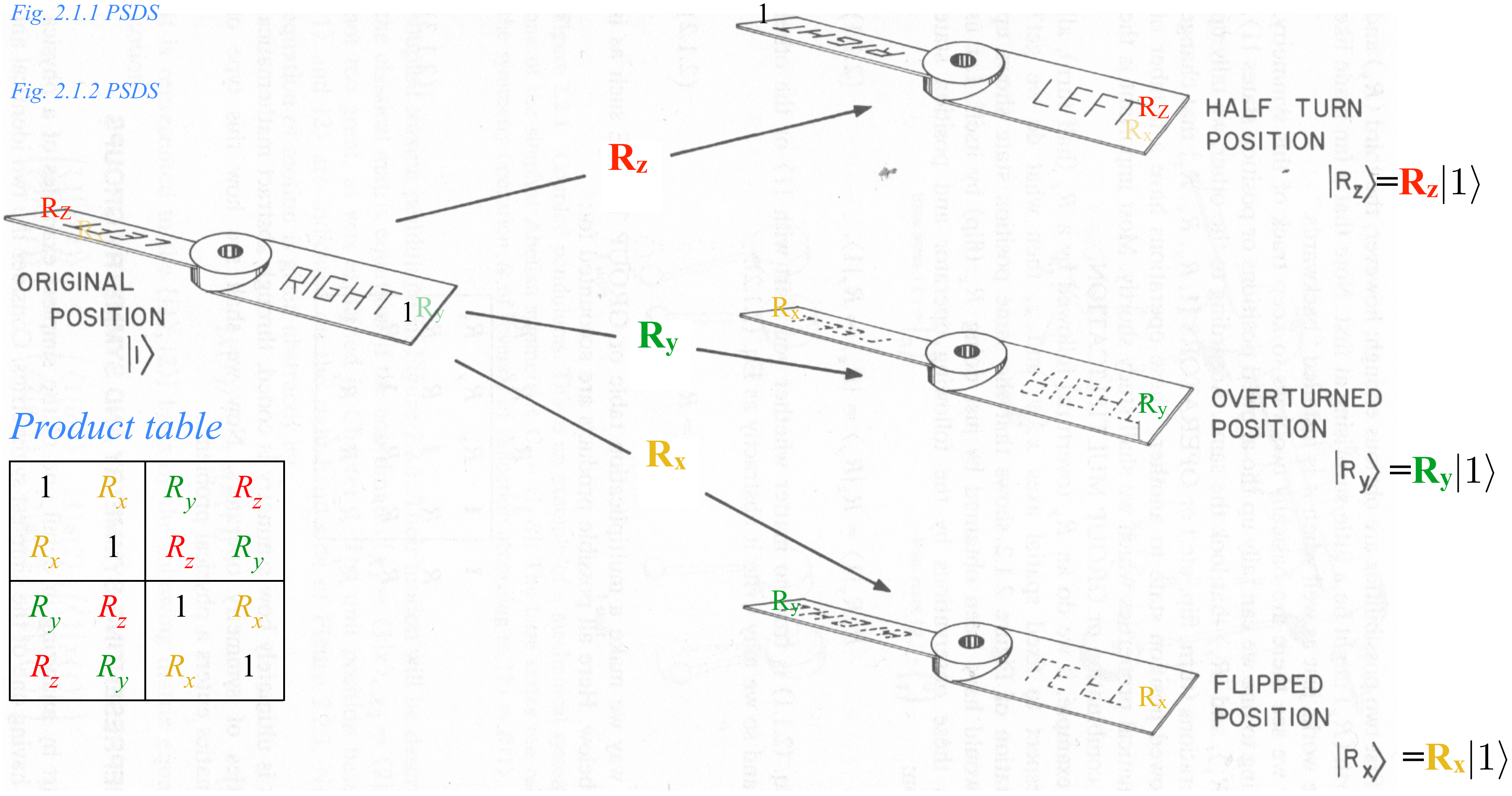
Fig. 2.1.1 PSDS

Fig. 2.1.2 PSDS



D_2 Product table

1	R_x	R_y	R_z
R_x	1	R_z	R_y
R_y	R_z	1	R_x
R_z	R_y	R_x	1

D_2 Symmetry (The 4-Group)

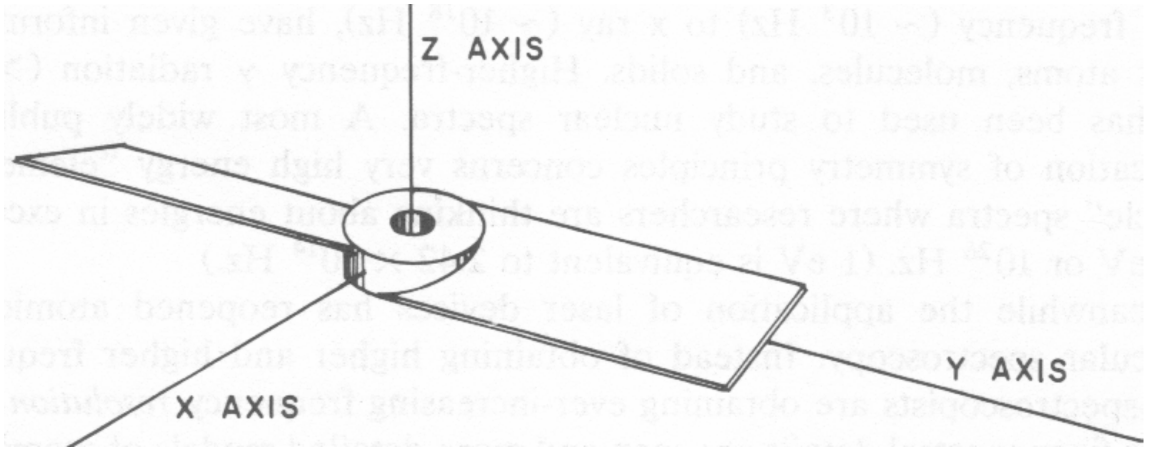


Fig. 2.1.1 PSDS

1 :	THE ORIGINAL POSITION	Don't touch the fan blade.
R_z:	THE HALF-TURN POSITION	Rotate it by 180° around its axle or the *z* axis.
R_y:	THE OVERTURNED POSITION	Overturn it 180° around the *y* axis.
R_x:	THE FLIPPED POSITION	Flip it 180° around the *x* axis.

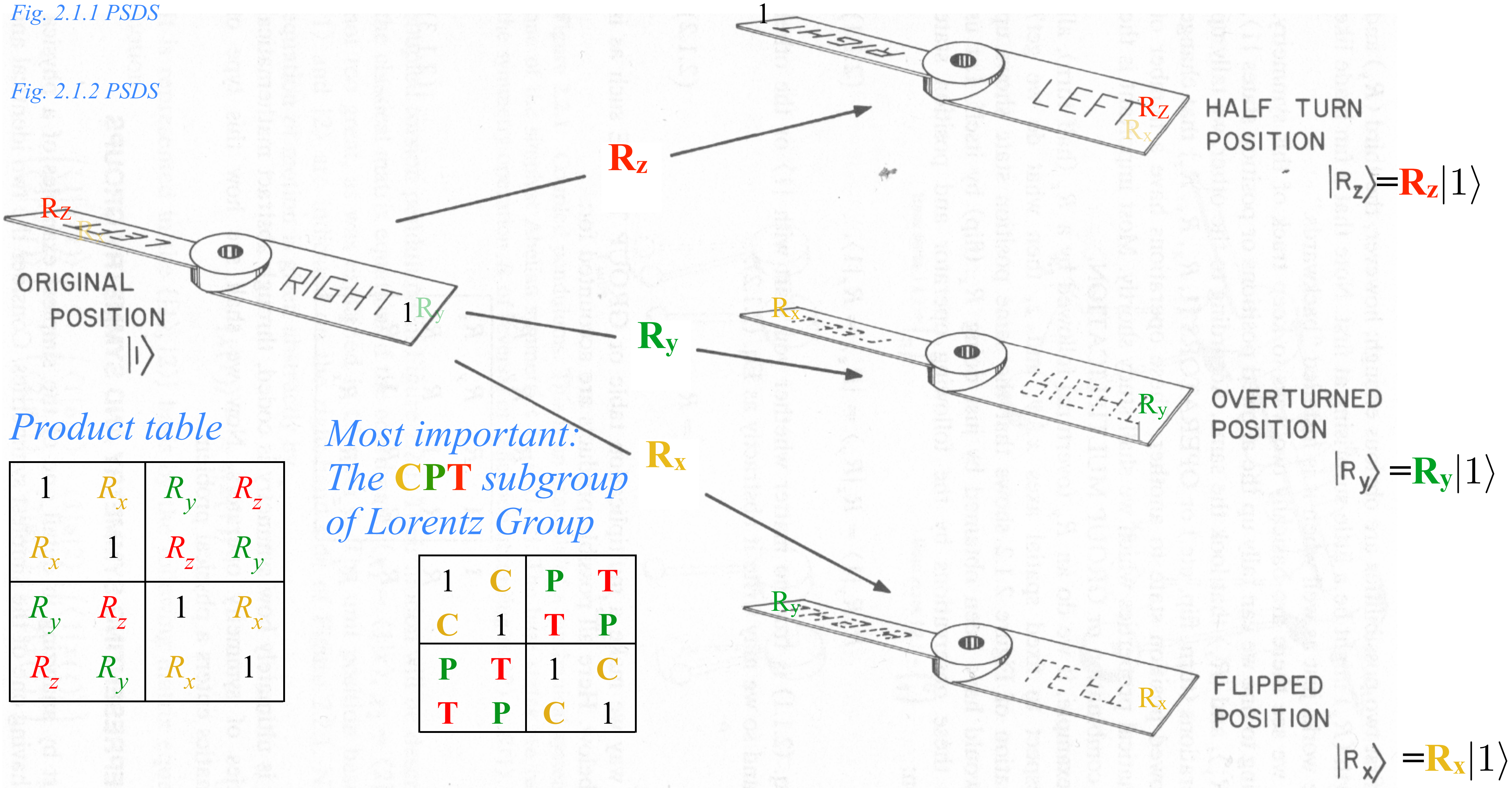


Fig. 2.1.2 PSDS

D_2 Product table

1	R_x	R_y	R_z
R_x	1	R_z	R_y
R_y	R_z	1	R_x
R_z	R_y	R_x	1

Most important:
*The **CPT** subgroup*
of Lorentz Group

1	**C**	**P**	**T**
C	1	**T**	**P**
P	**T**	1	**C**
T	**P**	**C**	1

- *Breaking C_N cyclic coupling into linear chains*
  - *Review of 1D-Bohr-ring related to infinite square well*
  - *Breaking C_{2N+2} to approximate linear N-chain (Examples $C_2 \rightleftarrows C_6 \rightleftarrows C_{14}$)*
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  - *Type-AB Avoided crossing view of band-gaps*

- *Finally! Symmetry groups that are not just C_N*
  - *The "4-Group(s)" D_2 and C_{2v}*
  - *Spectral decomposition of D_2*
    - *Some D_2 modes*
  - *Outer product properties and the Crystal-Point Symmetry Group Zoo*
  - *Polygonal geometry of $U(2) \supset C_N$ character spectral function*
    - *Algebra*
      - *Geometry*

The ***CPT*** *subgroup of Lorentz Group*

$$\chi^j(2\pi/n) = \frac{\sin\left(\pi(2j+1)/n\right)}{\sin\left(\pi/n\right)}$$

D_2 spectral decomposition: The old "1=1•1 trick" again

Two C_2 subgroup minimal equations:

$$\mathbf{R_x}^2 \mathbf{- 1 = 0}, \qquad \mathbf{R_y}^2 \mathbf{- 1 = 0}.$$

D_2 spectral decomposition: The old "1=1•1 trick" again

Two C_2 subgroup minimal equations and their projectors:

$$\mathbf{R}_x^2 - \mathbf{1} = \mathbf{0}, \qquad \mathbf{R}_y^2 - \mathbf{1} = \mathbf{0}.$$

$$\mathbf{P}_x^+ = \frac{\mathbf{1}+\mathbf{R}_x}{2} \qquad \mathbf{P}_y^+ = \frac{\mathbf{1}+\mathbf{R}_y}{2}$$

reducible projectors

$$\mathbf{P}_x^- = \frac{\mathbf{1}-\mathbf{R}_x}{2} \qquad \mathbf{P}_y^- = \frac{\mathbf{1}-\mathbf{R}_y}{2}$$

D_2 spectral decomposition: The old "1=1•1 trick" again

Two C_2 subgroup minimal equations and their projectors:

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reducible projectors

$$\mathbf{P}_x^- = \frac{\mathbf{1} - \mathbf{R}_x}{2} \qquad \mathbf{P}_y^- = \frac{\mathbf{1} - \mathbf{R}_y}{2}$$

$$\mathbf{1} = \mathbf{P}_x^+ + \mathbf{P}_x^- \qquad \mathbf{1} = \mathbf{P}_y^+ + \mathbf{P}_y^-$$

Completeness

D₂ spectral decomposition: The old "1=1•1 trick" again

Two C_2 subgroup minimal equations and their projectors:

$$\mathbf{R}_x{}^2 - \mathbf{1} = \mathbf{0}, \qquad \mathbf{R}_y{}^2 - \mathbf{1} = \mathbf{0}.$$

$$\mathbf{P}_x^+ = \frac{\mathbf{1}+\mathbf{R}_x}{2} \qquad \mathbf{P}_y^+ = \frac{\mathbf{1}+\mathbf{R}_y}{2}$$

$$\mathbf{P}_x^- = \frac{\mathbf{1}-\mathbf{R}_x}{2} \qquad \mathbf{P}_y^- = \frac{\mathbf{1}-\mathbf{R}_y}{2}$$

reducible projectors

$$\mathbf{1} = \mathbf{P}_x^+ + \mathbf{P}_x^- \qquad \mathbf{1} = \mathbf{P}_y^+ + \mathbf{P}_y^-$$

Completeness

$$\mathbf{R}_x = \mathbf{P}_x^+ - \mathbf{P}_x^- \qquad \mathbf{R}_y = \mathbf{P}_y^+ - \mathbf{P}_y^-$$

Spec.decomps

D₂ spectral decomposition: The old "1=1•1 trick" again

Two C_2 subgroup minimal equations and their projectors:

$$\mathbf{R_x}^2 - \mathbf{1} = \mathbf{0}, \qquad \mathbf{R_y}^2 - \mathbf{1} = \mathbf{0}.$$

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reducible projectors

$$\mathbf{P}_x^- = \frac{\mathbf{1}-\mathbf{R}_x}{2} \qquad \mathbf{P}_y^- = \frac{\mathbf{1}-\mathbf{R}_y}{2}$$

$$\mathbf{1} = \mathbf{P}_x^+ + \mathbf{P}_x^- \qquad \mathbf{1} = \mathbf{P}_y^+ + \mathbf{P}_y^-$$

Completeness

Spec.decomps

$$\mathbf{R}_x = \mathbf{P}_x^+ - \mathbf{P}_x^- \qquad \mathbf{R}_y = \mathbf{P}_y^+ - \mathbf{P}_y^-$$

The old "**1=1•1** trick" $\mathbf{1} = \mathbf{1}\cdot\mathbf{1} = \left(\mathbf{P}_x^+ + \mathbf{P}_x^-\right)\cdot\left(\mathbf{P}_y^+ + \mathbf{P}_y^-\right) = \mathbf{P}_x^+\cdot\mathbf{P}_y^+ + \mathbf{P}_x^-\cdot\mathbf{P}_y^+ + \mathbf{P}_x^+\cdot\mathbf{P}_y^- + \mathbf{P}_x^-\cdot\mathbf{P}_y^-$ *gives irrep projectors*

D_2 spectral decomposition: The old "1=1•1 trick" again

Two C_2 subgroup minimal equations and their projectors:

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Completeness

$$\mathbf{1} = \mathbf{P}_x^+ + \mathbf{P}_x^- \qquad \mathbf{1} = \mathbf{P}_y^+ + \mathbf{P}_y^-$$

Spec.decomps

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$$\mathbf{P}^{++} \equiv \mathbf{P}_x^+\cdot\mathbf{P}_y^+ = \frac{\left(\mathbf{1}+\mathbf{R}_x\right)\cdot\left(\mathbf{1}+\mathbf{R}_y\right)}{2\cdot 2} = \frac{1}{4}\left(\mathbf{1}+\mathbf{R}_x+\mathbf{R}_y+\mathbf{R}_z\right)$$

$$\mathbf{P}^{-+} \equiv \mathbf{P}_x^-\cdot\mathbf{P}_y^+ = \frac{\left(\mathbf{1}-\mathbf{R}_x\right)\cdot\left(\mathbf{1}+\mathbf{R}_y\right)}{2\cdot 2} = \frac{1}{4}\left(\mathbf{1}-\mathbf{R}_x+\mathbf{R}_y-\mathbf{R}_z\right)$$

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$$\mathbf{P}^{--} \equiv \mathbf{P}_x^-\cdot\mathbf{P}_y^- = \frac{\left(\mathbf{1}-\mathbf{R}_x\right)\cdot\left(\mathbf{1}-\mathbf{R}_y\right)}{2\cdot 2} = \frac{1}{4}\left(\mathbf{1}-\mathbf{R}_x-\mathbf{R}_y+\mathbf{R}_z\right)$$

D_2 spectral decomposition: The old "1=1•1 trick" again

Two C_2 subgroup minimal equations and their projectors:

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reducible projectors

$$\mathbf{1} = \mathbf{P}_x^+ + \mathbf{P}_x^- \qquad \mathbf{1} = \mathbf{P}_y^+ + \mathbf{P}_y^-$$

Completeness

$$\mathbf{R}_x = \mathbf{P}_x^+ - \mathbf{P}_x^- \qquad \mathbf{R}_y = \mathbf{P}_y^+ - \mathbf{P}_y^-$$

Spec.decomps

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(*completeness is first*)

$$\mathbf{1} = (+1)\mathbf{P}^{++} + (+1)\mathbf{P}^{-+} + (+1)\mathbf{P}^{+-} + (+1)\mathbf{P}^{--}$$

D_2 spectral decomposition: The old "1=1•1 trick" again

Two C_2 subgroup minimal equations and their projectors:

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$$\mathbf{P}_x^- = \frac{\mathbf{1}-\mathbf{R}_x}{2} \qquad \mathbf{P}_y^- = \frac{\mathbf{1}-\mathbf{R}_y}{2}$$

reducible projectors

$$\mathbf{1} = \mathbf{P}_x^+ + \mathbf{P}_x^- \qquad \mathbf{1} = \mathbf{P}_y^+ + \mathbf{P}_y^-$$

$$\mathbf{R}_x = \mathbf{P}_x^+ - \mathbf{P}_x^- \qquad \mathbf{R}_y = \mathbf{P}_y^+ - \mathbf{P}_y^-$$

Completeness
Spec.decomps

The old "**1=1•1** trick" $\mathbf{1} = \mathbf{1}\cdot\mathbf{1} = \left(\mathbf{P}_x^+ + \mathbf{P}_x^-\right)\cdot\left(\mathbf{P}_y^+ + \mathbf{P}_y^-\right) = \mathbf{P}_x^+\cdot\mathbf{P}_y^+ + \mathbf{P}_x^-\cdot\mathbf{P}_y^+ + \mathbf{P}_x^+\cdot\mathbf{P}_y^- + \mathbf{P}_x^-\cdot\mathbf{P}_y^-$ *gives irrep projectors*

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$$\mathbf{P}^{-+} \equiv \mathbf{P}_x^-\cdot\mathbf{P}_y^+ = \frac{\left(\mathbf{1}-\mathbf{R}_x\right)\cdot\left(\mathbf{1}+\mathbf{R}_y\right)}{2\cdot 2} = \frac{1}{4}\left(\mathbf{1}-\mathbf{R}_x+\mathbf{R}_y-\mathbf{R}_z\right)$$

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(*then R_x eigenvalues*)

$$\mathbf{1} = (+1)\mathbf{P}^{++} + (+1)\mathbf{P}^{-+} + (+1)\mathbf{P}^{+-} + (+1)\mathbf{P}^{--}$$

$$\mathbf{R}_x = (+1)\mathbf{P}^{++} + (-1)\mathbf{P}^{-+} + (+1)\mathbf{P}^{+-} + (-1)\mathbf{P}^{--}$$

D_2 spectral decomposition: The old "1=1•1 trick" again

Two C_2 subgroup minimal equations and their projectors:

$\mathbf{R_x}^2 \mathbf{- 1 = 0}$, $\mathbf{R_y}^2 \mathbf{- 1 = 0}$.

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reducible projectors

$$\mathbf{P}_x^- = \frac{\mathbf{1}-\mathbf{R}_x}{2} \qquad \mathbf{P}_y^- = \frac{\mathbf{1}-\mathbf{R}_y}{2}$$

Completeness

$$\mathbf{1} = \mathbf{P}_x^+ + \mathbf{P}_x^- \qquad \mathbf{1} = \mathbf{P}_y^+ + \mathbf{P}_y^-$$

Spec.decomps

$$\mathbf{R}_x = \mathbf{P}_x^+ - \mathbf{P}_x^- \qquad \mathbf{R}_y = \mathbf{P}_y^+ - \mathbf{P}_y^-$$

The old "**1=1•1** trick" $\mathbf{1} = \mathbf{1}\cdot\mathbf{1} = \left(\mathbf{P}_x^+ + \mathbf{P}_x^-\right)\cdot\left(\mathbf{P}_y^+ + \mathbf{P}_y^-\right) = \mathbf{P}_x^+\cdot\mathbf{P}_y^+ + \mathbf{P}_x^-\cdot\mathbf{P}_y^+ + \mathbf{P}_x^+\cdot\mathbf{P}_y^- + \mathbf{P}_x^-\cdot\mathbf{P}_y^-$ *gives irrep projectors*

$$\mathbf{P}^{++} \equiv \mathbf{P}_x^+\cdot\mathbf{P}_y^+ = \frac{\left(\mathbf{1}+\mathbf{R}_x\right)\cdot\left(\mathbf{1}+\mathbf{R}_y\right)}{2\cdot 2} = \frac{1}{4}\left(\mathbf{1}+\mathbf{R}_x+\mathbf{R}_y+\mathbf{R}_z\right)$$

$$\mathbf{P}^{-+} \equiv \mathbf{P}_x^-\cdot\mathbf{P}_y^+ = \frac{\left(\mathbf{1}-\mathbf{R}_x\right)\cdot\left(\mathbf{1}+\mathbf{R}_y\right)}{2\cdot 2} = \frac{1}{4}\left(\mathbf{1}-\mathbf{R}_x+\mathbf{R}_y-\mathbf{R}_z\right)$$

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(*...and so forth*)

$$\mathbf{1} = (+1)\mathbf{P}^{++} + (+1)\mathbf{P}^{-+} + (+1)\mathbf{P}^{+-} + (+1)\mathbf{P}^{--}$$

$$\mathbf{R}_x = (+1)\mathbf{P}^{++} + (-1)\mathbf{P}^{-+} + (+1)\mathbf{P}^{+-} + (-1)\mathbf{P}^{--}$$

$$\mathbf{R}_y = (+1)\mathbf{P}^{++} + (+1)\mathbf{P}^{-+} + (-1)\mathbf{P}^{+-} + (-1)\mathbf{P}^{--}$$

$$\mathbf{R}_z = (+1)\mathbf{P}^{++} + (-1)\mathbf{P}^{-+} + (-1)\mathbf{P}^{+-} + (+1)\mathbf{P}^{--}$$

D_2 spectral decomposition: The old "1=1•1 trick" again

Two C_2 subgroup minimal equations and their projectors:

$$\mathbf{R_x}^2 - \mathbf{1} = \mathbf{0}, \qquad \mathbf{R_y}^2 - \mathbf{1} = \mathbf{0}.$$

$$\mathbf{P}_x^+ = \frac{\mathbf{1}+\mathbf{R}_x}{2} \qquad \mathbf{P}_y^+ = \frac{\mathbf{1}+\mathbf{R}_y}{2}$$

reducible projectors

$$\mathbf{P}_x^- = \frac{\mathbf{1}-\mathbf{R}_x}{2} \qquad \mathbf{P}_y^- = \frac{\mathbf{1}-\mathbf{R}_y}{2}$$

Completeness

$$\mathbf{1} = \mathbf{P}_x^+ + \mathbf{P}_x^- \qquad \mathbf{1} = \mathbf{P}_y^+ + \mathbf{P}_y^-$$

Spec.decomps

$$\mathbf{R}_x = \mathbf{P}_x^+ - \mathbf{P}_x^- \qquad \mathbf{R}_y = \mathbf{P}_y^+ - \mathbf{P}_y^-$$

The old "**1=1•1** trick" $\mathbf{1} = \mathbf{1}\cdot\mathbf{1} = \left(\mathbf{P}_x^+ + \mathbf{P}_x^-\right)\cdot\left(\mathbf{P}_y^+ + \mathbf{P}_y^-\right) = \mathbf{P}_x^+\cdot\mathbf{P}_y^+ + \mathbf{P}_x^-\cdot\mathbf{P}_y^+ + \mathbf{P}_x^+\cdot\mathbf{P}_y^- + \mathbf{P}_x^-\cdot\mathbf{P}_y^-$ *gives irrep projectors*

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(*completeness is first*)

$$\mathbf{1} = (+1)\mathbf{P}^{++} + (+1)\mathbf{P}^{-+} + (+1)\mathbf{P}^{+-} + (+1)\mathbf{P}^{--}$$

$$\mathbf{R}_x = (+1)\mathbf{P}^{++} + (-1)\mathbf{P}^{-+} + (+1)\mathbf{P}^{+-} + (-1)\mathbf{P}^{--}$$

$$\mathbf{R}_y = (+1)\mathbf{P}^{++} + (+1)\mathbf{P}^{-+} + (-1)\mathbf{P}^{+-} + (-1)\mathbf{P}^{--}$$

$$\mathbf{R}_z = (+1)\mathbf{P}^{++} + (-1)\mathbf{P}^{-+} + (-1)\mathbf{P}^{+-} + (+1)\mathbf{P}^{--}$$

C_2^x	$\mathbf{1}$	$\mathbf{R}_x$
+	1	1
−	1	−1

×

C_2^y	$\mathbf{1}$	$\mathbf{R}_y$
+	1	1
−	1	−1

=

$C_2^x \times C_2^y$	$\mathbf{1}\cdot\mathbf{1}$	$\mathbf{R}_x\cdot\mathbf{1}$	$\mathbf{1}\cdot\mathbf{R}_y$	$\mathbf{R}_x\cdot\mathbf{R}_y$
$+\cdot+$	$1\cdot 1$	$1\cdot 1$	$1\cdot 1$	$1\cdot 1$
$-\cdot+$	$1\cdot 1$	$-1\cdot 1$	$1\cdot 1$	$-1\cdot 1$
$+\cdot-$	$1\cdot 1$	$1\cdot 1$	$1\cdot(-1)$	$1\cdot(-1)$
$-\cdot-$	$1\cdot 1$	$-1\cdot 1$	$1\cdot(-1)$	$-1\cdot(-1)$

Shortcut notation for getting D_2 character table

D_2 spectral decomposition: The old "1=1•1 trick" again

Two C_2 subgroup minimal equations and their projectors:

C_2^x	$\mathbf{1}$	$\mathbf{R}_x$
+	1	1
−	1	−1

×

C_2^y	$\mathbf{1}$	$\mathbf{R}_y$
+	1	1
−	1	−1

=

$C_2^x \times C_2^y$	$\mathbf{1}\cdot\mathbf{1}$	$\mathbf{R}_x\cdot\mathbf{1}$	$\mathbf{1}\cdot\mathbf{R}_y$	$\mathbf{R}_x\cdot\mathbf{R}_y$
$+\cdot+$	$1\cdot1$	$1\cdot1$	$1\cdot1$	$1\cdot1$
$-\cdot+$	$1\cdot1$	$-1\cdot1$	$1\cdot1$	$-1\cdot1$
$+\cdot-$	$1\cdot1$	$1\cdot1$	$1\cdot(-1)$	$1\cdot(-1)$
$-\cdot-$	$1\cdot1$	$-1\cdot1$	$1\cdot(-1)$	$-1\cdot(-1)$

=

D_2	$\mathbf{1}$	$\mathbf{R}_x$	$\mathbf{R}_y$	$\mathbf{R}_z$
$+\cdot+$	1	1	1	1
$-\cdot+$	1	-1	1	-1
$+\cdot-$	1	1	−1	−1
$-\cdot-$	1	-1	−1	1

$$\mathbf{R}_x^2 - \mathbf{1} = 0, \qquad \mathbf{R}_y^2 - \mathbf{1} = 0.$$

$$\mathbf{P}_x^+ = \frac{\mathbf{1}+\mathbf{R}_x}{2} \qquad \mathbf{P}_y^+ = \frac{\mathbf{1}+\mathbf{R}_y}{2}$$

reducible projectors

$$\mathbf{P}_x^- = \frac{\mathbf{1}-\mathbf{R}_x}{2} \qquad \mathbf{P}_y^- = \frac{\mathbf{1}-\mathbf{R}_y}{2}$$

$$\mathbf{1} = \mathbf{P}_x^+ + \mathbf{P}_x^- \qquad \mathbf{1} = \mathbf{P}_y^+ + \mathbf{P}_y^-$$

Completeness

Spec.decomps

$$\mathbf{R}_x = \mathbf{P}_x^+ - \mathbf{P}_x^- \qquad \mathbf{R}_y = \mathbf{P}_y^+ - \mathbf{P}_y^-$$

The old "**1=1•1** trick" $\mathbf{1}=\mathbf{1}\cdot\mathbf{1}=\left(\mathbf{P}_x^+ + \mathbf{P}_x^-\right)\cdot\left(\mathbf{P}_y^+ + \mathbf{P}_y^-\right) = \mathbf{P}_x^+\cdot\mathbf{P}_y^+ + \mathbf{P}_x^-\cdot\mathbf{P}_y^+ + \mathbf{P}_x^+\cdot\mathbf{P}_y^- + \mathbf{P}_x^-\cdot\mathbf{P}_y^-$ *gives irrep projectors*

$$\mathbf{P}^{++} \equiv \mathbf{P}_x^+\cdot\mathbf{P}_y^+ = \frac{\left(\mathbf{1}+\mathbf{R}_x\right)\cdot\left(\mathbf{1}+\mathbf{R}_y\right)}{2\cdot2} = \frac{1}{4}\left(\mathbf{1}+\mathbf{R}_x+\mathbf{R}_y+\mathbf{R}_z\right)$$

$$\mathbf{P}^{-+} \equiv \mathbf{P}_x^-\cdot\mathbf{P}_y^+ = \frac{\left(\mathbf{1}-\mathbf{R}_x\right)\cdot\left(\mathbf{1}+\mathbf{R}_y\right)}{2\cdot2} = \frac{1}{4}\left(\mathbf{1}-\mathbf{R}_x+\mathbf{R}_y-\mathbf{R}_z\right)$$

$$\mathbf{P}^{+-} \equiv \mathbf{P}_x^+\cdot\mathbf{P}_y^- = \frac{\left(\mathbf{1}+\mathbf{R}_x\right)\cdot\left(\mathbf{1}-\mathbf{R}_y\right)}{2\cdot2} = \frac{1}{4}\left(\mathbf{1}+\mathbf{R}_x-\mathbf{R}_y-\mathbf{R}_z\right)$$

$$\mathbf{P}^{--} \equiv \mathbf{P}_x^-\cdot\mathbf{P}_y^- = \frac{\left(\mathbf{1}-\mathbf{R}_x\right)\cdot\left(\mathbf{1}-\mathbf{R}_y\right)}{2\cdot2} = \frac{1}{4}\left(\mathbf{1}-\mathbf{R}_x-\mathbf{R}_y+\mathbf{R}_z\right)$$

(*completeness is first*)

$$\mathbf{1} = (+1)\mathbf{P}^{++} + (+1)\mathbf{P}^{-+} + (+1)\mathbf{P}^{+-} + (+1)\mathbf{P}^{--}$$

$$\mathbf{R}_x = (+1)\mathbf{P}^{++} + (-1)\mathbf{P}^{-+} + (+1)\mathbf{P}^{+-} + (-1)\mathbf{P}^{--}$$

$$\mathbf{R}_y = (+1)\mathbf{P}^{++} + (+1)\mathbf{P}^{-+} + (-1)\mathbf{P}^{+-} + (-1)\mathbf{P}^{--}$$

$$\mathbf{R}_z = (+1)\mathbf{P}^{++} + (-1)\mathbf{P}^{-+} + (-1)\mathbf{P}^{+-} + (+1)\mathbf{P}^{--}$$

C_2^x	$\mathbf{1}$	$\mathbf{R}_x$
+	1	1
−	1	−1

×

C_2^y	$\mathbf{1}$	$\mathbf{R}_y$
+	1	1
−	1	−1

=

$C_2^x \times C_2^y$	$\mathbf{1}\cdot\mathbf{1}$	$\mathbf{R}_x\cdot\mathbf{1}$	$\mathbf{1}\cdot\mathbf{R}_y$	$\mathbf{R}_x\cdot\mathbf{R}_y$
$+\cdot+$	$1\cdot1$	$1\cdot1$	$1\cdot1$	$1\cdot1$
$-\cdot+$	$1\cdot1$	$-1\cdot1$	$1\cdot1$	$-1\cdot1$
$+\cdot-$	$1\cdot1$	$1\cdot1$	$1\cdot(-1)$	$1\cdot(-1)$
$-\cdot-$	$1\cdot1$	$-1\cdot1$	$1\cdot(-1)$	$-1\cdot(-1)$

Shortcut notation for getting D_2 character table

D_2 spectral decomposition: The old "1=1•1 trick" again

Two C_2 subgroup minimal equations and their projectors:

C_2^x	$\mathbf{1}$	$\mathbf{R}_x$
$+$	1	1
$-$	1	-1

×

C_2^y	$\mathbf{1}$	$\mathbf{R}_y$
$+$	1	1
$-$	1	-1

=

$C_2^x \times C_2^y$	$\mathbf{1}\cdot\mathbf{1}$	$\mathbf{R}_x\cdot\mathbf{1}$	$\mathbf{1}\cdot\mathbf{R}_y$	$\mathbf{R}_x\cdot\mathbf{R}_y$
$+\cdot+$	$1\cdot1$	$1\cdot1$	$1\cdot1$	$1\cdot1$
$-\cdot+$	$1\cdot1$	$-1\cdot1$	$1\cdot1$	$-1\cdot1$
$+\cdot-$	$1\cdot1$	$1\cdot1$	$1\cdot(-1)$	$1\cdot(-1)$
$-\cdot-$	$1\cdot1$	$-1\cdot1$	$1\cdot(-1)$	$-1\cdot(-1)$

=

D_2	$\mathbf{1}$	$\mathbf{R}_x$	$\mathbf{R}_y$	$\mathbf{R}_z$
$++=A_1$	1	1	1	1
$-+=A_2$	1	-1	1	-1
$+-=B_1$	1	1	-1	-1
$--=B_2$	1	-1	-1	1

Note common notation

$$\mathbf{R}_x{}^2 - \mathbf{1} = 0, \qquad \mathbf{R}_y{}^2 - \mathbf{1} = 0.$$

$$\mathbf{P}_x^+ = \frac{\mathbf{1}+\mathbf{R}_x}{2} \qquad \mathbf{P}_y^+ = \frac{\mathbf{1}+\mathbf{R}_y}{2}$$

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$$\mathbf{P}_x^- = \frac{\mathbf{1}-\mathbf{R}_x}{2} \qquad \mathbf{P}_y^- = \frac{\mathbf{1}-\mathbf{R}_y}{2}$$

$$\mathbf{1} = \mathbf{P}_x^+ + \mathbf{P}_x^- \qquad \mathbf{1} = \mathbf{P}_y^+ + \mathbf{P}_y^-$$

Completeness Spec.decomps

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$$\mathbf{P}^{++} \equiv \mathbf{P}_x^+\cdot\mathbf{P}_y^+ = \frac{\left(\mathbf{1}+\mathbf{R}_x\right)\cdot\left(\mathbf{1}+\mathbf{R}_y\right)}{2\cdot2} = \frac{1}{4}\left(\mathbf{1}+\mathbf{R}_x+\mathbf{R}_y+\mathbf{R}_z\right)$$

$$\mathbf{P}^{-+} \equiv \mathbf{P}_x^-\cdot\mathbf{P}_y^+ = \frac{\left(\mathbf{1}-\mathbf{R}_x\right)\cdot\left(\mathbf{1}+\mathbf{R}_y\right)}{2\cdot2} = \frac{1}{4}\left(\mathbf{1}-\mathbf{R}_x+\mathbf{R}_y-\mathbf{R}_z\right)$$

$$\mathbf{P}^{+-} \equiv \mathbf{P}_x^+\cdot\mathbf{P}_y^- = \frac{\left(\mathbf{1}+\mathbf{R}_x\right)\cdot\left(\mathbf{1}-\mathbf{R}_y\right)}{2\cdot2} = \frac{1}{4}\left(\mathbf{1}+\mathbf{R}_x-\mathbf{R}_y-\mathbf{R}_z\right)$$

$$\mathbf{P}^{--} \equiv \mathbf{P}_x^-\cdot\mathbf{P}_y^- = \frac{\left(\mathbf{1}-\mathbf{R}_x\right)\cdot\left(\mathbf{1}-\mathbf{R}_y\right)}{2\cdot2} = \frac{1}{4}\left(\mathbf{1}-\mathbf{R}_x-\mathbf{R}_y+\mathbf{R}_z\right)$$

(*completeness is first*)

$$\mathbf{1} = (+1)\mathbf{P}^{++} + (+1)\mathbf{P}^{-+} + (+1)\mathbf{P}^{+-} + (+1)\mathbf{P}^{--}$$

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$$\mathbf{R}_y = (+1)\mathbf{P}^{++} + (+1)\mathbf{P}^{-+} + (-1)\mathbf{P}^{+-} + (-1)\mathbf{P}^{--}$$

$$\mathbf{R}_z = (+1)\mathbf{P}^{++} + (-1)\mathbf{P}^{-+} + (-1)\mathbf{P}^{+-} + (+1)\mathbf{P}^{--}$$

C_2^x	$\mathbf{1}$	$\mathbf{R}_x$
$+$	1	1
$-$	1	-1

×

C_2^y	$\mathbf{1}$	$\mathbf{R}_y$
$+$	1	1
$-$	1	-1

=

$C_2^x \times C_2^y$	$\mathbf{1}\cdot\mathbf{1}$	$\mathbf{R}_x\cdot\mathbf{1}$	$\mathbf{1}\cdot\mathbf{R}_y$	$\mathbf{R}_x\cdot\mathbf{R}_y$
$+\cdot+$	$1\cdot1$	$1\cdot1$	$1\cdot1$	$1\cdot1$
$-\cdot+$	$1\cdot1$	$-1\cdot1$	$1\cdot1$	$-1\cdot1$
$+\cdot-$	$1\cdot1$	$1\cdot1$	$1\cdot(-1)$	$1\cdot(-1)$
$-\cdot-$	$1\cdot1$	$-1\cdot1$	$1\cdot(-1)$	$-1\cdot(-1)$

Shortcut notation for getting D_2 character table

Breaking C_N cyclic coupling into linear chains
Review of 1D-Bohr-ring related to infinite square well
Breaking C_{2N+2} to approximate linear N-chain (Examples $C_2 \rightleftarrows C_6 \rightleftarrows C_{14}$)
Band-It simulation: Intro to scattering approach to quantum symmetry
How Band-It works: Match each Ψ and DΨ, Let L=0 at Right end

Breaking C_{2N} cyclic coupling down to C_N symmetry
Acoustical modes vs. Optical modes
Intro to other examples of band theory
Type-AB Avoided crossing view of band-gaps

Finally! Symmetry groups that are not just C_N
The "4-Group(s)" D_2 and C_{2v}
Spectral decomposition of D_2
Some D_2 modes
Outer product properties and the Crystal-Point Symmetry Group Zoo
Polygonal geometry of $U(2)\supset C_N$ character spectral function
Algebra
Geometry

The **CPT** *subgroup of Lorentz Group*

$$\chi^j(2\pi/n)=\frac{\sin(\pi(2j+1)/n)}{\sin(\pi/n)}$$

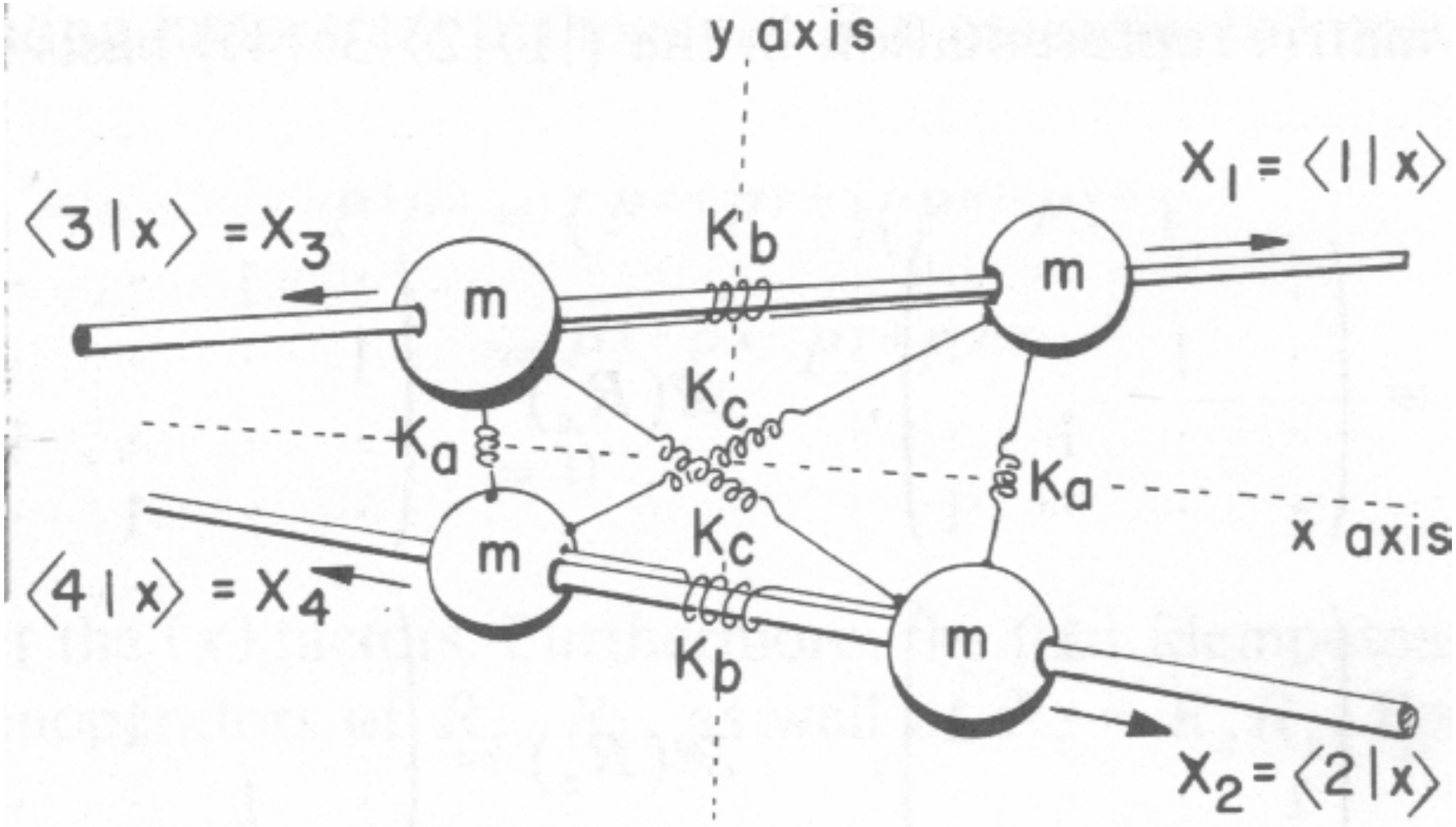


Fig. 2.8.1 PSDS

$$\begin{pmatrix} \langle 1|\ddot{x}\rangle \\ \langle 2|\ddot{x}\rangle \\ \langle 3|\ddot{x}\rangle \\ \langle 4|\ddot{x}\rangle \end{pmatrix} = \begin{pmatrix} A & a & b & c \\ a & A & c & b \\ b & c & A & a \\ c & b & a & A \end{pmatrix} \begin{pmatrix} \langle 1|x\rangle \\ \langle 2|x\rangle \\ \langle 3|x\rangle \\ \langle 4|x\rangle \end{pmatrix}$$

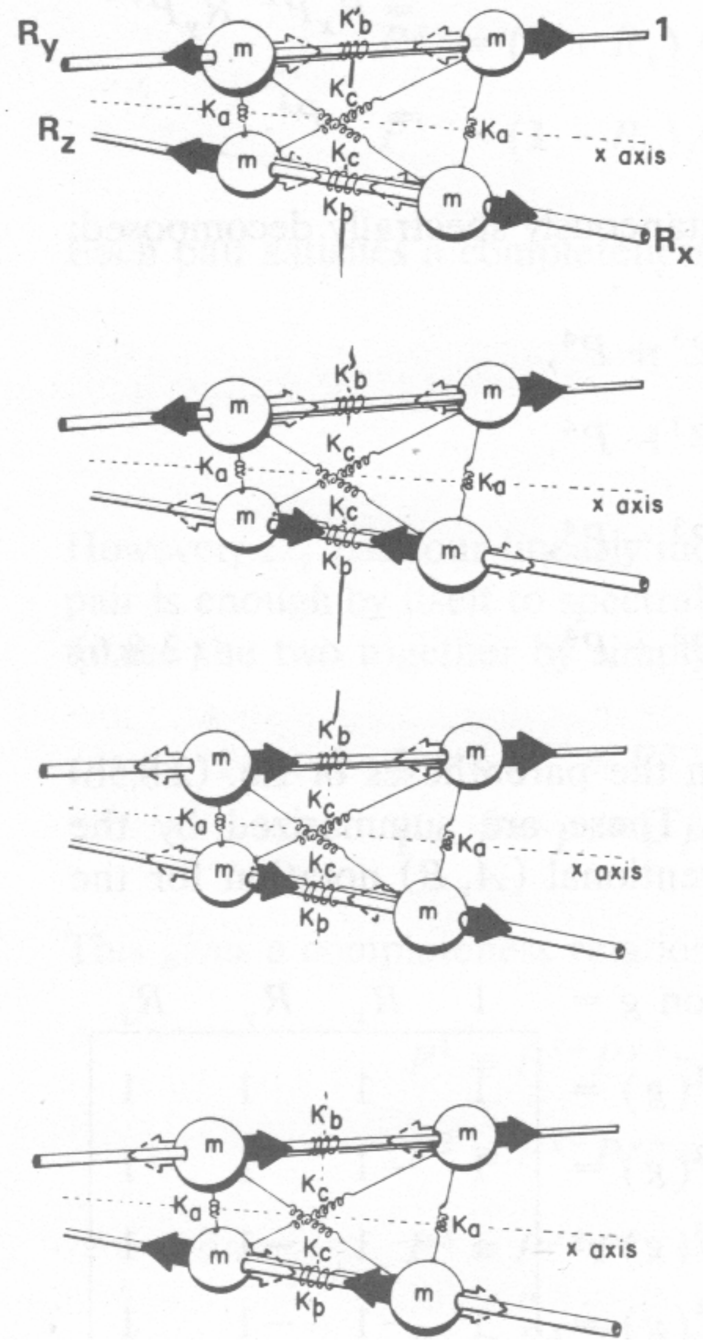
$$A = -(k_a \cos^2(a,b) + k_b + k_c \cos^2(b,c))/m,$$
$$a = -k_a \cos^2(a,b)/m,$$
$$b = -k_b/m,$$
$$c = -k_c \cos^2(b,c)/m.$$

$$|e^{A_1}\rangle \equiv |e^1\rangle = P^1|1\rangle\sqrt{4} = (|1\rangle + |2\rangle + |3\rangle + |4\rangle)/2,$$

$$|e^{B_2}\rangle \equiv |e^2\rangle = P^2|1\rangle\sqrt{4} = (|1\rangle - |2\rangle + |3\rangle - |4\rangle)/2,$$

$$|e^{B_1}\rangle \equiv |e^3\rangle = P^3|1\rangle\sqrt{4} = (|1\rangle + |2\rangle - |3\rangle - |4\rangle)/2,$$

$$|e^{A_2}\rangle \equiv |e^4\rangle = P^4|1\rangle\sqrt{4} = (|1\rangle - |2\rangle - |3\rangle + |4\rangle)/2,$$



$$\begin{pmatrix}1\\1\\1\\1\end{pmatrix}/2 \qquad (A+a+b+c)^{1/2}$$

$$\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}/2 \qquad (A-a+b-c)^{1/2}$$

$$\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}/2 \qquad (A+a-b-c)^{1/2}$$

$$\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}/2 \qquad (A-a-b+c)^{1/2}$$

Fig. 2.8.2 PSDS

Breaking C_N cyclic coupling into linear chains

Review of 1D-Bohr-ring related to infinite square well

Breaking C_{2N+2} to approximate linear N-chain (Examples $C_2 \rightleftarrows C_6 \rightleftarrows C_{14}$)

Band-It simulation: Intro to scattering approach to quantum symmetry

How *Band-It* works: Match each Ψ and DΨ, Let L=0 at Right end

Breaking C_{2N} cyclic coupling down to C_N symmetry

Acoustical modes vs. Optical modes

Intro to other examples of band theory

Type-AB Avoided crossing view of band-gaps

Finally! Symmetry groups that are not just C_N

The “4-Group(s)” D_2 and C_{2v}

Spectral decomposition of D_2

→ Some D_2 modes ←

Outer product properties and the Crystal-Point Symmetry Group Zoo

Polygonal geometry of $U(2) \supset C_N$ character spectral function

Algebra

Geometry

The **CPT** subgroup of Lorentz Group

$$\chi^j(2\pi/n) = \frac{\sin\left(\pi(2j+1)/n\right)}{\sin\left(\pi/n\right)}$$

Crystal-Point Group Zoo
having 32 groups
(Showing
16 Abelian
Crystal Groups)

Fig. 2.11.1 PSDS

The other 16
crystal-point groups
are
Non-Abelian

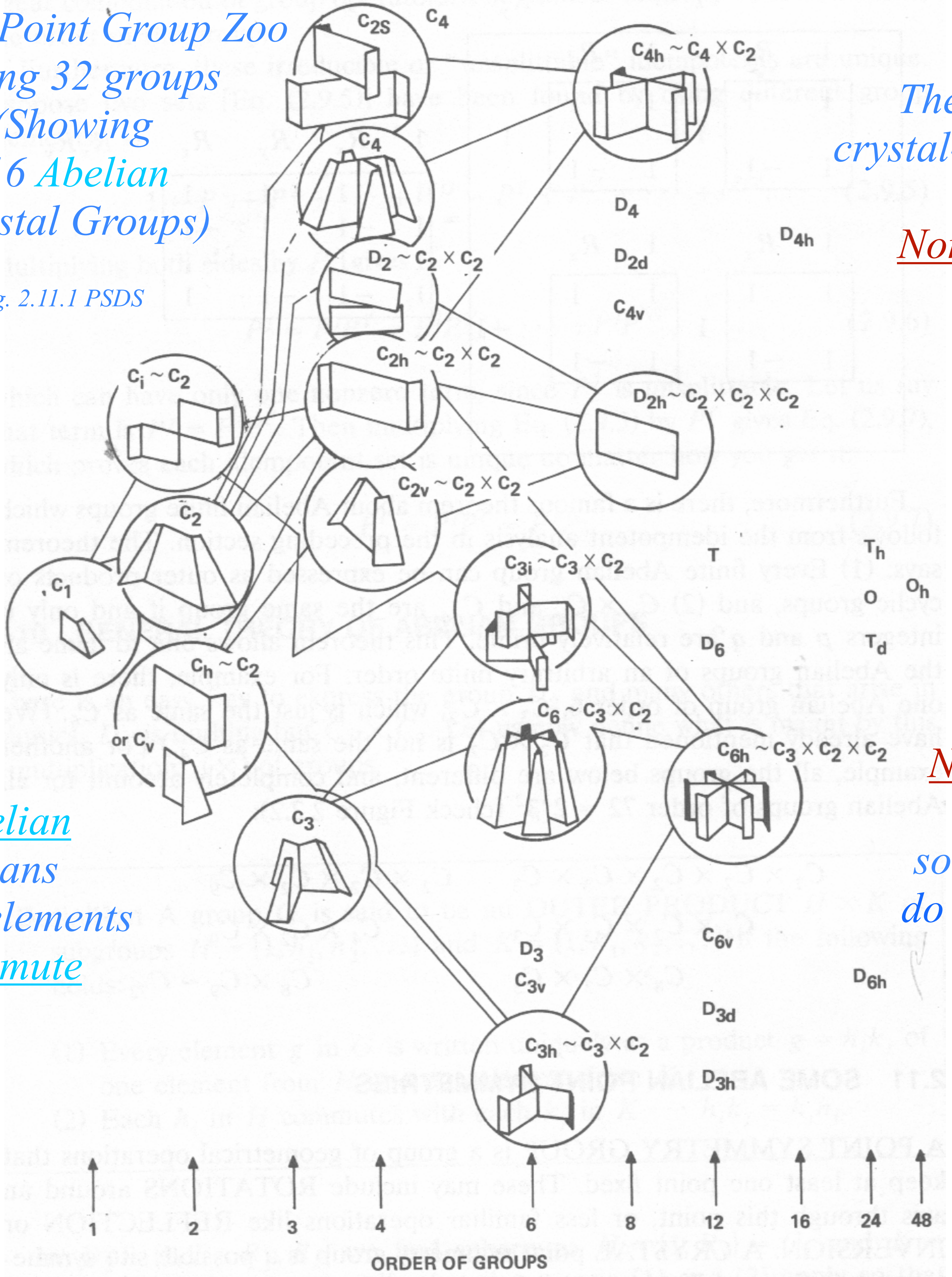


Figure 2.11.1 Abelian crystal point groups. Sixteen of the 32 crystal point groups are Abelian and are illustrated by models drawn in circles.

Abelian
means
all its elements
commute

Non-Abelian
means
some elements
do not commute

Crystal-Point Group Zoo
having 32 groups
(Showing
16 Abelian
Crystal Groups)

Fig. 2.11.1 PSDS

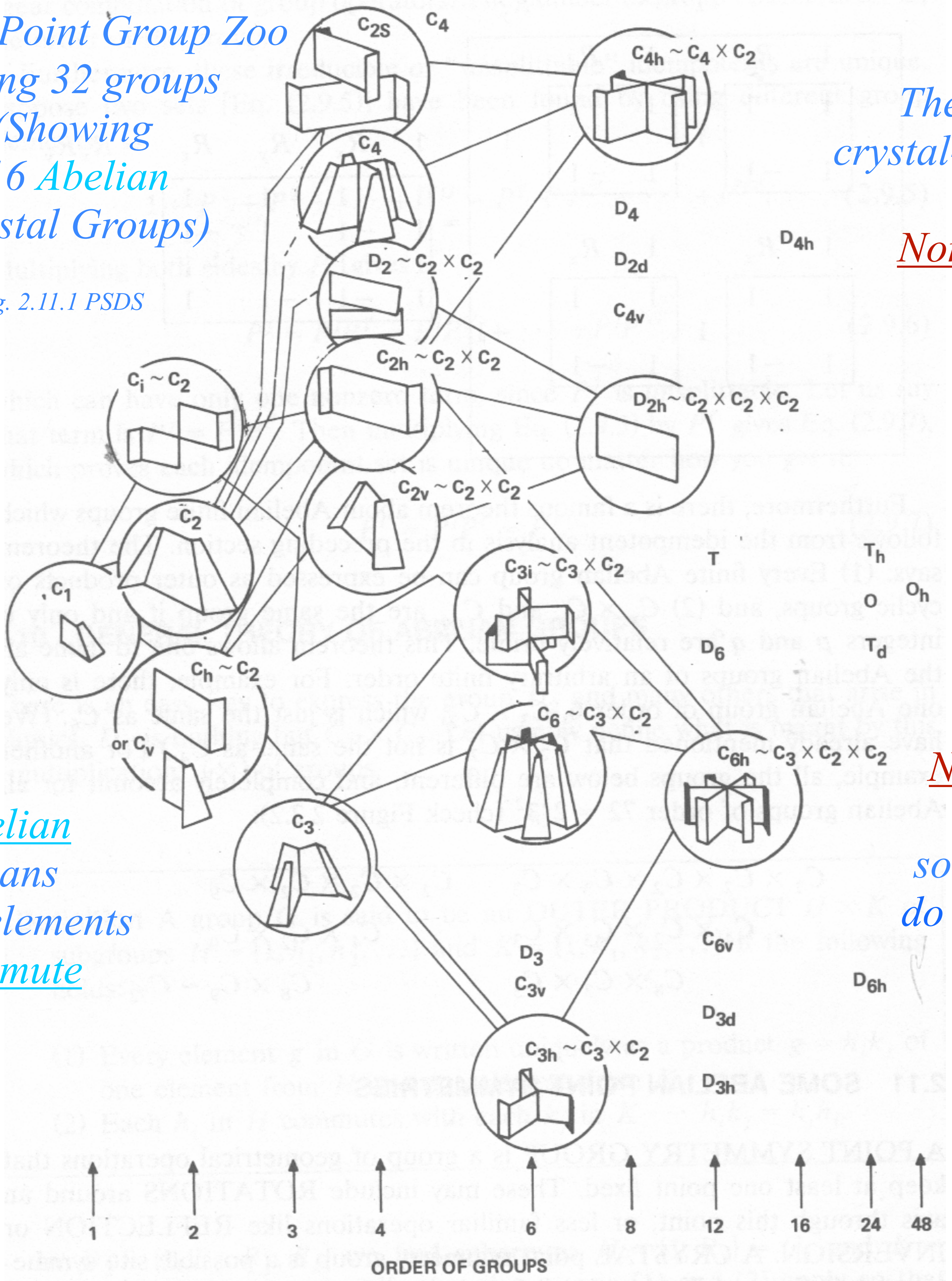


Figure 2.11.1 Abelian crystal point groups. Sixteen of the 32 crystal point groups are Abelian and are illustrated by models drawn in circles.

<u>Abelian</u>
means
<u>all</u> its elements
<u>commute</u>

The other 16
crystal-point groups
are
<u>Non</u>-Abelian

<u>Non</u>-Abelian
means
some elements
do <u>not</u> commute

From p. 93-101
Character Trace of
n-fold rotation
where: $\ell^j=2j+1$
is U(2) irrep dimension

$$\chi^j\left(\frac{2\pi}{n}\right) = \frac{\sin\frac{\pi}{n}(2j+1)}{\sin\frac{\pi}{n}} = \frac{\sin\frac{\pi\ell^j}{n}}{\sin\frac{\pi}{n}}$$

Crystal-Point Group Zoo
having 32 groups
(Showing
16 Abelian
Crystal Groups)

Fig. 2.11.1 PSDS

The other 16
crystal-point groups
are
Non-Abelian

From p. 93-101
Character Trace of
n-fold rotation
where: $\ell^j=2j+1$
is U(2) irrep dimension

$$\boxed{\chi^j\left(\frac{2\pi}{n}\right)=\frac{\sin\frac{\pi}{n}(2j+1)}{\sin\frac{\pi}{n}}=\frac{\sin\frac{\pi\ell^j}{n}}{\sin\frac{\pi}{n}}}$$

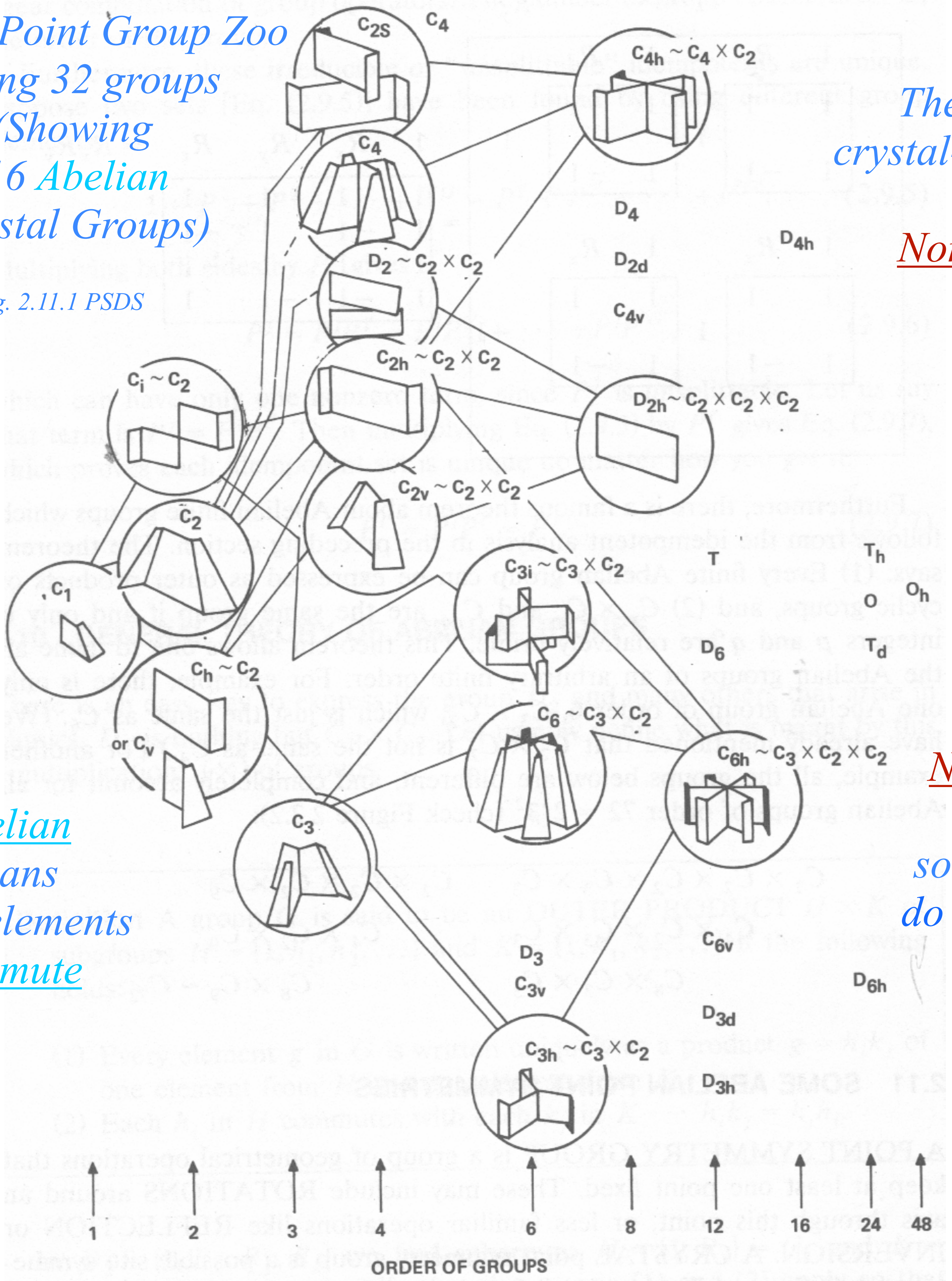


Figure 2.11.1 Abelian crystal point groups. Sixteen of the 32 crystal point groups are Abelian and are illustrated by models drawn in circles.

Abelian
means
all its elements
commute

Non-Abelian
means
some elements
do not commute

To be a crystal-point group
the Character Trace of
n-fold vector rotation
for: $\ell^1=2+1=3$
must be an integer

$$\chi^1\left(\frac{2\pi}{n}\right)=\frac{\sin\frac{\pi}{1}(2j+1)}{\sin\frac{\pi}{n}}=\frac{\sin\frac{3\pi}{n}}{\sin\frac{\pi}{n}}=\text{integer}$$

$$\frac{\sin\frac{3\pi}{2}}{\sin\frac{\pi}{2}}=-1\ (n=2\text{ ok})$$

$$\frac{\sin\frac{3\pi}{3}}{\sin\frac{\pi}{3}}=+1\ (n=3\text{ ok})$$

$$\frac{\sin\frac{3\pi}{4}}{\sin\frac{\pi}{4}}=+1\ (n=4\text{ ok})$$

$$\frac{\sin\frac{3\pi}{5}}{\sin\frac{\pi}{5}}=G^+\ (n=5\text{ NO!})$$

$$\frac{\sin\frac{3\pi}{6}}{\sin\frac{\pi}{6}}=+2\ (n=6\text{ ok})$$

...But,
n=7 to ∞
are not ok

Crystal-Point Group Zoo having 32 groups (Showing 16 Abelian Crystal Groups)

Fig. 2.11.1 PSDS

The other 16 crystal-point groups are Non-Abelian

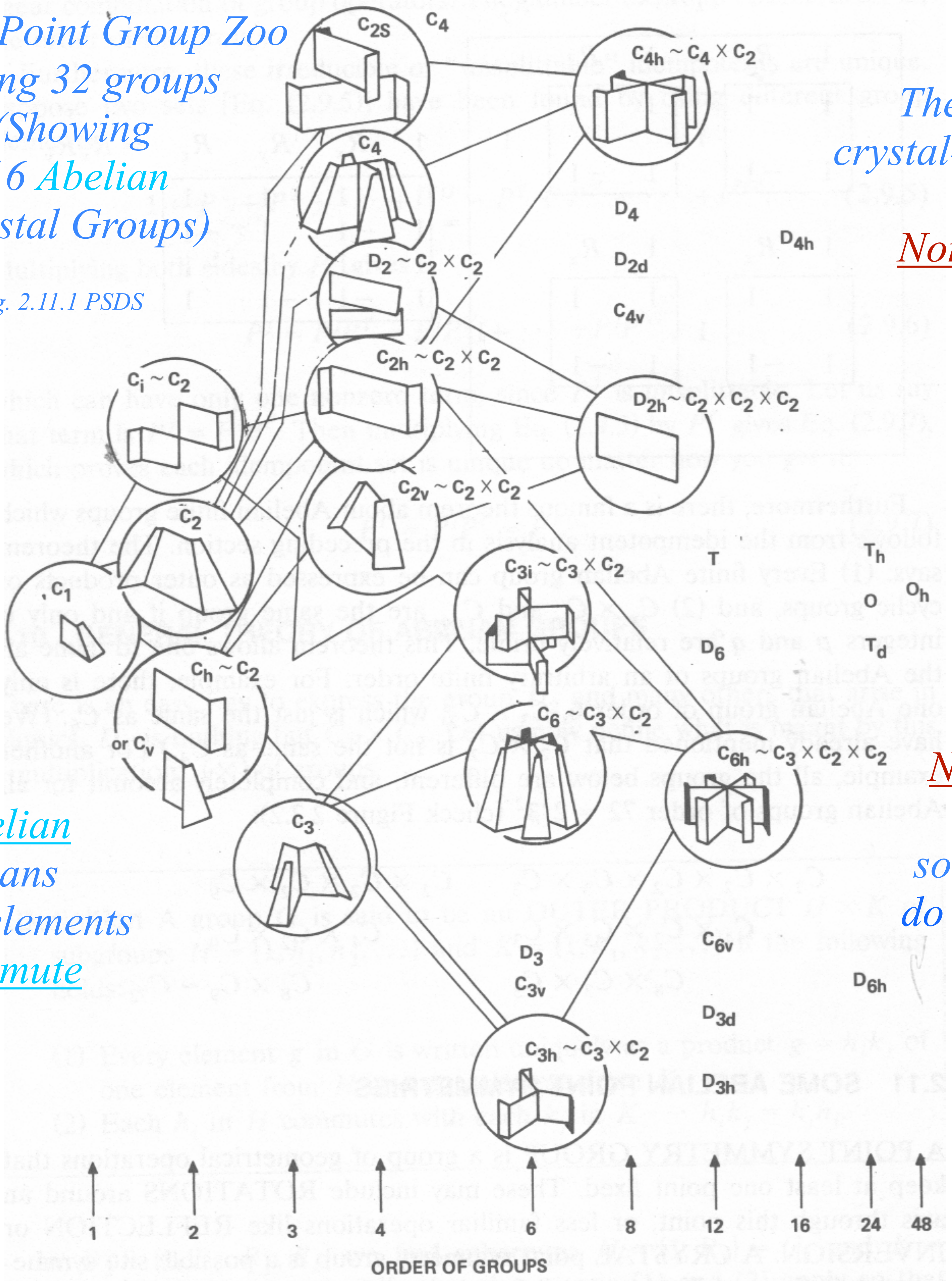


Figure 2.11.1 Abelian crystal point groups. Sixteen of the 32 crystal point groups are Abelian and are illustrated by models drawn in circles.

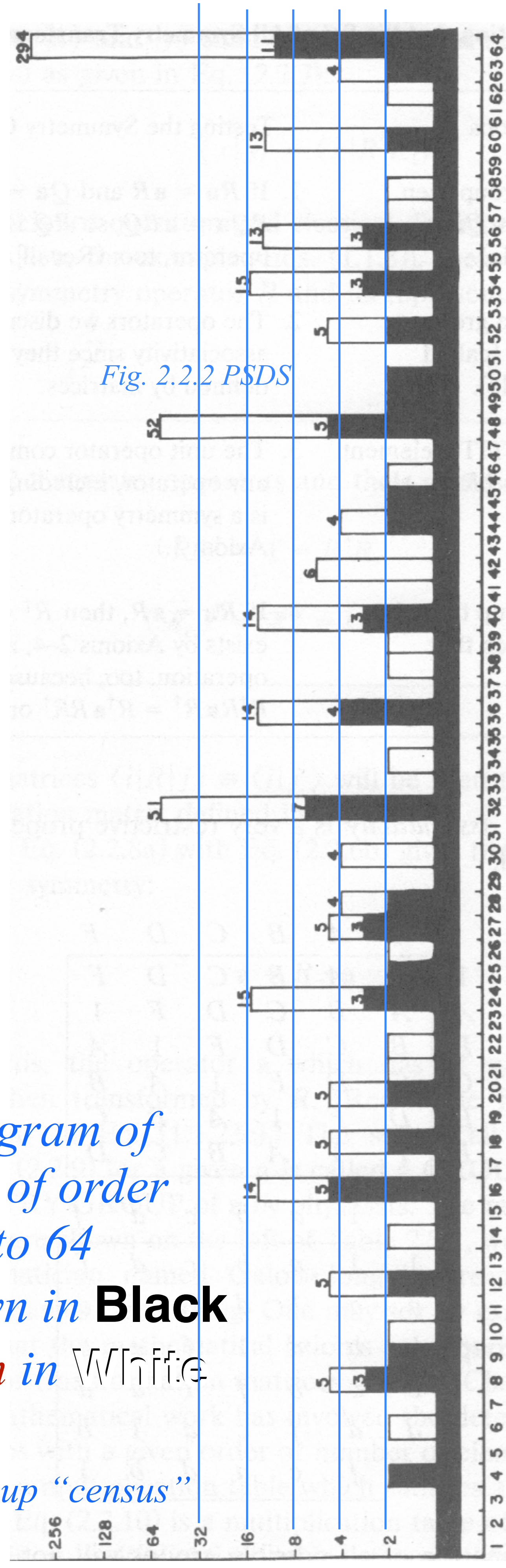


Fig. 2.2.2 PSDS

Group "census"

Order °G

Abelian means all its elements commute

Non-Abelian means some elements do not commute

Log-histogram of all groups of order °G=1 to 64

Abelian shown in **Black**

Non-Abelian in White

Crystal-Point Group Zoo
having 32 groups
(Showing
16 Non-Abelian
Crystal Groups)

Fig. 2.11.1 PSDS

The other 16
crystal-point groups
are
Abelian

Abelian
means
all its elements
commute

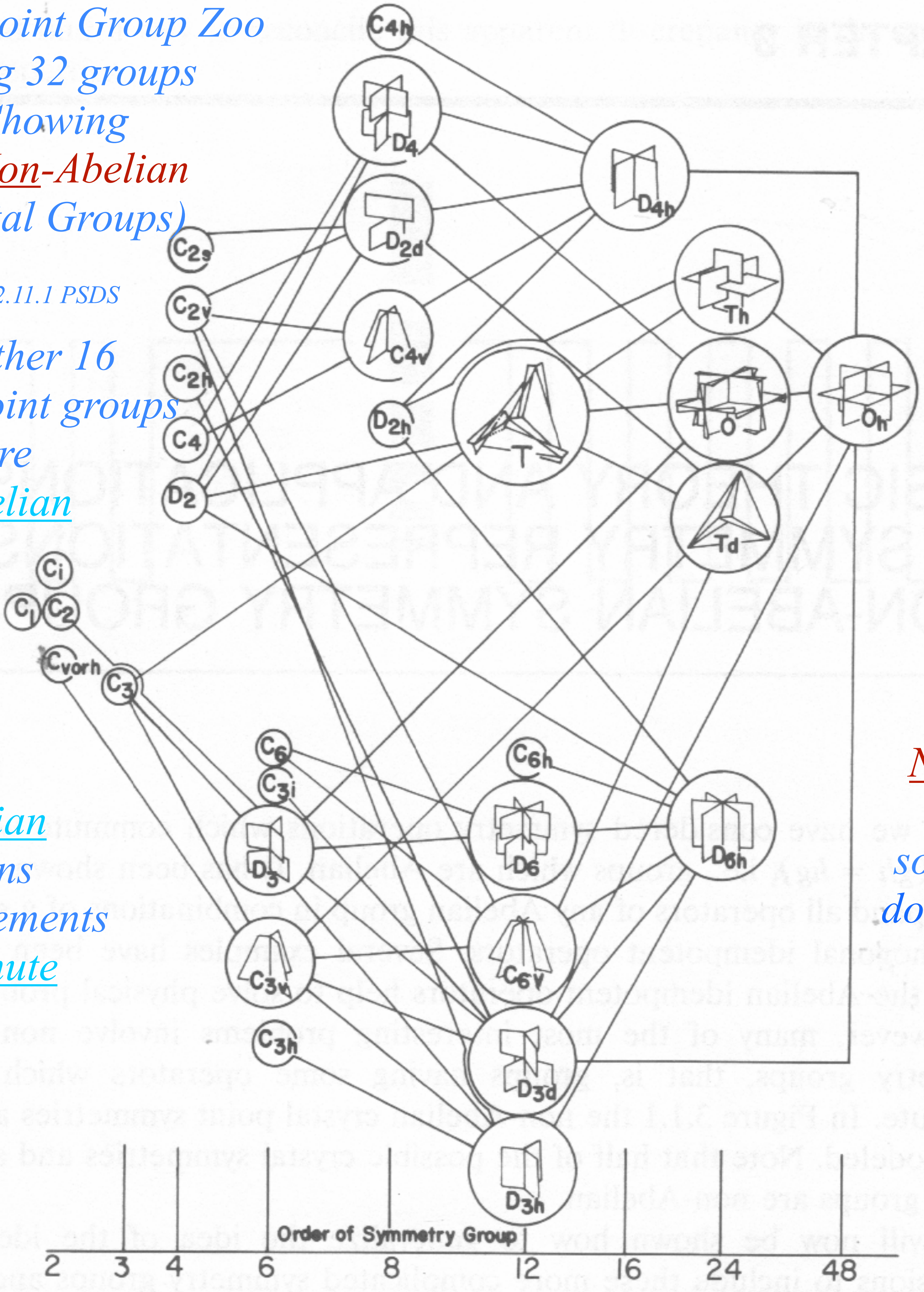
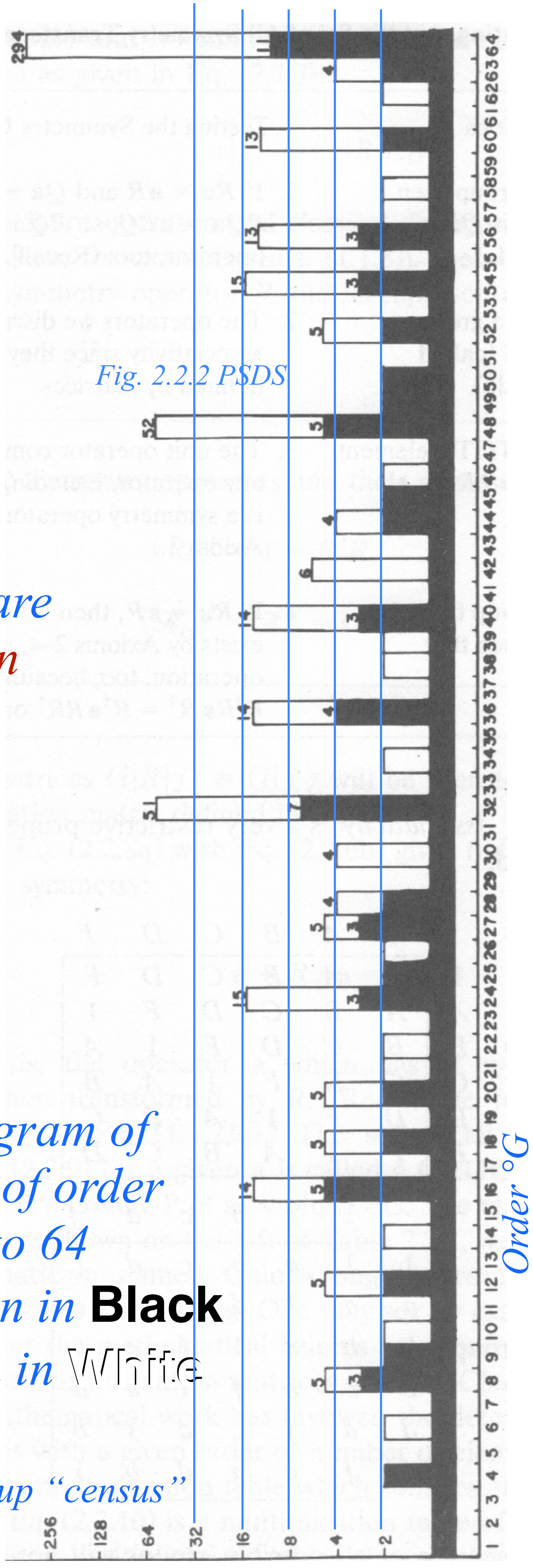


Figure 3.1.1 Crystal point symmetry groups. Models are sketched in circles for the 16 non-Abelian groups. (See also Figure 2.11.1.)

Clearly
most groups are
Non-Abelian

Non-Abelian
means
some elements
do not commute

Log-histogram of
all groups of order
°G=1 to 64
Abelian shown in **Black**
Non-Abelian in White



Group "census"

C_6 *is product* $\mathrm{C}_3\times\mathrm{C}_2$ *(but* C_4 *is NOT* $\mathrm{C}_2\times\mathrm{C}_2$*)*

C_3	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$(0)_3$	1	1	1
$(1)_3$	1	$e^{2\pi i/3}$	$e^{-2\pi i/3}$
$(2)_3$	1	$e^{-2\pi i/3}$	$e^{2\pi i/3}$

×

C_2	$\mathbf{1}$	$\mathbf{R}$
$(0)_2$	1	1
$(1)_2$	1	-1

=

$C_3\times C_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}\cdot\mathbf{R}$	$\mathbf{r}\cdot\mathbf{R}$	$\mathbf{r}^2\cdot\mathbf{R}$
$(0)_3\cdot(0)_2$	$1\cdot 1$	$1\cdot 1$	$1\cdot 1$	$1\cdot 1$	$1\cdot 1$	$1\cdot 1$
$(1)_3\cdot(0)_2$	$1\cdot 1$	$e^{2\pi i/3}\cdot 1$	$e^{-2\pi i/3}\cdot 1$	$1\cdot 1$	$e^{2\pi i/3}\cdot 1$	$e^{-2\pi i/3}\cdot 1$
$(2)_3\cdot(0)_2$	$1\cdot 1$	$e^{-2\pi i/3}\cdot 1$	$e^{2\pi i/3}\cdot 1$	$1\cdot 1$	$e^{-2\pi i/3}\cdot 1$	$e^{2\pi i/3}\cdot 1$
$(0)_3\cdot(1)_2$	$1\cdot 1$	$1\cdot 1$	$1\cdot 1$	$1\cdot(-1)$	$1\cdot(-1)$	$1\cdot(-1)$
$(1)_3\cdot(1)_2$	$1\cdot 1$	$1\cdot 1$	$e^{-2\pi i/3}\cdot 1$	$1\cdot(-1)$	$e^{2\pi i/3}\cdot(-1)$	$e^{-2\pi i/3}\cdot(-1)$
$(2)_3\cdot(1)_2$	$1\cdot 1$	$e^{-2\pi i/3}\cdot 1$	$1\cdot 1$	$1\cdot(-1)$	$e^{-2\pi i/3}\cdot(-1)$	$e^{2\pi i/3}\cdot(-1)$

C_6 *is product* $C_3\times C_2$ *(but* C_4 *is NOT* $C_2\times C_2$*)*

C_3	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$(0)_3$	1	1	1
$(1)_3$	1	$e^{2\pi i/3}$	$e^{-2\pi i/3}$
$(2)_3$	1	$e^{-2\pi i/3}$	$e^{2\pi i/3}$

×

C_2	$\mathbf{1}$	$\mathbf{R}$
$(0)_2$	1	1
$(1)_2$	1	-1

=

$C_3\times C_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}\cdot\mathbf{R}$	$\mathbf{r}\cdot\mathbf{R}$	$\mathbf{r}^2\cdot\mathbf{R}$
$(0)_3\cdot(0)_2$	$1\cdot 1$	$1\cdot 1$	$1\cdot 1$	$1\cdot 1$	$1\cdot 1$	$1\cdot 1$
$(1)_3\cdot(0)_2$	$1\cdot 1$	$e^{2\pi i/3}\cdot 1$	$e^{-2\pi i/3}\cdot 1$	$1\cdot 1$	$e^{2\pi i/3}\cdot 1$	$e^{-2\pi i/3}\cdot 1$
$(2)_3\cdot(0)_2$	$1\cdot 1$	$e^{-2\pi i/3}\cdot 1$	$e^{2\pi i/3}\cdot 1$	$1\cdot 1$	$e^{-2\pi i/3}\cdot 1$	$e^{2\pi i/3}\cdot 1$
$(0)_3\cdot(1)_2$	$1\cdot 1$	$1\cdot 1$	$1\cdot 1$	$1\cdot(-1)$	$1\cdot(-1)$	$1\cdot(-1)$
$(1)_3\cdot(1)_2$	$1\cdot 1$	$1\cdot 1$	$e^{-2\pi i/3}\cdot 1$	$1\cdot(-1)$	$e^{2\pi i/3}\cdot(-1)$	$e^{-2\pi i/3}\cdot(-1)$
$(2)_3\cdot(1)_2$	$1\cdot 1$	$e^{-2\pi i/3}\cdot 1$	$1\cdot 1$	$1\cdot(-1)$	$e^{-2\pi i/3}\cdot(-1)$	$e^{2\pi i/3}\cdot(-1)$

=

$C_3\times C_2 = C_6$	$\mathbf{1}$	$\mathbf{r} = h^2$	$\mathbf{r}^2 = h^4$	$\mathbf{R} = \mathbf{h}^3$	$\mathbf{r}\cdot\mathbf{R} = h$	$\mathbf{r}^2\cdot\mathbf{R} = h^5$
$(0)_3\cdot(0)_2 = (0)_6$	1	1	1	1	1	1
$(1)_3\cdot(0)_2 = (2)_6$	1	$e^{2\pi i/3}$	$e^{-2\pi i/3}$	1	$e^{2\pi i/3}$	$e^{-2\pi i/3}$
$(2)_3\cdot(0)_2 = (4)_6$	1	$e^{-2\pi i/3}$	$e^{2\pi i/3}$	1	$e^{-2\pi i/3}$	$e^{2\pi i/3}$
$(0)_3\cdot(1)_2 = (3)_6$	1	1	1	-1	-1	-1
$(1)_3\cdot(1)_2 = (5)_6$	1	$e^{2\pi i/3}$	$e^{-2\pi i/3}$	-1	$-e^{2\pi i/3}$	$-e^{-2\pi i/3}$
$(2)_3\cdot(1)_2 = (1)_6$	1	$e^{-2\pi i/3}$	$e^{2\pi i/3}$	-1	$-e^{-2\pi i/3}$	$-e^{2\pi i/3}$

Breaking C_N cyclic coupling into linear chains
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$$\chi^j(2\pi/n) = \frac{\sin\left(\pi(2j+1)/n\right)}{\sin\left(\pi/n\right)}$$

*The **CPT** subgroup of Lorentz Group*

Polygonal geometry of U(2)⊃C$_N$ character spectral function

Trace-character $\chi^j(\Theta)$ of U(2) rotation by C_n angle $\Theta=2\pi/n$

is an ($\ell^j=2j+1$)-term sum of $e^{-im\Theta}$ over allowed *m*-quanta $m=\{-j, -j+1, \ldots, j-1, j\}$.

$$\chi^{1/2}(\Theta) = traceD^{1/2}(\Theta) = trace\begin{pmatrix} e^{-i\theta/2} & \cdot \\ \cdot & e^{+i\theta/2} \end{pmatrix}$$

(spinor-j=1/2)

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(vector-j=1)

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(spinor-j=1/2)

$$\chi^{3/2}(\Theta) = e^{-i\frac{3\Theta}{2}} + \ldots + e^{i\frac{3\Theta}{2}} = 2\cos\frac{\Theta}{2} + 2\cos\frac{3\Theta}{2}$$

$$\chi^{5/2}(\Theta) = e^{-i\frac{5\Theta}{2}} + \ldots + e^{i\frac{5\Theta}{2}} = 2\cos\frac{\Theta}{2} + 2\cos\frac{3\Theta}{2} + 2\cos\frac{5\Theta}{2}$$

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$$\chi^{0}(\Theta) = e^{-i\Theta\cdot 0} = 1$$

(scalar-j=0)

$$\chi^{1}(\Theta) = e^{-i\Theta} + 1 + e^{i\Theta} = 1 + 2\cos\Theta$$

(vector-j=1)

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(spinor-$j=1/2$)

$$\chi^{3/2}(\Theta)=e^{-i\frac{3\Theta}{2}}+\dots\quad+e^{i\frac{3\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}$$

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$$\chi^{1}(\Theta)=e^{-i\Theta}+1+e^{i\Theta}=1+2\cos\Theta$$

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$$\chi^{2}(\Theta)=e^{-i2\Theta}+\dots e^{i2\Theta}=1+2\cos\Theta+2\cos 2\Theta$$

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$\chi^j(\Theta)$ is a geometric series with ratio $e^{i\Theta}$ between each successive term.

$$\chi^{j}(\Theta)=TraceD^{(j)}(\Theta)=e^{-i\Theta j}+e^{-i\Theta(j-1)}+e^{-i\Theta(j-2)}+\dots+e^{+i\Theta(j-2)}+e^{+i\Theta(j-1)}+e^{+i\Theta j}$$

Polygonal geometry of $U(2)\supset C_N$ character spectral function

Trace-character $\chi^j(\Theta)$ of U(2) rotation by C_n angle $\Theta=2\pi/n$

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$$\chi^{1}(\Theta) = e^{-i\Theta} + 1 + e^{i\Theta} = 1 + 2\cos\Theta$$

(vector-j=1)

$$\chi^{5/2}(\Theta) = e^{-i\frac{5\Theta}{2}} + ... + e^{i\frac{5\Theta}{2}} = 2\cos\frac{\Theta}{2} + 2\cos\frac{3\Theta}{2} + 2\cos\frac{5\Theta}{2}$$

$$\chi^{2}(\Theta) = e^{-i2\Theta} + ...e^{i2\Theta} = 1 + 2\cos\Theta + 2\cos 2\Theta$$

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$\chi^j(\Theta)$ is a geometric series with ratio $e^{i\Theta}$ between each successive term.

$$\chi^{j}(\Theta) = TraceD^{(j)}(\Theta) = e^{-i\Theta j} + e^{-i\Theta(j-1)} + e^{-i\Theta(j-2)} + ...+ e^{+i\Theta(j-2)} + e^{+i\Theta(j-1)} + e^{+i\Theta j}$$

$$\chi^{j}(\Theta)e^{-i\Theta} = e^{-i\Theta(j+1)} + e^{-i\Theta j} + e^{-i\Theta(j-1)} + e^{-i\Theta(j-2)} + ...+ e^{+i\Theta(j-2)} + e^{+i\Theta(j-1)}$$

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$$\chi^{1/2}(\Theta)=e^{-i\frac{\Theta}{2}}+e^{i\frac{\Theta}{2}} \quad =2\cos\frac{\Theta}{2}$$

(spinor-j=1/2)

$$\chi^{3/2}(\Theta)=e^{-i\frac{3\Theta}{2}}+\ldots \quad +e^{i\frac{3\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}$$

$$\chi^{5/2}(\Theta)=e^{-i\frac{5\Theta}{2}}+\ldots \quad +e^{i\frac{5\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}+2\cos\frac{5\Theta}{2}$$

$$\chi^{0}(\Theta)=e^{-i\Theta\cdot 0} \quad =1$$

(scalar-j=0)

$$\chi^{1}(\Theta)=e^{-i\Theta}+1+e^{i\Theta} \quad =1+2\cos\Theta$$

(vector-j=1)

$$\chi^{2}(\Theta)=e^{-i2\Theta}+\ldots e^{i2\Theta} \quad =1+2\cos\Theta+2\cos 2\Theta$$

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$\chi^j(\Theta)$ is a geometric series with ratio $e^{i\Theta}$ between each successive term.

$$\chi^j(\Theta)=TraceD^{(j)}(\Theta)=\cancel{e^{-i\Theta j}}+e^{-i\Theta(j-1)}+e^{-i\Theta(j-2)}+\ldots+e^{+i\Theta(j-2)}+\cancel{e^{+i\Theta(j-1)}}+e^{+i\Theta j}$$

$$\chi^j(\Theta)e^{-i\Theta}=e^{-i\Theta(j+1)}+\cancel{e^{-i\Theta j}}+e^{-i\Theta(j-1)}+e^{-i\Theta(j-2)}+\ldots+\cancel{e^{+i\Theta(j-2)}}+e^{+i\Theta(j-1)}$$

Subtracting gives:

$$\chi^j(\Theta)(1-e^{-i\Theta})=-e^{-i\Theta(j+1)} \qquad + \qquad e^{+i\Theta j}$$

Polygonal geometry of $U(2)\supset C_N$ character spectral function

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(spinor-$j=1/2$)

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$$\chi^{1/2}(\Theta)=e^{-i\frac{\Theta}{2}}+e^{i\frac{\Theta}{2}} \quad =2\cos\frac{\Theta}{2}$$

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$$\chi^{3/2}(\Theta)=e^{-i\frac{3\Theta}{2}}+\ldots \quad +e^{i\frac{3\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}$$

$$\chi^{1}(\Theta)=e^{-i\Theta}+1+e^{i\Theta} \quad =1+2\cos\Theta$$

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$$\chi^{5/2}(\Theta)=e^{-i\frac{5\Theta}{2}}+\ldots \quad +e^{i\frac{5\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}+2\cos\frac{5\Theta}{2}$$

$$\chi^{2}(\Theta)=e^{-i2\Theta}+\ldots e^{i2\Theta} \quad =1+2\cos\Theta+2\cos 2\Theta$$

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$\chi^j(\Theta)$ is a geometric series with ratio $e^{i\Theta}$ between each successive term.

$$\chi^{j}(\Theta)=TraceD^{(j)}(\Theta)=e^{-i\Theta j}+e^{-i\Theta(j-1)}+e^{-i\Theta(j-2)}+\ldots+e^{+i\Theta(j-2)}+e^{+i\Theta(j-1)}+e^{+i\Theta j}$$

$$\chi^{j}(\Theta)e^{-i\Theta}=e^{-i\Theta(j+1)}+e^{-i\Theta j}+e^{-i\Theta(j-1)}+e^{-i\Theta(j-2)}+\ldots+e^{+i\Theta(j-2)}+e^{+i\Theta(j-1)}$$

Subtracting/dividing gives $\chi^j(\Theta)$ formula.

$$\chi^{j}(\Theta)=\frac{e^{+i\Theta j}-e^{-i\Theta(j+1)}}{1-e^{-i\Theta}}=\frac{e^{+i\Theta(j+\frac{1}{2})}-e^{-i\Theta(j+\frac{1}{2})}}{e^{+i\frac{\Theta}{2}}-e^{-i\frac{\Theta}{2}}}=\frac{\sin\Theta(j+\frac{1}{2})}{\sin\frac{\Theta}{2}}$$

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$$\underset{\textit{(spinor-j=1/2)}}{\chi^{1/2}(\Theta) = traceD^{1/2}(\Theta)} = trace\begin{pmatrix} e^{-i\theta/2} & \cdot \\ \cdot & e^{+i\theta/2} \end{pmatrix} \qquad \underset{\textit{(vector-j=1)}}{\chi^{1}(\Theta) = traceD^{1}(\Theta)} = trace\begin{pmatrix} e^{-i\theta} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{-i\theta} \end{pmatrix}$$

$\chi^j(\Theta)$ involves a sum of $2\cos(m\,\Theta/2)$ for $m\geq 0$ up to $m=j$.

$$\chi^{1/2}(\Theta) = e^{-i\frac{\Theta}{2}} + e^{i\frac{\Theta}{2}} \qquad = 2\cos\frac{\Theta}{2} \qquad \textit{(spinor-j=1/2)}$$

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(scalar-j=0)

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$$\chi^j(\Theta)e^{-i\Theta} = e^{-i\Theta(j+1)} + \cancel{e^{-i\Theta j}} + e^{-i\Theta(j-1)} + e^{-i\Theta(j-2)} + \ldots + e^{+i\Theta(j-2)} + \cancel{e^{+i\Theta(j-1)}}$$

Subtracting/dividing gives $\chi^j(\Theta)$ formula.

$$\chi^j(\Theta) = \frac{e^{+i\Theta j} - e^{-i\Theta(j+1)}}{1 - e^{-i\Theta}} = \frac{e^{+i\Theta(j+\frac{1}{2})} - e^{-i\Theta(j+\frac{1}{2})}}{e^{+i\frac{\Theta}{2}} - e^{-i\frac{\Theta}{2}}} = \frac{\sin\Theta(j+\frac{1}{2})}{\sin\frac{\Theta}{2}}$$

For C_n angle $\Theta=2\pi/n$ this χ^j has a lot of geometric significance.

$$\boxed{\chi^j(\frac{2\pi}{n}) = \frac{\sin\frac{\pi}{n}(2j+1)}{\sin\frac{\pi}{n}} = \frac{\sin\frac{\pi\ell^j}{n}}{\sin\frac{\pi}{n}}}$$

Character Spectral Function
where: $\ell^j=2j+1$
is U(2) irrep dimension

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The **CPT** *subgroup of Lorentz Group*

$$\chi^j(2\pi/n)=\frac{\sin\left(\pi(2j+1)/n\right)}{\sin\left(\pi/n\right)}$$

Polygonal geometry of $U(2)\supset C_N$ character spectral function

$$\chi^j\left(\frac{2\pi}{n}\right)=\frac{\sin\frac{\pi}{n}(2j+1)}{\sin\frac{\pi}{n}}=\frac{\sin\frac{\pi\ell^j}{n}}{\sin\frac{\pi}{n}}$$

Character Spectral Function
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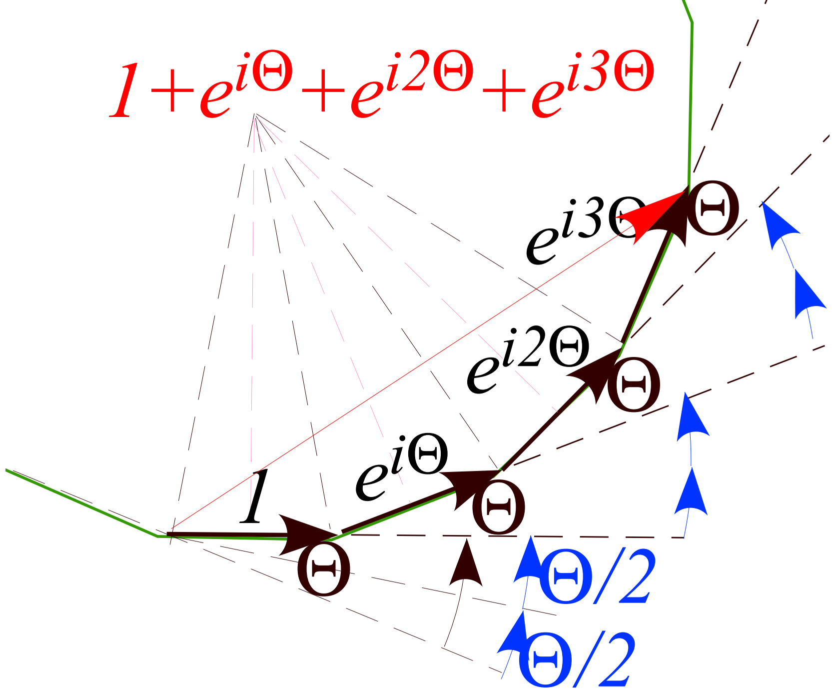
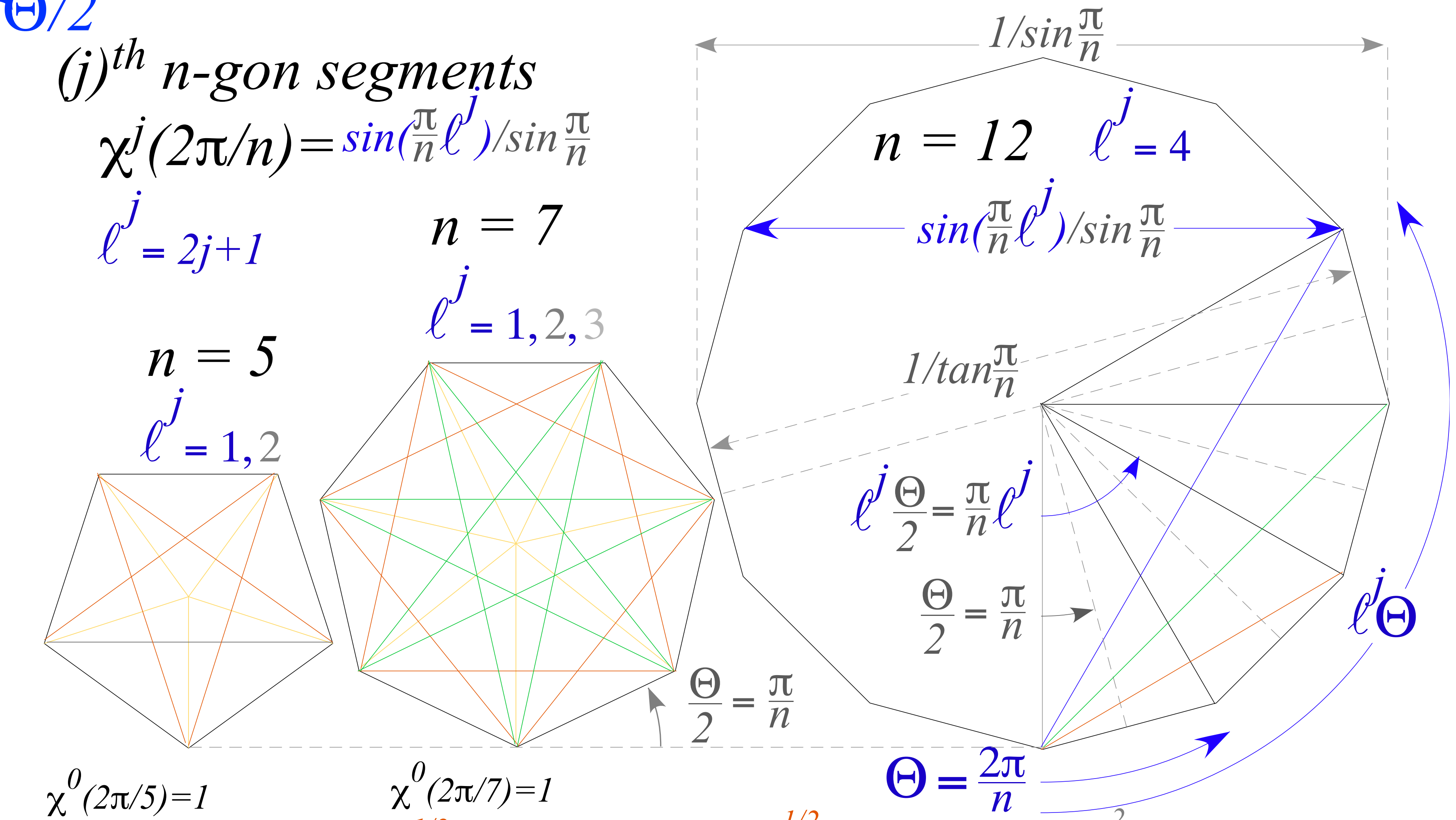


$(j)^{th}$ n-gon segments

$$\chi^j(2\pi/n)=\sin\left(\frac{\pi}{n}\ell^j\right)/\sin\frac{\pi}{n}$$

$$\ell^j=2j+1$$



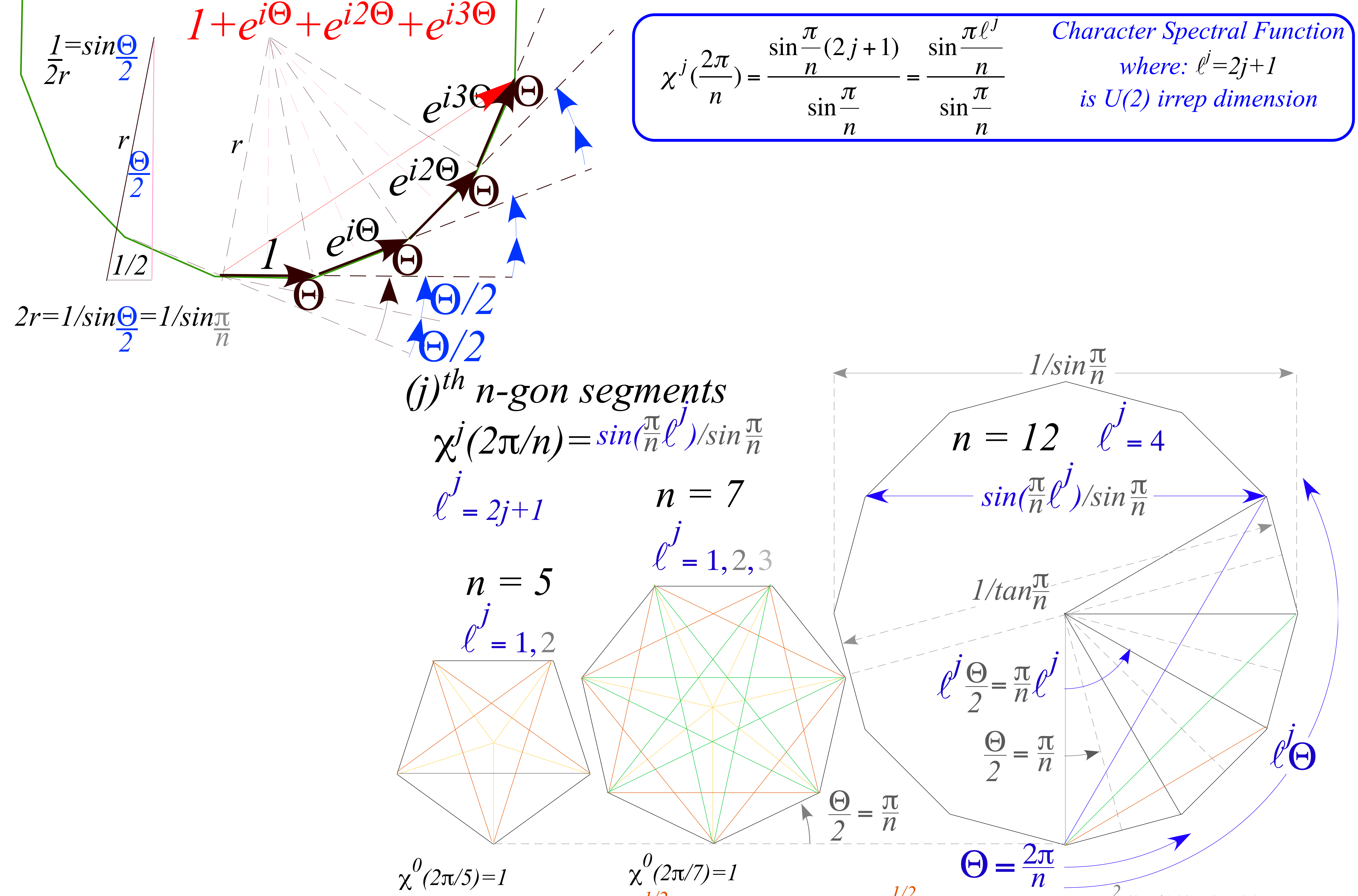
$\chi^0(2\pi/5)=1$
$\chi^{1/2}(2\pi/5)=1.618...$
$=(1+\sqrt{5})/2=$

$\chi^0(2\pi/7)=1$
$\chi^{1/2}(2\pi/7)=1.802...$
$\chi^1(2\pi/7)=2.247...$
$\chi^{3/2}(2\pi/7)=2.247...$

$\chi^{1/2}(2\pi/12)=1.932...$
$\chi^1(2\pi/12)=2.732...$
$\chi^{3/2}(2\pi/12)=3.346...$

$\chi^2(2\pi/12)=3.732...$
$\chi^{5/2}(2\pi/12)=3.864...$
$\chi^3(2\pi/12)=3.732...$

Polygonal geometry of U(2)⊃C_N character spectral function



$$\chi^j\left(\frac{2\pi}{n}\right)=\frac{\sin\frac{\pi}{n}(2j+1)}{\sin\frac{\pi}{n}}=\frac{\sin\frac{\pi\ell^j}{n}}{\sin\frac{\pi}{n}}$$

Character Spectral Function
where: $\ell^j=2j+1$
is U(2) irrep dimension

$(j)^{th}$ n-gon segments

$$\chi^j(2\pi/n)=\sin(\tfrac{\pi}{n}\ell^j)/\sin\tfrac{\pi}{n}$$

$\ell^j = 2j+1$

$\chi^0(2\pi/5)=1$
$\chi^{1/2}(2\pi/5)=1.618...$
$=(1+\surd 5)/2=$

$\chi^0(2\pi/7)=1$
$\chi^{1/2}(2\pi/7)=1.802...$
$\chi^1(2\pi/7)=2.247...$
$\chi^{3/2}(2\pi/7)=2.247...$

$\chi^{1/2}(2\pi/12)=1.932...$
$\chi^1(2\pi/12)=2.732...$
$\chi^{3/2}(2\pi/12)=3.346...$

$\chi^2(2\pi/12)=3.732...$
$\chi^{5/2}(2\pi/12)=3.864...$
$\chi^3(2\pi/12)=3.732...$

Polygonal geometry of $U(2)\supset C_N$ character spectral function

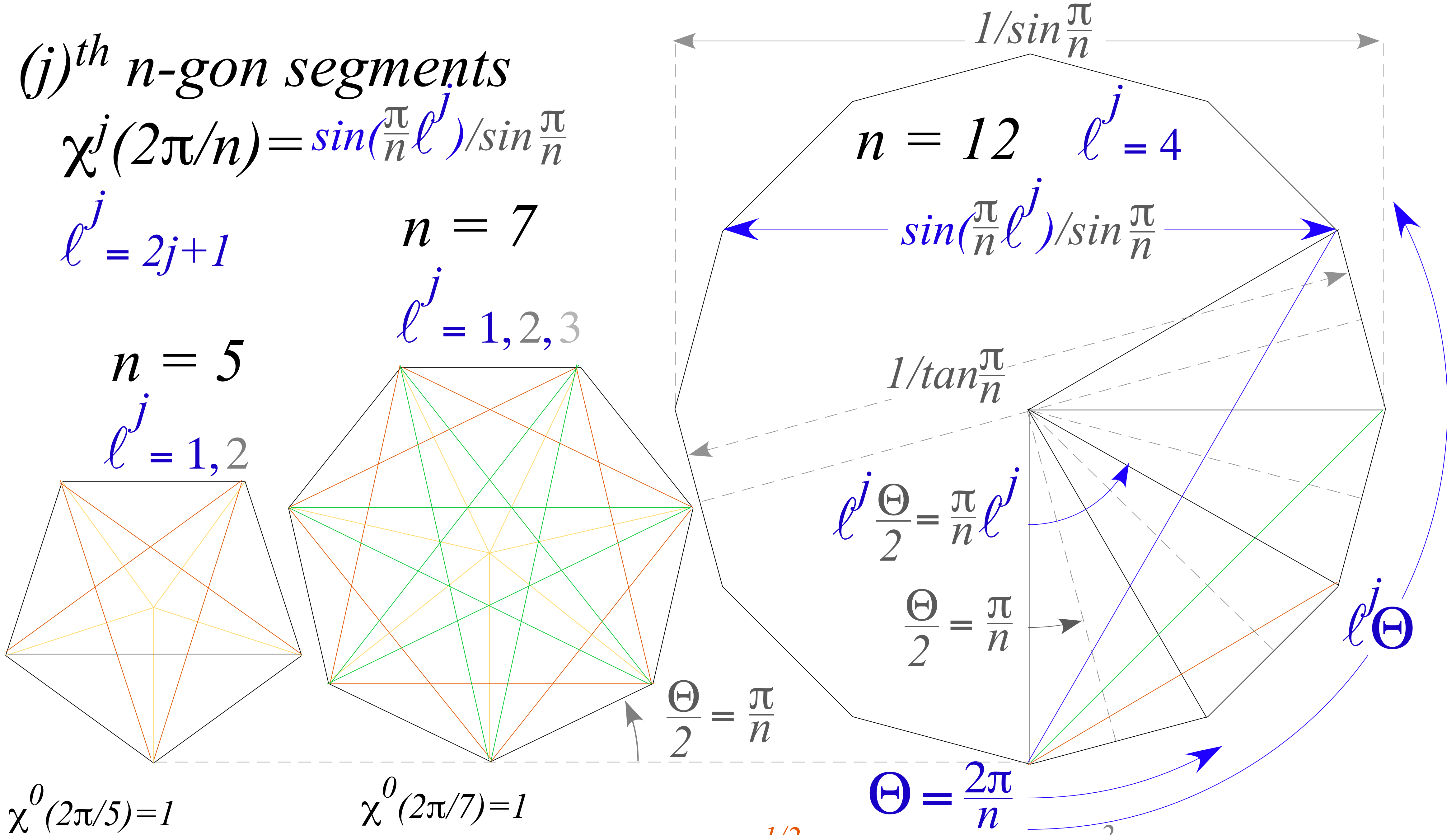
$$\chi^j\left(\frac{2\pi}{n}\right) = \frac{\sin\frac{\pi}{n}(2j+1)}{\sin\frac{\pi}{n}} = \frac{\sin\frac{\pi\ell^j}{n}}{\sin\frac{\pi}{n}}$$

Character Spectral Function
where: $\ell^j=2j+1$
is U(2) irrep dimension

$(j)^{th}$ n-gon segments

$$\chi^j(2\pi/n) = \sin\left(\frac{\pi}{n}\ell^j\right)/\sin\frac{\pi}{n}$$

$$\ell^j = 2j+1$$



$\chi^0(2\pi/5)=1$
$\chi^{1/2}(2\pi/5)=1.618...$
$=(1+\sqrt{5})/2=$

$\chi^0(2\pi/7)=1$
$\chi^{1/2}(2\pi/7)=1.802...$
$\chi^1(2\pi/7)=2.247...$
$\chi^{3/2}(2\pi/7)=2.247...$

$\chi^{1/2}(2\pi/12)=1.932...$
$\chi^1(2\pi/12)=2.732...$
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$\chi^2(2\pi/12)=3.732...$
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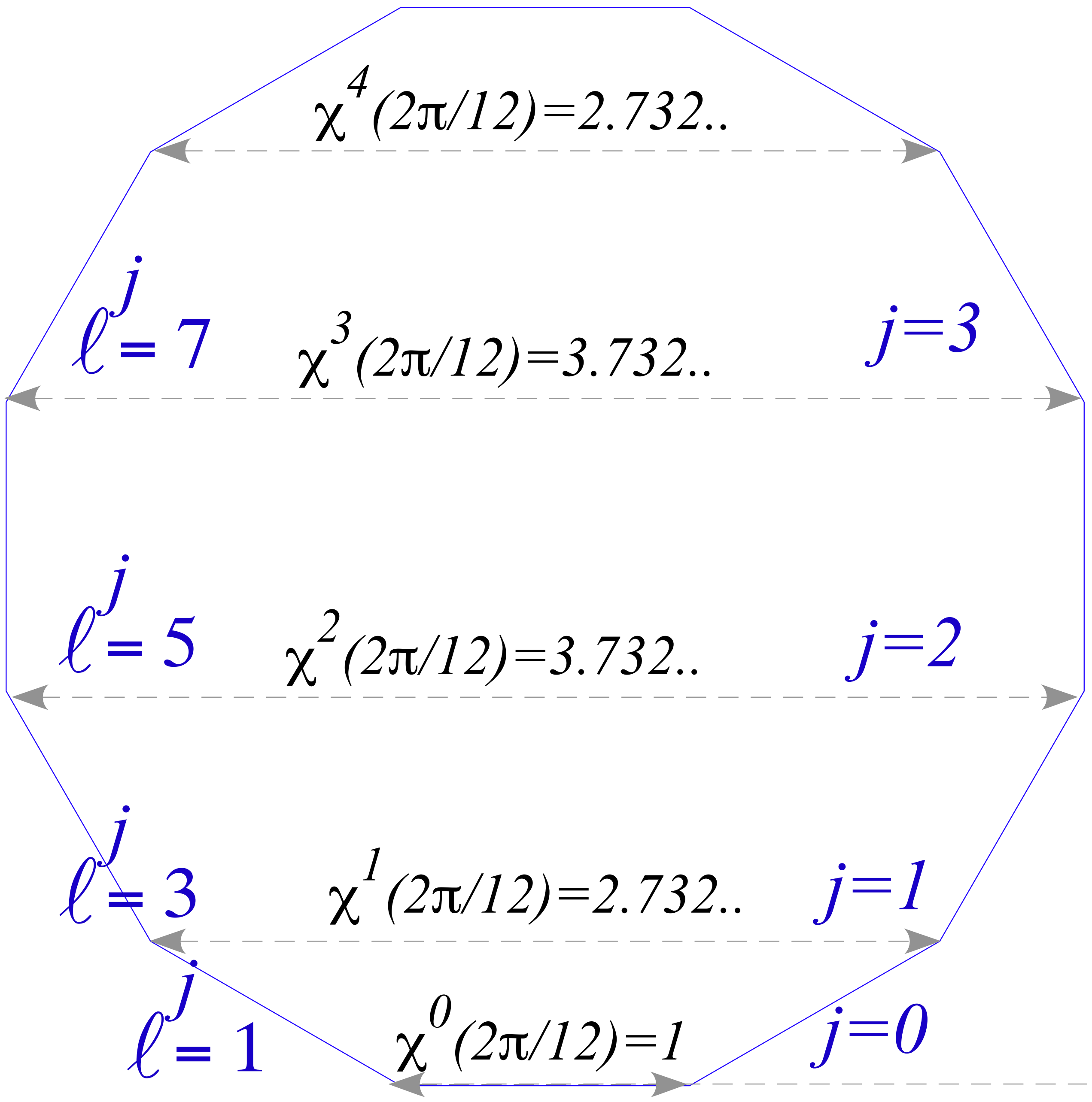
Polygonal geometry of $U(2)\supset C_N$ character spectral function

$$\chi^j\left(\frac{2\pi}{n}\right)=\frac{\sin\frac{\pi}{n}(2j+1)}{\sin\frac{\pi}{n}}=\frac{\sin\frac{\pi\ell^j}{n}}{\sin\frac{\pi}{n}}$$

Character Spectral Function
where: $\ell^j=2j+1$
is U(2) irrep dimension

Integer j for n=12



$\chi^4(2\pi/12)=2.732..$

$\ell^j=7$ $\chi^3(2\pi/12)=3.732..$ $j=3$

$\ell^j=5$ $\chi^2(2\pi/12)=3.732..$ $j=2$

$\ell^j=3$ $\chi^1(2\pi/12)=2.732..$ $j=1$

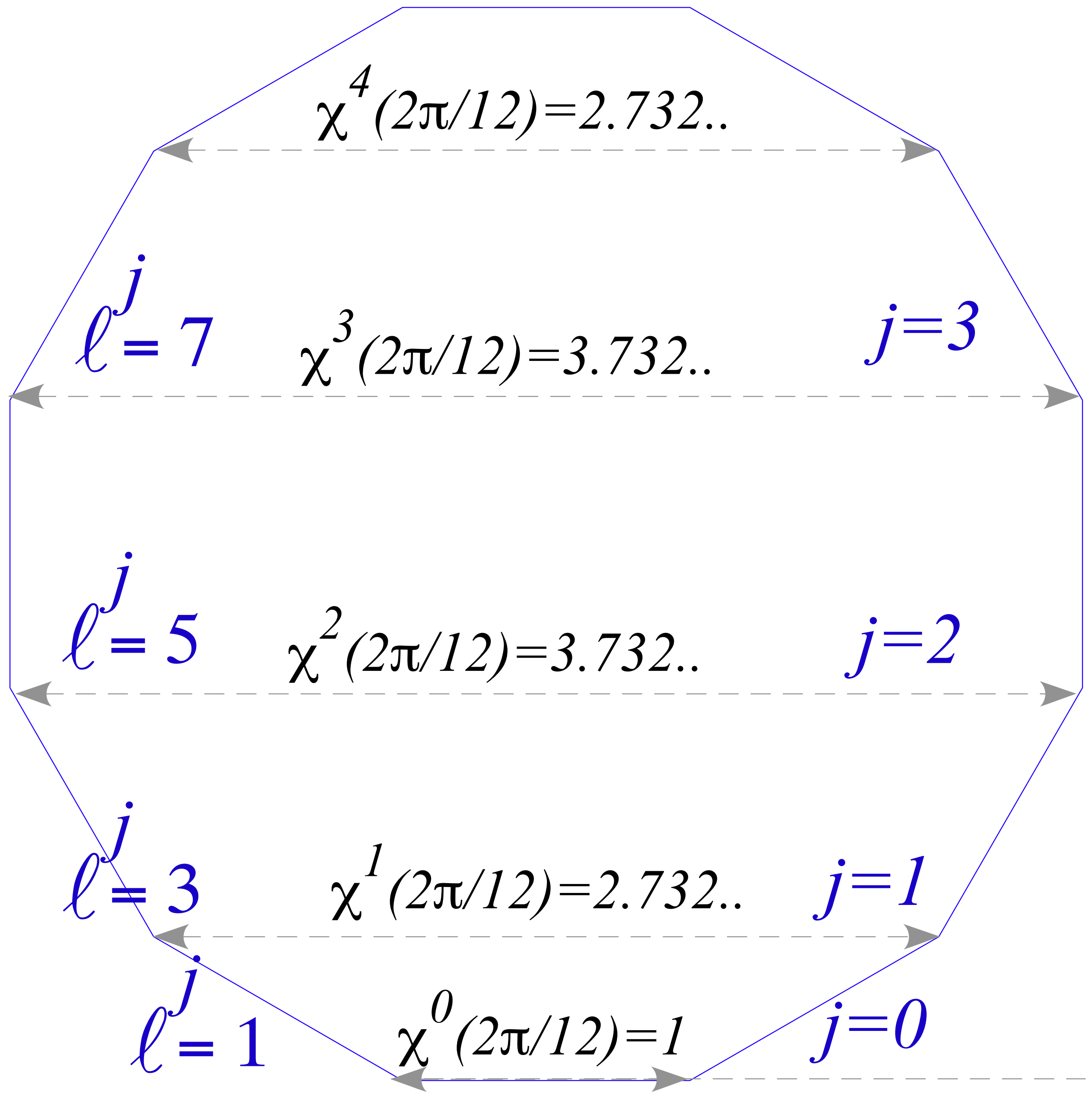
$\ell^j=1$ $\chi^0(2\pi/12)=1$ $j=0$

Polygonal geometry of U(2)⊃C$_N$ character spectral function

$$\chi^j\left(\frac{2\pi}{n}\right) = \frac{\sin\frac{\pi}{n}(2j+1)}{\sin\frac{\pi}{n}} = \frac{\sin\frac{\pi\ell^j}{n}}{\sin\frac{\pi}{n}}$$

Character Spectral Function
where: $\ell^j=2j+1$
is U(2) irrep dimension

Integer j for n=12



1/2-Integer j for n=12