Group Theory in Quantum Mechanics

Lecture 12 (2.23.17)

Symmetry and Dynamics of C_N cyclic systems

(Geometry of U(2) characters - Ch. 6-9 of Unit 3)
(Principles of Symmetry, Dynamics, and Spectroscopy - Sec. 3-7 of Ch. 2)

1st Step: Abelian symmetry analysis Expand C_6 symmetric **H** by C_6 group product table
 Regular representation and coupling parameters $\{r_0,r_1,r_2,r_3,r_4,r_5\}$ and Fourier dispersion
2nd Step: Find **H** eigenfunctions by spectral resolution of $C_6 = \{\mathbf{1}=\mathbf{r}^0,\mathbf{r}^1,\mathbf{r}^2,\mathbf{r}^3,\mathbf{r}^4,\mathbf{r}^5\}$
 Character tables of C_2, C_3, C_4, C_5,.... C_{144}
3rd Step: Dispersion functions and eigenvalues for varyious coupling parameter systems
 Ortho-complete eigenvalue/parameter relations
  Gauge shifts due to complex coupling

Wave dynamics of phase, mean phase, and group velocity by Expo-Cosine identity
Relating space-time (x,t) and per-space-time (k,ω)
 Wave coordinates
 Pulse-waves (PW) vs Continuous-waves (CW)
 Wave coordinates for Linear Dispersion
 Wave coordinates for Bohr-Schrodinger Dispersion
 Einstein-Lorentz-Minkowski laser coordinates

Some Topics for Lecture 13

Introduction to C_N beat dynamics and "Revivals" due to Bohr-dispersion
 ∞-Square well PE versus Bohr rotor
 SinNx/x wavepackets bandwidth and uncertainty
  SinNx/x explosion and revivals
 Bohr-rotor dynamics
  Gaussian wave-packet bandwidth and uncertainty
  Gaussian Bohr-rotor revivals
 Farey-Sums ,Ford-products, and Phase dynamics

1st Step: Abelian symmetry analysis Expand C_6 symmetric $\mathbf{H}$ by C_6 group product table

Regular representation and coupling parameters $\{r_0,r_1,r_2,r_3,r_4,r_5\}$ and Fourier dispersion

2nd Step: Find $\mathbf{H}$ eigenfunctions by spectral resolution of $C_6 =\{\mathbf{1}=\mathbf{r}^0,\mathbf{r}^1,\mathbf{r}^2,\mathbf{r}^3,\mathbf{r}^4,\mathbf{r}^5\}$

Character tables of C_2, C_3, C_4, C_5,…. C_{144}

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Wave coordinates

Pulse-waves (PW) vs Continuous-waves (CW)

Wave coordinates for Linear Dispersion

Wave coordinates for Bohr-Schrodinger Dispersion

Einstein-Lorentz-Minkowski laser coordinates

1^{st} Step in Abelian symmetry analysis

Expand C_6 symmetric **H** *matrix using C_6 group table($\overset{gg^\dagger}{form}$)*

$$\mathbf{H}= r_0\mathbf{r}^0+r_1\mathbf{r}^1+r_2\mathbf{r}^2+\ldots+r_{n-1}\mathbf{r}^{n-1}=\Sigma r_q\mathbf{r}^k$$

$g^\dagger$

	C_6	**1**	$\mathbf{r}^5$	$\mathbf{r}^4$	$\mathbf{r}^3$	$\mathbf{r}^2$	$\mathbf{r}$
	1	**1**	$\mathbf{r}^5$	$\mathbf{r}^4$	$\mathbf{r}^3$	$\mathbf{r}^2$	$\mathbf{r}$
	$\mathbf{r}$	$\mathbf{r}$	**1**	$\mathbf{r}^5$	$\mathbf{r}^4$	$\mathbf{r}^3$	$\mathbf{r}^2$
	$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}$	**1**	$\mathbf{r}^5$	$\mathbf{r}^4$	$\mathbf{r}^3$
g	$\mathbf{r}^3$	$\mathbf{r}^3$	$\mathbf{r}^2$	$\mathbf{r}$	**1**	$\mathbf{r}^5$	$\mathbf{r}^4$
	$\mathbf{r}^4$	$\mathbf{r}^4$	$\mathbf{r}^3$	$\mathbf{r}^2$	$\mathbf{r}$	**1**	$\mathbf{r}^5$
	$\mathbf{r}^5$	$\mathbf{r}^5$	$\mathbf{r}^4$	$\mathbf{r}^3$	$\mathbf{r}^2$	$\mathbf{r}$	**1**

$$\mathbf{H} = r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2 + r_3\mathbf{r}^3 + r_4\mathbf{r}^4 + r_5\mathbf{r}^5$$

$$\begin{pmatrix} r_0 & r_5 & r_4 & r_3 & r_2 & r_1 \\ r_1 & r_0 & r_5 & r_4 & r_3 & r_2 \\ r_2 & r_1 & r_0 & r_5 & r_4 & r_3 \\ r_3 & r_2 & r_1 & r_0 & r_5 & r_4 \\ r_4 & r_3 & r_2 & r_1 & r_0 & r_5 \\ r_5 & r_4 & r_3 & r_2 & r_1 & r_0 \end{pmatrix} = r_0\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix} + r_1\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix} + r_2\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix} + r_3\begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix} + r_4\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + r_5\begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

C_6 group table gives ***r****-matrices,...*

(known as a *regular* representation of the group)

1^{st} Step in Abelian symmetry analysis

*Expand C_6 symmetric **H** matrix using C_6 group table($\overset{gg^\dagger}{form}$)*

$$\mathbf{H} = r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2 + \ldots + r_{n-1}\mathbf{r}^{n-1} = \Sigma\, r_q\mathbf{r}^k$$

$$\begin{array}{cc|cccccc} & & & & g^\dagger & & & \\ & C_6 & \mathbf{1} & \mathbf{r}^5 & \mathbf{r}^4 & \mathbf{r}^3 & \mathbf{r}^2 & \mathbf{r} \\ \hline & \mathbf{1} & \mathbf{1} & \mathbf{r}^5 & \mathbf{r}^4 & \mathbf{r}^3 & \mathbf{r}^2 & \mathbf{r} \\ & \mathbf{r} & \mathbf{r} & \mathbf{1} & \mathbf{r}^5 & \mathbf{r}^4 & \mathbf{r}^3 & \mathbf{r}^2 \\ & \mathbf{r}^2 & \mathbf{r}^2 & \mathbf{r} & \mathbf{1} & \mathbf{r}^5 & \mathbf{r}^4 & \mathbf{r}^3 \\ g & \mathbf{r}^3 & \mathbf{r}^3 & \mathbf{r}^2 & \mathbf{r} & \mathbf{1} & \mathbf{r}^5 & \mathbf{r}^4 \\ & \mathbf{r}^4 & \mathbf{r}^4 & \mathbf{r}^3 & \mathbf{r}^2 & \mathbf{r} & \mathbf{1} & \mathbf{r}^5 \\ & \mathbf{r}^5 & \mathbf{r}^5 & \mathbf{r}^4 & \mathbf{r}^3 & \mathbf{r}^2 & \mathbf{r} & \mathbf{1} \end{array}$$

$$\mathbf{H} = r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2 + r_3\mathbf{r}^3 + r_4\mathbf{r}^4 + r_5\mathbf{r}^5$$

$$\begin{pmatrix} r_0 & r_5 & r_4 & r_3 & r_2 & r_1 \\ r_1 & r_0 & r_5 & r_4 & r_3 & r_2 \\ r_2 & r_1 & r_0 & r_5 & r_4 & r_3 \\ r_3 & r_2 & r_1 & r_0 & r_5 & r_4 \\ r_4 & r_3 & r_2 & r_1 & r_0 & r_5 \\ r_5 & r_4 & r_3 & r_2 & r_1 & r_0 \end{pmatrix} = r_0\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix} + r_1\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix} + r_2\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix} + r_3\begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix} + r_4\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + r_5\begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

*C_6 group table gives **r**-matrices,...*

Put “1” wherever $\mathbf{r}^3$ appears in product-table

(known as a *regular* representation of the group)

1st Step in Abelian symmetry analysis

*Expand C_6 symmetric **H** matrix using C_6 group table($\overset{gg^\dagger}{form}$)*

$g^\dagger$

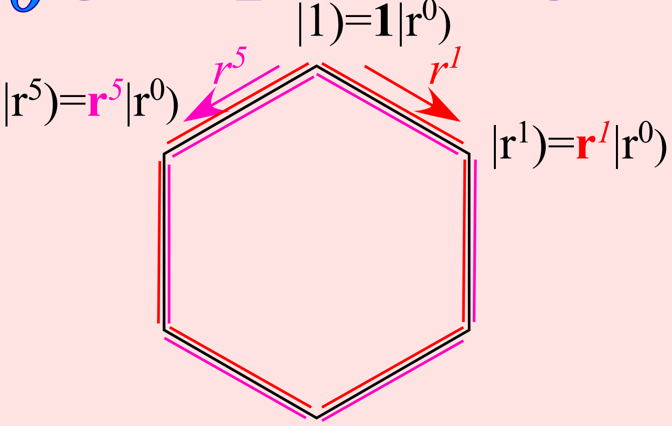
	C_6	$\mathbf{1}$	$\mathbf{r}^5$	$\mathbf{r}^4$	$\mathbf{r}^3$	$\mathbf{r}^2$	$\mathbf{r}$
	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^5$	$\mathbf{r}^4$	$\mathbf{r}^3$	$\mathbf{r}^2$	$\mathbf{r}$
	$\mathbf{r}$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^5$	$\mathbf{r}^4$	$\mathbf{r}^3$	$\mathbf{r}^2$
	$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^5$	$\mathbf{r}^4$	$\mathbf{r}^3$
g	$\mathbf{r}^3$	$\mathbf{r}^3$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^5$	$\mathbf{r}^4$
	$\mathbf{r}^4$	$\mathbf{r}^4$	$\mathbf{r}^3$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^5$
	$\mathbf{r}^5$	$\mathbf{r}^5$	$\mathbf{r}^4$	$\mathbf{r}^3$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$

$$\mathbf{H}= r_0\mathbf{r}^0+r_1\mathbf{r}^1+r_2\mathbf{r}^2+\ldots+r_{n-1}\mathbf{r}^{n-1}=\Sigma r_q\mathbf{r}^k$$

$$\mathbf{H} = r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2 + r_3\mathbf{r}^3 + r_4\mathbf{r}^4 + r_5\mathbf{r}^5$$

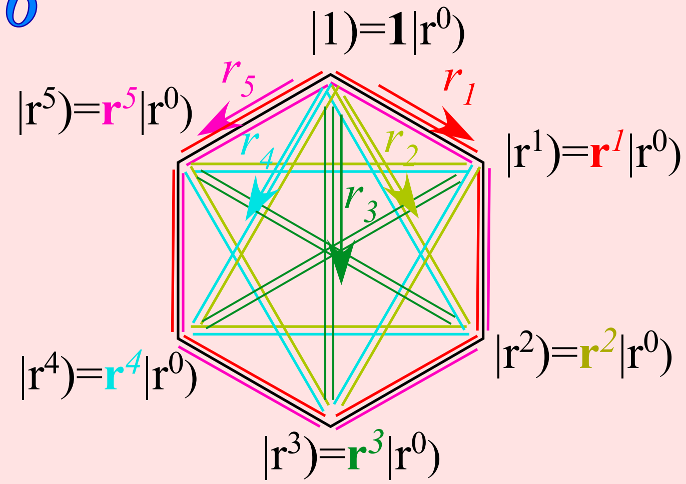
$$\begin{pmatrix} r_0 & r_5 & r_4 & r_3 & r_2 & r_1 \\ r_1 & r_0 & r_5 & r_4 & r_3 & r_2 \\ r_2 & r_1 & r_0 & r_5 & r_4 & r_3 \\ r_3 & r_2 & r_1 & r_0 & r_5 & r_4 \\ r_4 & r_3 & r_2 & r_1 & r_0 & r_5 \\ r_5 & r_4 & r_3 & r_2 & r_1 & r_0 \end{pmatrix} = r_0\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix} + r_1\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix} + r_2\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix} + r_3\begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix} + r_4\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + r_5\begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

*C_6 group table gives **r**-matrices,...C_6-allowed **H**-matrices...*



Nearest neighbor coupling

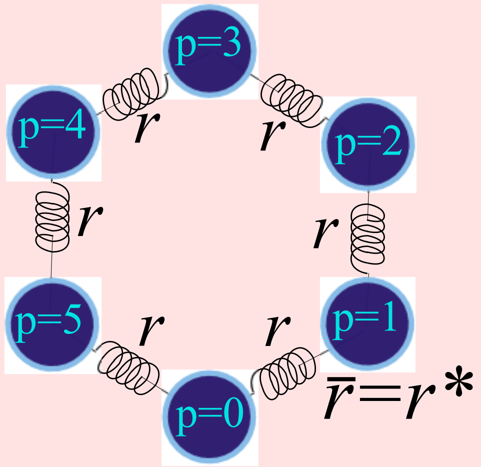
$$\begin{pmatrix} r_0 & r_5 & & & & r_1 \\ r_1 & r_0 & r_5 & & & \\ & r_1 & r_0 & r_5 & & \\ & & r_1 & r_0 & r_5 & \\ & & & r_1 & r_0 & r_5 \\ r_5 & & & & r_1 & r_0 \end{pmatrix}$$



ALL neighbor coupling

$$\begin{pmatrix} r_0 & r_5 & r_4 & r_3 & r_2 & r_1 \\ r_1 & r_0 & r_5 & r_4 & r_3 & r_2 \\ r_2 & r_1 & r_0 & r_5 & r_4 & r_3 \\ r_3 & r_2 & r_1 & r_0 & r_5 & r_4 \\ r_4 & r_3 & r_2 & r_1 & r_0 & r_5 \\ r_5 & r_4 & r_3 & r_2 & r_1 & r_0 \end{pmatrix}$$

(a) 1^{st} Neighbor C_6

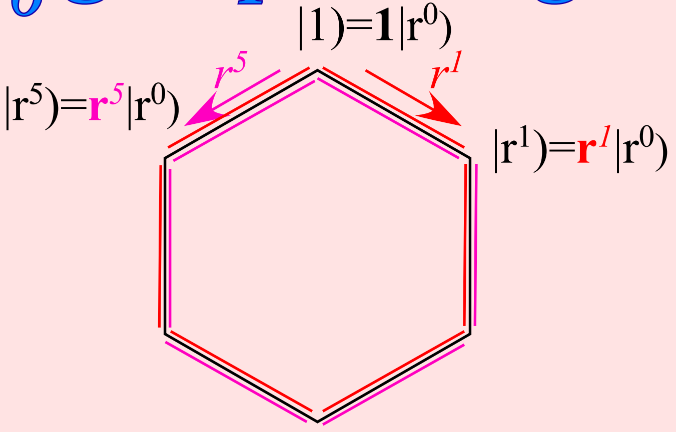


$\bar{r}=r^*$

$$\mathbf{H}^{B1(6)}=\begin{array}{cccccc c} p=0 & 1 & 2 & 3 & 4 & 5 & \\ H_1 & -r & \cdot & \cdot & \cdot & -\bar{r} & 0 \\ -\bar{r} & H_1 & -r & \cdot & \cdot & \cdot & 1 \\ \cdot & -\bar{r} & H_1 & -r & \cdot & \cdot & 2 \\ \cdot & \cdot & -\bar{r} & H_1 & -r & \cdot & 3 \\ \cdot & \cdot & \cdot & -\bar{r} & H_1 & -r & 4 \\ -r & \cdot & \cdot & \cdot & -\bar{r} & H_1 & 5 \end{array}$$

$$= H_1\mathbf{1} - r\mathbf{r} - \bar{r}\mathbf{r}^{-1}$$

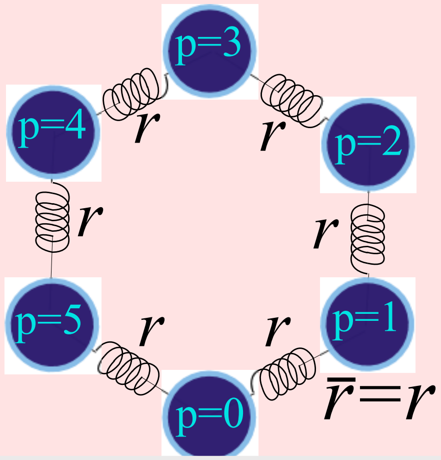
C_6 group table gives $\mathbf{r}$-matrices,..



Nearest neighbor coupling

$$\begin{pmatrix} r_0 & r_5 & & & & r_1 \\ r_1 & r_0 & r_5 & & & \\ & r_1 & r_0 & r_5 & & \\ & & r_1 & r_0 & r_5 & \\ & & & r_1 & r_0 & r_5 \\ r_5 & & & & r_1 & r_0 \end{pmatrix}$$

(a) 1st Neighbor C_6



$\bar{r}=r^*=r$

$$\mathbf{H}^{B1(6)}=2r\mathbf{1}-r\mathbf{r}^{1}-r\mathbf{r}^{-1}$$

$$\begin{array}{|cccccc|c} 0 & 1 & 2 & 3 & 4 & 5 & p \\ \hline 2r & -r & \cdot & \cdot & \cdot & -r & 0 \\ -r & 2r & -r & \cdot & \cdot & \cdot & 1 \\ \cdot & -r & 2r & -r & \cdot & \cdot & 2 \\ \cdot & \cdot & -r & 2r & -r & \cdot & 3 \\ \cdot & \cdot & \cdot & -r & 2r & -r & 4 \\ -r & \cdot & \cdot & \cdot & -r & 2r & 5 \end{array}$$

Conjugation symmetry

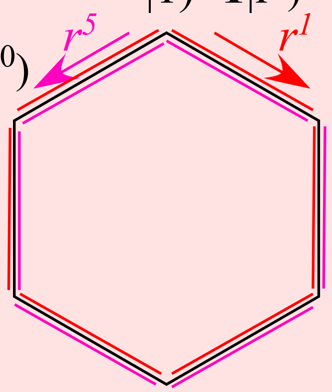
Hermitian Hamiltonian ($\mathbf{H}^*_{jk}=\mathbf{H}_{kj}$) requires $r_0^*=r_0$ and $r_1=r_5^*$.

Elementary Bloch model assumes both are real

(r_1=-r=r_5^*)

r_1 equals conjugate of r_5: ($r_1=r_5^*$)

C_6 group table gives $\mathbf{r}$-matrices,..

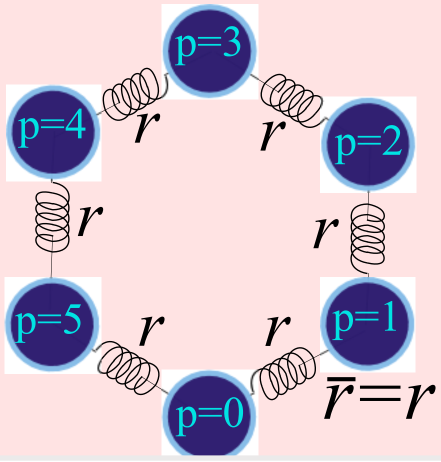


Elementary - Bloch - Model : Nearest neighbor coupling:

$$\mathbf{H}^{B1(6)}=r_0\mathbf{1}+r_1\mathbf{r}^1+r_5\mathbf{r}^5 \qquad =2r\mathbf{1}-r\mathbf{r}^1+-r\mathbf{r}^{-1}$$

$$\begin{pmatrix} r_0 & r_5 & \cdot & \cdot & \cdot & r_1 \\ r_1 & r_0 & r_5 & \cdot & \cdot & \cdot \\ \cdot & r_1 & r_0 & r_5 & \cdot & \cdot \\ \cdot & \cdot & r_1 & r_0 & r_5 & \cdot \\ \cdot & \cdot & \cdot & r_1 & r_0 & r_5 \\ r_5 & \cdot & \cdot & \cdot & r_1 & r_0 \end{pmatrix} = \begin{array}{|cccccc|c} 0 & 1 & 2 & 3 & 4 & 5 & p \\ \hline 2r & -r & \cdot & \cdot & \cdot & -r & 0 \\ -r & 2r & -r & \cdot & \cdot & \cdot & 1 \\ \cdot & -r & 2r & -r & \cdot & \cdot & 2 \\ \cdot & \cdot & -r & 2r & -r & \cdot & 3 \\ \cdot & \cdot & \cdot & -r & 2r & -r & 4 \\ -r & \cdot & \cdot & \cdot & -r & 2r & 5 \end{array}$$

(a) 1st Neighbor C_6

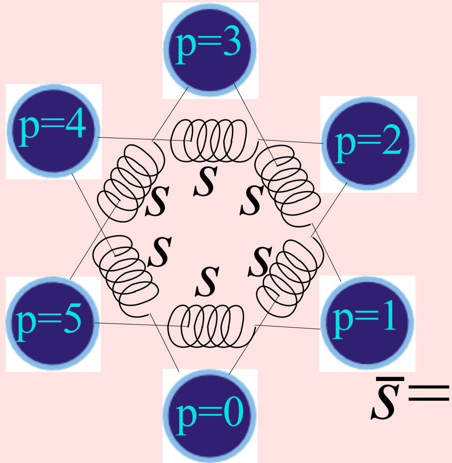


$\bar{r}=r^*=r$

$$\mathbf{H}^{B1(6)}=2r\mathbf{1}-r\mathbf{r}^{1}-r\mathbf{r}^{-1}$$

$$\begin{array}{cccccc|c}
0 & 1 & 2 & 3 & 4 & 5 & p \\
\hline
2r & -r & \cdot & \cdot & \cdot & -r & 0 \\
-r & 2r & -r & \cdot & \cdot & \cdot & 1 \\
\cdot & -r & 2r & -r & \cdot & \cdot & 2 \\
\cdot & \cdot & -r & 2r & -r & \cdot & 3 \\
\cdot & \cdot & \cdot & -r & 2r & -r & 4 \\
-r & \cdot & \cdot & \cdot & -r & 2r & 5
\end{array}$$

(b) 2nd Neighbor C_6



$\bar{s}=s^*=s$

$$\mathbf{H}^{B2(6)}=H_2\mathbf{1}-s\mathbf{r}^{2}-s\mathbf{r}^{-2}$$

$$\begin{array}{cccccc|c}
0 & 1 & 2 & 3 & 4 & 5 & p \\
\hline
H_2 & \cdot & -s & \cdot & -s & \cdot & 0 \\
\cdot & H_2 & \cdot & -s & \cdot & -s & 1 \\
-s & \cdot & H_2 & \cdot & -s & \cdot & 2 \\
\cdot & -s & \cdot & H_2 & \cdot & -s & 3 \\
-s & \cdot & -s & \cdot & H_2 & \cdot & 4 \\
\cdot & -s & \cdot & -s & \cdot & H_2 & 5
\end{array}$$

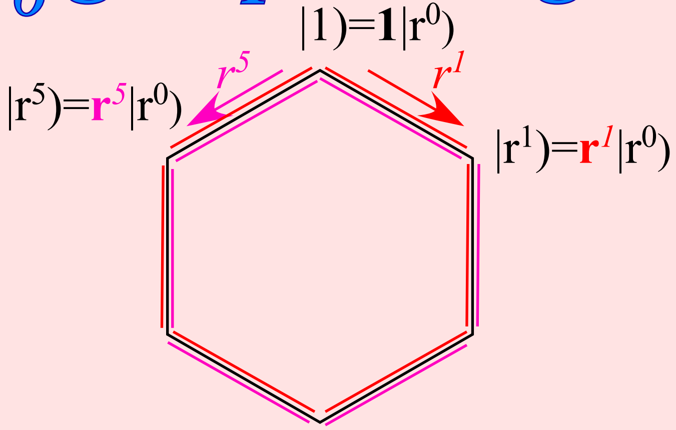
Conjugation symmetry

$(\mathbf{H}^*_{jk}=\mathbf{H}_{kj})$ requires $r_0^*=r_0$ and $r_2=r_4^*$.

r_1 equals conjugate of r_5: ($r_1=r_5^*=-r$)

($r_2=r_4^*=-s$) *We assume both are real*

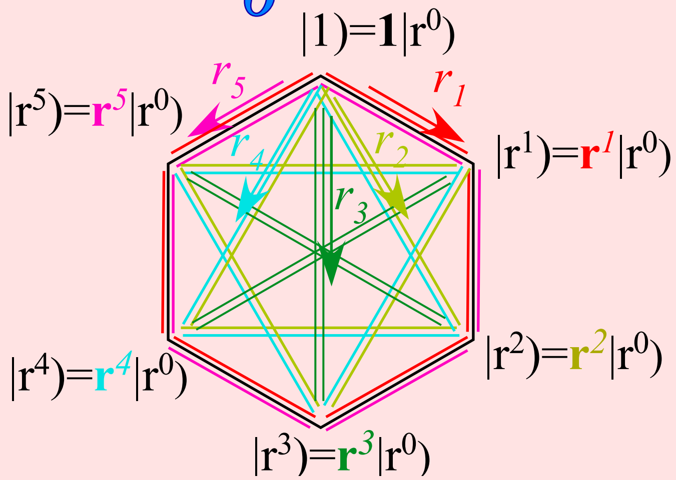
C_6 group table gives $\mathbf{r}$-matrices,..., and all C_6-allowed $\mathbb{H}$-matrices...



2^{nd} Nearest neighbor coupling:

$$\mathbf{H}^{B1(6)}=r_0\mathbf{1}+r_2\mathbf{r}^2+r_4\mathbf{r}^4$$

$$\left(\begin{array}{cccccc}
r_0 & \cdot & r_4 & \cdot & r_2 & \cdot \\
\cdot & r_0 & \cdot & r_4 & \cdot & r_2 \\
r_2 & \cdot & r_0 & \cdot & r_4 & \cdot \\
\cdot & r_2 & \cdot & r_0 & \cdot & r_4 \\
r_4 & \cdot & r_2 & \cdot & r_0 & \cdot \\
\cdot & r_4 & \cdot & r_2 & \cdot & r_0
\end{array}\right)$$

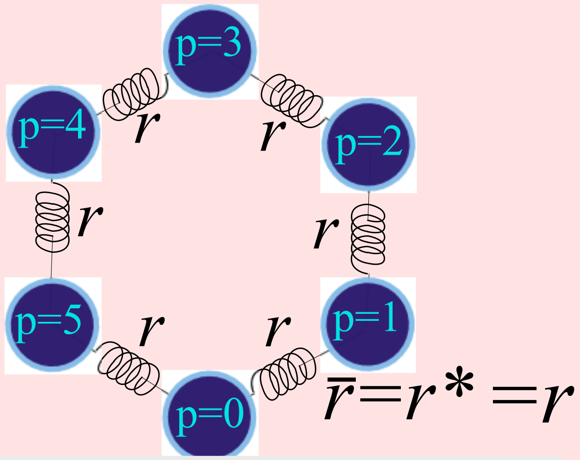


All – neighbor coupling:

$$\mathbf{H}^{A(6)}=r_0\mathbf{1}+r_1\mathbf{r}^1+r_2\mathbf{r}^2+r_3\mathbf{r}^3+r_4\mathbf{r}^4+r_5\mathbf{r}^5$$

$$\left(\begin{array}{cccccc}
r_0 & r_5 & r_4 & r_3 & r_2 & r_1 \\
r_1 & r_0 & r_5 & r_4 & r_3 & r_2 \\
r_2 & r_1 & r_0 & r_5 & r_4 & r_3 \\
r_3 & r_2 & r_1 & r_0 & r_5 & r_4 \\
r_4 & r_3 & r_2 & r_1 & r_0 & r_5 \\
r_5 & r_4 & r_3 & r_2 & r_1 & r_0
\end{array}\right)$$

(a) 1^{st} Neighbor C_6

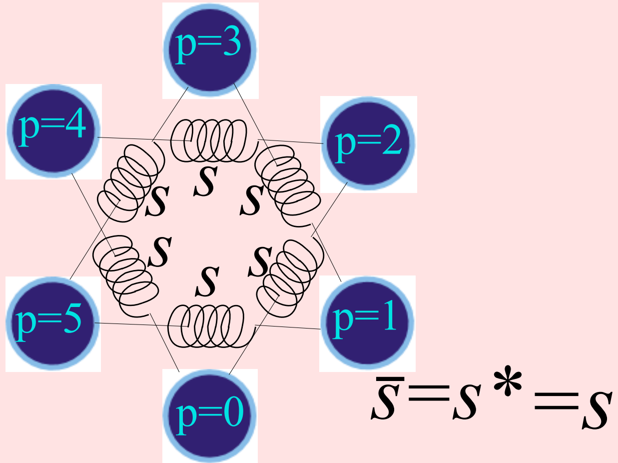


$\bar{r}=r^*=r$

$$\mathbf{H}^{B1(6)}=2r\mathbf{1}-r\mathbf{r}^{1}-r\mathbf{r}^{-1}$$

$$\begin{array}{cccccc|c}
0 & 1 & 2 & 3 & 4 & 5 & p \\
\hline
2r & -r & \cdot & \cdot & \cdot & -r & 0 \\
-r & 2r & -r & \cdot & \cdot & \cdot & 1 \\
\cdot & -r & 2r & -r & \cdot & \cdot & 2 \\
\cdot & \cdot & -r & 2r & -r & \cdot & 3 \\
\cdot & \cdot & \cdot & -r & 2r & -r & 4 \\
-r & \cdot & \cdot & \cdot & -r & 2r & 5
\end{array}$$

Neighbor C_6

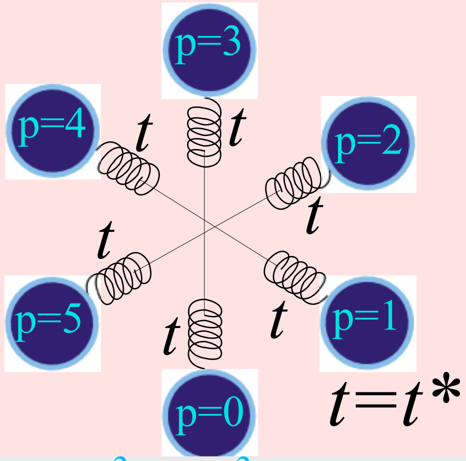


$\bar{s}=s^*=s$

$$\mathbf{H}^{B2(6)}=H_2\mathbf{1}-s\mathbf{r}^{2}-s\mathbf{r}^{-2}$$

$$\begin{array}{cccccc|c}
0 & 1 & 2 & 3 & 4 & 5 & p \\
\hline
H_2 & \cdot & -s & \cdot & -s & \cdot & 0 \\
\cdot & H_2 & \cdot & -s & \cdot & -s & 1 \\
-s & \cdot & H_2 & \cdot & -s & \cdot & 2 \\
\cdot & -s & \cdot & H_2 & \cdot & -s & 3 \\
-s & \cdot & -s & \cdot & H_2 & \cdot & 4 \\
\cdot & -s & \cdot & -s & \cdot & H_2 & 5
\end{array}$$

(c) 3^{rd} Neighbor C_6



$t=t^*$

$$\mathbf{H}^{B3(6)}=H_3\mathbf{1}-t\mathbf{r}^{3}-t\mathbf{r}^{-3}$$

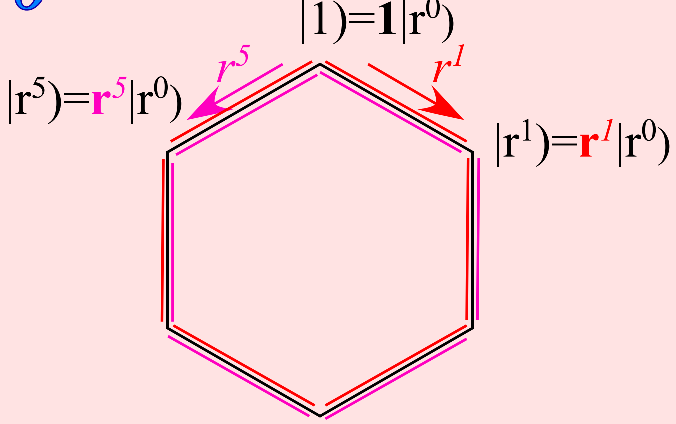
$$\begin{array}{cccccc|c}
0 & 1 & 2 & 3 & 4 & 5 & p \\
\hline
H_3 & \cdot & \cdot & -t & \cdot & \cdot & 0 \\
\cdot & H_3 & \cdot & \cdot & -t & \cdot & 1 \\
\cdot & \cdot & H_3 & \cdot & \cdot & -t & 2 \\
-t & \cdot & \cdot & H_3 & \cdot & \cdot & 3 \\
\cdot & -t & \cdot & \cdot & H_3 & \cdot & 4 \\
\cdot & \cdot & -t & \cdot & \cdot & H_3 & 5
\end{array}$$

r_1 equals conjugate of r_5: ($r_1=r_5^* = -r$)

($r_2=r_4^* = -s$)

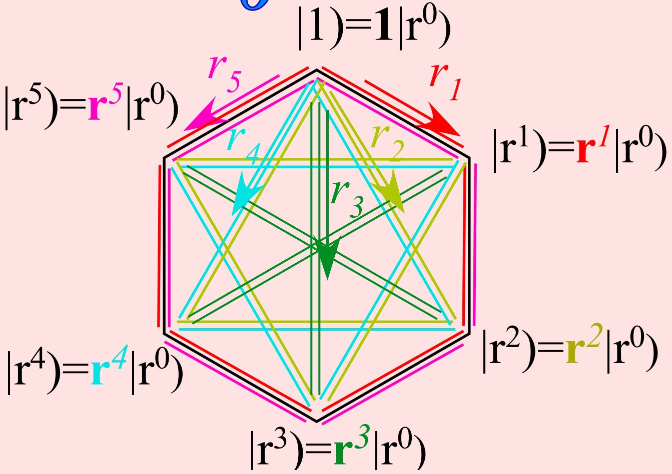
($r_3=r_3^* = t$) *must* be real

C_6 group table gives $\mathbf{r}$-matrices,..., and all C_6-allowed $\mathbb{H}$-matrices...



Nearest neighbor coupling

$$\begin{pmatrix}
r_0 & r_5 & & & & r_1 \\
r_1 & r_0 & r_5 & & & \\
 & r_1 & r_0 & r_5 & & \\
 & & r_1 & r_0 & r_5 & \\
 & & & r_1 & r_0 & r_5 \\
r_5 & & & & r_1 & r_0
\end{pmatrix}$$



ALL neighbor coupling

$$\begin{pmatrix}
r_0 & r_5 & r_4 & r_3 & r_2 & r_1 \\
r_1 & r_0 & r_5 & r_4 & r_3 & r_2 \\
r_2 & r_1 & r_0 & r_5 & r_4 & r_3 \\
r_3 & r_2 & r_1 & r_0 & r_5 & r_4 \\
r_4 & r_3 & r_2 & r_1 & r_0 & r_5 \\
r_5 & r_4 & r_3 & r_2 & r_1 & r_0
\end{pmatrix}$$

1^{st} Step: Abelian symmetry analysis Expand C_6 symmetric **H** *by C_6 group product table*

Regular representation and coupling parameters $\{r_0,r_1,r_2,r_3,r_4,r_5\}$ and Fourier dispersion

→ *2^{nd} Step: Find* **H** *eigenfunctions by spectral resolution of $C_6 = \{\mathbf{1}=\mathbf{r}^0,\mathbf{r}^1,\mathbf{r}^2,\mathbf{r}^3,\mathbf{r}^4,\mathbf{r}^5\}$*

Character tables of C_2, C_3, C_4, C_5,.... C_{144}

3^{rd} Step: Dispersion functions and eigenvalues for varyious coupling parameter systems

Ortho-complete eigenvalue/parameter relations

Gauge shifts due to complex coupling

Wave dynamics of phase, mean phase, and group velocity by Expo-Cosine identity

Relating space-time (x,t) and per-space-time (k,ω)

Wave coordinates

Pulse-waves (PW) vs Continuous-waves (CW)

Wave coordinates for Linear Dispersion

Wave coordinates for Bohr-Schrodinger Dispersion

Einstein-Lorentz-Minkowski laser coordinates

2^{nd} Step

H diagonalized by spectral resolution of $r, r^2, \ldots, r^6 = 1$

All $x = r^p$ *satisfy* $x^6 = 1$ *and use* 6^{th}*-roots-of-1 for eigenvalues*

$$D^m(\mathbf{r}) = e^{-2\pi i m/6} = \chi_1^m = \psi_1^{m*}$$

$$D^m(\mathbf{r}^p) = e^{-2\pi i m \cdot p/6} = \chi_p^m = \psi_p^{m*}$$

$\psi_1^0 = 1$

$\psi_1^1 = e^{2\pi i/6}$

$\psi_1^2 = \psi_2^1 = e^{4\pi i/6}$

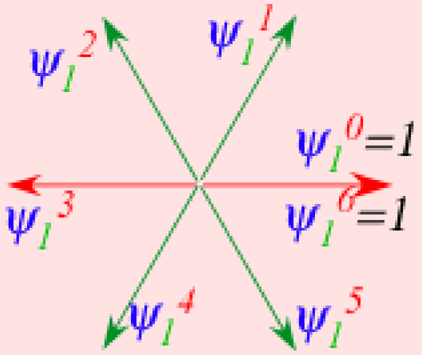
$\psi_1^3 = \psi_3^1 = -1$

$\psi_1^4 = \psi_4^1 = \psi_1^{-2} = e^{-4\pi i/6}$

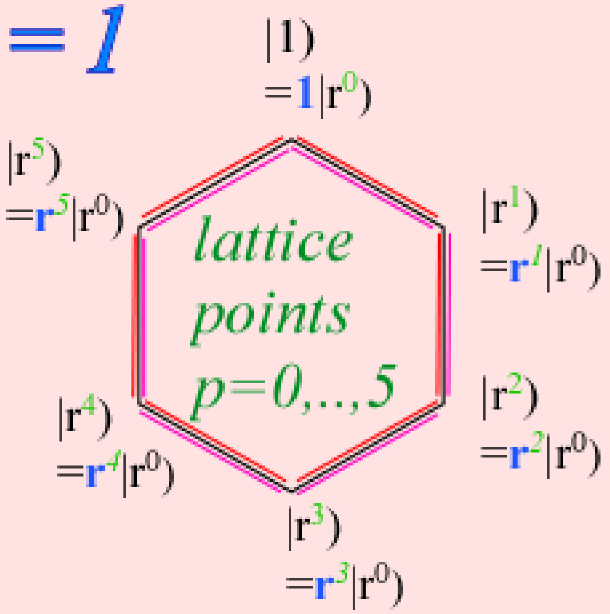
$\psi_1^5 = \psi_5^1 = \psi_1^{-1} = e^{-2\pi i/6}$

p=power (exponent)
or position point

m = momentum
or wave-number



6^{th}-roots of 1
m=0,...,5



Groups "know" their roots and will tell you them if you ask nicely!

You efficiently get:

- *invariant projectors*
- *irreducible projectors*
- *irreducible representations (irreps)*
- *H eigenvalues*
- *H eigenvectors*
- *T matrices*
- *dispersion functions*

2^{nd} Step (contd.)

H diagonalized by spectral resolution of $r, r^2, \ldots, r^6 = 1$

All $x = r^p$ satisfy $x^6 = 1$ and use *6^{th}-roots-of-1* *for eigenvalues*

$$D^m(\mathbf{r}) = e^{-2\pi i m/6} = \chi_1^{\,m} = \psi_1^{\,m*}$$

$$D^m(\mathbf{r}^p) = e^{-2\pi i m \cdot p/6} = \chi_p^{\,m} = \psi_p^{\,m*}$$

p=power (exponent)

or position point

m = momentum

or wave-number

$\psi_1^{\,0} = 1$

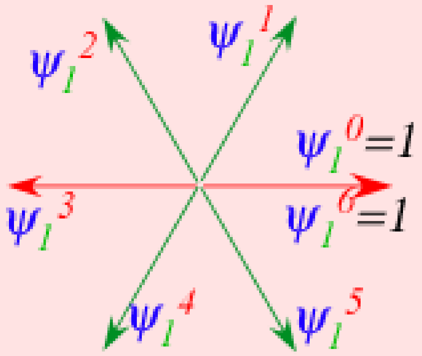
$\psi_1^{\,1} = e^{2\pi i/6}$

$\psi_1^{\,2} = \psi_2^{\,1} = e^{4\pi i/6}$

$\psi_1^{\,3} = \psi_3^{\,1} = -1$

$\psi_1^{\,4} = \psi_4^{\,1} = \psi_1^{\,-2} = e^{-4\pi i/6}$

$\psi_1^{\,5} = \psi_5^{\,1} = \psi_1^{\,-1} = e^{-2\pi i/6}$



top-row flip
not needed...

$\mathbf{P}^{(m)} = \mathbf{P}^{(m)\dagger}$

6 ring	$\mathbf{P}^{(0)}$	$\mathbf{P}^{(1)}$	$\mathbf{P}^{(2)}$	$\mathbf{P}^{(3)}$	$\mathbf{P}^{(4)}$	$\mathbf{P}^{(5)}$
$\mathbf{P}^{(0)}$	$\mathbf{P}^{(0)}$	.	.	.	.	.
$\mathbf{P}^{(1)}$	.	$\mathbf{P}^{(1)}$	.	.	.	.
$\mathbf{P}^{(2)}$	.	.	$\mathbf{P}^{(2)}$	.	.	.
$\mathbf{P}^{(3)}$	.	.	.	$\mathbf{P}^{(3)}$	.	.
$\mathbf{P}^{(4)}$	.	.	.	.	$\mathbf{P}^{(4)}$	.
$\mathbf{P}^{(5)}$	.	.	.	.	.	$\mathbf{P}^{(5)}$

$$\mathbf{r}^p = \chi_p^{\,0}\,\mathbf{P}^{(0)} + \chi_p^{\,1}\,\mathbf{P}^{(1)} + \chi_p^{\,2}\,\mathbf{P}^{(2)} + \chi_p^{\,3}\,\mathbf{P}^{(3)} + \chi_p^{\,4}\,\mathbf{P}^{(4)} + \chi_p^{\,5}\,\mathbf{P}^{(5)}$$

$$\begin{pmatrix} \chi_p^{\,0} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \chi_p^{\,1} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \chi_p^{\,2} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \chi_p^{\,3} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \chi_p^{\,4} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \chi_p^{\,5} \end{pmatrix} = \chi_p^{\,0} \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + \chi_p^{\,1} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + \chi_p^{\,2} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + \chi_p^{\,3} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + \chi_p^{\,4} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + \chi_p^{\,5} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

Projectors $\mathbf{P}^{(m)}$ *are eigenvalue "placeholders" having orthogonal-idempotent products, eigen-equations,*

$$\mathbf{P}^{(m)}\mathbf{P}^{(n)} = \delta^{mn}\mathbf{P}^{(m)} \qquad \mathbf{r}^p\,\mathbf{P}^{(n)} = \chi_p^{\,n}\mathbf{P}^{(n)}$$

and one completeness rule: $\mathbf{P}^{(0)} + \mathbf{P}^{(1)} + \mathbf{P}^{(2)} + \ldots + \mathbf{P}^{(5)} = \mathbf{1}$

2^{nd} Step (contd.)

H diagonalized by spectral resolution of $r, r^2, \ldots, r^6 = 1$

All $x = r^p$ satisfy $x^6 = 1$ and use ***6^{th}-roots-of-1*** *for eigenvalues*

$$D^m(\mathbf{r}) = e^{-2\pi i m/6} = \chi_1^m = \psi_1^{m*}$$

$$D^m(\mathbf{r}^p) = e^{-2\pi i m\cdot p/6} = \chi_p^m = \psi_p^{m*}$$

p=power (exponent)
or position point

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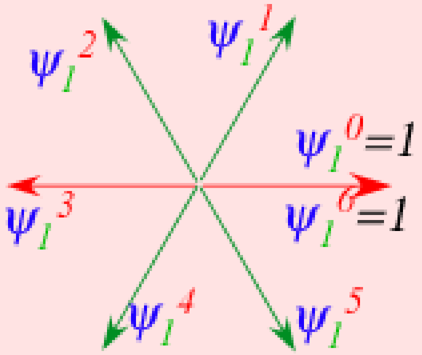
or wave-number

$\psi_1^0 = 1$
$\psi_1^1 = e^{2\pi i/6}$
$\psi_1^2 = \psi_2^1 = e^{4\pi i/6}$
$\psi_1^3 = \psi_3^1 = -1$
$\psi_1^4 = \psi_4^1 = \psi_1^{-2} = e^{-4\pi i/6}$
$\psi_1^5 = \psi_5^1 = \psi_1^{-1} = e^{-2\pi i/6}$



top-row flip
not needed...

$$\mathbf{P}^{(m)} = \mathbf{P}^{(m)\dagger}$$

$$\begin{array}{c|cccccc}
6\ ring & \mathbf{P}^{(0)} & \mathbf{P}^{(1)} & \mathbf{P}^{(2)} & \mathbf{P}^{(3)} & \mathbf{P}^{(4)} & \mathbf{P}^{(5)} \\
\hline
\mathbf{P}^{(0)} & \mathbf{P}^{(0)} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\mathbf{P}^{(1)} & \cdot & \mathbf{P}^{(1)} & \cdot & \cdot & \cdot & \cdot \\
\mathbf{P}^{(2)} & \cdot & \cdot & \mathbf{P}^{(2)} & \cdot & \cdot & \cdot \\
\mathbf{P}^{(3)} & \cdot & \cdot & \cdot & \mathbf{P}^{(3)} & \cdot & \cdot \\
\mathbf{P}^{(4)} & \cdot & \cdot & \cdot & \cdot & \mathbf{P}^{(4)} & \cdot \\
\mathbf{P}^{(5)} & \cdot & \cdot & \cdot & \cdot & \cdot & \mathbf{P}^{(5)}
\end{array}$$

$$\mathbf{r}^p = \chi_p^0\,\mathbf{P}^{(0)} + \chi_p^1\,\mathbf{P}^{(1)} + \chi_p^2\,\mathbf{P}^{(2)} + \chi_p^3\,\mathbf{P}^{(3)} + \chi_p^4\,\mathbf{P}^{(4)} + \chi_p^5\,\mathbf{P}^{(5)}$$

$$\begin{pmatrix}\chi_p^0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \chi_p^1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \chi_p^2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \chi_p^3 & & \end{pmatrix} = \chi_p^0\begin{pmatrix}1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + \chi_p^1\begin{pmatrix}\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + \chi_p^2\begin{pmatrix}\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + \chi_p^3\begin{pmatrix}\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix} + \chi_p^4\begin{pmatrix}\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + \chi_p^5\begin{pmatrix}\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot \\ & & & & \cdot & 1 \end{pmatrix}$$

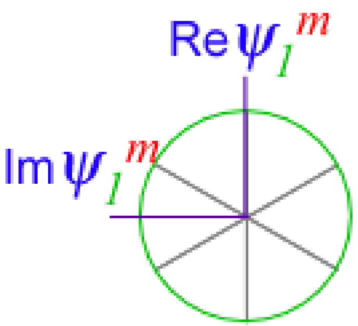
Inverse C_6 spectral resolution m-wave $\psi_p^m = D^{m*}(\mathbf{r}^p) = e^{+2\pi i m\cdot p/6}$:

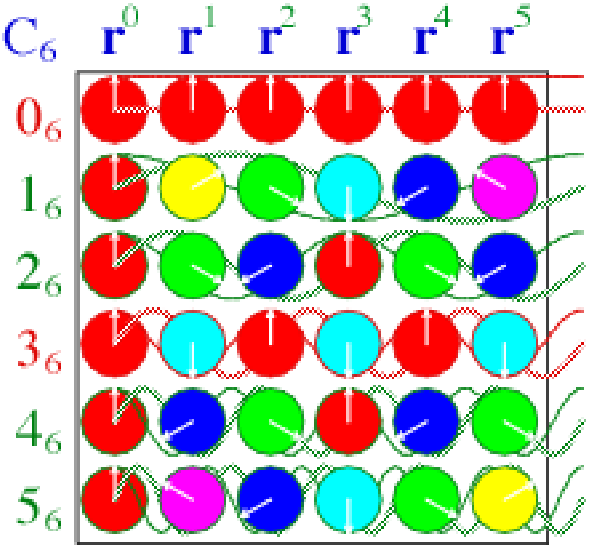
$$6\cdot\mathbf{P}^{(m)} = \psi_0^m\,\mathbf{r}^0 + \psi_1^m\,\mathbf{r}^1 + \psi_2^m\,\mathbf{r}^2 + \psi_3^m\,\mathbf{r}^3 + \psi_4^m\,\mathbf{r}^4 + \psi_5^m\,\mathbf{r}^5$$

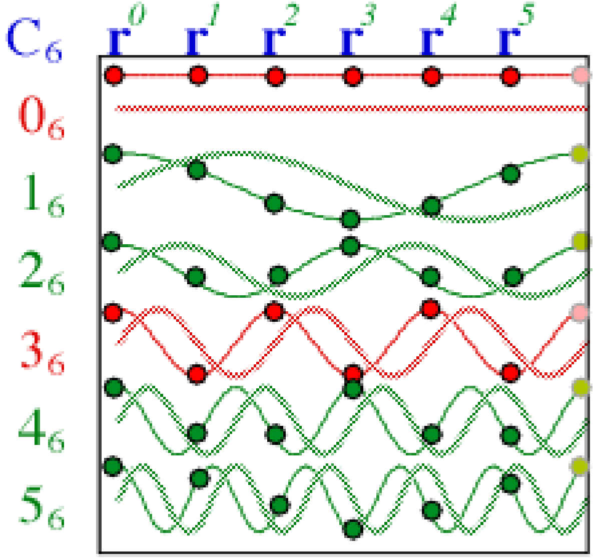
position p (or power of $\mathbf{r}^p$)

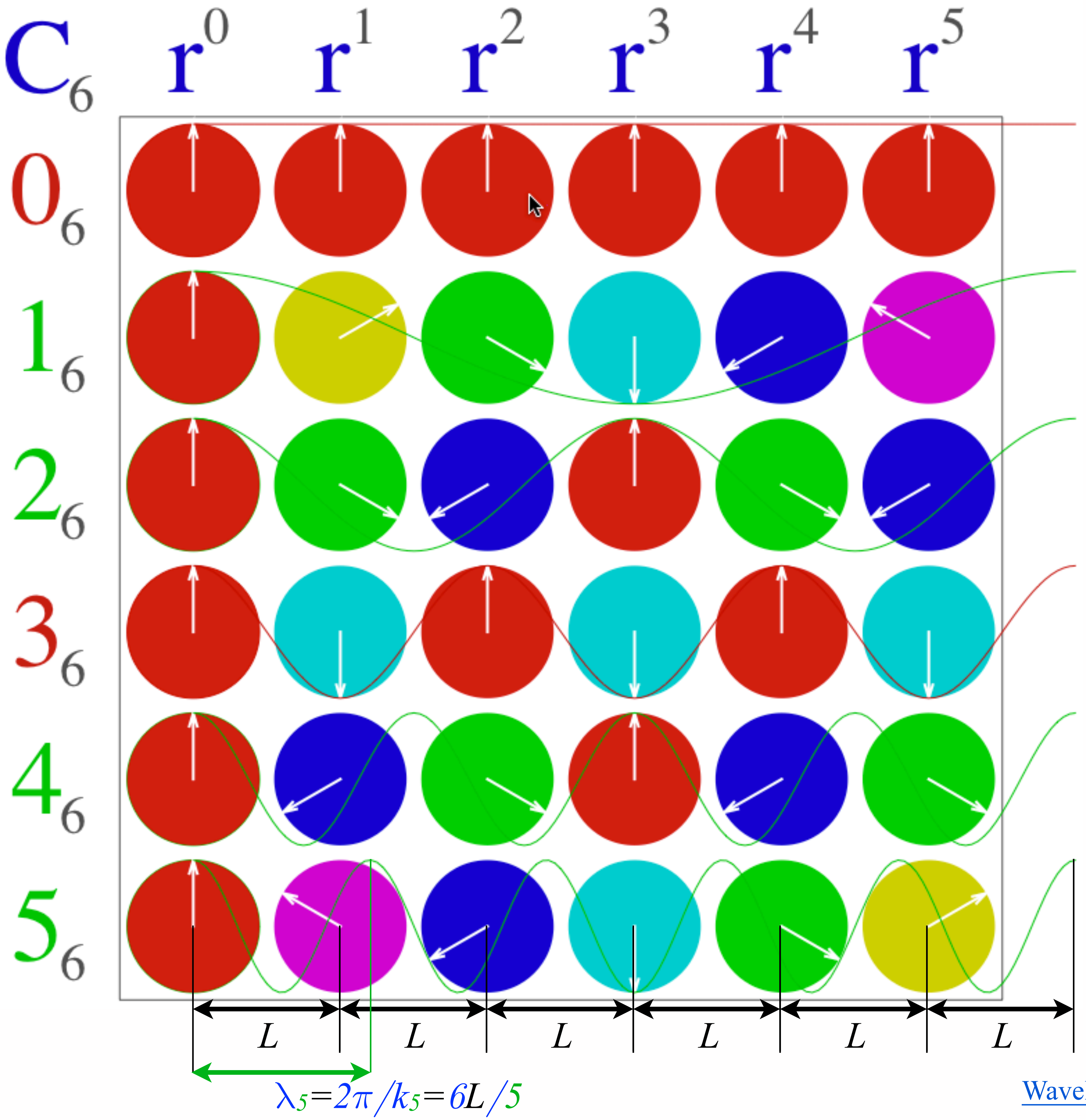
momentum m

	p=0	1	2	3	4	5
m=0	ψ_0^0	ψ_1^0	ψ_2^0	ψ_3^0	ψ_4^0	ψ_5^0
m=1	ψ_0^1	ψ_1^1	ψ_2^1	ψ_3^1	ψ_4^1	ψ_5^1
m=2	ψ_0^2	ψ_1^2	ψ_2^2	ψ_3^2	ψ_4^2	ψ_5^2
m=3	ψ_0^3	ψ_1^3	ψ_2^3	ψ_3^3	ψ_4^3	ψ_5^3
m=4	ψ_0^4	ψ_1^4	ψ_2^4	ψ_3^4	ψ_4^4	ψ_5^4
m=5	ψ_0^5	ψ_1^5	ψ_2^5	ψ_3^5	ψ_4^5	ψ_5^5









C_6 character

$$\chi_{mp}=e^{-imp2\pi/6}$$

is wave function conjugate

$$\psi^*_m(r_p)=\frac{e^{-imp2\pi/6}}{\sqrt{6}}$$ (with norm √6)

C_6 Plane wave function

$$\psi_m(r_p)=\frac{e^{ik_m\cdot r_p}}{\sqrt{6}}$$

$$=\frac{e^{imp2\pi/6}}{\sqrt{6}}$$

C_6 Lattice position vector

$$r_p=L\cdot p$$

Wavevector

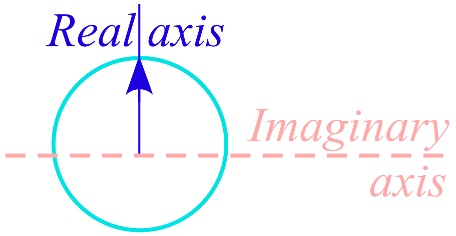
$$k_m=2\pi m/6L=2\pi/\lambda_m$$

Wavelength

$$\lambda_m=2\pi/k_m=6L/m$$

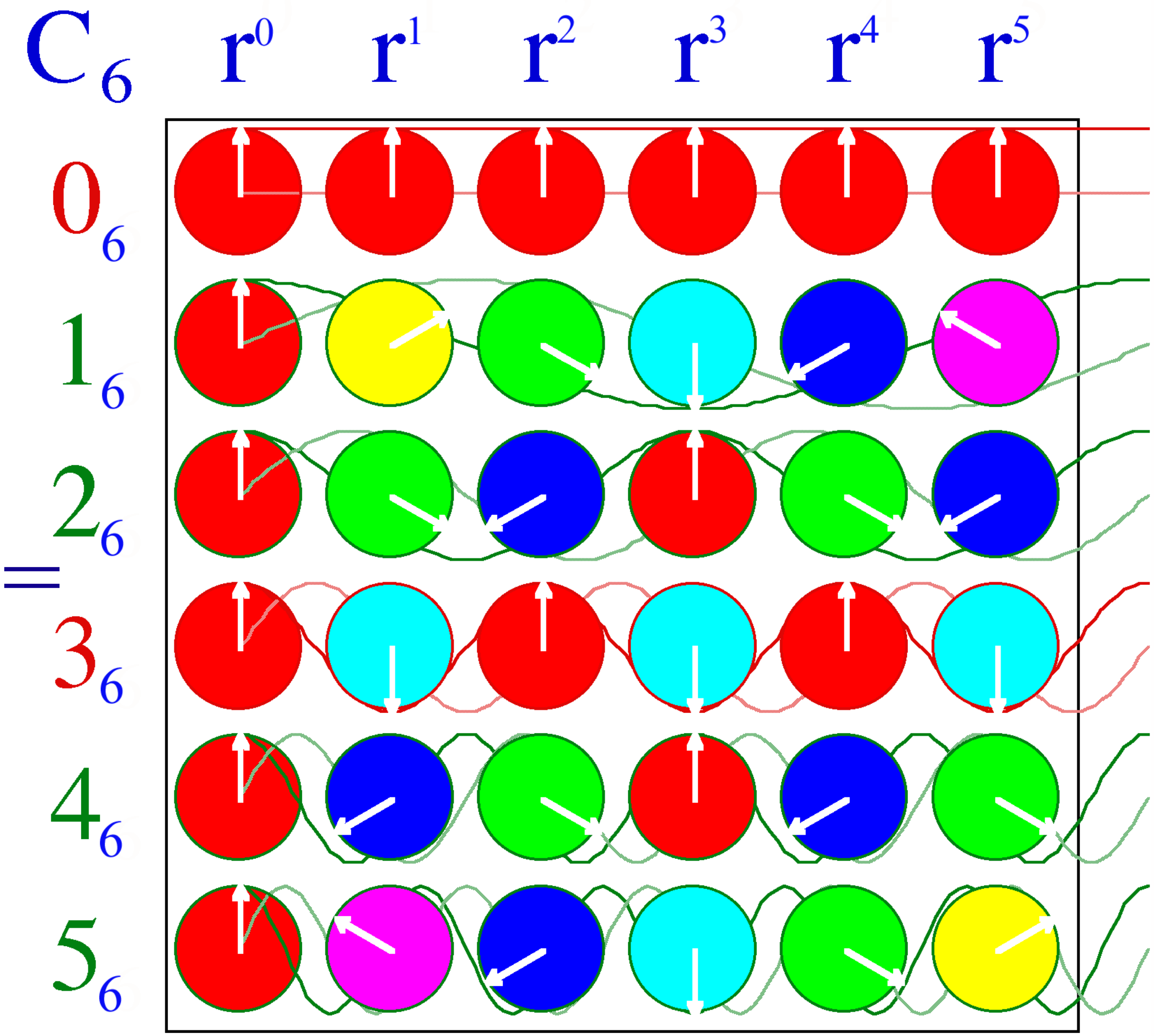
WaveIt C_6 Character Phasors Web Simulation

Backwards phasors
for
conjugate waves
(turn counter-clockwise)



$\chi_p^m(C_6)$	$\mathbf{r}^{p=0}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{r}^3$	$\mathbf{r}^4$	$\mathbf{r}^5$
$m=0_6$	1	1	1	1	1	1
1_6	1	ε^*	ε^{2*}	-1	ε^2	ε
2_6	1	ε^{2*}	ε^2	1	ε^{2*}	ε^2
$3_6=-3_6$	1	-1	1	-1	1	-1
$4_6=-2_6$	1	ε^2	ε^{2*}	1	ε^2	ε^{2*}
$5_6=-1_6$	1	ε	ε^2	-1	ε^{2*}	ε

$$\varepsilon=e^{i2\pi/6}$$

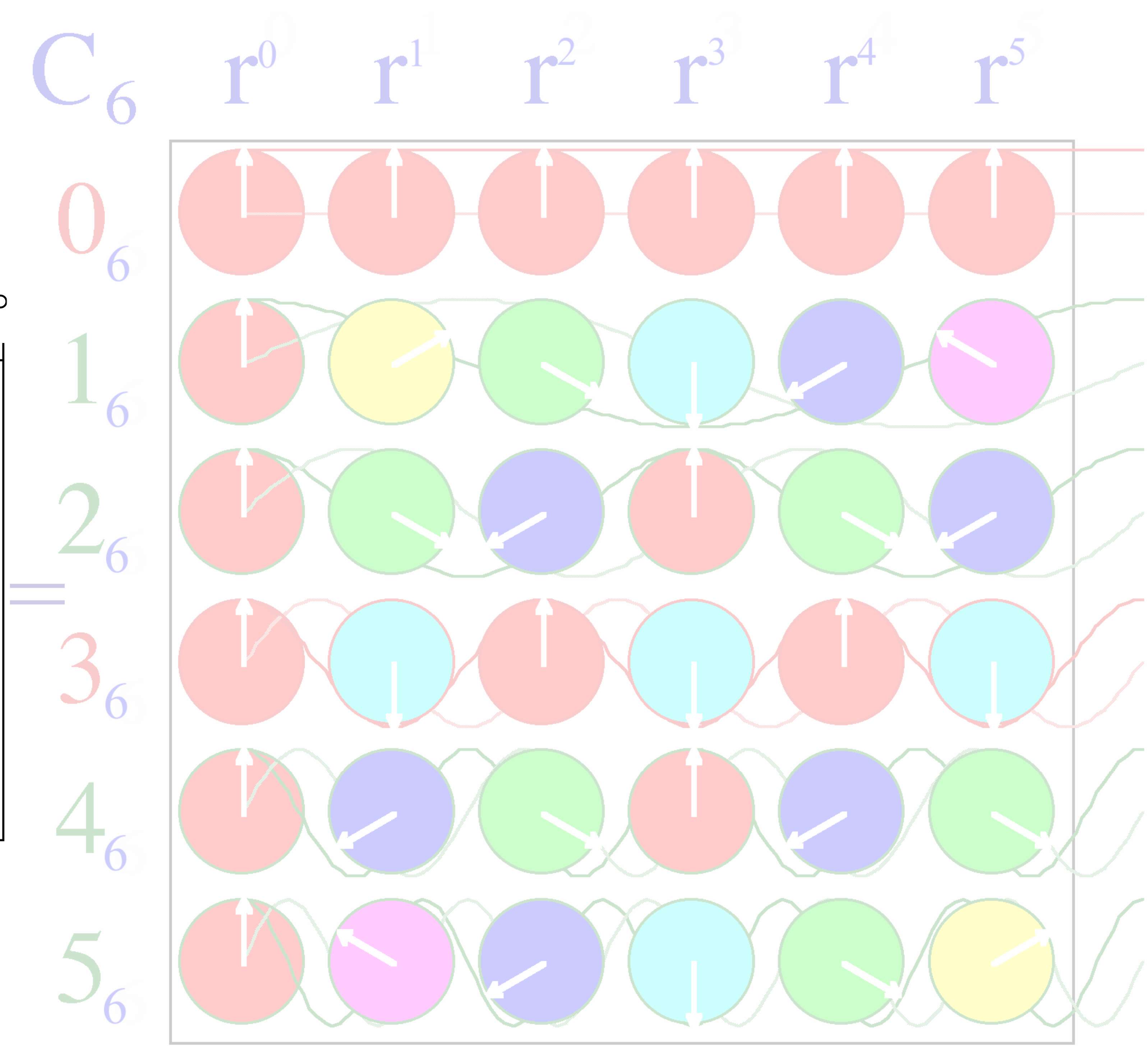


WaveIt *C₆ Character Phasors* Web Simulation

	0°	*60°*	*120°*	*180°*	*-120°*	*-60°*
α	1	1	1	1	1	1
β	1	ε^*	ε^{2*}	-1	ε^2	ε
γ	1	ε^{2*}	ε^2	1	ε^{2*}	ε^2
δ	1	-1	1	-1	1	-1
γ^*	1	ε^2	ε^{2*}	1	ε^2	ε^{2*}
β^*	1	ε	ε^2	-1	ε^{2*}	ε

$$\varepsilon = e^{i2\pi/6}$$

What you'll get
if you look up
C_6 characters in library



Wave phasor stuff? FUGggedd-aboudit!

WaveIt *C_6 Character Phasors* Web Simulation

1st Step: Abelian symmetry analysis Expand C_6 symmetric **H** *by C_6 group product table*

Regular representation and coupling parameters $\{r_0,r_1,r_2,r_3,r_4,r_5\}$ and Fourier dispersion

2nd Step: Find **H** *eigenfunctions by spectral resolution of $C_6 = \{\mathbf{1}=\mathbf{r}^0,\mathbf{r}^1,\mathbf{r}^2,\mathbf{r}^3,\mathbf{r}^4,\mathbf{r}^5\}$*

Character tables of C_2, C_3, C_4, C_5,…. C_{144}

3rd Step: Dispersion functions and eigenvalues for varyious coupling parameter systems

Ortho-complete eigenvalue/parameter relations

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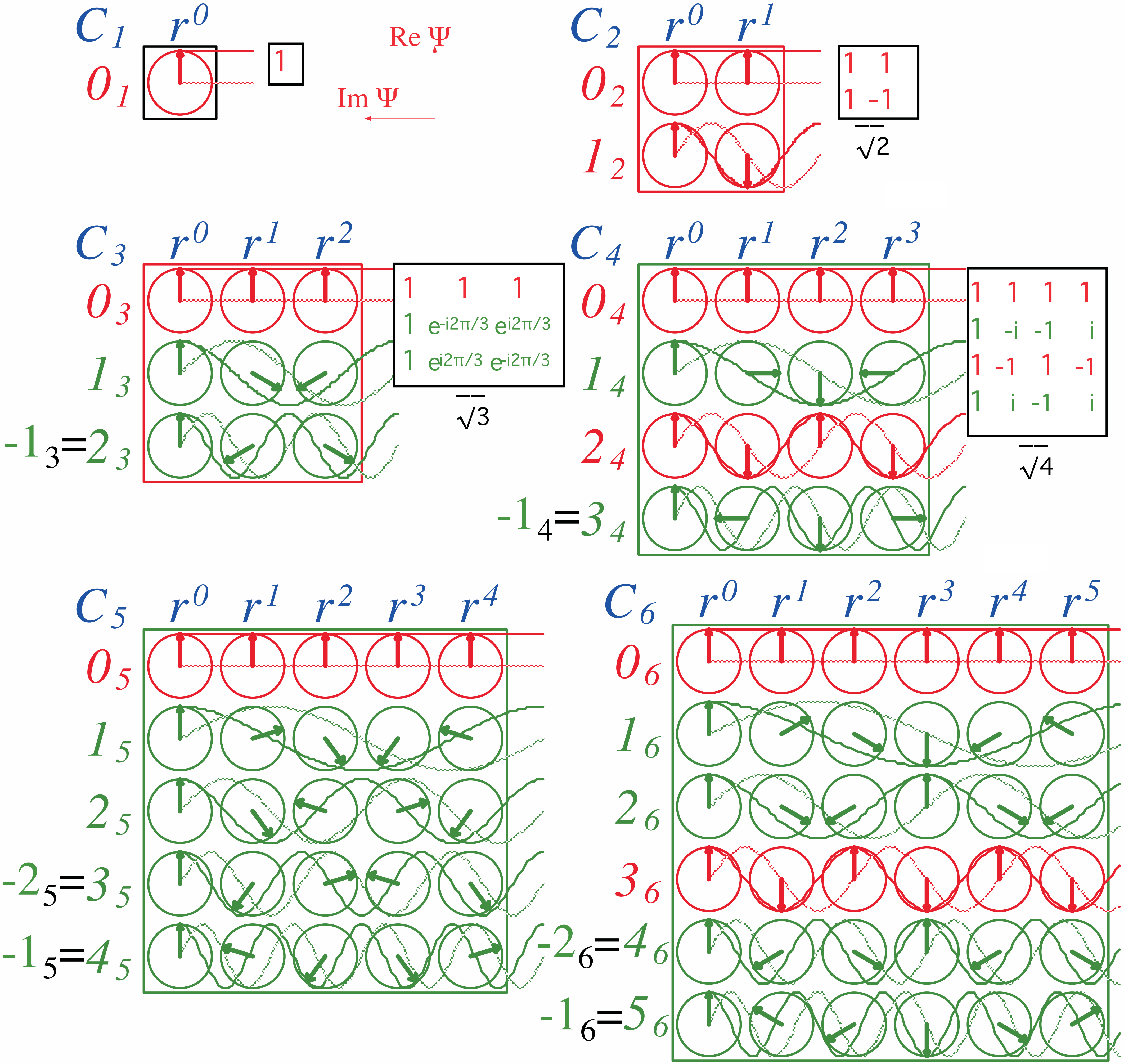
Wave coordinates

Pulse-waves (PW) vs Continuous-waves (CW)

Wave coordinates for Linear Dispersion

Wave coordinates for Bohr-Schrodinger Dispersion

Einstein-Lorentz-Minkowski laser coordinates



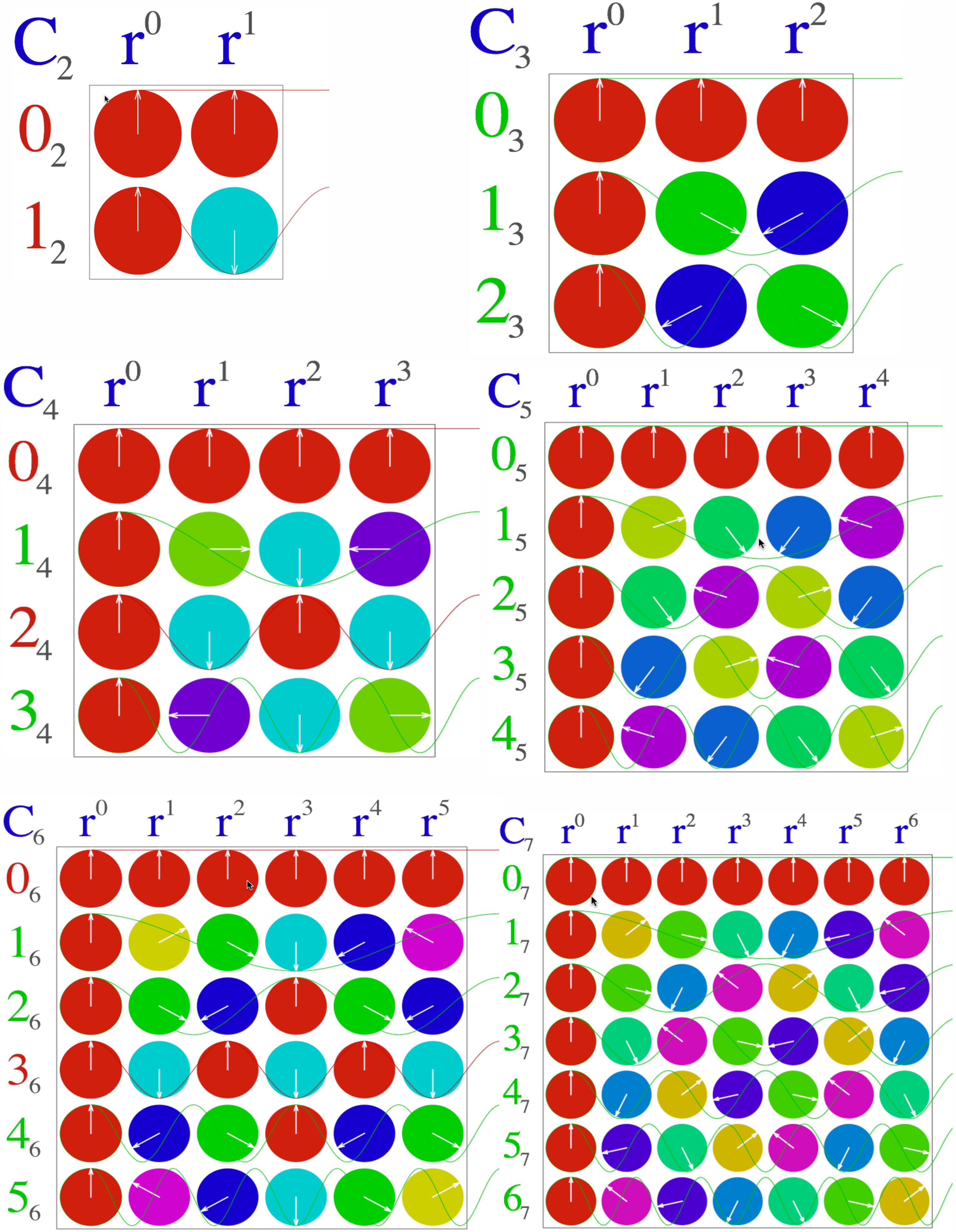
$$\begin{pmatrix}1\end{pmatrix}$$

$$\begin{pmatrix}1&1\\1&-1\end{pmatrix}\Big/\sqrt{2}$$

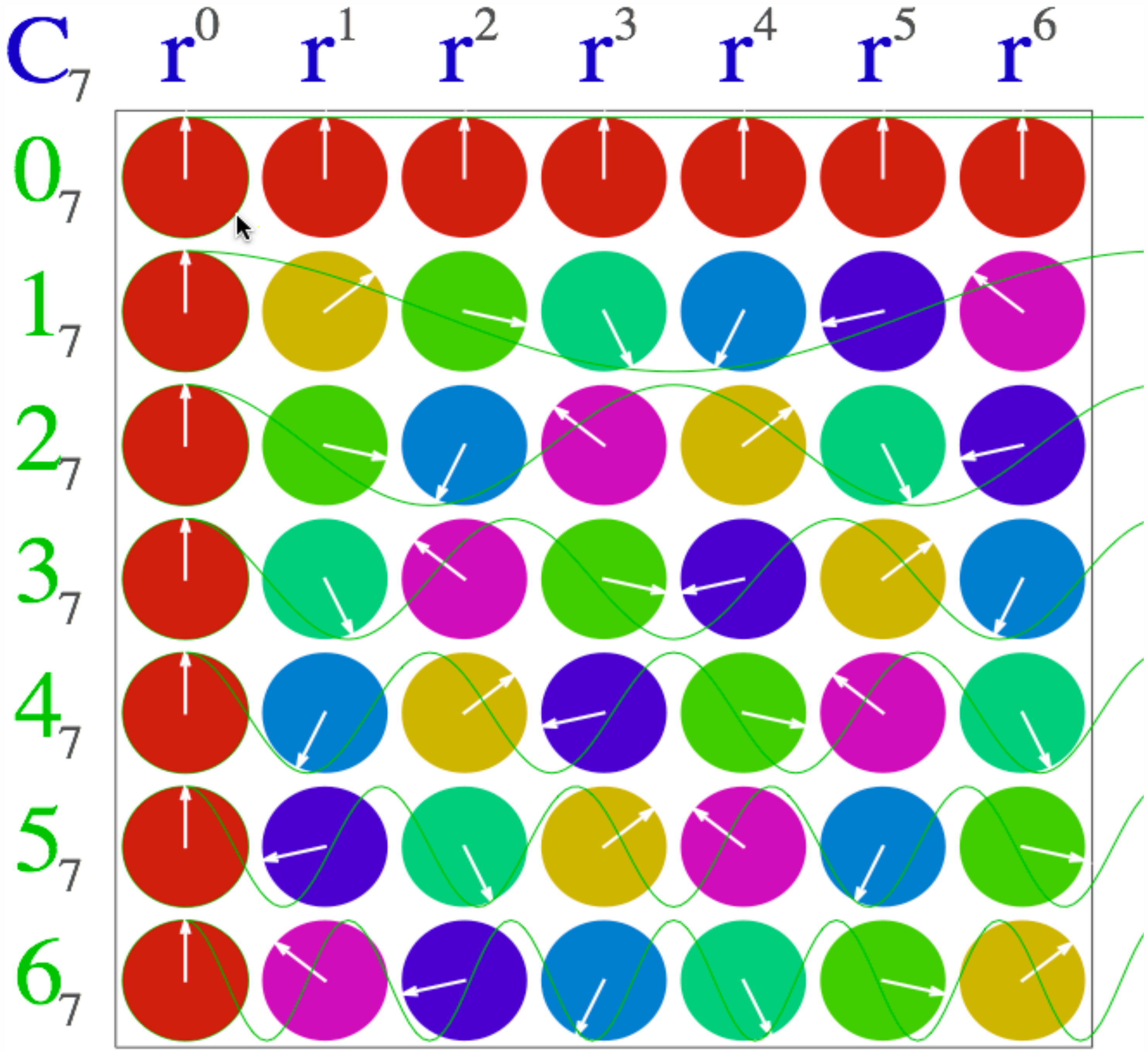
$$\begin{pmatrix}1&1&1\\1&e^{-i2\pi/3}&e^{i2\pi/3}\\1&e^{i2\pi/3}&e^{-i2\pi/3}\end{pmatrix}\Big/\sqrt{3}$$

$$\begin{pmatrix}1&1&1&1\\1&-i&-1&i\\1&-1&1&-1\\1&i&-1&i\end{pmatrix}\Big/\sqrt{4}$$

WaveIt App



WaveIt
C_6 Character Phasors
Web Simulation



WaveIt

C_7 *Character Phasors*

Web Simulation

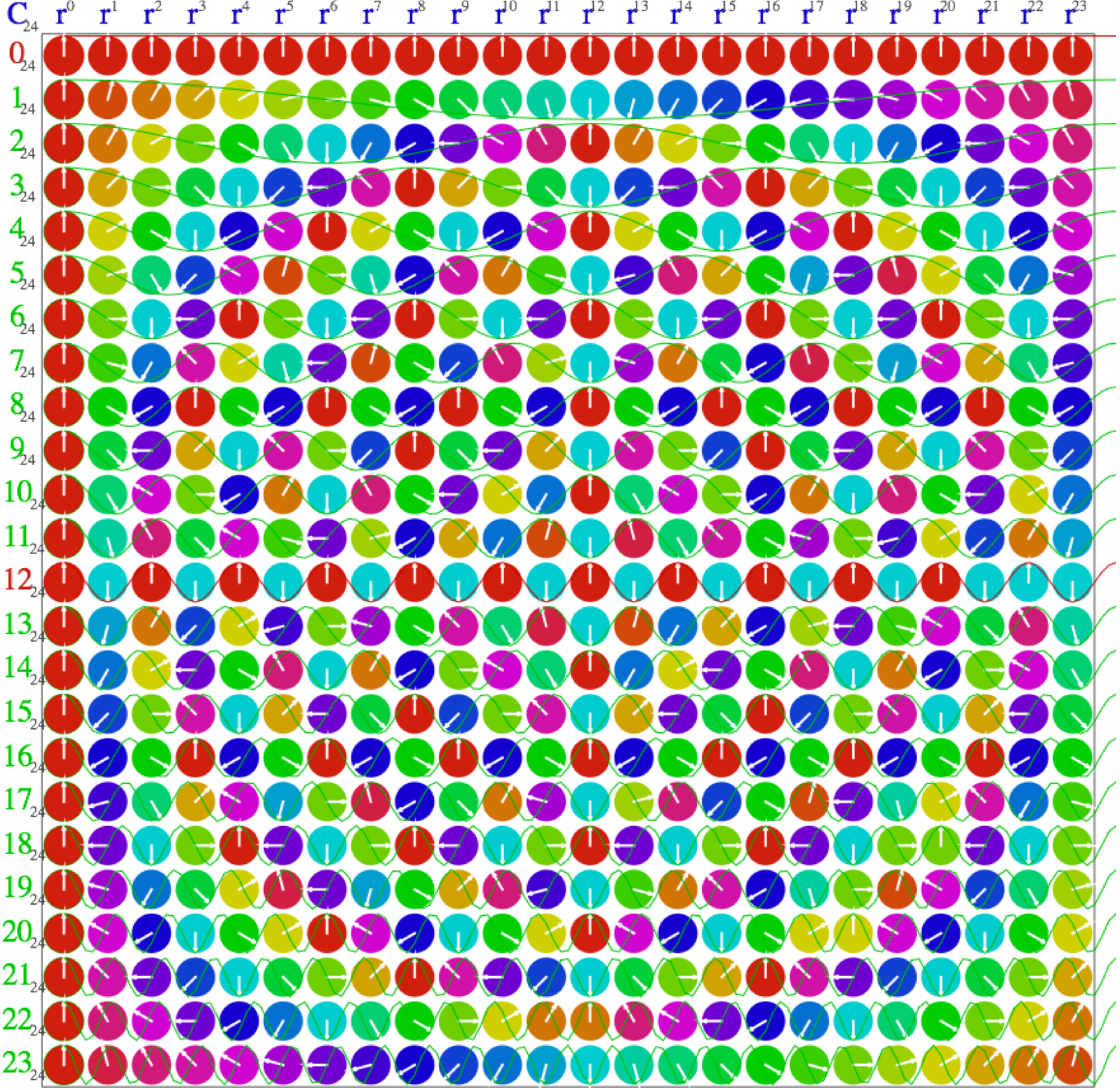
C_N *Lattice position vector*
$r_p=L\cdot p$

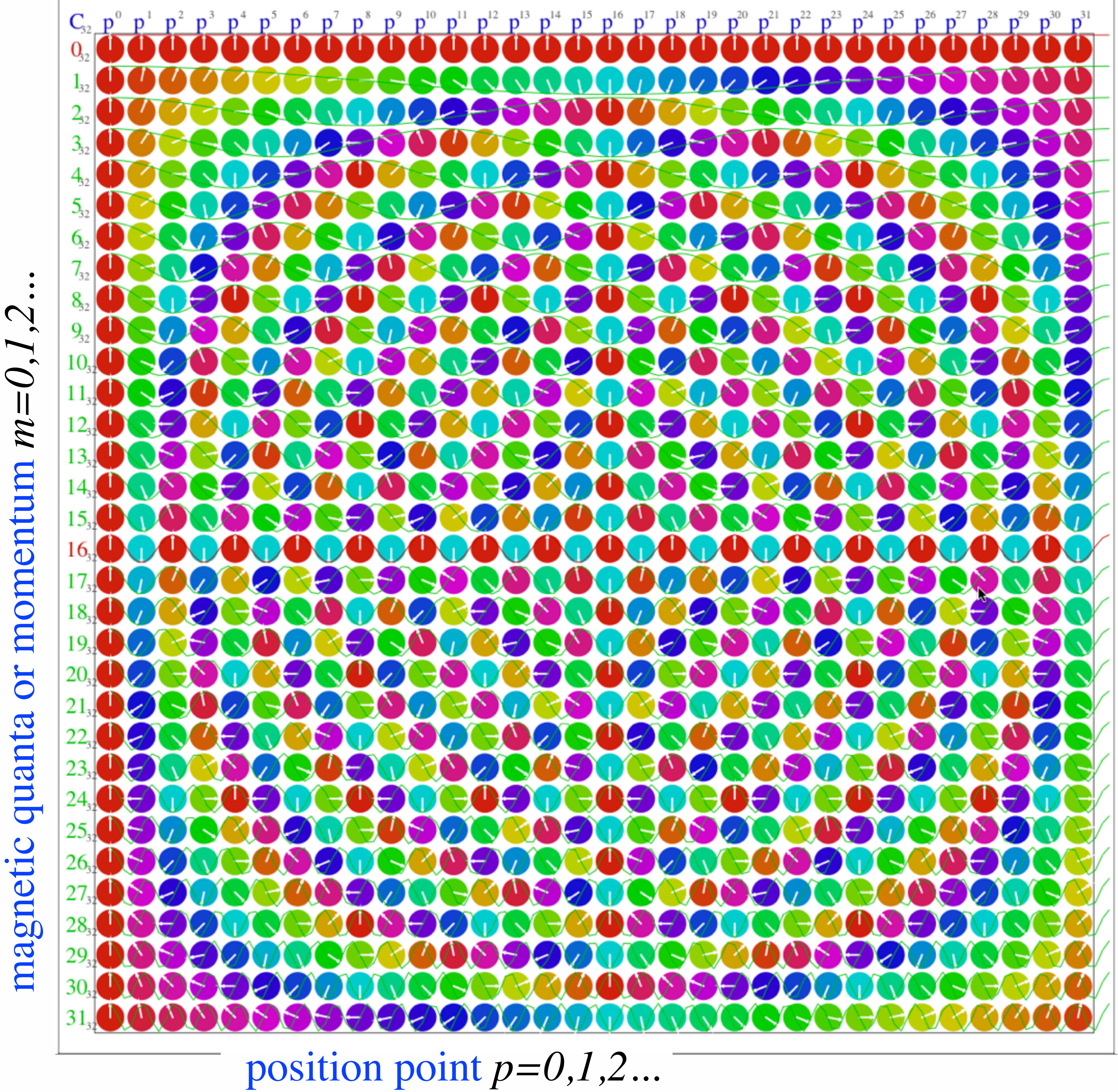
Wavevector
$k_m=2\pi/\lambda_m$
$=2\pi m/NL$

Wavelength
$\lambda_m=2\pi/k_m$
$=NL/m$

C_N Plane wave function

$$\psi_m(x_p)$$
$$=\frac{e^{ik_m\cdot x_p}}{\sqrt{N}}$$
$$=\frac{e^{imp2\pi/N}}{\sqrt{N}}$$





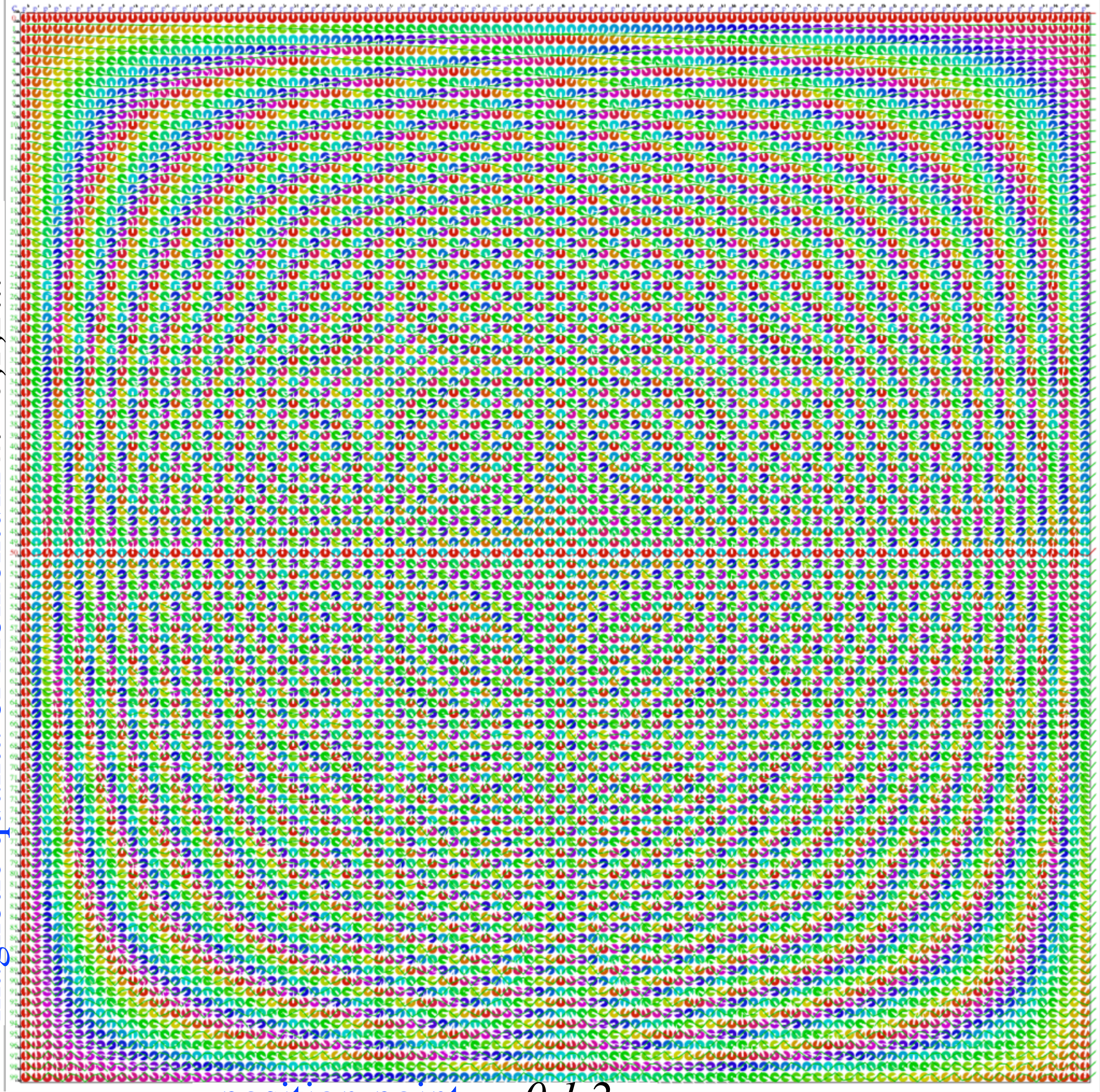
C_{32}

phasor character table

$$\chi_p^m = e^{-ik_m r^p}$$

$$= e^{\frac{-2\pi imp}{32}}$$

WaveIt C_{32} Character Phasors Web Simulation



magnetic quanta or momentum *m=0,1,2...*

position point *p=0,1,2...*

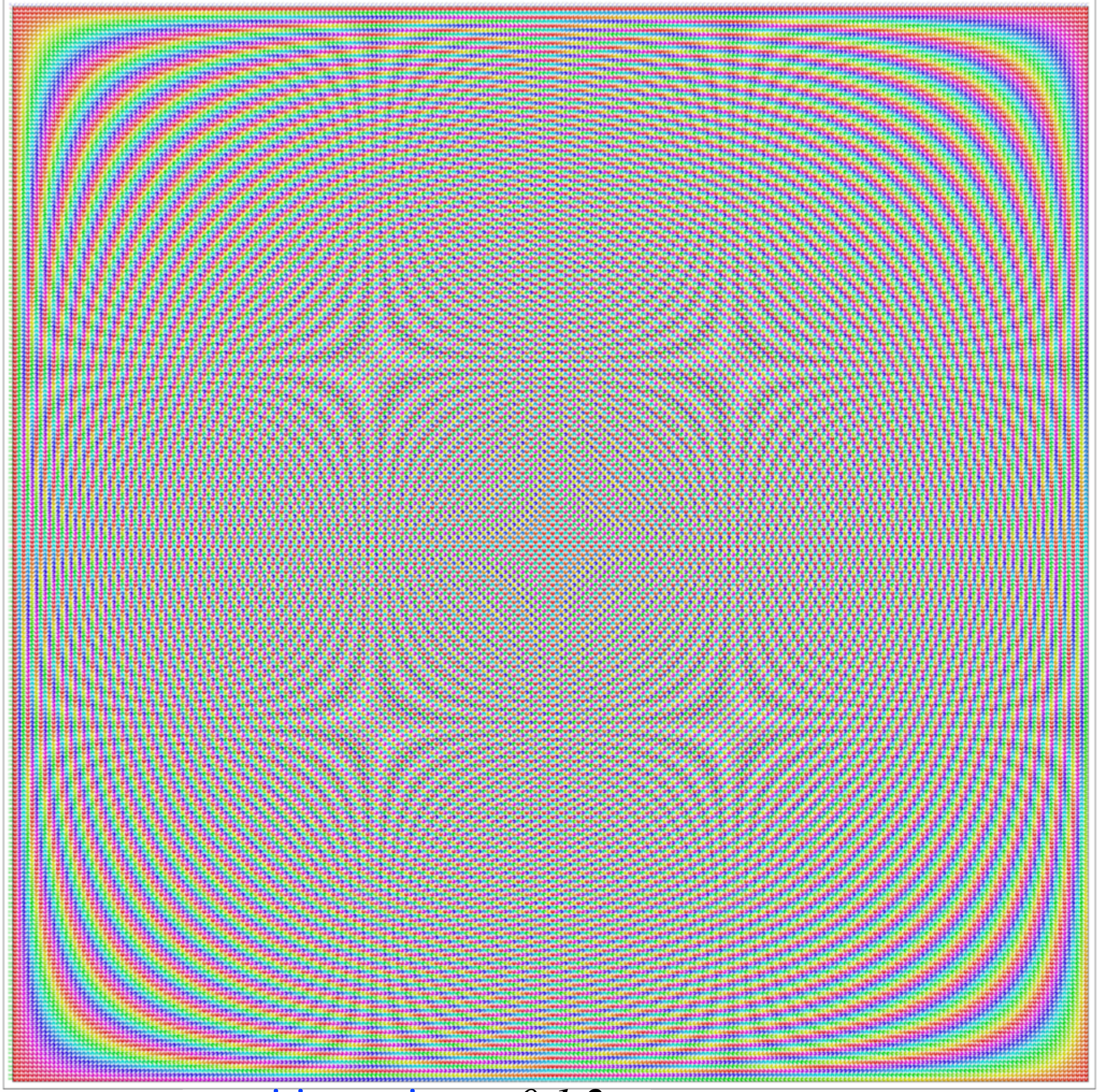
C_{100}

phasor
character
table

$$\chi_p^m = e^{-ik_m r^p}$$

$$= e^{\frac{-2\pi imp}{100}}$$

Invariant phase
"Uncertainty"
hyperbolas:
$m \cdot p = const.$



C_{256}

phasor
character
table

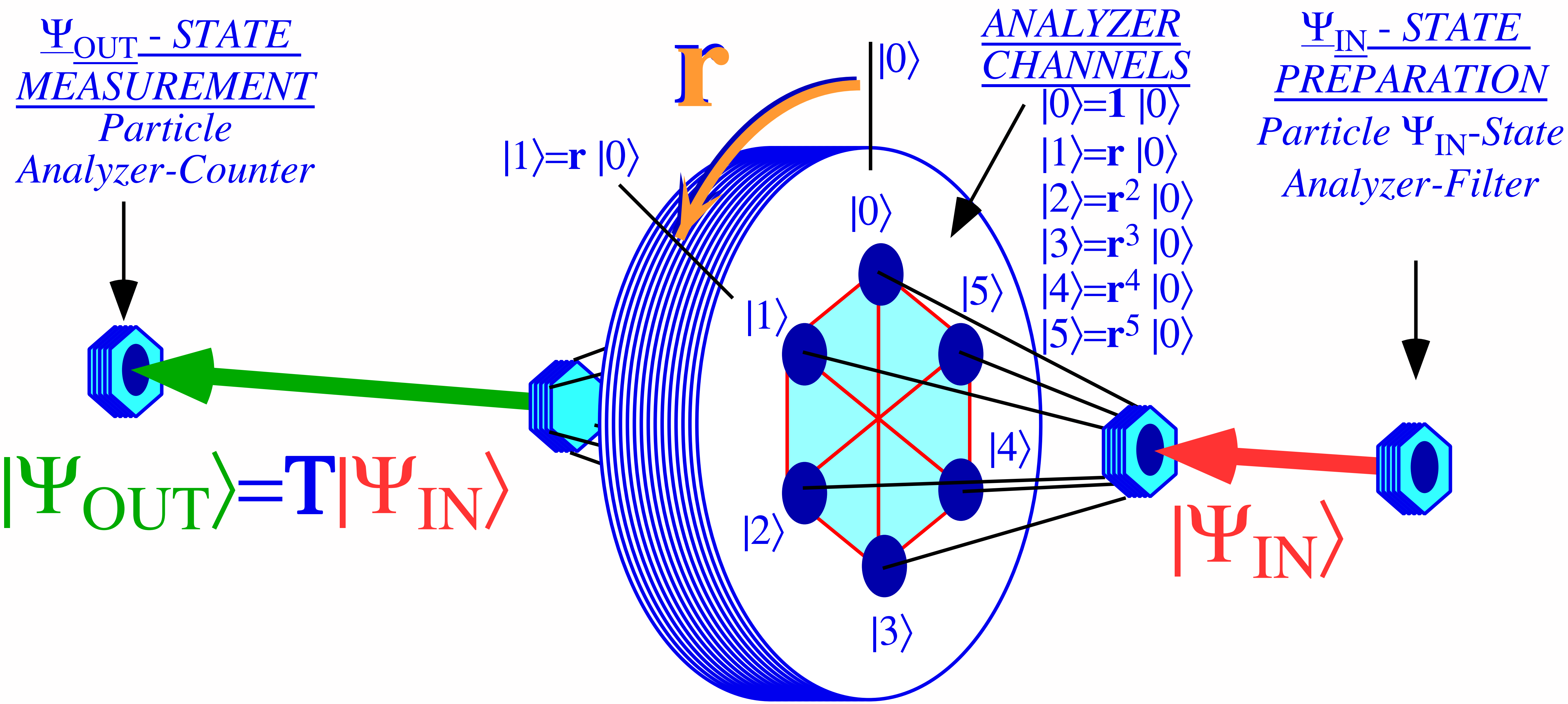
$$\chi_p^m = e^{-ik_m r^p}$$

$$= e^{\frac{-2\pi imp}{256}}$$

Invariant phase
"Uncertainty"
hyperbolas:
$m \cdot p = const.$

WaveIt C256 Character Phasors
Web Simulation

C_6 Beam analyzer used in Unit 3 Ch. 8 thru Ch. 9



Ψ_{OUT} - STATE MEASUREMENT
Particle
Analyzer-Counter

ANALYZER CHANNELS
$|0\rangle=\mathbf{1}\ |0\rangle$
$|1\rangle=\mathbf{r}\ |0\rangle$
$|2\rangle=\mathbf{r}^2\ |0\rangle$
$|3\rangle=\mathbf{r}^3\ |0\rangle$
$|4\rangle=\mathbf{r}^4\ |0\rangle$
$|5\rangle=\mathbf{r}^5\ |0\rangle$

Ψ_{IN} - STATE PREPARATION
Particle Ψ_{IN}-State
Analyzer-Filter

$$|\Psi_{OUT}\rangle=\mathbf{T}|\Psi_{IN}\rangle$$

QTforCA Fig. 8.1.1

1st Step: Abelian symmetry analysis Expand C_6 symmetric $\mathbf{H}$ by C_6 group product table

Regular representation and coupling parameters $\{r_0,r_1,r_2,r_3,r_4,r_5\}$ and Fourier dispersion

2nd Step: Find $\mathbf{H}$ eigenfunctions by spectral resolution of $C_6 = \{\mathbf{1}=\mathbf{r}^0,\mathbf{r}^1,\mathbf{r}^2,\mathbf{r}^3,\mathbf{r}^4,\mathbf{r}^5\}$

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→ *3rd Step: Dispersion functions and eigenvalues for varyious coupling parameter systems*

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Wave coordinates for Linear Dispersion

Wave coordinates for Bohr-Schrodinger Dispersion

Einstein-Lorentz-Minkowski laser coordinates

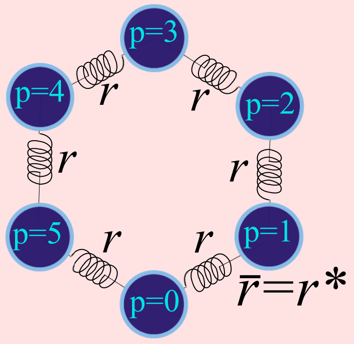
3^{rd} Step *Display all eigensolutions of all possible C_6 symmetric real H*

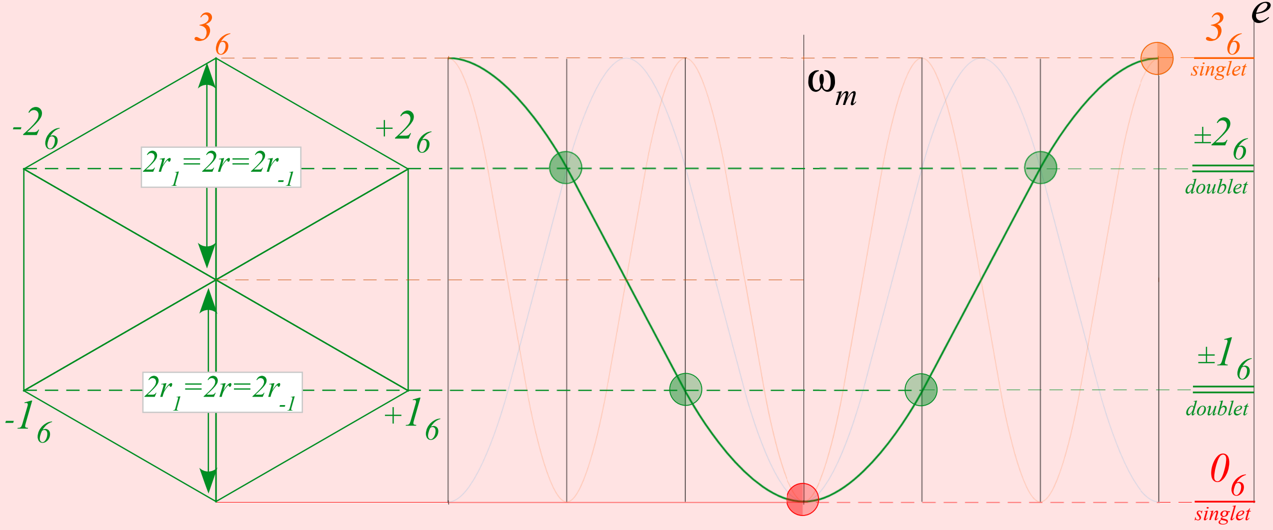
$$\mathbf{H}=\sum_{p=0}^{n-1} r_p \mathbf{r}^p=\sum_{p=0}^{n-1} r_p \sum_{m=0}^{n-1} \chi_p^m \mathbf{P}^{(m)}=\sum_{m=0}^{n-1} \omega^{(m)} \mathbf{P}^{(m)} \quad where: \omega^{(m)}=\sum_{p=0}^{n-1} r_p \chi_p^m=\omega(k_m)$$ *(Dispersion functions)*

3rd Step *Display all eigensolutions of all possible C_6 symmetric real H*

$$\mathbf{H}=\sum_{p=0}^{n-1} r_p \mathbf{r}^p=\sum_{p=0}^{n-1} r_p \sum_{m=0}^{n-1} \chi_p^m \mathbf{P}^{(m)}=\sum_{m=0}^{n-1} \omega^{(m)} \mathbf{P}^{(m)} \quad where: \omega^{(m)}=\sum_{p=0}^{n-1} r_p \chi_p^m=\omega(k_m)$$ *(Dispersion functions)*

Elementary
Bloch Model
H= H_1**1** - r**r** - r**r**$^{-1}$





eigenvalues of $\mathbf{H}^{B1(6)}$

$$\begin{array}{cccccc|c} p=0 & 1 & 2 & 3 & 4 & 5 & \\ H_1 & -r & \cdot & \cdot & \cdot & -r & 0 \\ -r & H_1 & -r & \cdot & \cdot & \cdot & 1 \\ \cdot & -r & H_1 & -r & \cdot & \cdot & 2 \\ \cdot & \cdot & -r & H_1 & -r & \cdot & 3 \\ \cdot & \cdot & \cdot & -r & H_1 & -r & 4 \\ -r & \cdot & \cdot & \cdot & -r & H_1 & 5 \end{array}$$

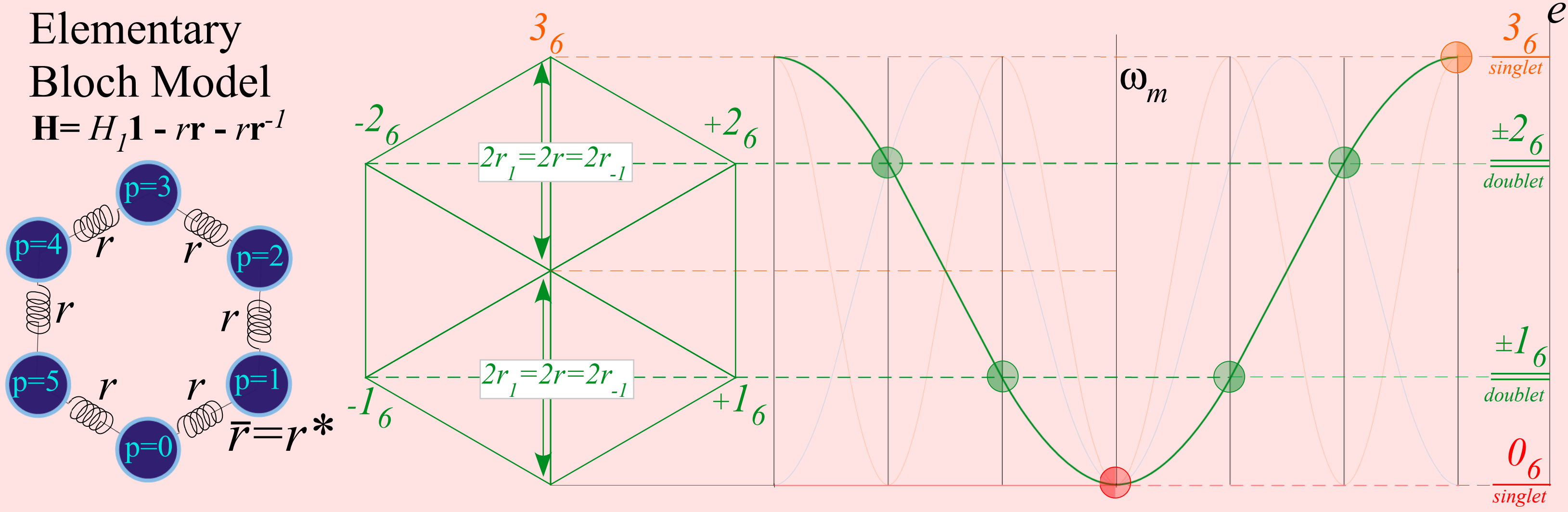
$$\omega^{B1(n)}(k_m)$$
$$= r_0\chi_0^m + r_1\chi_1^m + r_{-1}\chi_{-1}^m$$
$$= H_1 - 2r\cos(2\pi m/6)$$

r_1 equals conjugate of r_5: ($r_1 = r_5^* = -r$)

3^{rd} Step *Display all eigensolutions of all possible C_6 symmetric real H*

$$\mathbf{H}=\sum_{p=0}^{n-1} r_p\mathbf{r}^p=\sum_{p=0}^{n-1} r_p\sum_{m=0}^{n-1}\chi_p^m\mathbf{P}^{(m)}=\sum_{m=0}^{n-1}\omega^{(m)}\mathbf{P}^{(m)}\quad where:\ \omega^{(m)}=\sum_{p=0}^{n-1} r_p\chi_p^m=\omega(k_m)$$ *(Dispersion functions)*

Elementary Bloch Model
H= $H_1\mathbf{1} - r\mathbf{r} - r\mathbf{r}^{-1}$

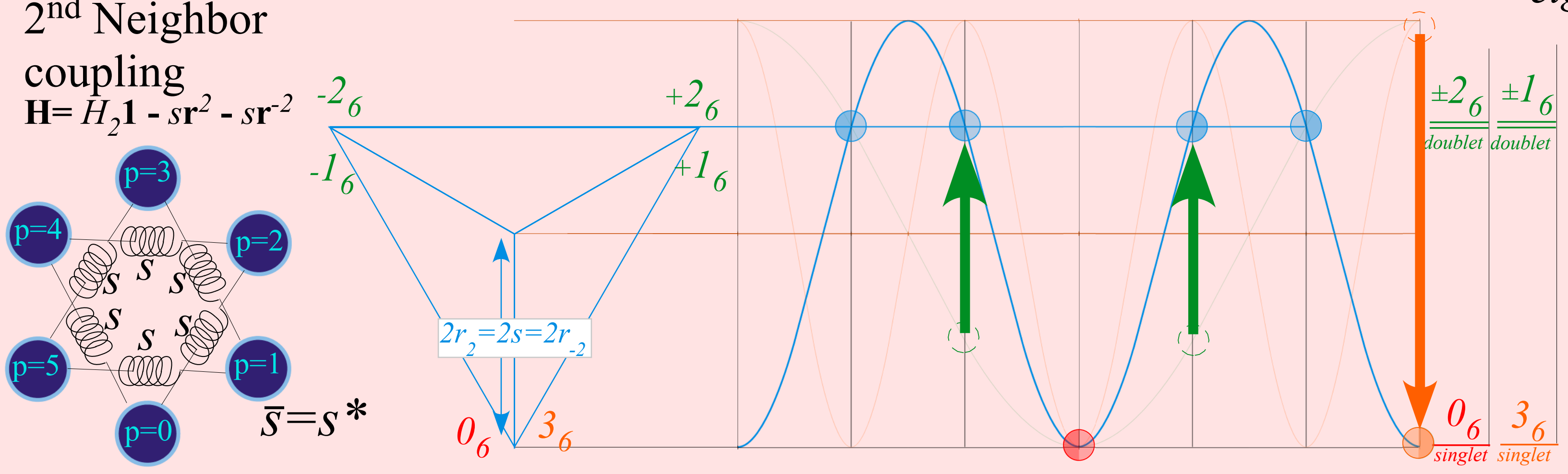


eigenvalues of $\mathbf{H}^{B1(6)}$

$$\begin{array}{c|cccccc|c} & p=0 & 1 & 2 & 3 & 4 & 5 & \\ \hline & H_1 & -r & \cdot & \cdot & \cdot & -r & 0 \\ & -r & H_1 & -r & \cdot & \cdot & \cdot & 1 \\ & \cdot & -r & H_1 & -r & \cdot & \cdot & 2 \\ & \cdot & \cdot & -r & H_1 & -r & \cdot & 3 \\ & \cdot & \cdot & \cdot & -r & H_1 & -r & 4 \\ & -r & \cdot & \cdot & \cdot & -r & H_1 & 5 \end{array}$$

$$\omega^{B1(n)}(k_m)$$
$$=r_0\chi_0^m+r_1\chi_1^m+r_{-1}\chi_{-1}^m$$
$$=H_1-2r\,cos(2\pi m/6)$$

2^{nd} Neighbor coupling
H= $H_2\mathbf{1} - s\mathbf{r}^2 - s\mathbf{r}^{-2}$



eigenvalues of $\mathbf{H}^{B2(6)}$

$$\begin{array}{c|cccccc|c} & p=0 & 1 & 2 & 3 & 4 & 5 & \\ \hline & H_2 & \cdot & -s & \cdot & -s & \cdot & 0 \\ & \cdot & H_2 & \cdot & -s & \cdot & -s & 1 \\ & -s & \cdot & H_2 & \cdot & -s & \cdot & 2 \\ & \cdot & -s & \cdot & H_2 & \cdot & -s & 3 \\ & -s & \cdot & -s & \cdot & H_2 & \cdot & 4 \\ & \cdot & -s & \cdot & -s & \cdot & H_2 & 5 \end{array}$$

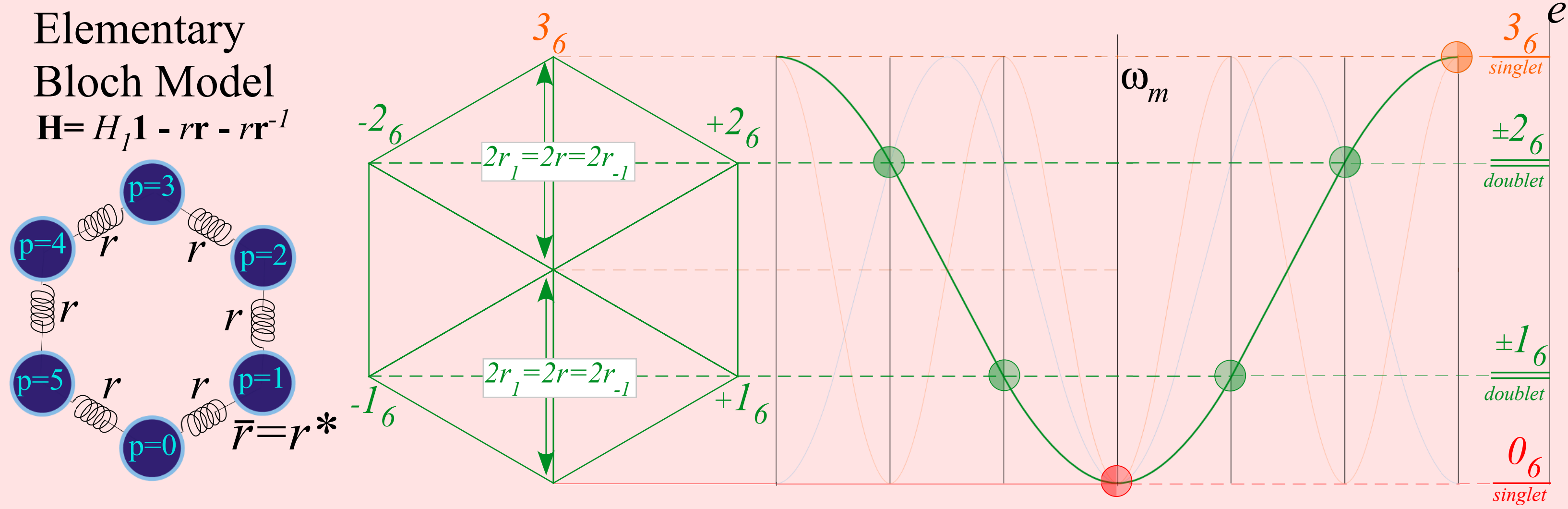
$$\omega^{B2(n)}(k_m)$$
$$=r_0\chi_0^m+r_2\chi_2^m+r_{-2}\chi_{-2}^m$$
$$=H_2-2s\,cos(4\pi m/6)$$

3^{rd} Step *Display all eigensolutions of all possible C_6 symmetric real H*

$$\mathbf{H}=\sum_{p=0}^{n-1} r_p \mathbf{r}^p=\sum_{p=0}^{n-1} r_p \sum_{m=0}^{n-1} \chi_p^m \mathbf{P}^{(m)}=\sum_{m=0}^{n-1} \omega^{(m)} \mathbf{P}^{(m)} \quad where: \omega^{(m)}=\sum_{p=0}^{n-1} r_p \chi_p^m=\omega(k_m)$$

(Dispersion functions)

Elementary
Bloch Model
H= $H_1\mathbf{1} - r\mathbf{r} - r\mathbf{r}^{-1}$

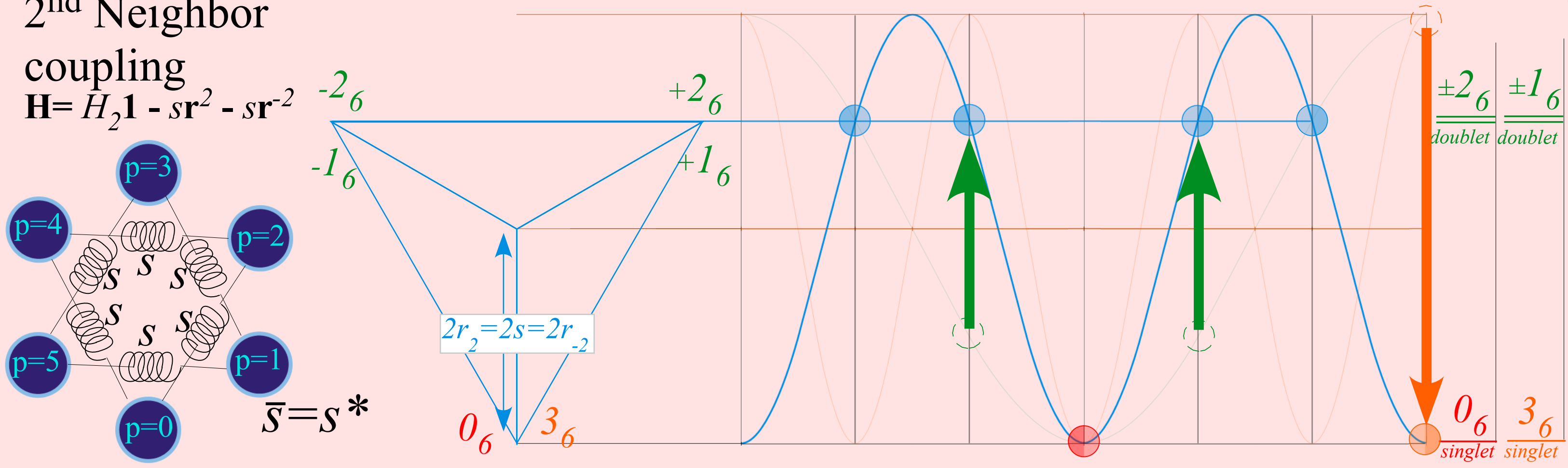


eigenvalues of $\mathbf{H}^{B1(6)}$

$$\begin{array}{cccccc|c} p=0 & 1 & 2 & 3 & 4 & 5 & \\ H_1 & -r & \cdot & \cdot & \cdot & -r & 0 \\ -r & H_1 & -r & \cdot & \cdot & \cdot & 1 \\ \cdot & -r & H_1 & -r & \cdot & \cdot & 2 \\ \cdot & \cdot & -r & H_1 & -r & \cdot & 3 \\ \cdot & \cdot & \cdot & -r & H_1 & -r & 4 \\ -r & \cdot & \cdot & \cdot & -r & H_1 & 5 \end{array}$$

$$\omega^{B1(n)}(k_m) = r_0\chi_0^m + r_1\chi_1^m + r_{-1}\chi_{-1}^m = H_1 - 2r\,cos(2\pi m/6)$$

2^{nd} Neighbor
coupling
H= $H_2\mathbf{1} - s\mathbf{r}^2 - s\mathbf{r}^{-2}$

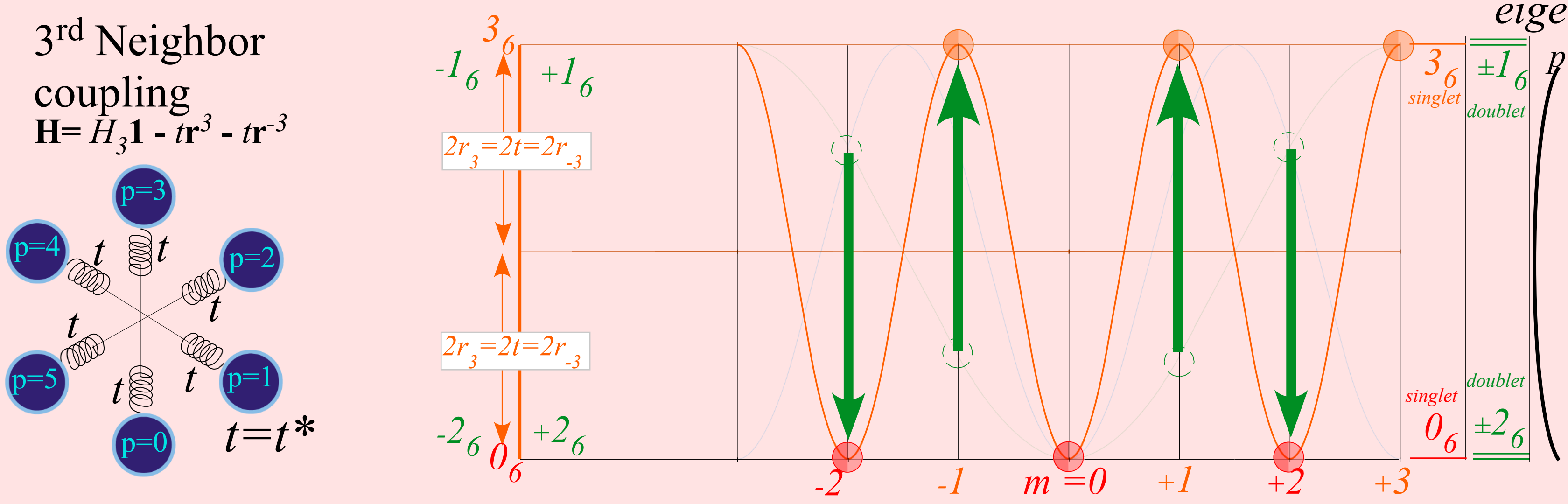


eigenvalues of $\mathbf{H}^{B2(6)}$

$$\begin{array}{cccccc|c} p=0 & 1 & 2 & 3 & 4 & 5 & \\ H_2 & \cdot & -s & \cdot & -s & \cdot & 0 \\ \cdot & H_2 & \cdot & -s & \cdot & -s & 1 \\ -s & \cdot & H_2 & \cdot & -s & \cdot & 2 \\ \cdot & -s & \cdot & H_2 & \cdot & -s & 3 \\ -s & \cdot & -s & \cdot & H_2 & \cdot & 4 \\ \cdot & -s & \cdot & -s & \cdot & H_2 & 5 \end{array}$$

$$\omega^{B2(n)}(k_m) = r_0\chi_0^m + r_2\chi_2^m + r_{-2}\chi_{-2}^m = H_2 - 2s\,cos(4\pi m/6)$$

3^{rd} Neighbor
coupling
H= $H_3\mathbf{1} - t\mathbf{r}^3 - t\mathbf{r}^{-3}$



eigenvalues of $\mathbf{H}^{B3(6)}$

$$\begin{array}{cccccc|c} p=0 & 1 & 2 & 3 & 4 & 5 & \\ H_3 & \cdot & \cdot & -t & \cdot & \cdot & 0 \\ \cdot & H_3 & \cdot & \cdot & -t & \cdot & 1 \\ \cdot & \cdot & H_3 & \cdot & \cdot & -t & 2 \\ -t & \cdot & \cdot & H_3 & \cdot & \cdot & 3 \\ \cdot & -t & \cdot & \cdot & H_3 & \cdot & 4 \\ \cdot & \cdot & -t & \cdot & \cdot & H_3 & 5 \end{array}$$

$$\omega^{B3(n)}(k_m) = r_0\chi_0^m + r_3\chi_3^m + r_{-3}\chi_{-3}^m = H_3 - 2t\,(-1)^m$$

1^{st} Step: Abelian symmetry analysis Expand C_6 symmetric **H** *by C_6 group product table*

Regular representation and coupling parameters $\{r_0,r_1,r_2,r_3,r_4,r_5\}$ and Fourier dispersion

2^{nd} Step: Find **H** *eigenfunctions by spectral resolution of $C_6 = \{\mathbf{1}=\mathbf{r}^0,\mathbf{r}^1,\mathbf{r}^2,\mathbf{r}^3,\mathbf{r}^4,\mathbf{r}^5\}$*

Character tables of C_2, C_3, C_4, C_5,…. C_{144}

3^{rd} Step: Dispersion functions and eigenvalues for varyious coupling parameter systems

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Wave coordinates for Linear Dispersion

Wave coordinates for Bohr-Schrodinger Dispersion

Einstein-Lorentz-Minkowski laser coordinates

Complete sets of C_6 coupling parameters and Fourier dispersion

$$\omega_m(\mathbf{H}^{GB(N)}) = \langle m|\sum_{p=0} r_p \mathbf{r}^p |m\rangle = \sum_{p=0} r_p \langle m|\mathbf{r}^p|m\rangle = \sum_{p=0} r_p e^{-i2\pi\frac{m\cdot p}{N}} = \sum_{p=0} |r_p| e^{-i(2\pi\frac{m\cdot p}{N}-\phi_p)}$$

C_6 Bloch $\mathbf{H}^{\mathrm{GB}(N)}$ eigenvalues are Fourier series

Complete sets of C_6 coupling parameters and Fourier dispersion

$$\omega_m(\mathbf{H}^{GB(N)}) = \langle m| \sum_{p=0} r_p \mathbf{r}^p |m\rangle = \sum_{p=0} r_p \langle m|\mathbf{r}^p|m\rangle = \sum_{p=0} r_p e^{-i2\pi\frac{m\cdot p}{N}} = \sum_{p=0} |r_p| e^{-i(2\pi\frac{m\cdot p}{N} - \phi_p)}$$

Setting gauge to zero ($\phi_p = 0$)

Real C_6 Bloch $\mathbf{H}^{GB(N)}$ eigenvalues are Fourier series with 4 (for $N=6$) Fourier parameters

$\{ r_0 = H, \quad r_1 = r = r_{-1}, \; r_2 = s = r_{-2}, \quad r_3 = t = r_{-3} \}$

Complete sets of C_6 coupling parameters and Fourier dispersion

$$\omega_m(\mathbf{H}^{GB(N)})=\langle m|\sum_{p=0} r_p\mathbf{r}^p|m\rangle=\sum_{p=0} r_p\langle m|\mathbf{r}^p|m\rangle=\sum_{p=0} r_p e^{-i2\pi\frac{m\cdot p}{N}}=\sum_{p=0}|r_p|e^{-i(2\pi\frac{m\cdot p}{N}-\phi_p)}$$

Setting gauge to zero ($\phi_p=0$)

Real C_6 Bloch $\mathbf{H}^{\mathrm{GB}(N)}$ eigenvalues are Fourier series with 4 (for $N=6$) Fourier parameters

$\{ r_0 = H,\quad r_1 = r = r_{-1}\ , r_2 = s = r_{-2},\quad r_3 = t = r_{-3} \}$

$$\omega_m(\mathbf{H}_{real}{}^{GB(6)})=r_0+r_1(e^{i\pi\frac{m\cdot 1}{3}}+e^{-i\pi\frac{m\cdot 1}{3}})\quad +r_2(e^{i\pi\frac{m\cdot 2}{3}}+e^{-i\pi\frac{m\cdot 2}{3}})\quad +r_3(e^{i\pi\frac{m\cdot 3}{3}})\qquad \text{(for real: } r_p=r_{-p}=r_p^*)$$

Complete sets of C_6 coupling parameters and Fourier dispersion

$$\omega_m(\mathbf{H}^{GB(N)}) = \langle m| \sum_{p=0} r_p \mathbf{r}^p |m\rangle = \sum_{p=0} r_p \langle m|\mathbf{r}^p|m\rangle = \sum_{p=0} r_p e^{-i2\pi\frac{m\cdot p}{N}} = \sum_{p=0} |r_p| e^{-i(2\pi\frac{m\cdot p}{N}-\phi_p)}$$

Setting gauge to zero ($\phi_p = 0$)

Real C_6 Bloch $\mathbf{H}^{GB(N)}$ eigenvalues are Fourier series with 4 (for $N=6$) Fourier parameters

$\{ r_0 = H, \quad r_1 = r = r_{-1} \ , r_2 = s = r_{-2}, \quad r_3 = t = r_{-3} \}$

$$\omega_m(\mathbf{H}^{GB(6)}_{real}) = r_0 + r_1(e^{i\pi\frac{m\cdot 1}{3}} + e^{-i\pi\frac{m\cdot 1}{3}}) \quad + r_2(e^{i\pi\frac{m\cdot 2}{3}} + e^{-i\pi\frac{m\cdot 2}{3}}) \quad + r_3(e^{i\pi\frac{m\cdot 3}{3}}) \qquad (\text{for real: } r_p = r_{-p} = r_p^*)$$

$$= H \ + \ 2r\cos\pi\frac{m\cdot 1}{3} \qquad + \qquad 2s\cos\pi\frac{m\cdot 2}{3} \qquad + \ t(-1)^m$$

Complete sets of C_6 coupling parameters and Fourier dispersion

$$\omega_m(\mathbf{H}^{GB(N)}) = \langle m|\sum_{p=0} r_p \mathbf{r}^p|m\rangle = \sum_{p=0} r_p \langle m|\mathbf{r}^p|m\rangle = \sum_{p=0} r_p e^{-i2\pi\frac{m\cdot p}{N}} = \sum_{p=0} |r_p| e^{-i(2\pi\frac{m\cdot p}{N}-\phi_p)}$$

Setting gauge to zero (ϕ_p=0)

Real C_6 Bloch $\mathbf{H}^{\mathrm{GB}(N)}$ eigenvalues are Fourier series with 4 (for N=6) Fourier parameters

$\{ r_0 = H,\quad r_1 = r = r_{-1}\ , r_2 = s = r_{-2},\quad r_3 = t = r_{-3} \}$

$$\omega_m(\mathbf{H}^{GB(6)}_{real}) = r_0 + r_1(e^{i\pi\frac{m\cdot 1}{3}} + e^{-i\pi\frac{m\cdot 1}{3}}) \quad + r_2(e^{i\pi\frac{m\cdot 2}{3}} + e^{-i\pi\frac{m\cdot 2}{3}}) \quad + r_3(e^{i\pi\frac{m\cdot 3}{3}}) \qquad \text{(for real: } r_p = r_{-p} = r_p^*)$$

$$= H \;+\; 2r\cos\pi\frac{m\cdot 1}{3} \quad + \quad 2s\cos\pi\frac{m\cdot 2}{3} \quad + \; t(-1)^m$$

giving 4 ω_m-levels:

$$\omega_m = \begin{cases} \omega_0 & = H + \ 2r + \ 2s + t \\ \omega_{\pm 1} & = H + \ \ r - \ \ s - t \\ \omega_{\pm 2} & = H - \ \ r - \ \ s + t \\ \omega_3 & = H - 2r + 2s - t \end{cases}$$

Complete sets of C_6 coupling parameters and Fourier dispersion

$$\omega_m(\mathbf{H}^{GB(N)}) = \langle m| \sum_{p=0} r_p \mathbf{r}^p |m\rangle = \sum_{p=0} r_p \langle m|\mathbf{r}^p|m\rangle = \sum_{p=0} r_p e^{-i2\pi\frac{m\cdot p}{N}} = \sum_{p=0} |r_p| e^{-i(2\pi\frac{m\cdot p}{N}-\phi_p)}$$

Setting gauge to zero (ϕ_p=0)

Real C_6 Bloch **H** $^{GB(N)}$ eigenvalues are Fourier series with 4 (for N=6) Fourier parameters

$\{\ r_0 = H,\quad r_1 = r = r_{-1}\ ,\ r_2 = s = r_{-2},\quad r_3 = t = r_{-3}\ \}$

$$\omega_m(\mathbf{H}^{GB(6)}_{real}) = r_0 + r_1(e^{i\pi\frac{m\cdot 1}{3}} + e^{-i\pi\frac{m\cdot 1}{3}}) \quad + r_2(e^{i\pi\frac{m\cdot 2}{3}} + e^{-i\pi\frac{m\cdot 2}{3}}) \quad + r_3(e^{i\pi\frac{m\cdot 3}{3}}) \qquad \text{(for real: } r_p = r_{-p} = r_p^*)$$

$$= H \ +\ 2r\cos\pi\frac{m\cdot 1}{3} \quad + \quad 2s\cos\pi\frac{m\cdot 2}{3} \quad + \ t(-1)^m$$

giving 4 ω_m-levels:

$$\omega_m = \begin{cases} \omega_0 & = H + \ 2r + \ 2s + t \\ \omega_{\pm 1} & = H + \ \ r - \ \ s - t \\ \omega_{\pm 2} & = H - \ \ r - \ \ s + t \\ \omega_3 & = H - 2r + 2s - t \end{cases}$$

...in terms of 4 solvable r_p-parameters:

$$r_p = \begin{cases} H = \frac{1}{4}(\omega_0 + \ \omega_1 \ + \ \omega_2 \ + \ \omega_3) \\ r \ = \frac{1}{6}(\omega_0 + \ \omega_1 \ - \ \omega_2 \ - \ \omega_3) \\ s \ = \frac{1}{6}(\omega_0 - \ \omega_1 \ - \ \omega_2 \ + \ \omega_3) \\ t \ = \frac{1}{6}(\omega_0 - 2\omega_1 \ + 2\omega_2 - \ \omega_3) \end{cases}$$

Complete sets of C_6 coupling parameters and Fourier dispersion

$$\omega_m(\mathbf{H}^{GB(N)}) = \langle m| \sum_{p=0} r_p \mathbf{r}^p |m\rangle = \sum_{p=0} r_p \langle m|\mathbf{r}^p|m\rangle = \sum_{p=0} r_p e^{-i2\pi\frac{m\cdot p}{N}} = \sum_{p=0} |r_p| e^{-i(2\pi\frac{m\cdot p}{N}-\phi_p)}$$

Setting gauge to zero (ϕ_p =0)

Real C_6 Bloch $\mathbf{H}^{GB(N)}$ eigenvalues are Fourier series with 4 (for N=6) Fourier parameters

$\{ r_0 = H,\quad r_1 = r = r_{-1},\ r_2 = s = r_{-2},\quad r_3 = t = r_{-3} \}$

$$\omega_m(\mathbf{H}^{GB(6)}_{real}) = r_0 + r_1(e^{i\pi\frac{m\cdot 1}{3}} + e^{-i\pi\frac{m\cdot 1}{3}}) + r_2(e^{i\pi\frac{m\cdot 2}{3}} + e^{-i\pi\frac{m\cdot 2}{3}}) + r_3(e^{i\pi\frac{m\cdot 3}{3}}) \quad \text{(for real: } r_p = r_{-p} = r_p^*)$$

$$= H + 2r\cos\pi\frac{m\cdot 1}{3} + 2s\cos\pi\frac{m\cdot 2}{3} + t(-1)^m$$

giving 4 ω_m-levels:

$$\omega_m = \begin{cases} \omega_0 = H + 2r + 2s + t \\ \omega_{\pm 1} = H + r - s - t \\ \omega_{\pm 2} = H - r - s + t \\ \omega_3 = H - 2r + 2s - t \end{cases}$$

...in terms of 4 solvable r_p-parameters:

$$r_p = \begin{cases} H = \frac{1}{4}(\omega_0 + \omega_1 + \omega_2 + \omega_3) \\ r = \frac{1}{6}(\omega_0 + \omega_1 - \omega_2 - \omega_3) \\ s = \frac{1}{6}(\omega_0 - \omega_1 - \omega_2 + \omega_3) \\ t = \frac{1}{6}(\omega_0 - 2\omega_1 + 2\omega_2 - \omega_3) \end{cases}$$

General Bloch $\mathbf{H}^{GB(N)}$ eigenvalues are Fourier series with six (for N=6) Fourier parameters

$\{ r_0 = H,\quad r_1 = re^{i\phi_1},\quad r_{-1} = re^{-i\phi_1},\quad r_2 = se^{i\phi_2},\quad r_{-2} = se^{-i\phi_2},\quad r_3 = t = r_{-3} \}$

Nonzero gauge ϕ_p , or complex: $r_{-p} = r_p^*$

$$\omega_m(\mathbf{H}_{complex}{}^{GZB(6)}) = H + 2r\cos\left(\pi\frac{m\cdot 1}{3} - \phi_1\right) + 2s\cos\left(\pi\frac{m\cdot 2}{3} - \phi_2\right) + t(-1)^m$$

1st Step: Abelian symmetry analysis Expand C_6 symmetric **H** *by C_6 group product table*

Regular representation and coupling parameters $\{r_0,r_1,r_2,r_3,r_4,r_5\}$ and Fourier dispersion

2nd Step: Find **H** *eigenfunctions by spectral resolution of $C_6 =\{\mathbf{1}=\mathbf{r}^0,\mathbf{r}^1,\mathbf{r}^2,\mathbf{r}^3,\mathbf{r}^4,\mathbf{r}^5\}$*

Character tables of C_2, C_3, C_4, C_5,…. C_{144}

3rd Step: Dispersion functions and eigenvalues for varyious coupling parameter systems

Ortho-complete eigenvalue/parameter relations

➡ *Gauge shifts due to complex coupling*

Wave dynamics of phase, mean phase, and group velocity by Expo-Cosine identity

Relating space-time (x,t) and per-space-time (k,ω)

Wave coordinates

Pulse-waves (PW) vs Continuous-waves (CW)

Wave coordinates for Linear Dispersion

Wave coordinates for Bohr-Schrodinger Dispersion

Einstein-Lorentz-Minkowski laser coordinates

Complex sets of C_6 coupling parameters and gauge shifts

$$\omega_m(\mathbf{H}^{GB(N)}) = \langle m | \sum_{p=0} r_p \mathbf{r}^p | m \rangle = \sum_{p=0} r_p \langle m | \mathbf{r}^p | m \rangle = \sum_{p=0} r_p e^{-i2\pi\frac{m\cdot p}{N}} = \sum_{p=0} |r_p| e^{-i(2\pi\frac{m\cdot p}{N} - \phi_p)}$$

Complex Bloch matrix $\mathbf{H}^{GB(N)}$ eigenvalues are Fourier series with 6 (for $N=6$) Fourier parameters

$\{ r_0 = H, \quad r_1 = re^{i\phi_1}, \quad r_{-1} = re^{-i\phi_1}, \quad r_2 = se^{i\phi_2}, \quad r_{-2} = se^{-i\phi_2}, \quad r_3 = t = r_{-3} \}$

Nonzero gauge ϕ_p ,

$$\omega_m(\mathbf{H}^{GZB(6)}_{complex}) = r_0 + r_1 e^{i\pi\frac{m\cdot 1}{3}} + r_{-1} e^{-i\pi\frac{m\cdot 1}{3}} + r_2 e^{i\pi\frac{m\cdot 2}{3}} + r_{-2} e^{-i\pi\frac{m\cdot 2}{3}} + r_3 e^{i\pi\frac{m\cdot 3}{3}}$$

or complex: $r_{-p} = r_p^*$.

Complex sets of C_6 coupling parameters and gauge shifts

$$\omega_m(\mathbf{H}^{GB(N)}) = \langle m| \sum_{p=0} r_p \mathbf{r}^p |m\rangle = \sum_{p=0} r_p \langle m|\mathbf{r}^p|m\rangle = \sum_{p=0} r_p e^{-i2\pi\frac{m\cdot p}{N}} = \sum_{p=0} |r_p| e^{-i(2\pi\frac{m\cdot p}{N} - \phi_p)}$$

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$$\omega_m(\mathbf{H}^{GZB(6)}_{complex}) = r_0 + r_1 e^{i\pi\frac{m\cdot 1}{3}} + r_{-1} e^{-i\pi\frac{m\cdot 1}{3}} + r_2 e^{i\pi\frac{m\cdot 2}{3}} + r_{-2} e^{-i\pi\frac{m\cdot 2}{3}} + r_3 e^{i\pi\frac{m\cdot 3}{3}}$$

or complex: $r_{-p} = r^*_p$.

$$\omega_m(\mathbf{H}^{GZB(6)}_{complex}) = H + 2r\cos\left(\pi\frac{m\cdot 1}{3} - \phi_1\right) + 2s\cos\left(\pi\frac{m\cdot 2}{3} - \phi_2\right) + t(-1)^m$$

or complex: $r_{-p} = r^*_p$

Complex sets of C_6 coupling parameters and gauge shifts

$$\omega_m(\mathbf{H}^{GB(N)}) = \langle m| \sum_{p=0} r_p \mathbf{r}^p |m\rangle = \sum_{p=0} r_p \langle m|\mathbf{r}^p|m\rangle = \sum_{p=0} r_p e^{-i2\pi\frac{m\cdot p}{N}} = \sum_{p=0} |r_p| e^{-i(2\pi\frac{m\cdot p}{N}-\phi_p)}$$

Complex Bloch matrix $\mathbf{H}^{GB(N)}$ eigenvalues are Fourier series with 6 (for N=6) Fourier parameters

$\{ r_0 = H, \quad r_1 = re^{i\phi_1}, \quad r_{-1} = re^{-i\phi_1}, \quad r_2 = se^{i\phi_2}, \quad r_{-2} = se^{-i\phi_2}, \quad r_3 = t = r_{-3} \}$

Nonzero gauge ϕ_p,

$$\omega_m(\mathbf{H}^{GZB(6)}_{complex}) = r_0 + r_1 e^{i\pi\frac{m\cdot 1}{3}} + r_{-1} e^{-i\pi\frac{m\cdot 1}{3}} + r_2 e^{i\pi\frac{m\cdot 2}{3}} + r_{-2} e^{-i\pi\frac{m\cdot 2}{3}} + r_3 e^{i\pi\frac{m\cdot 3}{3}}$$

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$$\omega_m(\mathbf{H}^{GZB(6)}_{complex}) = H + 2r\cos\left(\pi\frac{m\cdot 1}{3} - \phi_1\right) + 2s\cos\left(\pi\frac{m\cdot 2}{3} - \phi_2\right) + t(-1)^m$$

or complex: $r_{-p} = r_p^*$

giving 6 ω_m-levels:

$$\omega_m = \begin{cases} \omega_0 = r_0 + r_1 + r_{-1} + r_2 + r_{-2} + r_3 \\ \omega_{+1} = r_0 + r_1 e^{\frac{i\pi}{3}} + r_{-1} e^{\frac{-i\pi}{3}} + r_2 e^{\frac{i2\pi}{3}} + r_{-2} e^{\frac{-i2\pi}{3}} - r_3 \\ \omega_{-1} = r_0 + r_1 e^{\frac{-i\pi}{3}} + r_{-1} e^{\frac{i\pi}{3}} + r_2 e^{\frac{-i2\pi}{3}} + r_{-2} e^{\frac{i2\pi}{3}} - r_3 \\ \omega_{+2} = r_0 + r_1 e^{\frac{i2\pi}{3}} + r_{-1} e^{\frac{-i2\pi}{3}} - r_2 e^{\frac{i\pi}{3}} - r_{-2} e^{\frac{-i\pi}{3}} + r_3 \\ \omega_{-2} = r_0 + r_1 e^{\frac{-i2\pi}{3}} + r_{-1} e^{\frac{i2\pi}{3}} - r_2 e^{\frac{-i\pi}{3}} - r_{-2} e^{\frac{i\pi}{3}} + r_3 \\ \omega_3 = r_0 - r_1 - r_{-1} + r_2 + r_{-2} - r_3 \end{cases}$$

Complex sets of C_6 coupling parameters and gauge shifts

$$\omega_m(\mathbf{H}^{GB(N)}) = \langle m| \sum_{p=0} r_p \mathbf{r}^p |m\rangle = \sum_{p=0} r_p \langle m|\mathbf{r}^p|m\rangle = \sum_{p=0} r_p e^{-i2\pi\frac{m\cdot p}{N}} = \sum_{p=0} |r_p| e^{-i(2\pi\frac{m\cdot p}{N}-\phi_p)}$$

Complex Bloch matrix $\mathbf{H}^{\mathrm{GB}(N)}$ eigenvalues are Fourier series with 6 (for N=6) Fourier parameters

$\{ r_0 = H,\quad r_1 = re^{i\phi_1},\quad r_{-1} = re^{-i\phi_1},\quad r_2 = se^{i\phi_2},\quad r_{-2} = se^{-i\phi_2},\quad r_3 = t = r_{-3} \}$ Nonzero gauge ϕ_p ,

$$\omega_m(\mathbf{H}^{GZB(6)}_{complex}) = r_0 + r_1 e^{i\pi\frac{m\cdot 1}{3}} + r_{-1} e^{-i\pi\frac{m\cdot 1}{3}} + r_2 e^{i\pi\frac{m\cdot 2}{3}} + r_{-2} e^{-i\pi\frac{m\cdot 2}{3}} + r_3 e^{i\pi\frac{m\cdot 3}{3}}$$

or complex: $r_{-p} = r_p^*$.

$$\omega_m(\mathbf{H}^{GZB(6)}_{complex}) = H + 2r\cos\left(\pi\frac{m\cdot 1}{3} - \phi_1\right) + 2s\cos\left(\pi\frac{m\cdot 2}{3} - \phi_2\right) + t(-1)^m$$

or complex: $r_{-p} = r_p^*$

giving 6 ω_m-levels: ...in terms of 6 solvable r_p-parameters:

$$\omega_m = \begin{cases} \omega_0 = r_0 + r_1 + r_{-1} + r_2 + r_{-2} + r_3 \\ \omega_{+1} = r_0 + r_1 e^{\frac{i\pi}{3}} + r_{-1} e^{\frac{-i\pi}{3}} + r_2 e^{\frac{i2\pi}{3}} + r_{-2} e^{\frac{-i2\pi}{3}} - r_3 \\ \omega_{-1} = r_0 + r_1 e^{\frac{-i\pi}{3}} + r_{-1} e^{\frac{i\pi}{3}} + r_2 e^{\frac{-i2\pi}{3}} + r_{-2} e^{\frac{i2\pi}{3}} - r_3 \\ \omega_{+2} = r_0 + r_1 e^{\frac{i2\pi}{3}} + r_{-1} e^{\frac{-i2\pi}{3}} - r_2 e^{\frac{i\pi}{3}} - r_{-2} e^{\frac{-i\pi}{3}} + r_3 \\ \omega_{-2} = r_0 + r_1 e^{\frac{-i2\pi}{3}} + r_{-1} e^{\frac{i2\pi}{3}} - r_2 e^{\frac{-i\pi}{3}} - r_{-2} e^{\frac{i\pi}{3}} + r_3 \\ \omega_3 = r_0 - r_1 - r_{-1} + r_2 + r_{-2} - r_3 \end{cases}$$

$$r_p = \begin{cases} r_0 = ? \\ r_1 = ? \\ r_{-1} = ? \\ r_2 = ? \\ r_{-2} = ? \\ r_3 = ? \end{cases}$$

Left as an exercise...

Geometric solution shown next...

3rd Step (contd.)

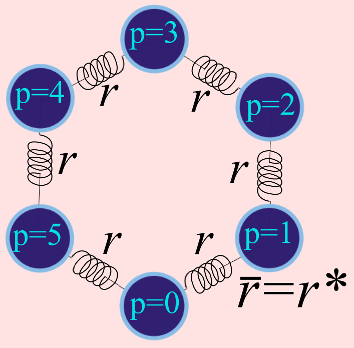
...eigensolutions for all possible C_6 symmetric complex H

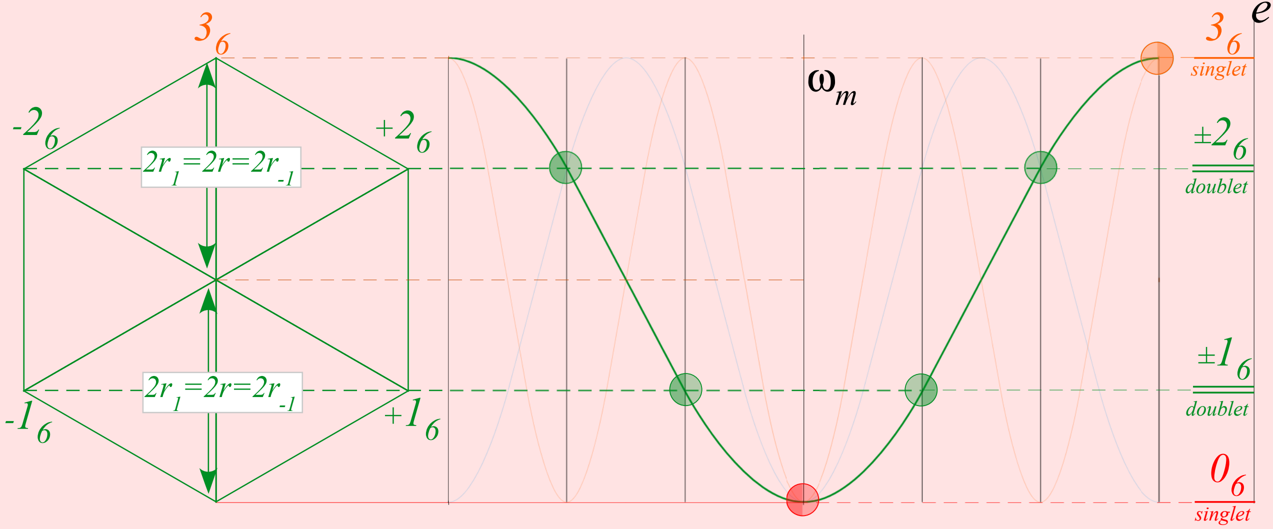
$$\mathbf{H} = \sum_{p=0}^{n-1} r_p \mathbf{r}^p = \sum_{p=0}^{n-1} r_p \sum_{m=0}^{n-1} \chi_p^m \mathbf{P}^{(m)} = \sum_{m=0}^{n-1} \omega^{(m)} \mathbf{P}^{(m)} \quad where: \ \omega^{(m)} = \sum_{p=0}^{n-1} r_p \chi_p^m = \omega(k_m)$$

(Dispersion function)

Elementary Bloch Model

$\mathbf{H} = H_1\mathbf{1} - r\mathbf{r} - r\mathbf{r}^{-1}$





eigenvalues of $\mathbf{H}^{B1(6)}$

$$\begin{array}{cccccc|c} p=0 & 1 & 2 & 3 & 4 & 5 & \\ H_1 & r_{-1} & \cdot & \cdot & \cdot & r_1 & 0 \\ r_1 & H_1 & r_{-1} & \cdot & \cdot & \cdot & 1 \\ \cdot & r_1 & H_1 & r_{-1} & \cdot & \cdot & 2 \\ \cdot & \cdot & r_1 & H_1 & r_{-1} & \cdot & 3 \\ \cdot & \cdot & \cdot & r_1 & H_1 & r_{-1} & 4 \\ r_{-1} & \cdot & \cdot & \cdot & r_1 & H_1 & 5 \end{array}$$

$$\omega^{B1(n)}(k_m) = r_0\chi_0^m + r_1\chi_1^m + r_{-1}\chi_{-1}^m = H_1 - 2r\cos(2\pi m/6)$$

Nearest neighbor coupling

$$\begin{pmatrix} r_0 & r_{-1} & & & & r_1 \\ r_1 & r_0 & r_{-1} & & & \\ & r_1 & r_0 & r_{-1} & & \\ & & r_1 & r_0 & r_{-1} & \\ & & & r_1 & r_0 & r_{-1} \\ r_{-1} & & & & r_1 & r_0 \end{pmatrix}$$

3rd Step (contd.)

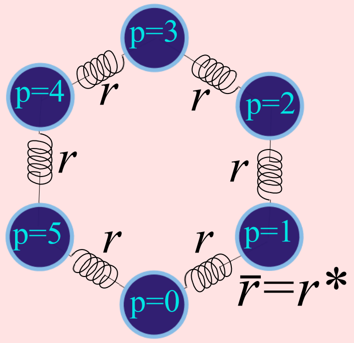
...eigensolutions for all possible C_6 symmetric complex H

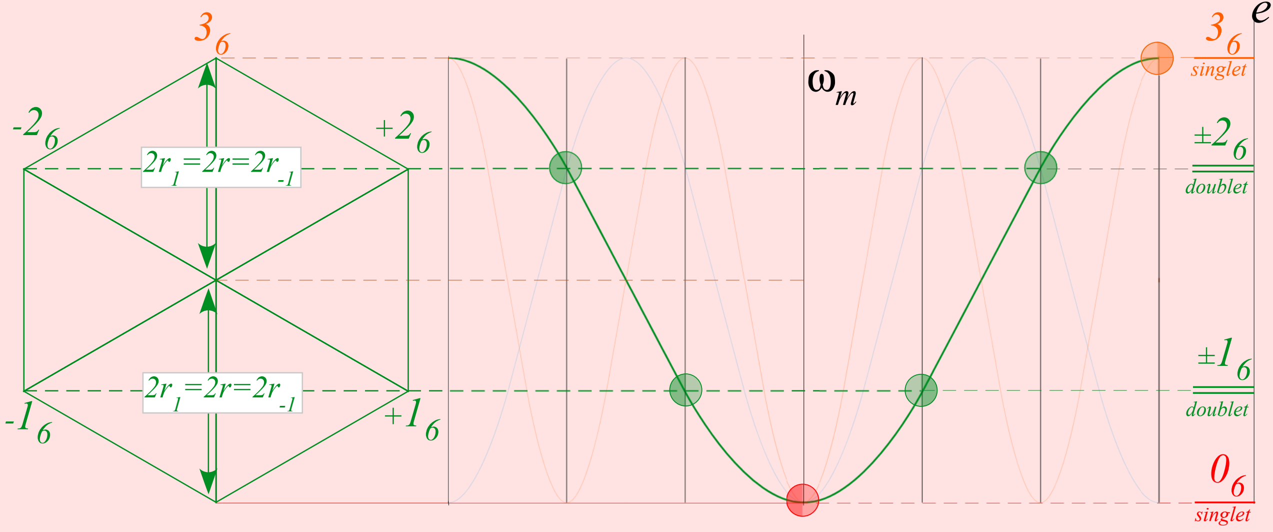
$$\mathbf{H} = \sum_{p=0}^{n-1} r_p \mathbf{r}^p = \sum_{p=0}^{n-1} r_p \sum_{m=0}^{n-1} \chi_p^m \mathbf{P}^{(m)} = \sum_{m=0}^{n-1} \omega^{(m)} \mathbf{P}^{(m)} \quad where: \ \omega^{(m)} = \sum_{p=0}^{n-1} r_p \chi_p^m = \omega(k_m)$$

(Dispersion function)

Elementary Bloch Model

$\mathbf{H}= H_1\mathbf{1} - r\mathbf{r} - r\mathbf{r}^{-1}$





eigenvalues of $\mathbf{H}^{B1(6)}$

$$\begin{array}{c|cccccc|c} & p=0 & 1 & 2 & 3 & 4 & 5 & \\ \hline & H_1 & r_{-1} & \cdot & \cdot & \cdot & r_1 & 0 \\ & r_1 & H_1 & r_{-1} & \cdot & \cdot & \cdot & 1 \\ & \cdot & r_1 & H_1 & r_{-1} & \cdot & \cdot & 2 \\ & \cdot & \cdot & r_1 & H_1 & r_{-1} & \cdot & 3 \\ & \cdot & \cdot & \cdot & r_1 & H_1 & r_{-1} & 4 \\ & r_{-1} & \cdot & \cdot & \cdot & r_1 & H_1 & 5 \end{array}$$

$$\omega^{B1(n)}(k_m) = r_0\chi_0^m + r_1\chi_1^m + r_{-1}\chi_{-1}^m = H_1 - 2r\,cos(2\pi m/6)$$

Nearest neighbor coupling

$$\mathbf{H}^{B1(6)}=\begin{pmatrix} r_0 & r_{-1} & & & & r_1 \\ r_1 & r_0 & r_{-1} & & & \\ & r_1 & r_0 & r_{-1} & & \\ & & r_1 & r_0 & r_{-1} & \\ & & & r_1 & r_0 & r_{-1} \\ r_{-1} & & & & r_1 & r_0 \end{pmatrix}$$

For Hermitian $\mathbf{H}^{B1(6)}=(\mathbf{H}^{B1(6)})^\dagger$

complex components

$r_1=-re^{i\phi}$ *imply*

conjugate components

$r^*_1=r_{-1}=-re^{-i\phi}$

3rd Step (contd.)

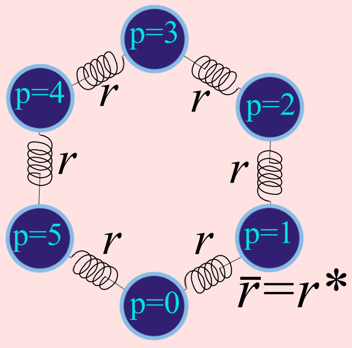
...eigensolutions for all possible C_6 symmetric complex H

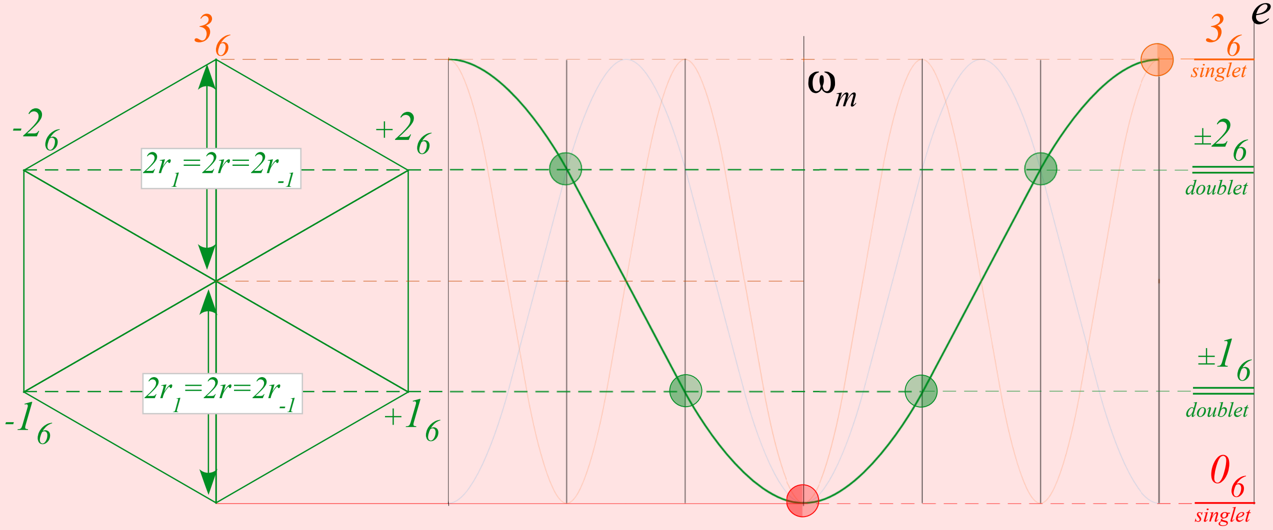
$$\mathbf{H}=\sum_{p=0}^{n-1} r_p\mathbf{r}^p=\sum_{p=0}^{n-1} r_p\sum_{m=0}^{n-1}\chi_p^m\mathbf{P}^{(m)}=\sum_{m=0}^{n-1}\omega^{(m)}\mathbf{P}^{(m)}\quad where:\ \omega^{(m)}=\sum_{p=0}^{n-1} r_p\chi_p^m=\omega(k_m)$$

(Dispersion function)

Elementary
Bloch Model

$\mathbf{H}= H_1\mathbf{1} - r\mathbf{r} - r\mathbf{r}^{-1}$





eigenvalues of $\mathbf{H}^{B1(6)}$

$$\begin{array}{c|cccccc|c}
 & p=0 & 1 & 2 & 3 & 4 & 5 & \\
\hline
 & H_1 & r_{-1} & \cdot & \cdot & \cdot & r_1 & 0\\
 & r_1 & H_1 & r_{-1} & \cdot & \cdot & \cdot & 1\\
 & \cdot & r_1 & H_1 & r_{-1} & \cdot & \cdot & 2\\
 & \cdot & \cdot & r_1 & H_1 & r_{-1} & \cdot & 3\\
 & \cdot & \cdot & \cdot & r_1 & H_1 & r_{-1} & 4\\
 & r_{-1} & \cdot & \cdot & \cdot & r_1 & H_1 & 5
\end{array}$$

$$\begin{aligned}\omega^{B1(n)}(k_m)\\ &= r_0\chi_0^m + r_1\chi_1^m + r_{-1}\chi_{-1}^m\\ &= H_1 - 2r\,cos(2\pi m/6)\end{aligned}$$

Nearest neighbor coupling

$$\mathbf{H}^{B1(6)}=\begin{pmatrix} r_0 & r_{-1} & & & & r_1\\ r_1 & r_0 & r_{-1} & & & \\ & r_1 & r_0 & r_{-1} & & \\ & & r_1 & r_0 & r_{-1} & \\ & & & r_1 & r_0 & r_{-1}\\ r_{-1} & & & & r_1 & r_0 \end{pmatrix}$$

For Hermitian $\mathbf{H}^{B1(6)}=(\mathbf{H}^{B1(6)})^\dagger$
complex components
$r_1=-re^{i\phi}$ *imply*
conjugate components
$r^*_1=r_{-1}=-re^{-i\phi}$

$$\begin{aligned}\omega^{B1(6)}(k_m) &= r_0\chi^m{}_0 + r_1\chi^m{}_1 + r_{-1}\chi^m{}_{-1}\\ &= r_0 - r\,e^{i\phi}e^{i2\pi m/6} - r\,e^{-i\phi}e^{-i2\pi m/6}\\ &= r_0 - 2r\,cos(2\pi m/6+\phi)\end{aligned}$$

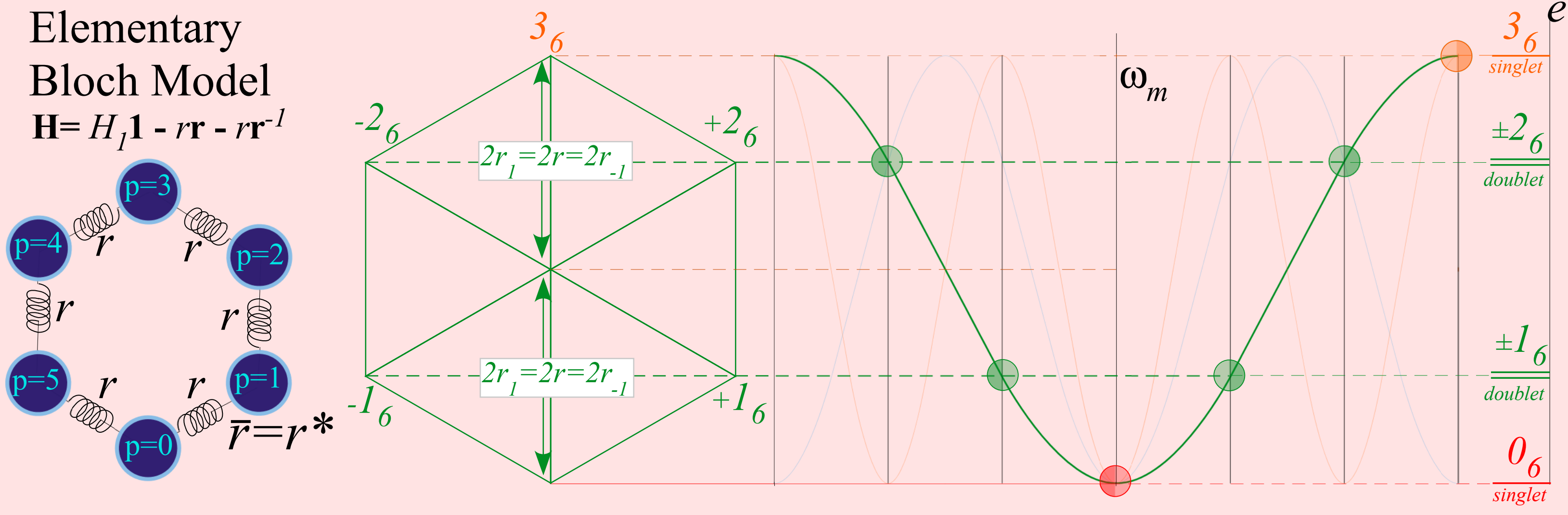
3^{rd} Step (contd.)

...eigensolutions for all possible C_6 symmetric complex H

$$\mathbf{H} = \sum_{p=0}^{n-1} r_p \mathbf{r}^p = \sum_{p=0}^{n-1} r_p \sum_{m=0}^{n-1} \chi_p^m \mathbf{P}^{(m)} = \sum_{m=0}^{n-1} \omega^{(m)} \mathbf{P}^{(m)} \quad where: \omega^{(m)} = \sum_{p=0}^{n-1} r_p \chi_p^m = \omega(k_m)$$

(Dispersion function)

Elementary
Bloch Model
$\mathbf{H} = H_1\mathbf{1} - r\mathbf{r} - r\mathbf{r}^{-1}$



eigenvalues of $\mathbf{H}^{B1(6)}$

$$\begin{array}{ccccccc} p=0 & 1 & 2 & 3 & 4 & 5 & \\ H_1 & r_{-1} & \cdot & \cdot & \cdot & r_1 & 0 \\ r_1 & H_1 & r_{-1} & \cdot & \cdot & \cdot & 1 \\ \cdot & r_1 & H_1 & r_{-1} & \cdot & \cdot & 2 \\ \cdot & \cdot & r_1 & H_1 & r_{-1} & \cdot & 3 \\ \cdot & \cdot & \cdot & r_1 & H_1 & r_{-1} & 4 \\ r_{-1} & \cdot & \cdot & \cdot & r_1 & H_1 & 5 \end{array}$$

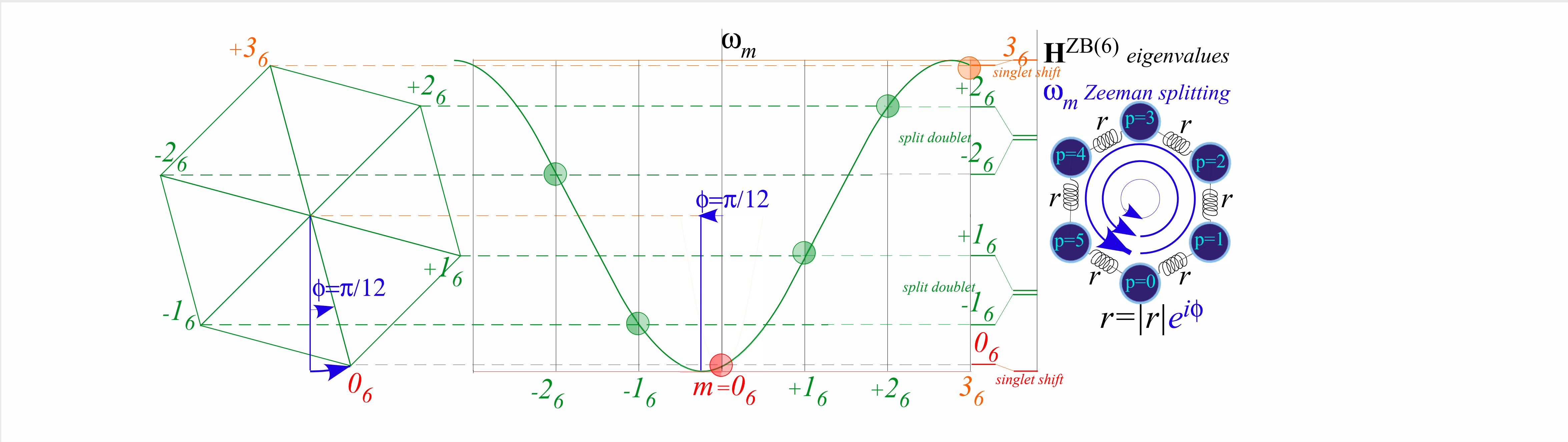
$$\omega^{B1(n)}(k_m) = r_0\chi_0^m + r_1\chi_1^m + r_{-1}\chi_{-1}^m = H_1 - 2r\cos(2\pi m/6)$$

Nearest neighbor coupling

$$\mathbf{H}^{B1(6)} = \begin{pmatrix} r_0 & r_{-1} & & & & r_1 \\ r_1 & r_0 & r_{-1} & & & \\ & r_1 & r_0 & r_{-1} & & \\ & & r_1 & r_0 & r_{-1} & \\ & & & r_1 & r_0 & r_{-1} \\ r_{-1} & & & & r_1 & r_0 \end{pmatrix}$$

For Hermitian $\mathbf{H}^{B1(6)} = (\mathbf{H}^{B1(6)})^\dagger$
complex components
$r_1 = -re^{i\phi}$ *imply*
conjugate components
$r^*_1 = r_{-1} = -re^{-i\phi}$

$$\omega^{B1(6)}(k_m) = r_0\chi^m{}_0 + r_1\chi^m{}_1 + r_{-1}\chi^m{}_{-1}$$
$$= r_0 - r\,e^{i\phi}e^{i2\pi m/6} - r\,e^{-i\phi}e^{-i2\pi m/6}$$
$$= r_0 - 2r\cos(2\pi m/6 + \phi)$$



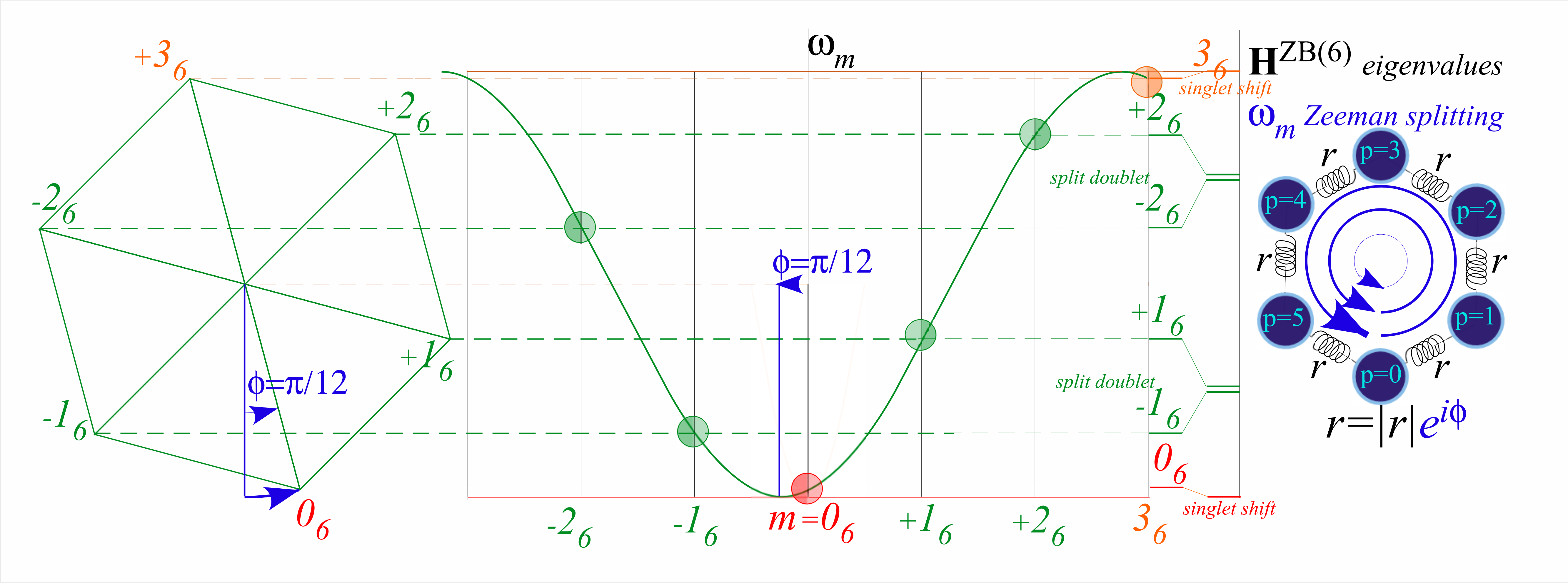
3^{rd} Step (contd.)

...eigensolutions for all possible C_6 symmetric complex H

$$\mathbf{H}=\sum_{p=0}^{n-1} r_p \mathbf{r}^p=\sum_{p=0}^{n-1} r_p \sum_{m=0}^{n-1} \chi_p^m \mathbf{P}^{(m)}=\sum_{m=0}^{n-1} \omega^{(m)} \mathbf{P}^{(m)} \quad where: \omega^{(m)}=\sum_{p=0}^{n-1} r_p \chi_p^m=\omega(k_m)$$

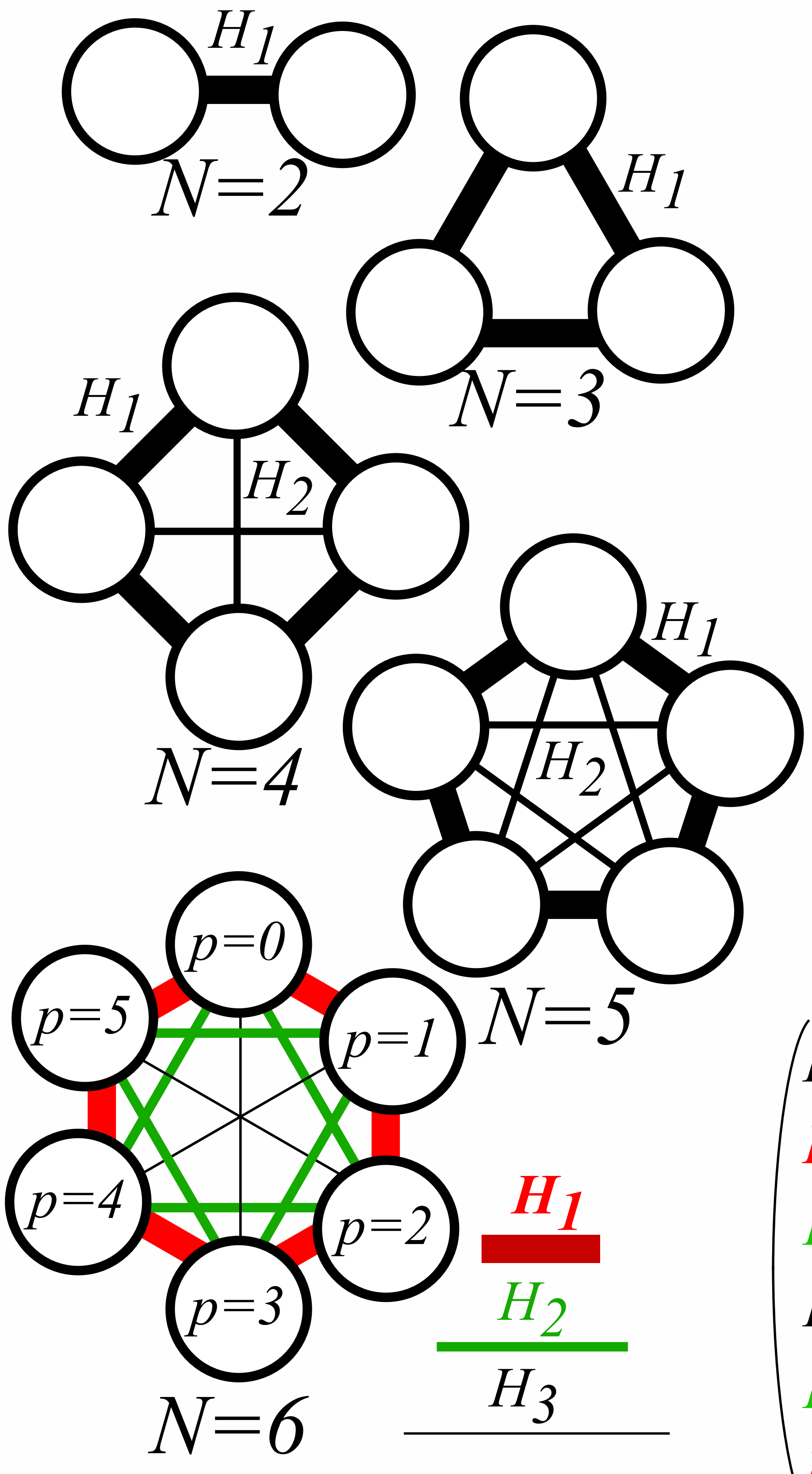
(Dispersion function)

In this C-Type situation m-eigenstates are required to be moving waves $e^{ik_m \cdot x_p}$



Simulating Complex Systems With Simpler Ones

Discrete Rotor Waves
Bohr-Rotors Made of Quantum Dots

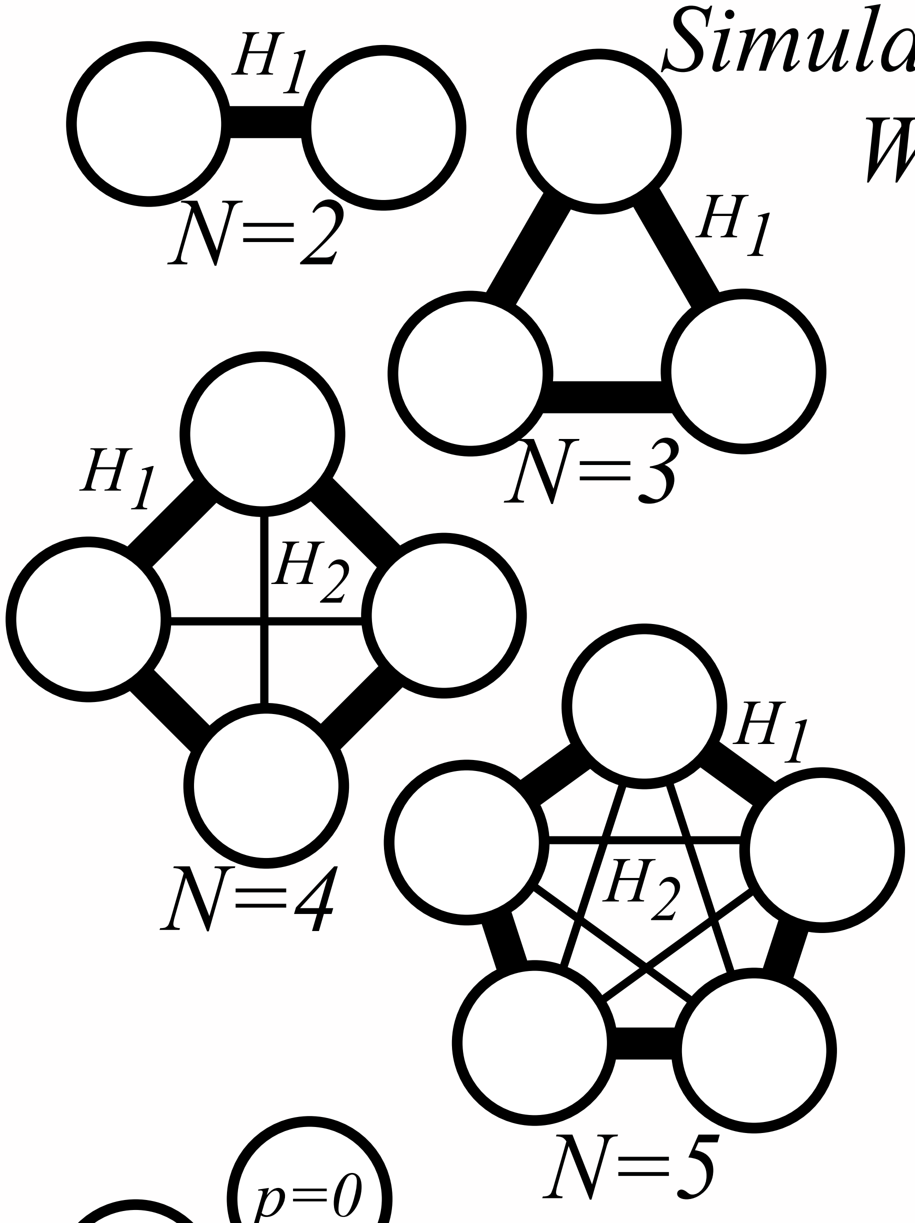


$$\begin{pmatrix} H_0 & \mathbf{H_1} & H_2 & H_3 & H_2 & \mathbf{H_1} \\ \mathbf{H1} & H_0 & \mathbf{H_1} & H_2 & H_3 & H_2 \\ H_2 & \mathbf{H_1} & H_0 & \mathbf{H_1} & H_2 & H_3 \\ H_3 & H_2 & \mathbf{H_1} & H_0 & \mathbf{H_1} & H_2 \\ H_2 & H_3 & H_2 & \mathbf{H_1} & H_0 & \mathbf{H_1} \\ \mathbf{H_1} & H_2 & H_3 & H_2 & \mathbf{H_1} & H_0 \end{pmatrix}$$

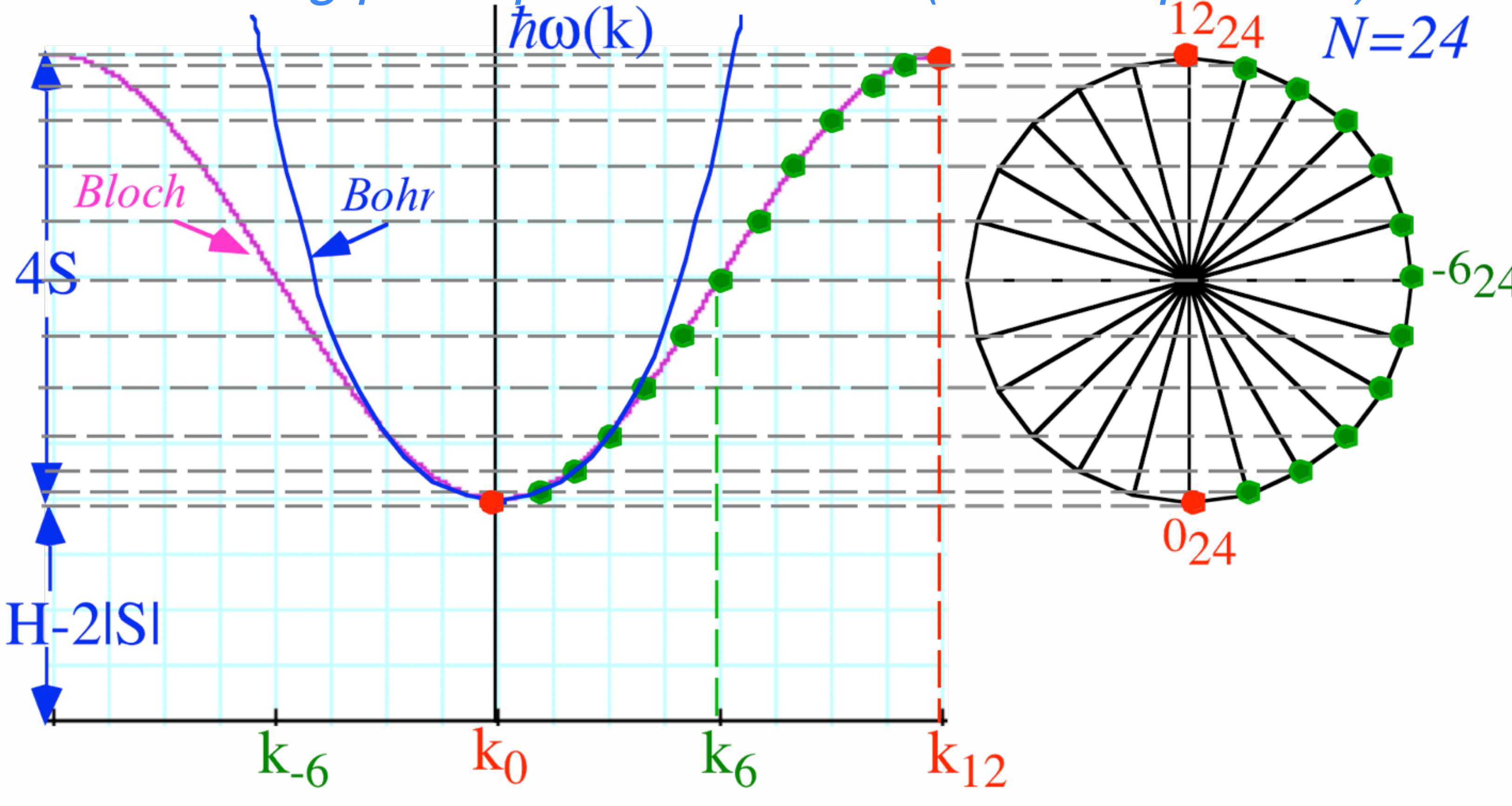
Simulating Complex Systems With Simpler Ones

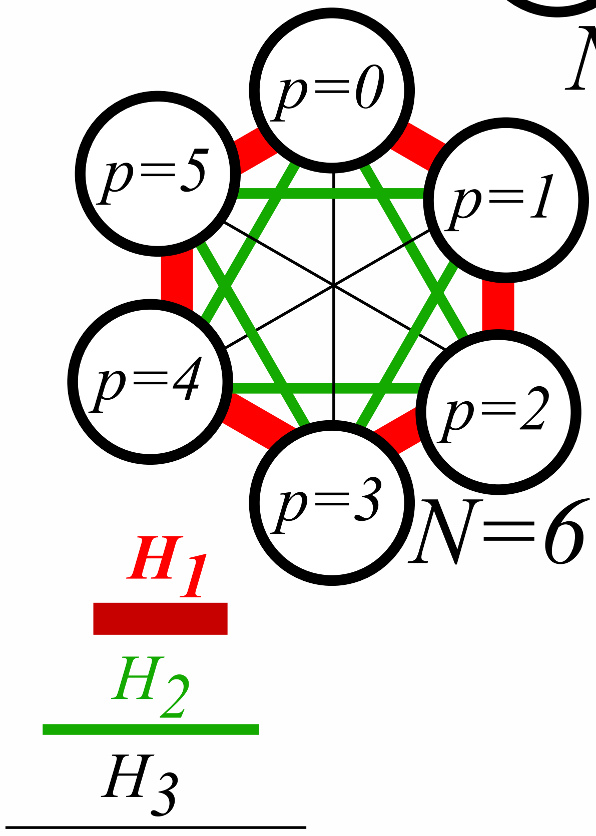
Made of Quantum Dots

[Harter, J. Mol. Spec. 210, 166-182 (2001)]



Making pure quadratic $\omega=ck^2$ (Bohr dispersion)





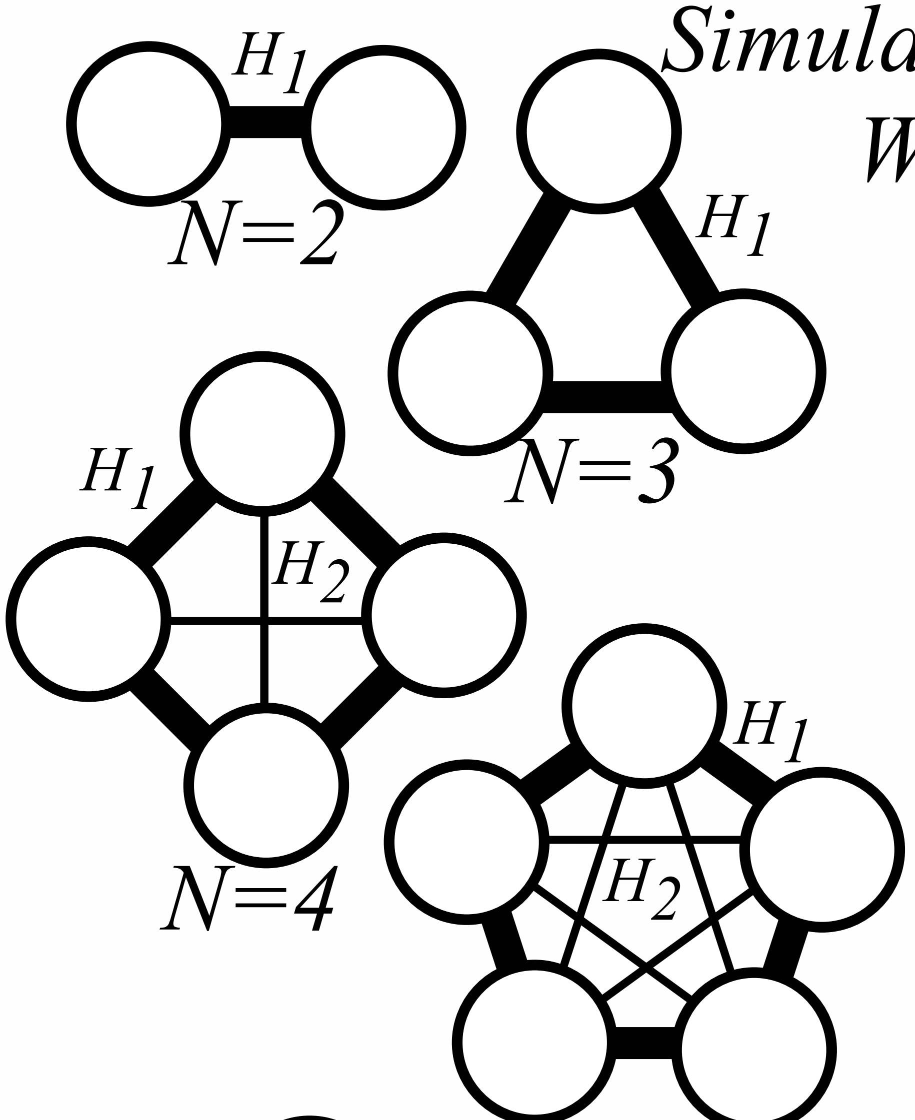
Hexagonal 2D Rotor

$$\begin{pmatrix} H_0 & \boldsymbol{H_1} & H_2 & H_3 & H_2 & \boldsymbol{H_1} \\ \boldsymbol{H1} & H_0 & \boldsymbol{H_1} & H_2 & H_3 & H_2 \\ H_2 & \boldsymbol{H_1} & H_0 & \boldsymbol{H_1} & H_2 & H_3 \\ H_3 & H_2 & \boldsymbol{H_1} & H_0 & \boldsymbol{H_1} & H_2 \\ H_2 & H_3 & H_2 & \boldsymbol{H_1} & H_0 & \boldsymbol{H_1} \\ \boldsymbol{H_1} & H_2 & H_3 & H_2 & \boldsymbol{H_1} & H_0 \end{pmatrix}$$

Simulating Complex Systems With Simpler Ones

Made of Quantum Dots

[Harter, J. Mol. Spec. 210, 166-182 (2001)]



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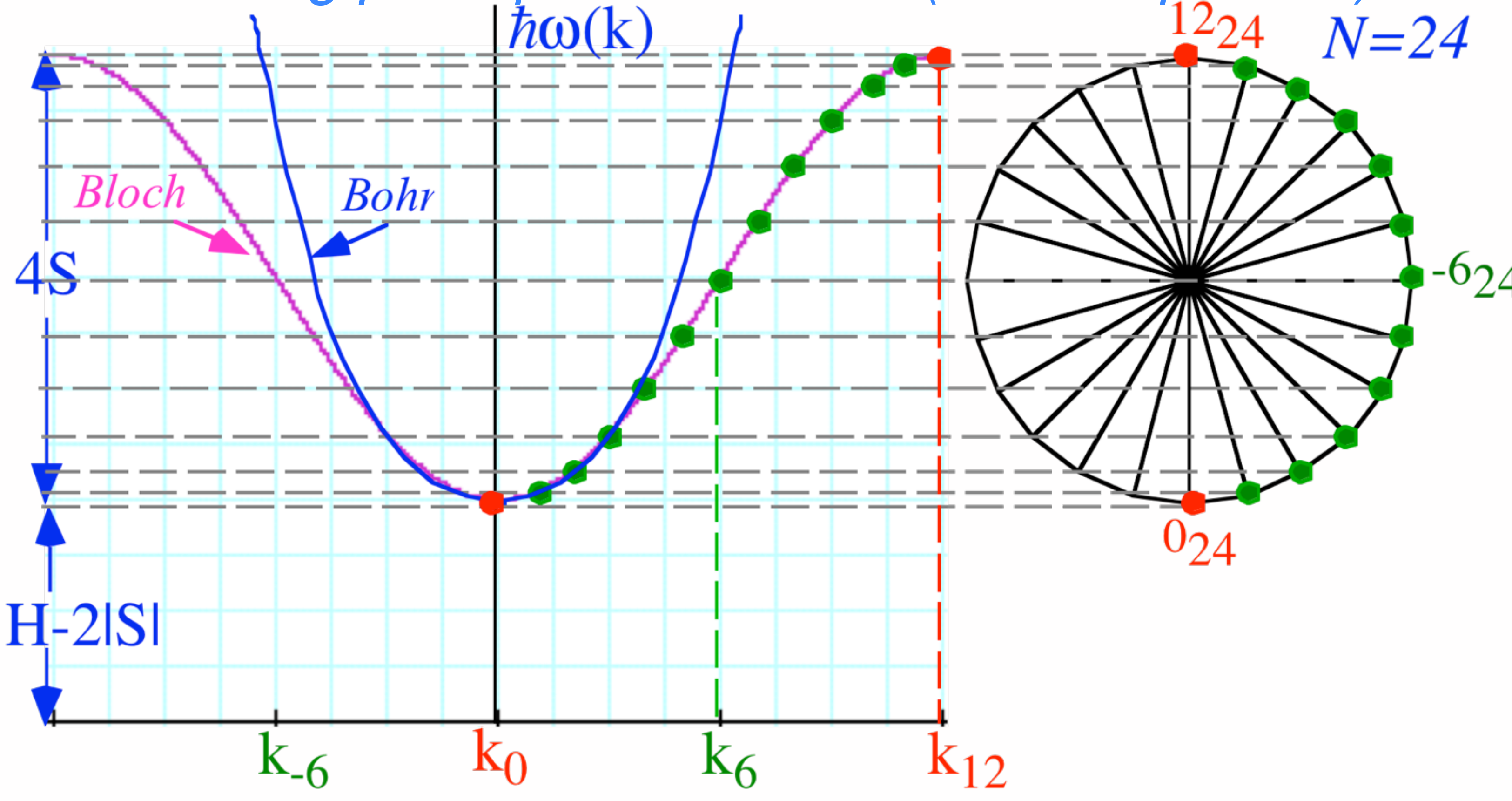
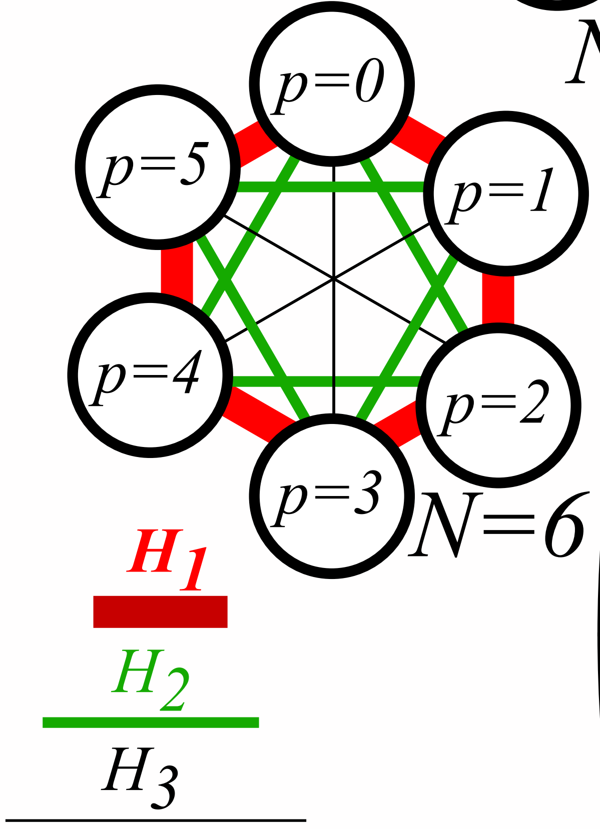


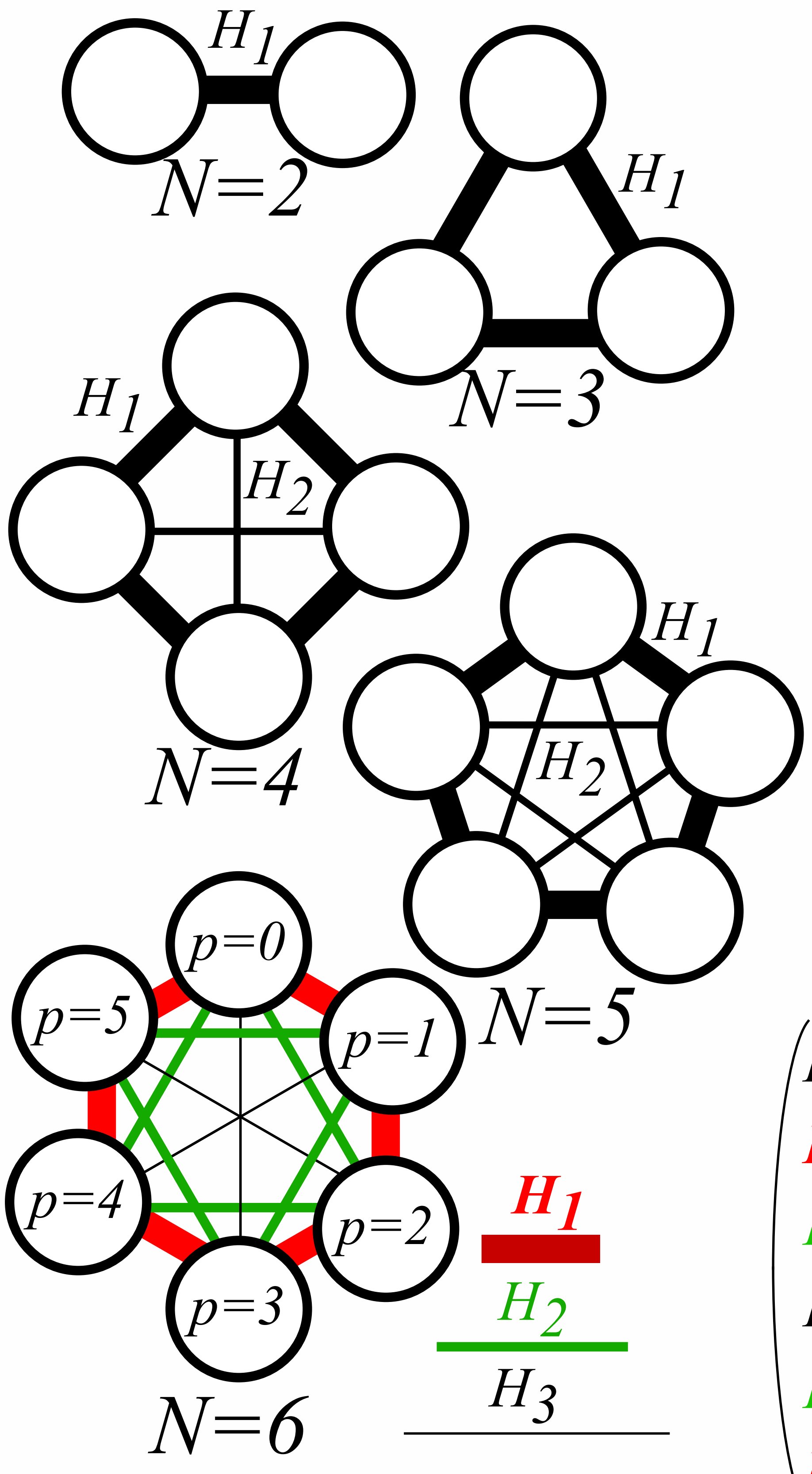


Hexagonal 2D Rotor

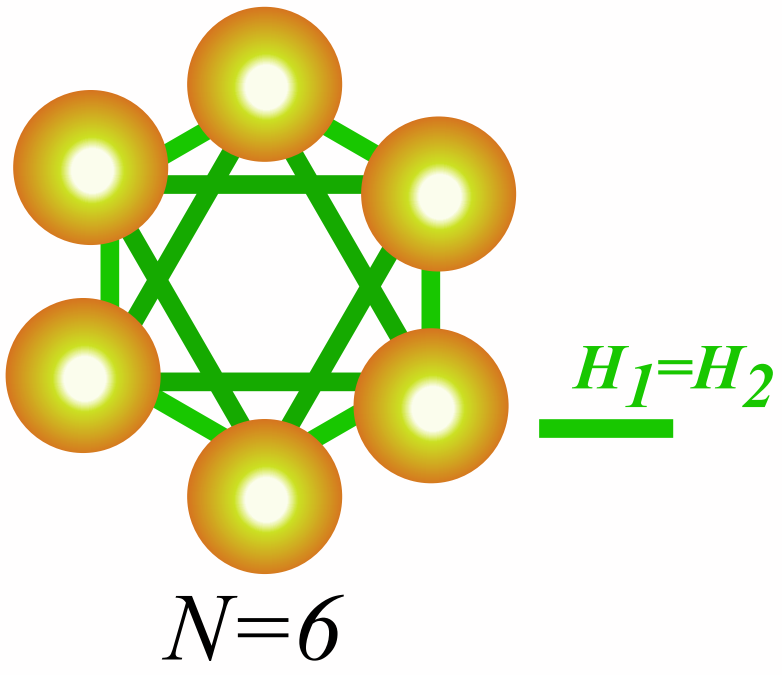
$$\begin{pmatrix} H_0 & H_1 & H_2 & H_3 & H_2 & H_1 \\ H1 & H_0 & H_1 & H_2 & H_3 & H_2 \\ H_2 & H_1 & H_0 & H_1 & H_2 & H_3 \\ H_3 & H_2 & H_1 & H_0 & H_1 & H_2 \\ H_2 & H_3 & H_2 & H_1 & H_0 & H_1 \\ H_1 & H_2 & H_3 & H_2 & H_1 & H_0 \end{pmatrix}$$

	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8
N=2	1/2	-1/2							
N=3	2/3	-1/3							
N=4	3/2	-1	1/2						
N=5	2	-1.1708	0.1708						
N=6	19/6	-2	2/3	-1/2					
N=7	4	-2.393	0.51	-0.1171					
N=8	11/2	-3.4142	1	-0.5858	1/2				
N=9	20/3	-4.0165	0.9270	-1/3	0.0895				
N=10	17/2	-5.2361	1.4472	-0.7639	0.5528	-1/2			
N=11	10	-6.0442	1.4391	-0.5733	0.2510	-0.0726			
N=12	73/6	-7.4641	2	-1	2/3	-0.5359	1/2		
N=13	14	-8.4766	2.0500	-0.8511	0.4194	-0.2028	0.06116		
N=14	33/2	-10.098	2.6560	-1.2862	0.8180	-0.6160	0.5260	-1/2	
N=15	57/3	-11.314	2.7611	-1.1708	0.6058	-1/3	0.1708	-0.0528	
N=16	43/2	-13.137	3.4142	-1.6199	1	-0.7232	0.5858	-0.5198	1/2
N=17	24	-14.557	3.5728	-1.5340	0.81413	-0.4732	0.2781	-0.1479	0.0465

Simulating Complex Systems With Simpler Ones



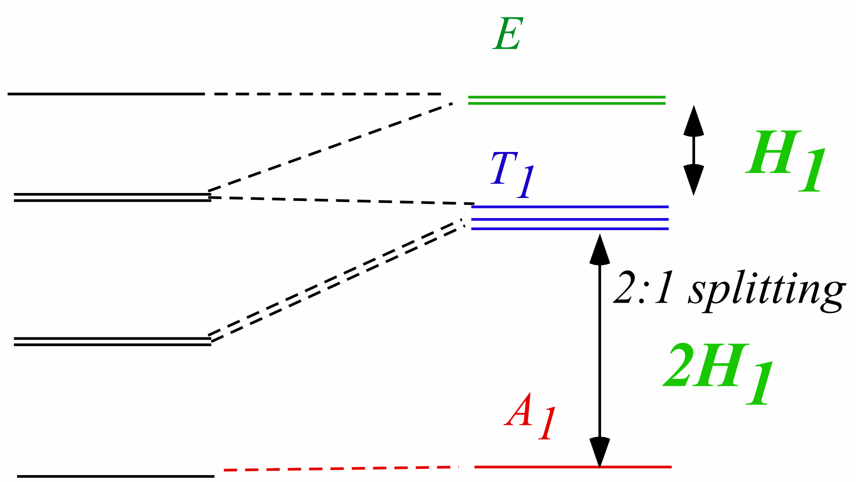
Discrete Rotor Waves
Bohr-Rotors Made of Quantum Dots



$$\begin{pmatrix} H_0 & H_1 & H_1 & 0 & H_1 & H_1 \\ H_1 & H_0 & H_1 & H_1 & 0 & H_1 \\ H_1 & H_1 & H_0 & H_1 & H_1 & 0 \\ 0 & H_1 & H_1 & H_0 & H_1 & H_1 \\ H_1 & 0 & H_1 & H_1 & H_0 & H_1 \\ H_1 & H_1 & 0 & H_1 & H_1 & H_0 \end{pmatrix}$$

Hexagonal becomes Octahedral

$$\begin{pmatrix} H_0 & H_1 & H_2 & H_3 & H_2 & H_1 \\ H1 & H_0 & H_1 & H_2 & H_3 & H_2 \\ H_2 & H_1 & H_0 & H_1 & H_2 & H_3 \\ H_3 & H_2 & H_1 & H_0 & H_1 & H_2 \\ H_2 & H_3 & H_2 & H_1 & H_0 & H_1 \\ H_1 & H_2 & H_3 & H_2 & H_1 & H_0 \end{pmatrix}$$



1^{st} Step: Abelian symmetry analysis Expand C_6 symmetric **H** *by C_6 group product table*

Regular representation and coupling parameters $\{r_0,r_1,r_2,r_3,r_4,r_5\}$ and Fourier dispersion

2^{nd} Step: Find **H** *eigenfunctions by spectral resolution of $C_6 =\{\mathbf{1}=\mathbf{r}^0,\mathbf{r}^1,\mathbf{r}^2,\mathbf{r}^3,\mathbf{r}^4,\mathbf{r}^5\}$*

Character tables of C_2, C_3, C_4, C_5,…. C_{144}

3^{rd} Step: Dispersion functions and eigenvalues for varyious coupling parameter systems

Ortho-complete eigenvalue/parameter relations

Gauge shifts due to complex coupling

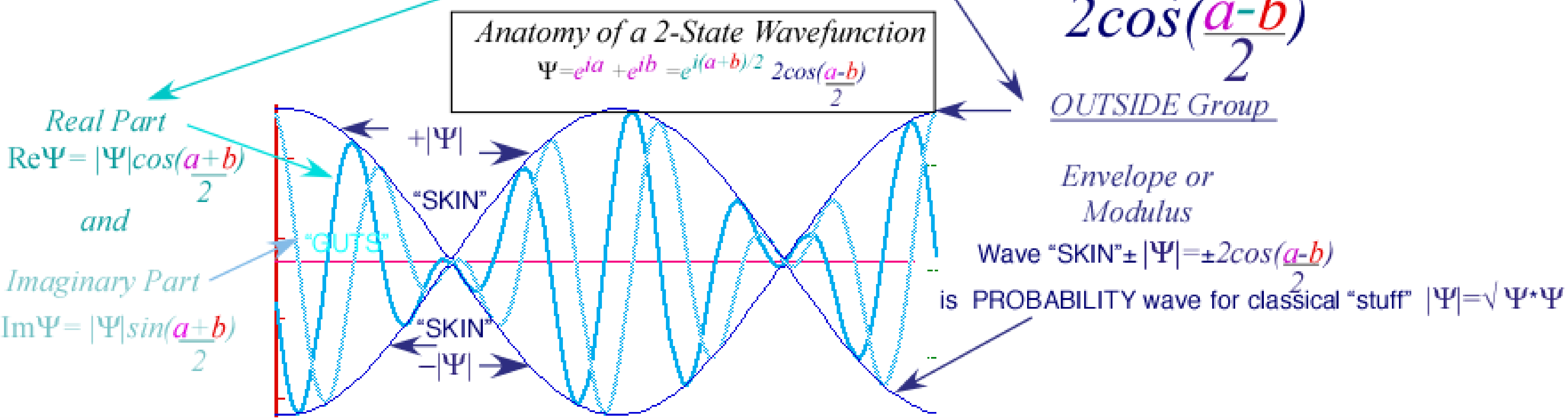
→ *Wave dynamics of phase, mean phase, and group velocity by Expo-Cosine identity*

Relating space-time (x,t) and per-space-time (k,ω)

- *Wave coordinates*
- *Pulse-waves (PW) vs Continuous-waves (CW)*
- *Wave coordinates for Linear Dispersion*
- *Wave coordinates for Bohr-Schrodinger Dispersion*
- *Einstein-Lorentz-Minkowski laser coordinates*

Interfering Plane Waves: The Expo-Cosine Identity

$$\Psi = e^{ia} + e^{ib} = \underbrace{e^{i(a+b)/2}}_{\text{INSIDE Phase}} \underbrace{\left(e^{i(a-b)/2} + e^{-i(a-b)/2}\right)}_{2\cos\left(\frac{a-b}{2}\right)}$$



OUTSIDE Group

Real Part

$\mathrm{Re}\Psi = |\Psi|\cos(\frac{a+b}{2})$

and

Imaginary Part

$\mathrm{Im}\Psi = |\Psi|\sin(\frac{a+b}{2})$

Envelope or
Modulus

Wave "SKIN" $\pm|\Psi| = \pm 2\cos(\frac{a-b}{2})$

is PROBABILITY wave for classical "stuff" $|\Psi| = \sqrt{\Psi^*\Psi}$

Fundamental wave dynamics based on Euler Expo-cosine Identity

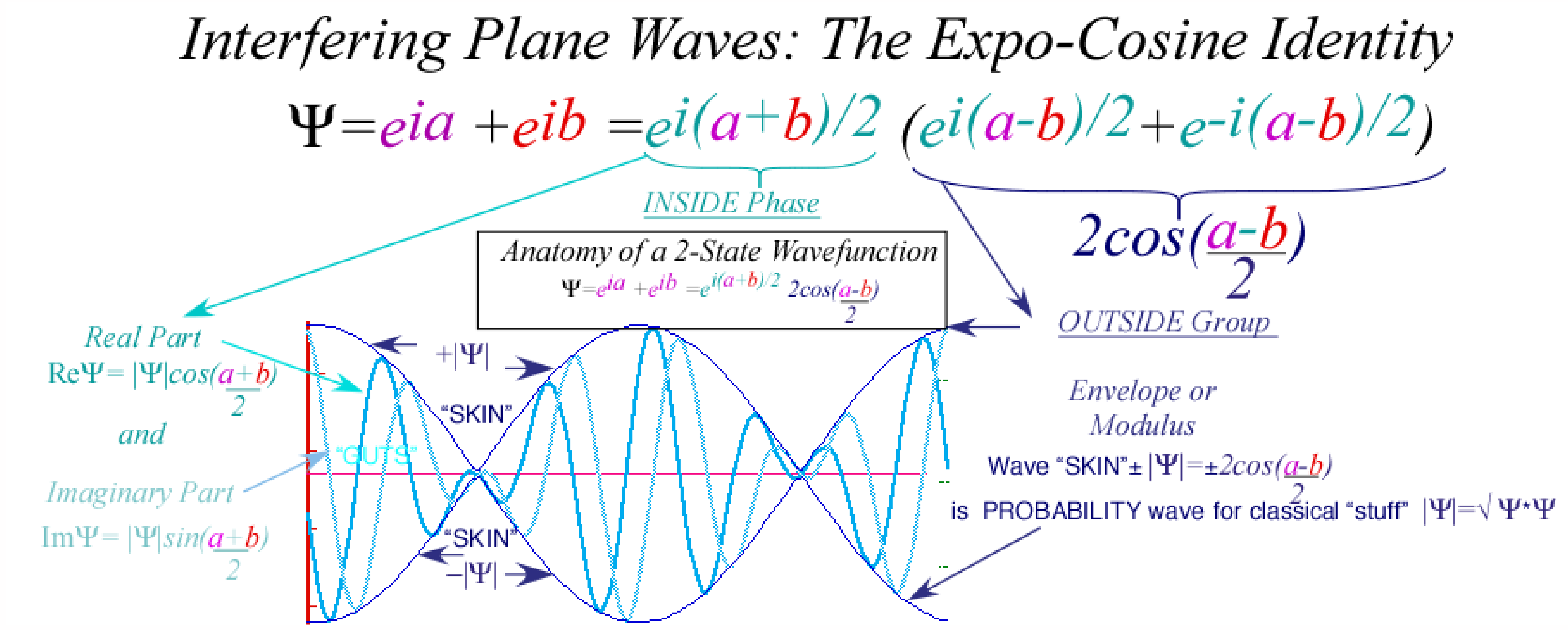
$$(e^{ia}+e^{ib})/2 = e^{i(a+b)/2}(e^{i(a-b)/2}+e^{-i(a-b)/2})/2 = e^{i(a+b)/2} \bullet \cos(a-b)/2$$

$a=k_1x-\omega_1t \qquad b=k_2x-\omega_2t$

Balanced (50-50) plane wave combination:

$$\Psi_{50_1-50_2}(x,t) = (1/2)\psi_{k_1}(x,t) + (1/2)\psi_{k_2}(x,t)$$

$$(1/2)e^{i(k_1x-\omega_1t)} + (1/2)e^{i(k_2x-\omega_2t)} = e^{i(k_px-\omega_pt)} \bullet \cos(k_gx-\omega_gt)$$



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$\omega_p=(\omega_1+\omega_2)/2$

$k_p=(k_1+k_2)/2$

Overall or Mean phase

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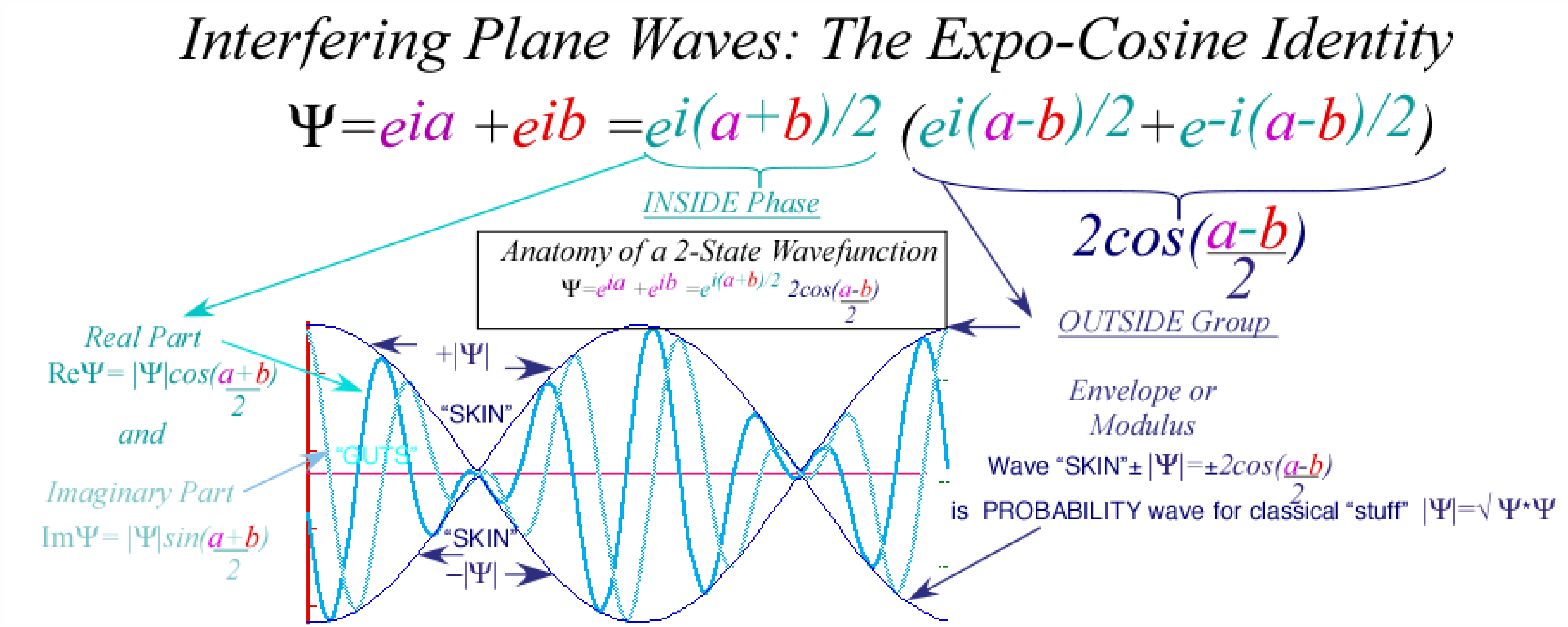

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Anatomy of a 2-State Wavefunction

$\Psi=e^{ia}+e^{ib}=e^{i(a+b)/2}\,2\cos(\frac{a-b}{2})$

Wave "SKIN"$\pm|\Psi|=\pm 2\cos(\frac{a-b}{2})$

is PROBABILITY wave for classical "stuff" $|\Psi|=\sqrt{\Psi^*\Psi}$

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1^{st} plane phase velocity	*2^{nd} plane phase velocity*	*Phase or Carrier velocity*	*Group or Envelope velocity*
$V_1=\frac{\omega_1}{k_1}$	$V_2=\frac{\omega_2}{k_2}$	$V_p=\frac{\omega_p}{k_p}=\frac{\omega_1+\omega_2}{k_1+k_2}$	$V_g=\frac{\omega_g}{k_g}=\frac{\omega_1-\omega_2}{k_1-k_2}$

Define **K***-vectors in per-spacetime*

$\mathbf{K}_1=(\omega_1, k_1)$	$\mathbf{K}_2=(\omega_2, k_2)$	$\mathbf{K}_{phase}=(\omega_p, k_p)$ $=(\mathbf{K}_1+\mathbf{K}_2)/2$	$\mathbf{K}_{group}=(\omega_g, k_g)$ $=(\mathbf{K}_1-\mathbf{K}_2)/2$

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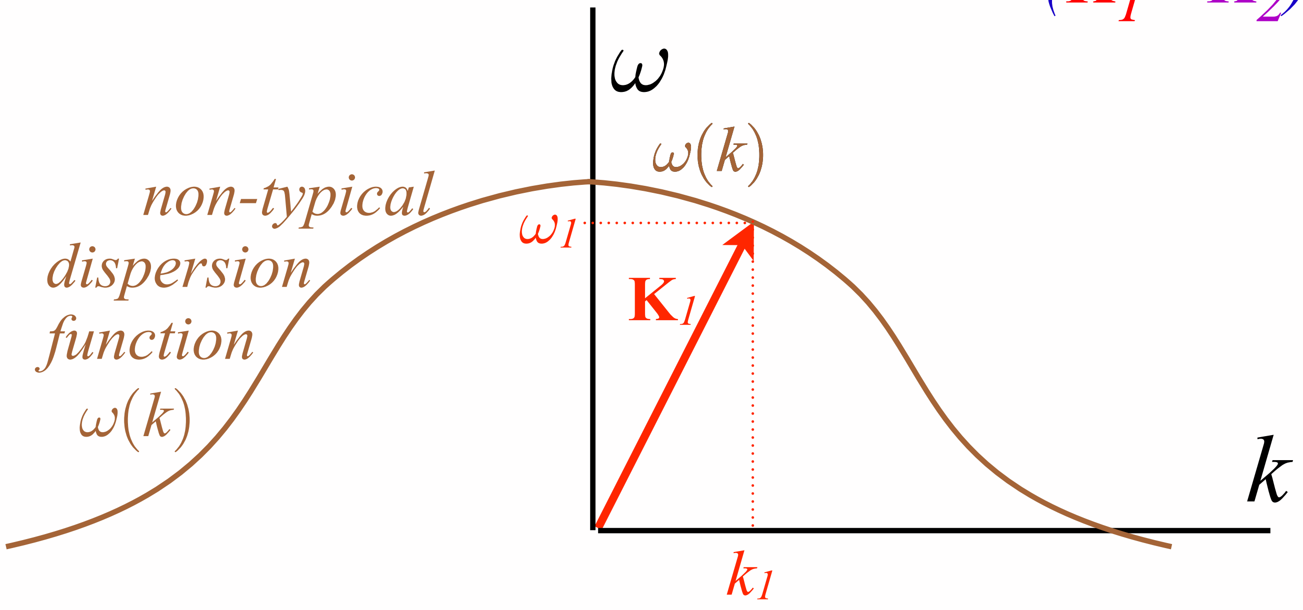

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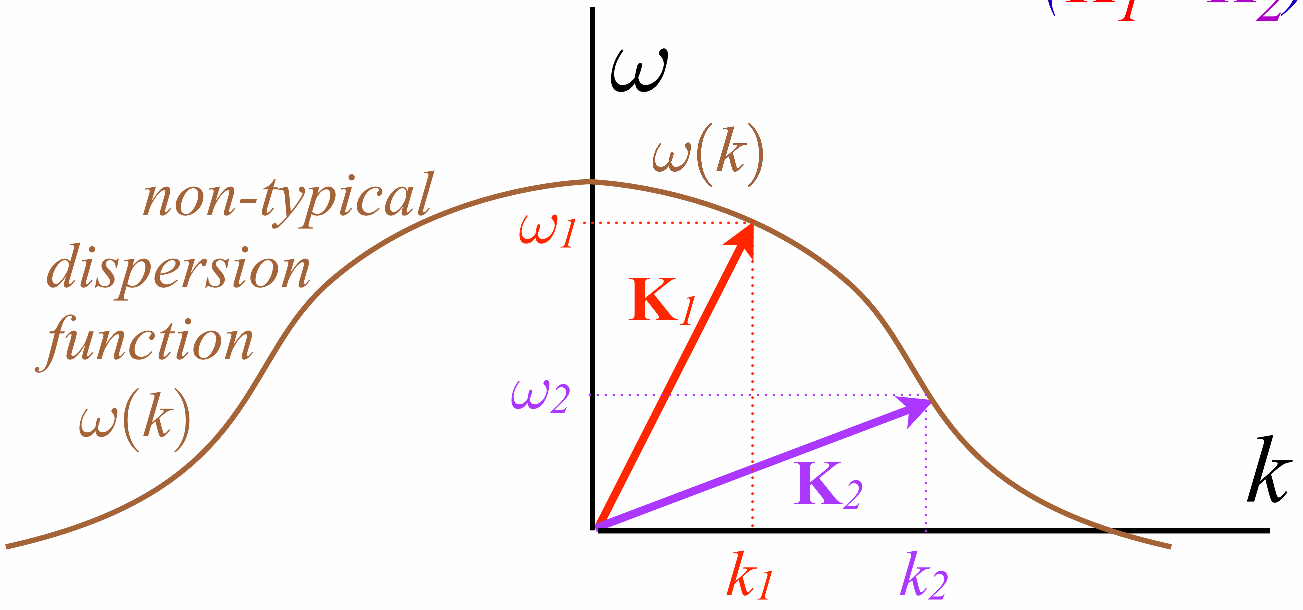

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1st plane phase velocity $\quad$ *2nd plane phase velocity* $\quad$ *Phase or Carrier velocity* $\quad$ *Group or Envelope velocity*

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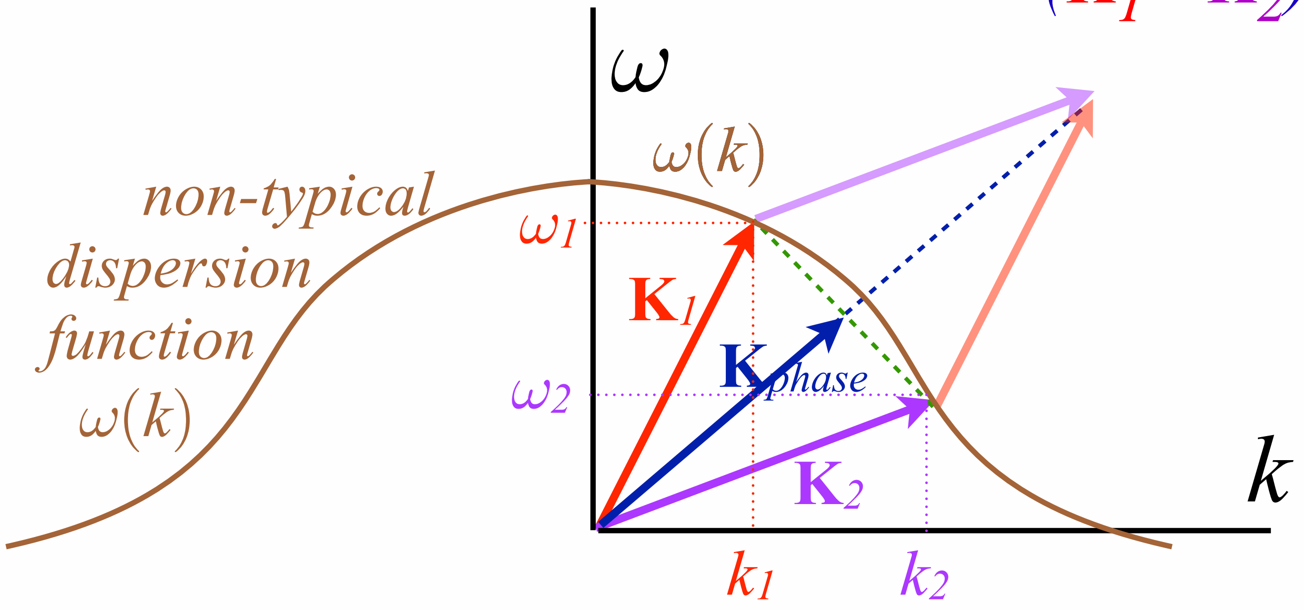

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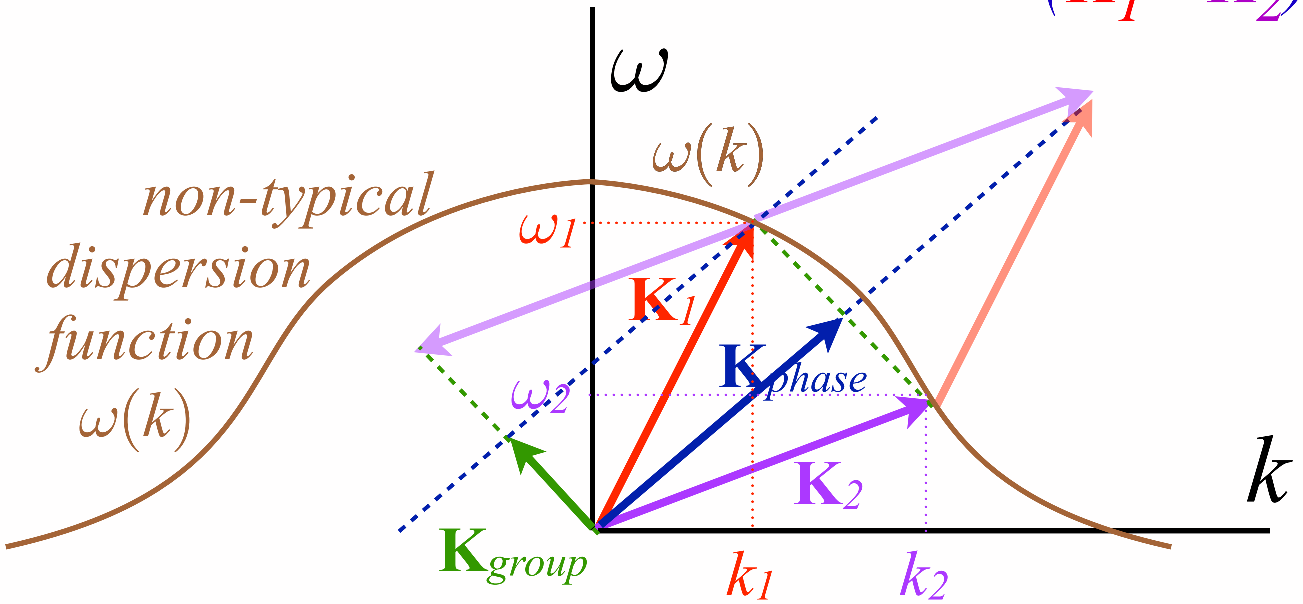

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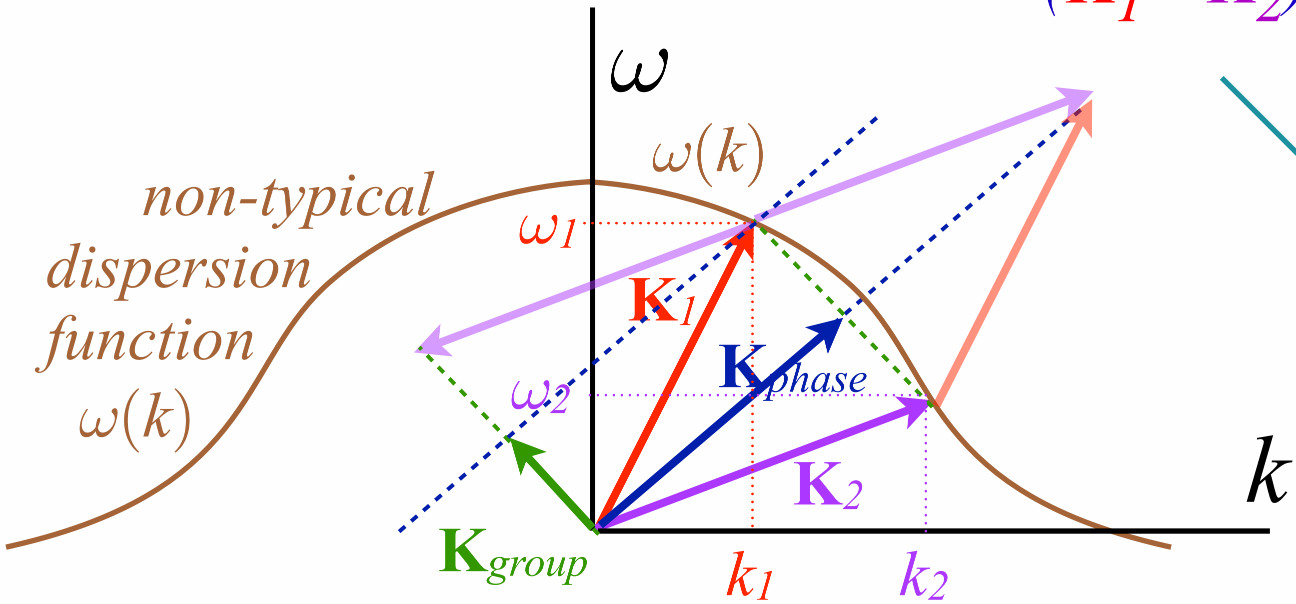
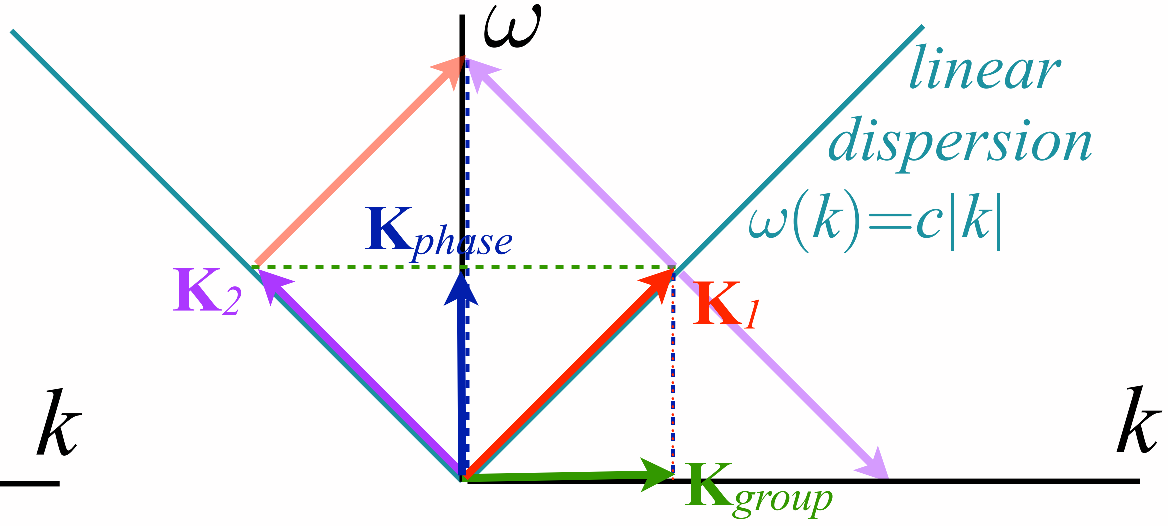
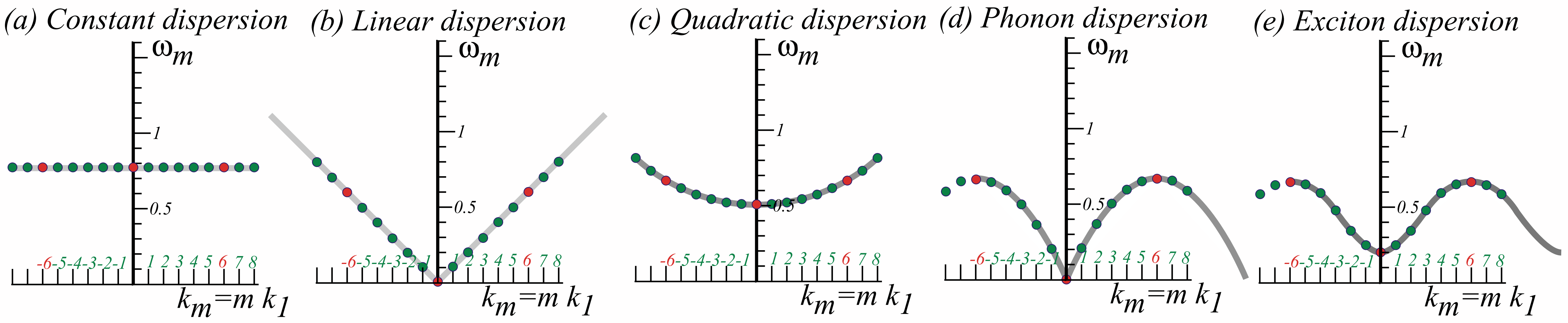


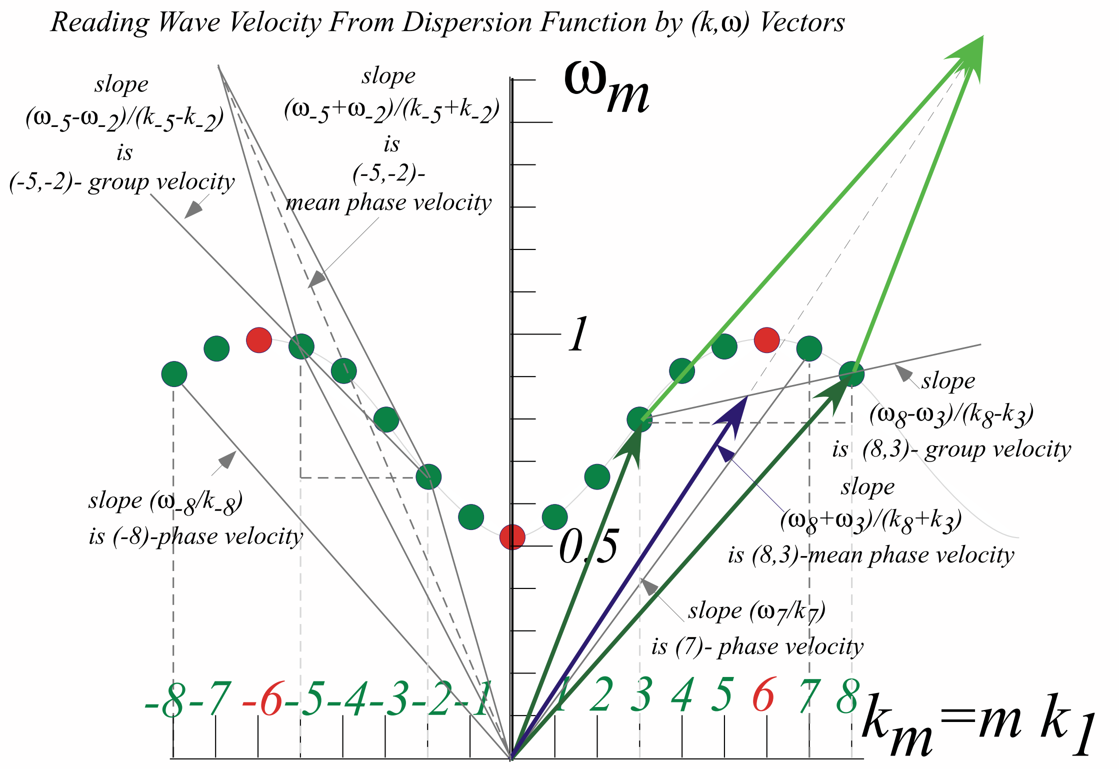


Archetypical Examples of Dispersion Functions


(a) Constant dispersion *(b) Linear dispersion* *(c) Quadratic dispersion* *(d) Phonon dispersion* *(e) Exciton dispersion*

Applications:

Uncoupled pendulums	*Weakly coupled pendulums (No gravity)*	*Weakly coupled pendulums (With gravity)*	*Strongly coupled pendulums (No gravity)*	*Strongly coupled pendulums (With gravity)*
Movie marquis *Xmas lights*	*Light in vacuum (Exactly)* *Sound (Approximately)*	*Light in fiber (Approx)* *Non-relativistic* *Schrodinger matter wave*	*Acoustic mode in solids*	*Optical mode in solids* *Relativistic matter* *(If exact hyperbola)*


Reading Wave Velocity From Dispersion Function by (k,ω) Vectors

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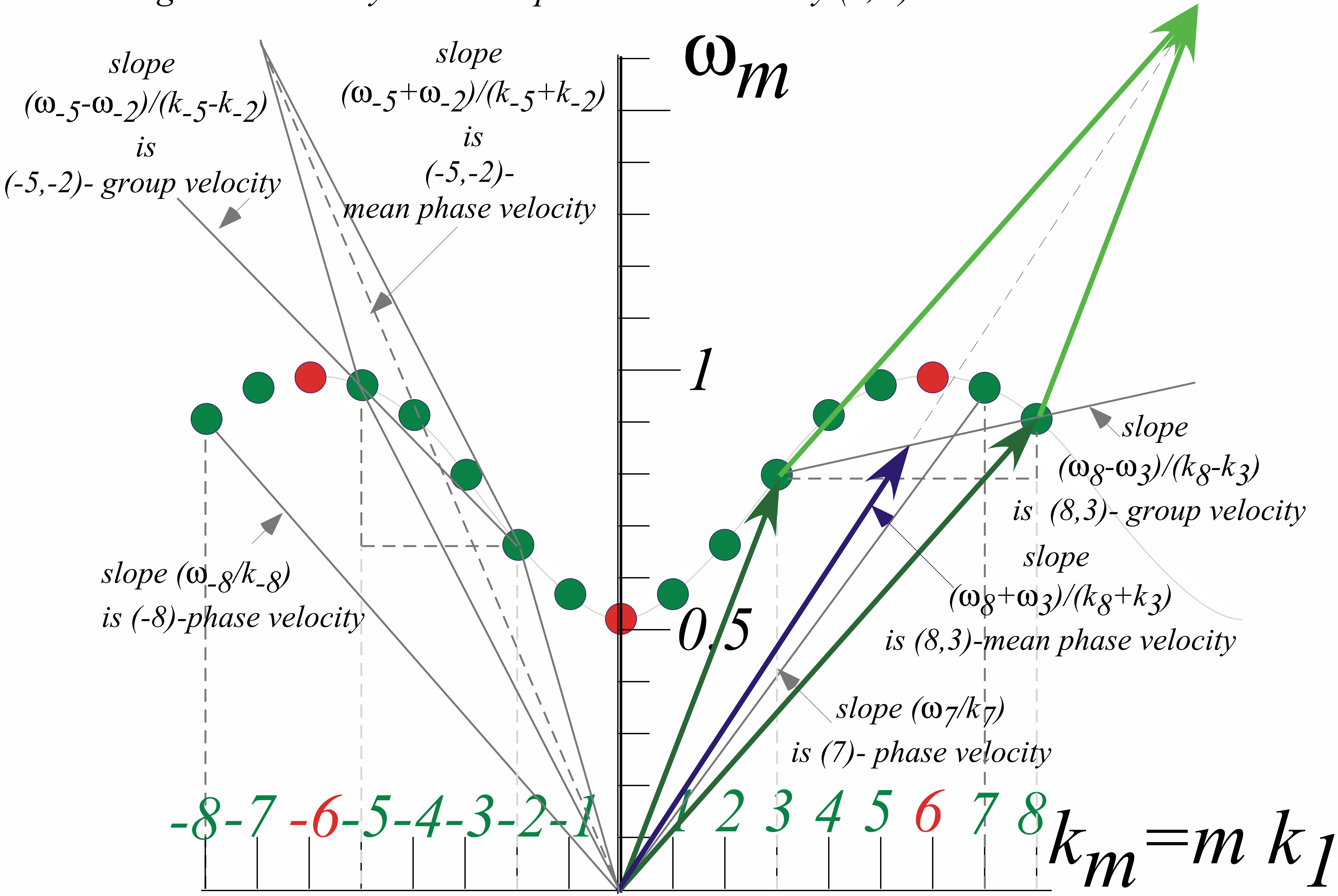


slope
$(\omega_{-5}-\omega_{-2})/(k_{-5}-k_{-2})$
is
(-5,-2)- group velocity

slope
$(\omega_{-5}+\omega_{-2})/(k_{-5}+k_{-2})$
is
(-5,-2)-
mean phase velocity

slope (ω_{-8}/k_{-8})
is (-8)-phase velocity

slope
$(\omega_8-\omega_3)/(k_8-k_3)$
is (8,3)- group velocity

slope
$(\omega_8+\omega_3)/(k_8+k_3)$
is (8,3)-mean phase velocity

slope (ω_7/k_7)
is (7)- phase velocity

*1^{st} Step: Abelian symmetry analysis Expand C_6 symmetric **H** by C_6 group product table*

Regular representation and coupling parameters $\{r_0,r_1,r_2,r_3,r_4,r_5\}$ and Fourier dispersion

*2^{nd} Step: Find **H** eigenfunctions by spectral resolution of $C_6 =\{\mathbf{1}=\mathbf{r}^0,\mathbf{r}^1,\mathbf{r}^2,\mathbf{r}^3,\mathbf{r}^4,\mathbf{r}^5\}$*

Character tables of C_2, C_3, C_4, C_5,.... C_{144}

3^{rd} Step: Dispersion functions and eigenvalues for varyious coupling parameter systems

Ortho-complete eigenvalue/parameter relations

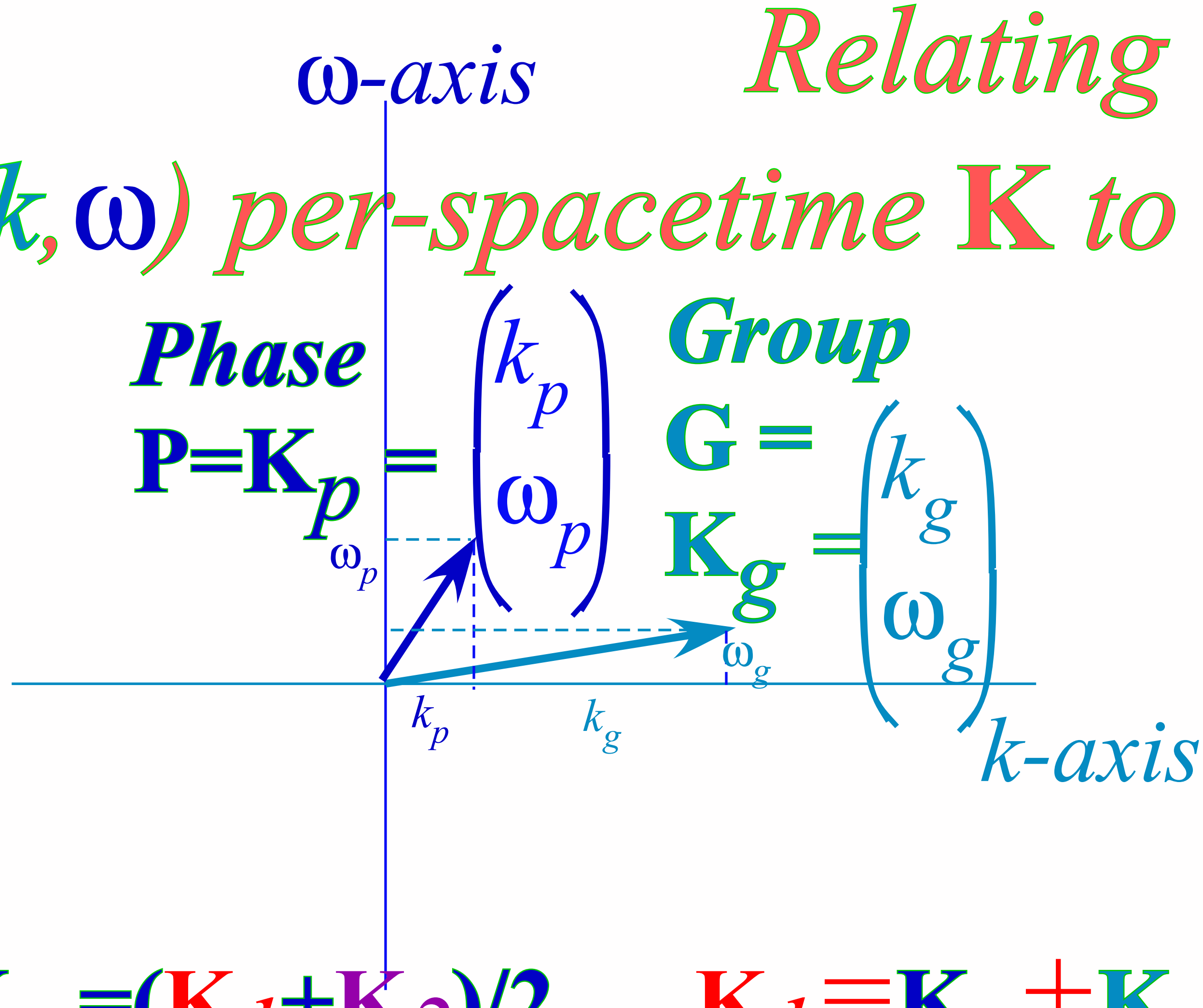
Gauge shifts due to complex coupling

Wave dynamics of phase, mean phase, and group velocity by Expo-Cosine identity

→ *Relating space-time (x,t) and per-space-time (k,ω)*

- *Wave coordinates*
- *Pulse-waves (PW) vs Continuous-waves (CW)*
- *Wave coordinates for Linear Dispersion*
- *Wave coordinates for Bohr-Schrodinger Dispersion*
- *Einstein-Lorentz-Minkowski laser coordinates*

Relating (k,ω) per-spacetime **K** to (x,t) spacetime

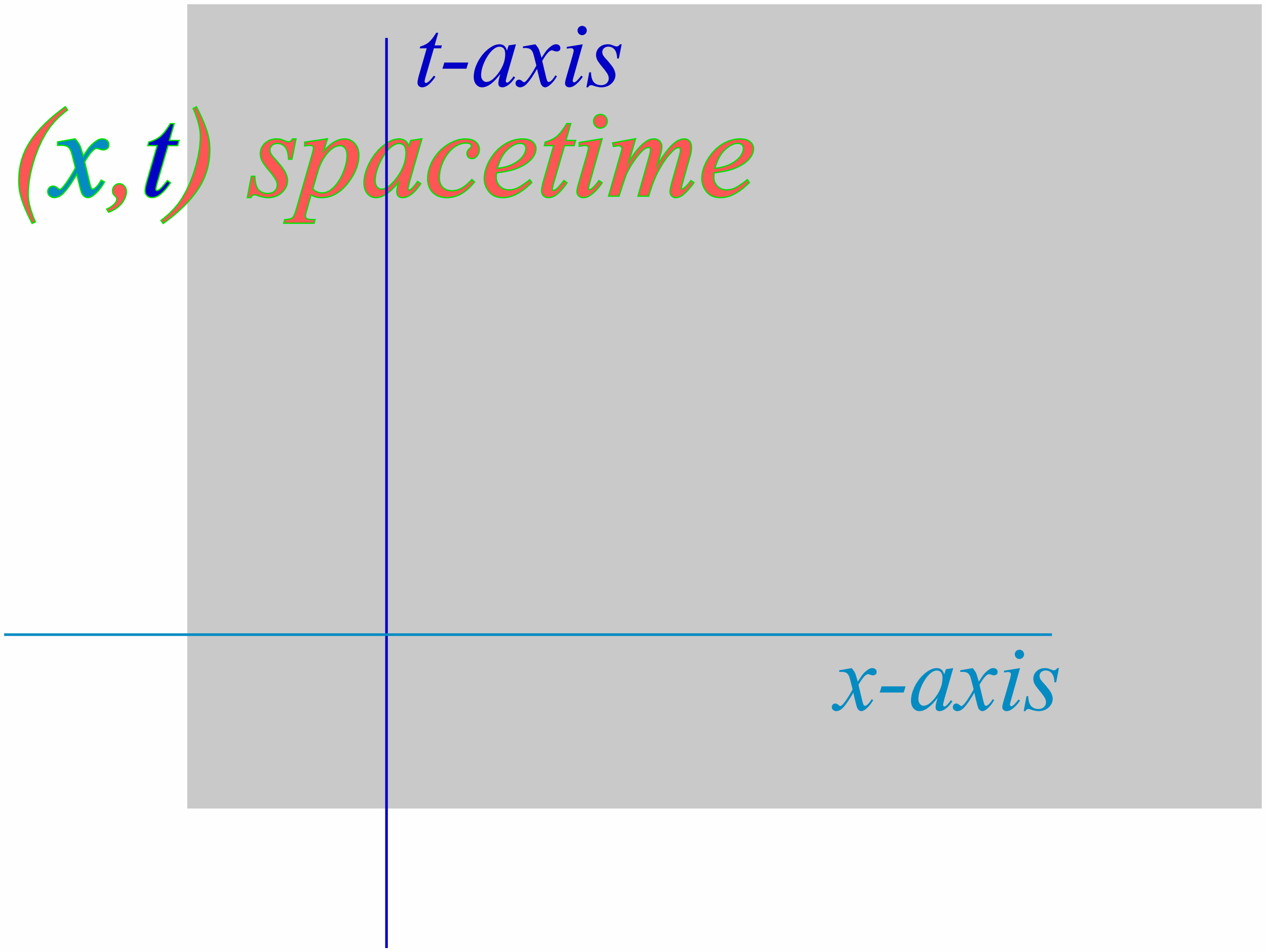


Phase

$$\mathbf{P}=\mathbf{K}_p=\begin{pmatrix} k_p \\ \omega_p \end{pmatrix}$$

Group

$$\mathbf{G}=\mathbf{K}_g=\begin{pmatrix} k_g \\ \omega_g \end{pmatrix}$$



$$\mathbf{K}_p=(\mathbf{K}_1+\mathbf{K}_2)/2 \qquad \mathbf{K}_1=\mathbf{K}_p+\mathbf{K}_g$$

$$\mathbf{K}_g=(\mathbf{K}_1-\mathbf{K}_2)/2 \qquad \mathbf{K}_2=\mathbf{K}_p-\mathbf{K}_g$$

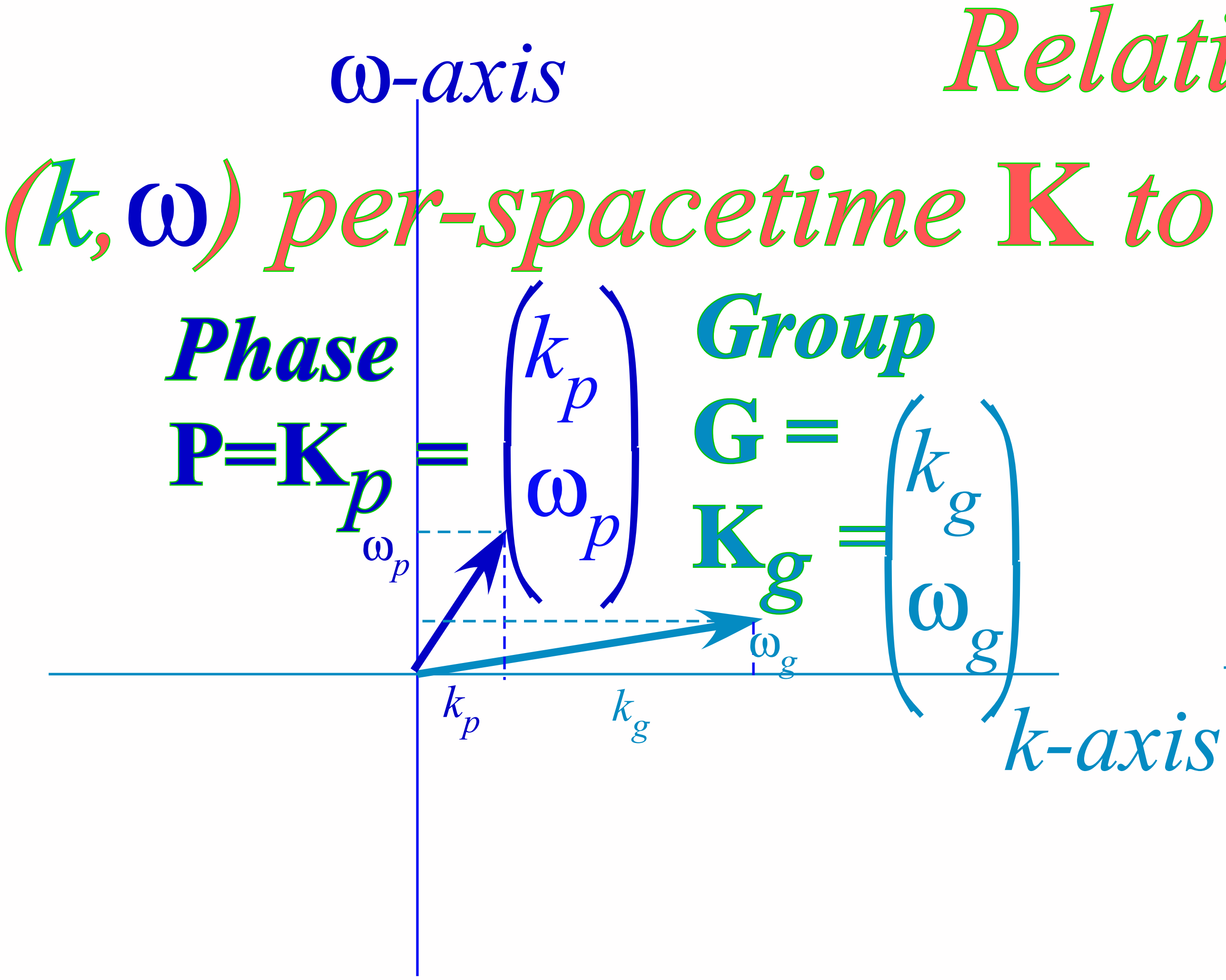
$\omega_p=(\omega_1+\omega_2)/2$
$k_p=(k_1+k_2)/2$
Overall or
Mean phase

$\omega_g=(\omega_1-\omega_2)/2$
$k_g=(k_1-k_2)/2$
Relative or
Group phase

Find tracks in space-time of a balanced (50-50) plane wave combination:

$$\Psi_{50_1\text{-}50_2}(x,t)=1/2e^{i(k_1x-\omega_1t)}+1/2e^{i(k_2x-\omega_2t)}=e^{i(k_px-\omega_pt)}\bullet\cos(k_gx-\omega_gt)$$

Relating
(k,ω) per-spacetime **K** to (x,t) spacetime-lattice



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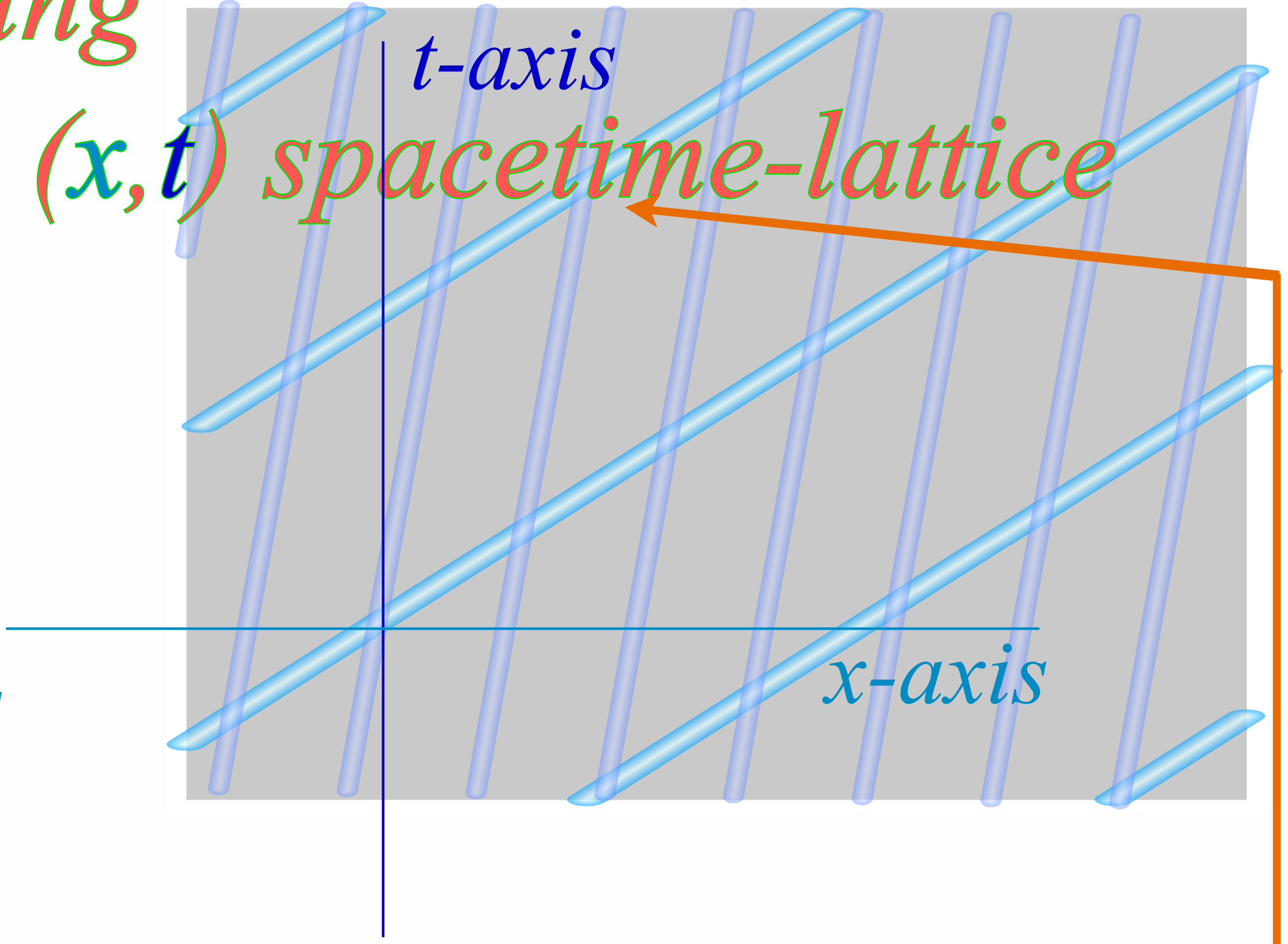

$\omega_p=(\omega_1+\omega_2)/2$

$k_p=(k_1+k_2)/2$

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$\omega_g=(\omega_1-\omega_2)/2$

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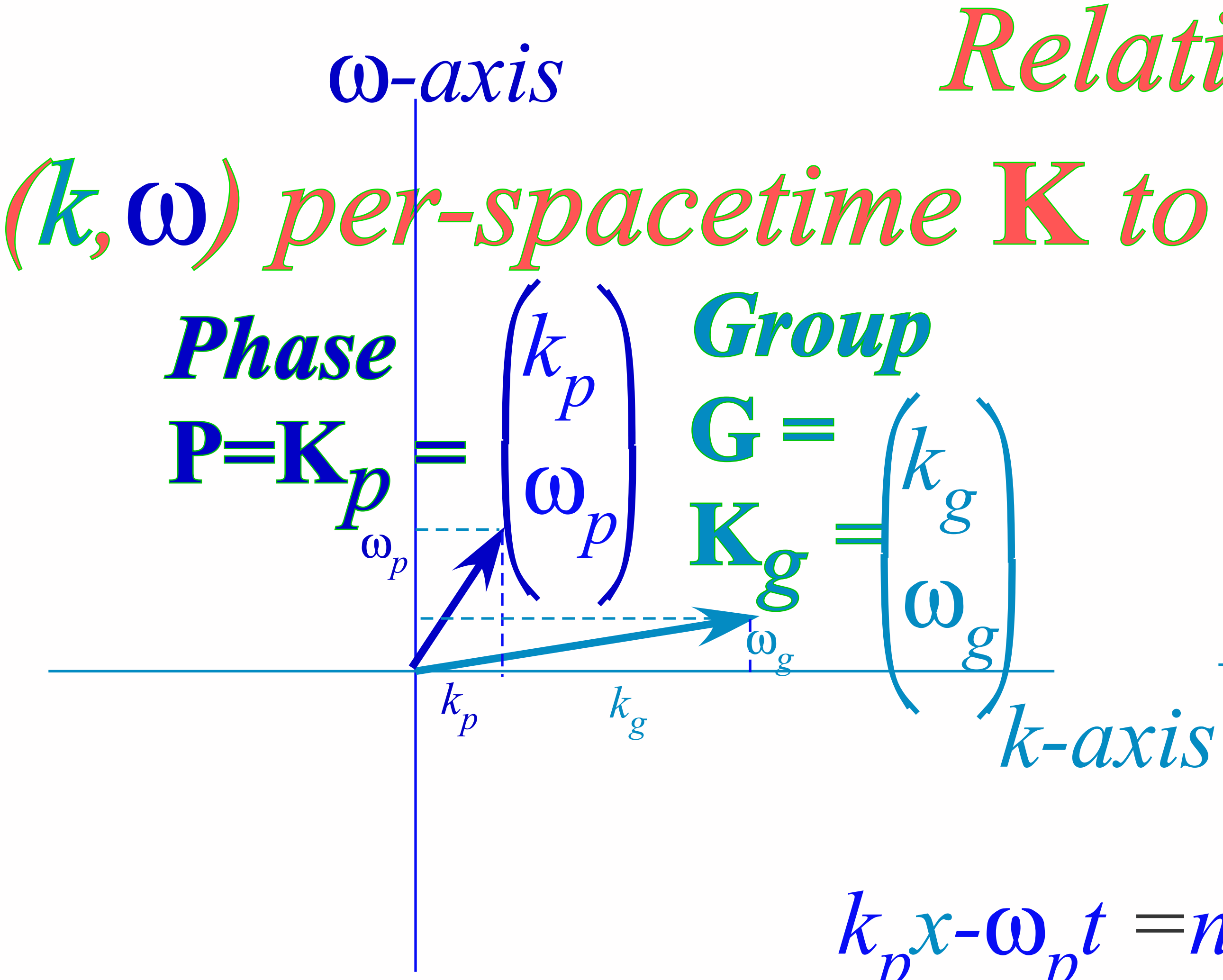
Real part has ZEROS that make: (x,t) spacetime-lattice

Relating
(k,ω) per-spacetime **K** *to (x,t) spacetime-lattice*



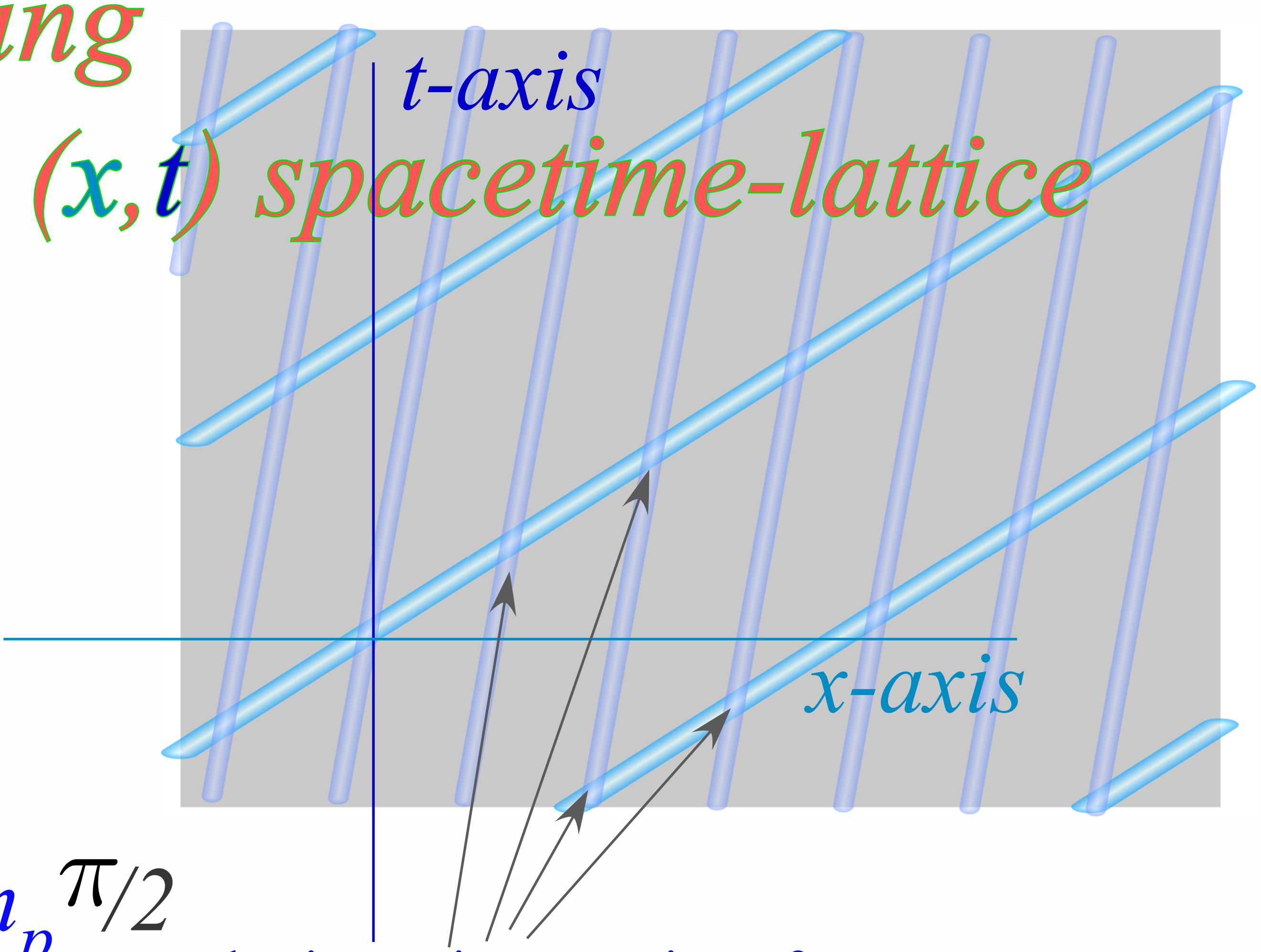
Phase

$\mathbf{P}=\mathbf{K}_p=\begin{pmatrix}k_p\\ \omega_p\end{pmatrix}$

Group

$\mathbf{G}=$

$\mathbf{K}_\mathbf{g}=\begin{pmatrix}k_g\\ \omega_g\end{pmatrix}$



$$k_p x-\omega_p t = n_p \pi/2$$

$$k_g x-\omega_g t = n_g \pi/2$$

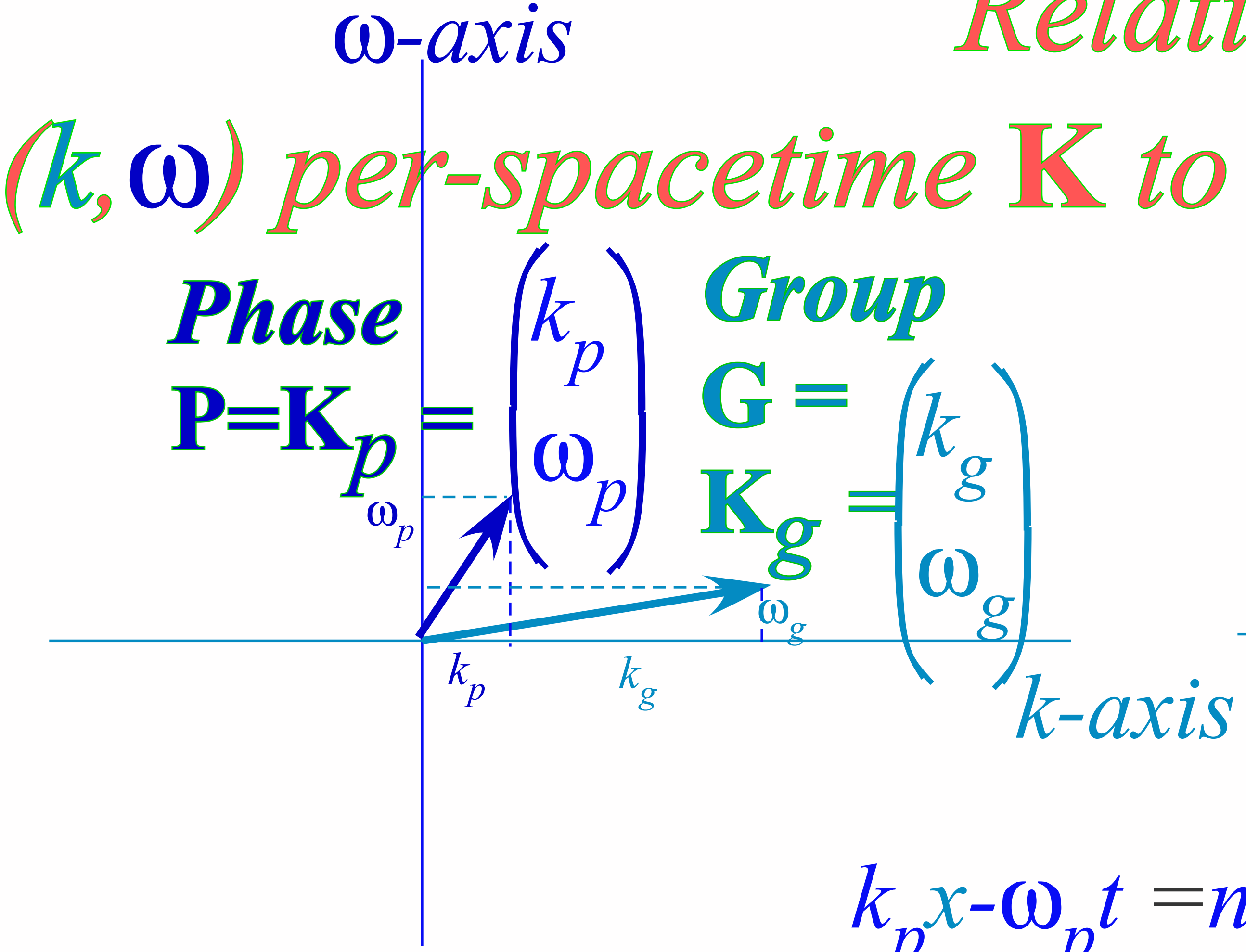
lattice point equations for:

$n_p=\pm 1,\pm 2...$ and $n_g=\pm 1,\pm 2...$

$Re[\Psi_{50_1-50_2}(x,t)]=$ $= \cos(k_p x-\omega_p t) \bullet \cos(k_g x-\omega_g t)=0$

Real part has ZEROS that make: *(x,t) CW spacetime-lattice*

Relating
(k,ω) per-spacetime $\mathbf{K}$ to (x,t) spacetime-lattice



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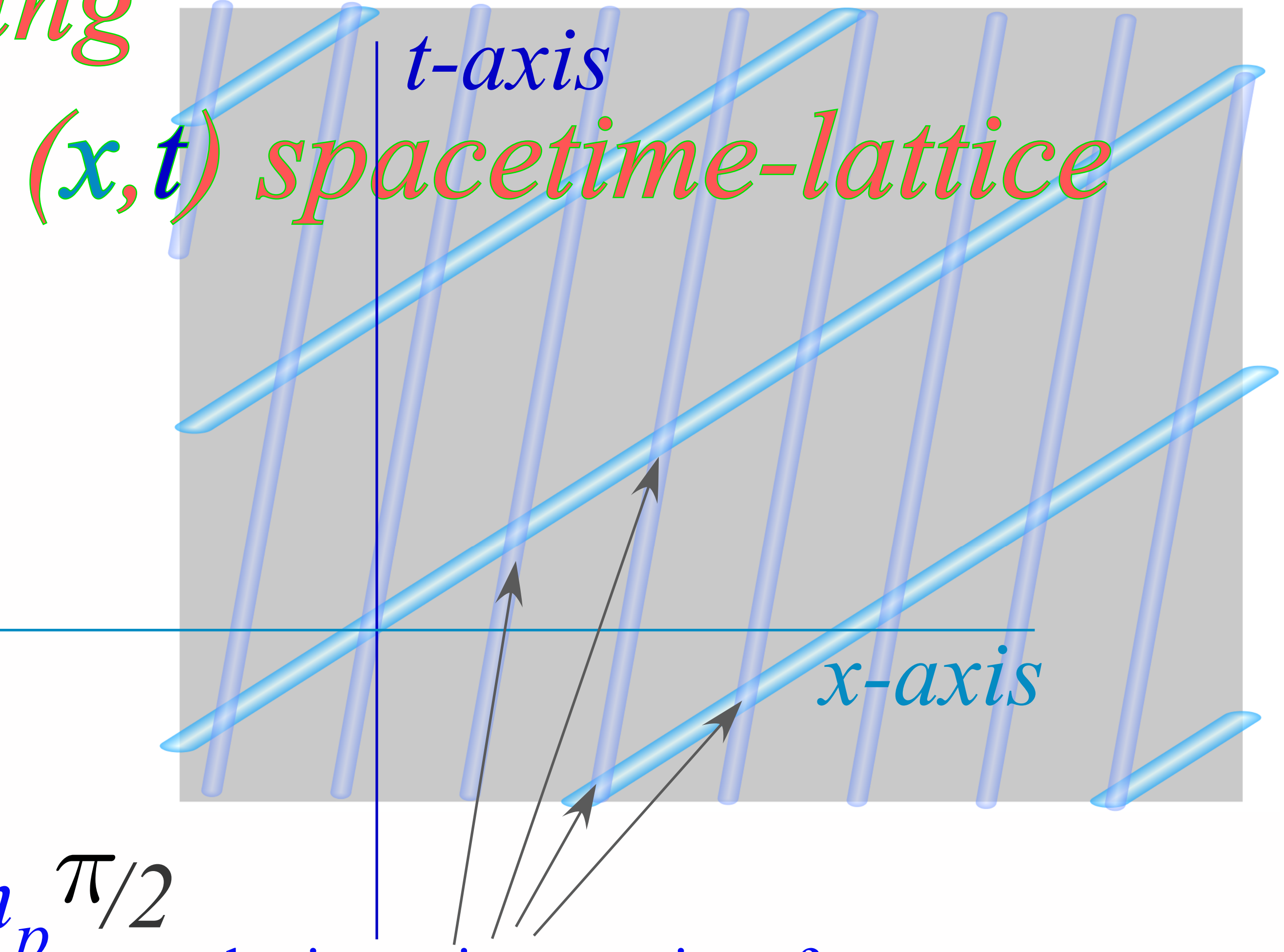

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Relating
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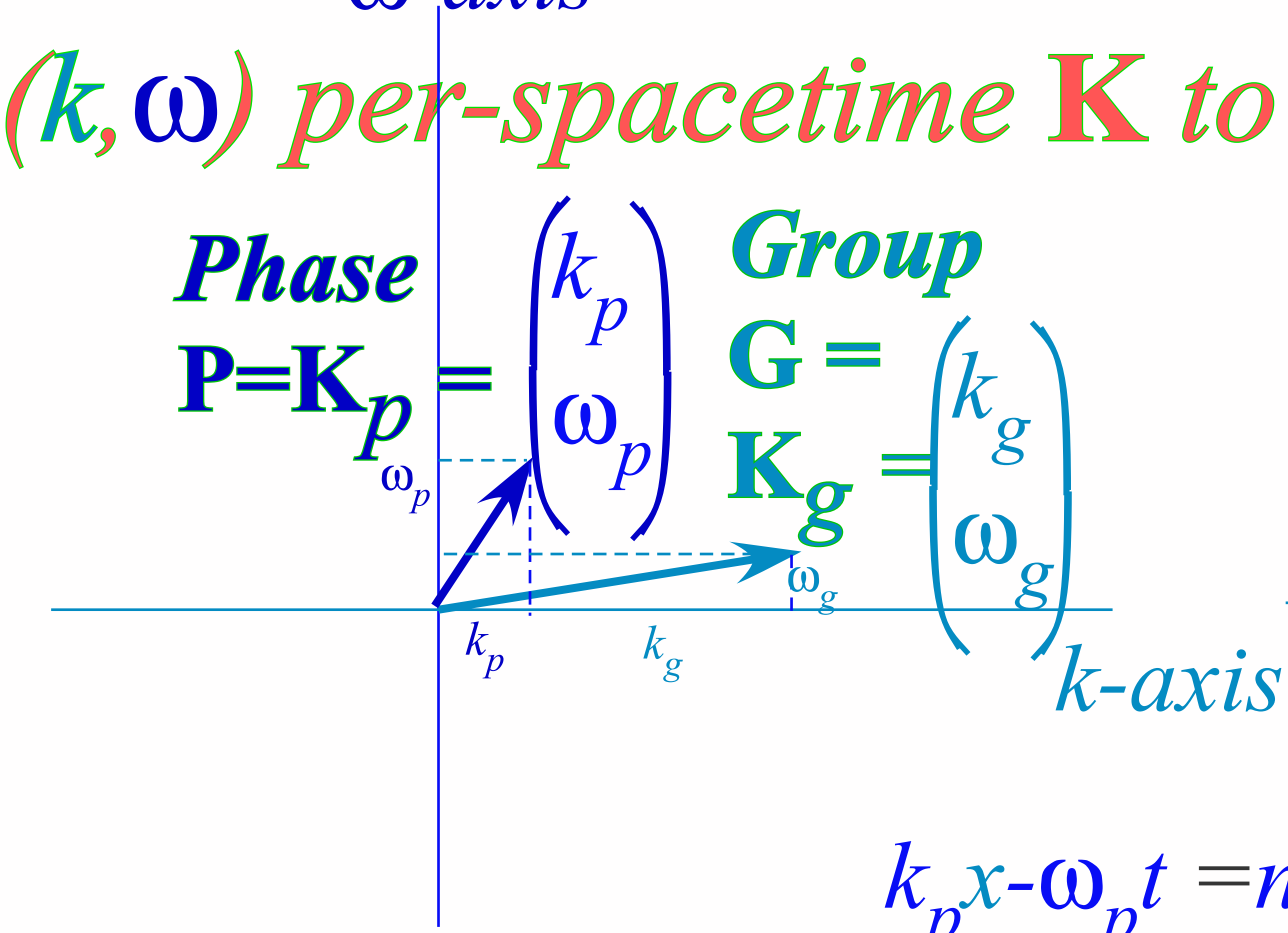

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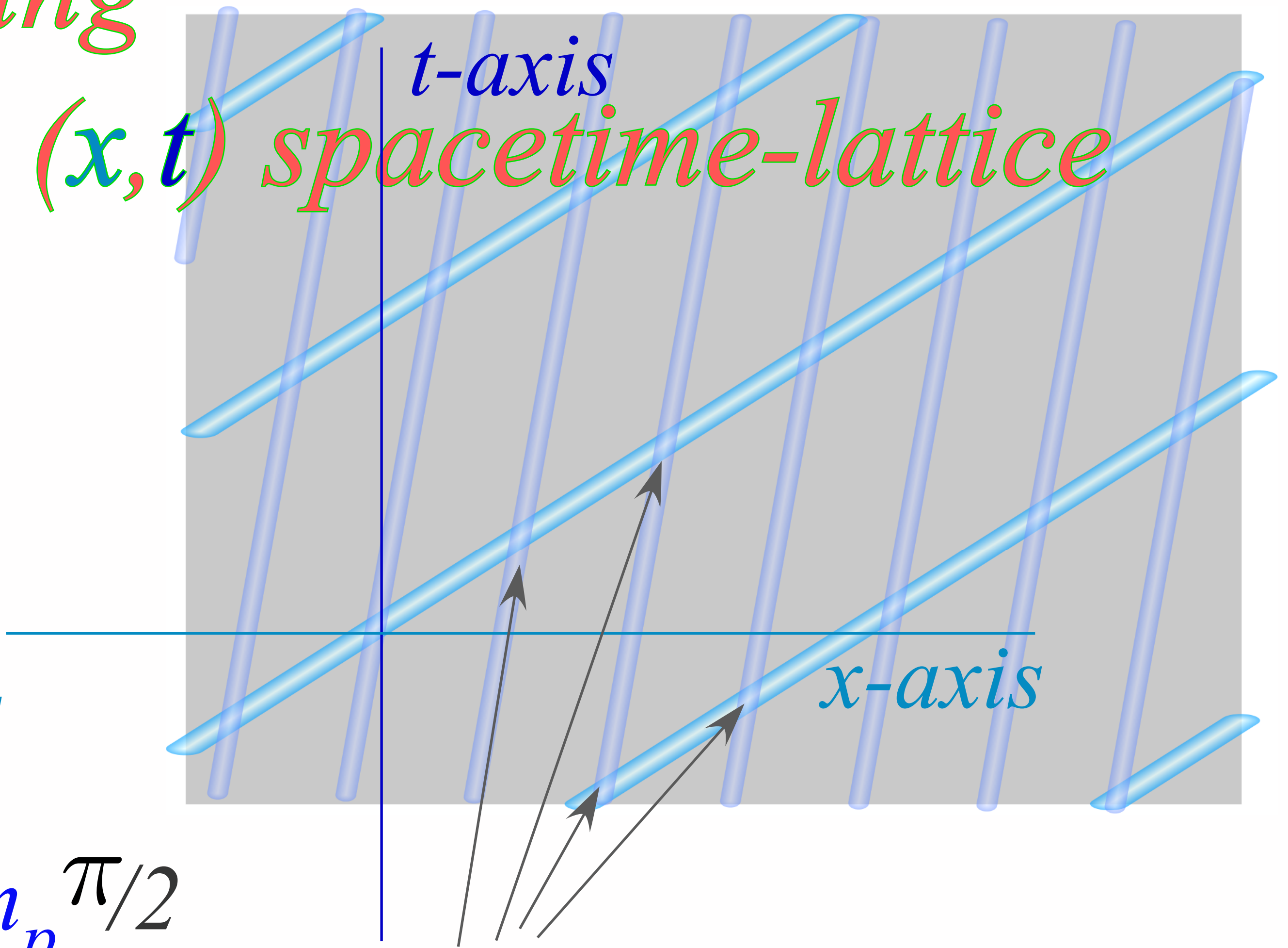

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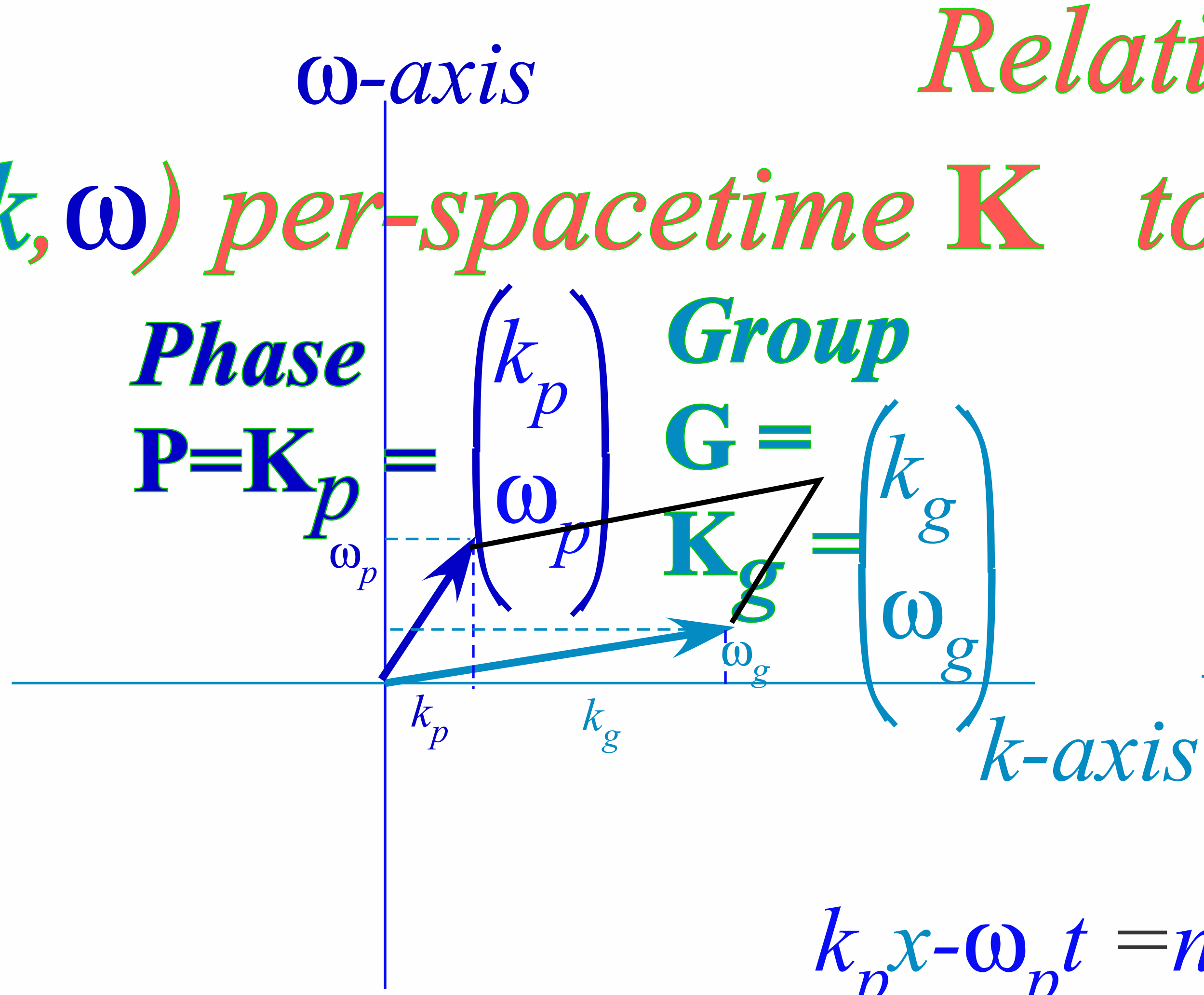
$n_p = \pm 1, \pm 2\ldots$ and $n_g = \pm 1, \pm 2\ldots$

$$\begin{pmatrix} k_p & -\omega_p \\ k_g & -\omega_g \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} n_p \\ n_g \end{pmatrix}\pi/2 \xrightarrow[\text{inverted}]{} \begin{pmatrix} x \\ t \end{pmatrix} = \frac{\begin{pmatrix} -\omega_g & \omega_p \\ -k_g & k_p \end{pmatrix}}{det\left|\mathbf{K}_g \mathrm{x} \mathbf{K}_p\right|}\begin{pmatrix} n_p \\ n_g \end{pmatrix}\pi/2 = \frac{-n_p \begin{pmatrix} \omega_g \\ k_g \end{pmatrix} + n_g \begin{pmatrix} \omega_p \\ k_p \end{pmatrix}}{det\left|\mathbf{K}_g \mathrm{x} \mathbf{K}_p\right| 2/\pi}$$

$Re[\Psi_{50_1-50_2}(x,t)] = \qquad = \cos(k_p x - \omega_p t) \bullet \cos(k_g x - \omega_g t) = 0$

Real part has ZEROS that make: (x,t) CW spacetime-lattice

Relating (k,ω) per-spacetime K to (x,t) spacetime-grid



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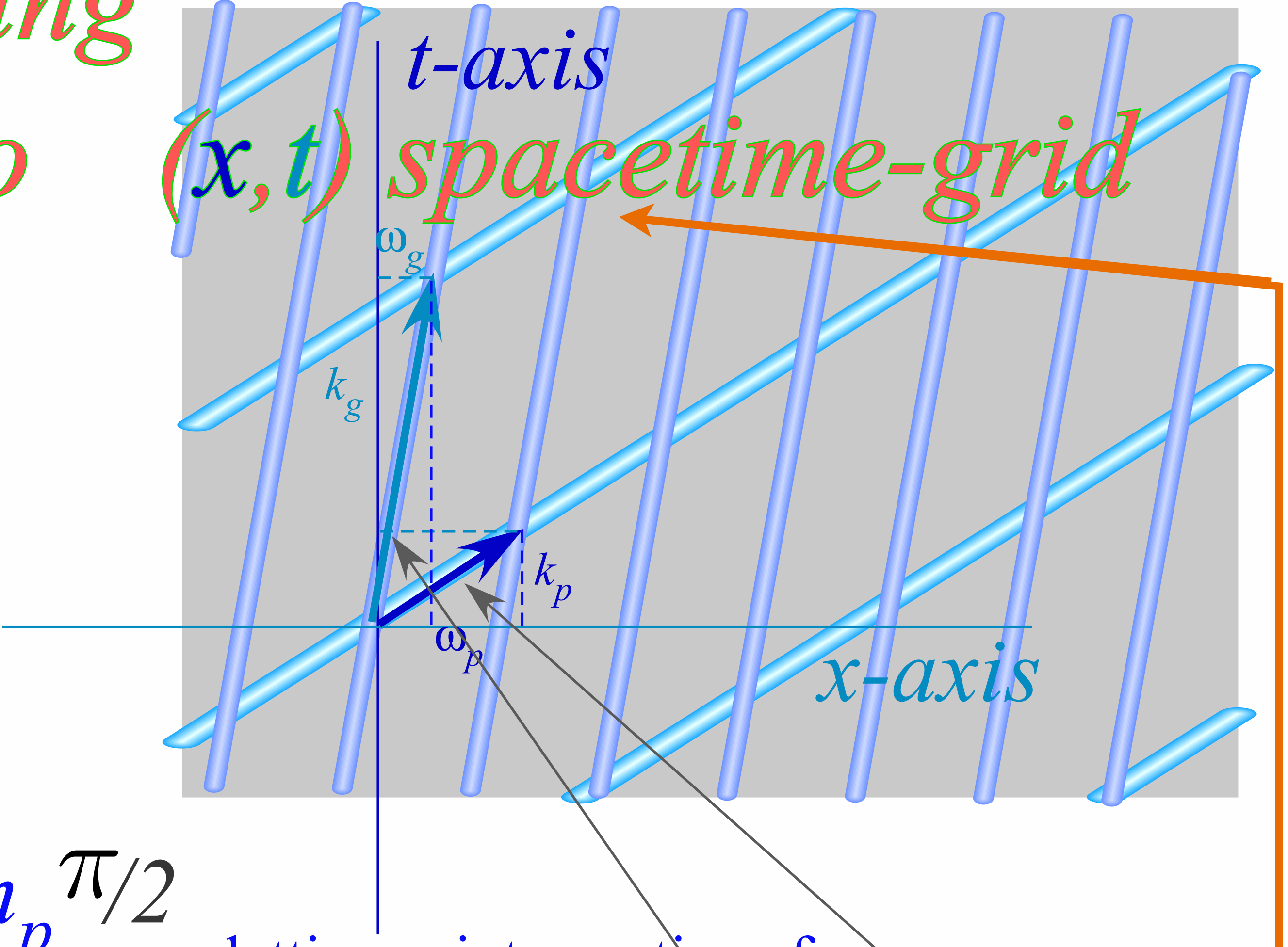

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$$\begin{pmatrix}k_p & -\omega_p\\ k_g & -\omega_g\end{pmatrix}\begin{pmatrix}x\\ t\end{pmatrix}=\begin{pmatrix}n_p\\ n_g\end{pmatrix}\pi/2$$

inverted

$$\begin{pmatrix}x\\ t\end{pmatrix}=\frac{\begin{pmatrix}-\omega_g & \omega_p\\ -k_g & k_p\end{pmatrix}}{det\left|\mathbf{K}_g \times \mathbf{K}_p\right|}\begin{pmatrix}n_p\\ n_g\end{pmatrix}\pi/2=-n_p\frac{\begin{pmatrix}\omega_g\\ k_g\end{pmatrix}+n_g\begin{pmatrix}\omega_p\\ k_p\end{pmatrix}}{det\left|\mathbf{K}_g \times \mathbf{K}_p\right|2/\pi}$$

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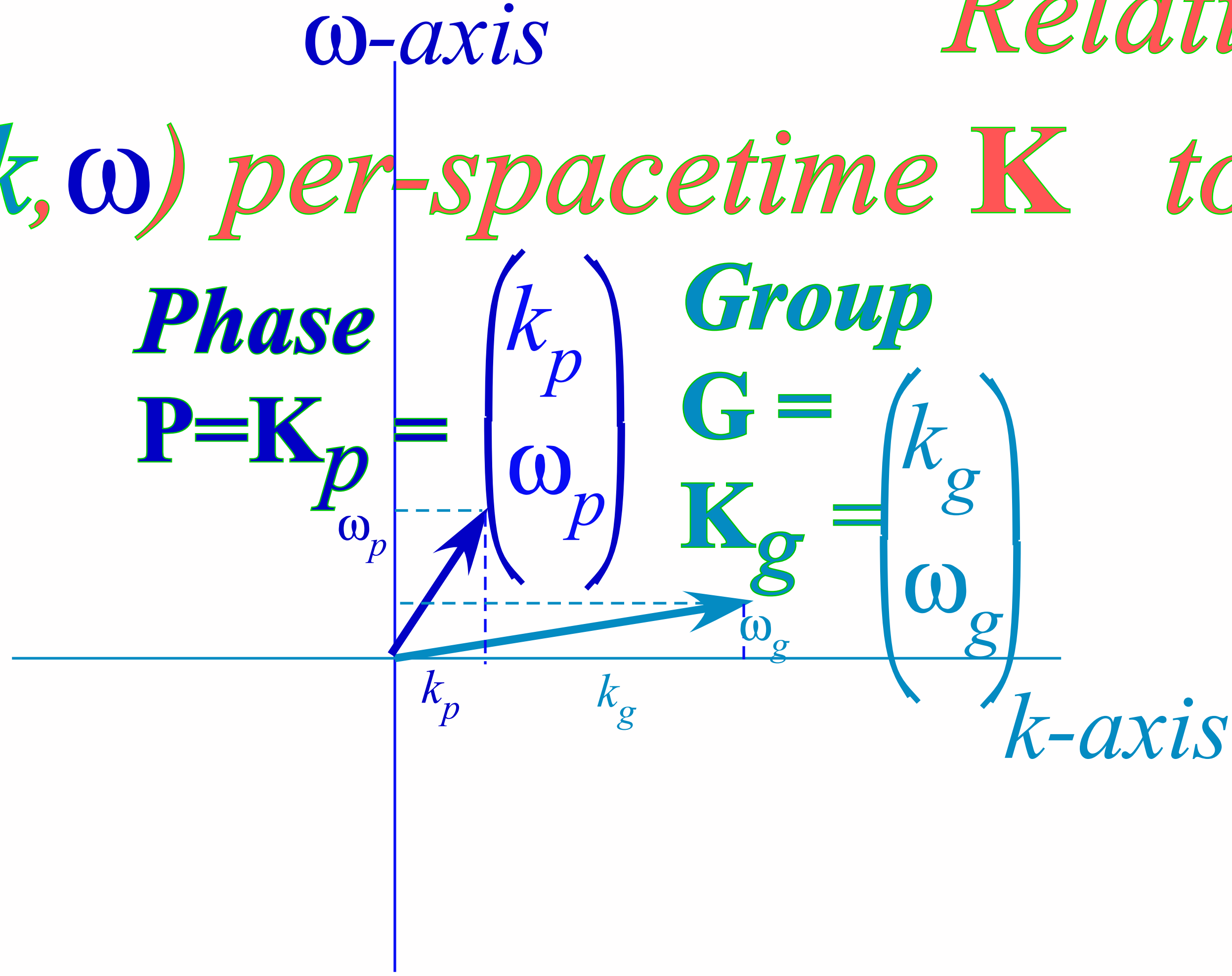
Relating *(k,ω) per-spacetime* **K** *to* *(x,t) spacetime-grid*



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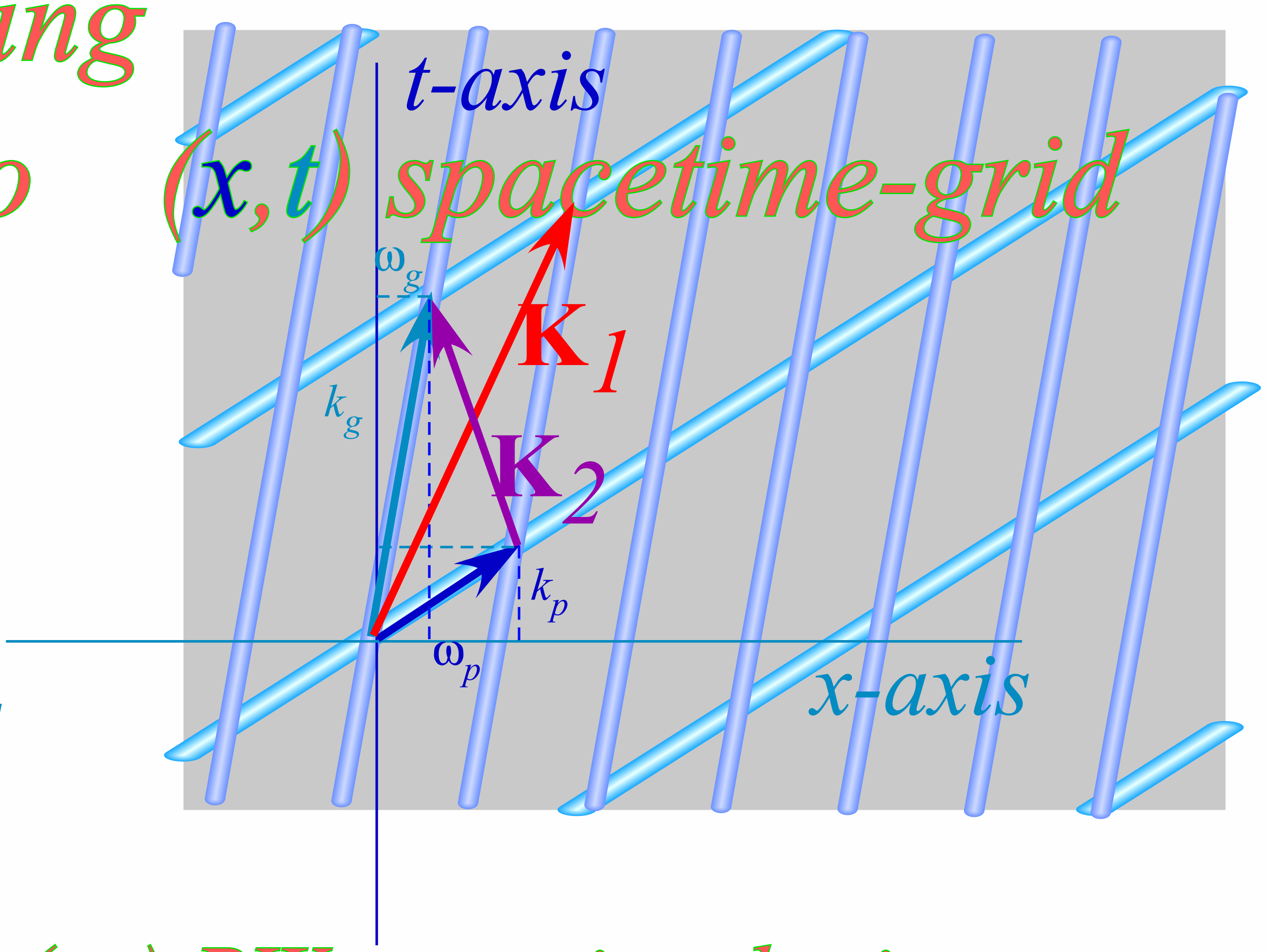

while primitive $\mathbf{K}_1$ *and* $\mathbf{K}_2$ *make:* *(x,t) PW spacetime-lattice*

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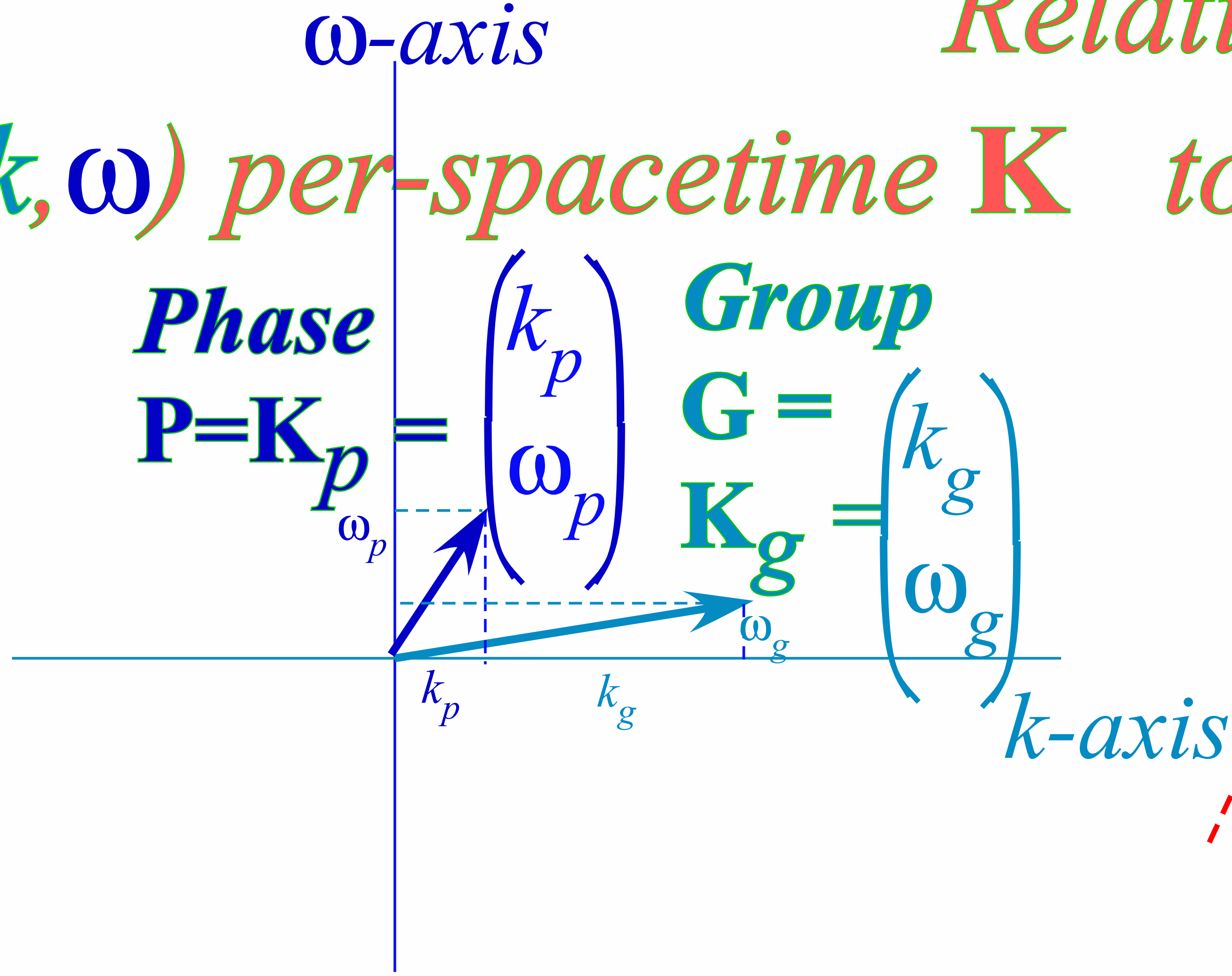

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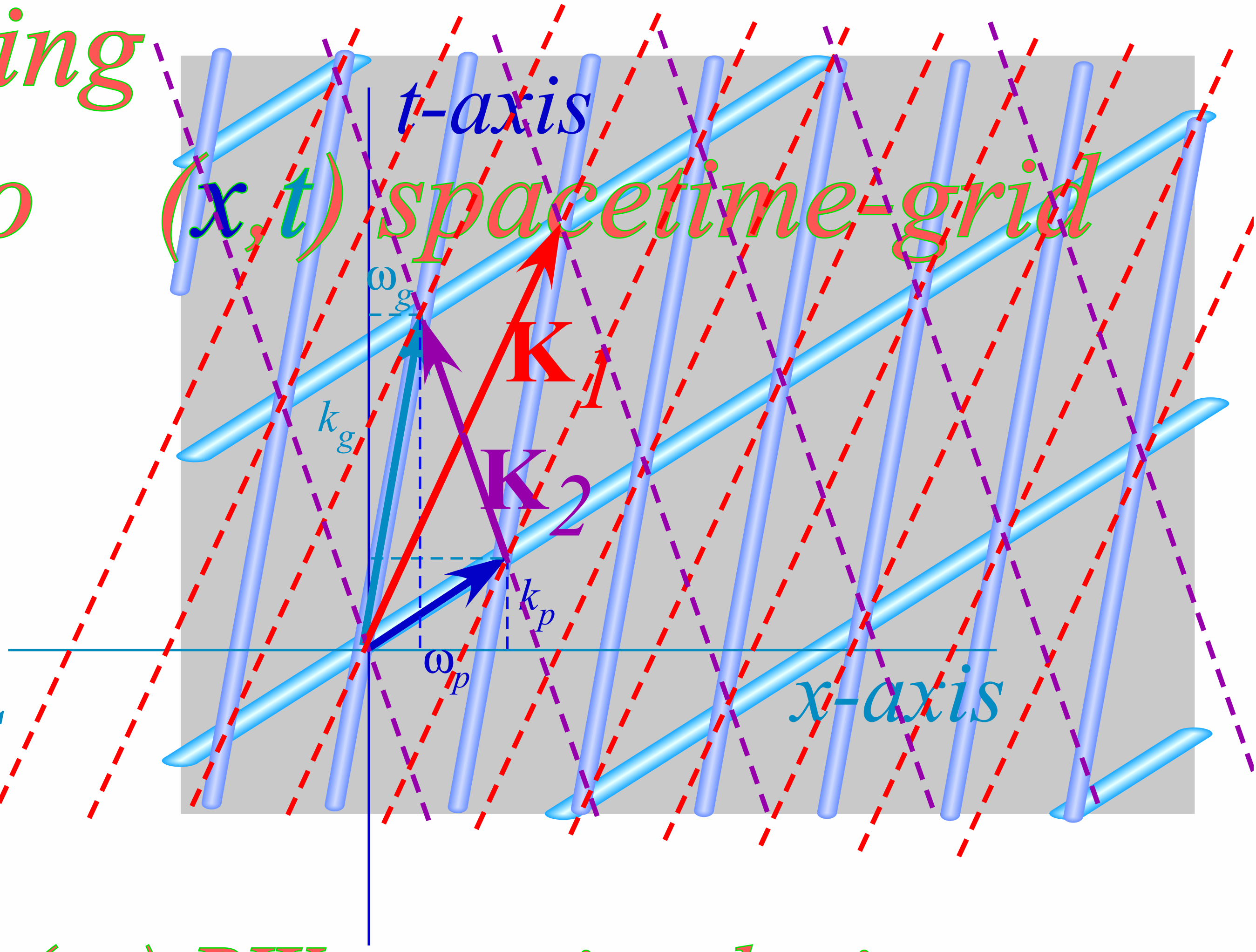

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Interfering Plane Waves: The Expo-Cosine Identity

$$\Psi = e^{ia} + e^{ib} = \underbrace{e^{i(a+b)/2}}_{\text{INSIDE Phase}} \underbrace{\left(e^{i(a-b)/2} + e^{-i(a-b)/2}\right)}_{2\cos\left(\frac{a-b}{2}\right)}$$

OUTSIDE Group

Anatomy of a 2-State Wavefunction

$$\Psi = e^{ia} + e^{ib} = e^{i(a+b)/2}\, 2\cos\left(\frac{a-b}{2}\right)$$

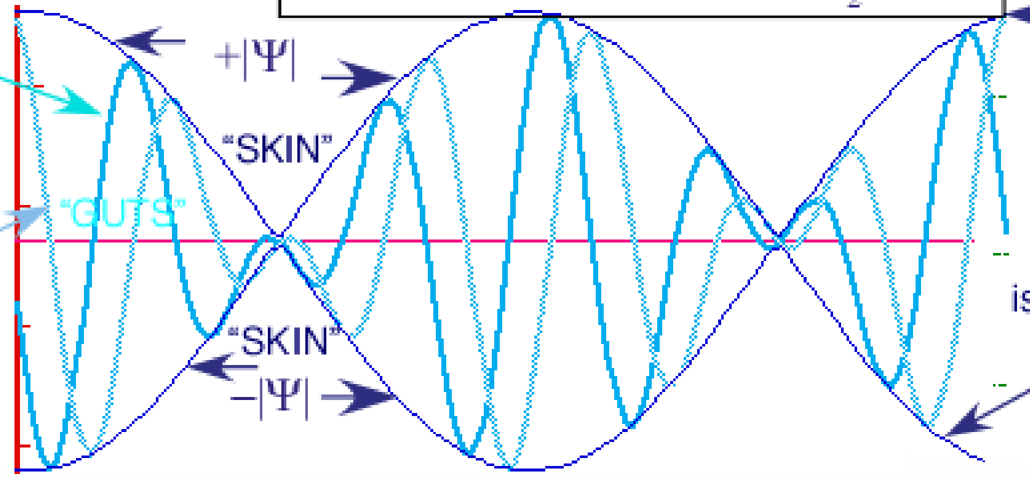
Real Part

$\mathrm{Re}\Psi = |\Psi|\cos\left(\frac{a+b}{2}\right)$

and

Imaginary Part

$\mathrm{Im}\Psi = |\Psi|\sin\left(\frac{a+b}{2}\right)$



Linear Dispersion

Envelope or Modulus

Wave "SKIN" ± $|\Psi| = \pm 2\cos\left(\frac{a-b}{2}\right)$

is PROBABILITY wave for classical "stuff" $|\Psi| = \sqrt{\Psi^*\Psi}$

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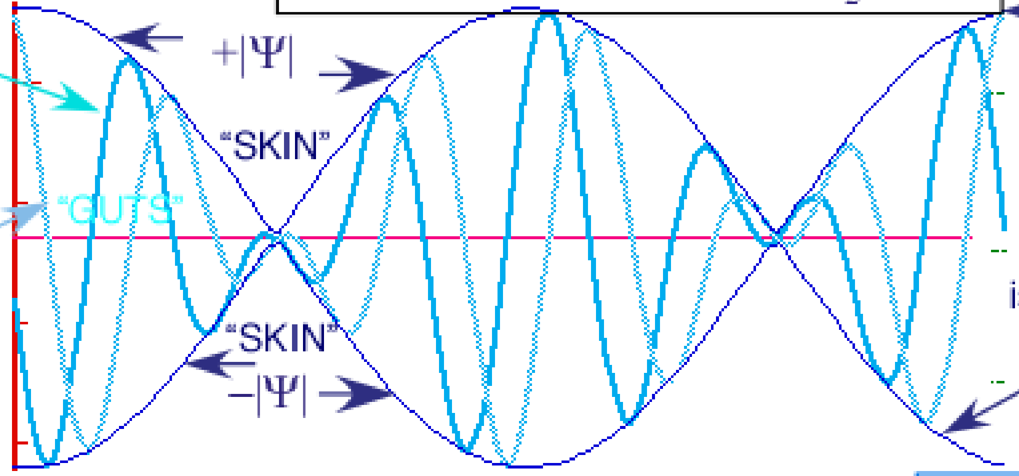

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Input phases

$a = k_a x - \omega_a t$

1st base vector

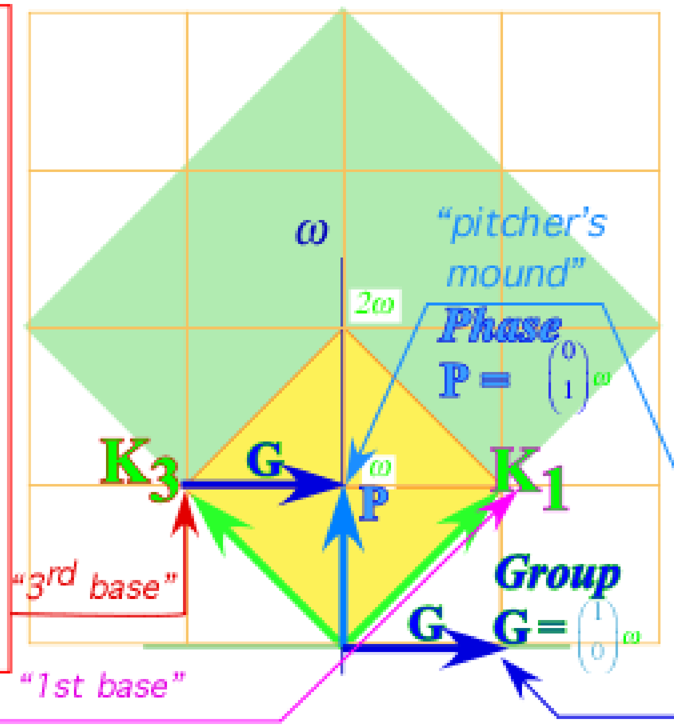
$$\mathbf{K_1} = \begin{pmatrix} ck_a \\ \omega_a \end{pmatrix} = \omega_a \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$b = k_b x - \omega_b t$

3rd base vector

$$\mathbf{K_3} = \begin{pmatrix} ck_b \\ \omega_b \end{pmatrix} = \omega_b \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Linear Dispersion



"dugout"

(Here: $\omega_a = \omega = \omega_b$)

$\frac{1}{2}$-Sum

Phase vector

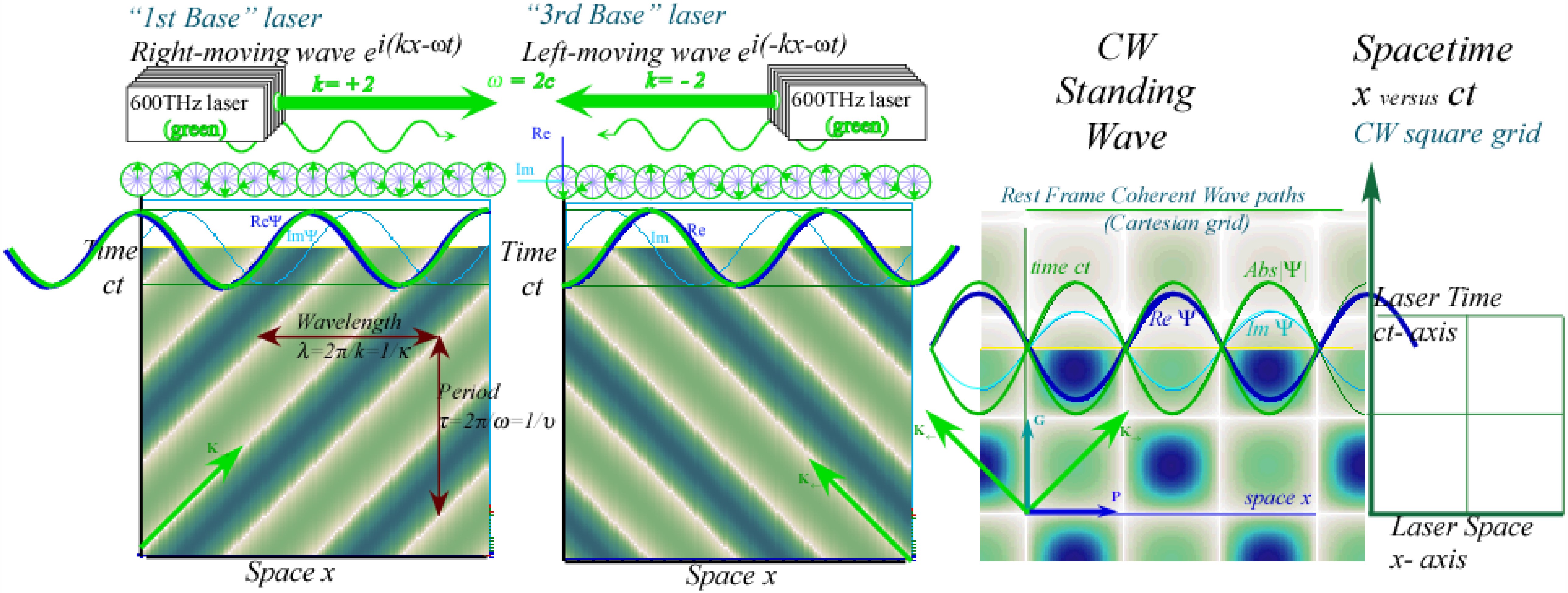
$$\mathbf{P} = \frac{1}{2}\begin{pmatrix} ck_a + ck_b \\ \omega_a + \omega_b \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \omega_a - \omega_b \\ \omega_a + \omega_b \end{pmatrix} = \omega \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\frac{1}{2}$-Difference

Group vector

$$\mathbf{G} = \frac{1}{2}\begin{pmatrix} ck_a - ck_b \\ \omega_a - \omega_b \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \omega_a + \omega_b \\ \omega_a - \omega_b \end{pmatrix} = \omega \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Zeros of head-on CW sum gives (x,ct)-grid



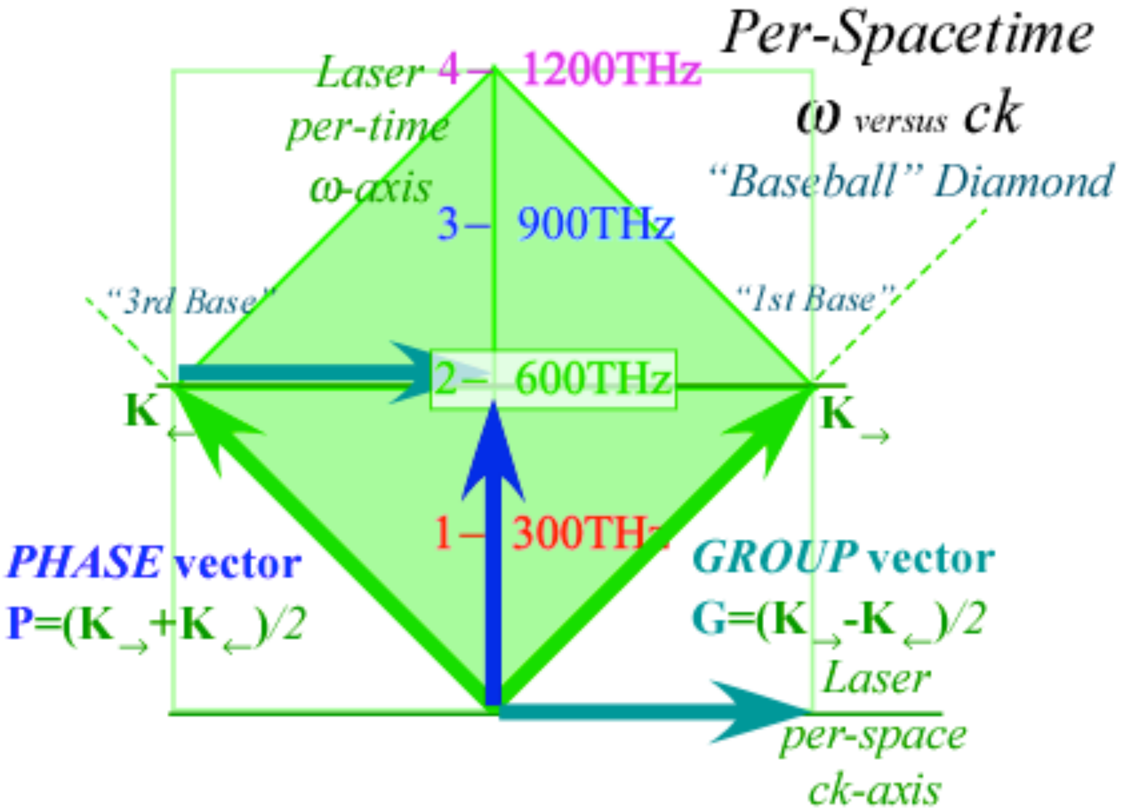
Find zeros by factoring sum:

$$\Psi = \overbrace{e^{ia}}^{(kx-\omega t)} + \overbrace{e^{ib}}^{(-kx-\omega t)}$$

$$= \underbrace{e^{i(a+b)/2}}_{\text{Phase factor:}} \underbrace{\left(e^{i(a-b)/2}+e^{-i(a-b)/2}\right)}_{\text{Group factor:}}$$

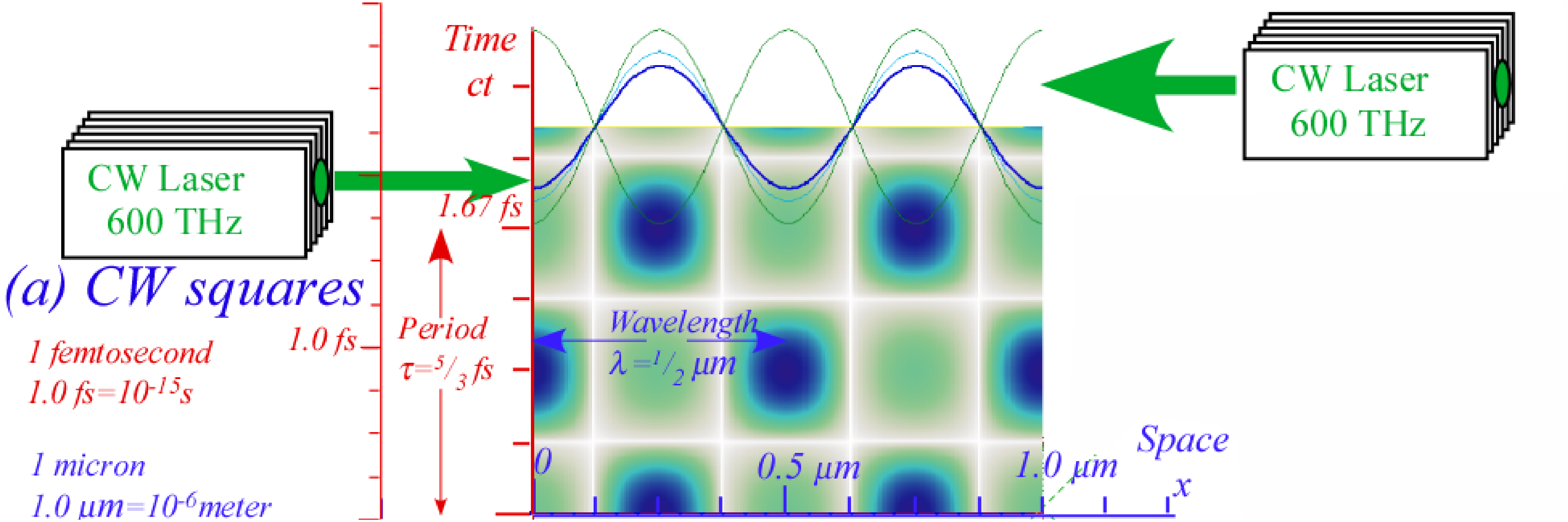
Phase factor: $\exp i\left(\frac{a+b}{2}\right)=e^{-i\omega t}$

Group factor: $2\cos\left(\frac{a-b}{2}\right)=2\cos(kx)$



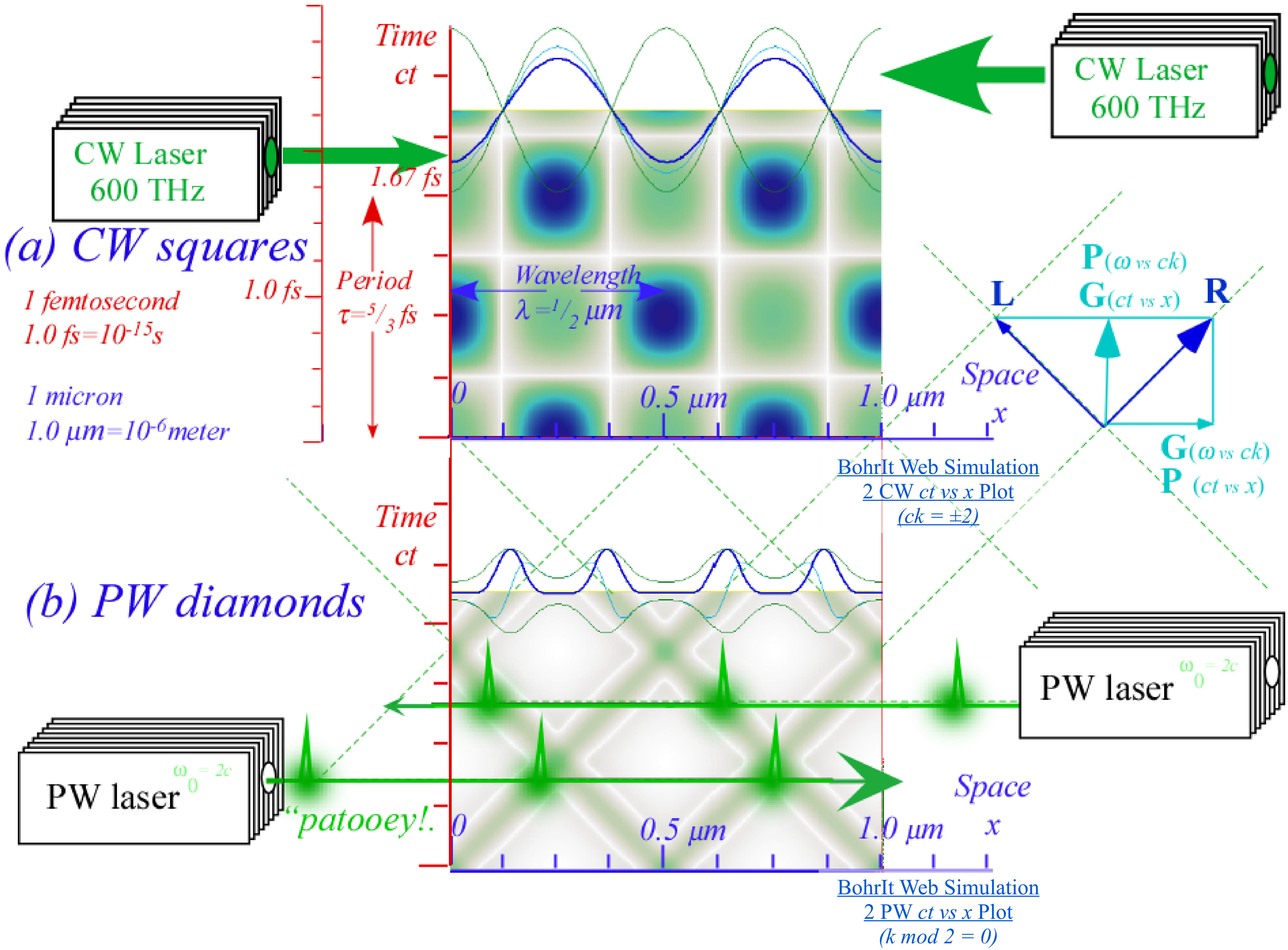
BohrIt Web Simulation
2 CW *ct vs x* Plot
(ck = ±2)

Wave coordinates for Linear Dispersion



BohrIt Web Simulation
2 CW *ct vs x* Plot
(ck = ±2)

Wave coordinates for Linear Dispersion



(a) CW squares

1 femtosecond
$1.0\ fs = 10^{-15} s$

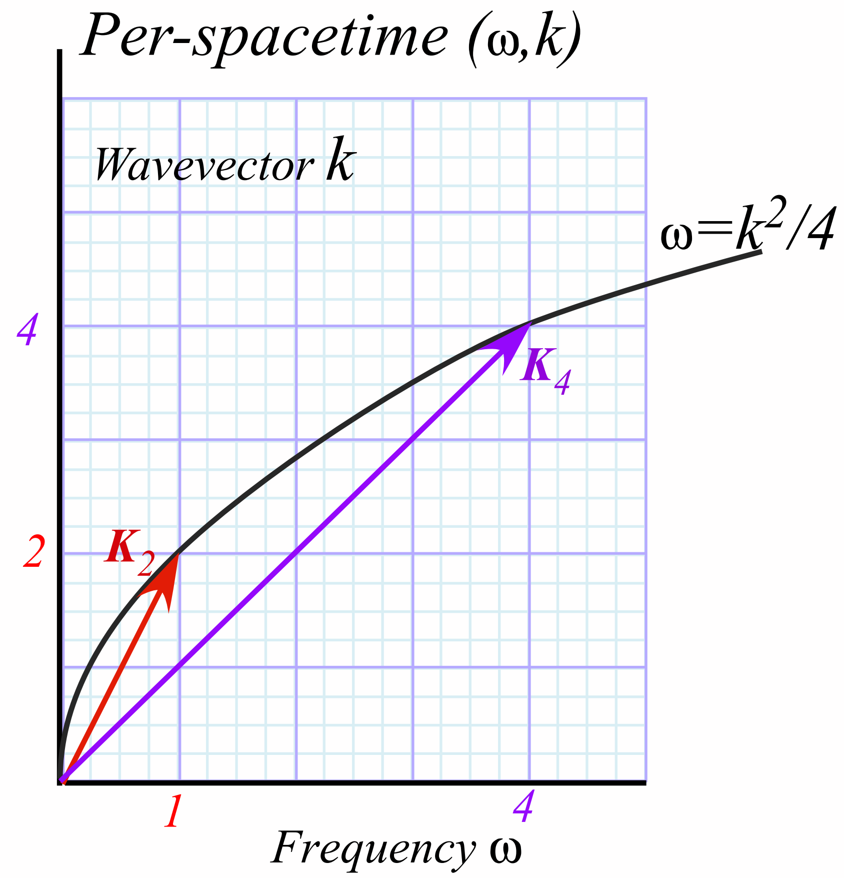
1 micron
$1.0\ \mu m = 10^{-6} meter$

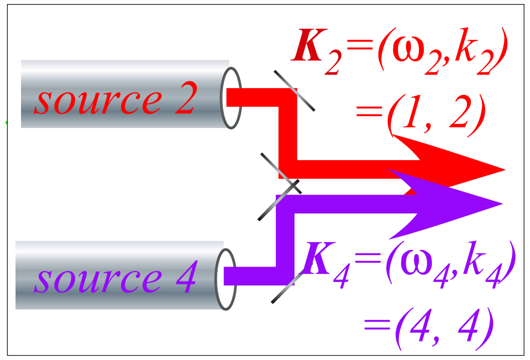
(b) PW diamonds

BohrIt Web Simulation
2 CW *ct vs x* Plot
(ck = ±2)

BohrIt Web Simulation
2 PW *ct vs x* Plot
(k mod 2 = 0)

Wave coordinates for Bohr Dispersion $\omega = k^2/4$

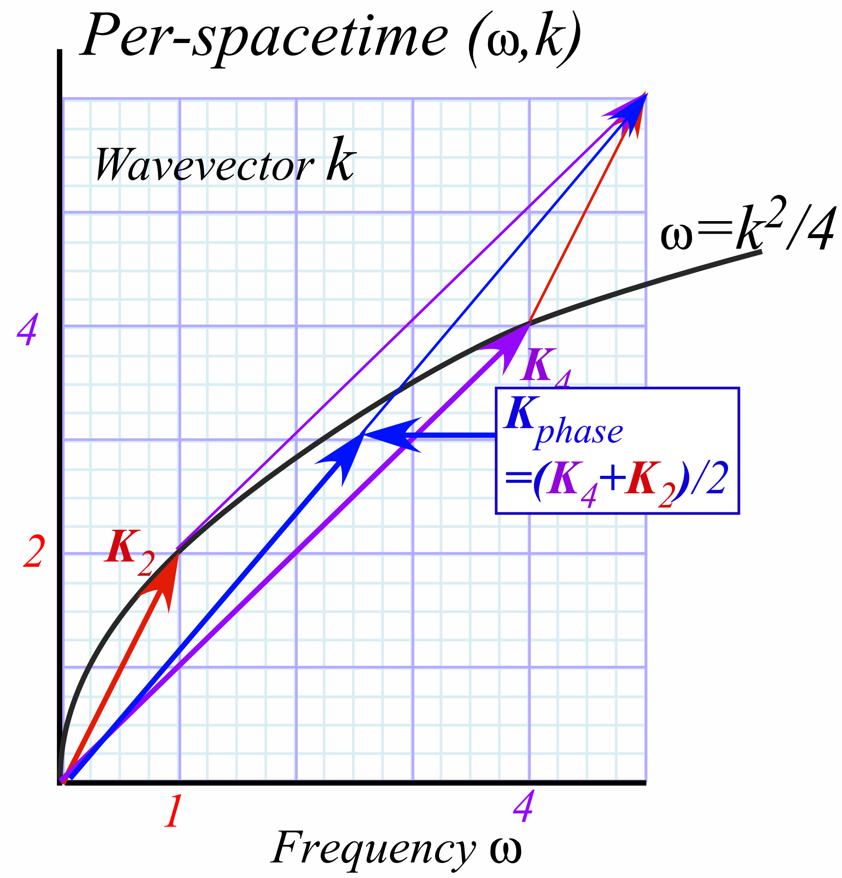




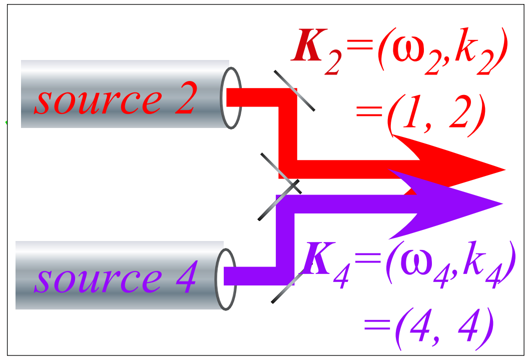
Suppose we are given two "mystery† sources"

†Bohr-Schrodinger "matter-waves"

Wave coordinates for Bohr Dispersion $\omega = k^2/4$



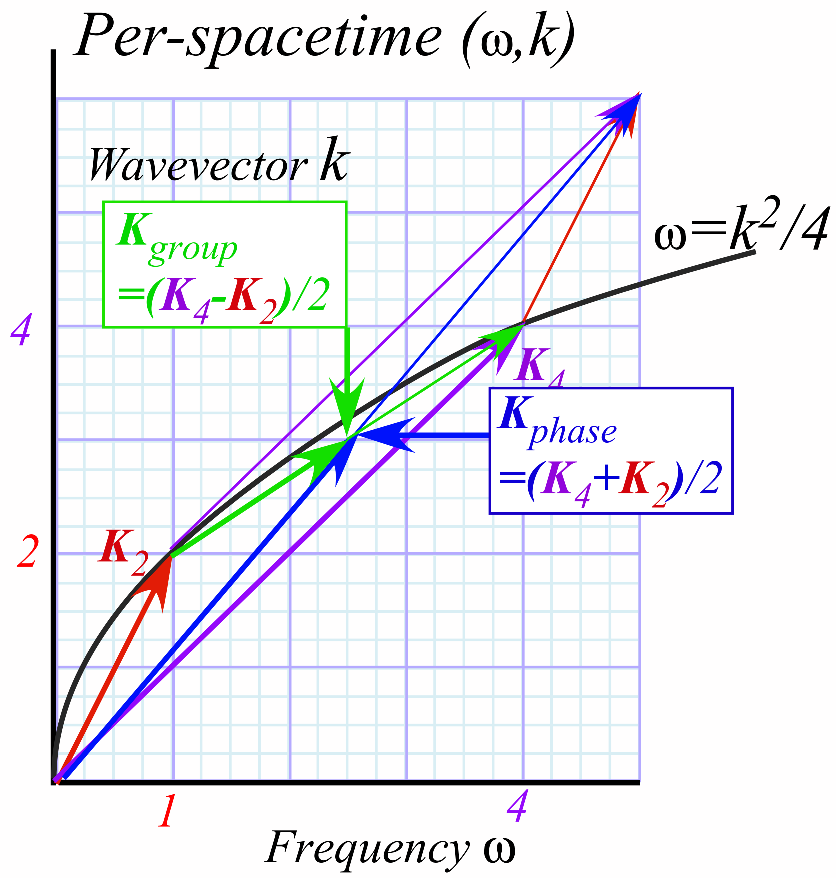
$$\mathbf{K}_{phase} = \frac{\mathbf{K}_4 + \mathbf{K}_2}{2}$$



Suppose we are given two "mystery[†] sources"

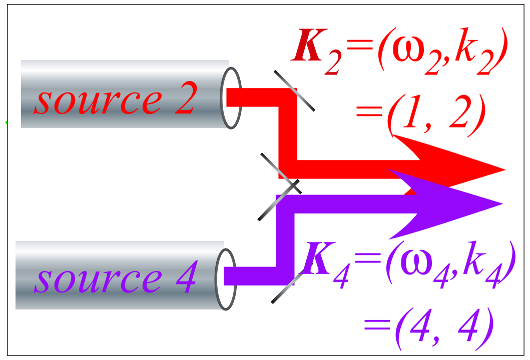
[†] Bohr-Schrodinger "matter-waves"

Wave coordinates for Bohr Dispersion $\omega = k^2/4$



$$\mathbf{K}_{phase} = \frac{\mathbf{K}_4 + \mathbf{K}_2}{2}$$

$$\mathbf{K}_{group} = \frac{\mathbf{K}_4 - \mathbf{K}_2}{2}$$



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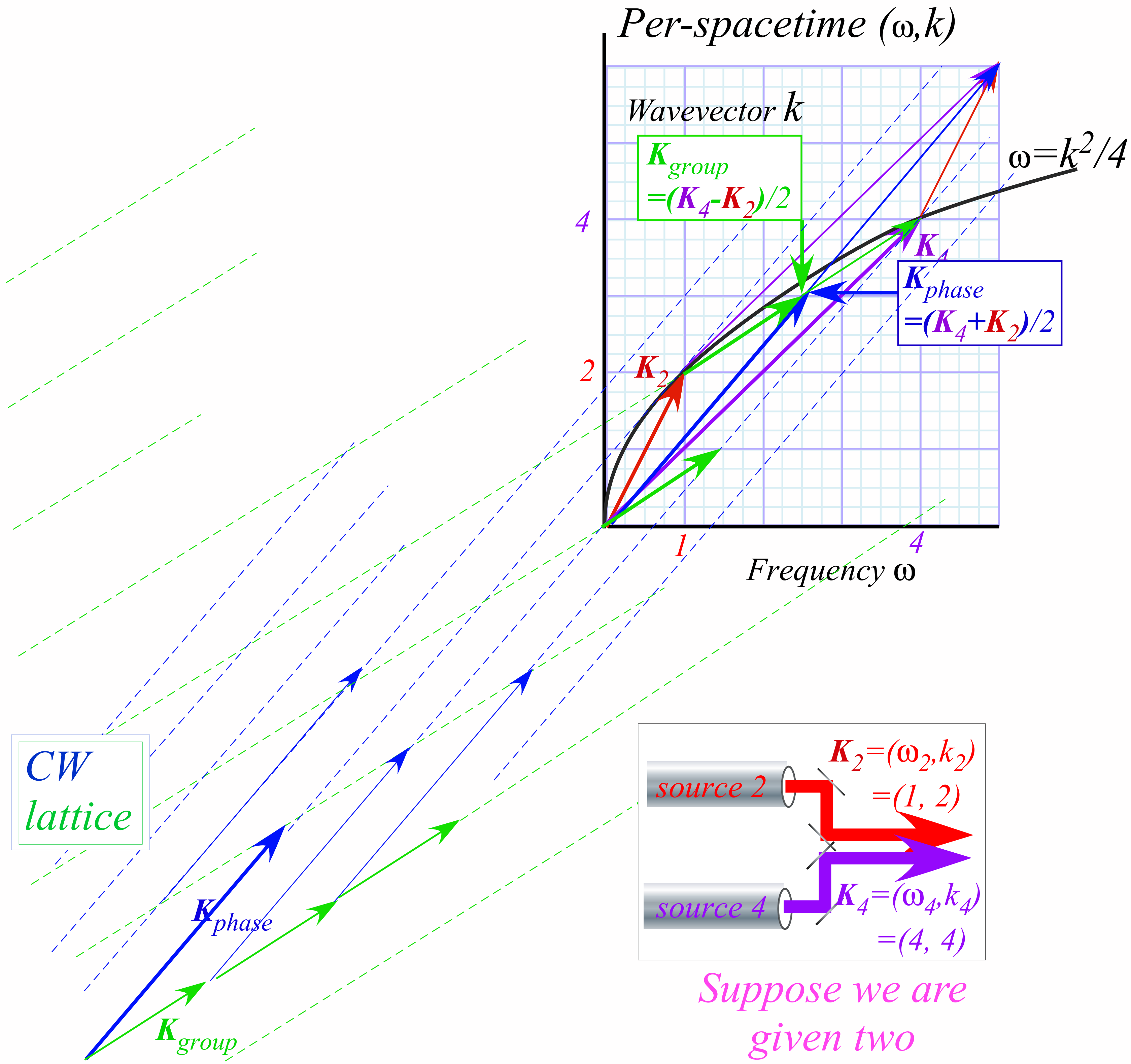


$$\mathbf{K}_{phase} = \frac{\mathbf{K}_4 + \mathbf{K}_2}{2}$$

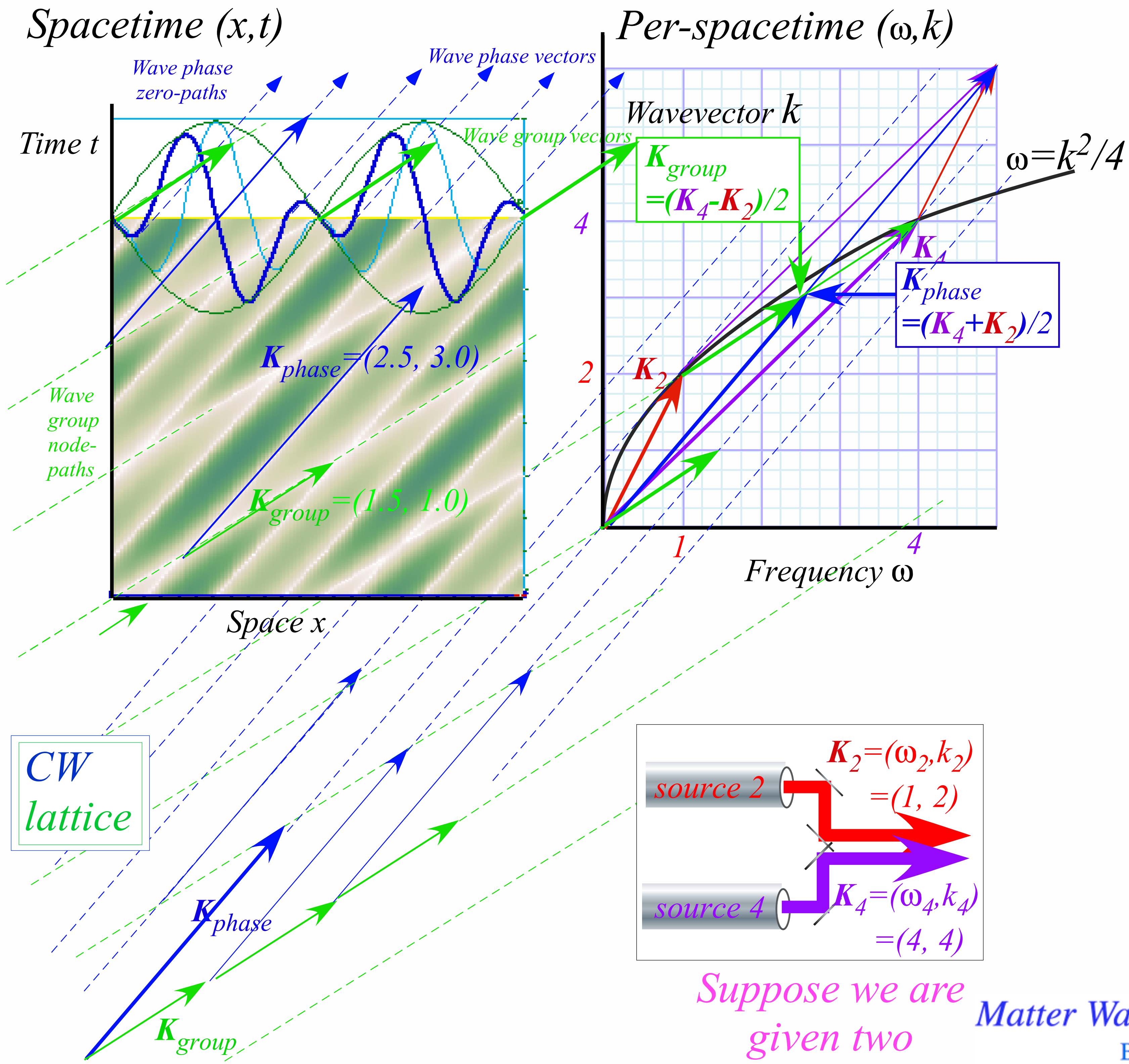
$$\mathbf{K}_{group} = \frac{\mathbf{K}_4 - \mathbf{K}_2}{2}$$

Suppose we are given two “mystery† sources”

†Bohr-Schrodinger “matter-waves”

Continuous Wave
or
Coherent Wave
lattice

Wave coordinates for Bohr Dispersion $\omega = k^2/4$



$$\mathbf{K}_{phase} = \frac{\mathbf{K}_4 + \mathbf{K}_2}{2}$$

$$\mathbf{K}_{group} = \frac{\mathbf{K}_4 - \mathbf{K}_2}{2}$$

Continuous Wave
or
Coherent Wave
lattice

Suppose we are given two "mystery† sources

†Bohr-Schrodinger "matter-waves"

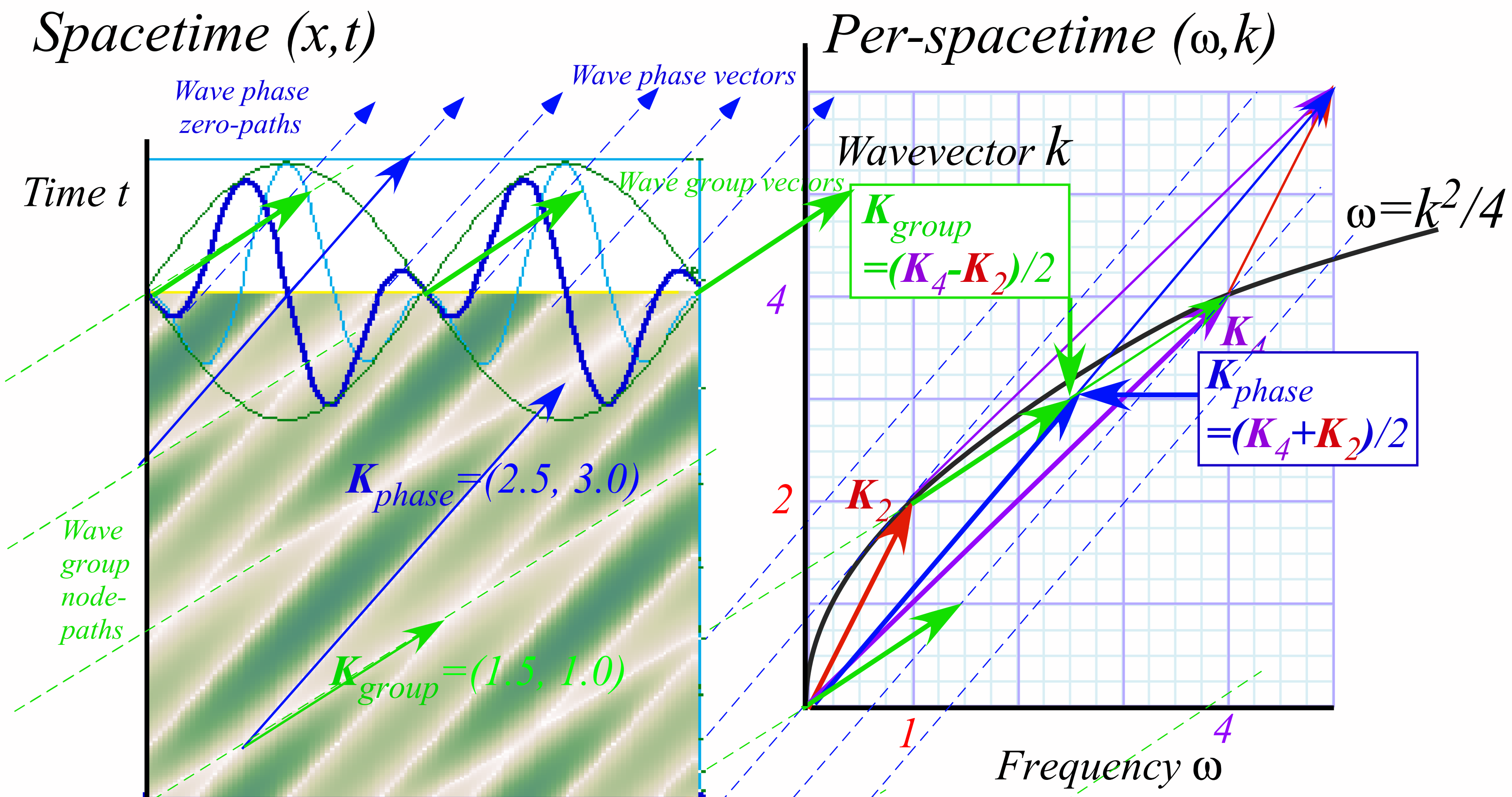
Matter Wave: Bohr-Schrödinger Approximation

Bohr-Schrödinger {Quadratic dispersion}

CW k=+1,+2
CW k=+2,+3
CW k=-1,+2

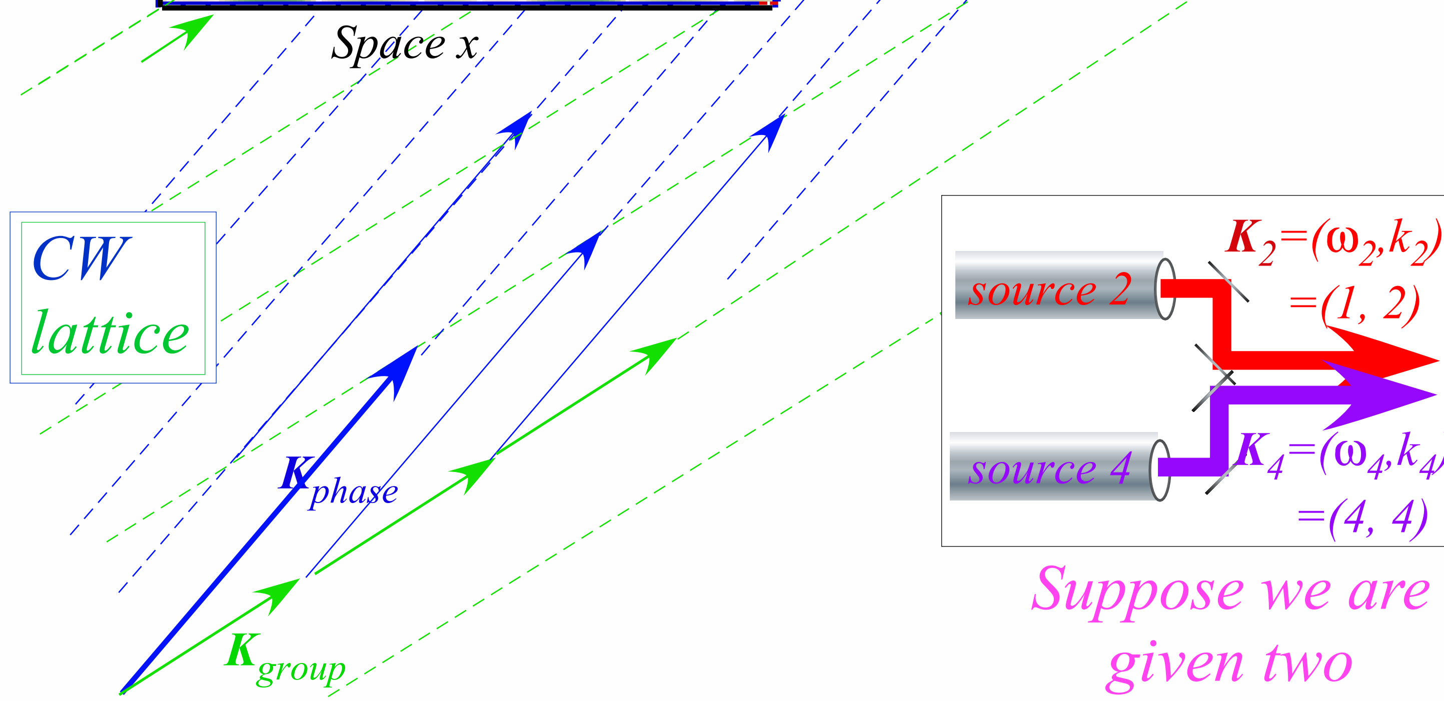
BohrIt Web Simulation
2 Copropigating Bohr Waves
(k = 2,4) ct/4 vs x Plot

Wave coordinates for Bohr Dispersion $\omega = k^2/4$



$$\mathbf{K}_{phase} = \frac{\mathbf{K}_4 + \mathbf{K}_2}{2}$$

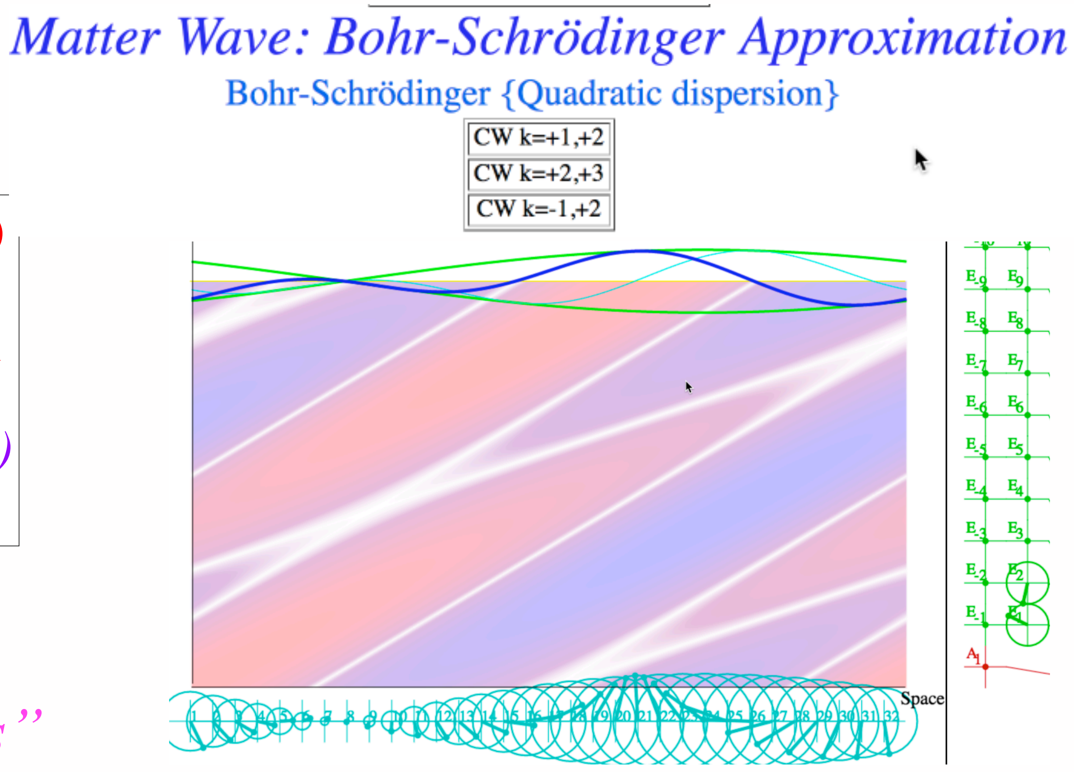
$$\mathbf{K}_{group} = \frac{\mathbf{K}_4 - \mathbf{K}_2}{2}$$



Continuous Wave
or
Coherent Wave
lattice

Suppose we are
given two
"mystery† sources"

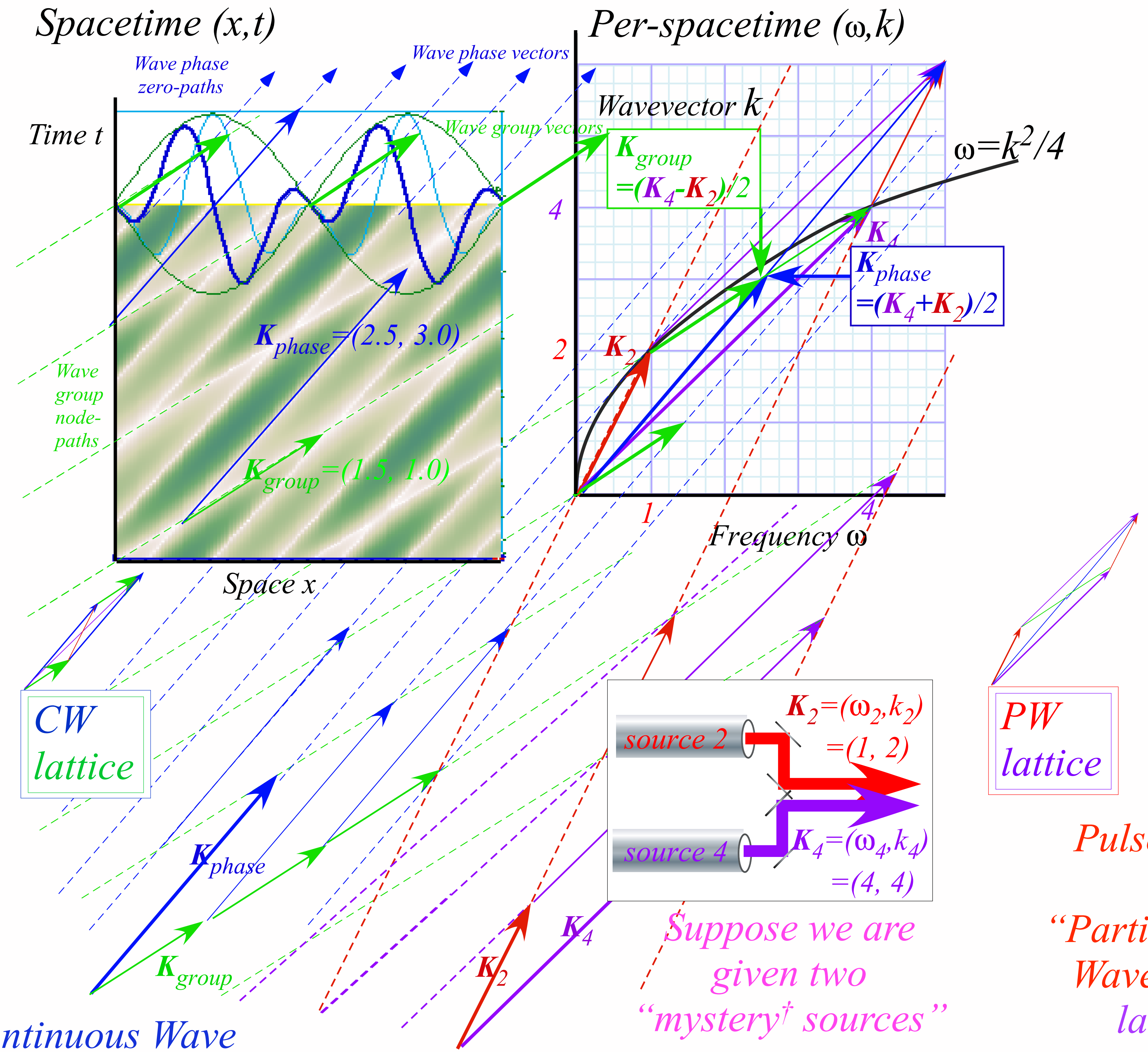
†Bohr-Schrodinger
"matter-waves"


Matter Wave: Bohr-Schrödinger Approximation

Bohr-Schrödinger {Quadratic dispersion}

BohrIt Web Simulation
2 Copropigating Bohr Waves
(k = 2,4) ct/4 vs x Plot

Wave coordinates for Bohr Dispersion $\omega = k^2/4$



Suppose we are given two “mystery[†] *sources”*

Pulse Wave
or
“Particle-like”
Wavepacket
lattice

$$\mathbf{K}_{phase} = \frac{\mathbf{K}_4 + \mathbf{K}_2}{2}$$

$$\mathbf{K}_{group} = \frac{\mathbf{K}_4 - \mathbf{K}_2}{2}$$

$$\mathbf{K}_4 = \mathbf{K}_{phase} + \mathbf{K}_{group}$$

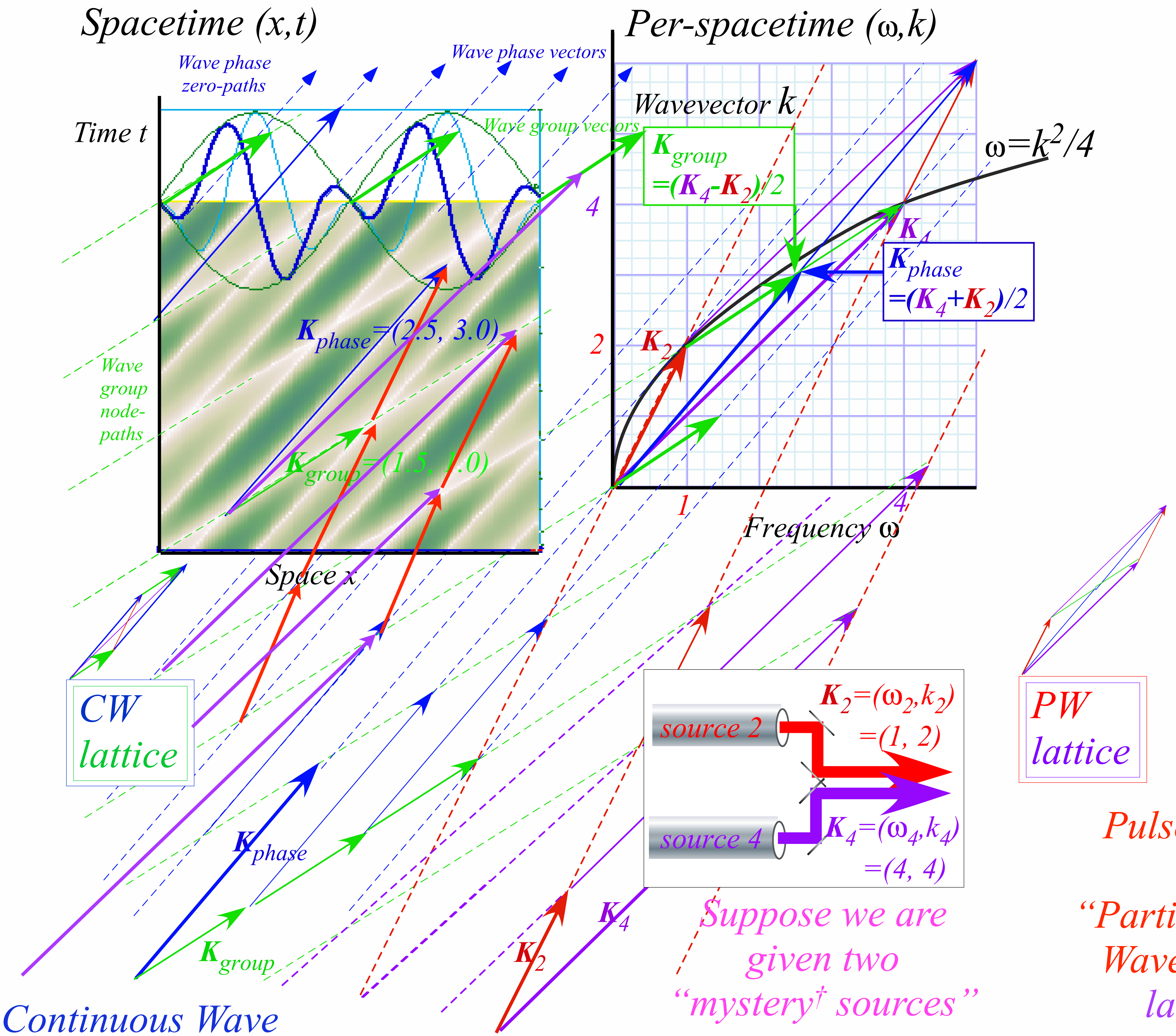
$$\mathbf{K}_2 = \mathbf{K}_{phase} - \mathbf{K}_{group}$$

Continuous Wave
or
Coherent Wave
lattice

[†] *Bohr-Schrodinger “matter-waves”*

BohrIt Web Simulation
2 Copropigating Bohr Waves
(k = 2,4) ct/4 vs x Plot

Wave coordinates for Bohr Dispersion $\omega = k^2/4$



$$\mathbf{K}_{phase} = \frac{\mathbf{K}_4 + \mathbf{K}_2}{2}$$

$$\mathbf{K}_{group} = \frac{\mathbf{K}_4 - \mathbf{K}_2}{2}$$

$$\mathbf{K}_4 = \mathbf{K}_{phase} + \mathbf{K}_{group}$$

$$\mathbf{K}_2 = \mathbf{K}_{phase} - \mathbf{K}_{group}$$

Continuous Wave
or
Coherent Wave
lattice

Suppose we are
given two
"mystery† sources"

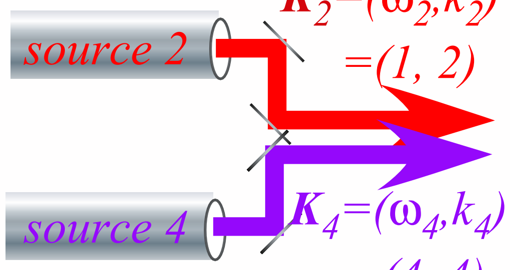
Pulse Wave
or
"Particle-like"
Wavepacket
lattice

†Bohr-Schrodinger
"matter-waves"

BohrIt Web Simulation
2 Copropigating Bohr Waves
(k = 2,4) ct/4 vs x Plot

Wave coordinates for Bohr Dispersion $\omega = k^2/4$

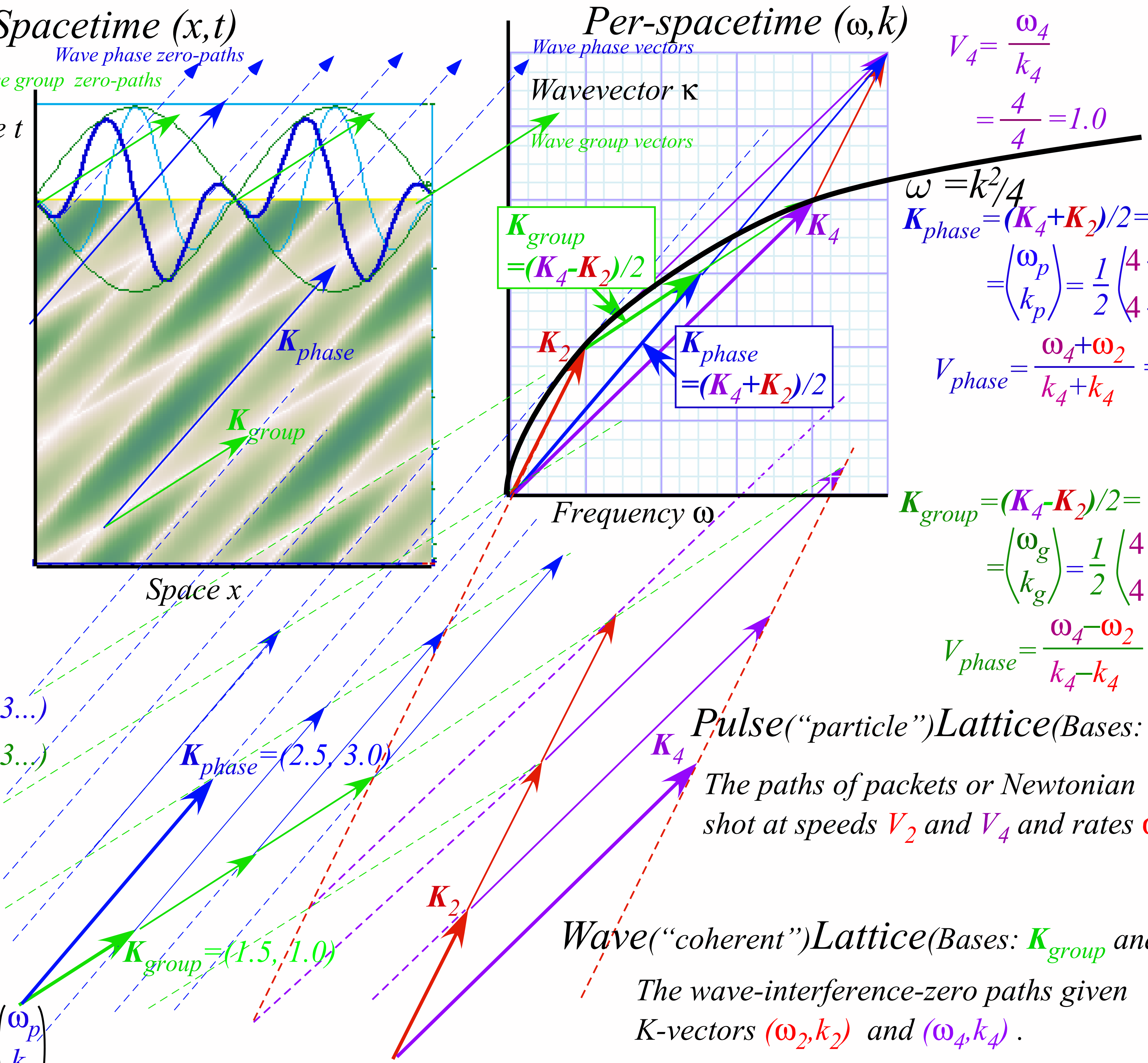
Suppose we are
given two
"mystery† sources"



$K_2=(\omega_2,k_2)$
$=(1,\ 2)$

$K_4=(\omega_4,k_4)$
$=(4,\ 4)$

†Shrodinger matter waves



$V_4 = \frac{\omega_4}{k_4} = \frac{4}{4} = 1.0$

$V_2 = \frac{\omega_2}{k_2} = \frac{1}{2} = 0.$

$$\mathbf{K}_{phase} = (\mathbf{K}_4 + \mathbf{K}_2)/2 = \begin{pmatrix} \omega_4+\omega_2 \\ k_4+k_4 \end{pmatrix}/2 = \begin{pmatrix} \omega_p \\ k_p \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4+1 \\ 4+2 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 3.0 \end{pmatrix}$$

$$V_{phase} = \frac{\omega_4+\omega_2}{k_4+k_4} = \frac{2.5}{3.0} = 0.83$$

$$\mathbf{K}_{group} = (\mathbf{K}_4 - \mathbf{K}_2)/2 = \begin{pmatrix} \omega_4-\omega_2 \\ k_4-k_4 \end{pmatrix}/2 = \begin{pmatrix} \omega_g \\ k_g \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4-1 \\ 4-2 \end{pmatrix} = \begin{pmatrix} 1.5 \\ 1.0 \end{pmatrix}$$

$$V_{phase} = \frac{\omega_4-\omega_2}{k_4-k_4} = \frac{1.5}{1.0} = 1.5$$

Wave("coherent")Lattice

Bases: $\mathbf{K}_{group}$ and $\mathbf{K}_{phase}$

$k_p x - \omega_p t = n_p = N_p\ /2 \quad (N_p=\pm1,\pm3...)$

$k_g x - \omega_g t = n_g = N_g\ /2 \quad (N_g=\pm1,\pm3...)$

$$\begin{pmatrix} k_p & -\omega_p \\ k_g & -\omega_g \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} n_p \\ n_g \end{pmatrix}$$

$$\begin{pmatrix} x \\ t \end{pmatrix} = \frac{\begin{pmatrix} -\omega_g & \omega_p \\ -k_g & k_p \end{pmatrix}\begin{pmatrix} n_p \\ n_g \end{pmatrix}}{\omega_p k_g - \omega_g k_p} = \frac{-n_p\begin{pmatrix} \omega_g \\ k_g \end{pmatrix} + n_g\begin{pmatrix} \omega_p \\ k_p \end{pmatrix}}{\omega_p k_g - \omega_g k_p} = \frac{n_p}{D}\mathbf{K}_{group} + \frac{n_g}{D}\mathbf{K}_{phase}$$

Pulse("particle")Lattice(Bases: $\mathbf{K}_2$ and $\mathbf{K}_4$)

The paths of packets or Newtonian "corpuscles
shot at speeds V_2 and V_4 and rates ω_2 and ω_2

Wave("coherent")Lattice(Bases: $\mathbf{K}_{group}$ and $\mathbf{K}_{phase}$)

The wave-interference-zero paths given
K-vectors (ω_2,k_2) and (ω_4,k_4) .

BohrIt Web Simulation
2 Copropigating Bohr Waves
(k = 2,4) ct/4 vs x Plot

1^{st} Step: Abelian symmetry analysis Expand C_6 symmetric **H** by C_6 group product table

Regular representation and coupling parameters $\{r_0,r_1,r_2,r_3,r_4,r_5\}$ and Fourier dispersion

2^{nd} Step: Find **H** eigenfunctions by spectral resolution of $C_6 =\{\mathbf{1}=\mathbf{r}^0,\mathbf{r}^1,\mathbf{r}^2,\mathbf{r}^3,\mathbf{r}^4,\mathbf{r}^5\}$

Character tables of C_2, C_3, C_4, C_5,…. C_{144}

3^{rd} Step: Dispersion functions and eigenvalues for varyious coupling parameter systems

Ortho-complete eigenvalue/parameter relations

Gauge shifts due to complex coupling

Wave dynamics of phase, mean phase, and group velocity by Expo-Cosine identity

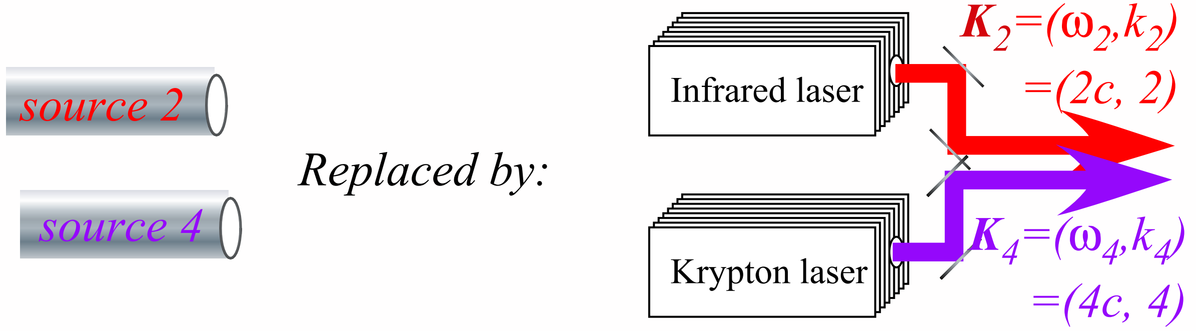
Relating space-time (x,t) and per-space-time (k,ω)

- *Wave coordinates*
- *Pulse-waves (PW) vs Continuous-waves (CW)*
- ➡ *Wave coordinates for Linear Dispersion*
- *Wave coordinates for Bohr-Schrodinger Dispersion*
- ➡ *Einstein-Lorentz-Minkowski laser coordinates*

(Back to) Wave coordinates for Linear Dispersion

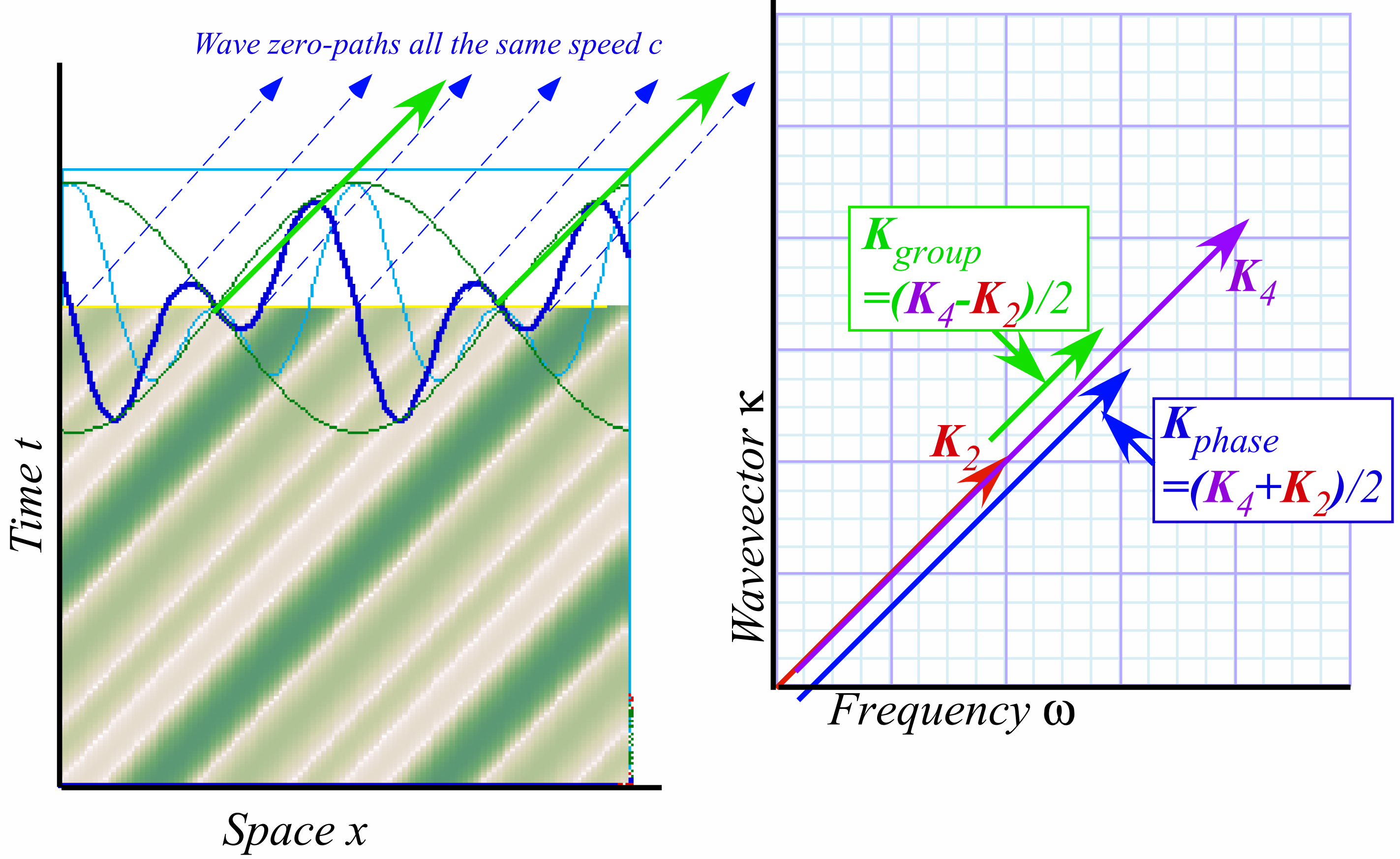
For co-propagating laser * sources...
...the wave-coordinate lattice collapses to lines..

*simple linear
ω=ck dispersion



(a) Spacetime (x,t)

(b) Per-spacetime (ω,k)



But, for counter-propagating laser sources...
...the wave coordinate lattice is the Lorentz-Einstein-Minkowski frame!!

1st Step: Abelian symmetry analysis Expand C_6 symmetric **H** *by C_6 group product table*

Regular representation and coupling parameters $\{r_0,r_1,r_2,r_3,r_4,r_5\}$ and Fourier dispersion

2nd Step: Find **H** *eigenfunctions by spectral resolution of $C_6 = \{\mathbf{1}=\mathbf{r}^0,\mathbf{r}^1,\mathbf{r}^2,\mathbf{r}^3,\mathbf{r}^4,\mathbf{r}^5\}$*

Character tables of C_2, C_3, C_4, C_5,…. C_{144}

3rd Step: Dispersion functions and eigenvalues for varyious coupling parameter systems

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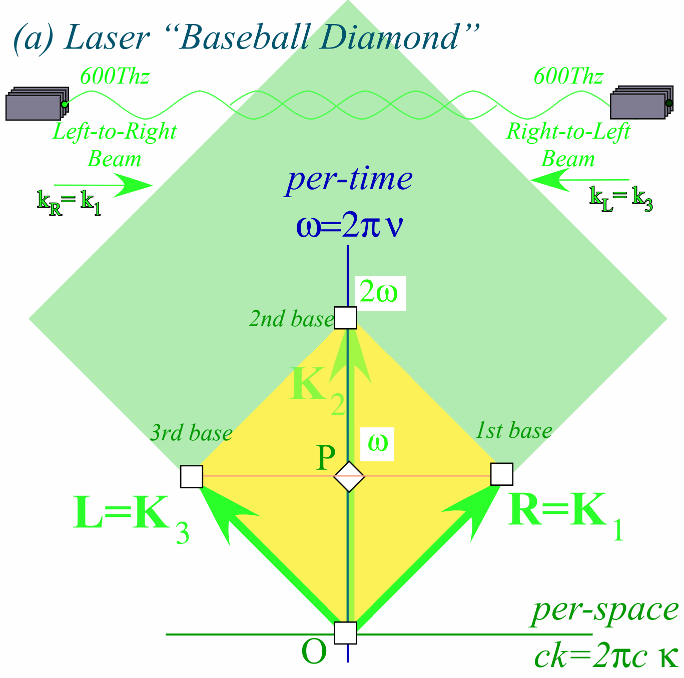
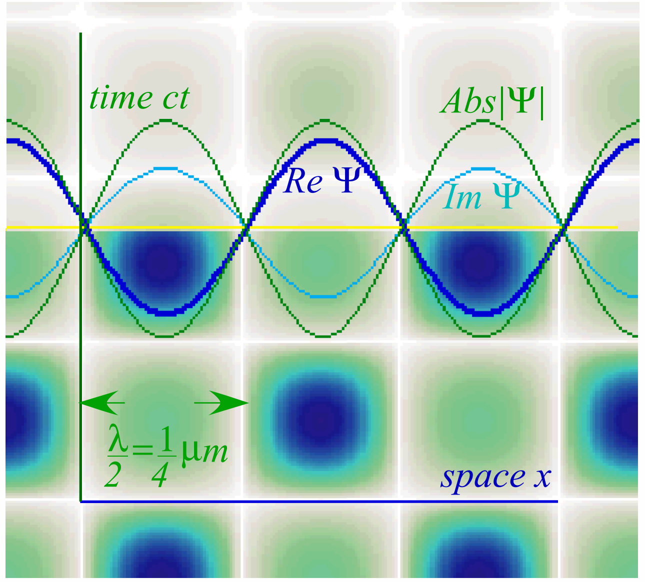
- *Wave coordinates for Bohr-Schrodinger Dispersion*
- → *Einstein-Lorentz-Minkowski laser coordinates*

(Back to) Wave coordinates for Linear Dispersion
u=0 space-time coordinates

(a) Laser "Baseball Diamond"



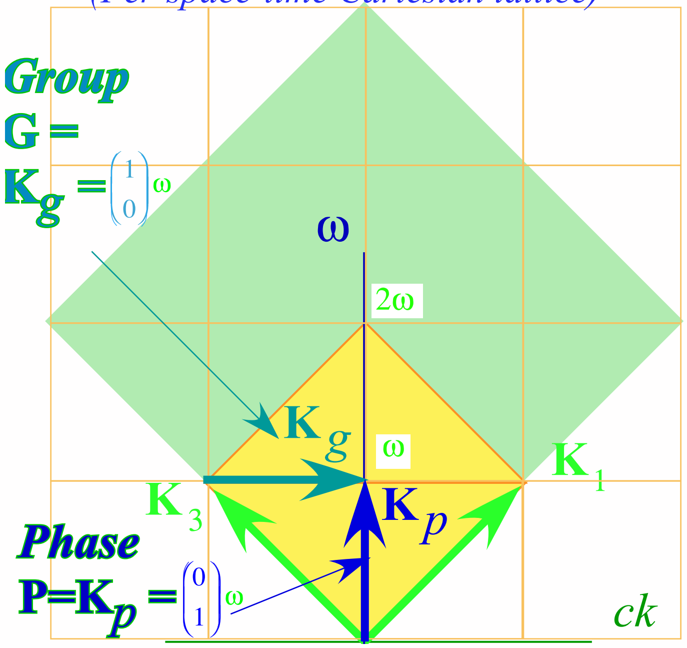
(c) Laser Coherent Wave (CW) paths
(Space-time Cartesian grid)



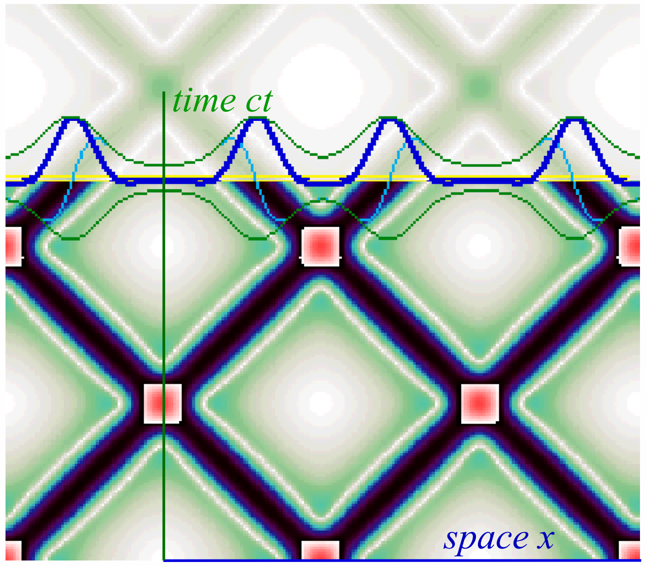
[BohrIt Web Simulation
2 CW *ct vs x* Plot
(ck = ±2)]()

[BohrIt Web Simulation
2 CW *ct vs x* Plot
(ck = ±2)]()

(b) Laser group and phase wavevectors
(Per-space-time Cartesian lattice)



(d) Laser Pulse Wave (PW) Paths
(Space-time Diamond grid)

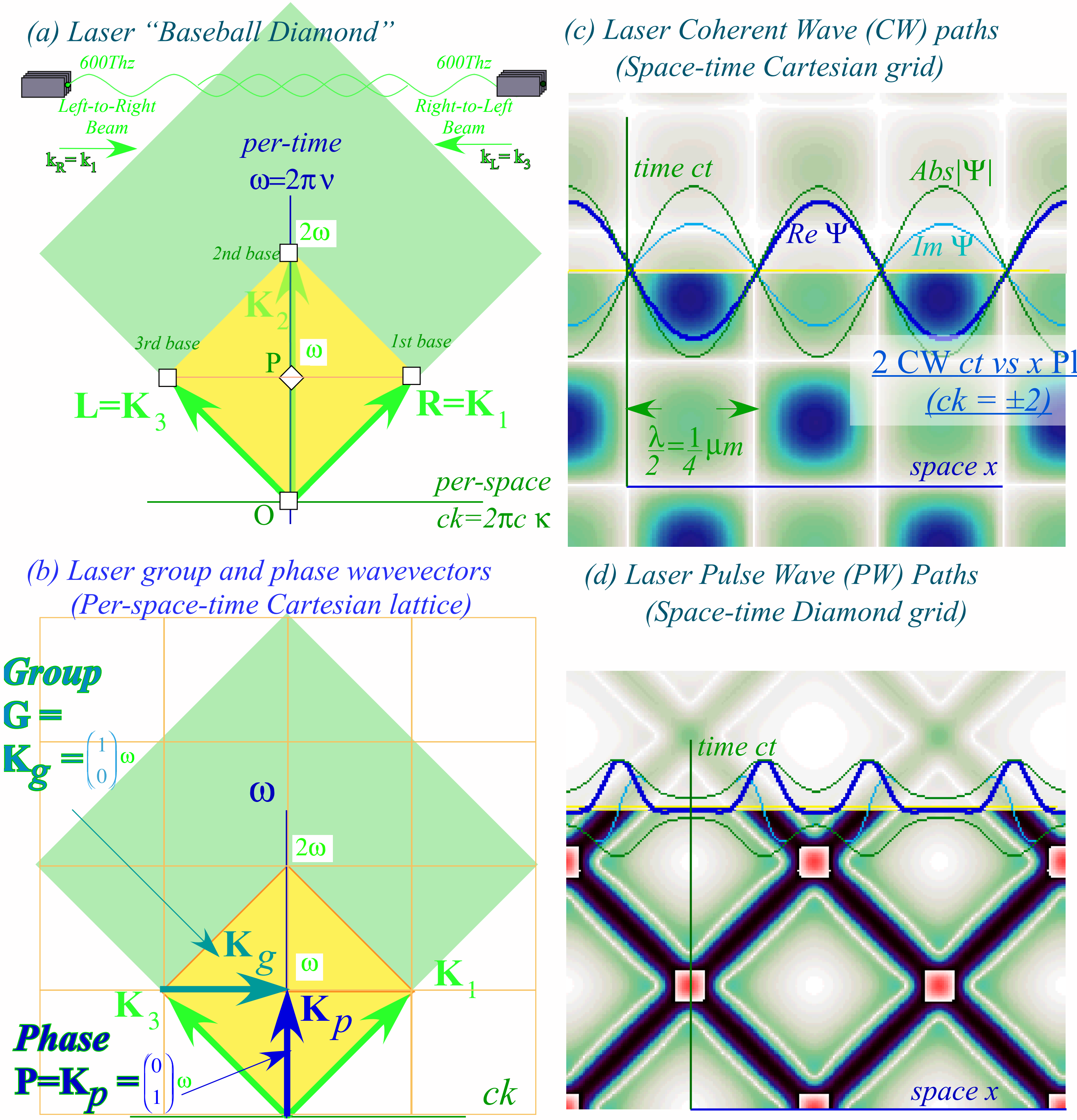


BohrIt Web Simulation
2 PW *ct vs x* Plot
(k mod 2 = 0)

u=0 space-time pulse waves

CMwith a BANG! Fig. 8.2.1

(Back to) Wave coordinates for Linear Dispersion u=0 space-time coordinates



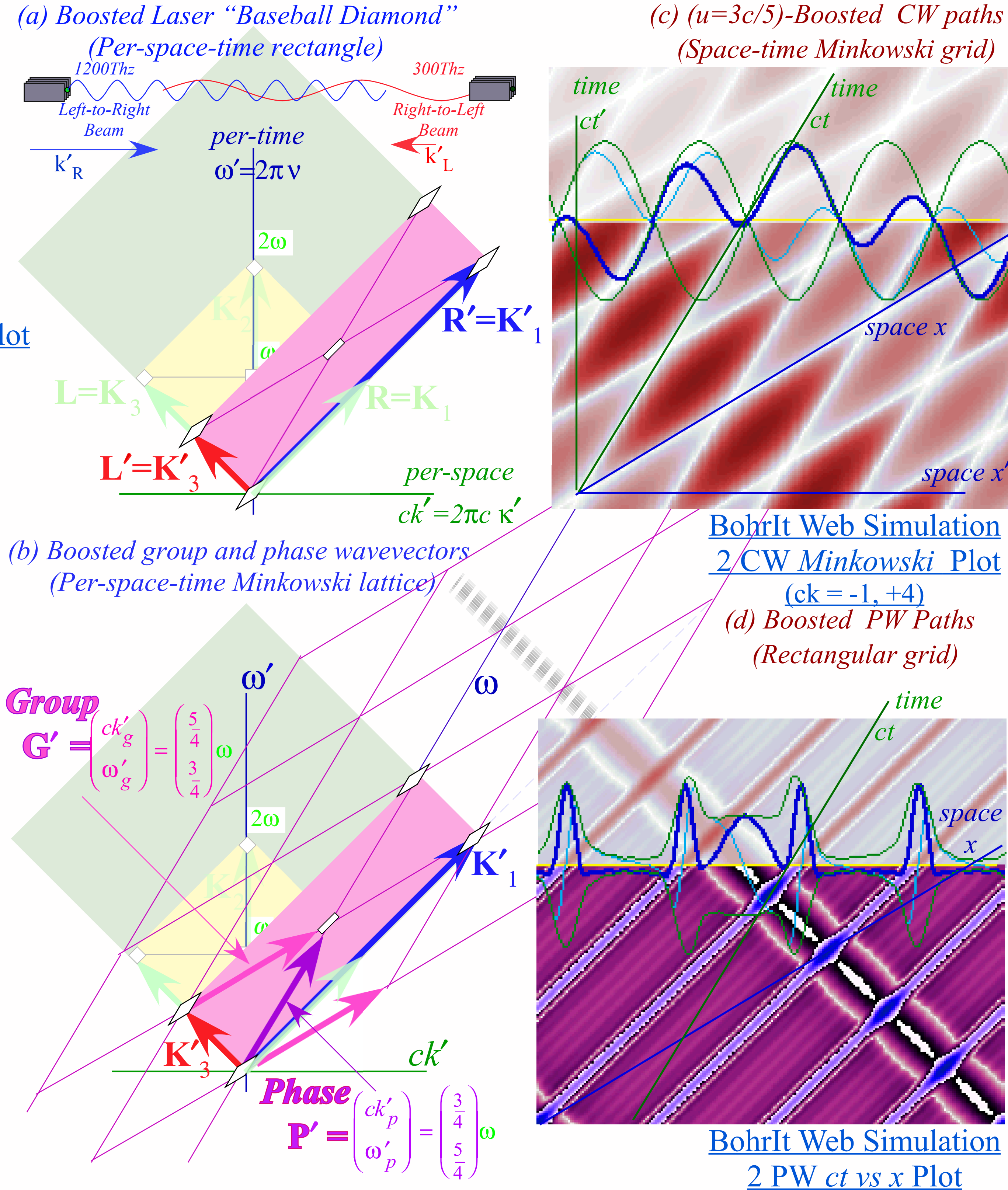
2 CW *ct vs x* Plot
(ck = ±2)

BohrIt Web Simulation
2 PW *ct vs x* Plot
(k mod 2 = 0)

u=0 space-time pulse waves

CMwith a BANG! Fig. 8.2.1

Doppler shifted by factor of 2 Gives u=3c/5 Einstein-Lorentz-Minkowski coordinates



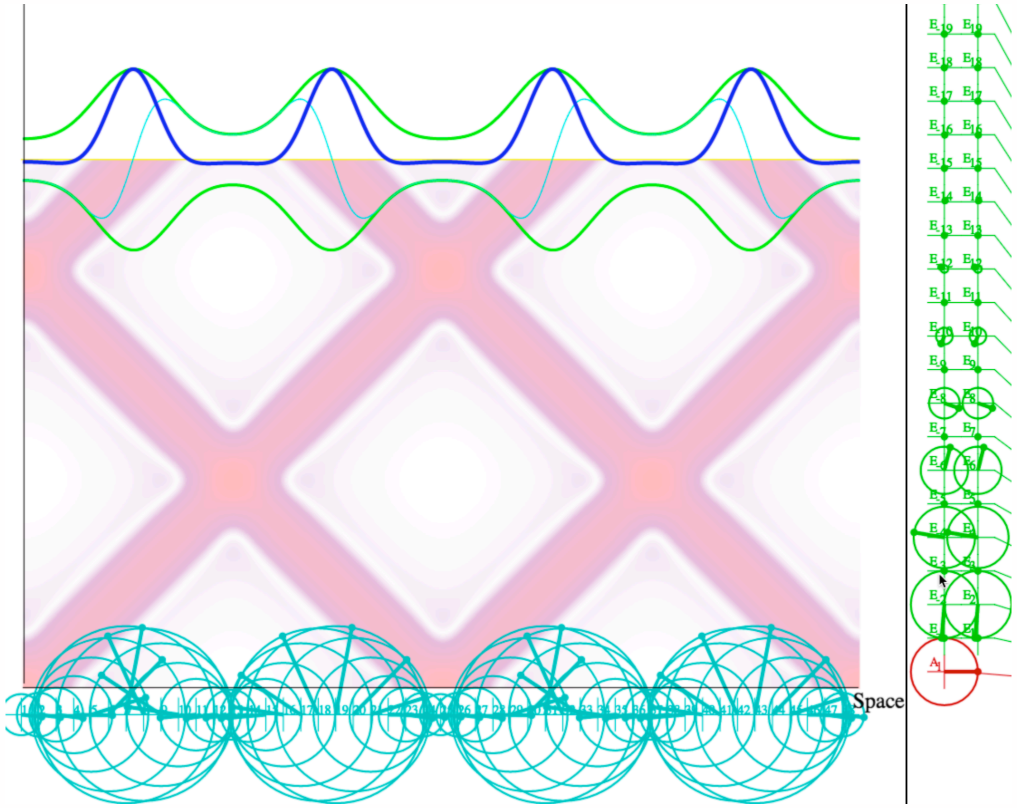
BohrIt Web Simulation
2 CW Minkowski Plot
(ck = -1, +4)

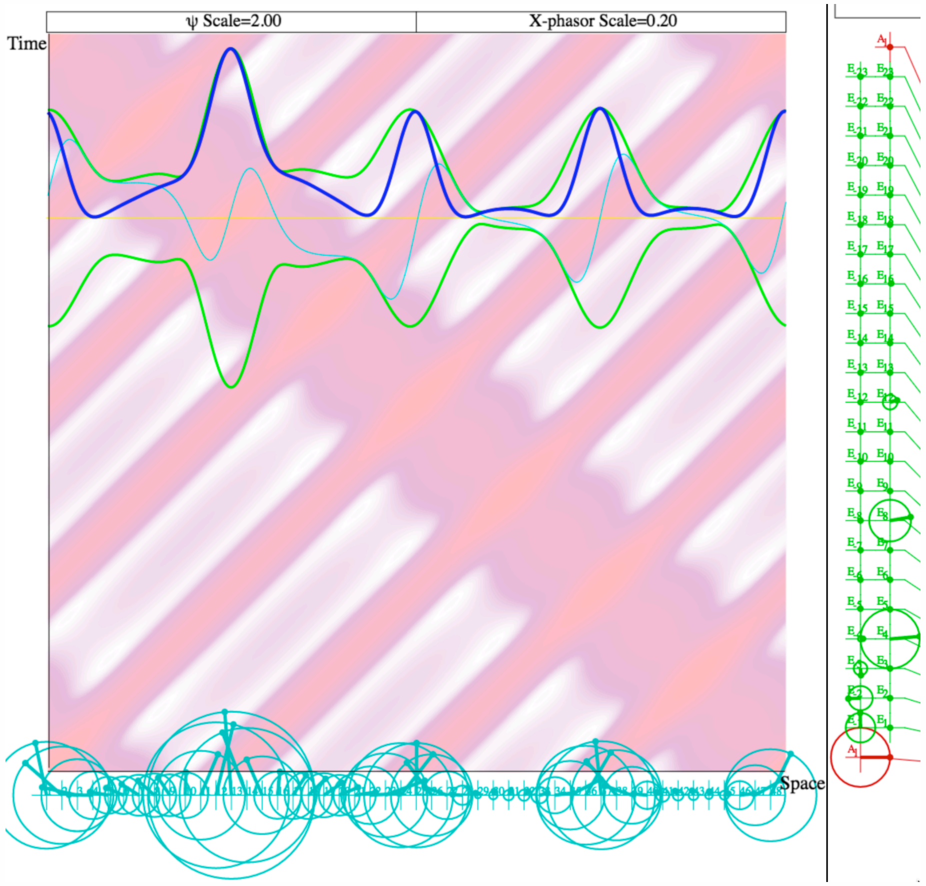
BohrIt Web Simulation
2 PW *ct vs x* Plot
(k = -n, 4n, n natural)

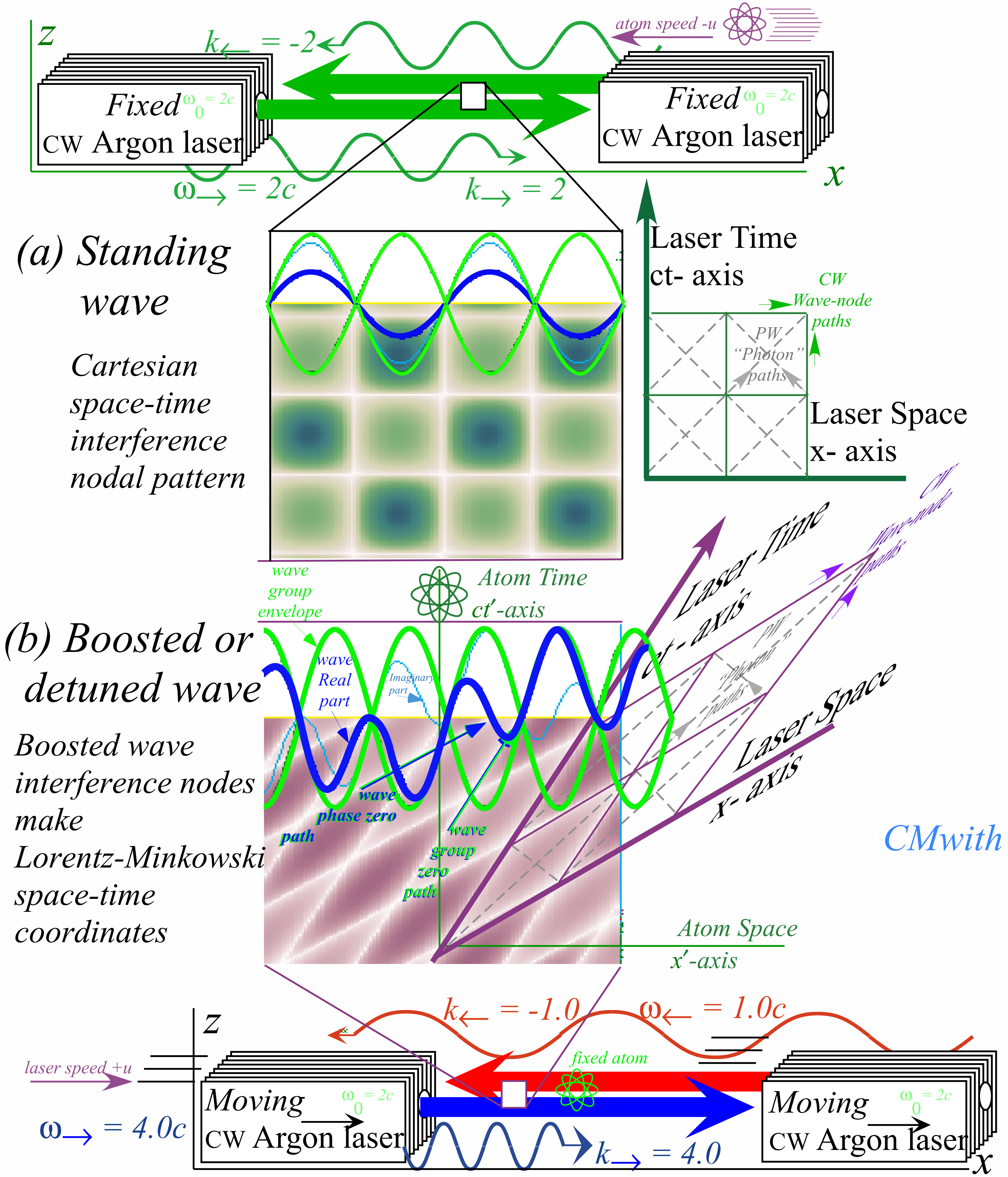
u=3c/5 space-time pulse waves

CMwith a BANG! Fig. 8.2.2

http://www.uark.edu/ua/modphys/markup/BohrItWeb.html







(a) Standing wave

Cartesian space-time interference nodal pattern

(b) Boosted or detuned wave

Boosted wave interference nodes make Lorentz-Minkowski space-time coordinates

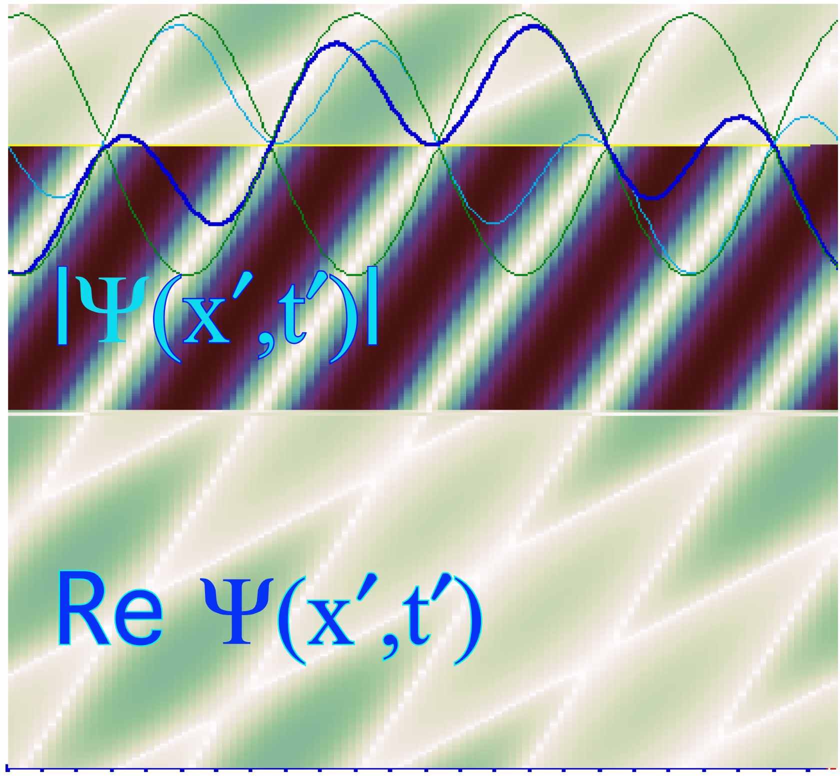
[BohrIt Web Simulation
2 CW *ct vs x* Plot
(ck = ±2)]

CMwith a BANG! Fig. 8.2.3

[BohrIt Web Simulation
2 CW *Minkowski* Plot
(ck = -1, +4)]

Phase lines may not show up in Magnitude ($|\Psi(x',t')|$) orProbability ($\Psi(x',t')^*\Psi(x',t')$) plots.

Unbiased $\Psi=\psi_{-1}+\psi_{+4}$

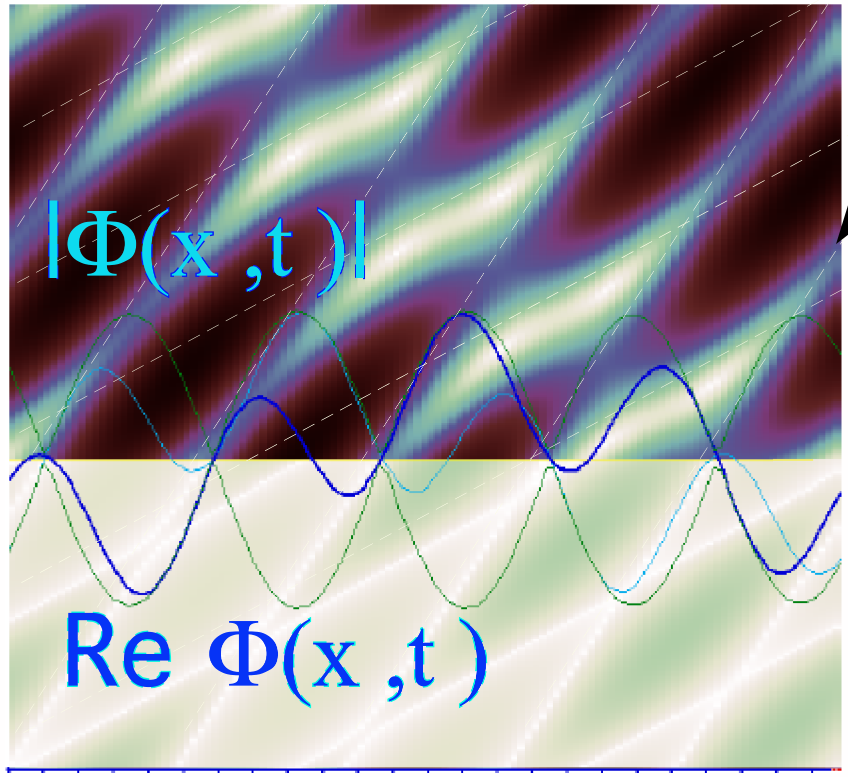


Only the group wave paths appear

The "inside phase" $e^{i[]}$ *gets killed in* ($\Psi(x',t')^*\Psi(x',t')$)
because $(e^{i[]})^* = e^{-i[]}$
and $(e^{i[]})^* \bullet e^{i[]} = 1$

Phase structure begins to show up if ground-state (k=0) component is added.

DC biased $\Phi=\psi_0+\Psi$

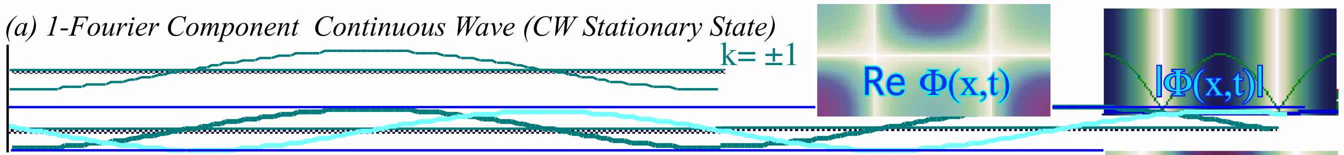


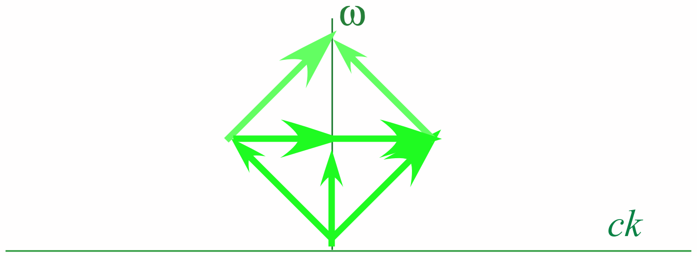
Group and phase paths begin to appear

BohrIt Web Simulation
3 CW *Minkowski* Plot
(ck = -1, ,0,3)

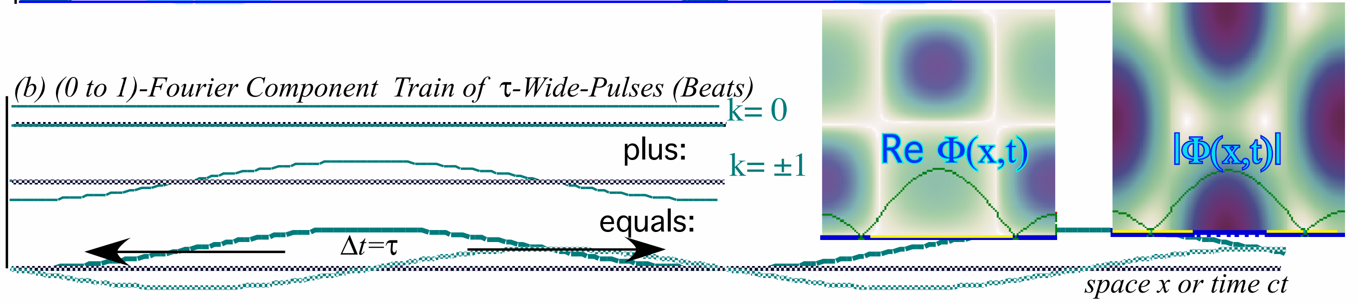
Each counter-propagating pair of beams makes a wave-interference-lattice. "Packets" or pulses made by adding more pairs. Finally, pulse lattice apppears.

(a) 1-Fourier Component Continuous Wave (CW Stationary State)

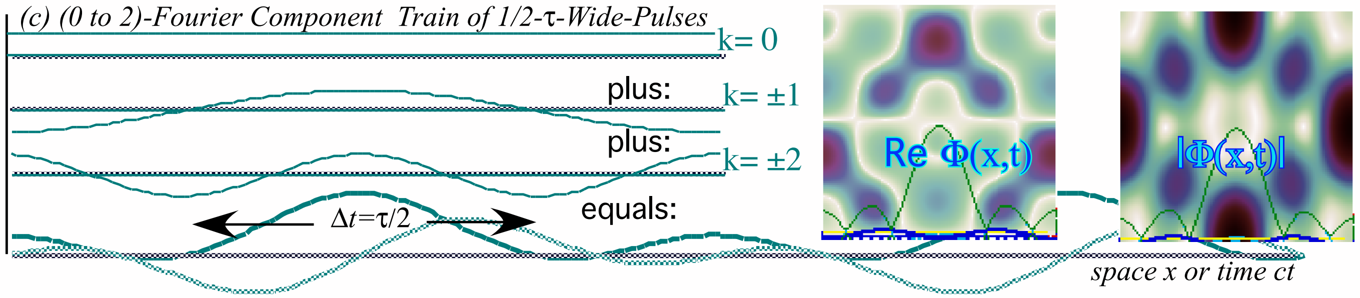


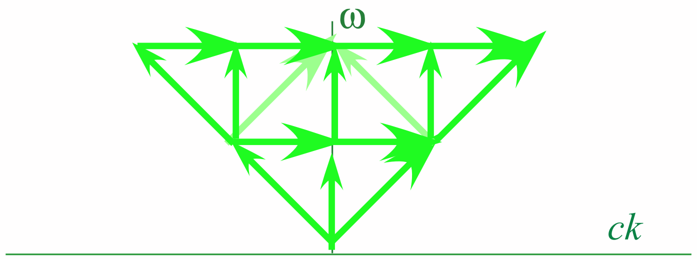


(b) (0 to 1)-Fourier Component Train of τ-Wide-Pulses (Beats)

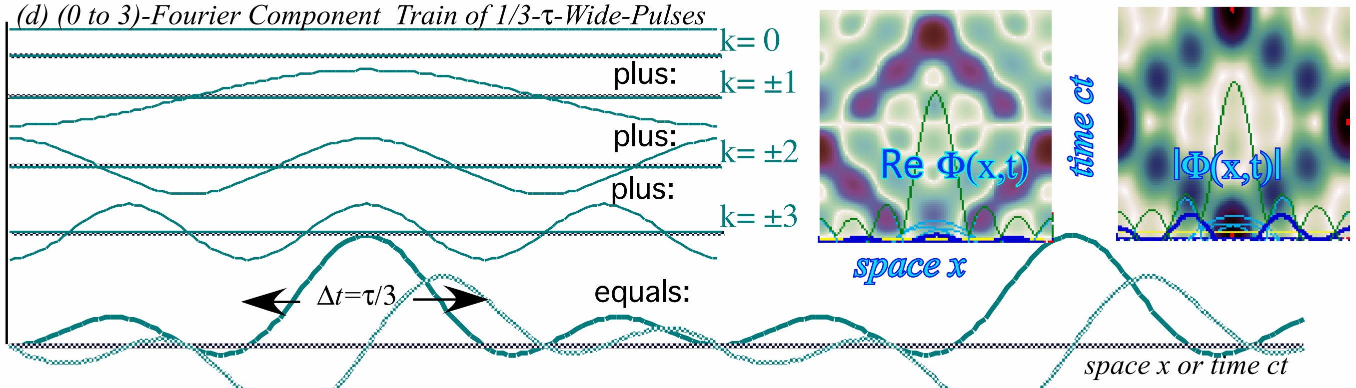


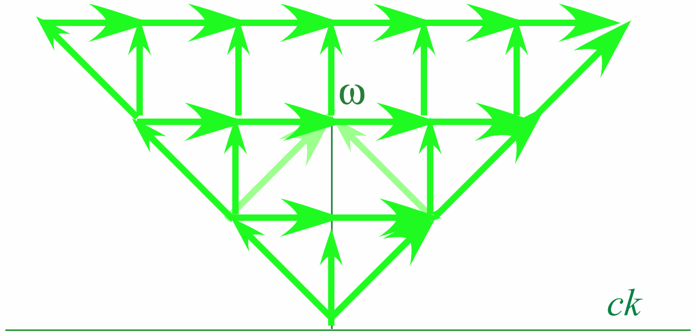
(c) (0 to 2)-Fourier Component Train of 1/2-τ-Wide-Pulses



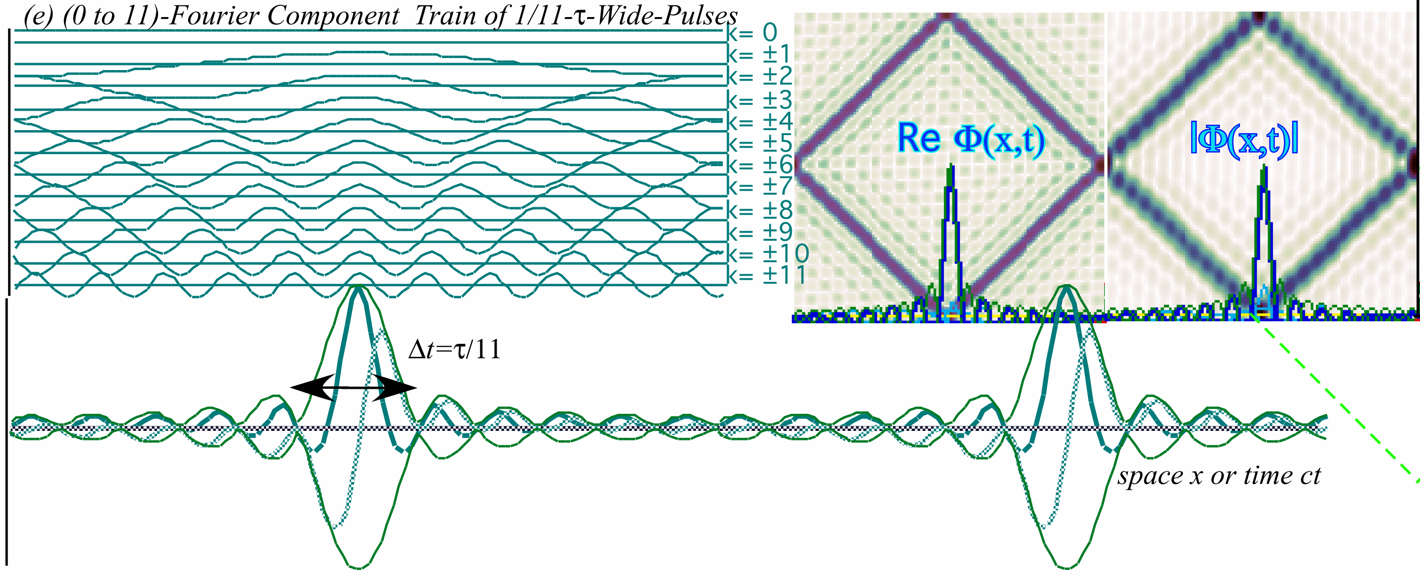


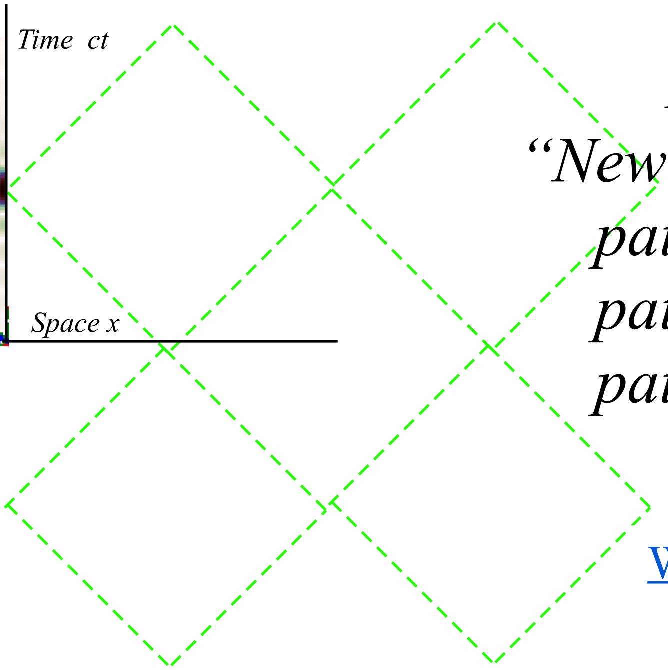
(d) (0 to 3)-Fourier Component Train of 1/3-τ-Wide-Pulses





(e) (0 to 11)-Fourier Component Train of 1/11-τ-Wide-Pulses





It's
"Newton-like"
patooey!
patooey!
patooey!
...

[WaveIt Web Simulation](#)
[12 CW (k=0,...11)](#)

Polygonal geometry of U(2)⊃C_N character spectral function
Algebra
Geometry

Introduction to wave dynamics of phase, mean phase, and group velocity
Expo-Cosine identity
Relating space-time and per-space-time
- *Wave coordinates*
- *Pulse-waves (PW) vs Continuous -waves (CW)*

Introduction to C_N beat dynamics and "Revivals" due to Bohr-dispersion
- → *∞-Square well PE versus Bohr rotor*
- *SinNx/x wavepackets bandwidth and uncertainty*
  - *SinNx/x revivals*
- *Gaussian wave-packet bandwidth and uncertainty*
  - *Gaussian revivals*
- *Farey-Sums and Ford-products*
- *Phase dynamics*

∞-Square well PE versus Bohr rotor

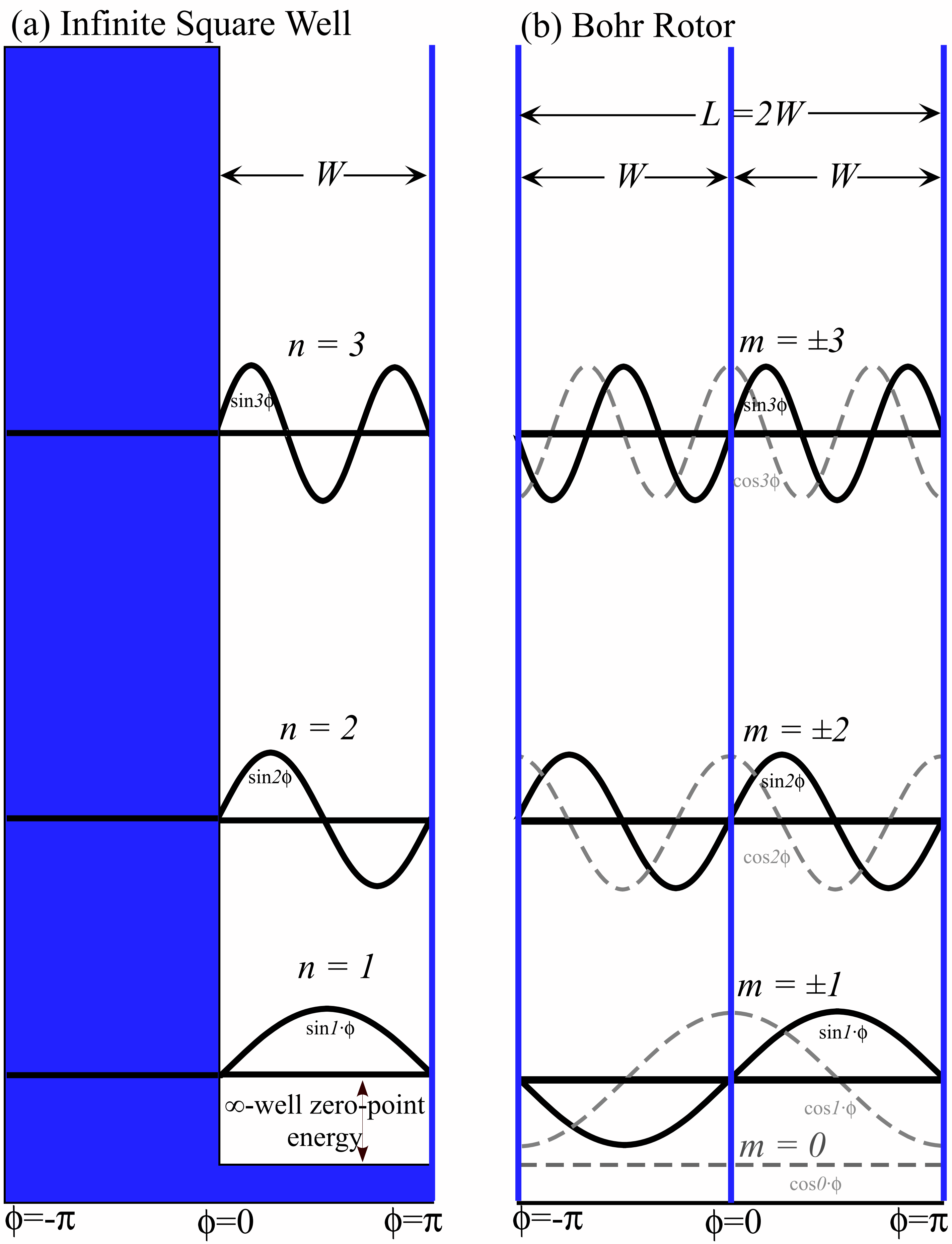
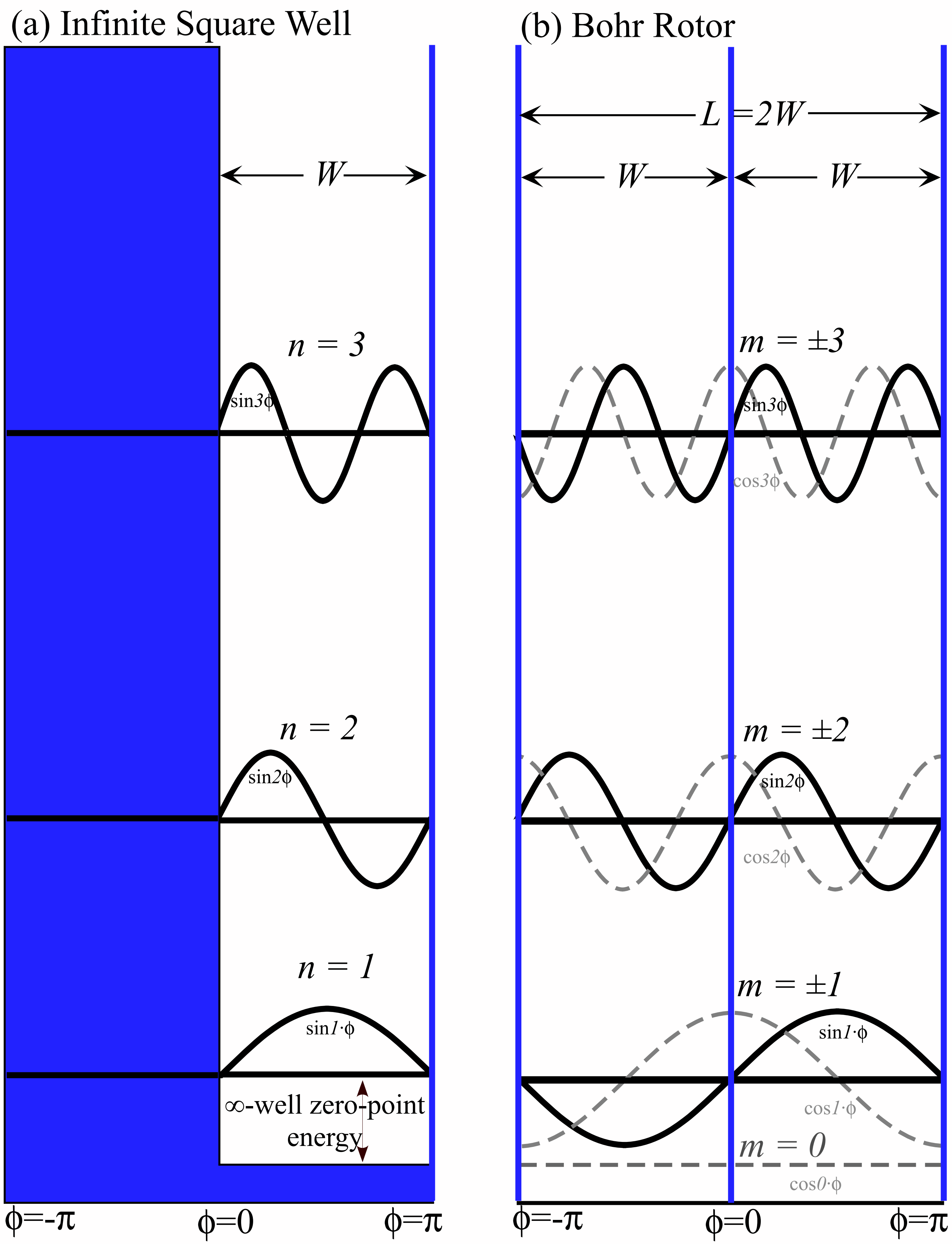


Fig. 12.2.6 Comparison of eigensolutions for (a) Infinite square well, and (b) Bohr rotor.

$m=0, \pm1, \pm2, \pm3$,...are momentum quanta in wavevector formula: $k_m=2\pi m/L$

($k_m=m$ if: $L=2\pi$)

∞-Square well PE versus Bohr rotor



(a) Infinite Square Well

(b) Bohr Rotor

Fig. 12.2.6 Comparison of eigensolutions for (a) Infinite square well, and (b) Bohr rotor.

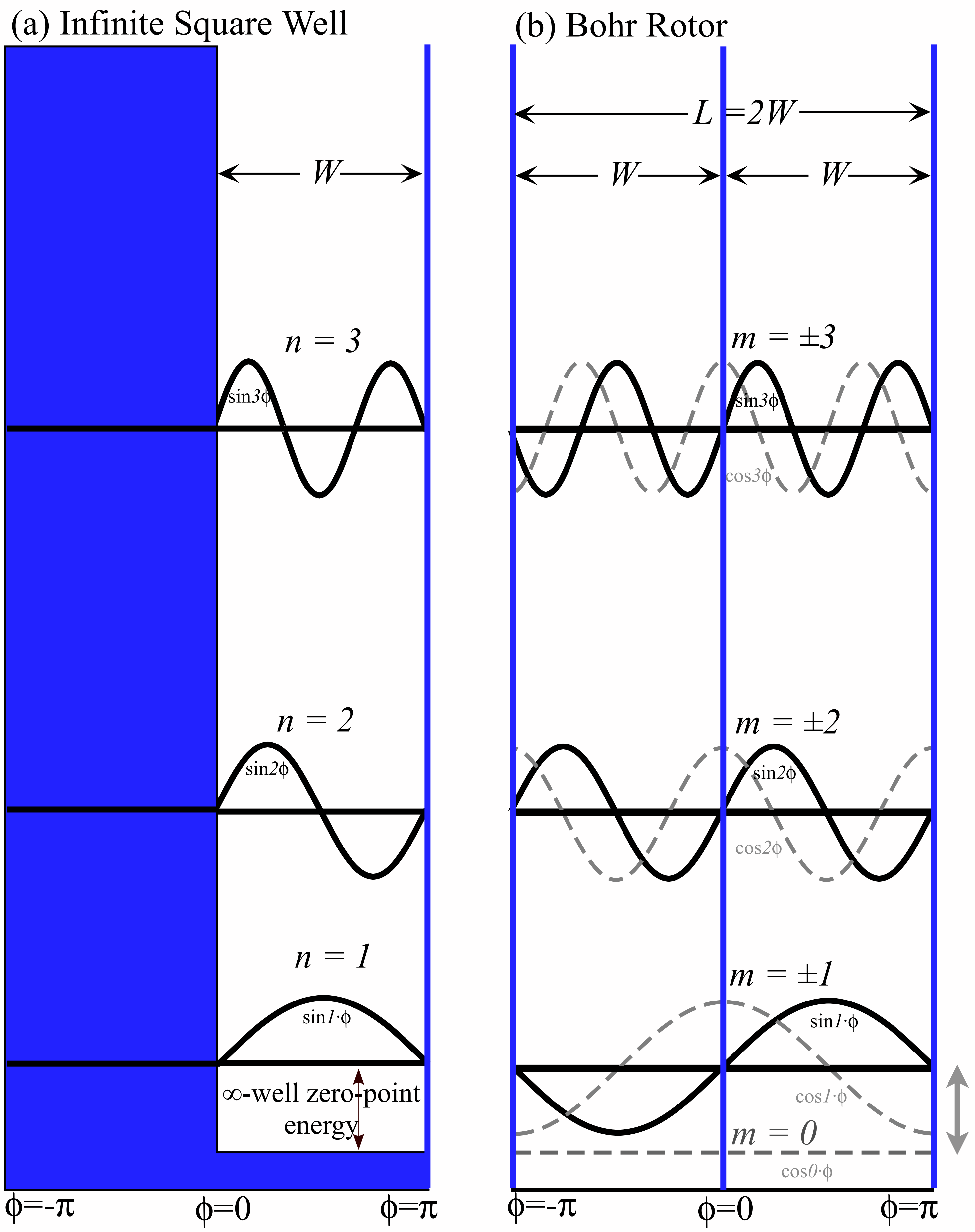
$m=0, \pm 1, \pm 2, \pm 3,\ldots$ are momentum quanta in wavevector formula: $k_m=2\pi m/L$

$(k_m=m \quad \text{if: } L=2\pi)$

$$E_m = (\hbar k_m)^2/2M = m^2[h^2/2ML^2]$$

$$= m^2\, h\upsilon_1 = m^2\, \hbar\omega_1$$

∞-Square well PE versus Bohr rotor



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$$E_m=(\hbar k_m)^2/2M = m^2[h^2/2ML^2]$$
$$= m^2 h\upsilon_1 = m^2 \hbar\omega_1$$

fundamental Bohr ∠−frequency

$\omega_1=2\pi\upsilon_1$

lowest *transition (beat) frequency*

$\upsilon_1 =(E_1-E_0)/h$ *(E_0 is defined as zero)*

Polygonal geometry of U(2)⊃C_N character spectral function

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- ➡ *SinNx/x wavepackets bandwidth and uncertainty*
  - *SinNx/x explosion and revivals*
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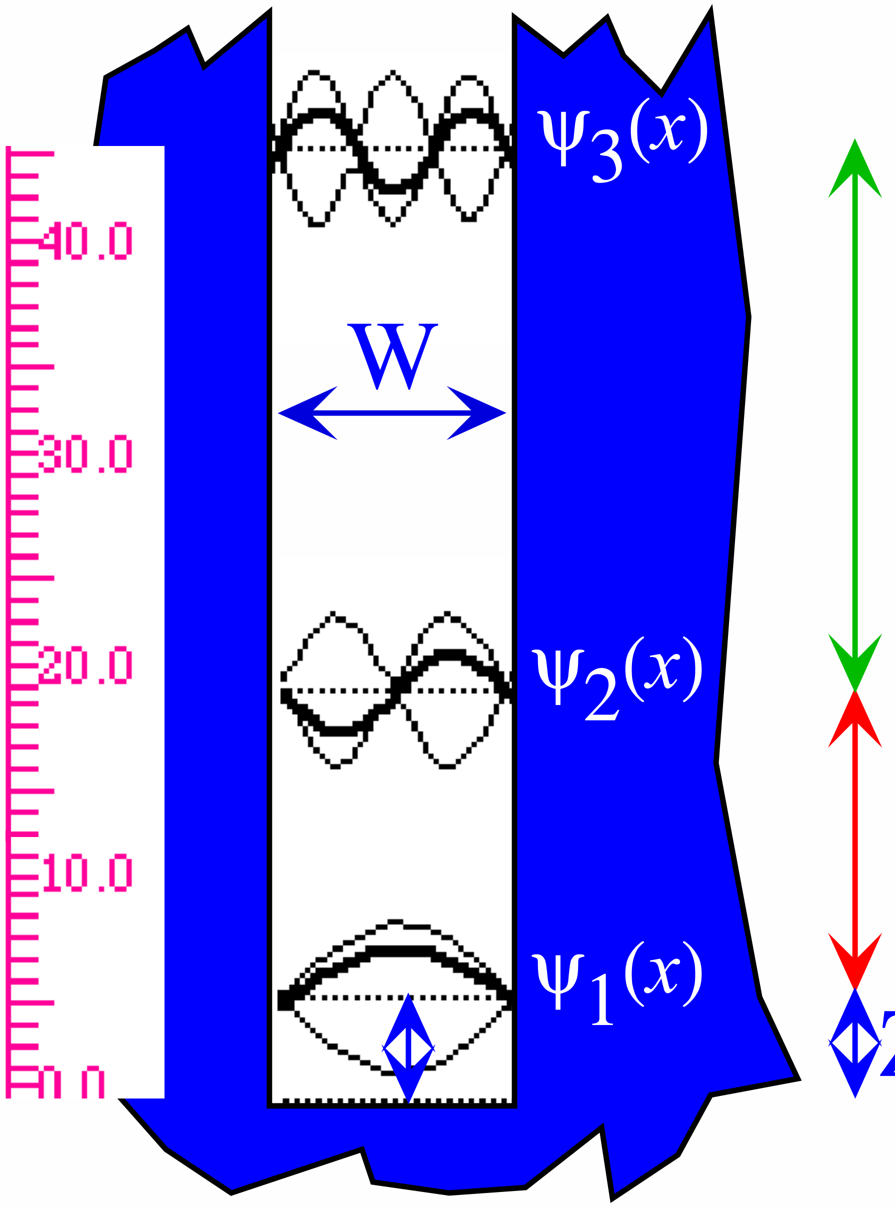
∞-Square well PE versus Bohr rotor



n=3 $3^2\varepsilon_1$

n=2 $2^2\varepsilon_1$

n=1 $1^2\varepsilon_1$

2nd transition energy $5\varepsilon_1$

1st transition ("beat") energy $3\varepsilon_1$

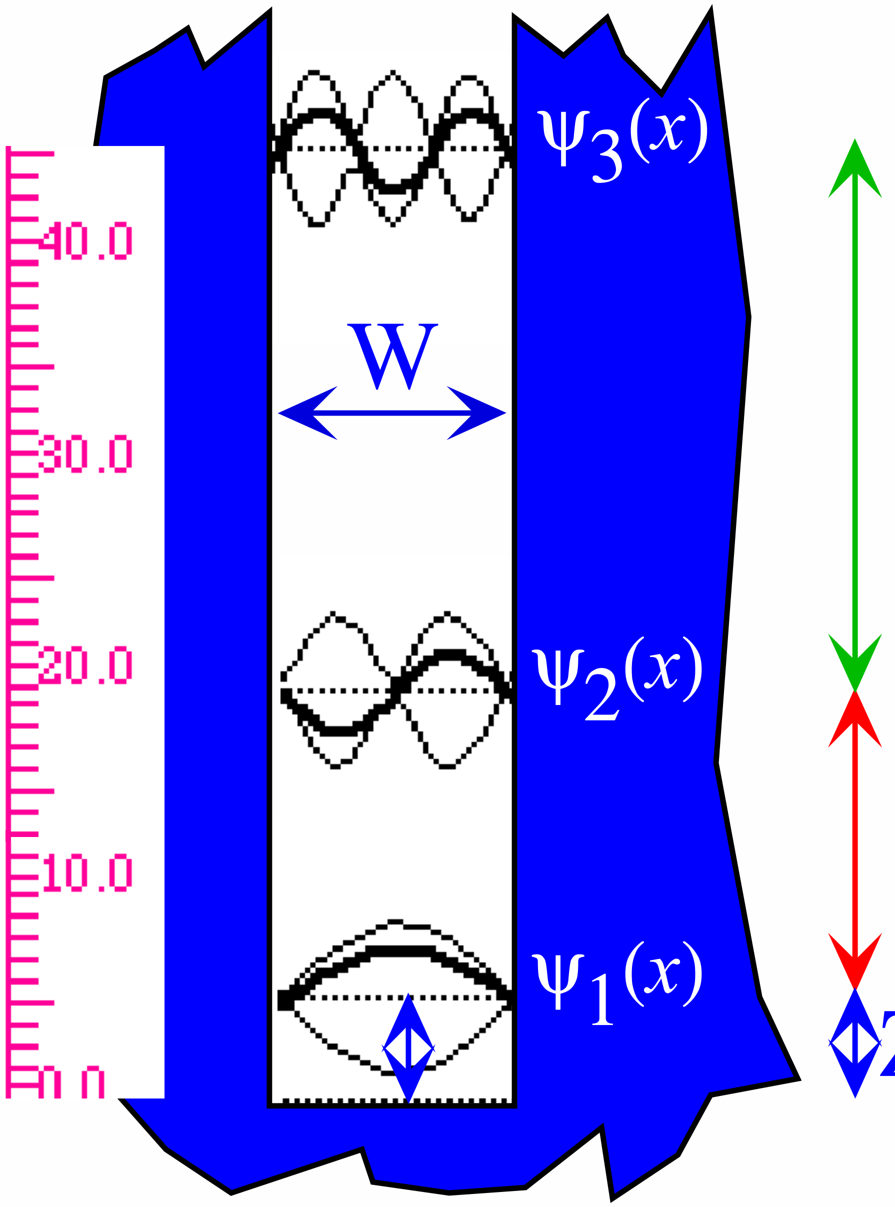
Zero-point energy $\varepsilon_1 = \frac{h^2}{8MW^2}$

∞-Square well PE versus Bohr rotor

$kW = n\pi \quad or: \ k = n\pi/W$

$$\langle x|\varepsilon_n\rangle = \psi_n(x) = A\sin(k_n x) = A\sin\left(\frac{n\pi x}{W}\right) \quad (n=1,2,3,\ldots\infty)$$

$$\varepsilon_n = \frac{\hbar^2}{2M}k^2 = \frac{\hbar^2 n^2 \pi^2}{2MW^2} = \left(1^2, 2^2, 3^2, \ldots\text{or } n^2\right)\frac{h^2}{8MW^2}$$



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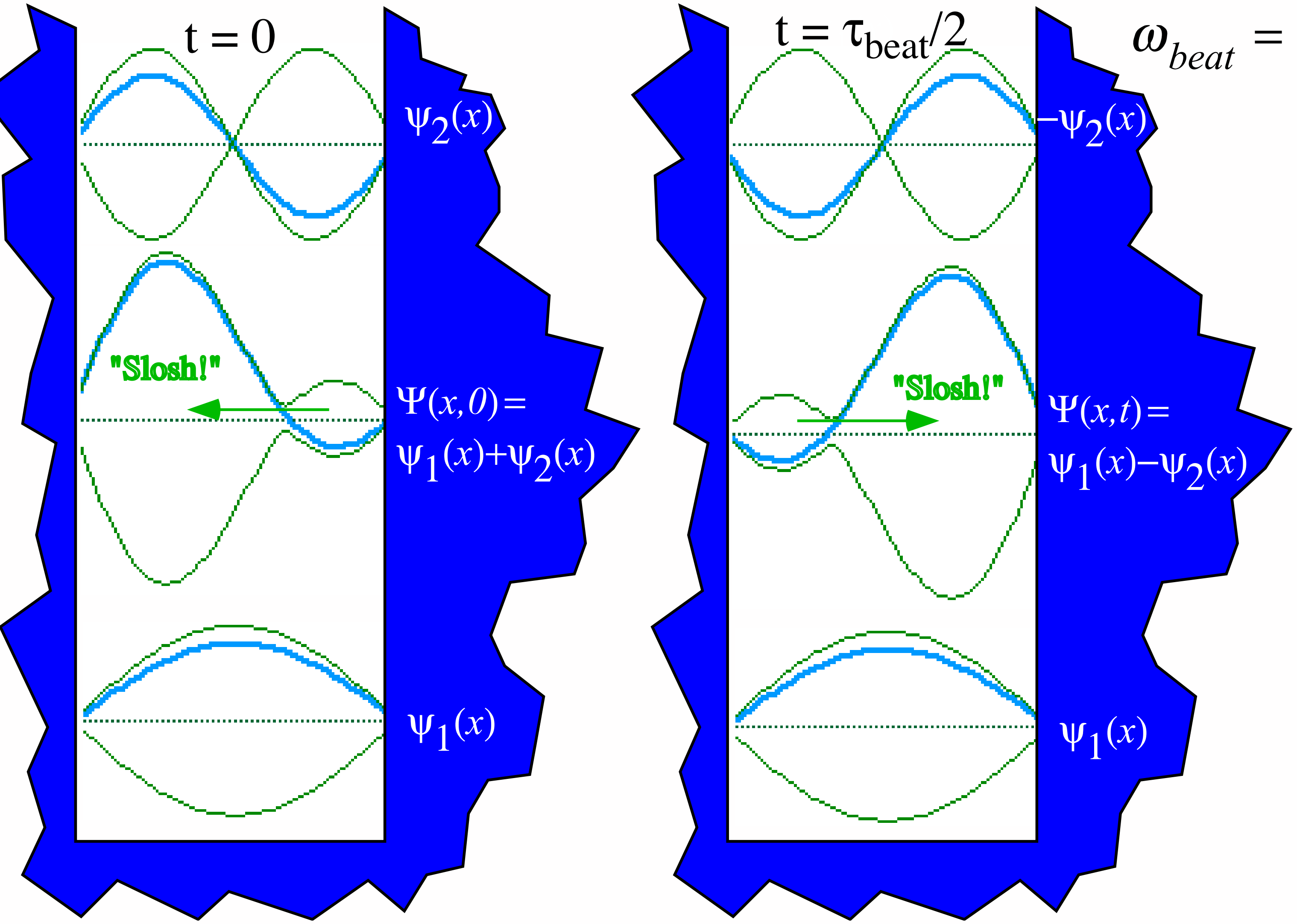

$$\omega_{beat} = \omega_2 \text{-} \omega_1 = \frac{\varepsilon_2 - \varepsilon_1}{\hbar} = \frac{2^2 - 1^2}{\hbar}\frac{h^2}{8MW^2} = 3\frac{2\pi\, h}{8MW^2} = 3\omega_1$$

Fig. 12.1.2 Exercise in prison. Infinite square well eigensolution combination "sloshes" back and forth.

SinNx/x wavepackets bandwidth and uncertainty

$$\delta\left(x-a\right)=\left\langle x\middle|a\right\rangle=\Sigma_{n=1}^{\infty}\left\langle x\middle|\varepsilon_{n}\right\rangle\left\langle\varepsilon_{n}\middle|a\right\rangle$$

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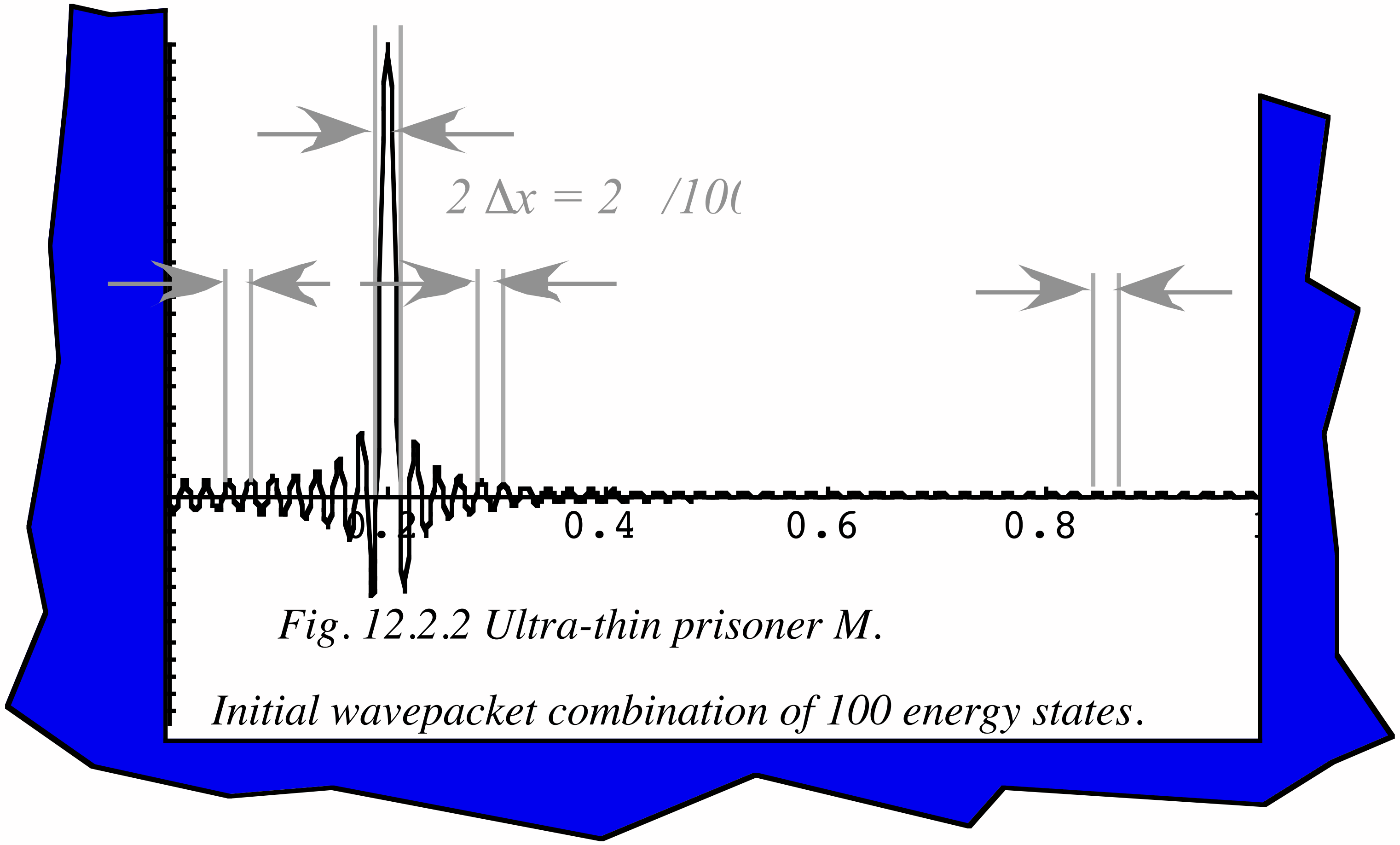

Fig. 12.2.2 Ultra-thin prisoner M.

Initial wavepacket combination of 100 energy states.

SinNx/x wavepackets bandwidth and uncertainty

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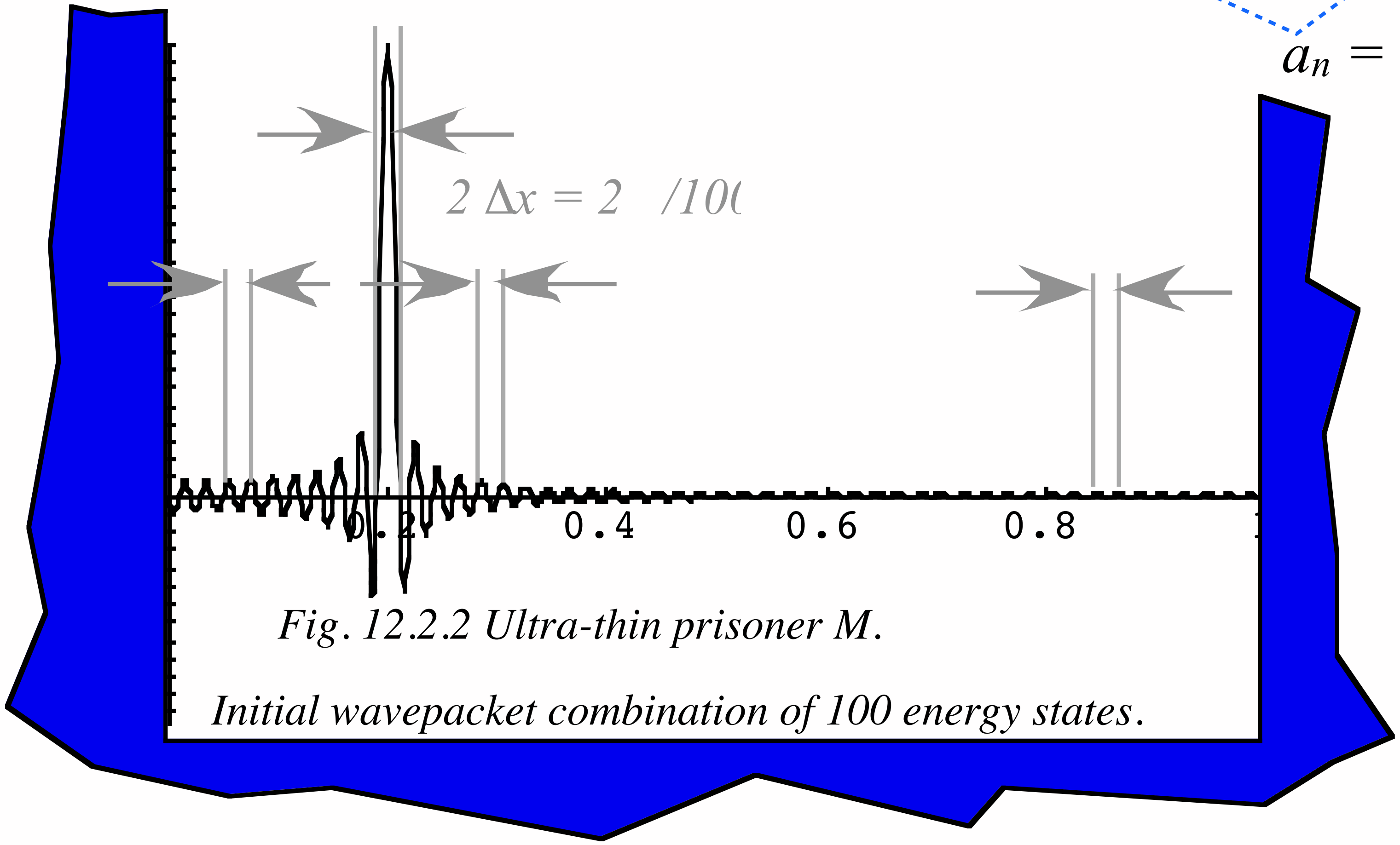

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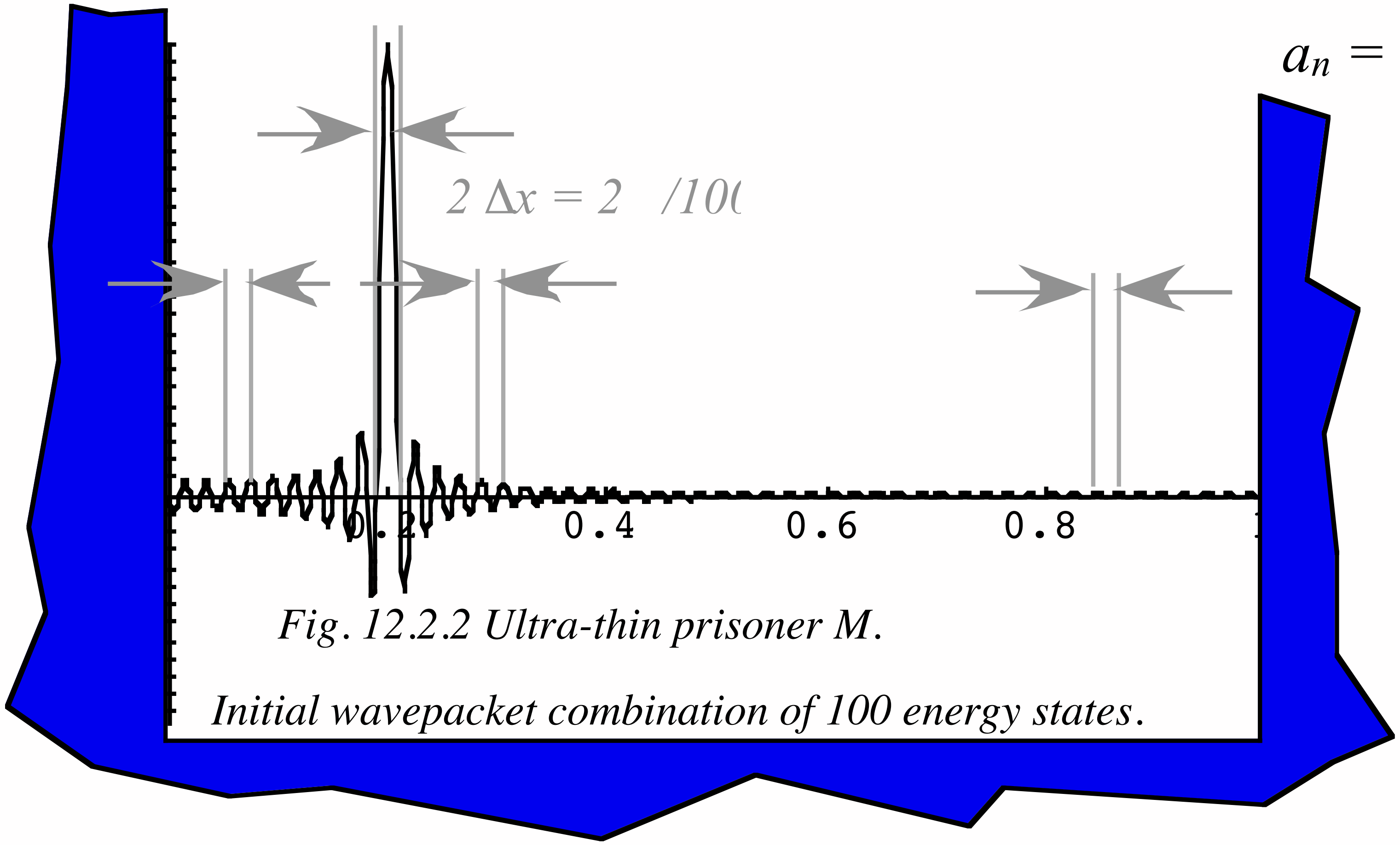

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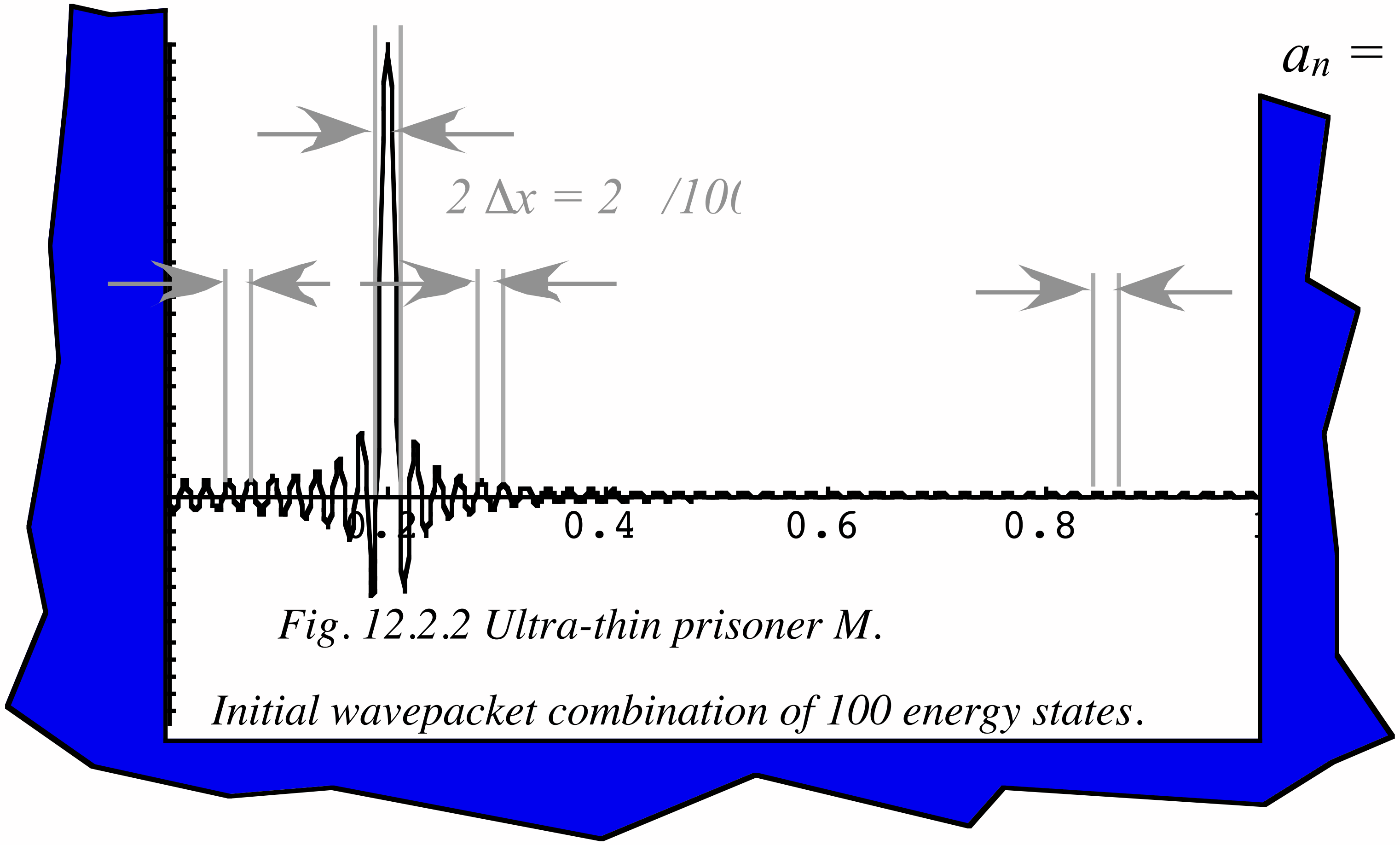

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$$\rightarrow\frac{2}{W}\int_{0}^{K_{\max}}dk\frac{\Delta n}{\Delta k}\sin ka\sin kx$$

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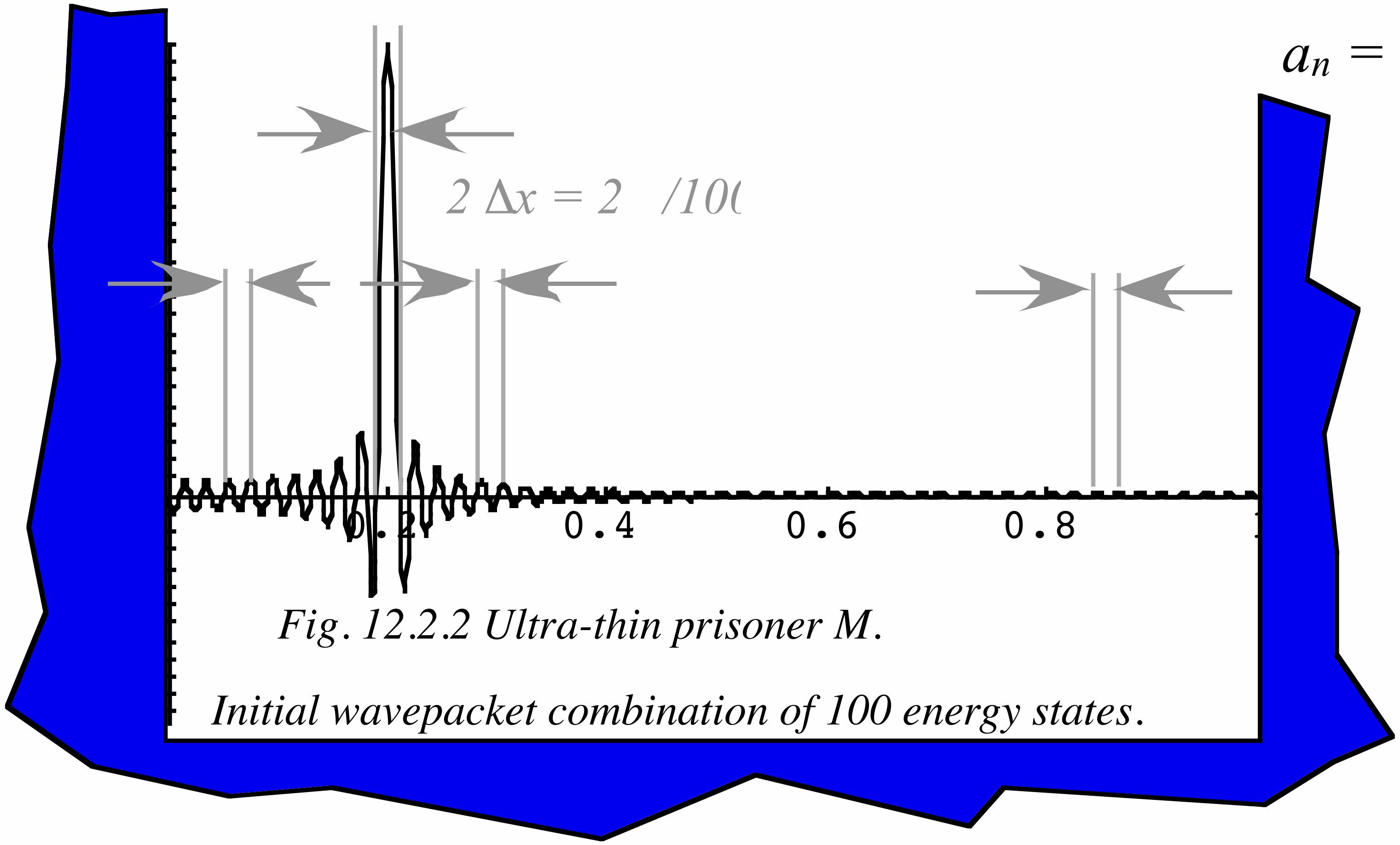

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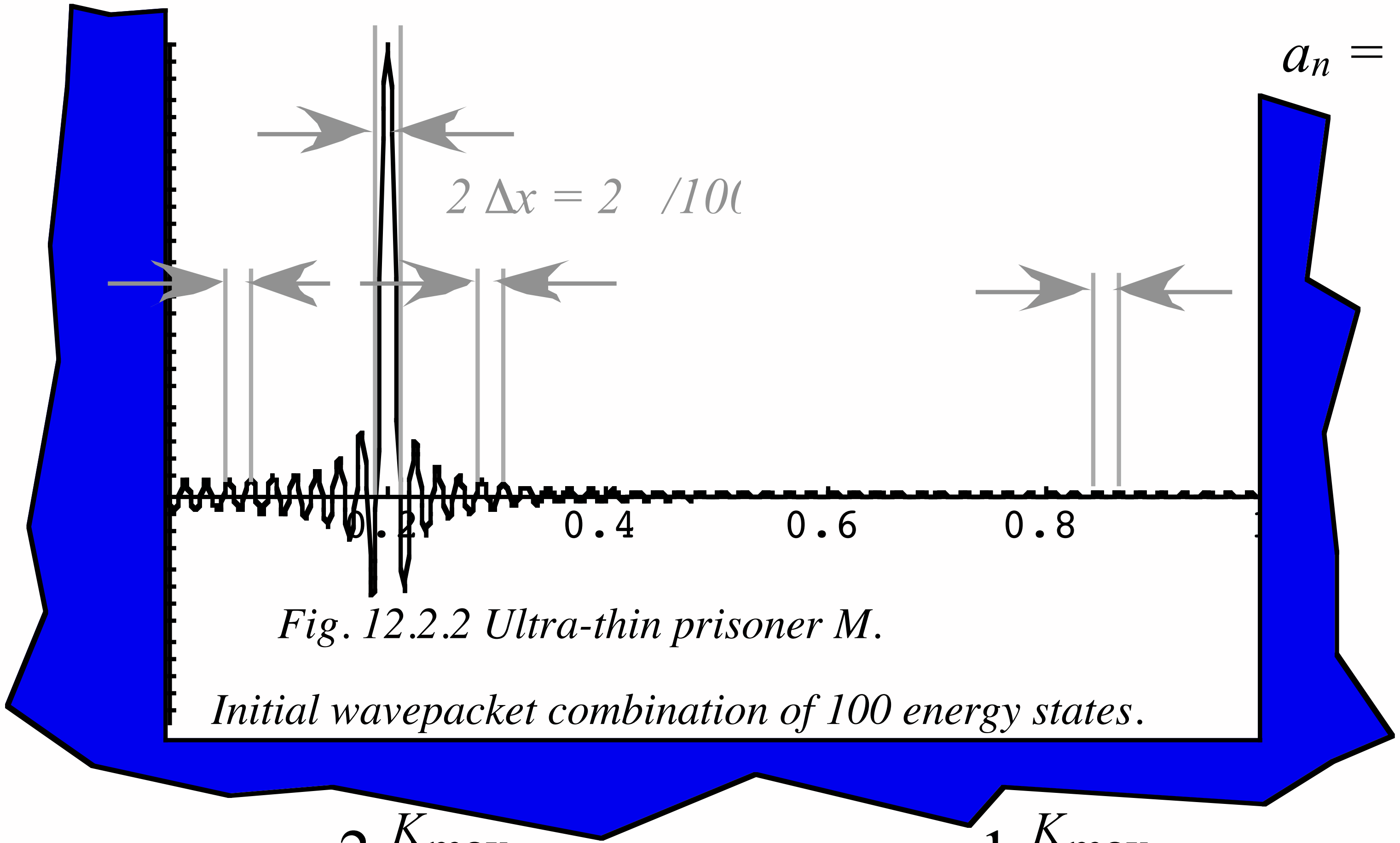

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$$\Psi\left(x\right)\cong\frac{2}{\pi}\int_{0}^{K_{\max}}dk\,\sin ka\,\sin kx=\frac{1}{\pi}\int_{0}^{K_{\max}}dk\left(\cos k\left(x-a\right)-\cos k\left(x+a\right)\right)$$

SinNx/x wavepackets bandwidth and uncertainty

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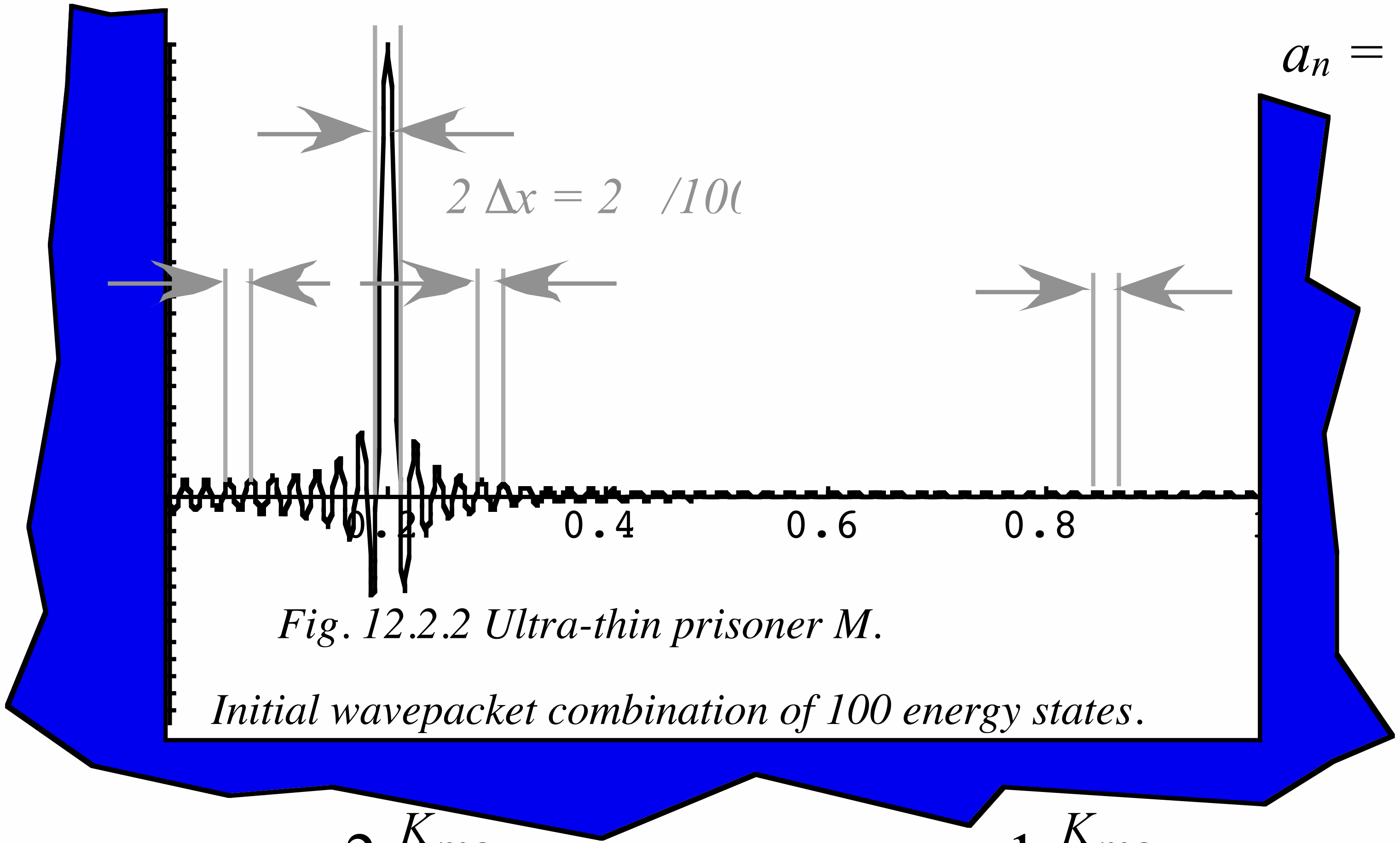

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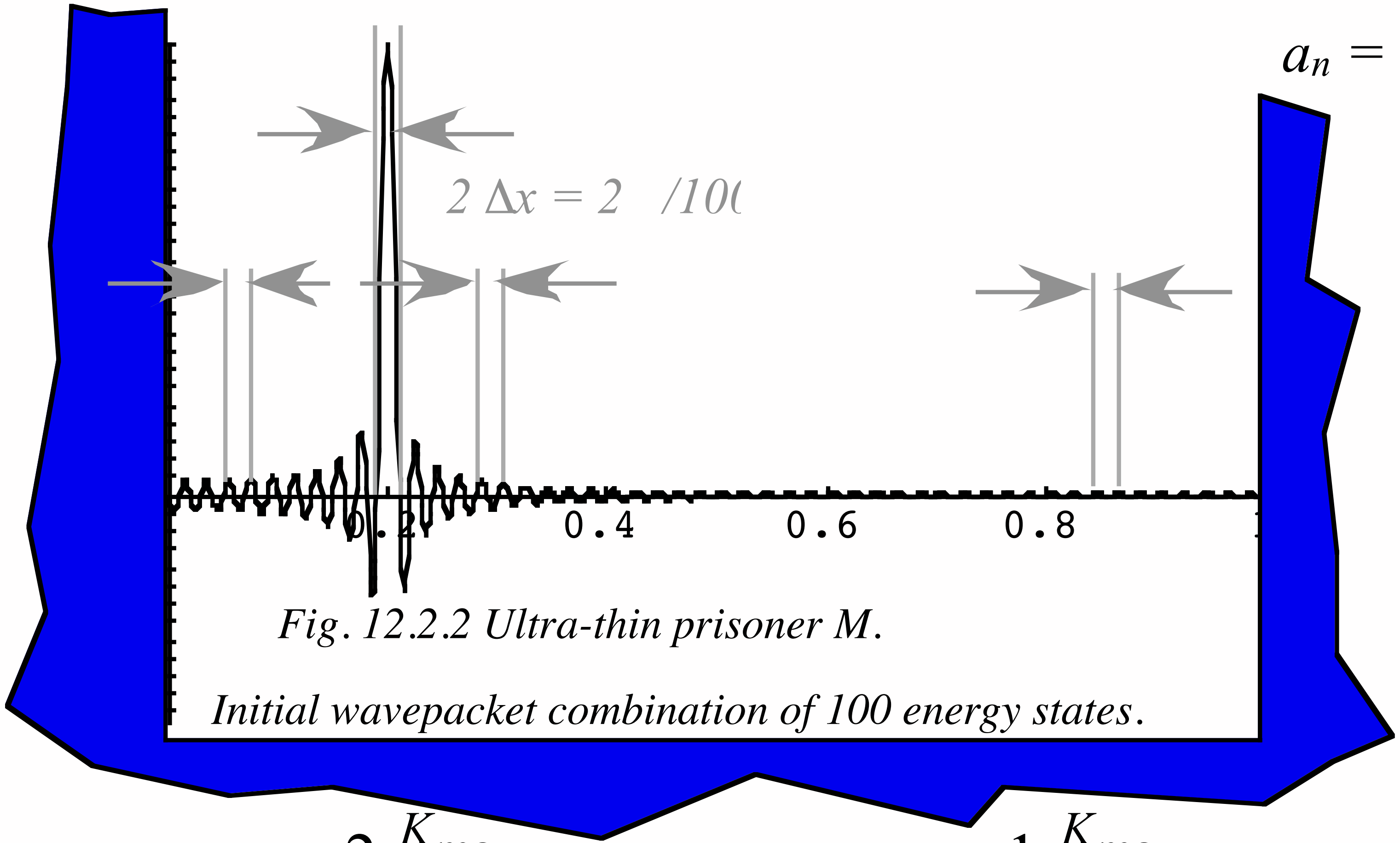
$$\cong\frac{\sin K_{\max}(x\text{-}a)}{\pi(x\text{-}a)}-\frac{\sin K_{\max}(x+a)}{\pi(x+a)}\cong\frac{\sin K_{\max}(x\text{-}a)}{\pi(x\text{-}a)}\quad\text{for: } x\approx a$$

SinNx/x wavepackets bandwidth and uncertainty

$$\delta(x-a) = \langle x|a\rangle = \sum_{n=1}^{\infty}\langle x|\varepsilon_n\rangle\langle\varepsilon_n|a\rangle = \sum_{n=1}^{\infty} a_n \sin k_n x$$



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"Last-in-first-out" effect. Last $K_{\max}$-value dominates and "inside" K get "smothered" by interference with neighbors.

SinNx/x wavepackets bandwidth and uncertainty

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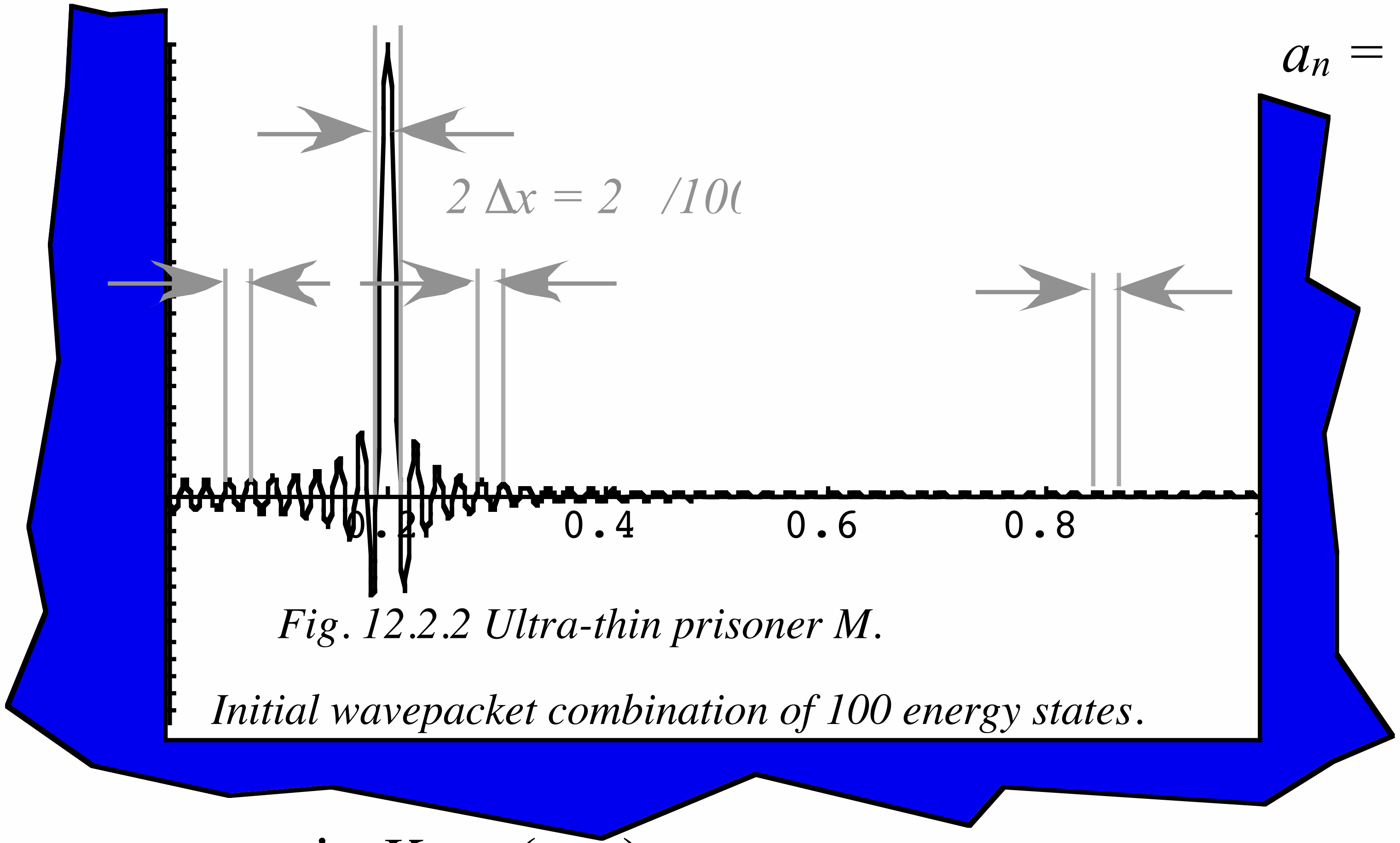

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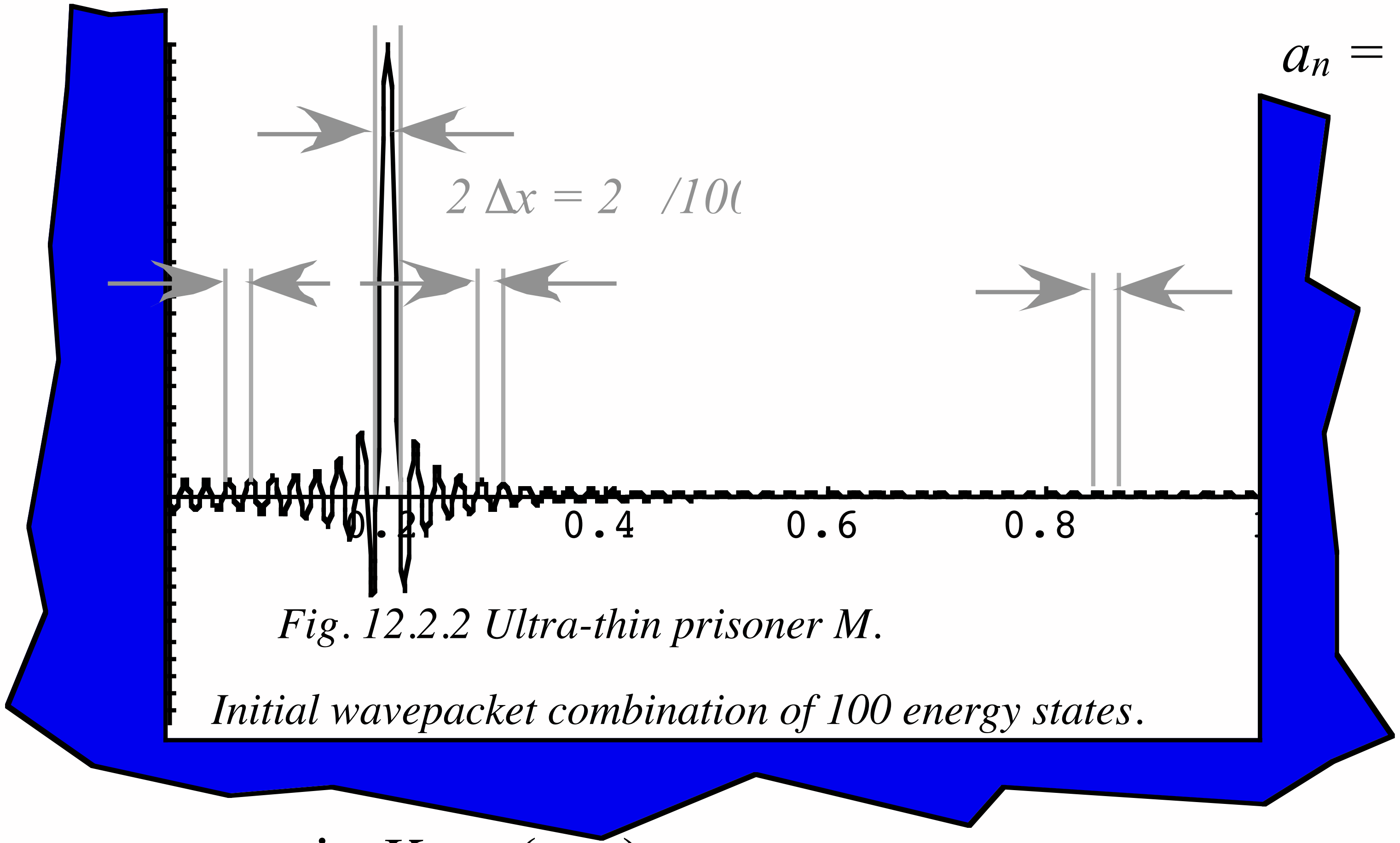

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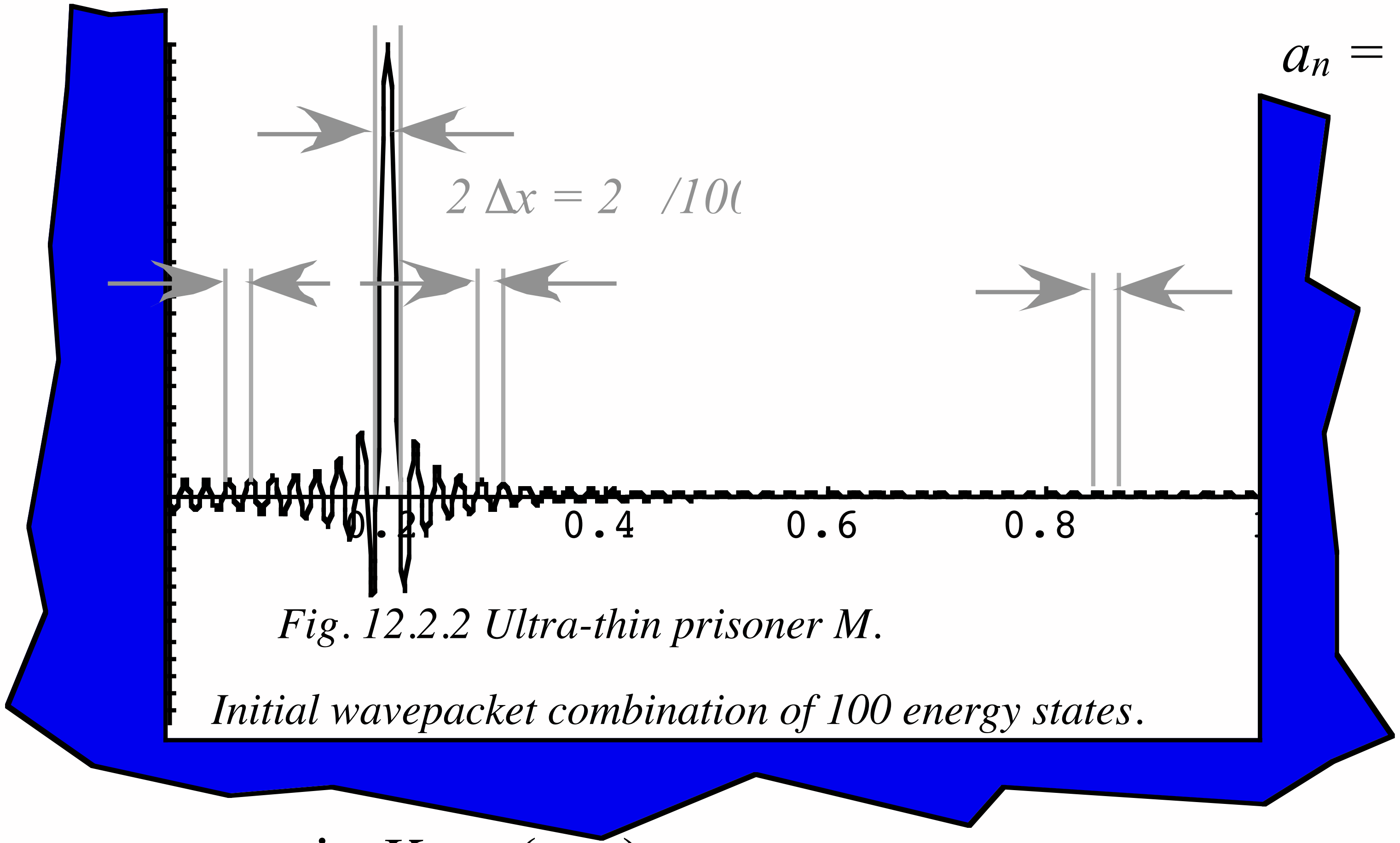

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$\sin K_{\max}(\Delta x)=0$, which implies: $(\Delta x)K_{\max}=\pm\pi$

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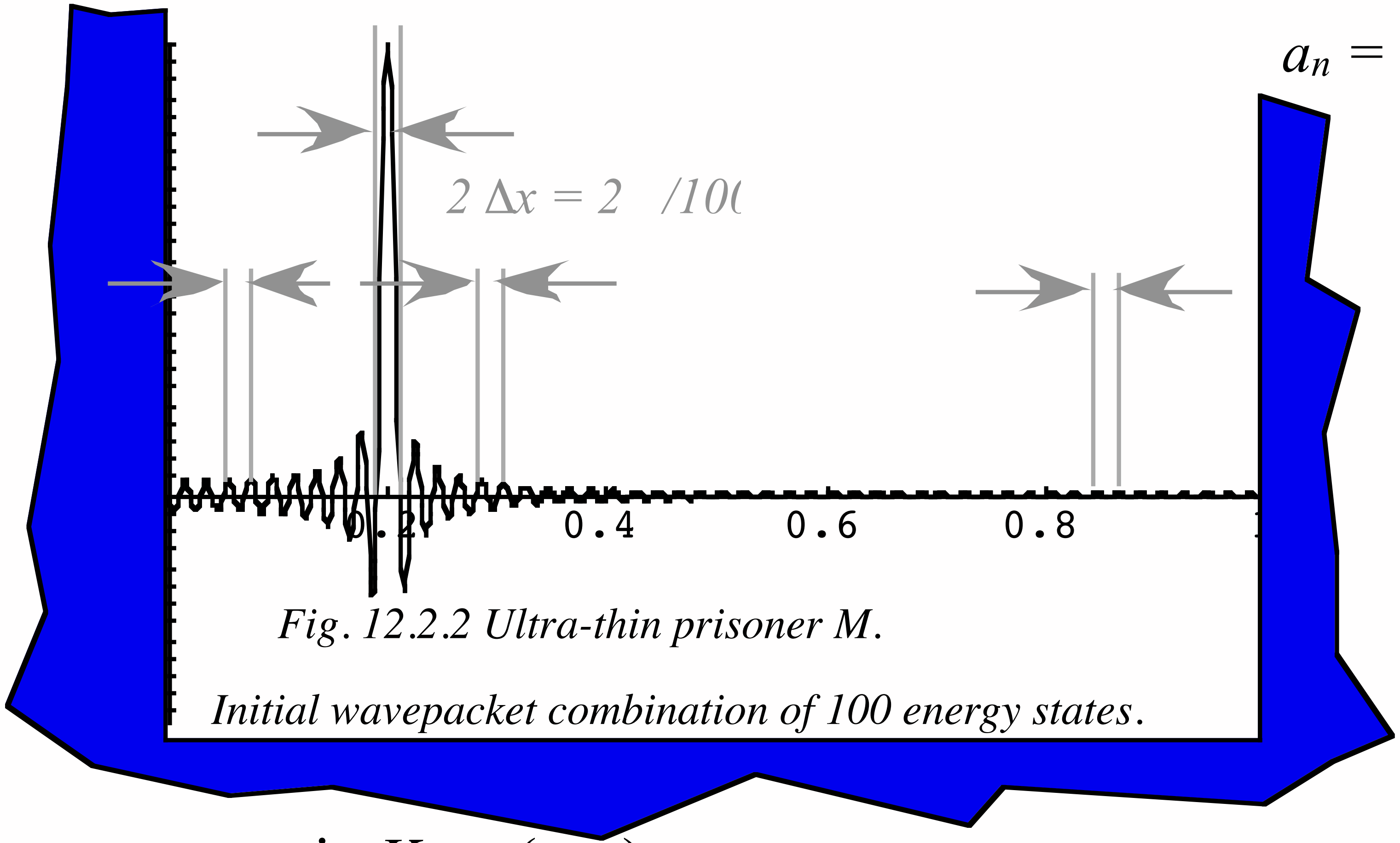

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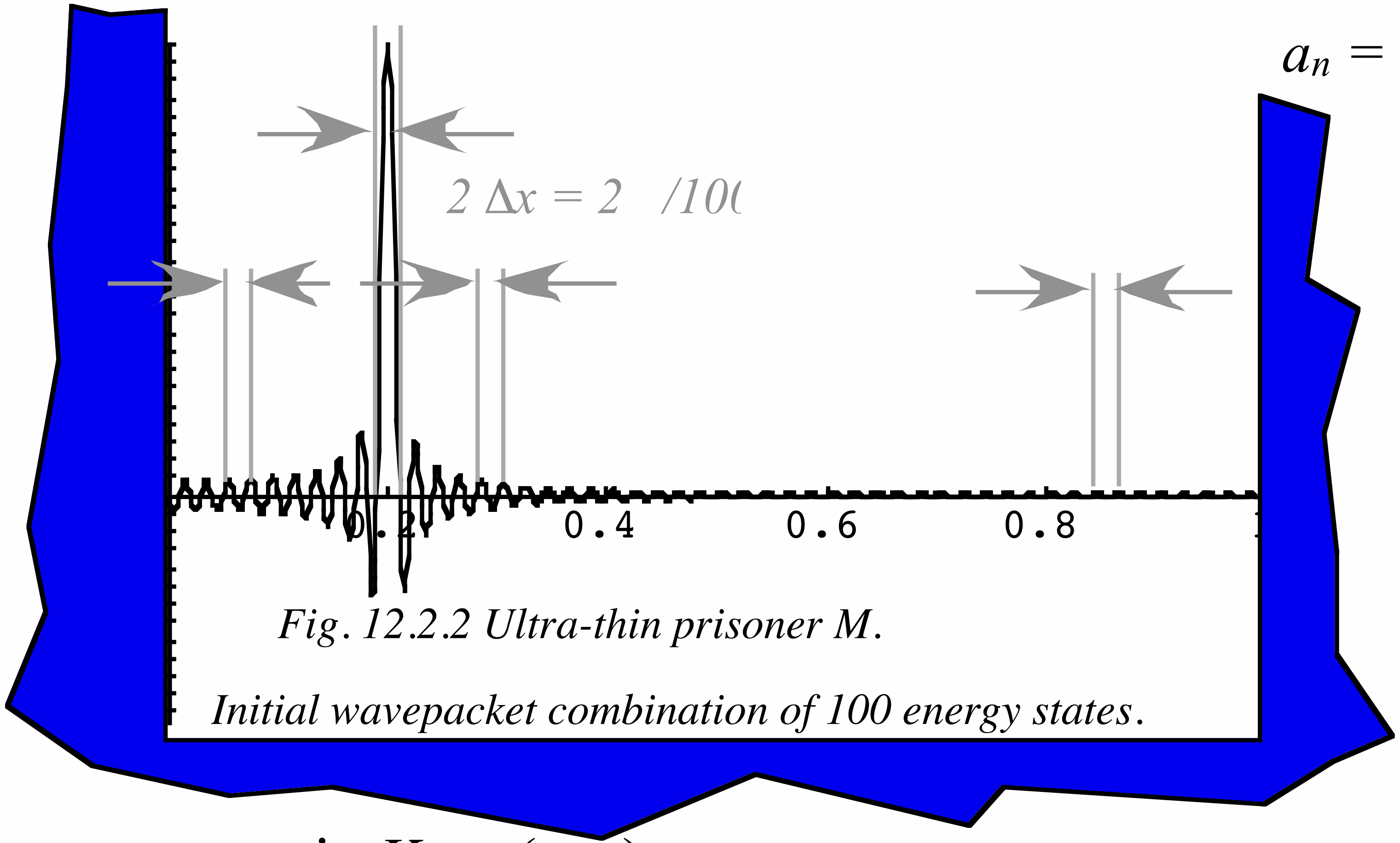

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$\Psi(x)$ peaks at $(x=a)$ and goes to zero on either side at $(x=a\pm\Delta x)$ with *half-width Δx*

$\sin K_{\max}(\Delta x)=0$, which implies: $(\Delta x)K_{\max} = \pm\pi$, or: $\Delta x = \pm\pi / K_{\max}$

$$\Delta x \cdot |K_{\max}| = \Delta x \cdot \Delta k = \pi$$

"Last-in-first-out" effect. Last $K_{\max}$-value dominates and “inside” K get "smothered" by interference with neighbors.

SinNx/x wavepackets bandwidth and uncertainty

$$\delta(x-a)=\langle x|a\rangle=\Sigma_{n=1}^{\infty}\langle x|\varepsilon_n\rangle\langle\varepsilon_n|a\rangle=\Sigma_{n=1}^{\infty}a_n\sin k_n x$$

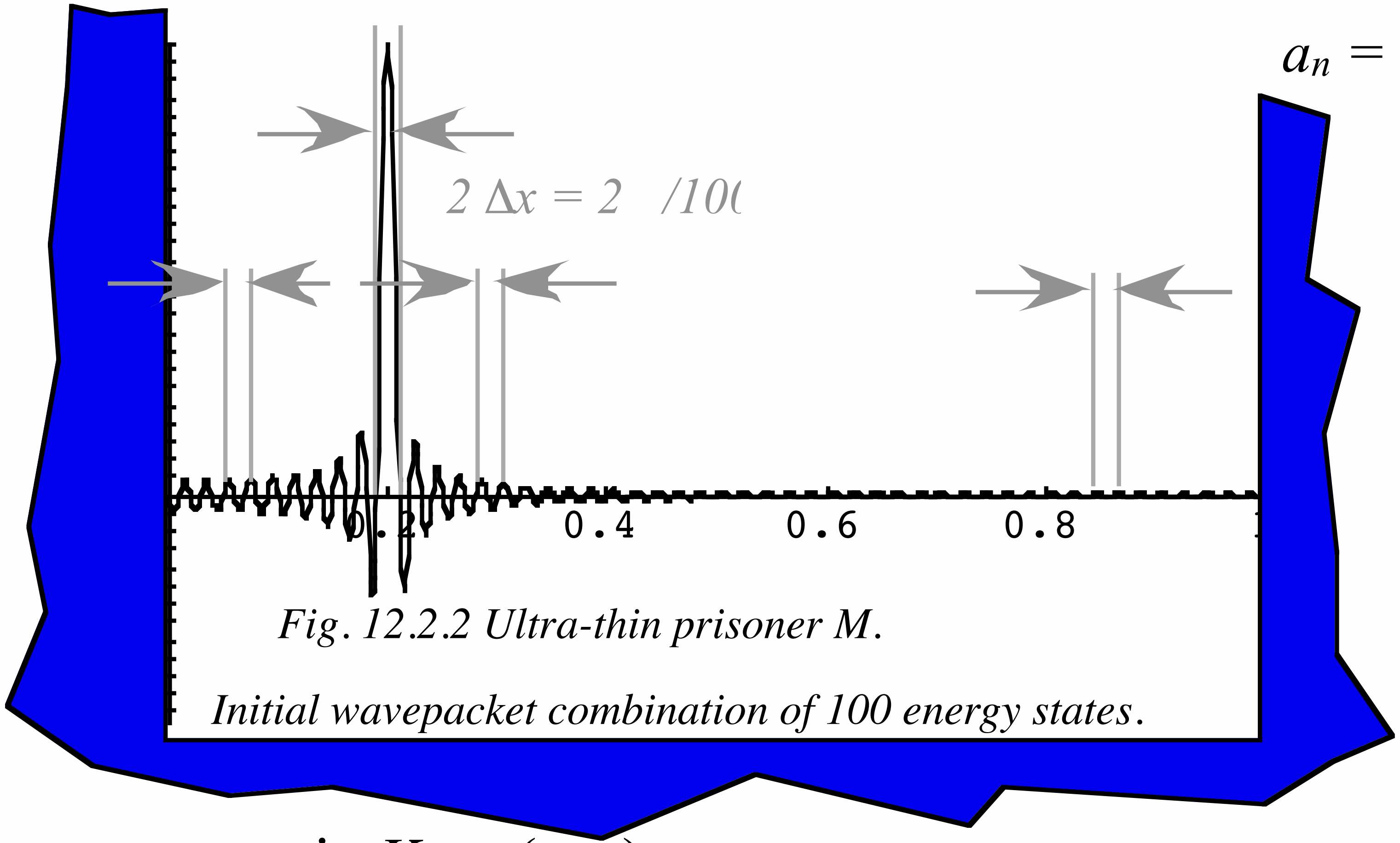

Fig. 12.2.2 Ultra-thin prisoner M.

Initial wavepacket combination of 100 energy states.

$a_n = \langle\varepsilon_n|a\rangle = (2/W)\sin k_n a \qquad (k_n = n\pi/W)$

$$\Psi(x)=\frac{2}{W}\sum_{n}^{N_{\max}}\sin k_n a\sin k_n x$$

$$\rightarrow\frac{2}{W}\int_{0}^{K_{\max}}dk\frac{\Delta n}{\Delta k}\sin ka\sin kx$$

$$=\frac{2}{W}\frac{W}{\pi}\int_{0}^{K_{\max}}dk\sin ka\sin kx$$

$$\Psi(x)\cong\frac{\sin K_{\max}(x\text{-}a)}{\pi(x\text{-}a)} \quad \text{for: } x\approx a$$

$\Psi(x)$ peaks at $(x=a)$ and goes to zero on either side at $(x=a\pm\Delta x)$ with *half-width* Δx

$\sin K_{\max}(\Delta x)=0$, which implies: $(\Delta x)K_{\max}=\pm\pi$, or: $\Delta x=\pm\pi/K_{\max}$

$\Delta x\cdot|K_{\max}|=\Delta x\cdot\Delta k=\pi$ or: $\boxed{\Delta x\cdot\Delta p=\pi\hbar=h/2 \qquad \infty\text{-Well uncertainty relation}}$

"Last-in-first-out" effect. Last $K_{\max}$-value dominates and "inside" K get "smothered" by interference with neighbors.

Polygonal geometry of $U(2)\supset C_N$ character spectral function
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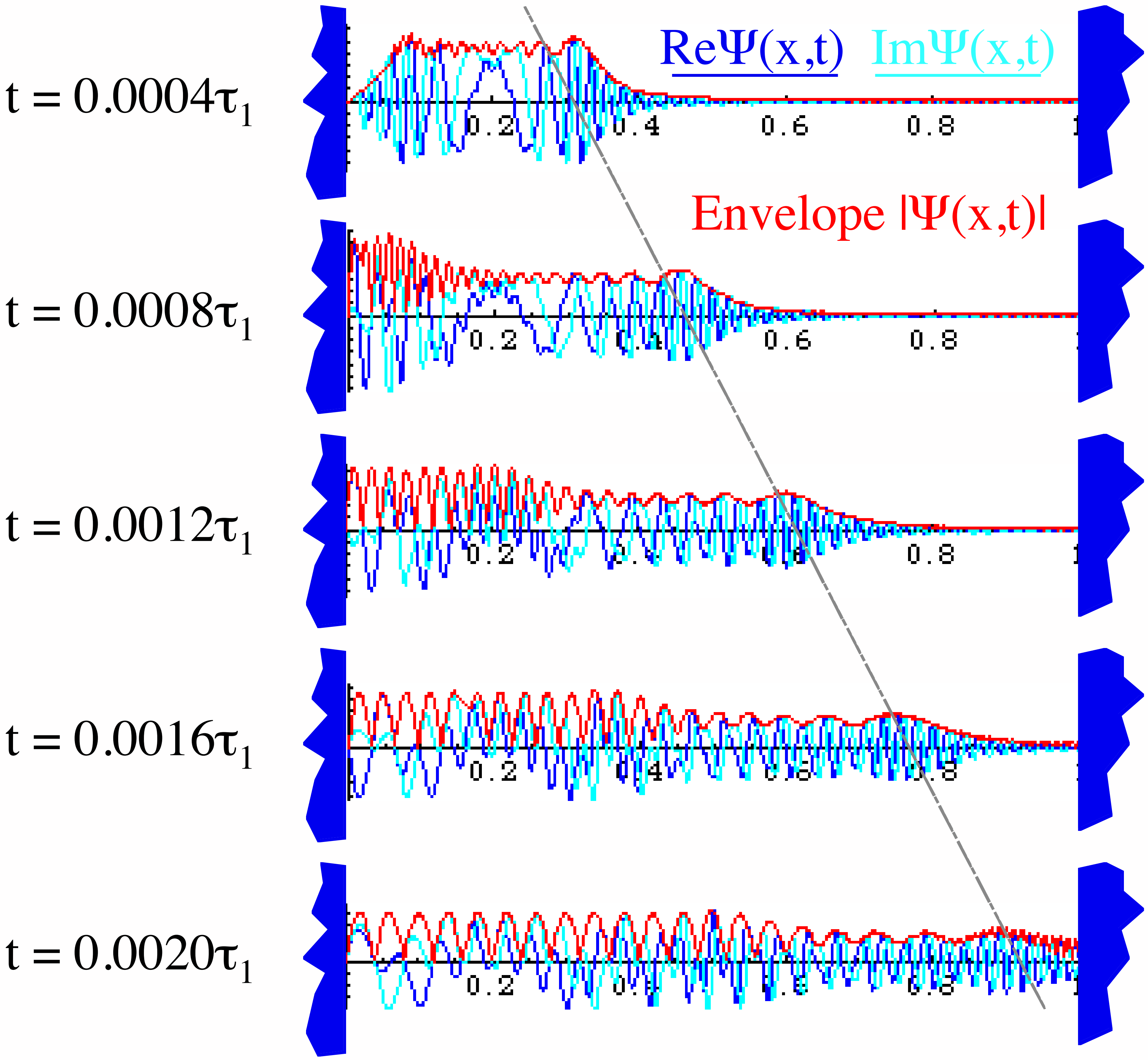
        Gaussian revivals
    Farey-Sums and Ford-products
    Phase dynamics

Wavepacket explodes!

Time given in units of period τ_1 (slowest phasor of ground level).

fundamental zero-point period $\tau_1 = 1/\nu_1$



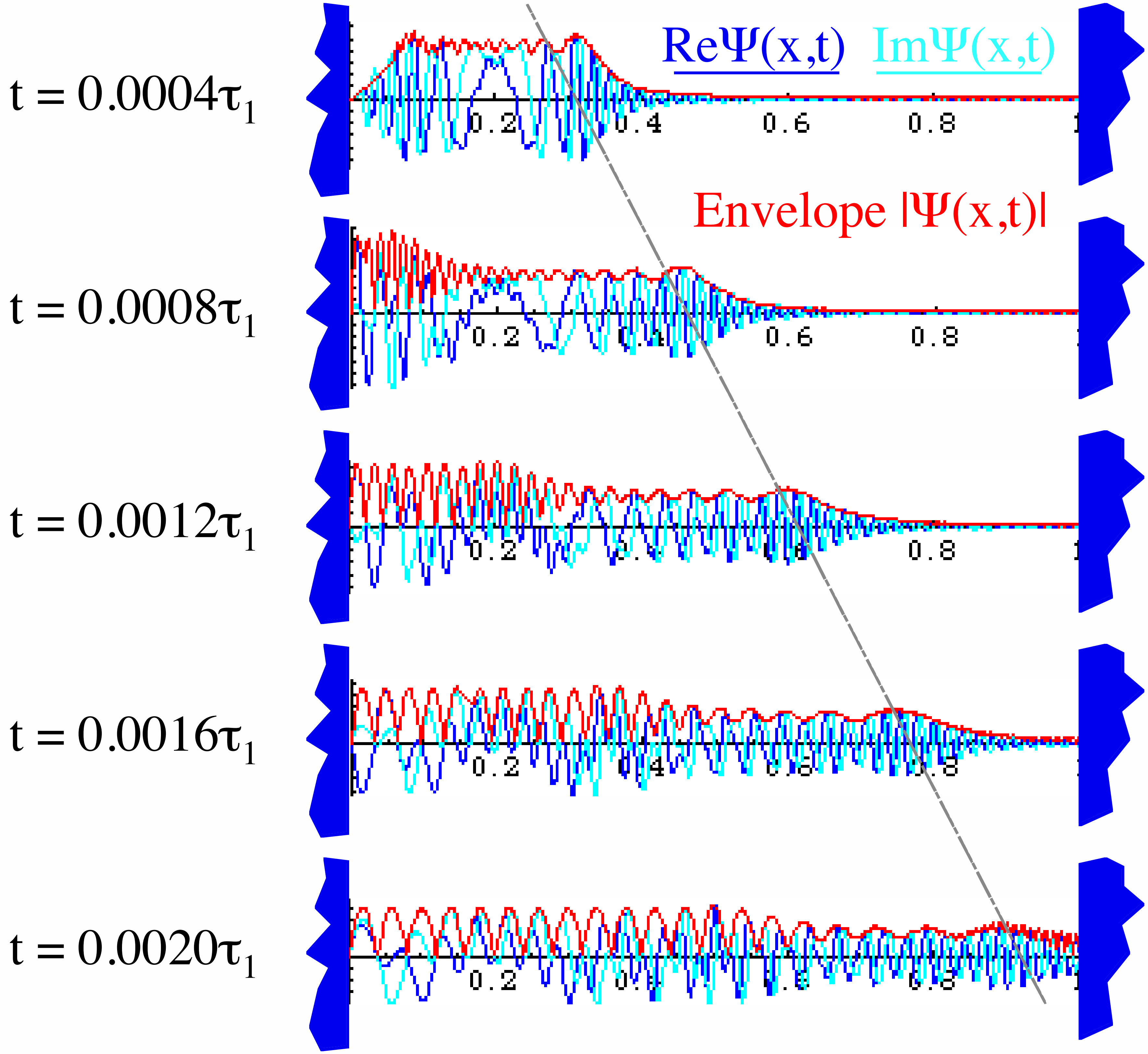
Wavepacket explodes!

Time given in units of period τ1 (slowest phasor of ground level).

fundamental zero-point period $\tau_1 = 1/\nu_1$ is

$$\tau_1 = \frac{2\pi}{\omega_1} = \frac{2\pi\hbar}{\varepsilon_1}$$

$$= \frac{h}{h^2/8MW^2} = \frac{8MW^2}{h}$$



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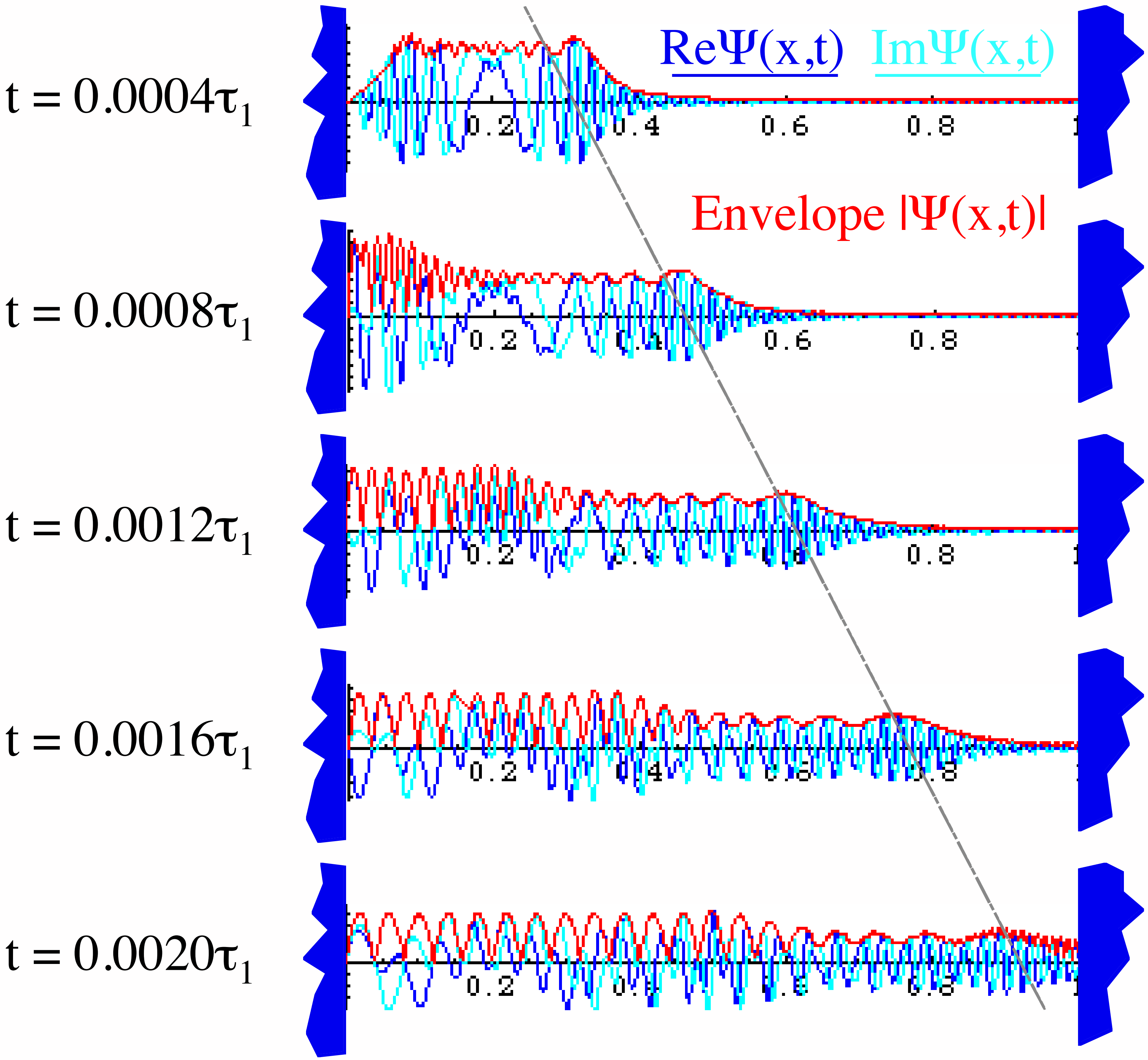

ε_n-*level classical velocity*:

$$V_n = \frac{d\omega_n}{dk} = \frac{1}{\hbar}\frac{d\varepsilon_n}{dk}$$

$$= \frac{1}{\hbar}\frac{\hbar^2}{2M}\frac{dk^2}{dk}$$

$$= \frac{\hbar 2k_n}{2M} = \frac{\hbar n\pi}{MW} = \frac{hn}{2MW}$$

Wavepacket explodes!

Time given in units of period τ_1 (slowest phasor of ground level).

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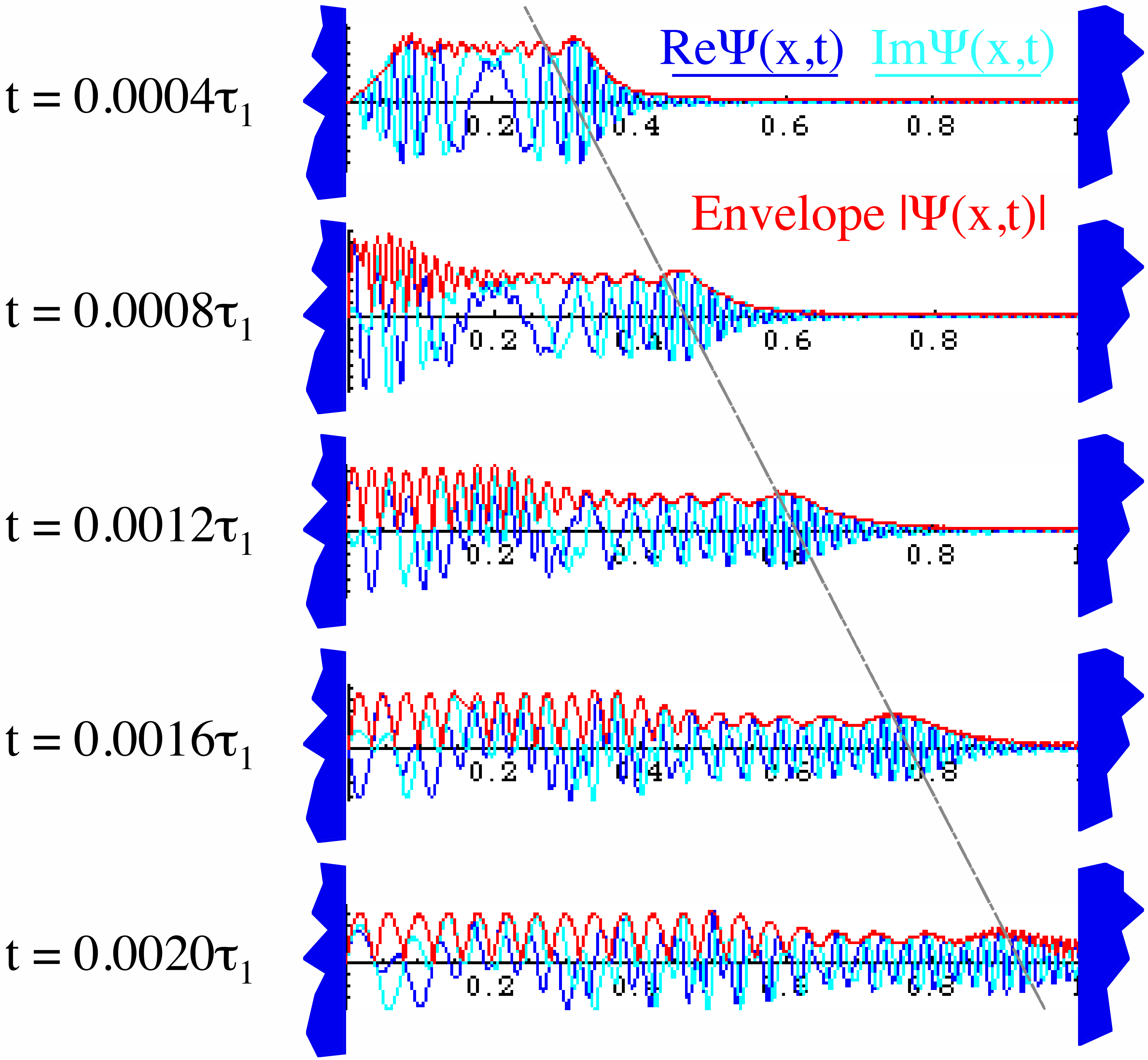


ε_n-level classical velocity:

$$V_n = \frac{d\omega_n}{dk} = \frac{1}{\hbar}\frac{d\varepsilon_n}{dk}$$

$$= \frac{1}{\hbar}\frac{\hbar^2}{2M}\frac{dk^2}{dk}$$

$$= \frac{\hbar 2k_n}{2M} = \frac{\hbar n\pi}{MW} = \frac{hn}{2MW}$$

ε_n-level classical round trip time $T_n(2W)$

$$T_n(2W) = \frac{2W}{V_n} = 2W\frac{2MW}{hn} = \frac{4MW^2}{hn}$$

$$= \frac{1}{2n}\frac{8MW^2}{h} = \frac{\tau_1}{2n}$$

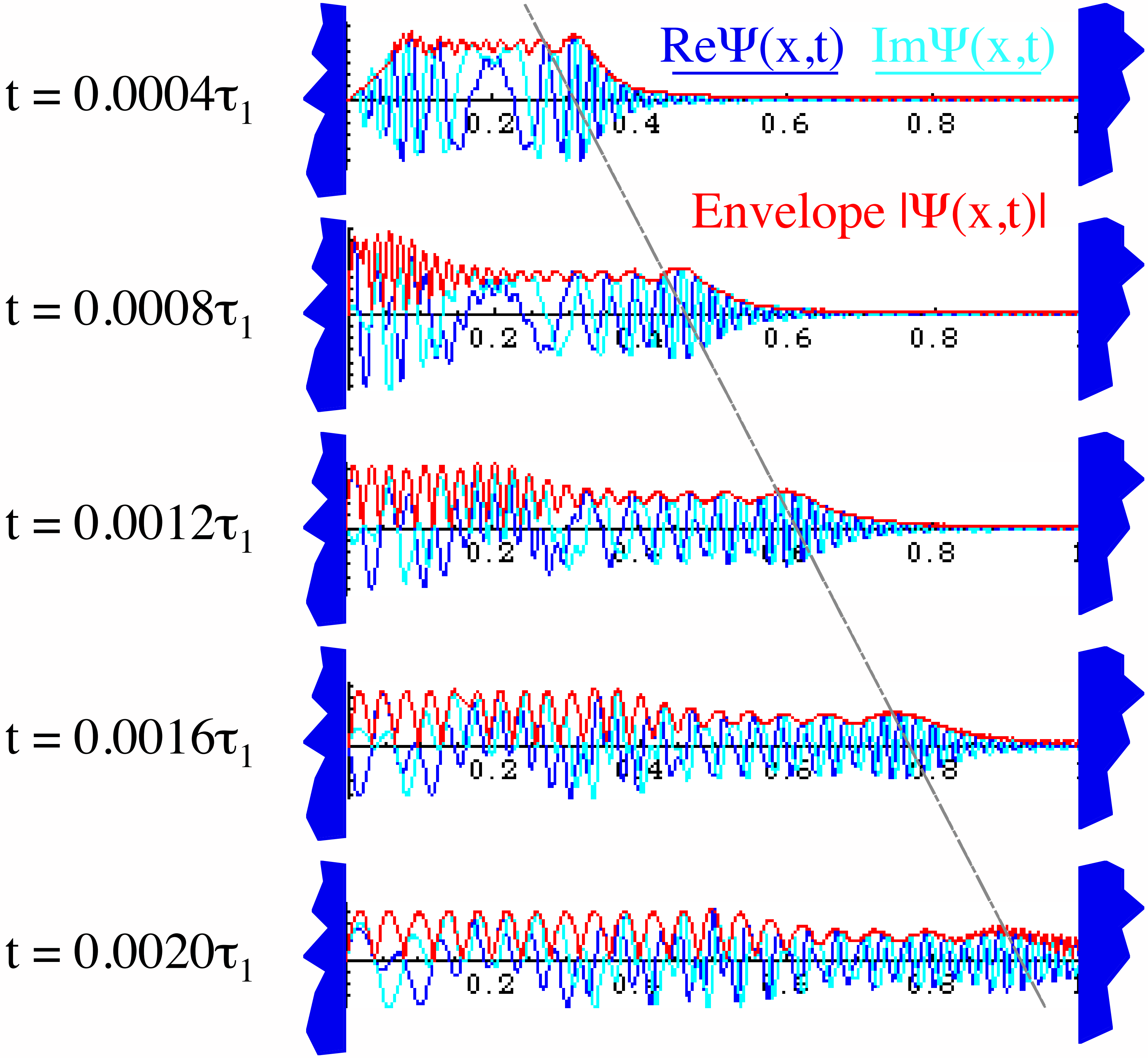
Wavepacket explodes!

Time given in units of period τ1 (slowest phasor of ground level).

fundamental zero-point period τ1 =*1/*ν1 is

$$\tau_1 = \frac{2\pi}{\omega_1} = \frac{2\pi\hbar}{\varepsilon_1}$$

$$= \frac{h}{h^2/8MW^2} = \frac{8MW^2}{h}$$



εn-level classical velocity:

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$$= \frac{1}{2n}\frac{8MW^2}{h} = \frac{\tau_1}{2n}$$

εn-level 1-way time Tn(W)

$$T_n(W) = T_n(2W)/2 = \frac{\tau_1}{4n}$$

$$\left(= 0.0025\,\tau_1 \quad \text{for: } n{=}100\right)$$

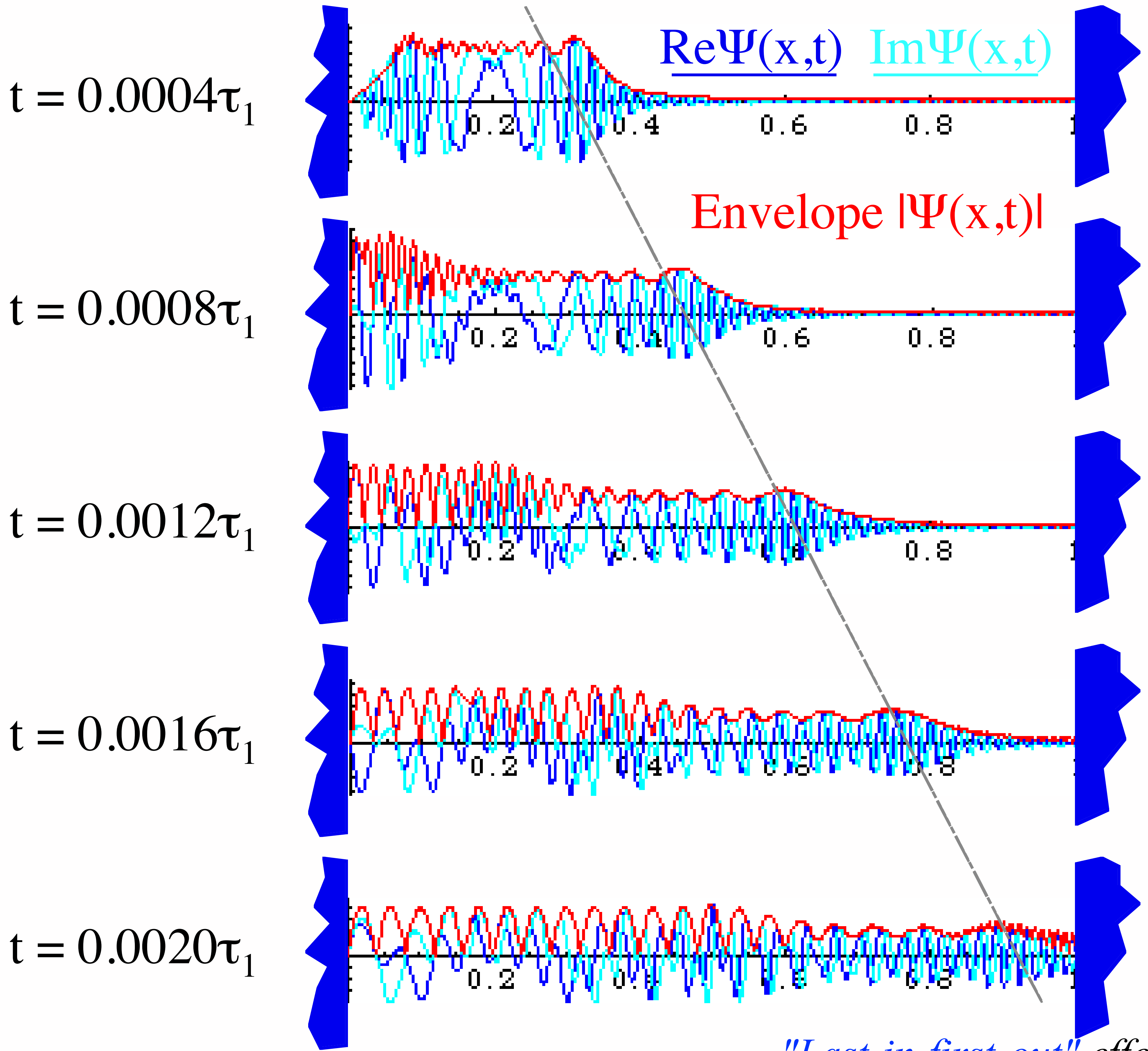
Wavepacket explodes!

Time given in units of period τ_1 (slowest phasor of ground level).

fundamental zero-point period $\tau_1 = 1/\nu_1$ is

$$\tau_1 = \frac{2\pi}{\omega_1} = \frac{2\pi\hbar}{\varepsilon_1}$$

$$= \frac{h}{h^2/8MW^2} = \frac{8MW^2}{h}$$



"Last-in-first-out" effect

ε_n-*level classical velocity*:

$$V_n = \frac{d\omega_n}{dk} = \frac{1}{\hbar}\frac{d\varepsilon_n}{dk}$$

$$= \frac{1}{\hbar}\frac{\hbar^2}{2M}\frac{dk^2}{dk}$$

$$= \frac{\hbar 2k_n}{2M} = \frac{\hbar n\pi}{MW} = \frac{hn}{2MW}$$

ε_n-*level classical round trip time* $T_n(2W)$

$$T_n(2W) = \frac{2W}{V_n} = 2W\frac{2MW}{hn} = \frac{4MW^2}{hn}$$

$$= \frac{1}{2n}\frac{8MW^2}{h} = \frac{\tau_1}{2n}$$

ε_n-*level 1-way time* $T_n(W)$

$$T_n(W) = T_n(2W)/2 = \frac{\tau_1}{4n}$$

$$\left(= 0.0025\,\tau_1 \quad \text{for: } n\text{=}100\right)$$

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  - *Gaussian revivals*
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- *Phase dynamics*

Wavepacket explodes! (Then revives)

Zero-point period τ_1 is just enough time for "particle" in ε_n-level to make *2n* round trips.

$$\tau_1 = 2n\, T_n(2W) = \frac{8ML^2}{h}$$

In time τ_1 ground ε_1-level particle does 2 round trips,

ε_2-level particle makes 4 round trips,

ε_3-level particle makes 6 round trips,..,

At time τ_1, *M* undergoes a *full revival* and "unexplodes" into his original spike at $x=0.2W$,

Wavepacket explodes! (Then revives)

Zero-point period τ_1 is just enough time for "particle" in ε_n-level to make *2n* round trips.

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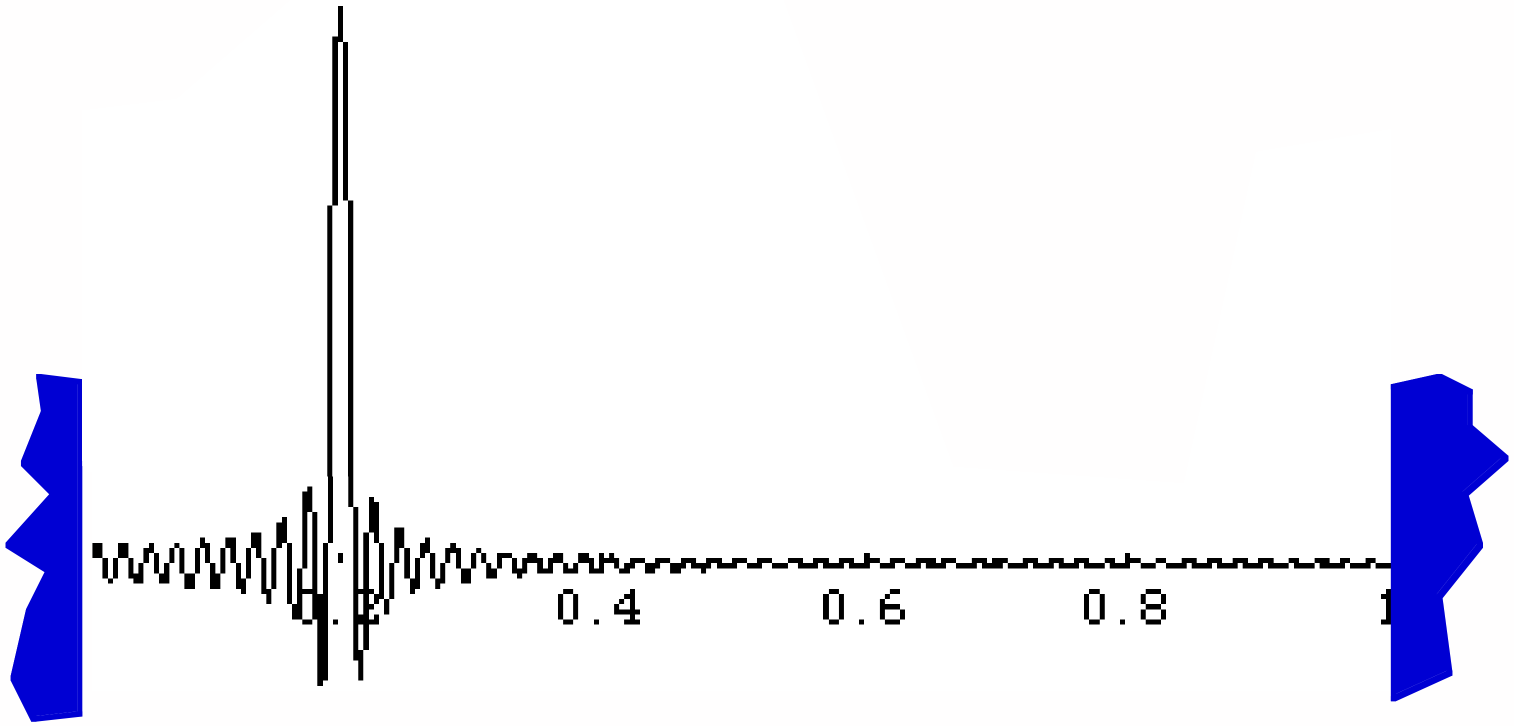
ε_2-level particle makes 4 round trips,

ε_3-level particle makes 6 round trips,..,

At time τ_1, *M* undergoes a *full revival* and "unexplodes" into his original spike at $x=0.2W$,

$t = 1.0000\tau_1$

$= 3.0\tau_{beat}$



Wavepacket explodes! (Then revives)

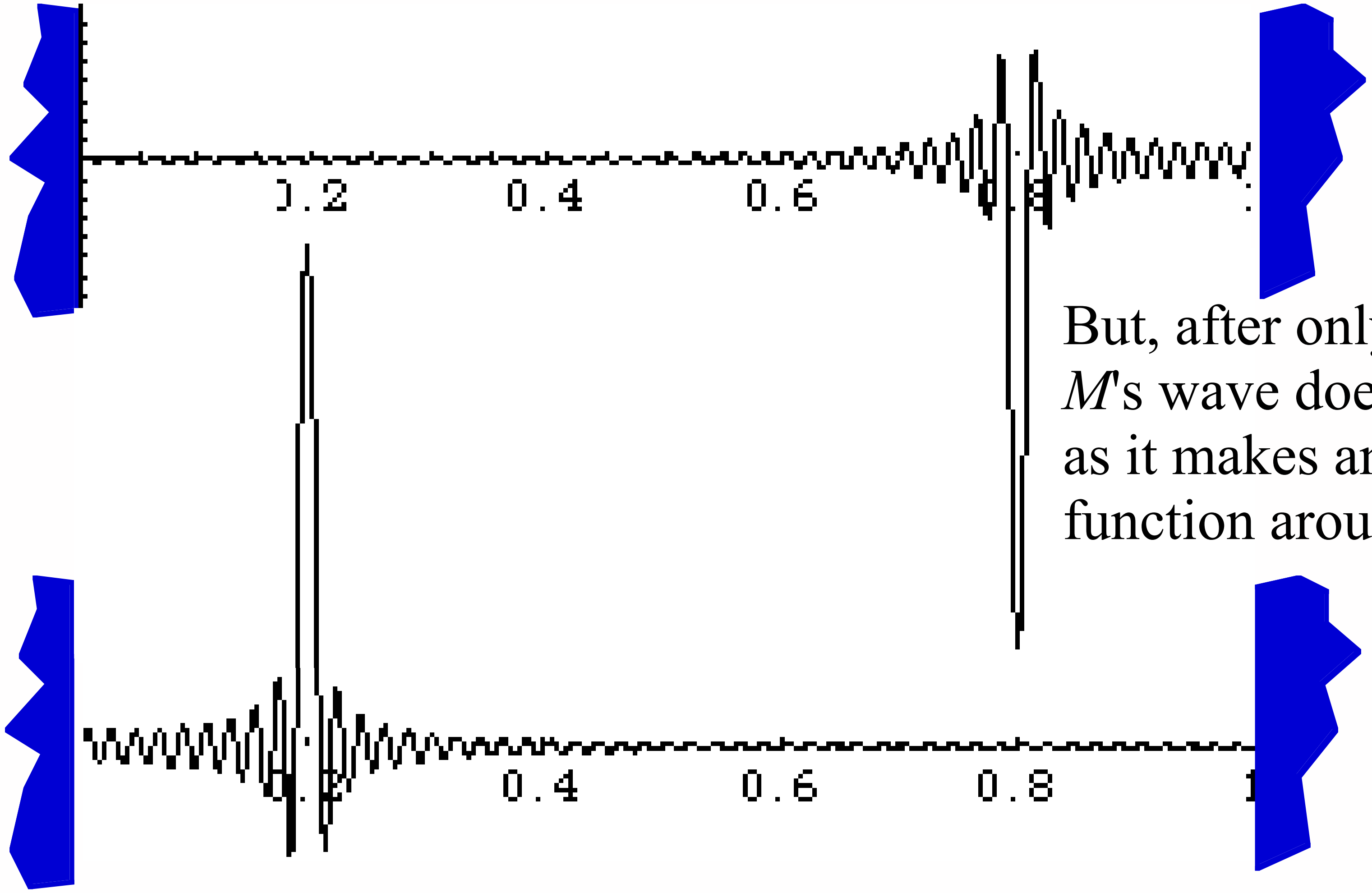
Zero-point period τ_1 is just enough time for "particle" in ε_n-level to make *2n* round trips.

$$\tau_1 = 2n\, T_n(2W) = \frac{8ML^2}{h}$$

In time τ_1 ground ε_1-level particle does 2 round trips,
ε_2-level particle makes 4 round trips,
ε_3-level particle makes 6 round trips,..,

At time τ_1, *M* undergoes a *full revival* and "unexplodes" into his original spike at *x=0.2W*,

$t = 0.5000\tau_1$
$= 1.5\tau_{\text{beat}}$



But, after only 50 round-trips *M*'s wave does a *partial revival* as it makes an upside down-delta function around *x=0.8W*.

$t = 1.0000\tau_1$
$= 3.0\tau_{\text{beat}}$

At fractional times τ_1/n M undergoes a number of *fractional revivals*

$t = \tau_1/3$

$t = \tau_1/5$

$t = \tau_1/7$

$t = \tau_1/9$

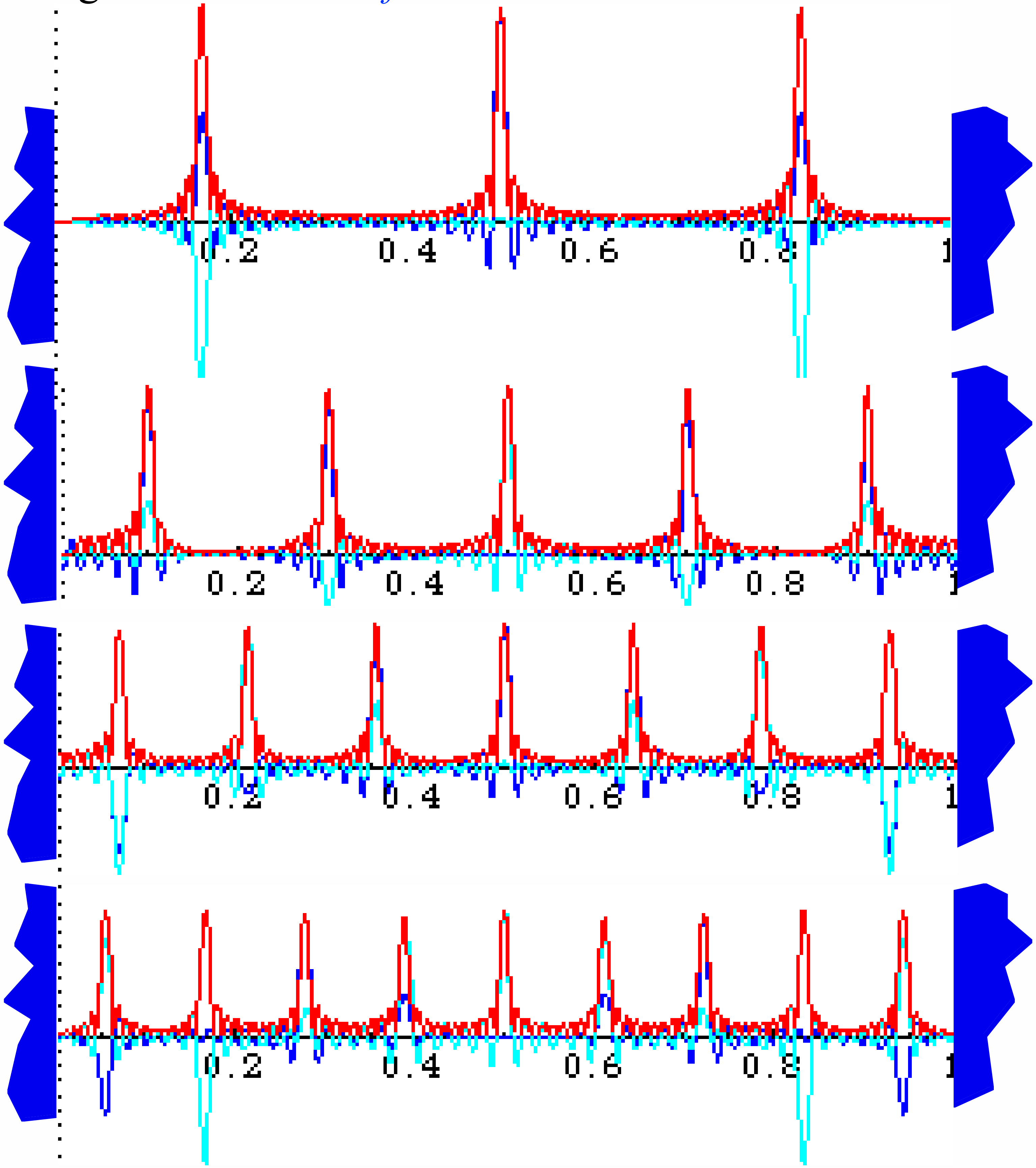


Fig. 12.2.5 The "Dance of the deltas." Mini-Revivals for prisoner M's wavepacket envelope function.

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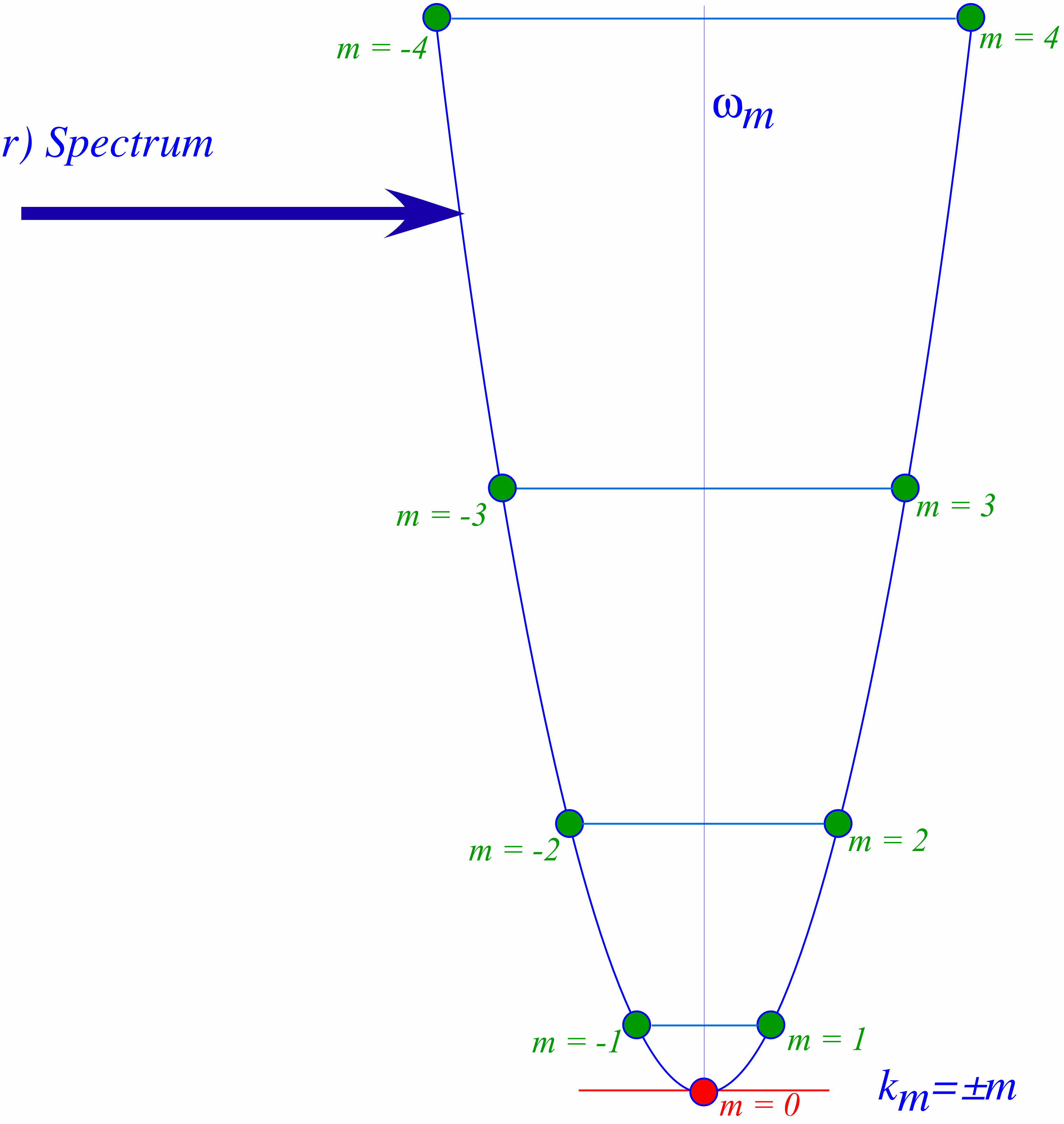
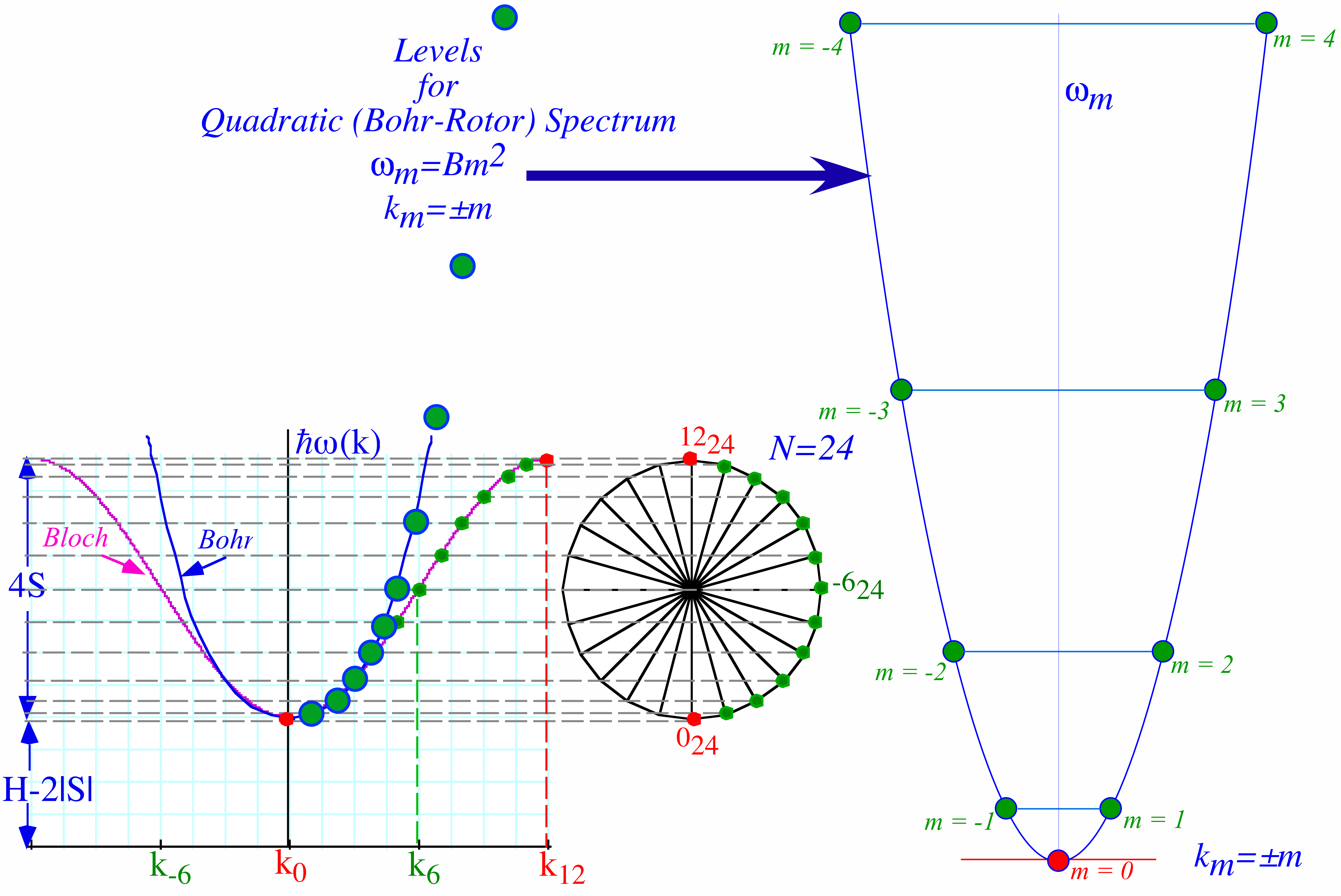
  - *Gaussian wave-packet bandwidth and uncertainty*
  - *Gaussian Bohr-rotor revivals*
- *Farey-Sums and Ford-products*
- *Phase dynamics*

Levels
for
Quadratic (Bohr-Rotor) Spectrum

$\omega_m = Bm^2$

$k_m = \pm m$





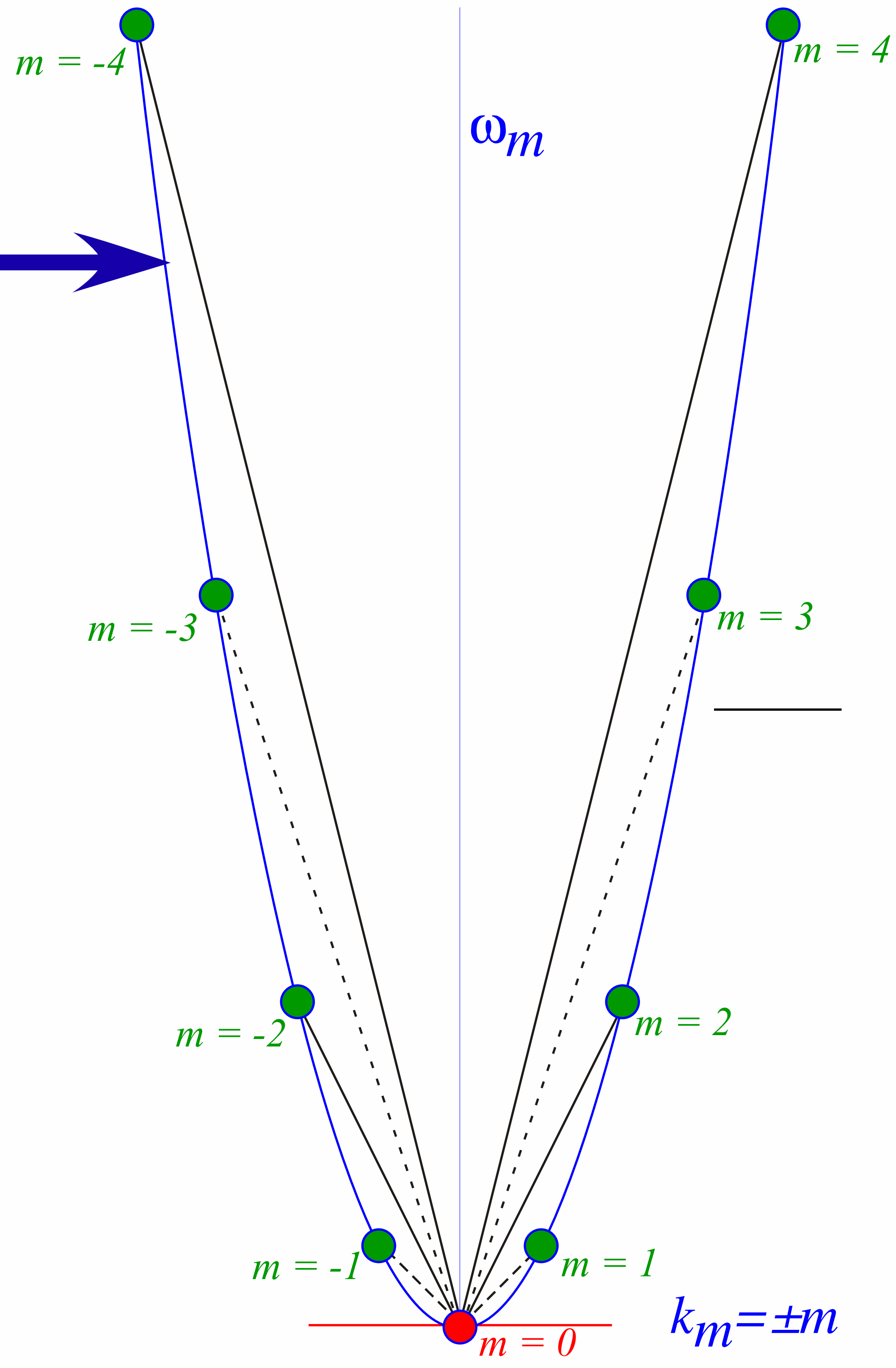
Levels
for
Quadratic (Bohr-Rotor) Spectrum
$\omega_m = Bm^2$
$k_m = \pm m$

Possible wave velocities for Quadratic (Bohr-Rotor) Spectrum

$$\omega_m = Bm^2$$

$$k_m = \pm m$$

$$\boxed{V_{phase} = \frac{\omega_m}{k_m} = \frac{Bm^2}{m} = mB}$$



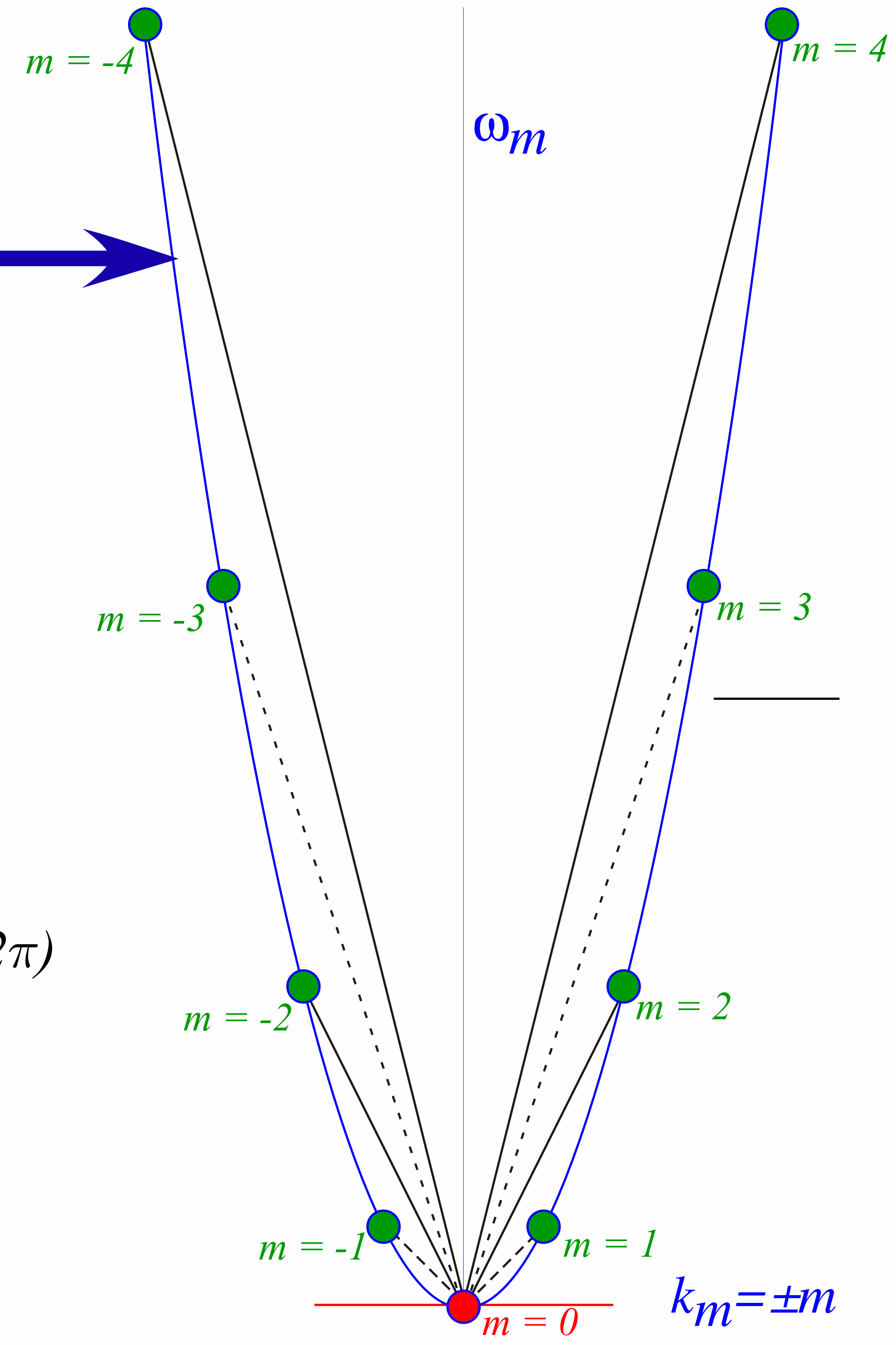
Possible wave velocities
for
Quadratic (Bohr-Rotor) Spectrum

$\omega_m = Bm^2$

$k_m = \pm m$

$$\boxed{V_{phase} = \frac{\omega_m}{k_m} = \frac{Bm^2}{m} = mB}$$

$m = 0, \pm 1, \pm 2, \pm 3, \ldots$ are momentum quanta
in wavevector formula: $k_m = 2\pi m / L \quad (k_m = m \quad if{:}\ L = 2\pi)$



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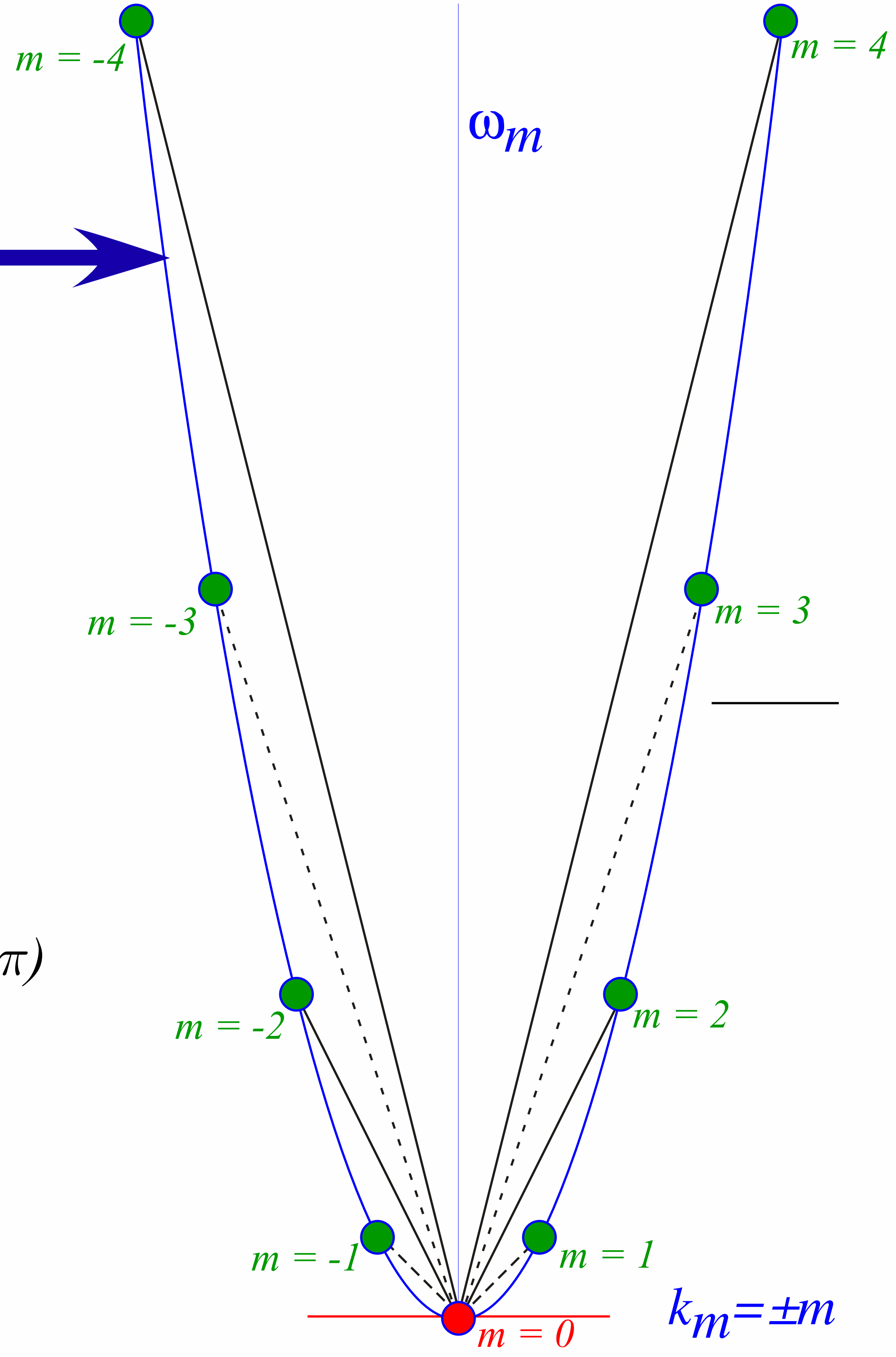

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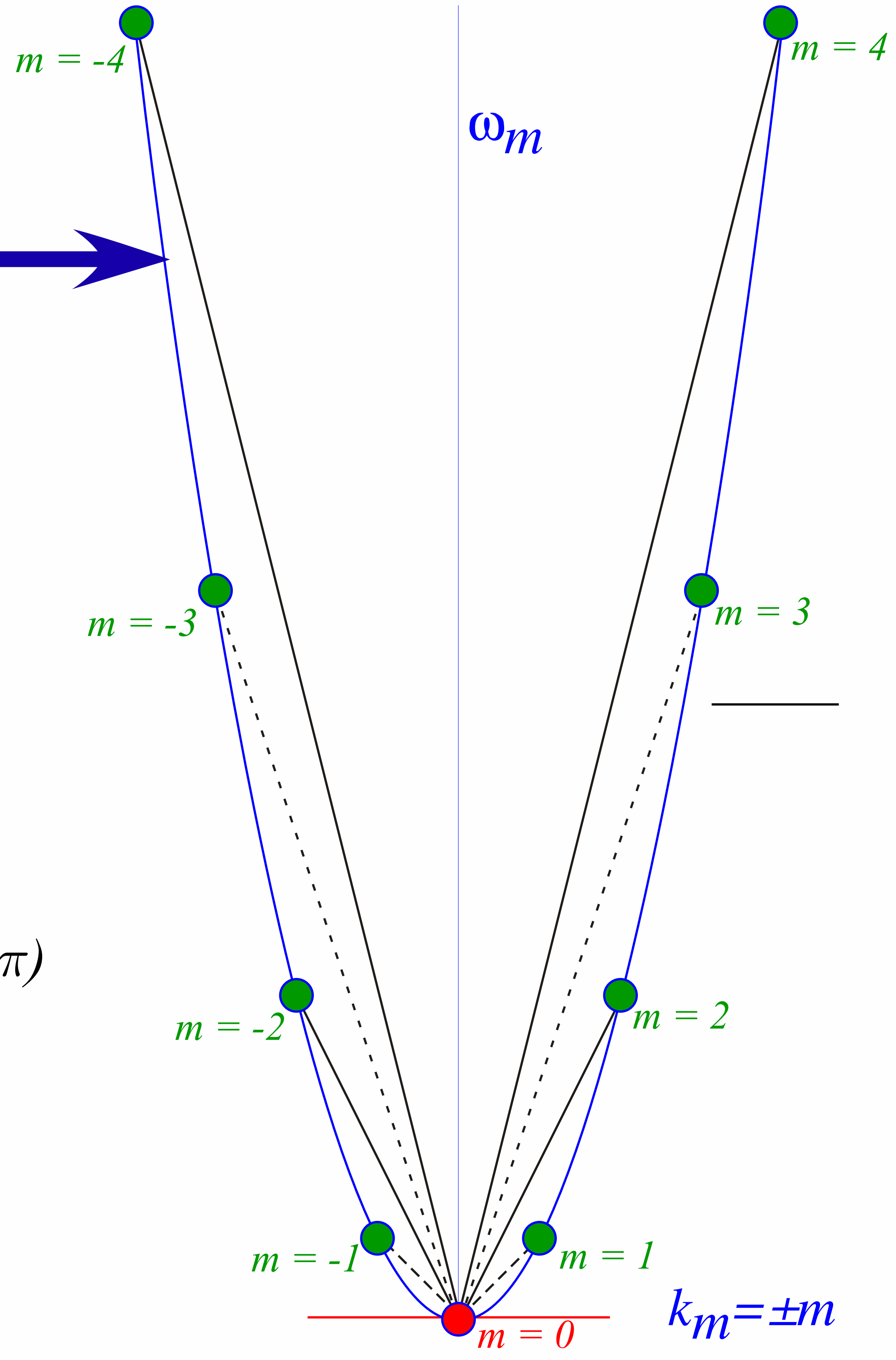
$$E_m = (\hbar k_m)^2/2M = m^2[h^2/2ML^2] = m^2\, h\upsilon_1 = m^2\, \hbar\omega_1$$

Possible wave velocities for Quadratic (Bohr-Rotor) Spectrum

$$\omega_m=Bm^2$$

$$k_m=\pm m$$

$$\boxed{V_{phase}=\frac{\omega_m}{k_m}=\frac{Bm^2}{m}=mB}$$



$m=0, \pm 1, \pm 2, \pm 3$,...are momentum quanta in wavevector formula: $k_m=2\pi m/L$ ($k_m=m$ if: $L=2\pi$)

$$E_m = (\hbar k_m)^2/2M = m^2[h^2/2ML^2] = m^2\, h\upsilon_1 = m^2\, \hbar\omega_1$$

fundamental Bohr ∠−frequency $\omega_1=2\pi\upsilon_1$

and lowest *transition (beat) frequency* $\upsilon_1 = (E_1-E_0)/h$

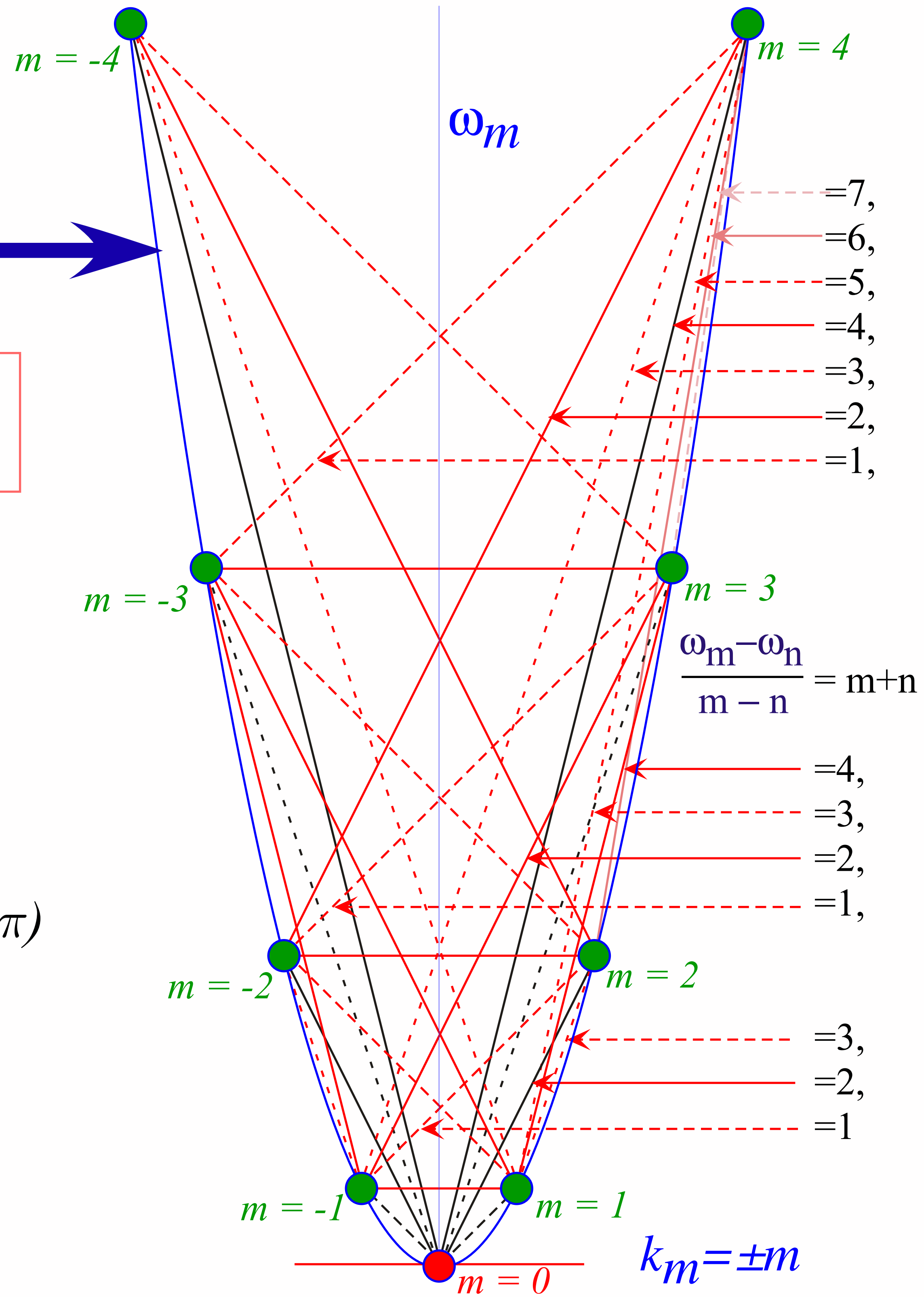
Possible wave velocities
for
Quadratic (Bohr-Rotor) Spectrum

$$\omega_m = Bm^2$$

$$k_m = \pm m$$

$$V_{phase} = \frac{\quad}{k_m} = \frac{Bm^2}{m} = mB$$

$$V_{group} = \frac{\omega_m - \omega_n}{k_m - k_n} = \frac{m^2 - n^2}{m \pm n} B = (m \pm n)B$$



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fundamental Bohr ∠−frequency $\omega_1 = 2\pi\upsilon_1$
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Possible wave velocities
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$\omega_m = Bm^2$

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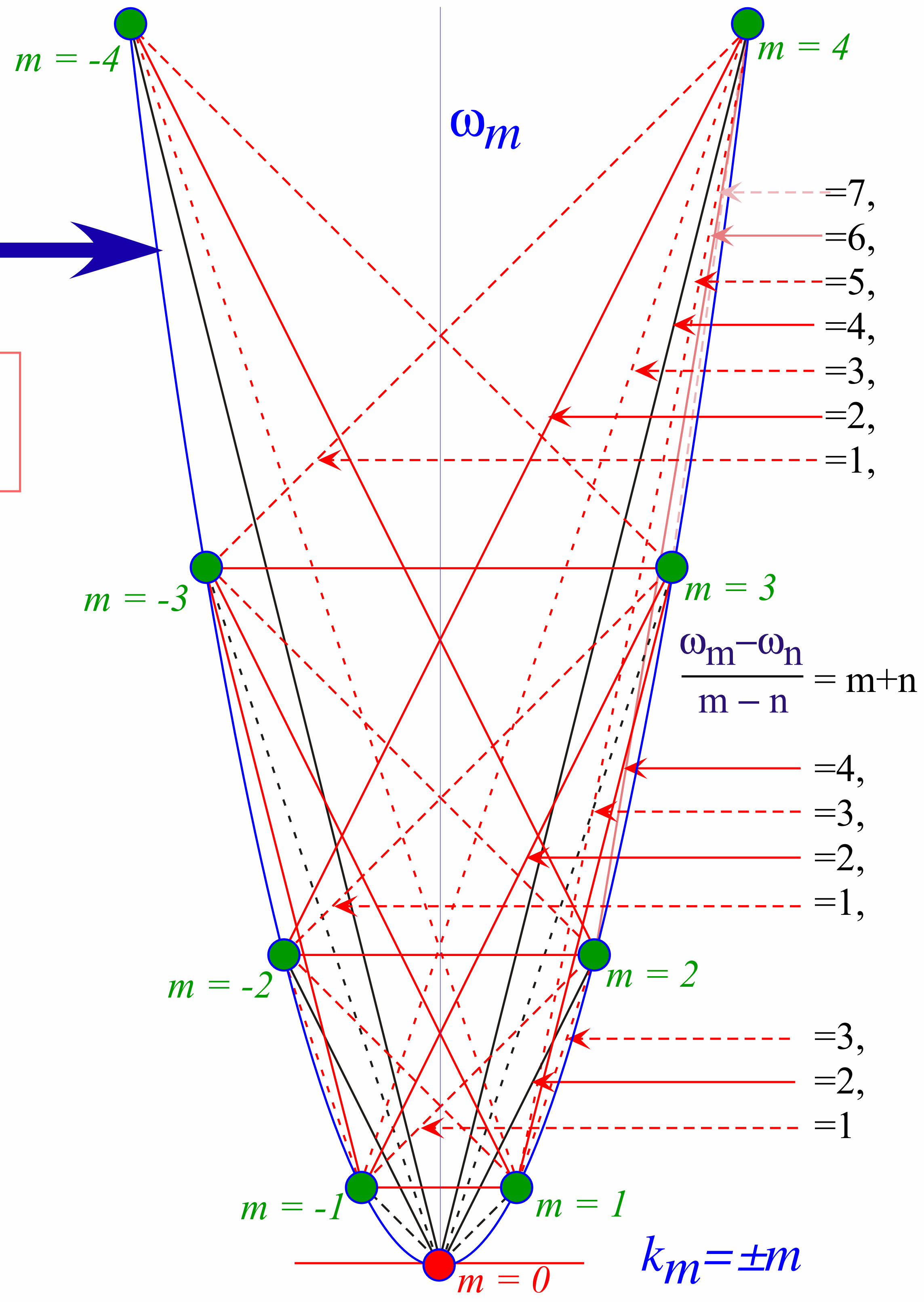
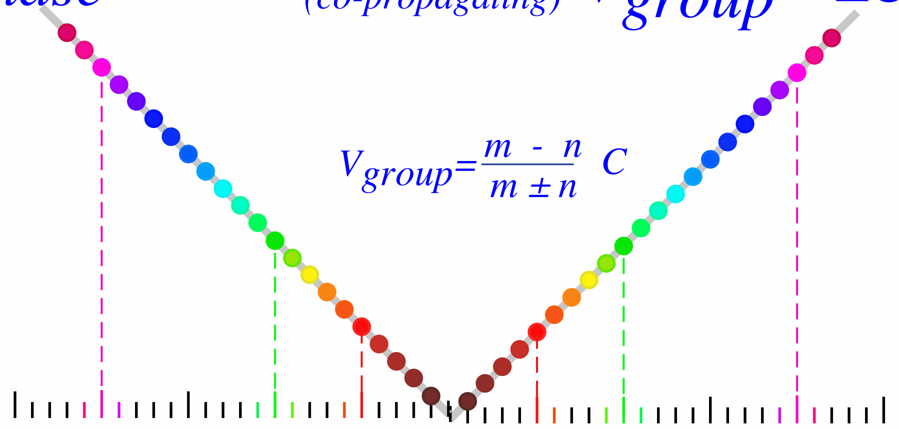


Possible wave velocities
for
Linear (Optical) Spectrum

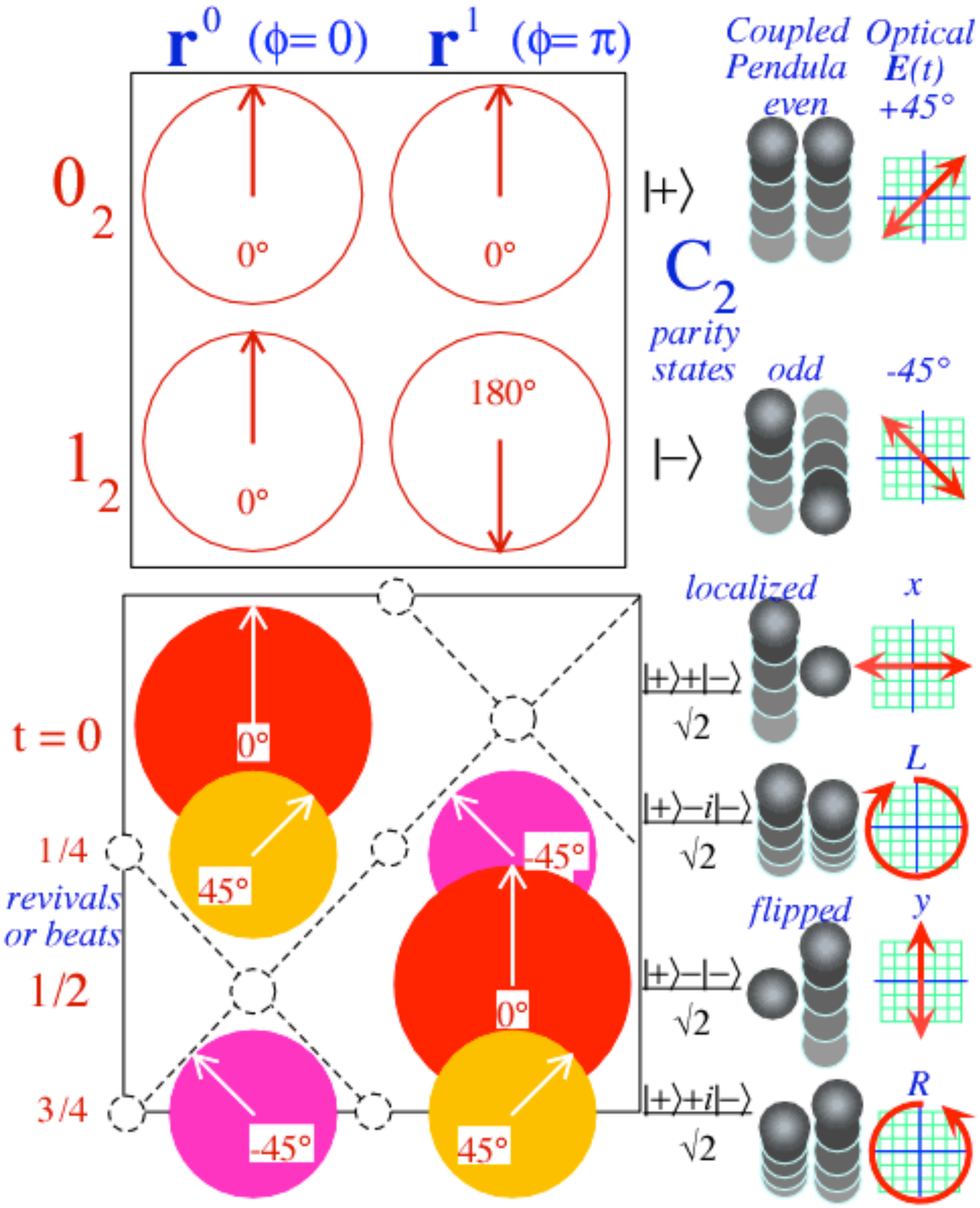
$\omega_m = C|m|^1$

$k_m = m$

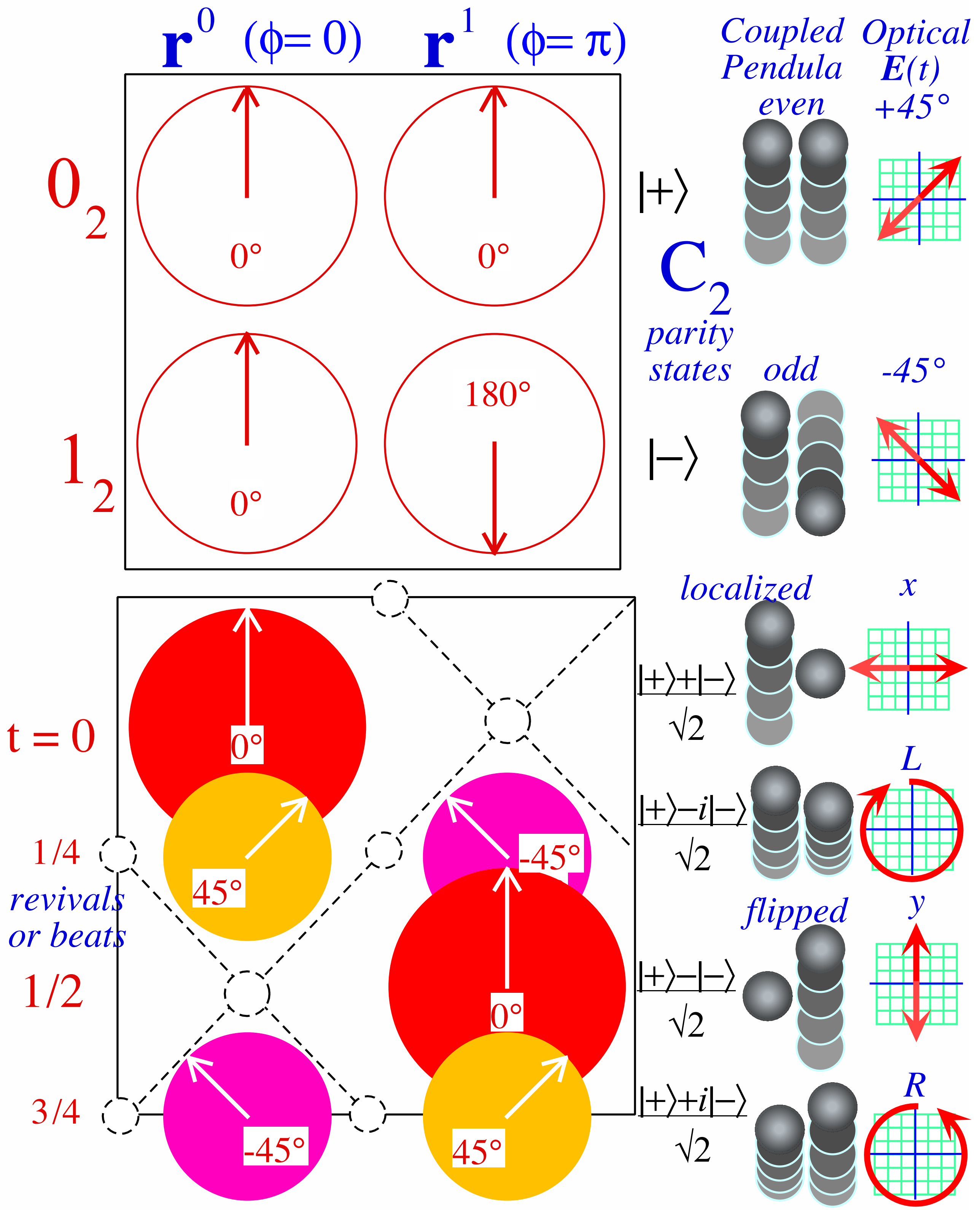
$V_{phase} = \pm C$ (co-propagating) $V_{group} = \pm C$

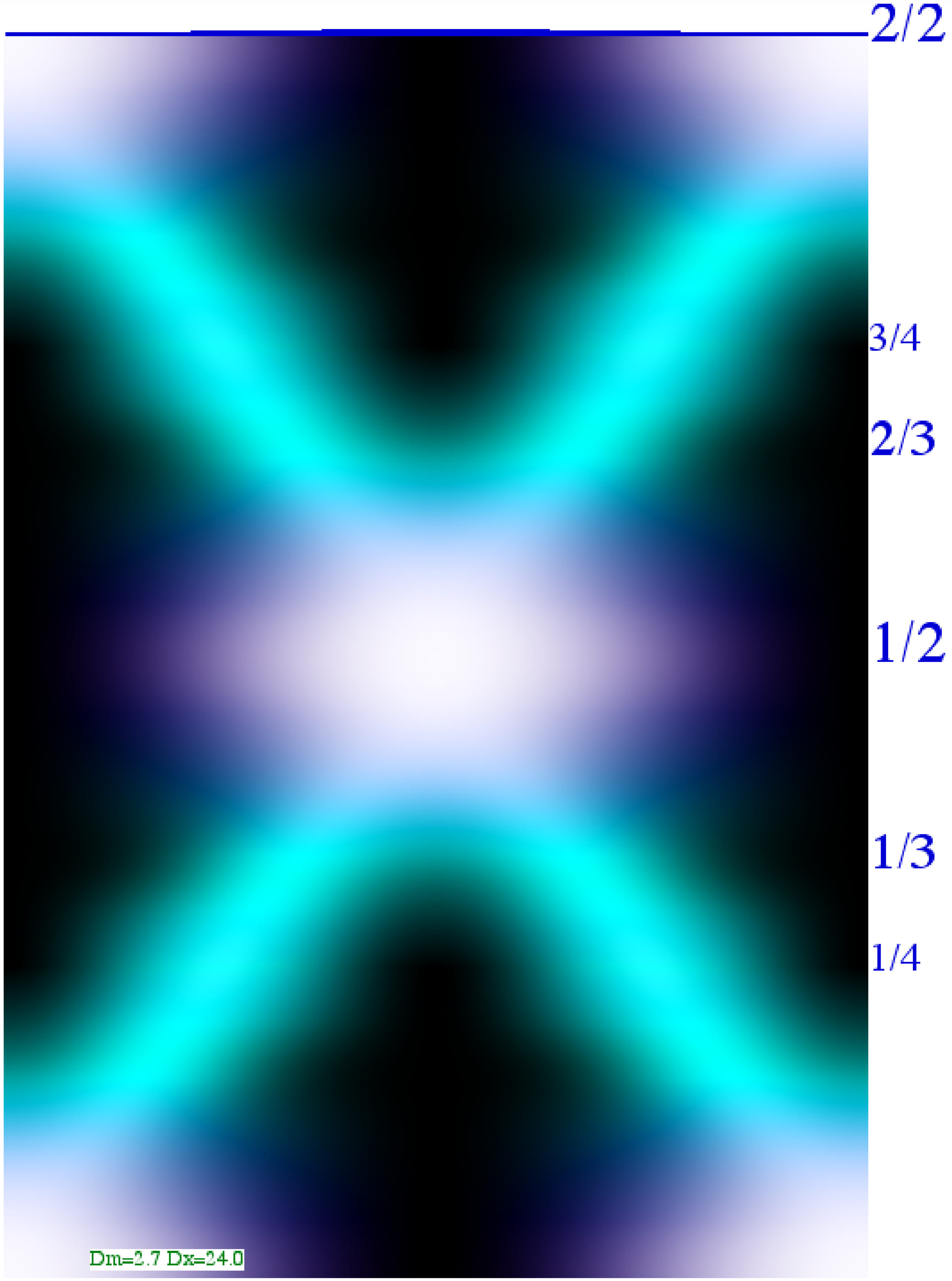


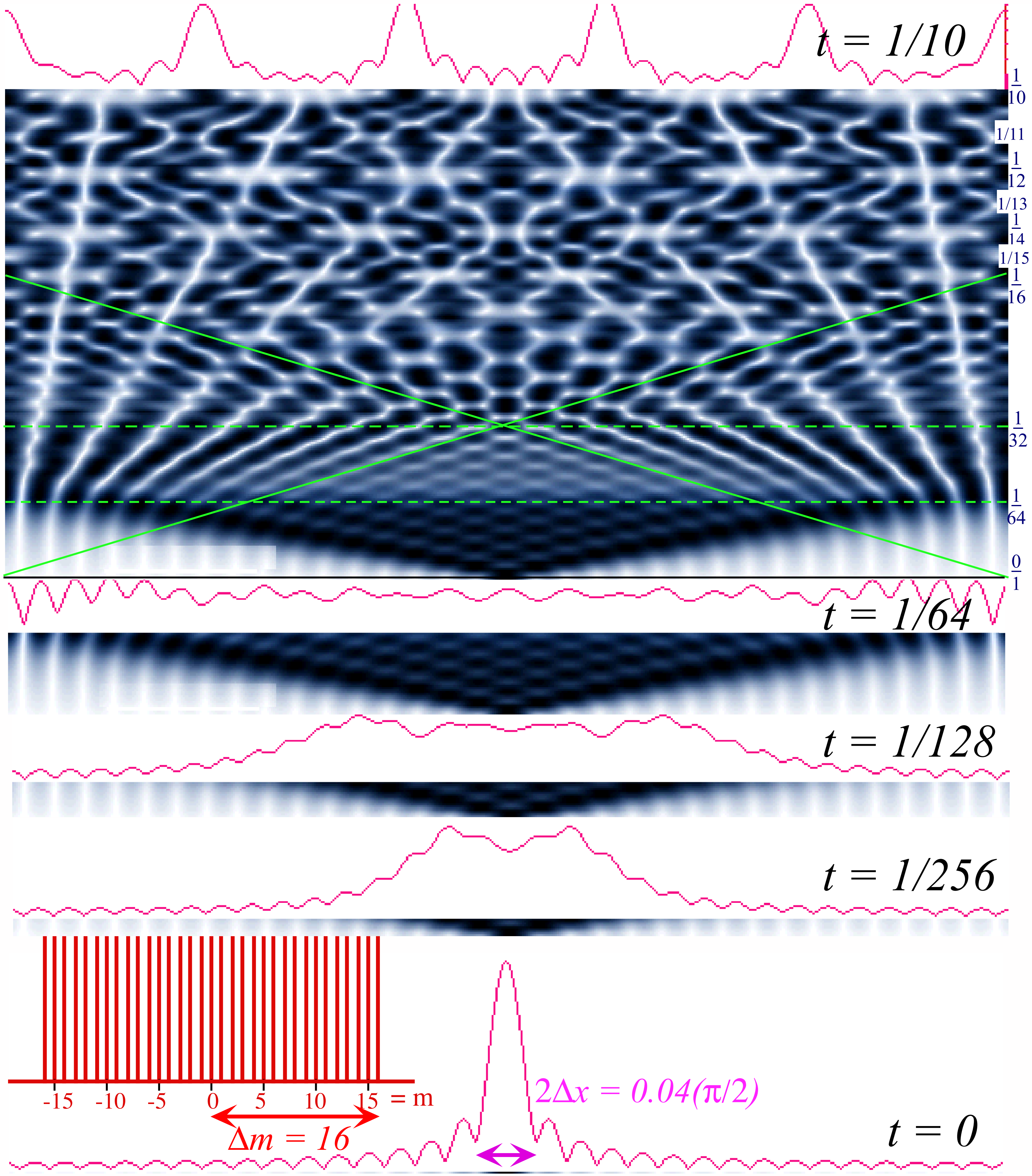
C_2 Fourier transformation matrix and dynamics



Fundamental Beats and 2-Level Transitions: The “Mother of all symmetry” is C_2

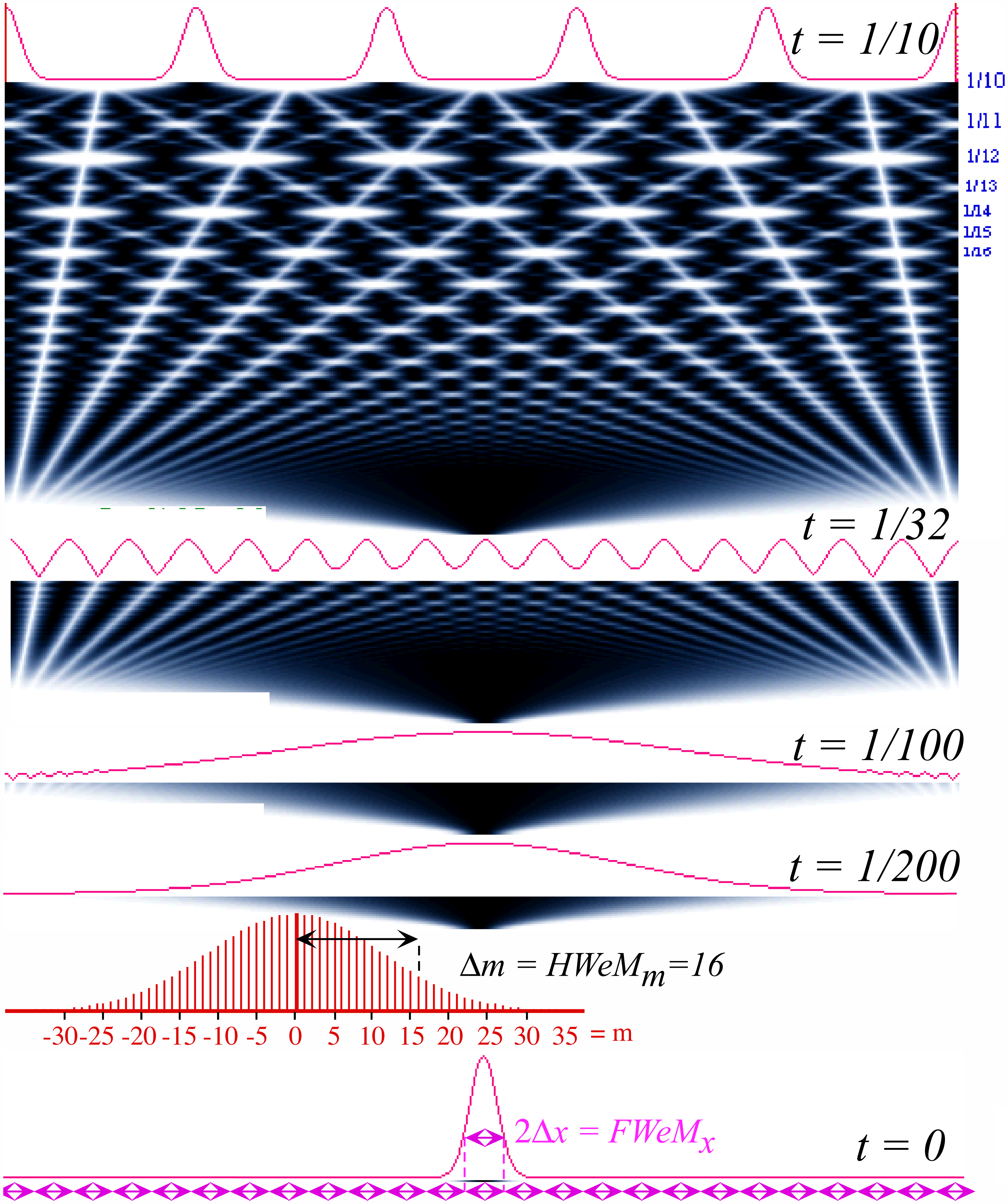






WaveIt Web Simulation

“Boxcar” distribution



WaveIt Web Simulation

Gaussian distribution

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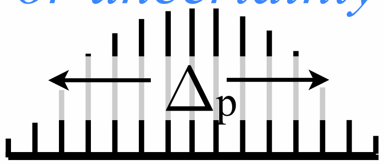
Gaussian wave-packet bandwidth and uncertainty

Let constant Δ_p *be momentum-m "spread" or uncertainty*

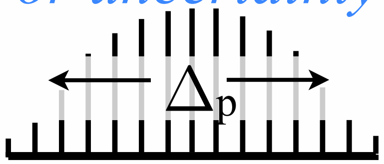


Suppose we excite a Gaussian combination of Bohr momentum-*m* plane waves:

$$\Psi(\phi,t{=}0)=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_p}\right)^2} e^{im\phi}$$

Gaussian wave-packet bandwidth and uncertainty

Let constant Δ_p *be momentum-**m** "spread" or uncertainty*



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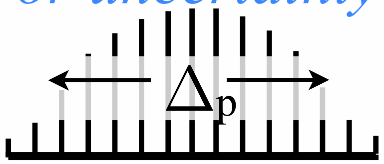
$$\Psi(\phi,t=0)=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_p}\right)^2} e^{im\phi} = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_p}\right)^2+im\phi}$$

Complete the square in exponent to simplify ϕ*-angle wavefunction.*

Gaussian wave-packet bandwidth and uncertainty

*Let constant Δ_{p} be momentum-**m** "spread" or uncertainty*



Suppose we excite a Gaussian combination of Bohr momentum-*m* plane waves:

$$\Psi(\phi,t=0) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_{\mathrm{p}}}\right)^2} e^{im\phi} = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_{\mathrm{p}}}\right)^2 + im\phi}$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_{\mathrm{p}}}\right)^2 + im\phi + \left(\frac{\Delta_{\mathrm{p}}}{2}\phi\right)^2 - \left(\frac{\Delta_{\mathrm{p}}}{2}\phi\right)^2}$$

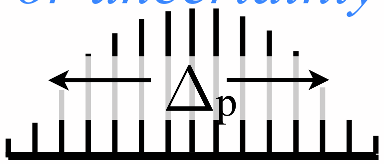
Complete the square in exponent to simplify ϕ-angle wavefunction.

Add and subtract : $\left(\frac{\Delta_{\mathrm{p}}}{2}\phi\right)^2 - \left(\frac{\Delta_{\mathrm{p}}}{2}\phi\right)^2$

in exponent...

Gaussian wave-packet bandwidth and uncertainty

*Let constant Δ_{p} be momentum-**m** "spread" or uncertainty*



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Extract binomial : $-\left(\frac{m}{\Delta_{\mathrm{p}}} - i\frac{\Delta_{\mathrm{p}}}{2}\phi\right)^2$

Gaussian wave-packet bandwidth and uncertainty

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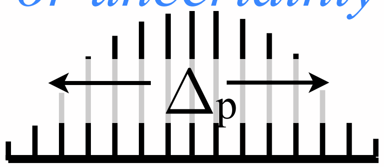

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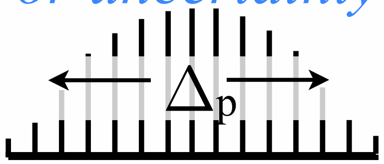

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Gaussian wave-packet bandwidth and uncertainty

Let constant Δ_p be momentum-m "spread" or uncertainty

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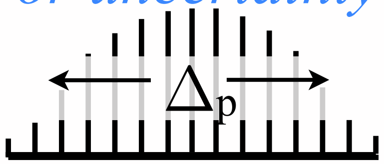

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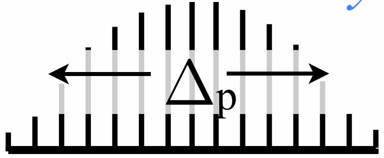
Let constant Δ_p *be momentum-m "spread" or uncertainty*



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Gaussian integral:

$$\sqrt{\int_{-\infty}^{\infty} e^{-x^2}dx}\sqrt{\int_{-\infty}^{\infty} e^{-y^2}dy} = \sqrt{\iint e^{-(x^2+y^2)}\,dx\,dy}$$

$$= \sqrt{\int_0^{2\pi}\int_0^{\infty} e^{-r^2}r\,dr\,d\theta} = \sqrt{2\pi\int_0^{\infty} e^{-r^2}\frac{dr^2}{2}} = \sqrt{\pi}$$

where:

$$A\left(\Delta_\text{p},\phi\right) = \sum_{m=-\infty}^{\infty} e^{-\left(\frac{m}{\Delta_\text{p}} - i\frac{\Delta_\text{p}}{2}\phi\right)^2} \xrightarrow[\Delta_\text{p}>>1]{} \int_{-\infty}^{\infty} dk\; e^{-\left(\frac{k}{\Delta_\text{p}} - i\frac{\Delta_\text{p}}{2}\phi\right)^2}$$

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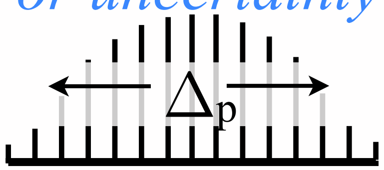

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$$\Psi(\phi,t=0)=\frac{\Delta_{\text{p}}}{2\sqrt{\pi}} e^{-\left(\frac{\Delta_{\text{p}}}{2}\phi\right)^2}$$

It is a Gaussian distribution, too

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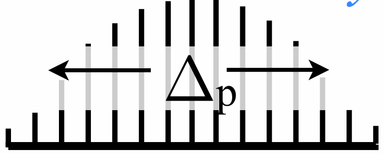

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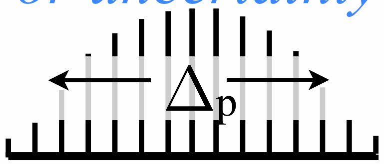

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Gaussian uncertainty relation

(Compare to $\Delta x\cdot\Delta k=\pi$ *for ∞-Well)*

where:

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$$E_m=(\hbar k_m)^2/2M=m^2[h^2/2ML^2]=m^2\,h\upsilon_1=m^2\,\hbar\omega_1$$

fundamental Bohr ∠−frequency $\omega_1=2\pi\upsilon_1$ and lowest *transition (beat) frequency* $\upsilon_1=(E_1-E_0)/h$

Polygonal geometry of $U(2)\supset C_N$ character spectral function
Algebra
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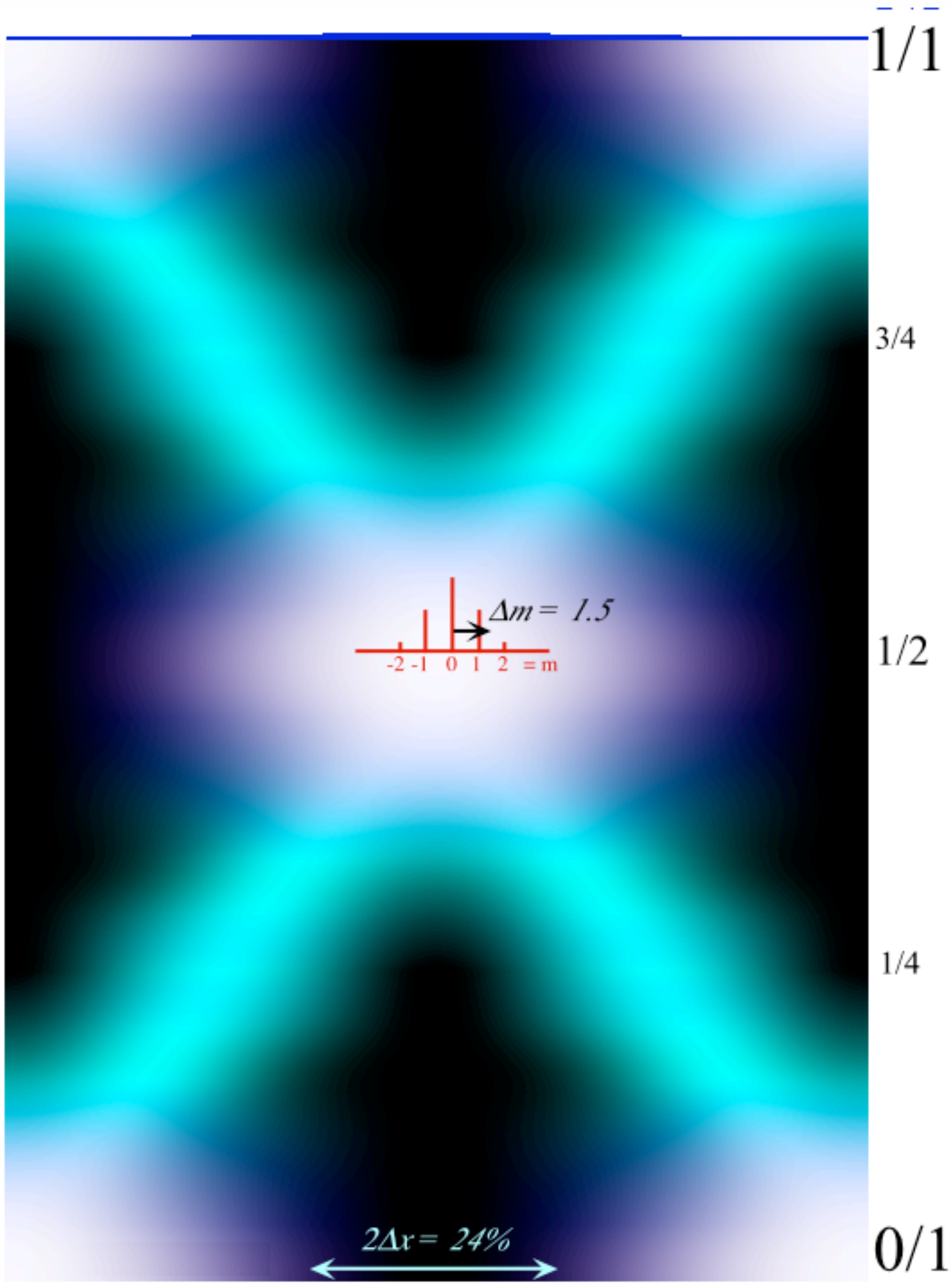
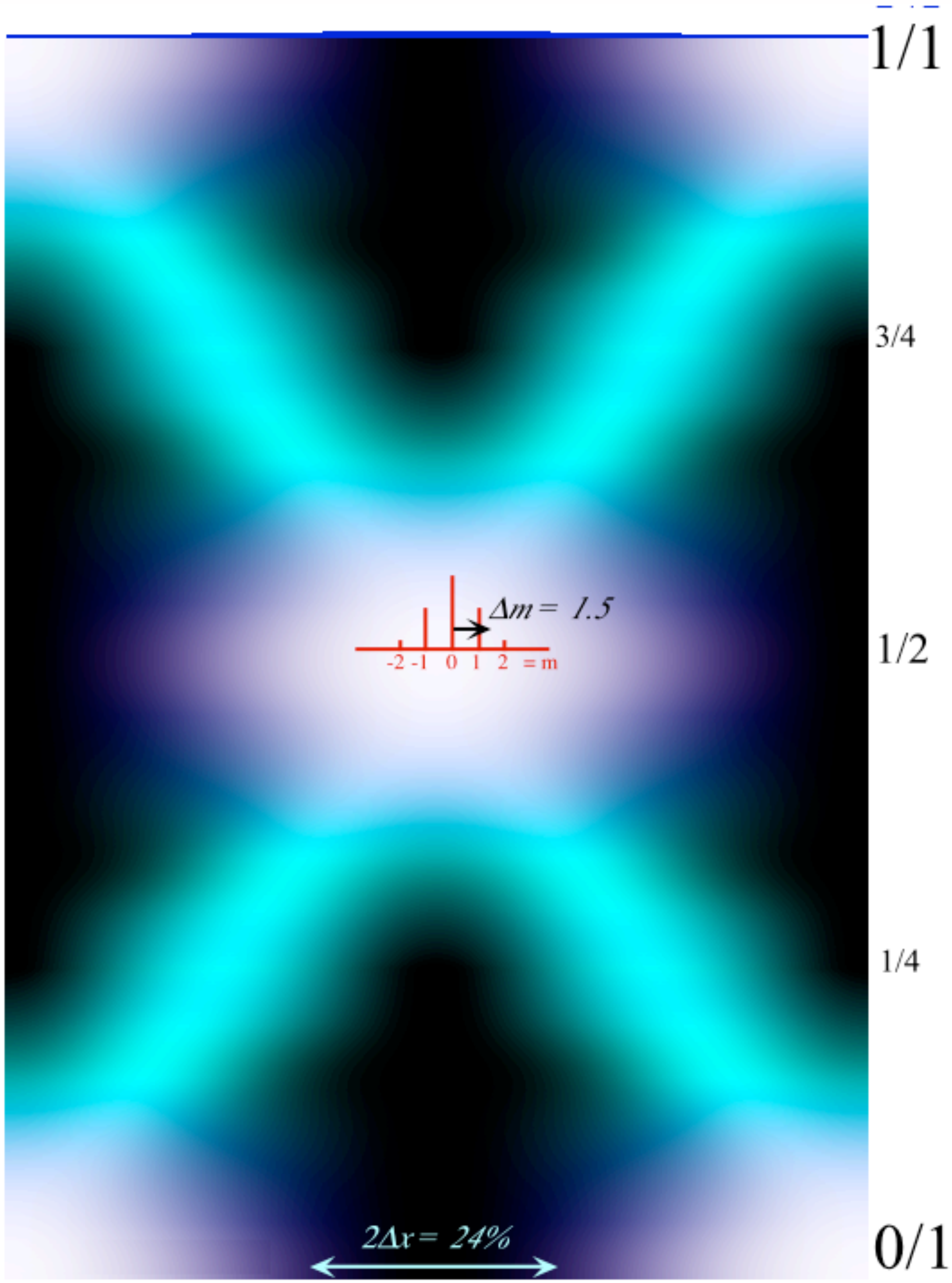

1/1
3/4
$\Delta m = 1.5$
-2 -1 0 1 2 = m
1/2
1/4
$2\Delta x = 24\%$
0/1


1/1
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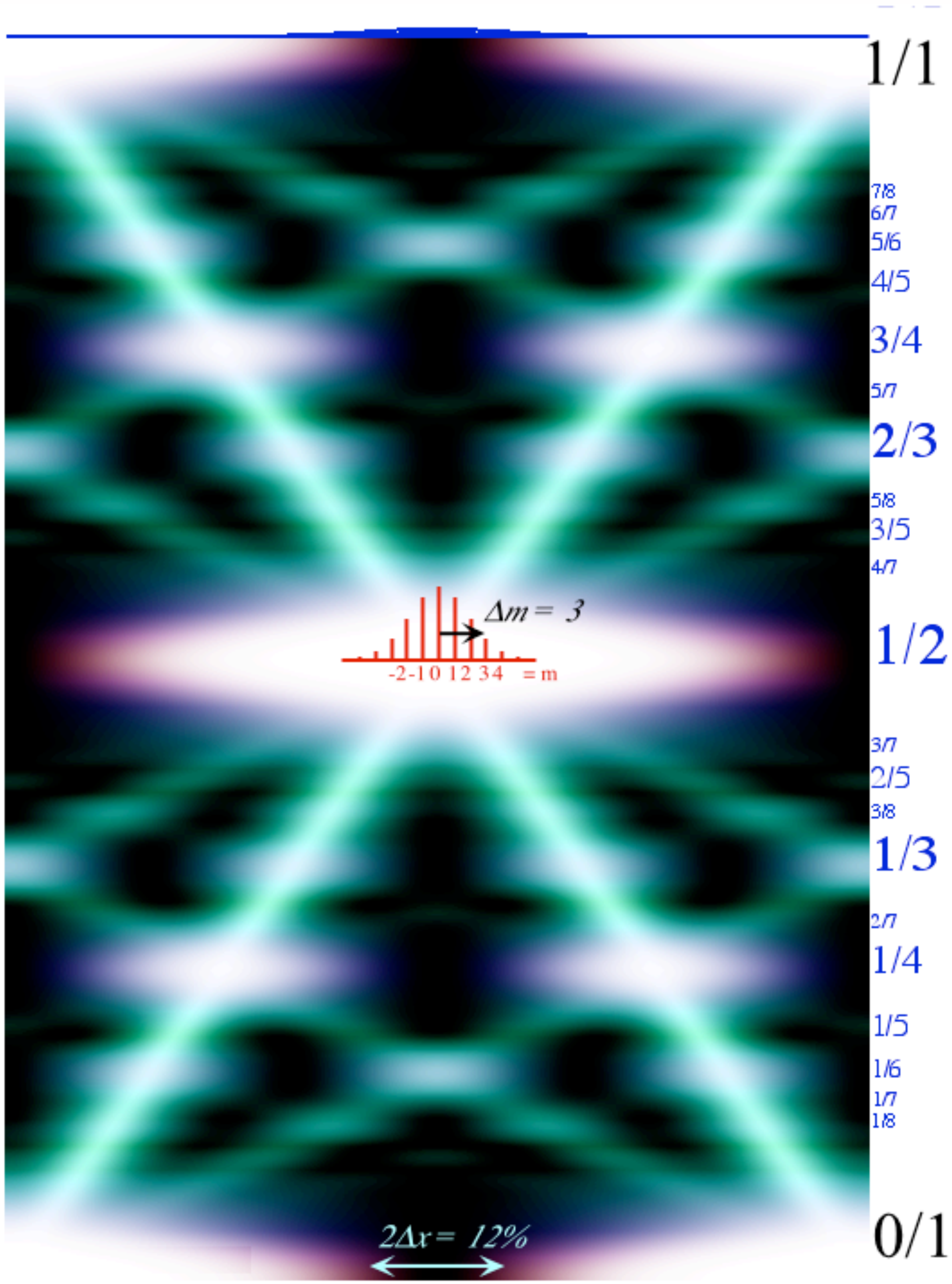
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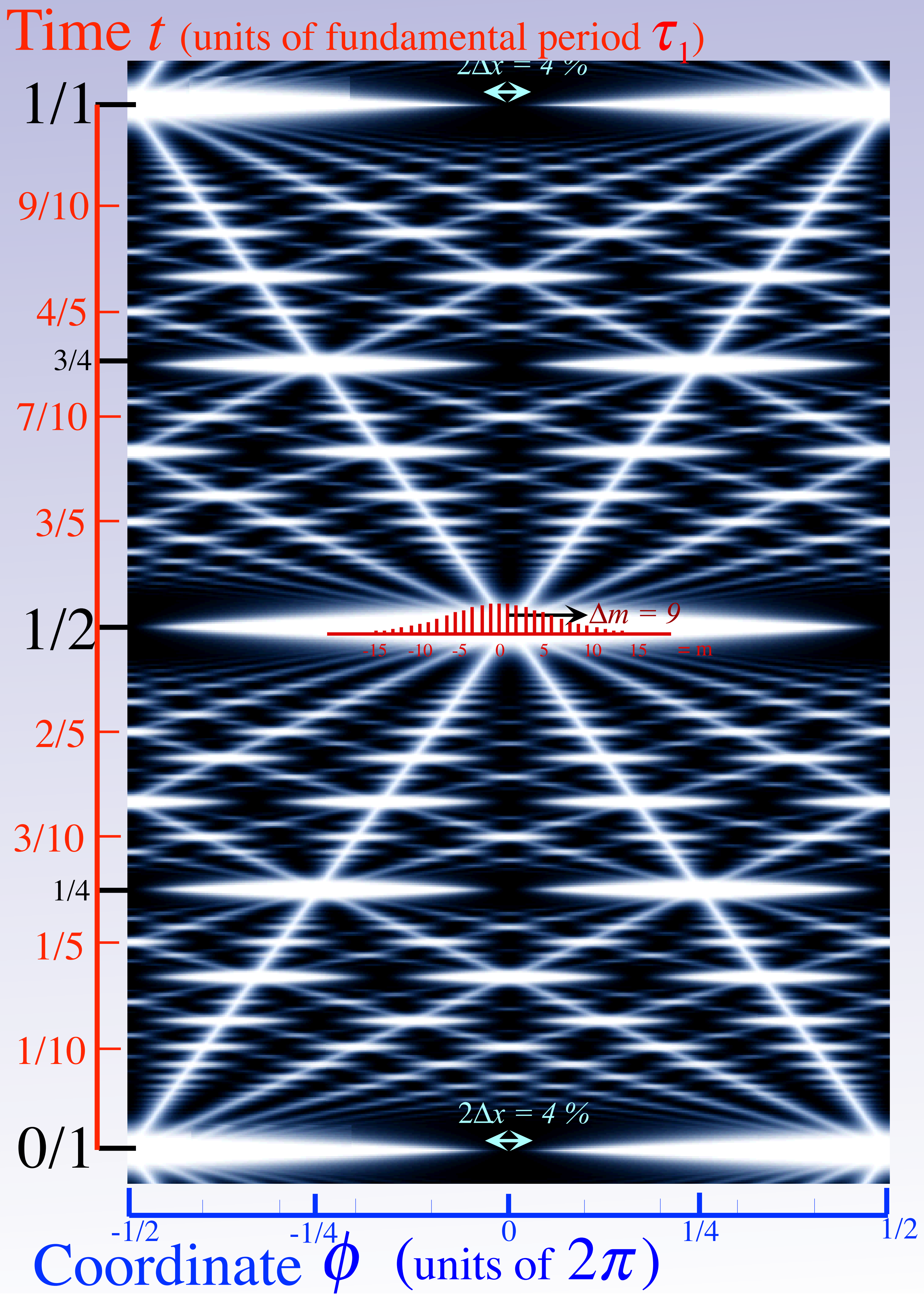
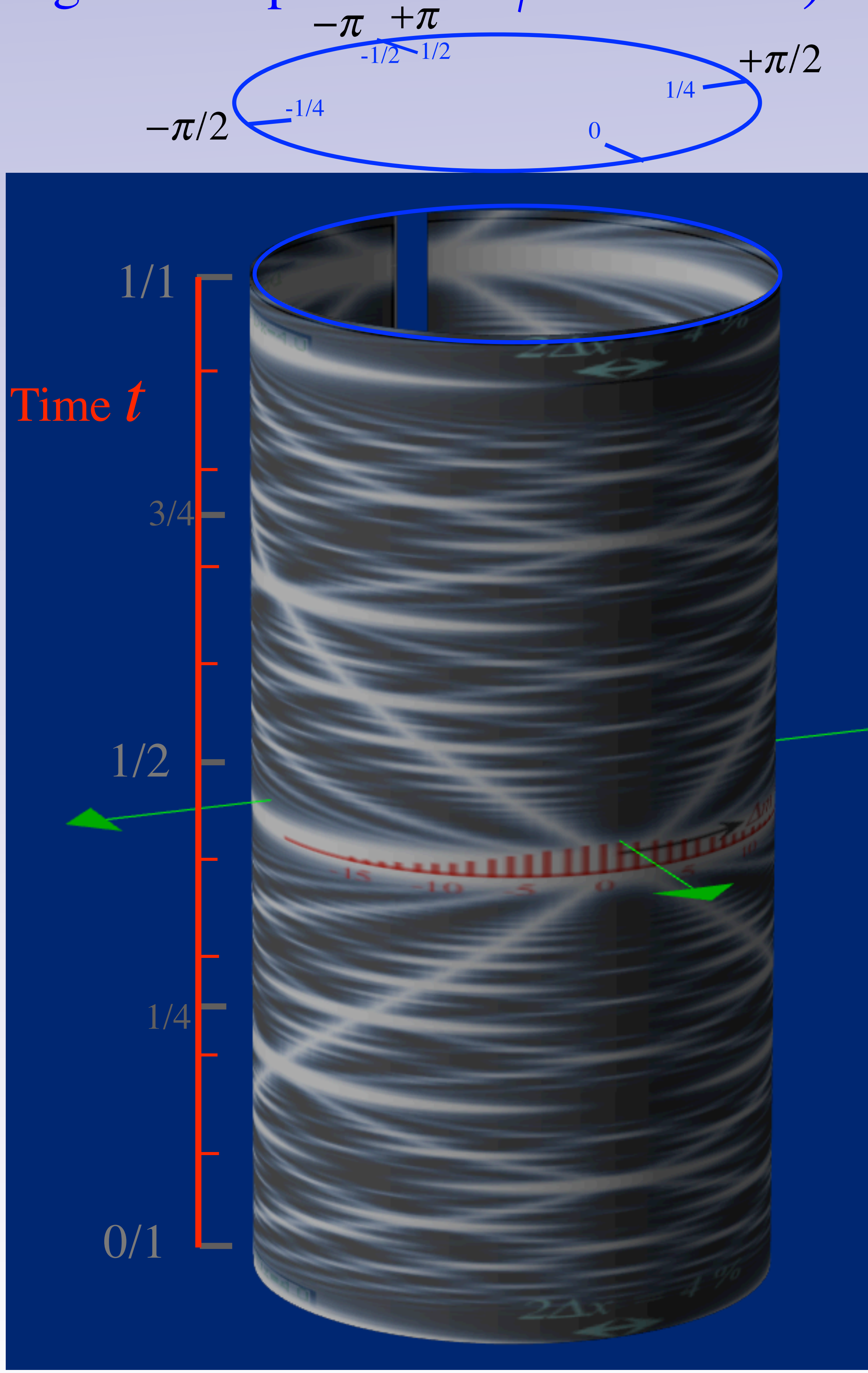
3/5
4/7
1/2
3/7
2/5
3/8
1/3
2/7
1/4
1/5
1/6
1/7
1/8
0/1
Δm = 3
-2-1 0 1 2 3 4 = m
2Δx = 12%



(Imagine "wrap-around" ϕ-coordinate)

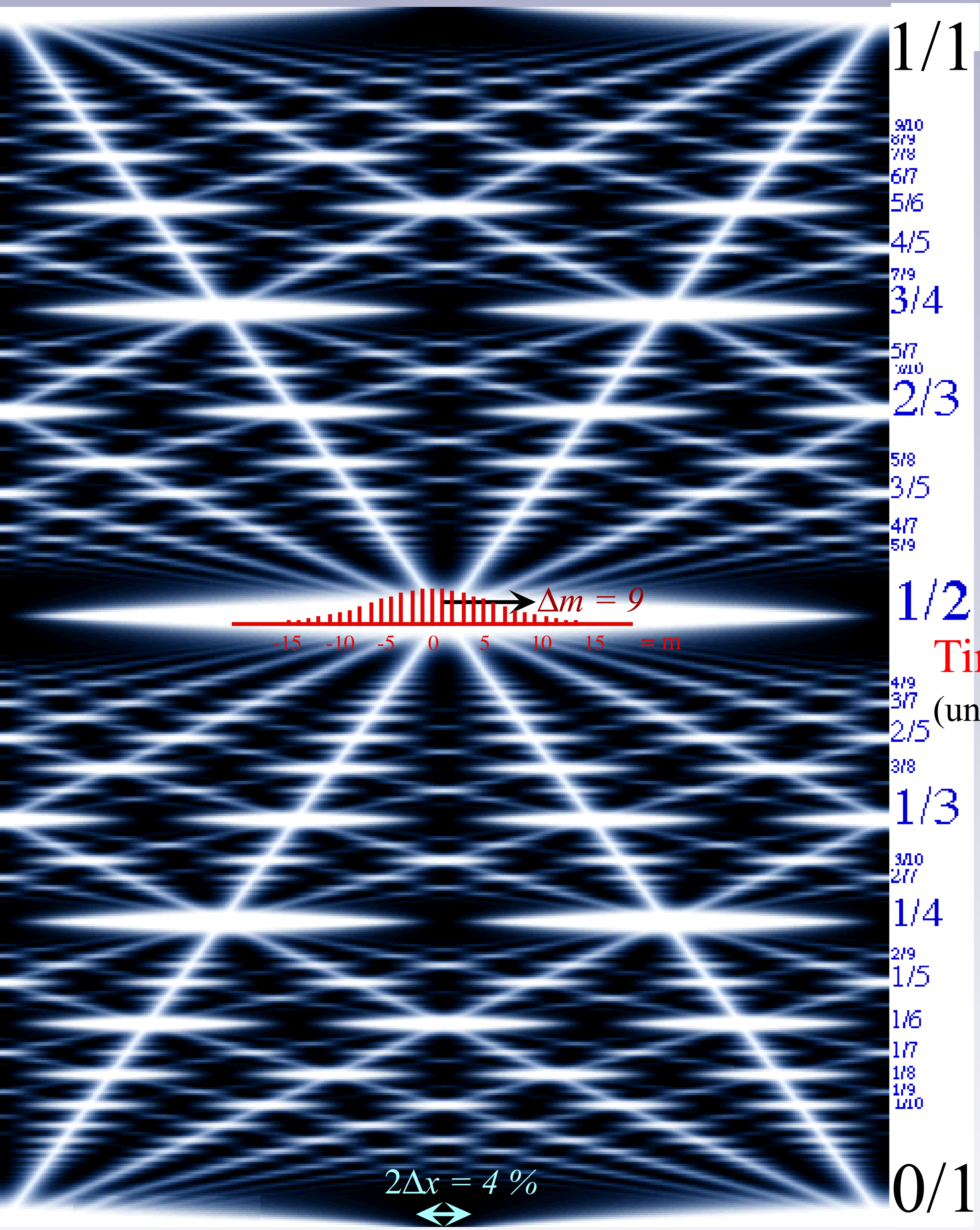


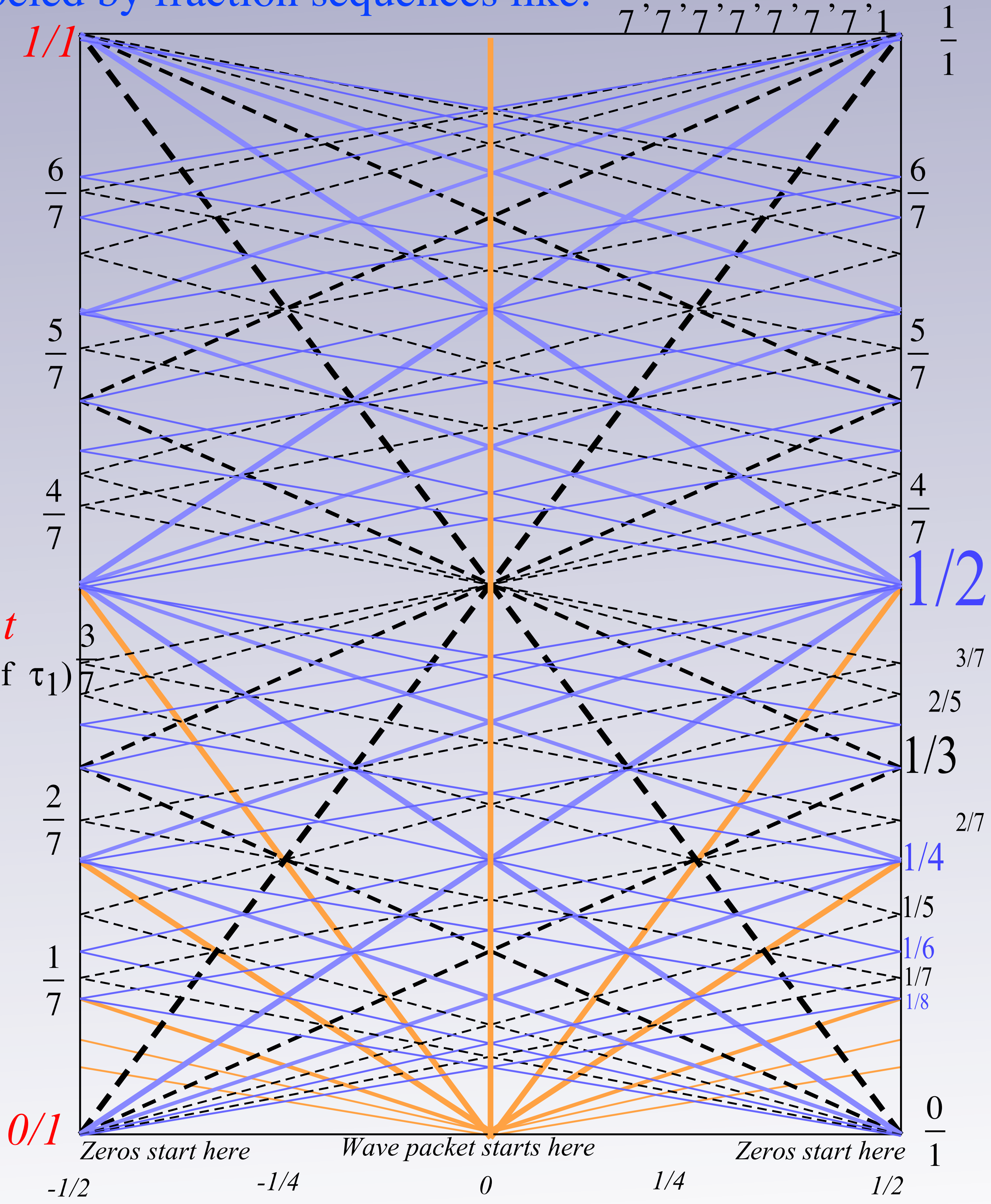
[*Harter, J. Mol. Spec. 210, 166-182 (2001)*]

N-level-system and revival-beat wave dynamics

(9 or10-levels (0, ±1, ±2, ±3, ±4,..., ±9, ±10, ±11...) excited)

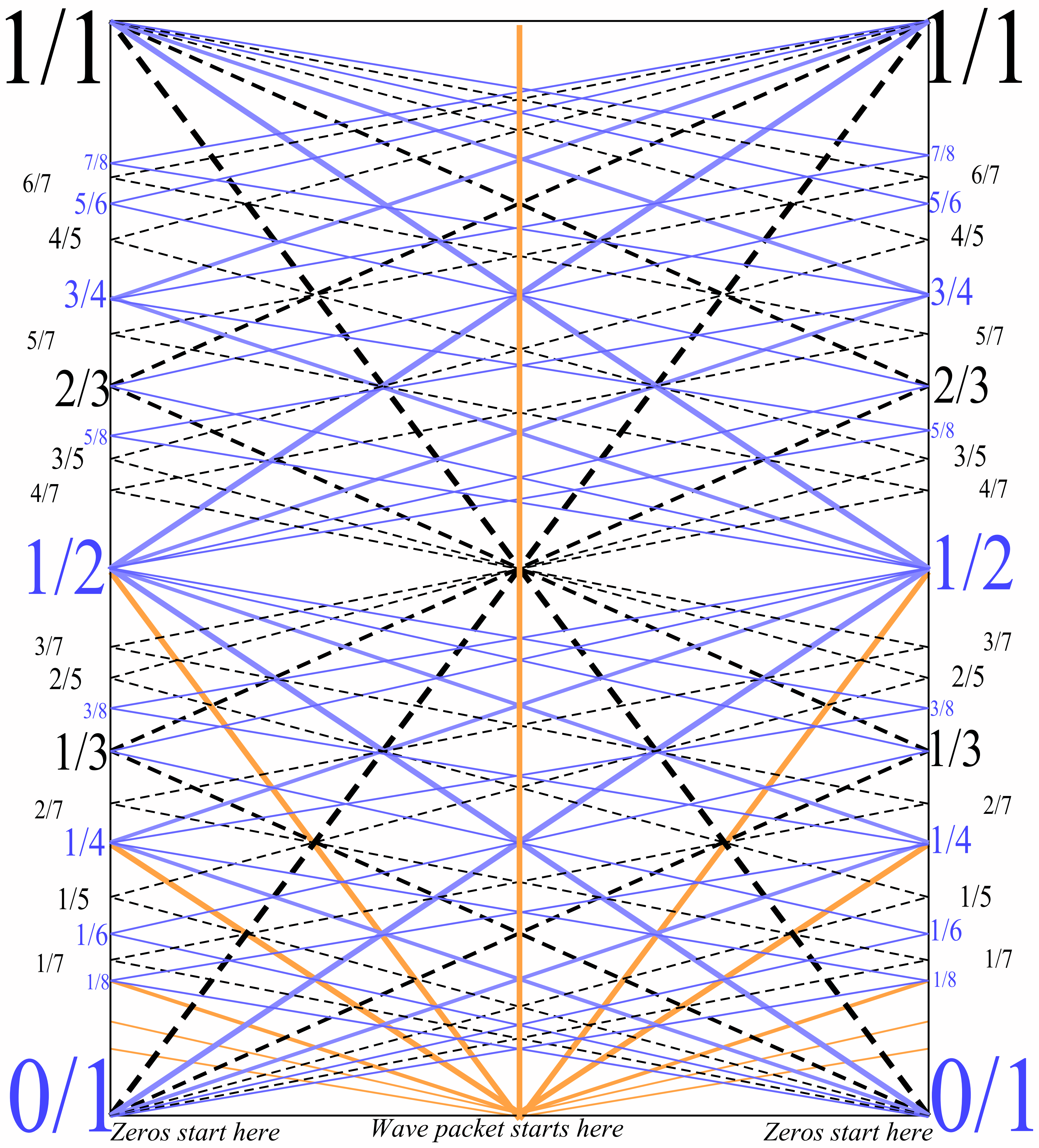
Zeros (clearly) and "particle-packets" (faintly) have paths labeled by fraction sequences like: $\frac{0}{7}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \frac{1}{1}$





Coordinate φ
(units of 2π)

[Harter, J. Mol. Spec. 210, 166-182 (2001)]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1/4
1/4
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1/5
1/6
1/6
1/7
1/7
1/8
1/8
0/1
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Zeros start here
Wave packet starts here
Zeros start here

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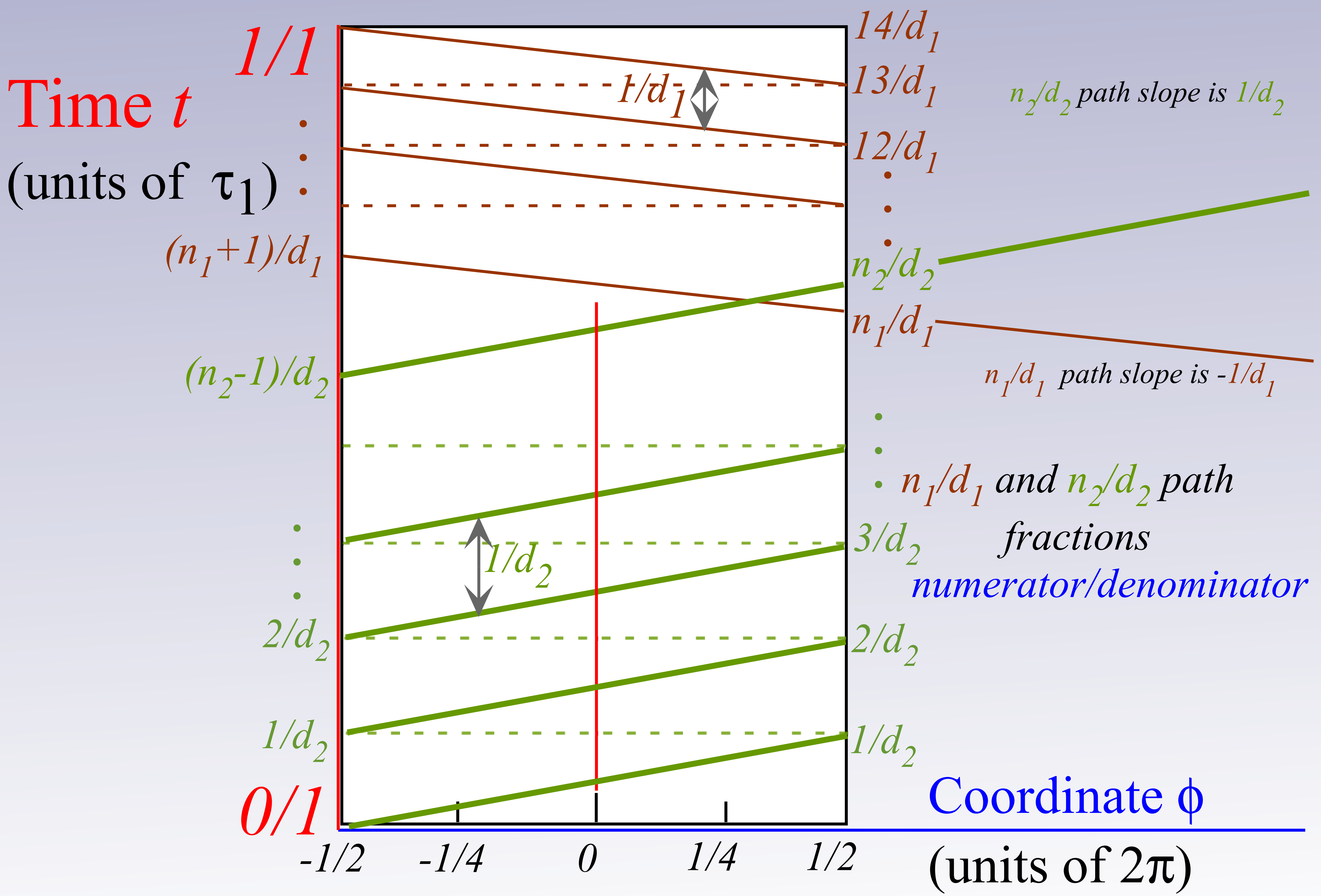
Farey Sum algebra of revival-beat wave dynamics

Label by *numerators N* and *denominators D* of rational fractions *N/D*



n_2/d_2 *path slope is* $1/d_2$

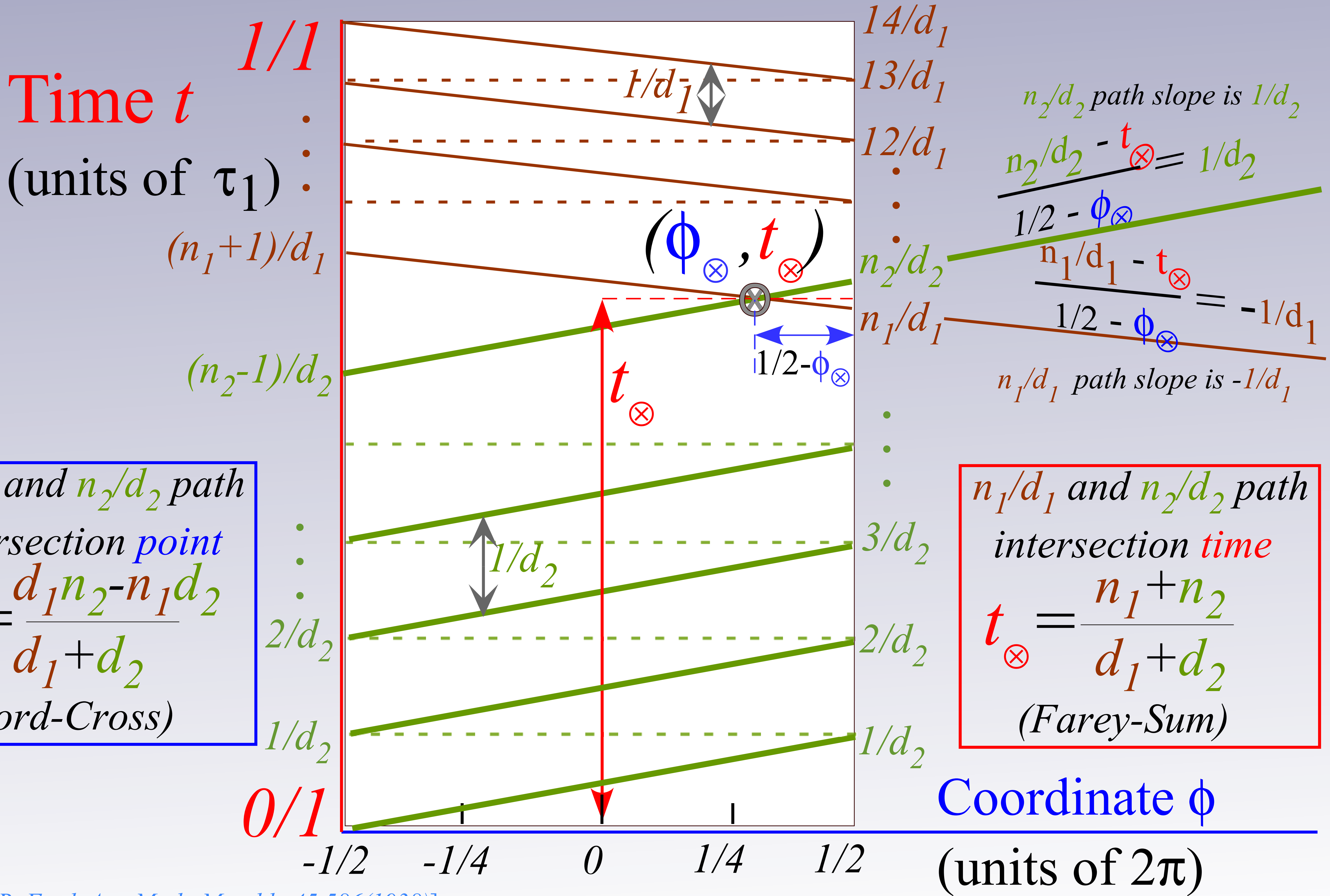
n_1/d_1 *path slope is* $-1/d_1$

n_1/d_1 *and* n_2/d_2 *path fractions*

numerator/denominator

Farey Sum algebra of revival-beat wave dynamics

Label by *numerators N* and *denominators D* of rational fractions *N/D*



$$\frac{n_2/d_2 - t_\otimes}{1/2 - \phi_\otimes} = 1/d_2$$

$$\frac{n_1/d_1 - t_\otimes}{1/2 - \phi_\otimes} = -1/d_1$$

n_1/d_1 and n_2/d_2 path intersection point

$$\phi_\otimes = \frac{d_1 n_2 - n_1 d_2}{d_1 + d_2}$$

(Ford-Cross)

n_1/d_1 and n_2/d_2 path intersection time

$$t_\otimes = \frac{n_1 + n_2}{d_1 + d_2}$$

(Farey-Sum)

[Lester. R. Ford, Am. Math. Monthly 45,586(1938)]

[John Farey, Phil. Mag.(1816)]

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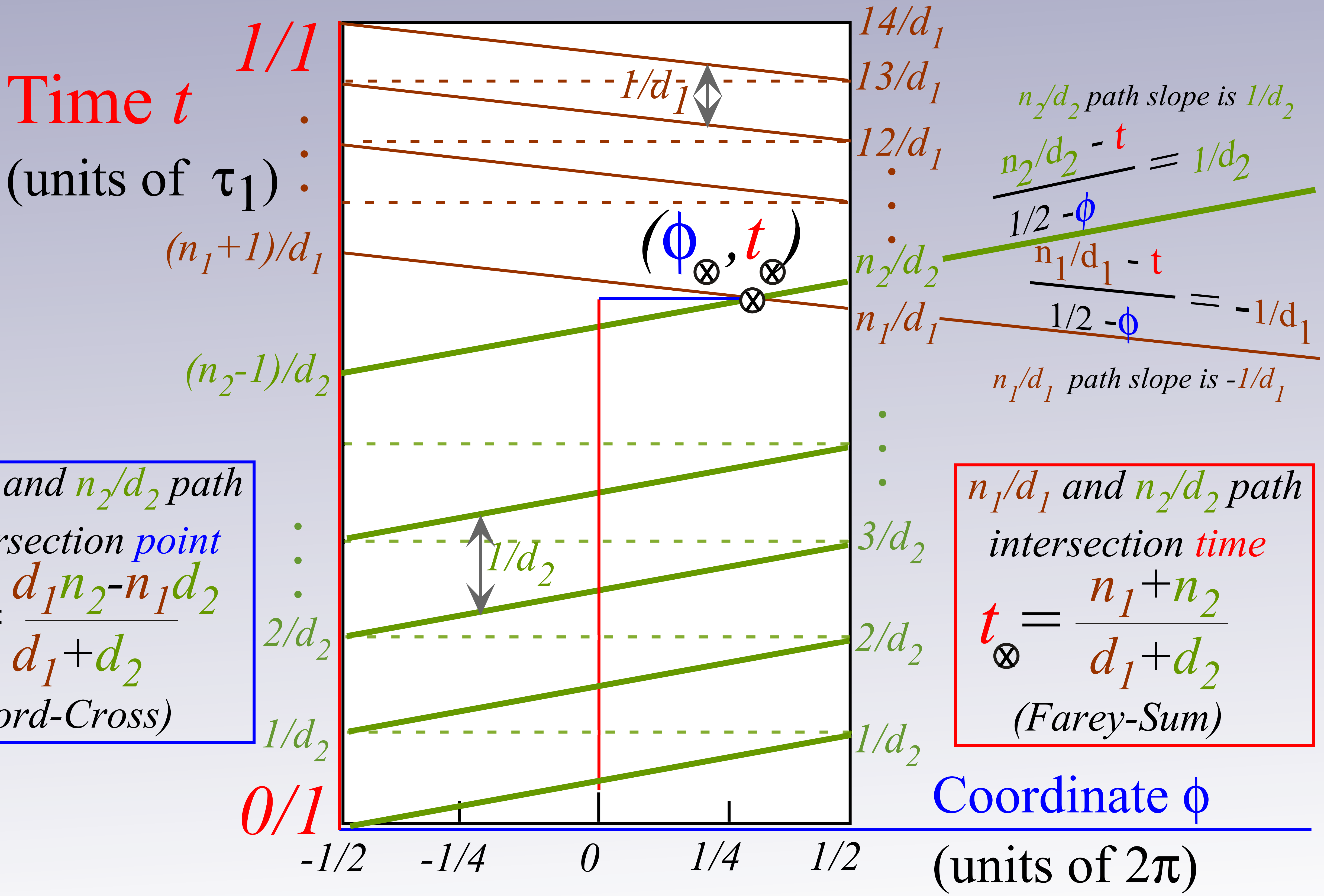

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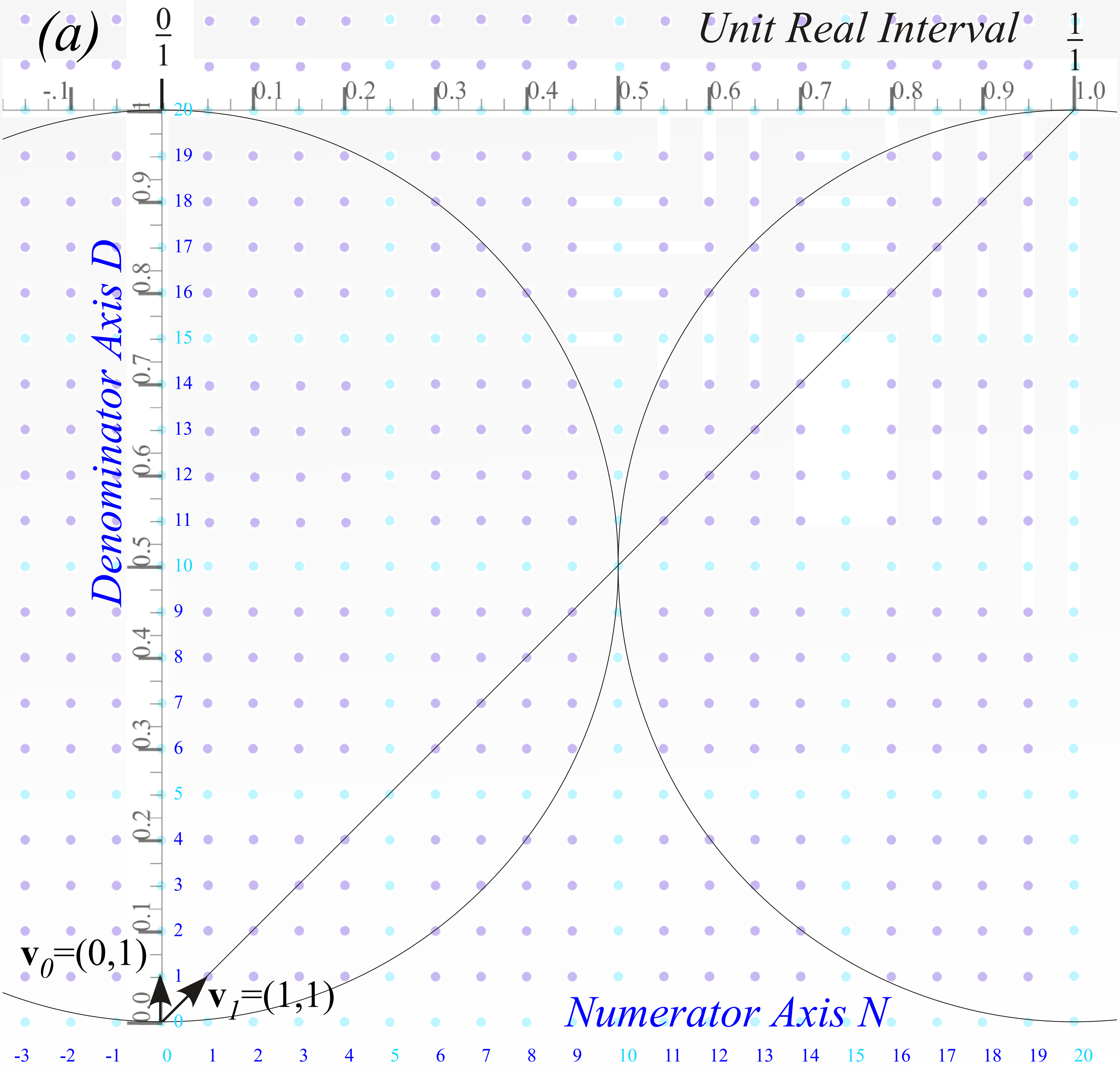
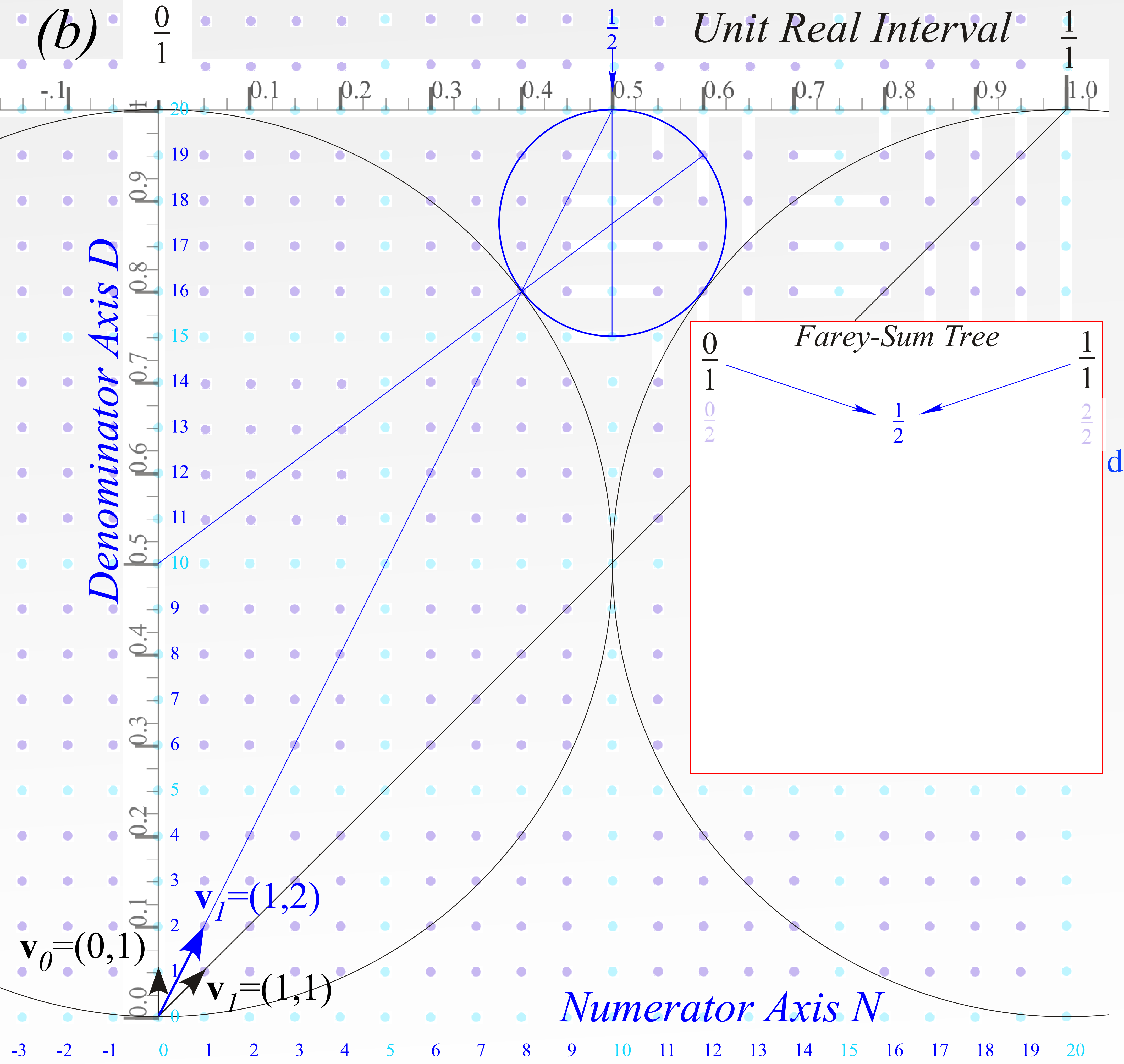


Farey Sum related to vector sum and *Ford Circles*

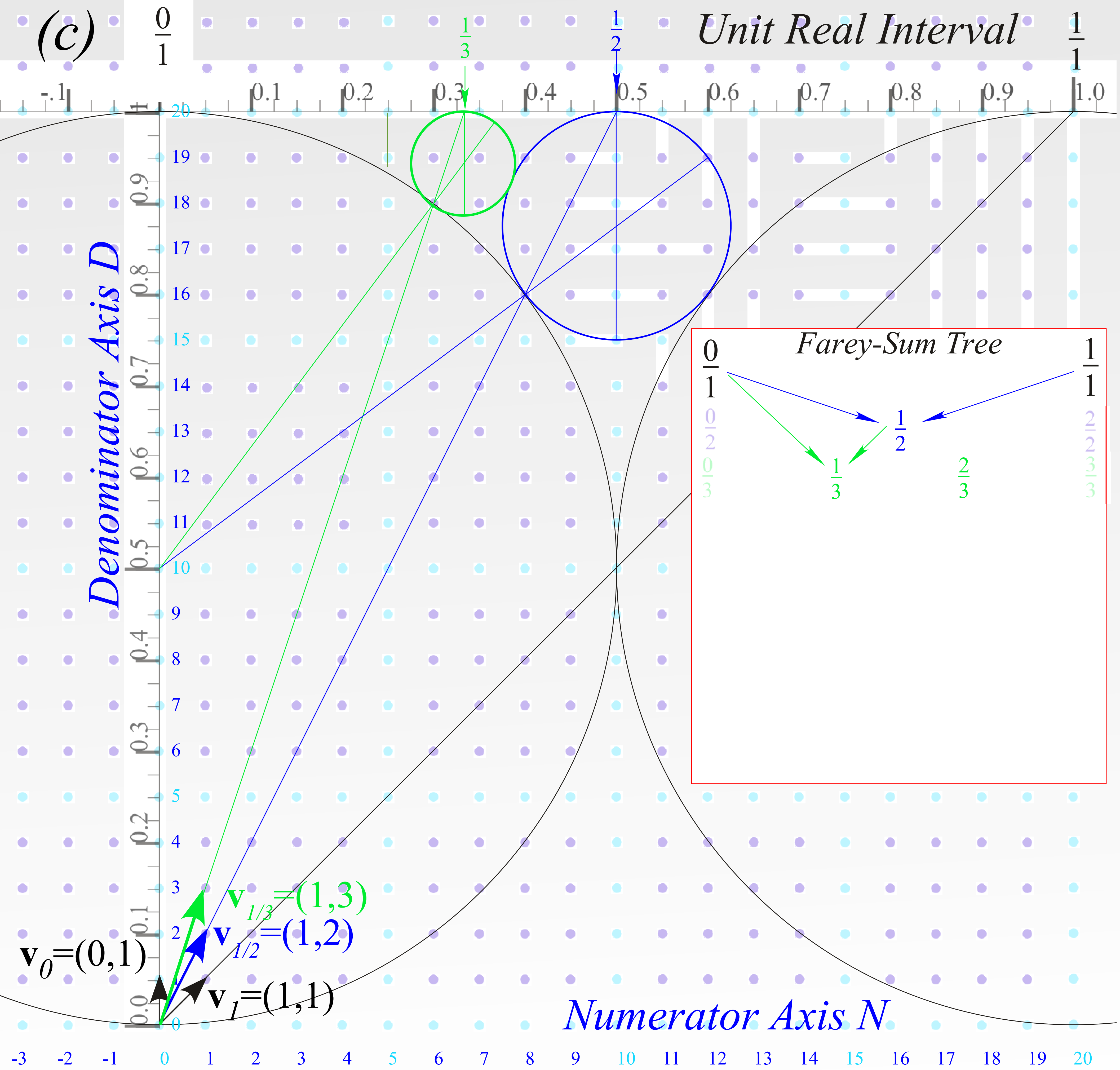
1/1-circle has diameter *1*



Farey Sum related to vector sum and *Ford Circles*

1/1-circle has diameter *1*

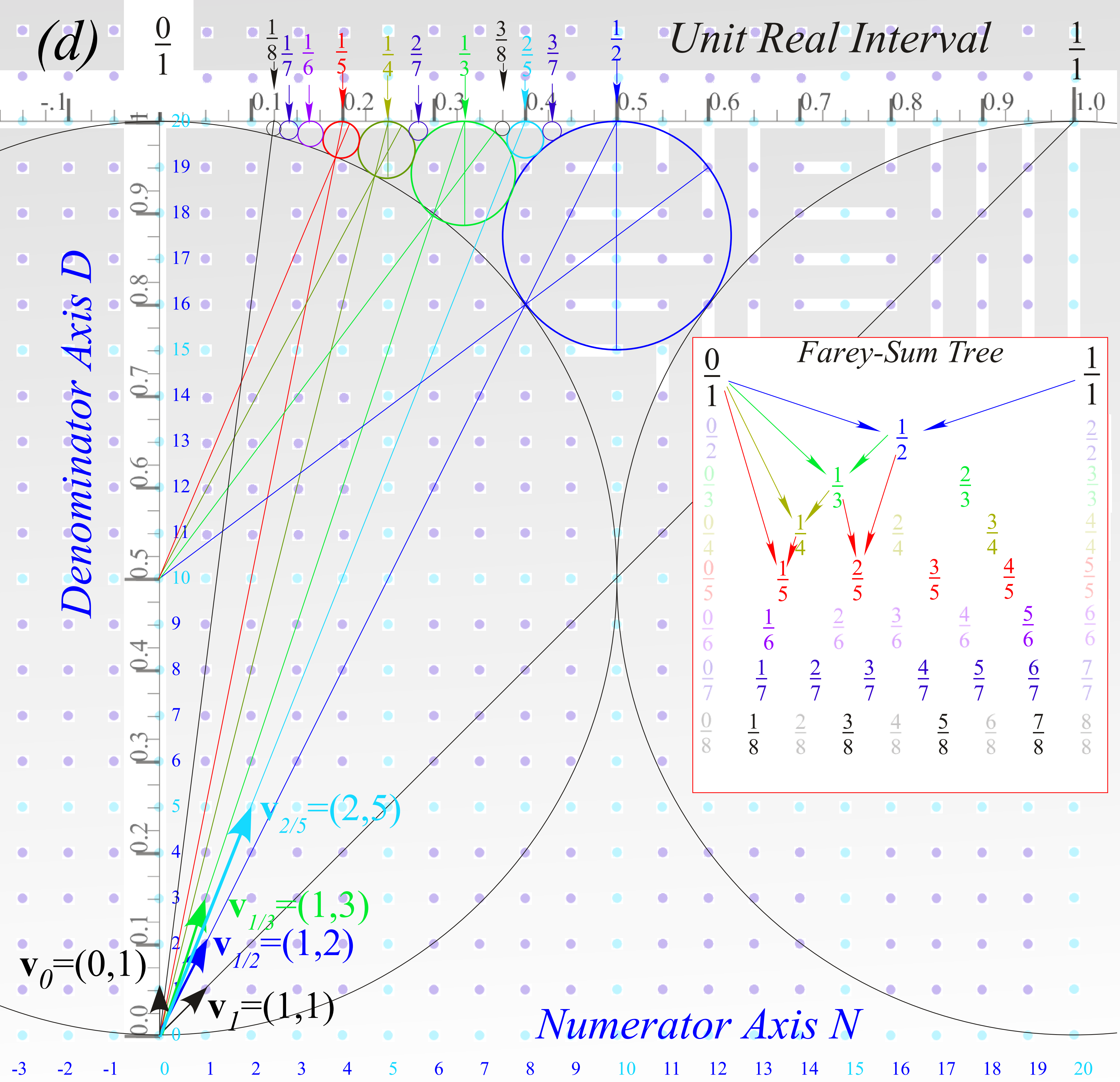
1/2-circle has diameter $1/2^2=1/4$



Farey Sum related to vector sum and *Ford Circles*

1/2-circle has diameter $1/2^2=1/4$

1/3-circles have diameter $1/3^2=1/9$



Farey Sum related to vector sum and *Ford Circles*

1/2-circle has diameter $1/2^2=1/4$

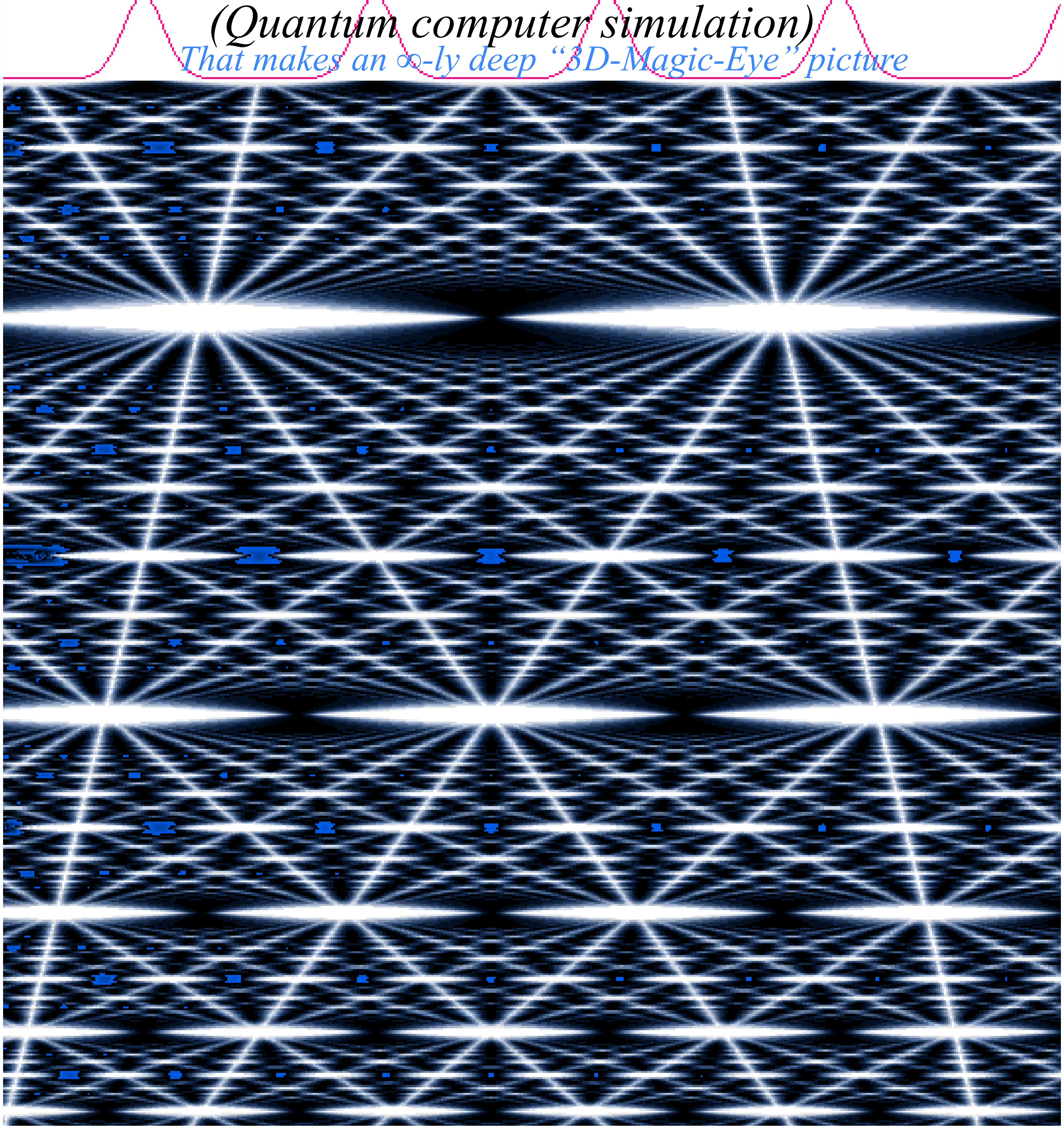
1/3-circles have diameter $1/3^2=1/9$

n/d-circles have diameter $1/d^2$

$D \leq 1$	$\mathbf{\frac{0}{1}}$																						$\mathbf{\frac{1}{1}}$
$D \leq 2$	$\frac{0}{1}$											$\mathbf{\frac{1}{2}}$											$\frac{1}{1}$
$D \leq 3$	$\frac{0}{1}$							$\mathbf{\frac{1}{3}}$				$\frac{1}{2}$				$\mathbf{\frac{2}{3}}$							$\frac{1}{1}$
$D \leq 4$	$\frac{0}{1}$					$\mathbf{\frac{1}{4}}$		$\frac{1}{3}$				$\frac{1}{2}$				$\frac{2}{3}$		$\mathbf{\frac{3}{4}}$					$\frac{1}{1}$
$D \leq 5$	$\frac{0}{1}$				$\mathbf{\frac{1}{5}}$	$\frac{1}{4}$		$\frac{1}{3}$		$\mathbf{\frac{2}{5}}$		$\frac{1}{2}$		$\mathbf{\frac{3}{5}}$		$\frac{2}{3}$		$\frac{3}{4}$	$\mathbf{\frac{4}{5}}$				$\frac{1}{1}$
$D \leq 6$	$\frac{0}{1}$			$\mathbf{\frac{1}{6}}$	$\frac{1}{5}$	$\frac{1}{4}$		$\frac{1}{3}$		$\frac{2}{5}$		$\frac{1}{2}$		$\frac{3}{5}$		$\frac{2}{3}$		$\frac{3}{4}$	$\frac{4}{5}$	$\mathbf{\frac{5}{6}}$			$\frac{1}{1}$
$D \leq 7$	$\frac{0}{1}$		$\mathbf{\frac{1}{7}}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\mathbf{\frac{2}{7}}$	$\frac{1}{3}$		$\frac{2}{5}$	$\mathbf{\frac{3}{7}}$	$\frac{1}{2}$	$\mathbf{\frac{4}{7}}$	$\frac{3}{5}$		$\frac{2}{3}$	$\mathbf{\frac{5}{7}}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\mathbf{\frac{6}{7}}$		$\frac{1}{1}$
$D \leq 8$	$\frac{0}{1}$	$\mathbf{\frac{1}{8}}$	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{2}{7}$	$\frac{1}{3}$	$\mathbf{\frac{3}{8}}$	$\frac{2}{5}$	$\frac{3}{7}$	$\frac{1}{2}$	$\frac{4}{7}$	$\frac{3}{5}$	$\mathbf{\frac{5}{8}}$	$\frac{2}{3}$	$\frac{5}{7}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	$\mathbf{\frac{7}{8}}$	$\frac{1}{1}$

(Quantum computer simulation)

That makes an ∞-ly deep “3D-Magic-Eye” picture



Polygonal geometry of $U(2) \supset C_N$ character spectral function
Algebra
Geometry

Introduction to wave dynamics of phase, mean phase, and group velocity
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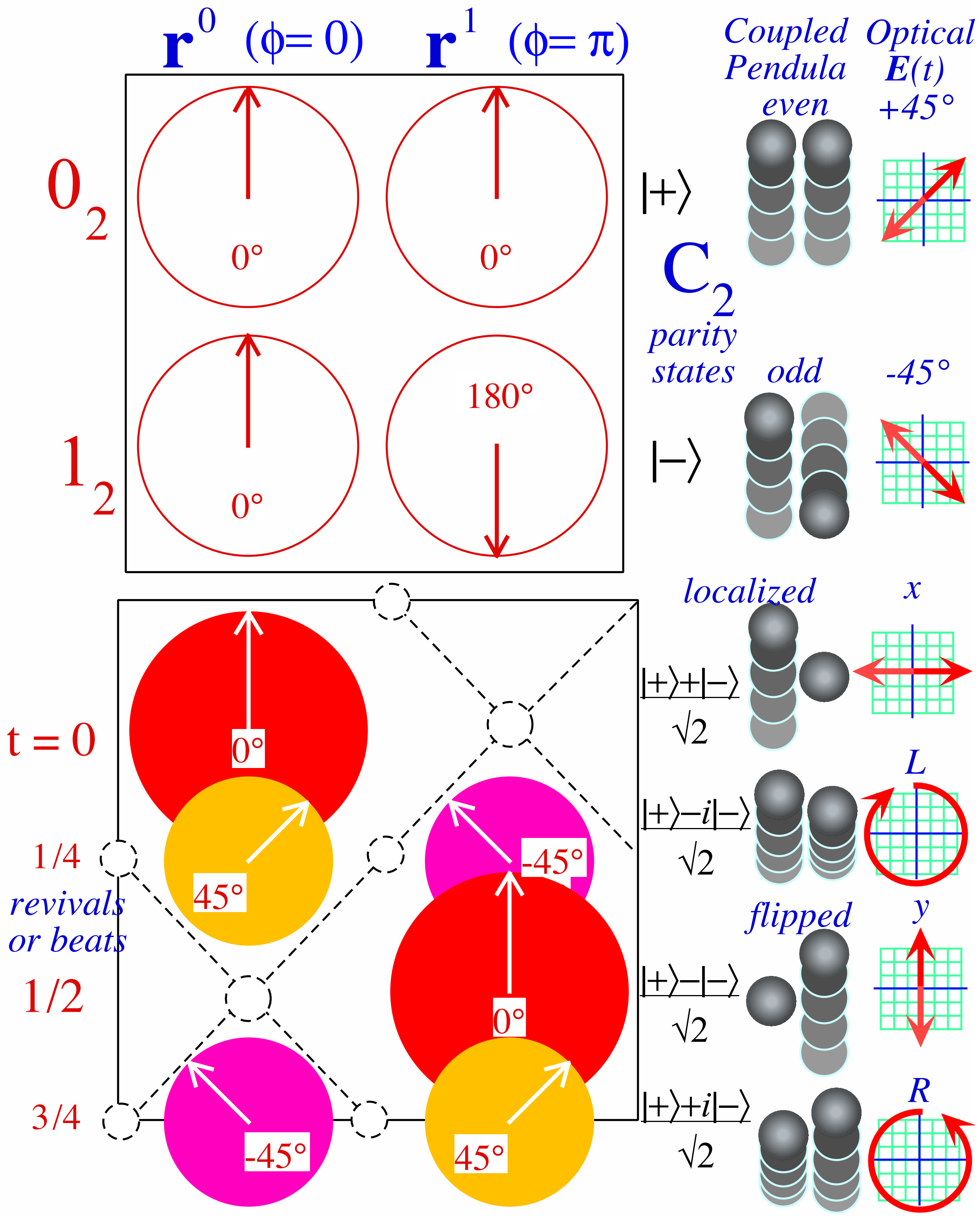
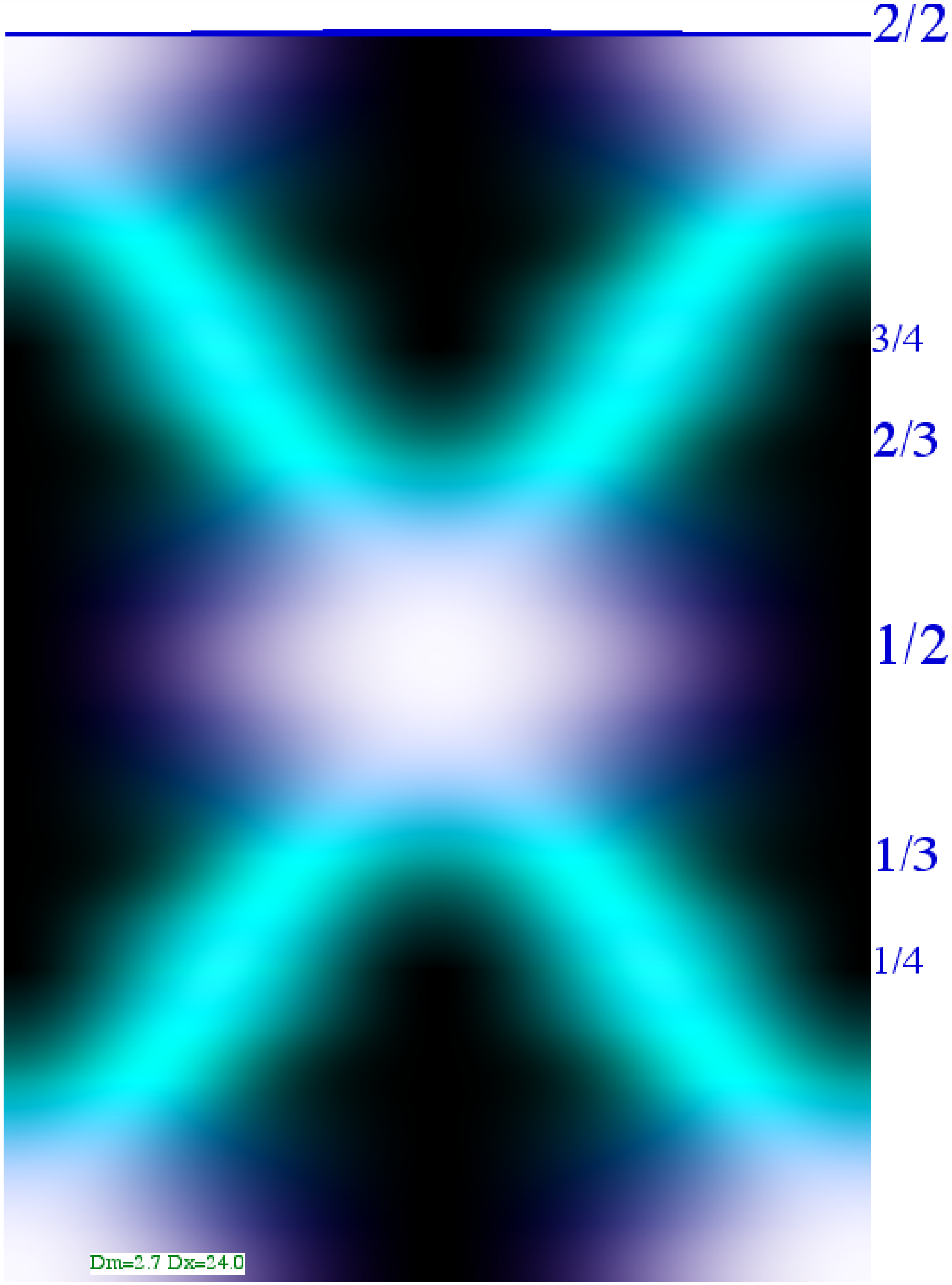
- *Wave coordinates*
- *Pulse-waves (PW) vs Continuous -waves (CW)*

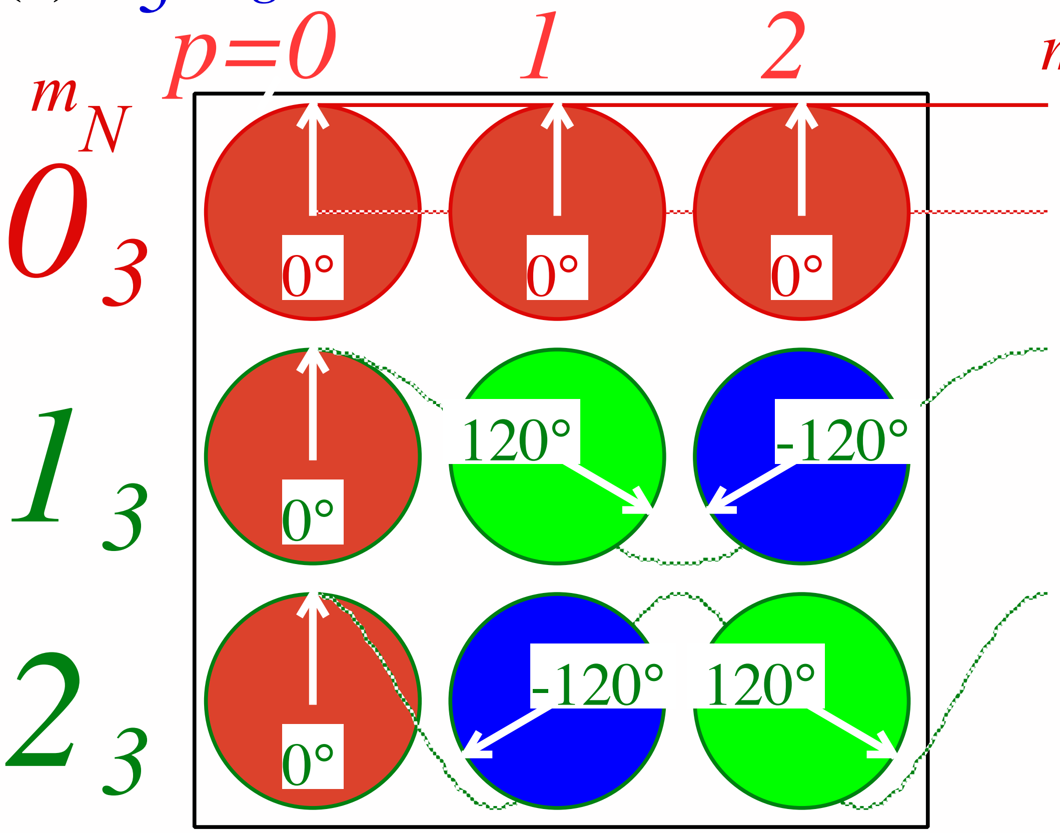
Introduction to C_N beat dynamics and "Revivals" due to Bohr-dispersion
- *∞-Square well PE versus Bohr rotor*
- *SinNx/x wavepackets bandwidth and uncertainty*
  - *SinNx/x explosion and revivals*
- *Bohr-rotor dynamics*
  - *Gaussian wave-packet bandwidth and uncertainty*
  - *Gaussian Bohr-rotor revivals*
- *Farey-Sums and Ford-products*
- → *Phase dynamics* ←

Fundamental Beats and 2-Level Transitions: The "Mother of all symmetry" is C_2

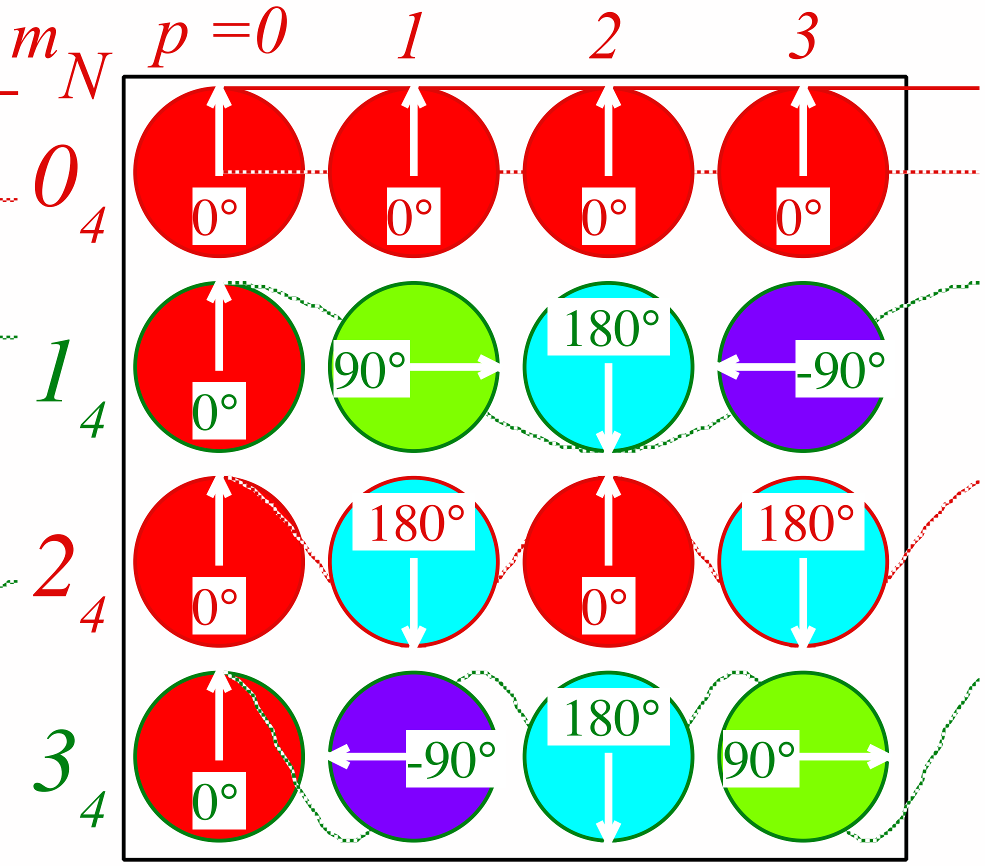




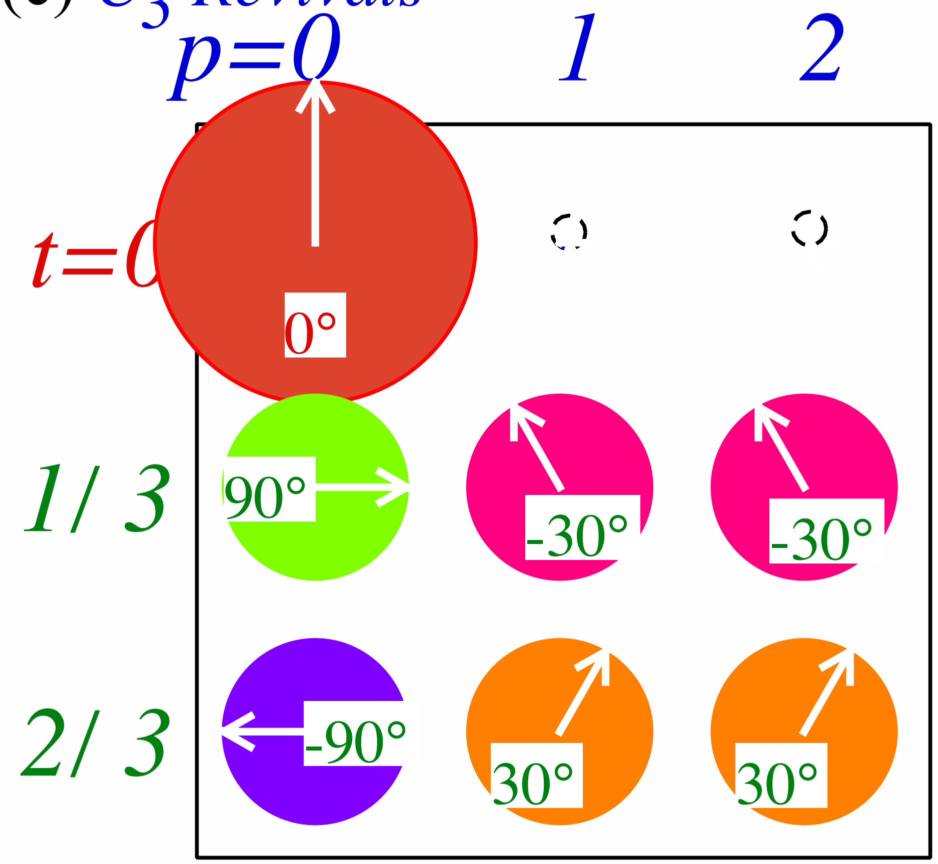
(a) C_3 *Eigenstate Characters*



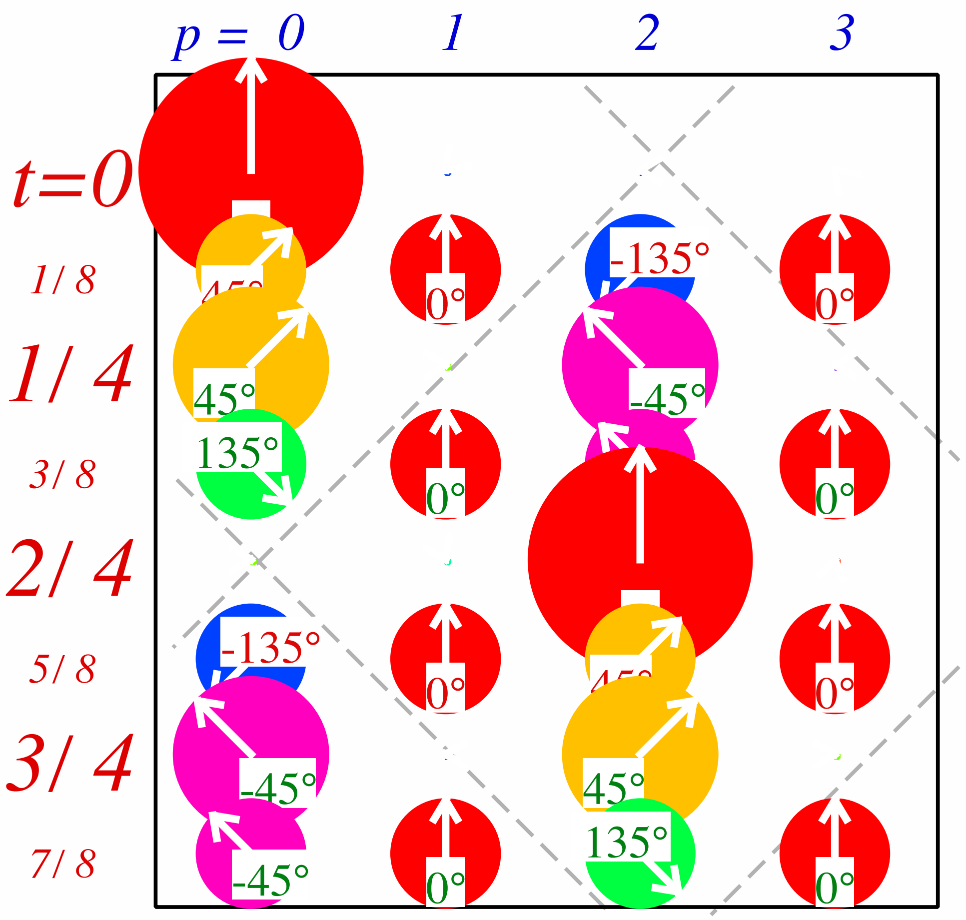
(b) C_4 *Eigenstate Characters*



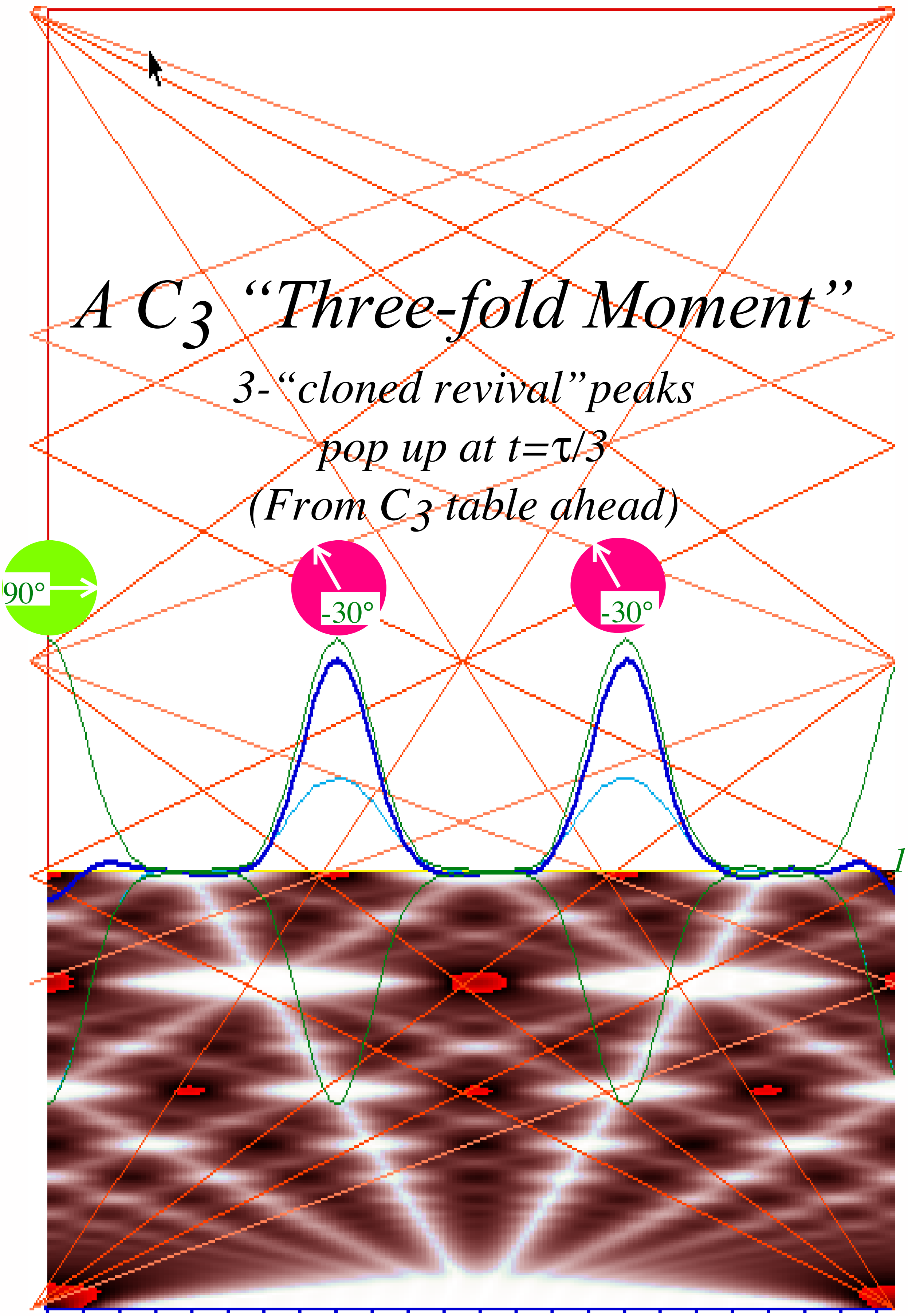
(c) C_3 *Revivals*



(d) C_4 *Revivals*

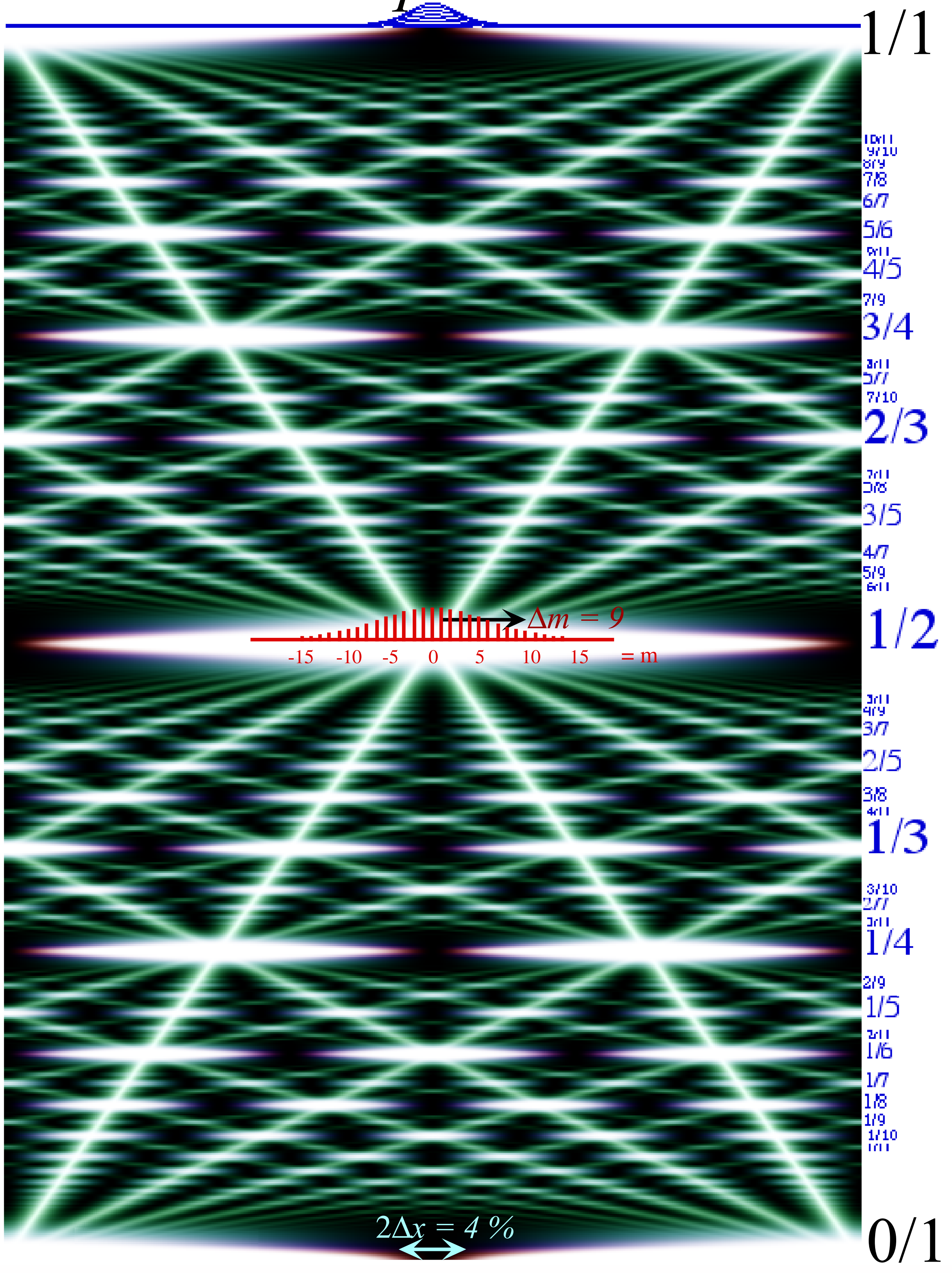


Revivals: All excited transitions take turns in a quantum rotor

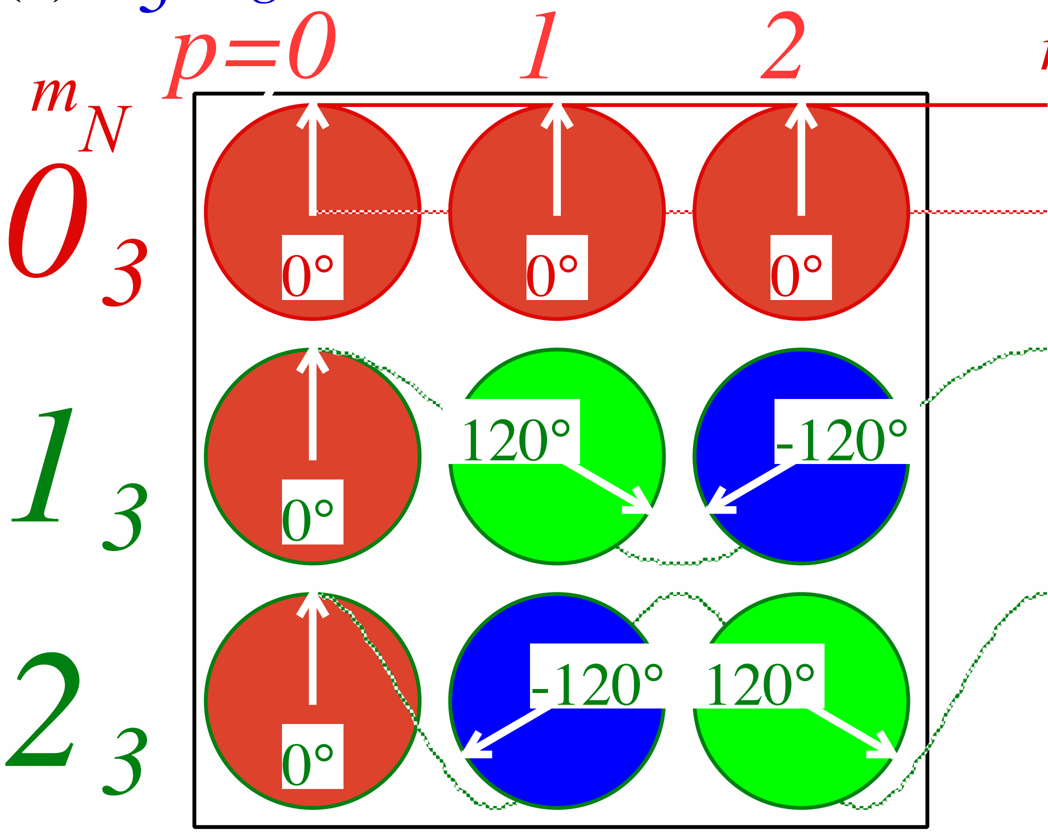


A C_3 "Three-fold Moment"

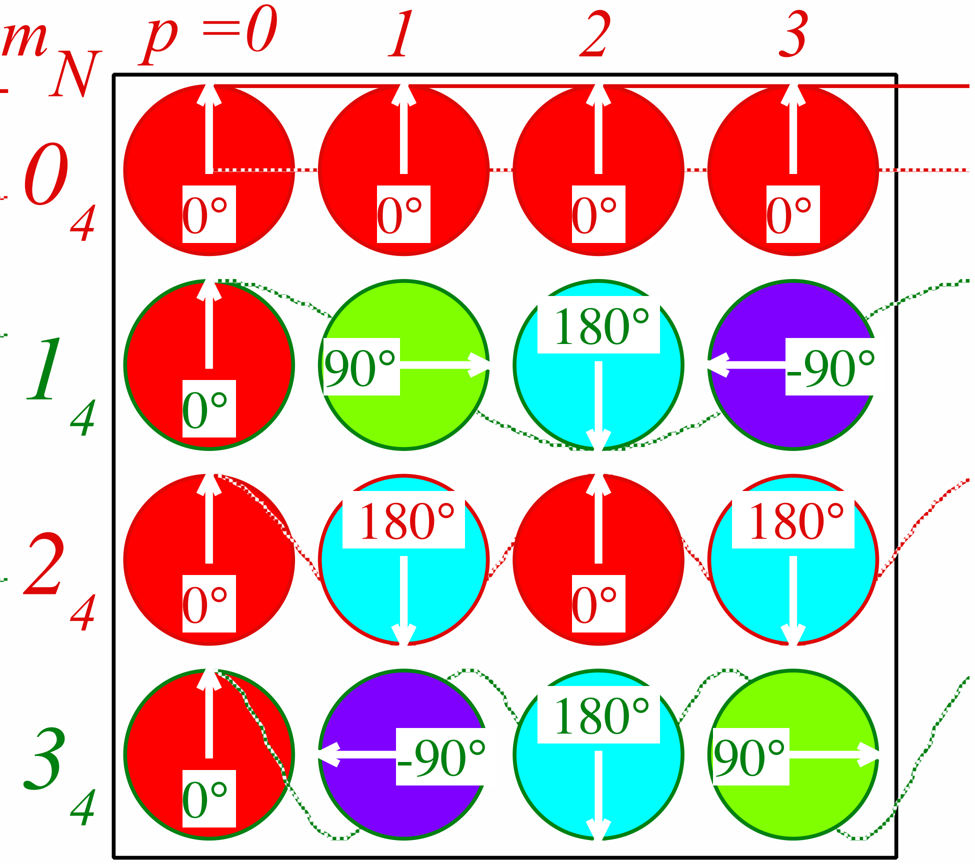
3-"cloned revival" peaks pop up at $t=\tau/3$ (From C_3 table ahead)



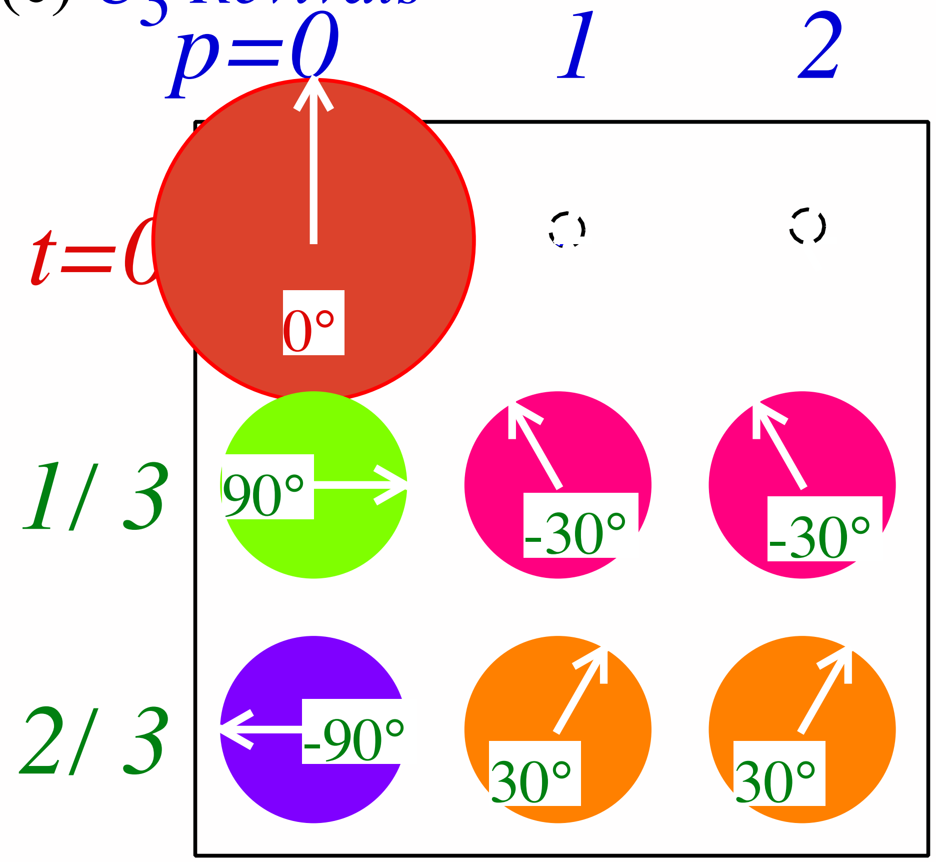
(a) *C_3 Eigenstate Characters*



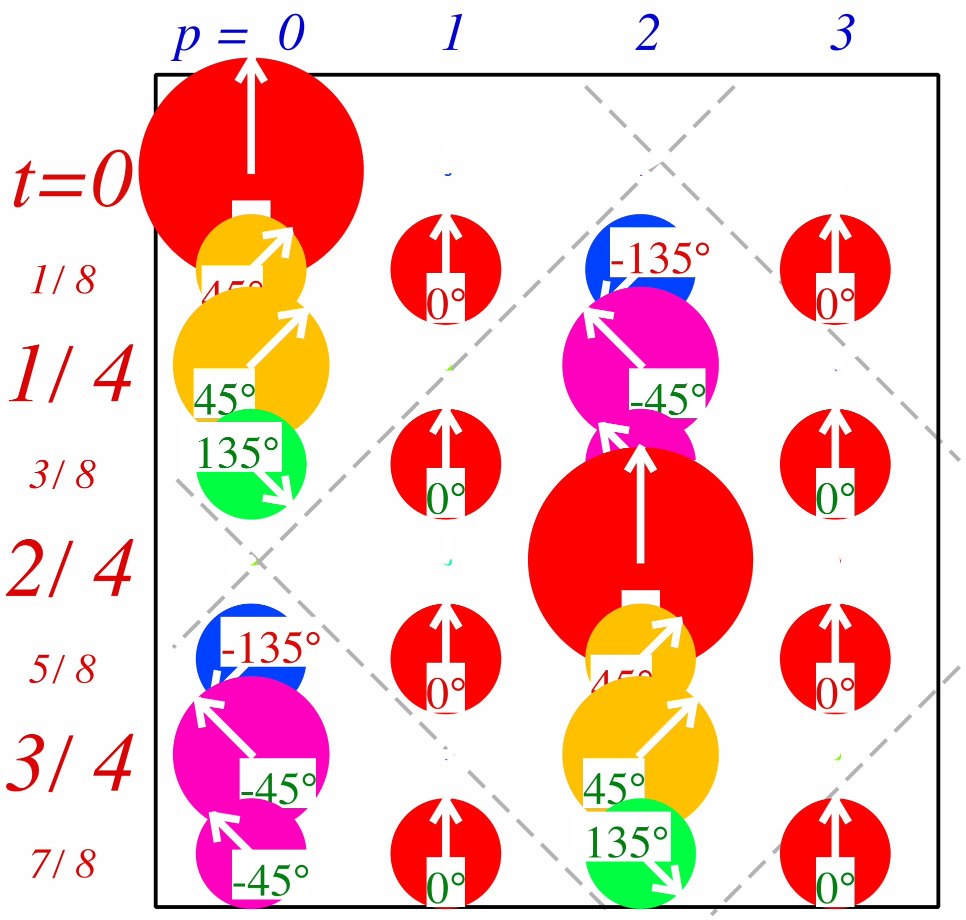
(b) *C_4 Eigenstate Characters*



(c) *C_3 Revivals*



(d) *C_4 Revivals*



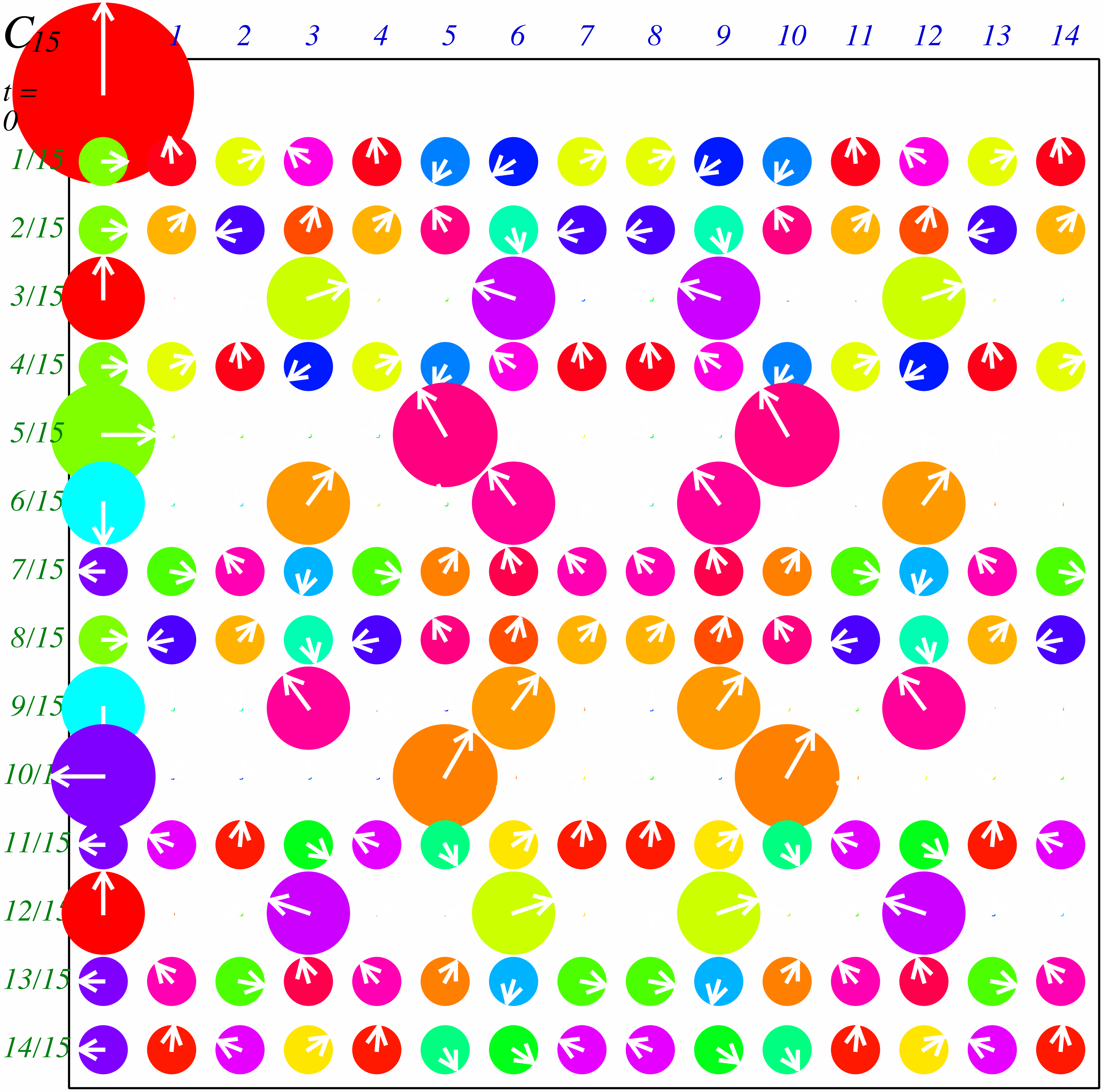
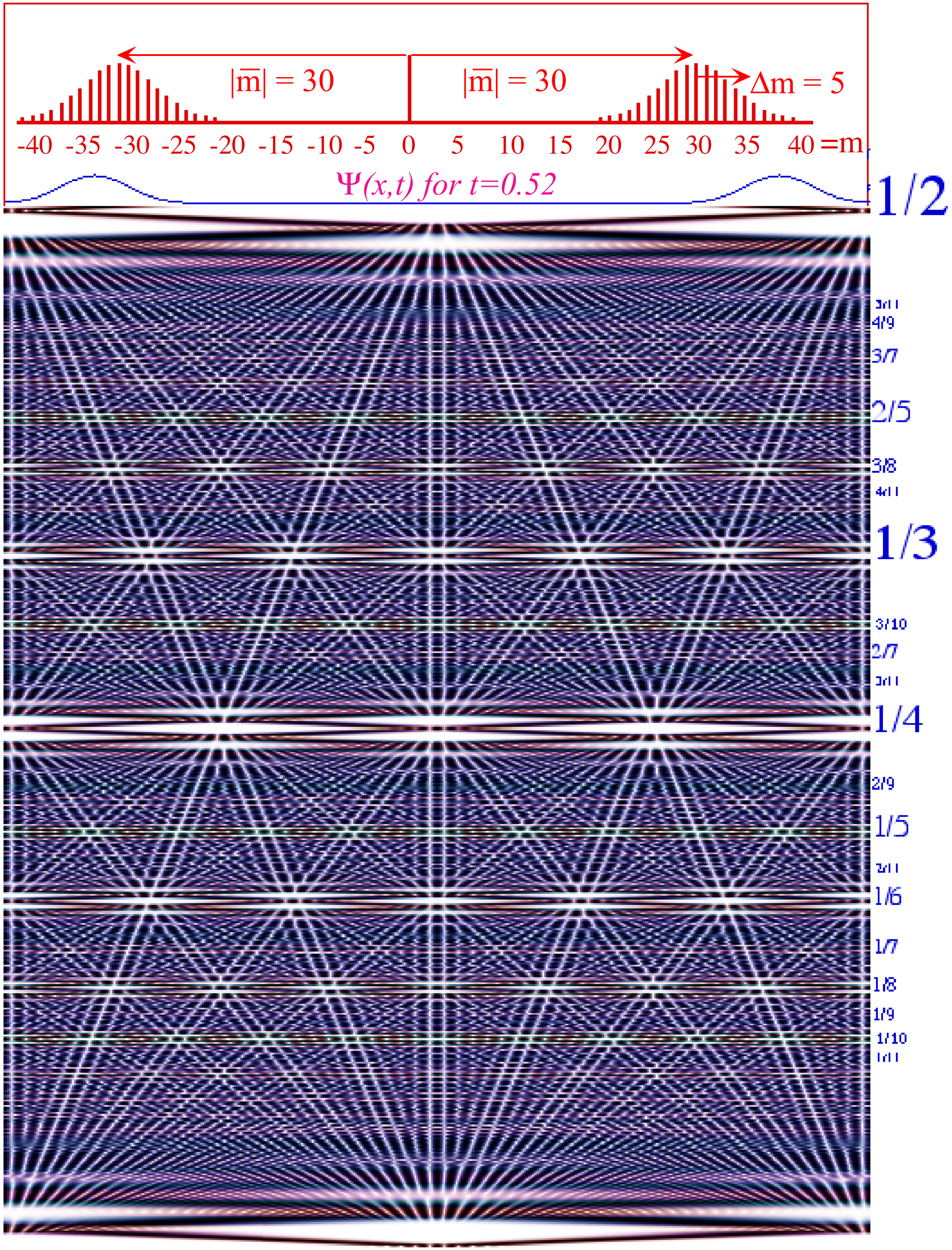
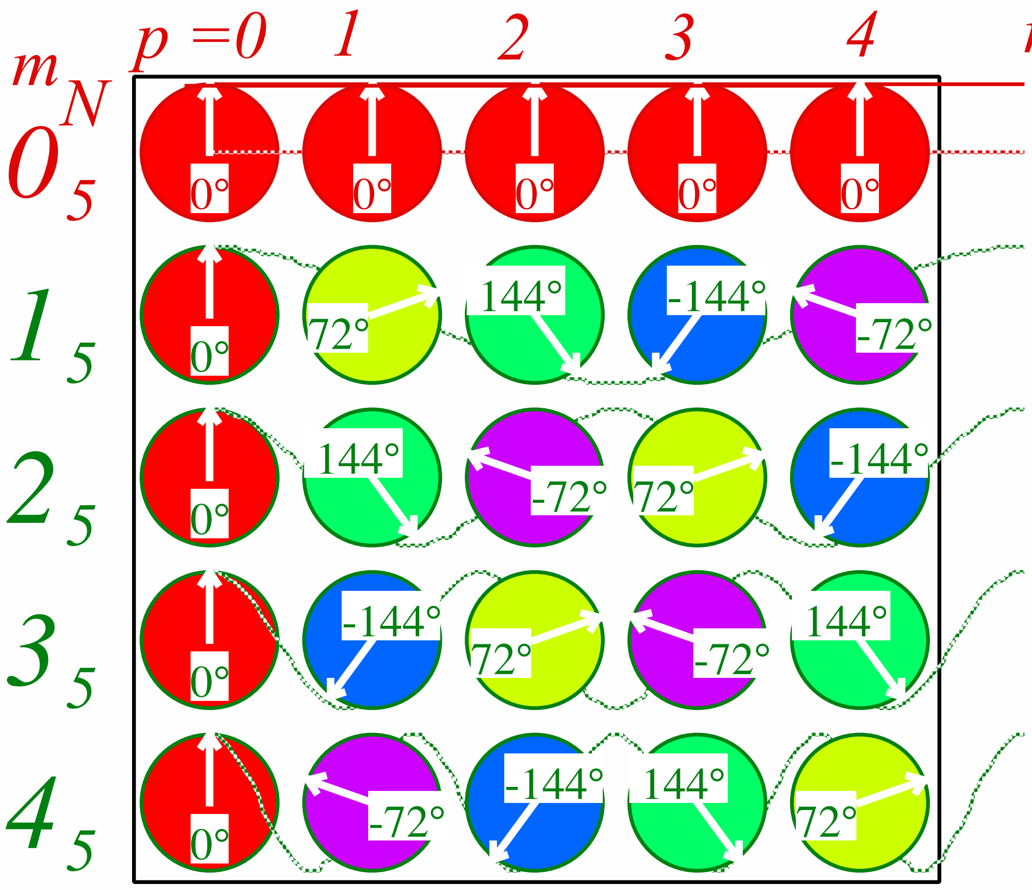


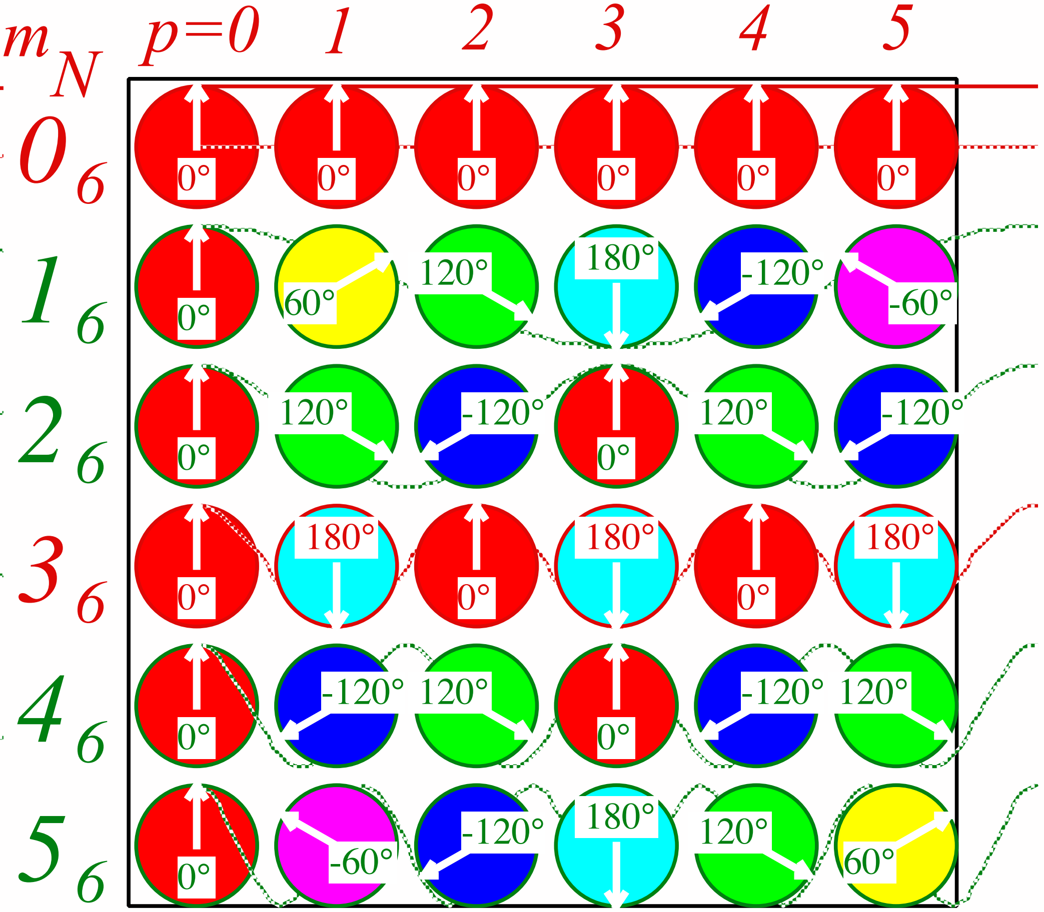
Fig. 9.4.4 Bohr space-time revival pattern for C_{15} Bohr system.


|m̄| = 30
|m̄| = 30
Δm = 5
-40 -35 -30 -25 -20 -15 -10 -5 0 5 10 15 20 25 30 35 40 =m
Ψ(x,t) for t=0.52
1/2
5/11
4/9
3/7
2/5
3/8
4/11
1/3
3/10
2/7
3/11
1/4
2/9
1/5
2/11
1/6
1/7
1/8
1/9
1/10
1/11

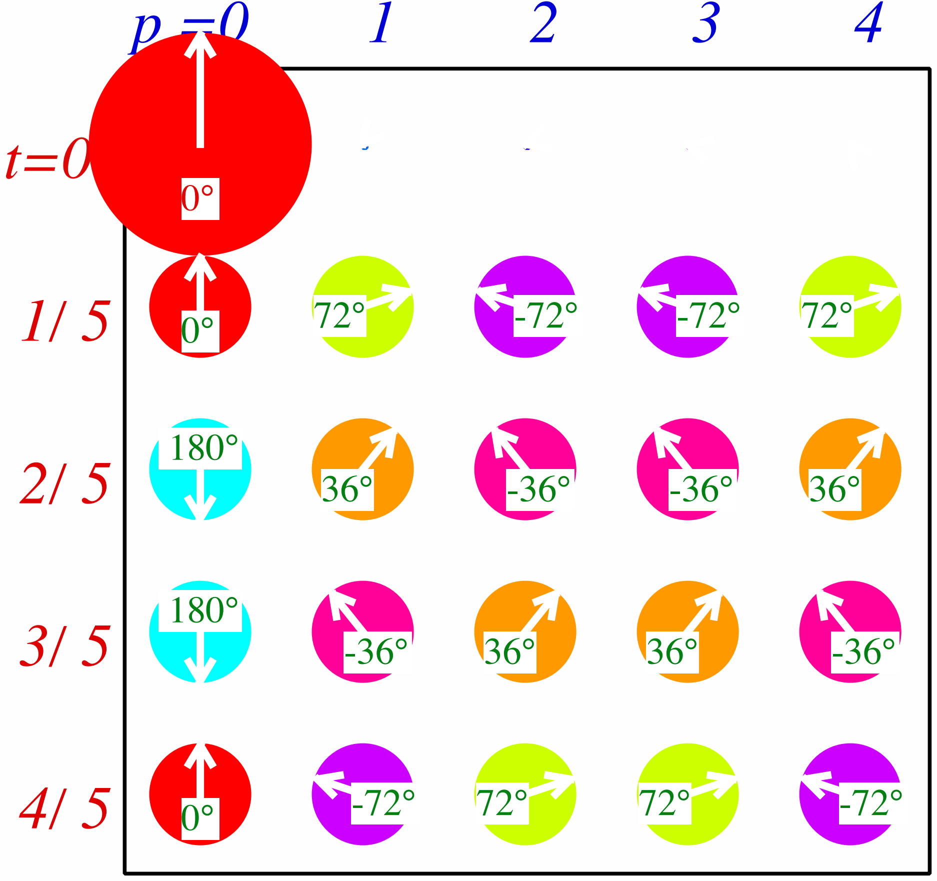
(a) C_5 Eigenstate Characters



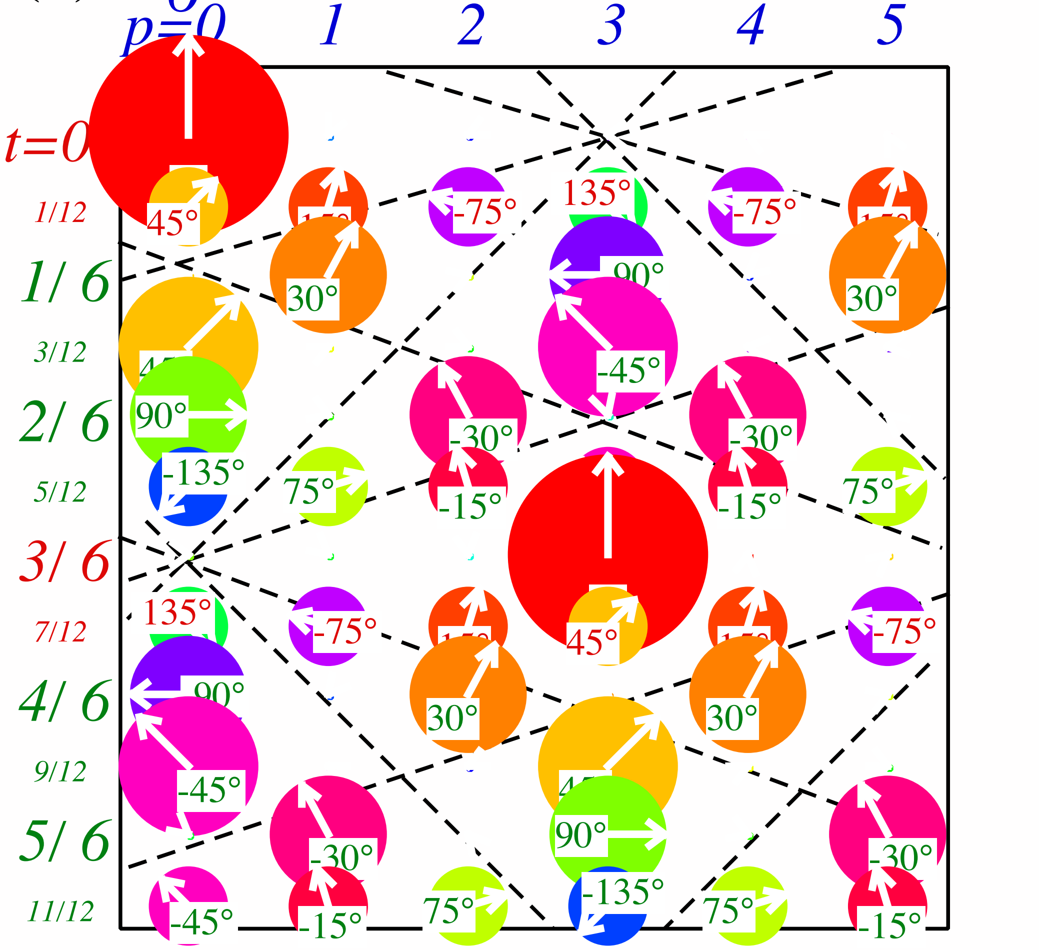
(b) C_6 Eigenstate Characters



(c) C_5 Revivals



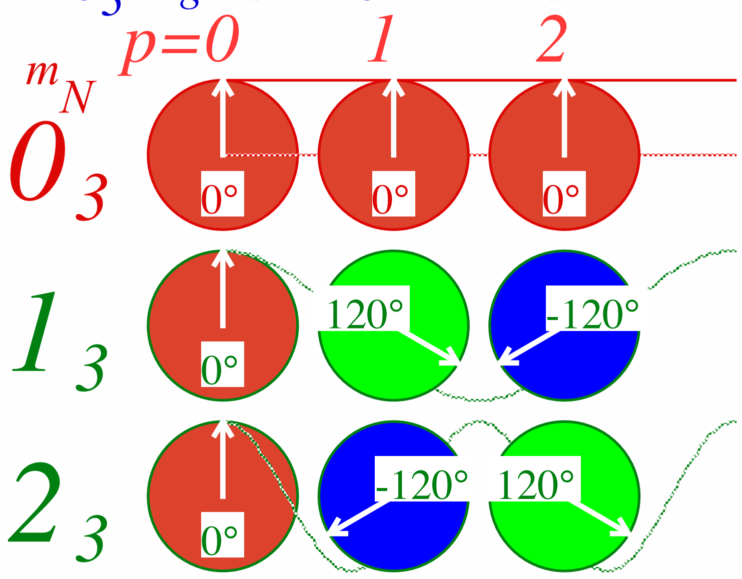
(d) C_6 Revivals



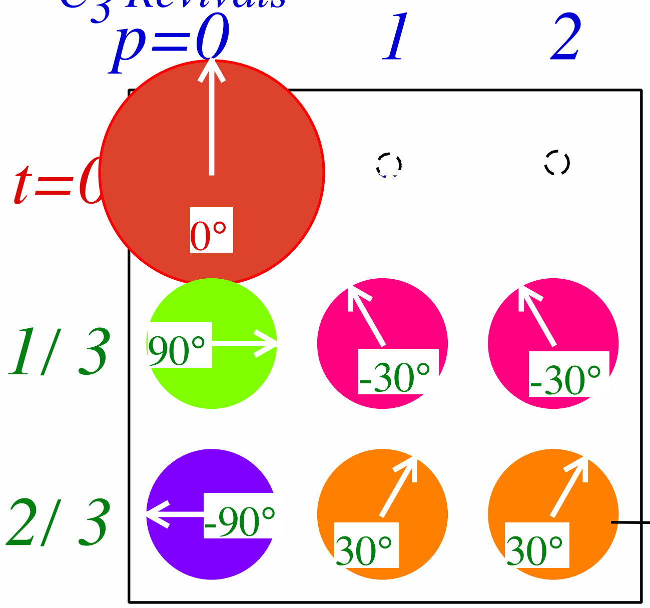
Simulating Complex Systems With Simpler Ones

Discrete 3-State or Trigonal System (Tesla's 3-Phase AC)

C_3 Eigenstate Characters



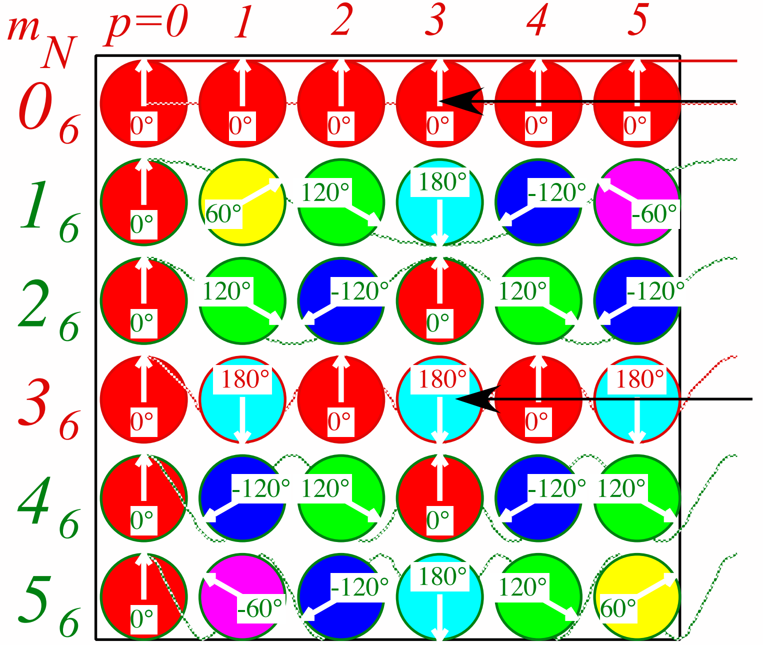
C_3 Revivals



Note 3-phase sub-symmetry

Discrete 6-State or Hexagonal System (6-Phase AC)

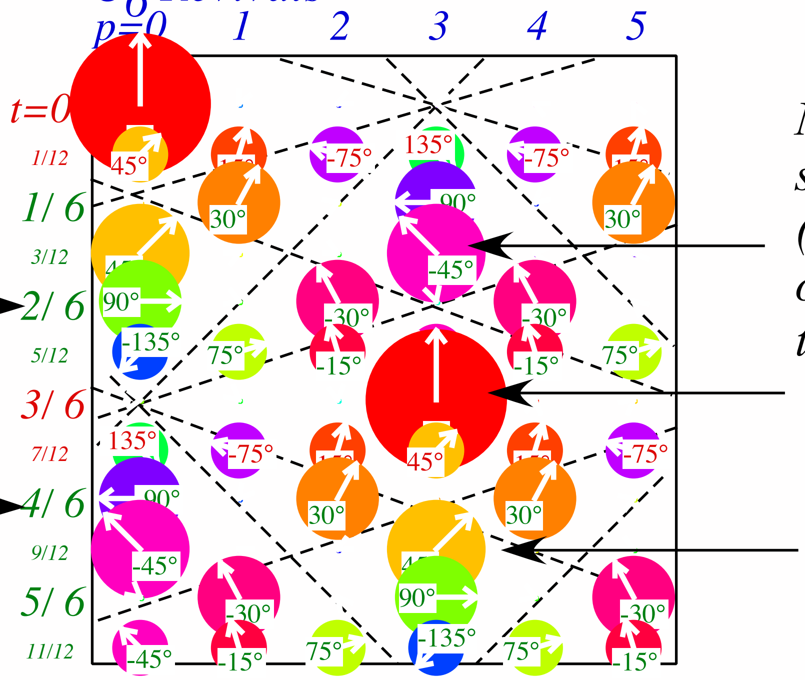
C_6 Eigenstate Characters



Note 2-phase AC

C_2

C_6 Revivals



Note 2-phase sub-symmetry (The "Mother of all symmetry" is C_2)

➡ *Polygonal geometry of $U(2) \supset C_N$ character spectral function*

Algebra

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- *Pulse-waves (PW) vs Continuous -waves (CW)*

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- *Farey-Sums and Ford-products*
- *Phase dynamics*

Polygonal geometry of U(2)⊃C$_N$ character spectral function

Trace-character $\chi^j(\Theta)$ of U(2) rotation by C_n angle $\Theta=2\pi/n$

is an ($\ell^j=2j+1$)-term sum of $e^{-im\Theta}$ over allowed m-quanta $m=\{-j, -j+1, \ldots, j-1, j\}$.

$$\underset{\textit{(spinor-j=1/2)}}{\chi^{1/2}(\Theta)} = trace D^{1/2}(\Theta) = trace\begin{pmatrix} e^{-i\theta/2} & \cdot \\ \cdot & e^{+i\theta/2} \end{pmatrix}$$

$$\underset{\textit{(vector-j=1)}}{\chi^{1}(\Theta)} = trace D^{1}(\Theta) = trace\begin{pmatrix} e^{-i\theta} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{-i\theta} \end{pmatrix}$$

Polygonal geometry of U(2)⊃C_N character spectral function

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(spinor-j=1/2)

$$\chi^{1}(\Theta)=traceD^{1}(\Theta)=trace\begin{pmatrix} e^{-i\theta} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{-i\theta} \end{pmatrix}$$

(vector-j=1)

$\chi^j(\Theta)$ involves a sum of $2\cos(m\,\Theta/2)$ for $m\geq 0$ up to $m=j$.

$$\chi^{1/2}(\Theta)=e^{-i\frac{\Theta}{2}}+e^{i\frac{\Theta}{2}}=2\cos\frac{\Theta}{2}$$

(spinor-j=1/2)

$$\chi^{3/2}(\Theta)=e^{-i\frac{3\Theta}{2}}+\ldots+e^{i\frac{3\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}$$

$$\chi^{5/2}(\Theta)=e^{-i\frac{5\Theta}{2}}+\ldots+e^{i\frac{5\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}+2\cos\frac{5\Theta}{2}$$

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(spinor-j=1/2)

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$$\chi^{5/2}(\Theta)=e^{-i\frac{5\Theta}{2}}+\ldots+e^{i\frac{5\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}+2\cos\frac{5\Theta}{2}$$

$$\chi^{0}(\Theta)=e^{-i\Theta\cdot 0}=1$$

(scalar-j=0)

$$\chi^{1}(\Theta)=e^{-i\Theta}+1+e^{i\Theta}=1+2\cos\Theta$$

(vector-j=1)

$$\chi^{2}(\Theta)=e^{-i2\Theta}+\ldots e^{i2\Theta}=1+2\cos\Theta+2\cos 2\Theta$$

(tensor-j=2)

Polygonal geometry of $U(2)\supset C_N$ character spectral function
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$$\chi^{1/2}(\Theta)=e^{-i\frac{\Theta}{2}}+e^{i\frac{\Theta}{2}} \qquad =2\cos\frac{\Theta}{2}$$

(spinor-j=1/2)

$$\chi^{0}(\Theta)=e^{-i\Theta\cdot 0} \qquad =1$$

(scalar-j=0)

$$\chi^{3/2}(\Theta)=e^{-i\frac{3\Theta}{2}}+\ldots \qquad +e^{i\frac{3\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}$$

$$\chi^{1}(\Theta)=e^{-i\Theta}+1+e^{i\Theta} \quad =1+2\cos\Theta$$

(vector-j=1)

$$\chi^{5/2}(\Theta)=e^{-i\frac{5\Theta}{2}}+\ldots \qquad +e^{i\frac{5\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}+2\cos\frac{5\Theta}{2}$$

$$\chi^{2}(\Theta)=e^{-i2\Theta}+\ldots e^{i2\Theta}=1+2\cos\Theta+2\cos 2\Theta$$

(tensor-j=2)

$\chi^j(\Theta)$ is a geometric series with ratio $e^{i\Theta}$ between each successive term.

$$\chi^{j}(\Theta)=TraceD^{(j)}(\Theta)=e^{-i\Theta j}+e^{-i\Theta(j-1)}+e^{-i\Theta(j-2)}+\ldots+e^{+i\Theta(j-2)}+e^{+i\Theta(j-1)}+e^{+i\Theta j}$$

Polygonal geometry of U(2)⊃CN character spectral function

Trace-character $\chi^j(\Theta)$ of U(2) rotation by C_n angle $\Theta=2\pi/n$

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(spinor-j=1/2)

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$$\chi^{0}(\Theta)=e^{-i\Theta\cdot 0}=1$$

(scalar-j=0)

$$\chi^{1}(\Theta)=e^{-i\Theta}+1+e^{i\Theta}=1+2\cos\Theta$$

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$$\chi^{2}(\Theta)=e^{-i2\Theta}+\ldots e^{i2\Theta}=1+2\cos\Theta+2\cos 2\Theta$$

(tensor-j=2)

$\chi^j(\Theta)$ is a geometric series with ratio $e^{i\Theta}$ between each successive term.

$$\chi^j(\Theta)=TraceD^{(j)}(\Theta)=e^{-i\Theta j}+e^{-i\Theta(j-1)}+e^{-i\Theta(j-2)}+\ldots+e^{+i\Theta(j-2)}+e^{+i\Theta(j-1)}+e^{+i\Theta j}$$

$$\chi^j(\Theta)e^{-i\Theta}=e^{-i\Theta(j+1)}+e^{-i\Theta j}+e^{-i\Theta(j-1)}+e^{-i\Theta(j-2)}+\ldots+e^{+i\Theta(j-2)}+e^{+i\Theta(j-1)}$$

Polygonal geometry of U(2)⊃C$_N$ character spectral function

Trace-character $\chi^j(\Theta)$ of U(2) rotation by C_n angle $\Theta=2\pi/n$

is an ($\ell^j=2j+1$)-term sum of $e^{-im\Theta}$ over allowed m-quanta $m=\{-j, -j+1, \ldots, j-1, j\}$.

$$\chi^{1/2}(\Theta)=traceD^{1/2}(\Theta)=trace\begin{pmatrix} e^{-i\theta/2} & \cdot \\ \cdot & e^{+i\theta/2} \end{pmatrix}$$

(spinor-j=1/2)

$$\chi^{1}(\Theta)=traceD^{1}(\Theta)=trace\begin{pmatrix} e^{-i\theta} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{-i\theta} \end{pmatrix}$$

(vector-j=1)

$\chi^j(\Theta)$ involves a sum of $2\cos(m\,\Theta/2)$ for $m\geq 0$ up to $m=j$.

$$\chi^{1/2}(\Theta)=e^{-i\frac{\Theta}{2}}+e^{i\frac{\Theta}{2}}=2\cos\frac{\Theta}{2}$$

(spinor-j=1/2)

$$\chi^{0}(\Theta)=e^{-i\Theta\cdot 0}=1$$

(scalar-j=0)

$$\chi^{3/2}(\Theta)=e^{-i\frac{3\Theta}{2}}+\ldots+e^{i\frac{3\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}$$

$$\chi^{1}(\Theta)=e^{-i\Theta}+1+e^{i\Theta}=1+2\cos\Theta$$

(vector-j=1)

$$\chi^{5/2}(\Theta)=e^{-i\frac{5\Theta}{2}}+\ldots+e^{i\frac{5\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}+2\cos\frac{5\Theta}{2}$$

$$\chi^{2}(\Theta)=e^{-i2\Theta}+\ldots e^{i2\Theta}=1+2\cos\Theta+2\cos 2\Theta$$

(tensor-j=2)

$\chi^j(\Theta)$ is a geometric series with ratio $e^{i\Theta}$ between each successive term.

$$\chi^{j}(\Theta)=TraceD^{(j)}(\Theta)=e^{-i\Theta j}+e^{-i\Theta(j-1)}+e^{-i\Theta(j-2)}+\ldots+e^{+i\Theta(j-2)}+e^{+i\Theta(j-1)}+e^{+i\Theta j}$$

$$\chi^{j}(\Theta)e^{-i\Theta}=e^{-i\Theta(j+1)}+e^{-i\Theta j}+e^{-i\Theta(j-1)}+e^{-i\Theta(j-2)}+\ldots+e^{+i\Theta(j-2)}+e^{+i\Theta(j-1)}$$

Subtracting gives:

$$\chi^{j}(\Theta)(1-e^{-i\Theta})=-e^{-i\Theta(j+1)}+e^{+i\Theta j}$$

Polygonal geometry of $U(2)\supset C_N$ character spectral function

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$$\chi^{j}(\Theta)e^{-i\Theta}=e^{-i\Theta(j+1)}+\cancel{e^{-i\Theta j}}+\cancel{e^{-i\Theta(j-1)}}+\cancel{e^{-i\Theta(j-2)}}+\ldots+\cancel{e^{+i\Theta(j-2)}}+\cancel{e^{+i\Theta(j-1)}}$$

Subtracting/dividing gives $\chi^j(\Theta)$ formula.

$$\chi^{j}(\Theta)=\frac{e^{+i\Theta j}-e^{-i\Theta(j+1)}}{1-e^{-i\Theta}}=\frac{e^{+i\Theta(j+\frac{1}{2})}-e^{-i\Theta(j+\frac{1}{2})}}{e^{+i\frac{\Theta}{2}}-e^{-i\frac{\Theta}{2}}}=\frac{\sin\Theta(j+\frac{1}{2})}{\sin\frac{\Theta}{2}}$$

Polygonal geometry of U(2)⊃C_N character spectral function

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Subtracting/dividing gives $\chi^j(\Theta)$ formula.

$$\chi^{j}(\Theta)=\frac{e^{+i\Theta j}-e^{-i\Theta(j+1)}}{1-e^{-i\Theta}}=\frac{e^{+i\Theta(j+\frac{1}{2})}-e^{-i\Theta(j+\frac{1}{2})}}{e^{+i\frac{\Theta}{2}}-e^{-i\frac{\Theta}{2}}}=\frac{\sin\Theta(j+\frac{1}{2})}{\sin\frac{\Theta}{2}}$$

For C_n angle $\Theta=2\pi/n$ this χ^j has a lot of geometric significance.

$$\boxed{\chi^{j}(\frac{2\pi}{n})=\frac{\sin\frac{\pi}{n}(2j+1)}{\sin\frac{\pi}{n}}=\frac{\sin\frac{\pi\ell^{j}}{n}}{\sin\frac{\pi}{n}}}$$

Character Spectral Function

where: $\ell^j=2j+1$

is U(2) irrep dimension

Polygonal geometry of $U(2)\supset C_N$ character spectral function
Algebra
→ *Geometry*

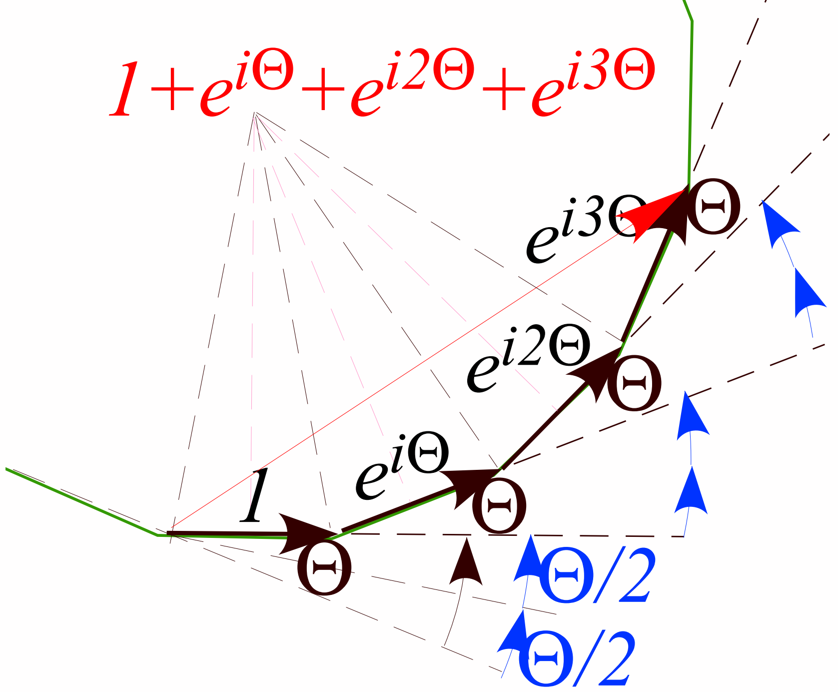
Introduction to wave dynamics of phase, mean phase, and group velocity
Expo-Cosine identity
Relating space-time and per-space-time
  Wave coordinates
  Pulse-waves (PW) vs Continuous -waves (CW)

Introduction to C_N beat dynamics and "Revivals"
  Farey-Sums and Ford-products
  Phase dynamics

Polygonal geometry of $U(2)\supset C_N$ character spectral function

$$\chi^j\left(\frac{2\pi}{n}\right)=\frac{\sin\frac{\pi}{n}(2j+1)}{\sin\frac{\pi}{n}}=\frac{\sin\frac{\pi\ell^J}{n}}{\sin\frac{\pi}{n}}$$

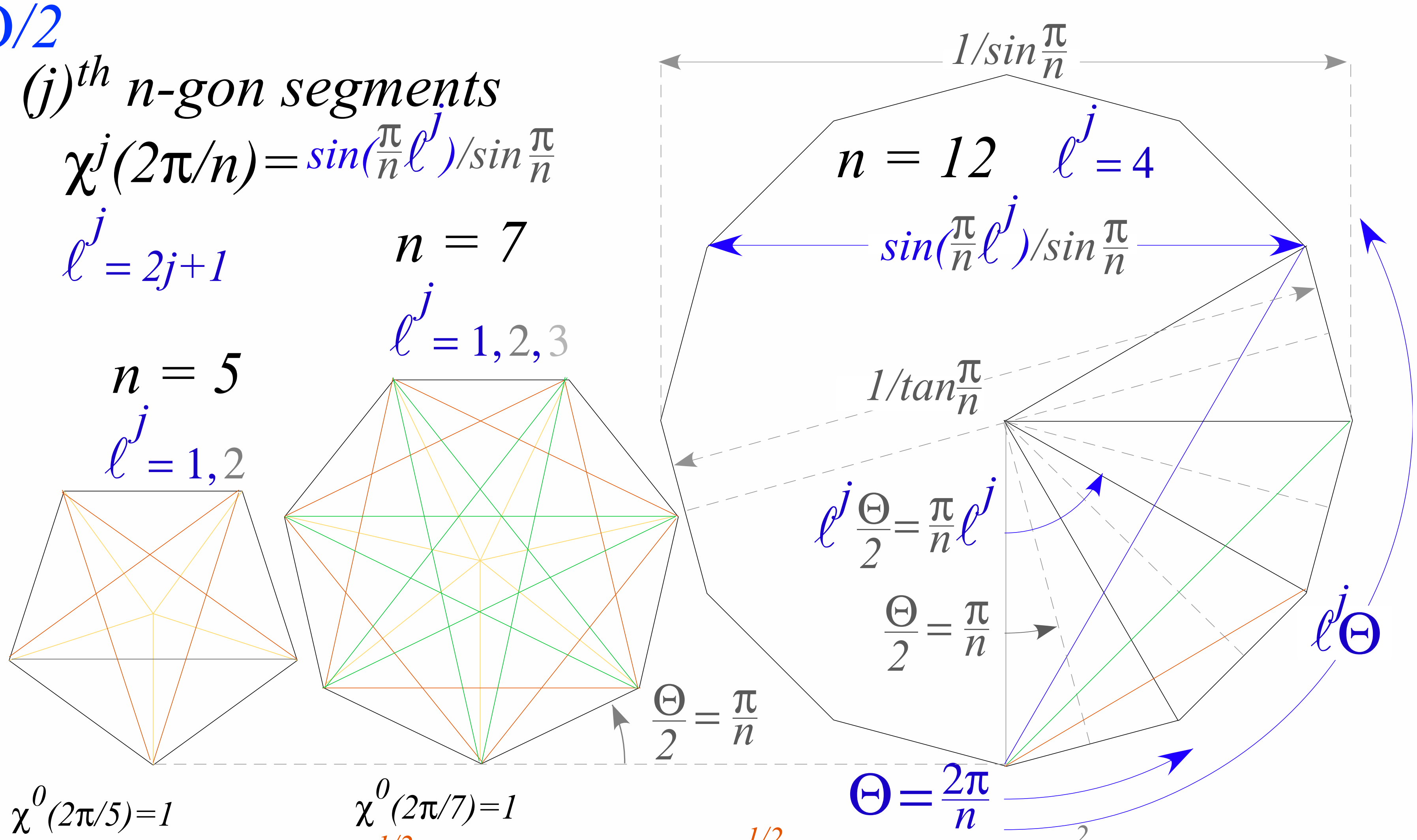
Character Spectral Function
where: $\ell^j=2j+1$
is U(2) irrep dimension



$(j)^{th}$ *n-gon segments*

$$\chi^j(2\pi/n)=\sin\left(\frac{\pi}{n}\ell^j\right)/\sin\frac{\pi}{n}$$

$\ell^j=2j+1$



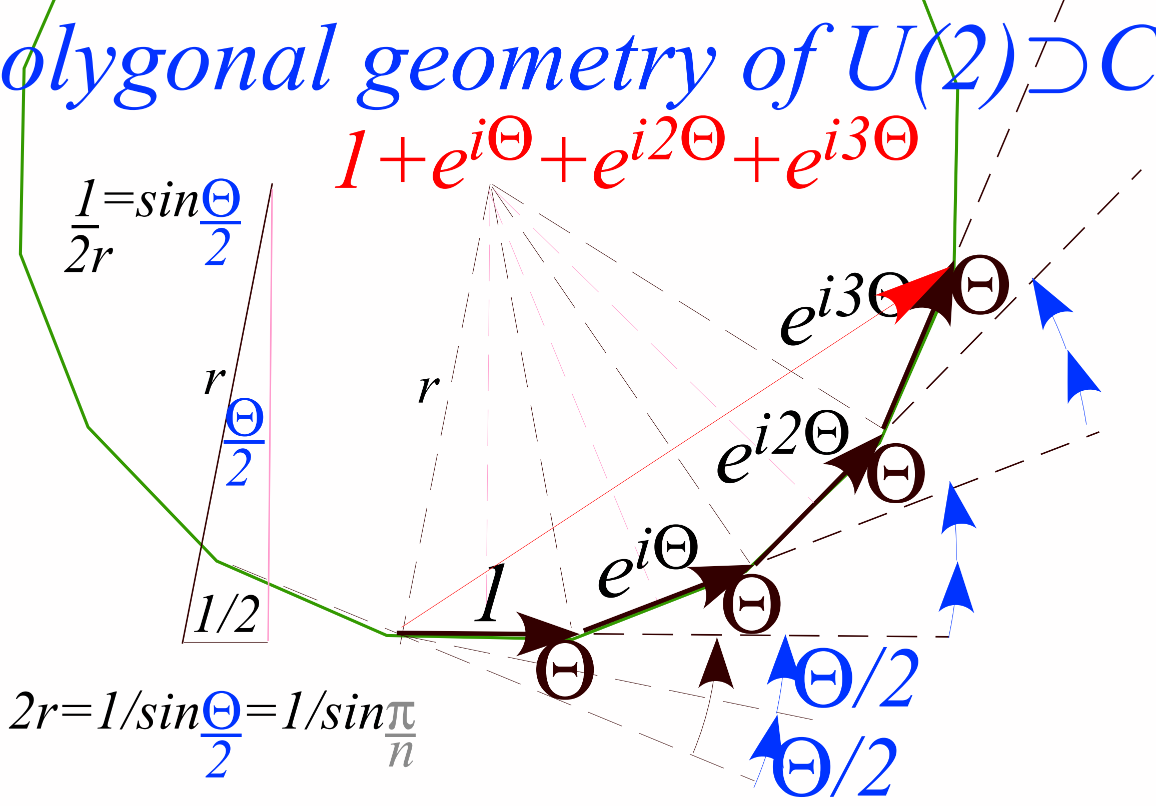
$\chi^0(2\pi/5)=1$
$\chi^{1/2}(2\pi/5)=1.618...$
$=(1+\sqrt{5})/2=$

$\chi^0(2\pi/7)=1$
$\chi^{1/2}(2\pi/7)=1.802...$
$\chi^1(2\pi/7)=2.247...$
$\chi^{3/2}(2\pi/7)=2.247...$

$\chi^{1/2}(2\pi/12)=1.932...$
$\chi^1(2\pi/12)=2.732...$
$\chi^{3/2}(2\pi/12)=3.346...$

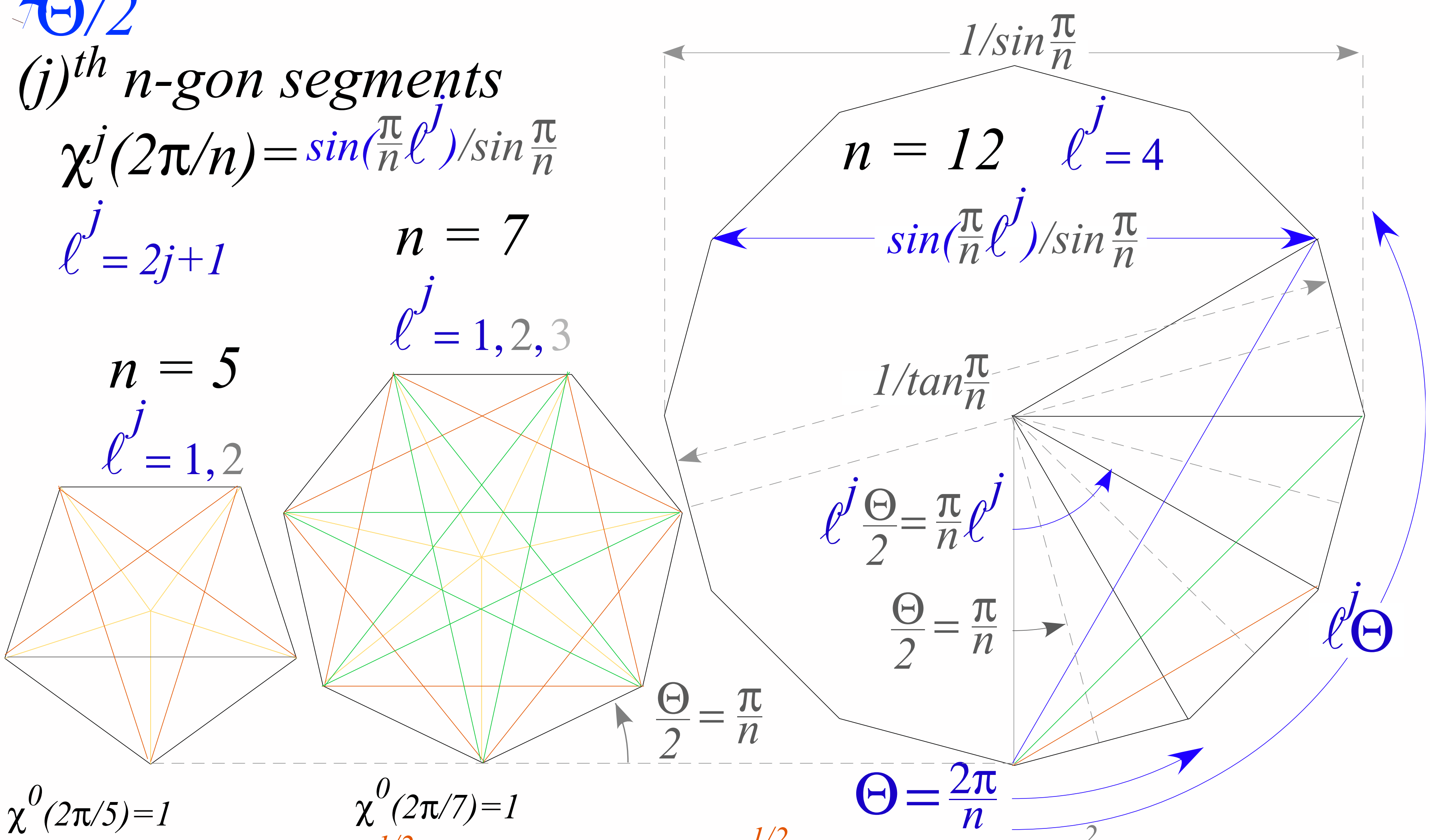
$\chi^2(2\pi/12)=3.732...$
$\chi^{5/2}(2\pi/12)=3.864...$
$\chi^3(2\pi/12)=3.732...$

Polygonal geometry of $U(2)\supset C_N$ character spectral function



$$\chi^j(\frac{2\pi}{n}) = \frac{\sin\frac{\pi}{n}(2j+1)}{\sin\frac{\pi}{n}} = \frac{\sin\frac{\pi\ell^J}{n}}{\sin\frac{\pi}{n}}$$

Character Spectral Function
where: $\ell^j=2j+1$
is U(2) irrep dimension



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Polygonal geometry of U(2)⊃C_N character spectral function

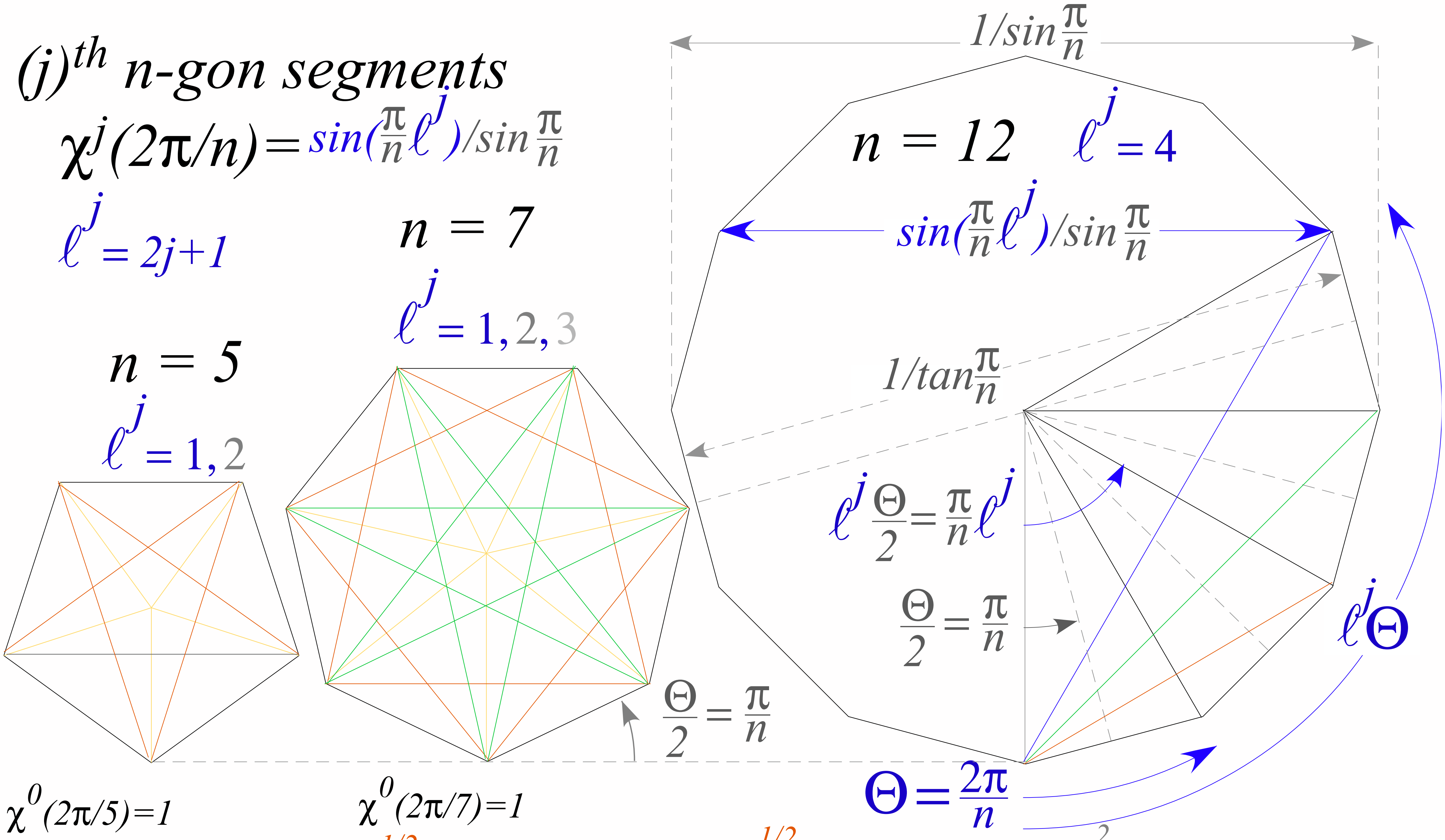
$$\chi^j(\frac{2\pi}{n})=\frac{\sin\frac{\pi}{n}(2j+1)}{\sin\frac{\pi}{n}}=\frac{\sin\frac{\pi\ell^J}{n}}{\sin\frac{\pi}{n}}$$

Character Spectral Function
where: $\ell^j=2j+1$
is U(2) irrep dimension

$(j)^{th}$ n-gon segments

$$\chi^j(2\pi/n)=\sin(\frac{\pi}{n}\ell^j)/\sin\frac{\pi}{n}$$

$$\ell^j=2j+1$$



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Polygonal geometry of $U(2) \supset C_N$ character spectral function

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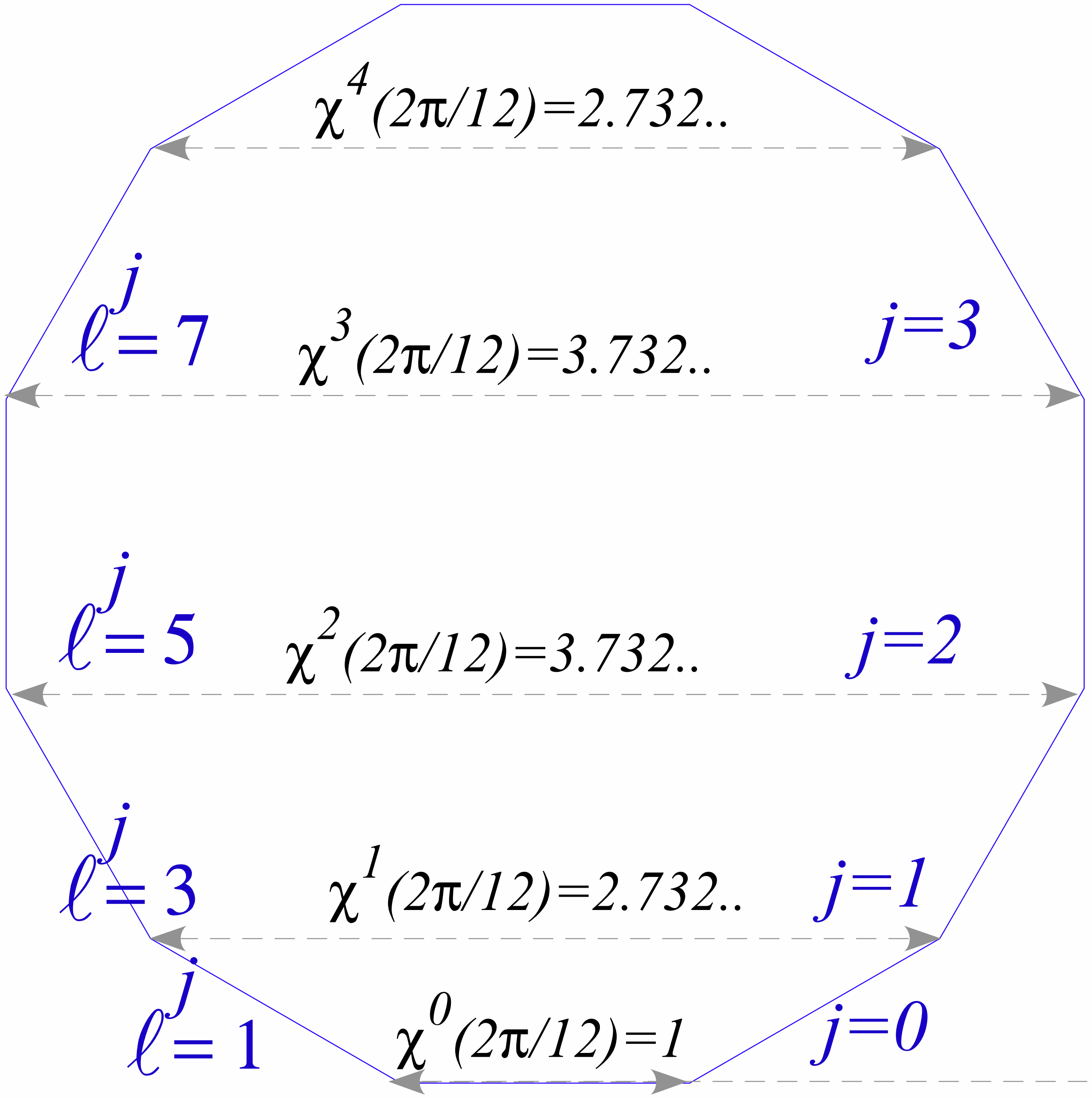
Character Spectral Function
where: $\ell^j=2j+1$
is $U(2)$ irrep dimension

Integer j for n=12



Polygonal geometry of U(2)⊃C_N character spectral function

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Character Spectral Function
where: $\ell^j=2j+1$
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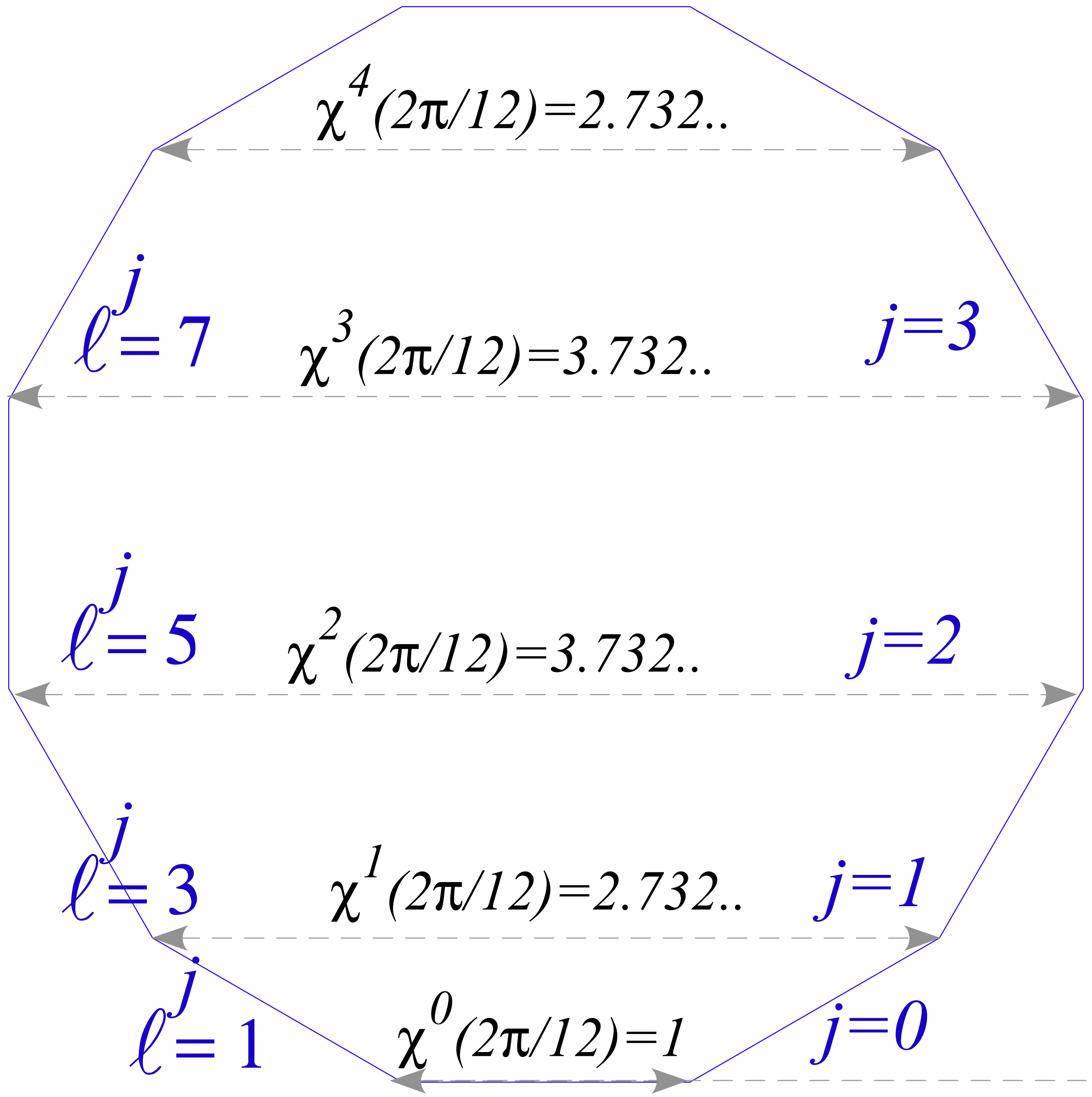
Integer j for n=12



1/2-Integer j for n=12