Group Theory in Quantum Mechanics
Lecture 11 (2.21.17)
Representations of cyclic groups $C_3 \subset C_6 \supset C_2$

(Quantum Theory for Computer Age - Ch. 6-9 of Unit 3)
(Principles of Symmetry, Dynamics, and Spectroscopy - Sec. 3-7 of Ch. 2)

Review of C_2 spectral resolution for 2D oscillator (Lecture 6 : p. 11, p. 17, and p. 11)

C_3 **g†g**-*product-table and basic group representation theory*

C_3 **H**-and-$\mathbf{r}^p$-*matrix representations and conjugation symmetry*

C_3 Spectral resolution: 3rd roots of unity and ortho-completeness relations

C_3 *character table and modular labeling*

Ortho-completeness inversion for operators and states

Comparing wave function operator algebra to bra-ket algebra

Modular quantum number arithmetic

C_3-group jargon and structure of various tables

C_3 Eigenvalues and wave dispersion functions

Standing waves vs Moving waves

WebApps used

[WaveIt App]

[MolVibes]

Review of C_2 spectral resolution for 2D oscillator Lecture 6

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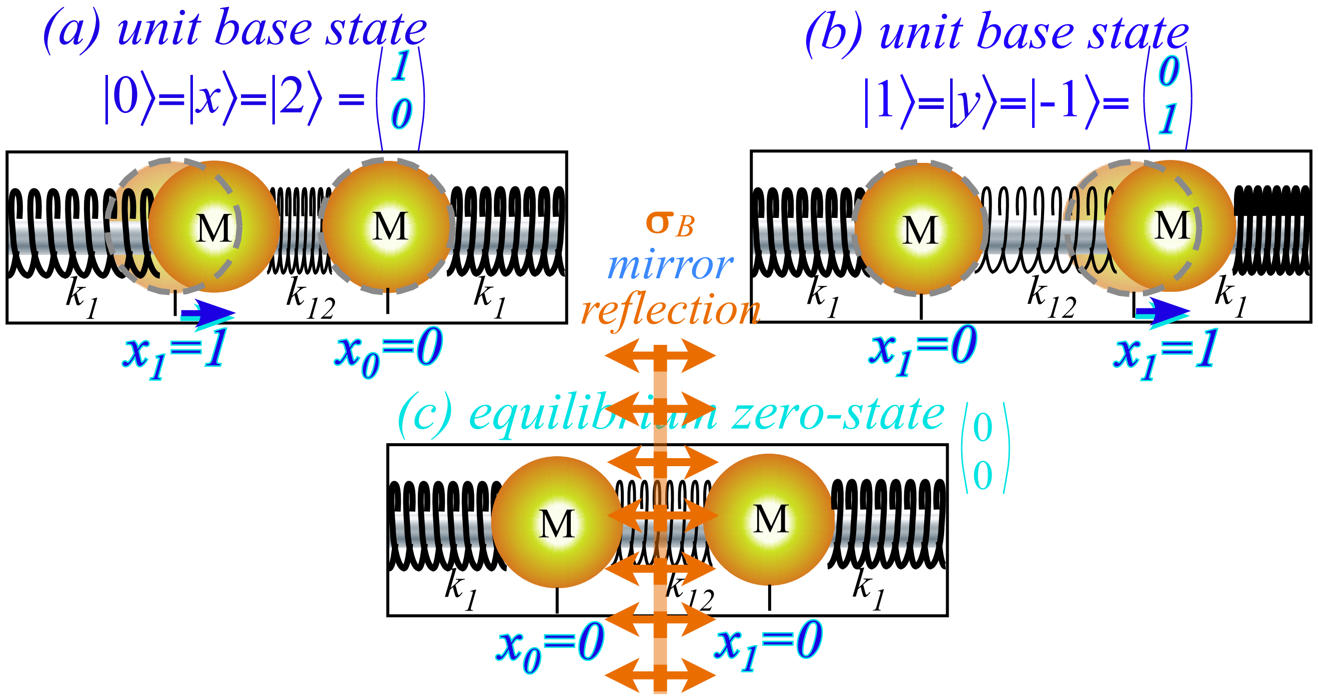
C_6 Spectral resolution: 6th roots of unity and higher

Complete sets of coupling parameters and Fourier dispersion

Gauge shifts due to complex coupling

C_2 Symmetric two-dimensional harmonic oscillators (2DHO)

2D HO "binary" bases and coord. $\{x_0,x_1\}$



2D HO Matrix operator equations

$$\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_2\end{pmatrix} = -\begin{pmatrix}\frac{k_1+k_{12}}{M} & \frac{-k_{12}}{M}\\ \frac{-k_{12}}{M} & \frac{k_1+k_{12}}{M}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$= -\begin{pmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\,|\mathbf{x}\rangle$$

More conventional coordinate notation $\{x_0,x_1\} \rightarrow \{x_1,x_2\}$

C_2 (Bilateral σ_B reflection) symmetry conditions:

$K_{11} \equiv K \equiv K_{22}$ and: $K_{12} \equiv k \equiv K_{12} = -k_{12}$ (Let: $M= 1$)

$$\begin{pmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{pmatrix} = \begin{pmatrix}K & k\\ k & K\end{pmatrix} = K\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + k\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$\mathbf{K} = K\mathbf{1} + k\sigma_B$$

K-matrix is made of its symmetry operators in *group* $C_2=\{\mathbf{1},\sigma_B\}$ with *product table*:

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

Symmetry *product table* gives C_2 group representations in *group basis* $\{|0\rangle=\mathbf{1}|0\rangle\equiv|\mathbf{1}\rangle\ ,\ |1\rangle=\boldsymbol{\sigma}_B|0\rangle\equiv|\boldsymbol{\sigma}_B\rangle\}$

$$\begin{pmatrix}\langle\mathbf{1}|\mathbf{1}|\mathbf{1}\rangle & \langle\mathbf{1}|\mathbf{1}|\sigma_B\rangle\\ \langle\sigma_B|\mathbf{1}|\mathbf{1}\rangle & \langle\sigma_B|\mathbf{1}|\sigma_B\rangle\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad \begin{pmatrix}\langle\mathbf{1}|\boldsymbol{\sigma}_B|\mathbf{1}\rangle & \langle\mathbf{1}|\boldsymbol{\sigma}_B|\sigma_B\rangle\\ \langle\sigma_B|\boldsymbol{\sigma}_B|\mathbf{1}\rangle & \langle\sigma_B|\boldsymbol{\sigma}_B|\sigma_B\rangle\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

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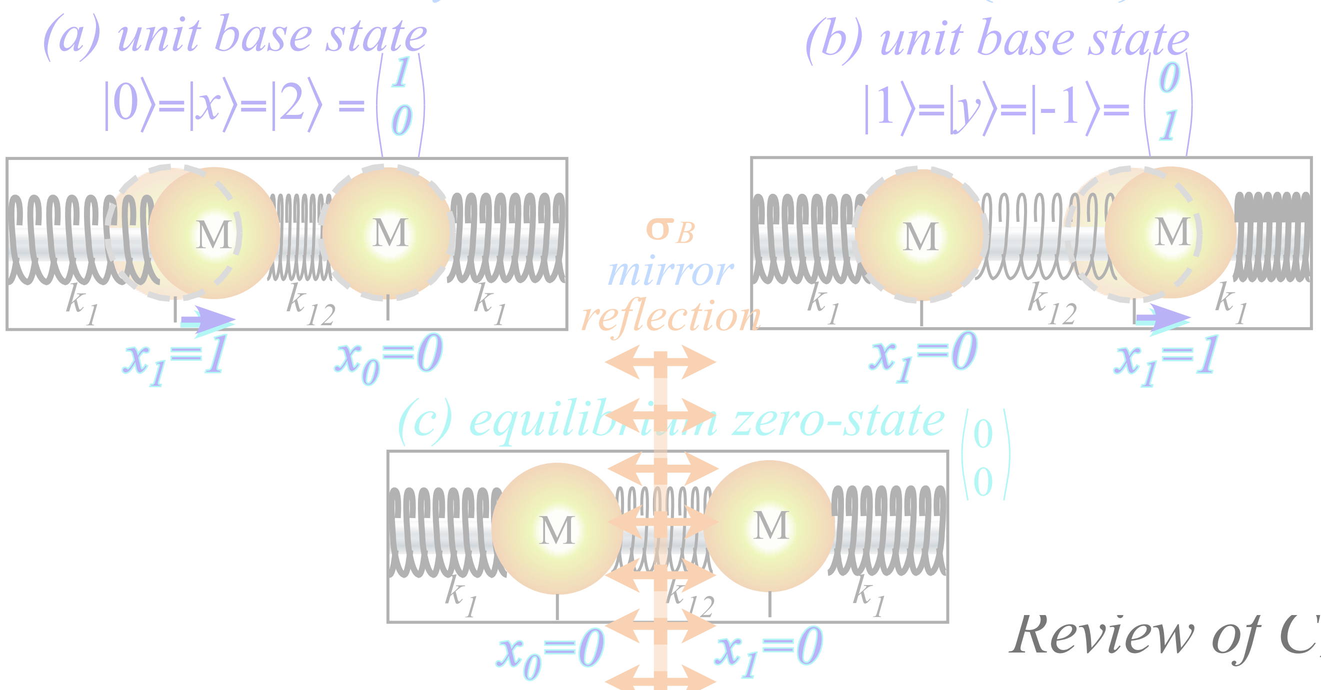

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Review of C_2 spectral resolution for 2D oscillator Lecture 6 p.17

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$\mathbf{1}$	$\mathbf{1}$	σ_B
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$$\begin{pmatrix} \langle \mathbf{1}|\mathbf{1}|\mathbf{1}\rangle & \langle \mathbf{1}|\mathbf{1}|\sigma_B\rangle \\ \langle \sigma_B|\mathbf{1}|\mathbf{1}\rangle & \langle \sigma_B|\mathbf{1}|\sigma_B\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} \langle \mathbf{1}|\sigma_B|\mathbf{1}\rangle & \langle \mathbf{1}|\sigma_B|\sigma_B\rangle \\ \langle \sigma_B|\sigma_B|\mathbf{1}\rangle & \langle \sigma_B|\sigma_B|\sigma_B\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Minimal equation of σ_B is: $\sigma_B{}^2 = 1$

or: $\sigma_B{}^2 - 1 = 0 = (\sigma_B - 1)(\sigma_B + 1)$

with eigenvalues:

$\{\chi^+(\sigma_B) = +1,\ \chi^-(\sigma_B) = -1\}$

Spectral decomposition of $C_2(\sigma_B)$ into $\{\mathbf{P}^+, \mathbf{P}^-\}$

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

$\mathbf{P}^\pm$-projectors:

$$\mathbf{P}^+ = \frac{\mathbf{1} + \sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\mathbf{P}^+ = \frac{\mathbf{1} - \sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

C_2 Symmetric 2DHO eigensolutions

$$\mathbf{K} = K\mathbf{1} - k_{12}\sigma_B$$

$$K\begin{pmatrix}1&0\\0&1\end{pmatrix} - k_{12}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}k_1+k_{12} & -k_{12}\\ -k_{12} & k_1+k_{12}\end{pmatrix}$$

K-matrix is made of its symmetry operators in ***group*** $C_2=\{\mathbf{1},\sigma_B\}$ with ***product table***:

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+,\mathbf{P}^-\}$ projectors:

$$\mathbf{P}^+ = \frac{\mathbf{1}+\sigma_B}{2} = \begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\ \frac{1}{2}&\frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix} = |+\rangle\langle+|$$

factored projectors

$$\mathbf{P}^- = \frac{\mathbf{1}-\sigma_B}{2} = \begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\\ -\frac{1}{2}&\frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix} = |-\rangle\langle-|$$

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

Eigenvalues of σ_B :

$\{\chi^+(\sigma_B)=+1,\ \chi^-(\sigma_B)=-1\}$

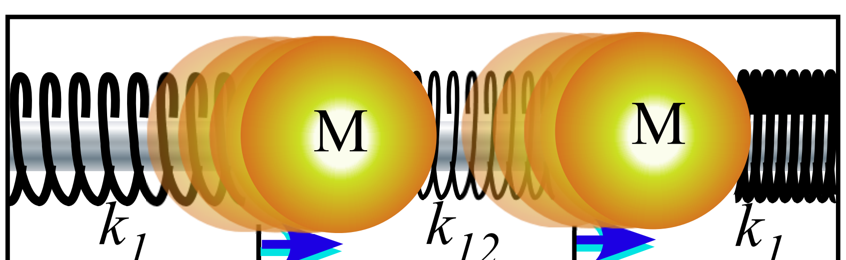
Diagonalizing transformation (D-tran) of K-matrix:

$$\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}k_1+k_{12}&-k_{12}\\ -k_{12}&k_1+k_{12}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix} = \begin{pmatrix}k_1&0\\0&k_1+2k_{12}\end{pmatrix}$$

Eigenvalues of $\mathbf{K}=K\cdot\mathbf{1} - k_{12}\cdot\sigma_B$:

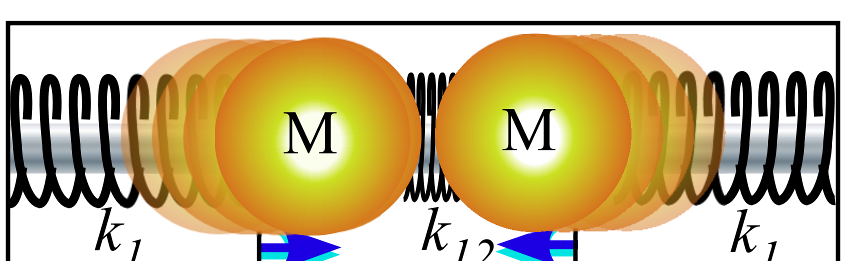
$\varepsilon^+(\mathbf{K}) = K - k_{12} = k_1$, $\quad\varepsilon^-(\mathbf{K}) = K + k_{12} = k_1+2k_{12}$

Even mode $|+\rangle = |0_2\rangle = \begin{pmatrix}1\\1\end{pmatrix}/\sqrt{2}$



$x_0=1/\sqrt{2}\quad x_1=1/\sqrt{2}$

Odd mode $|-\rangle = |1_2\rangle = \begin{pmatrix}1\\-1\end{pmatrix}/\sqrt{2}$

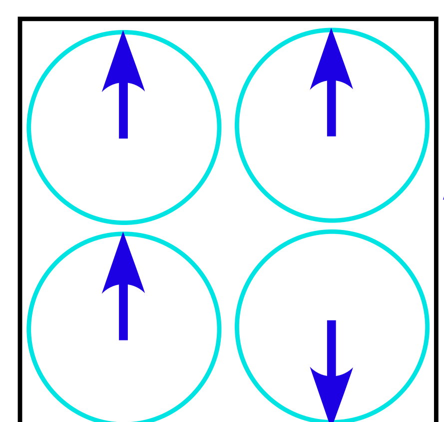


$x_0=1/\sqrt{2}\quad x_1=-1/\sqrt{2}$

C_2 mode phase character tables

p is position

	$p=0$	$p=1$
$m=0_2$	1	1
$m=1_2$	1	-1



norm: $1/\sqrt{2}$

m is wave-number or "momentum"

(D-tran)

$$\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix} = \begin{pmatrix}\langle x_1|+\rangle & \langle x_1|-\rangle\\ \langle x_2|+\rangle & \langle x_2|-\rangle\end{pmatrix}$$

(D-tran is its own inverse in this case!)

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C_3	$\mathbf{r}^0=\mathbf{1}$	$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^2=\mathbf{r}^{-1}$
$\mathbf{r}^0=\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{r}^2=\mathbf{r}^{-1}$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
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C_3 $\mathbf{g}^\dagger\mathbf{g}$-*product-table*
Pairs each operator $\mathbf{g}$ in the 1st row
with its inverse $\mathbf{g}^\dagger=\mathbf{g}^{-1}$ in the 1st column
so all *unit* $\mathbf{1}=\mathbf{g}^{-1}\mathbf{g}$ *elements* lie on diagonal.

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A C_3 **H**-*matrix* is then constructed directly from the $\mathbf{g}^\dagger\mathbf{g}$-*table* and so is each $\mathbf{r}^p$-*matrix representation*.

$$\mathbf{H}=\begin{pmatrix} r_0 & r_1 & r_2 \\ r_2 & r_0 & r_1 \\ r_1 & r_2 & r_0 \end{pmatrix} = r_0\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + r_1\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + r_2\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= r_0\cdot\mathbf{1} \qquad + r_1\cdot\mathbf{r}^1 \qquad + r_2\cdot\mathbf{r}^2$$

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$\mathbf{r}^2=\mathbf{r}^{-1}$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
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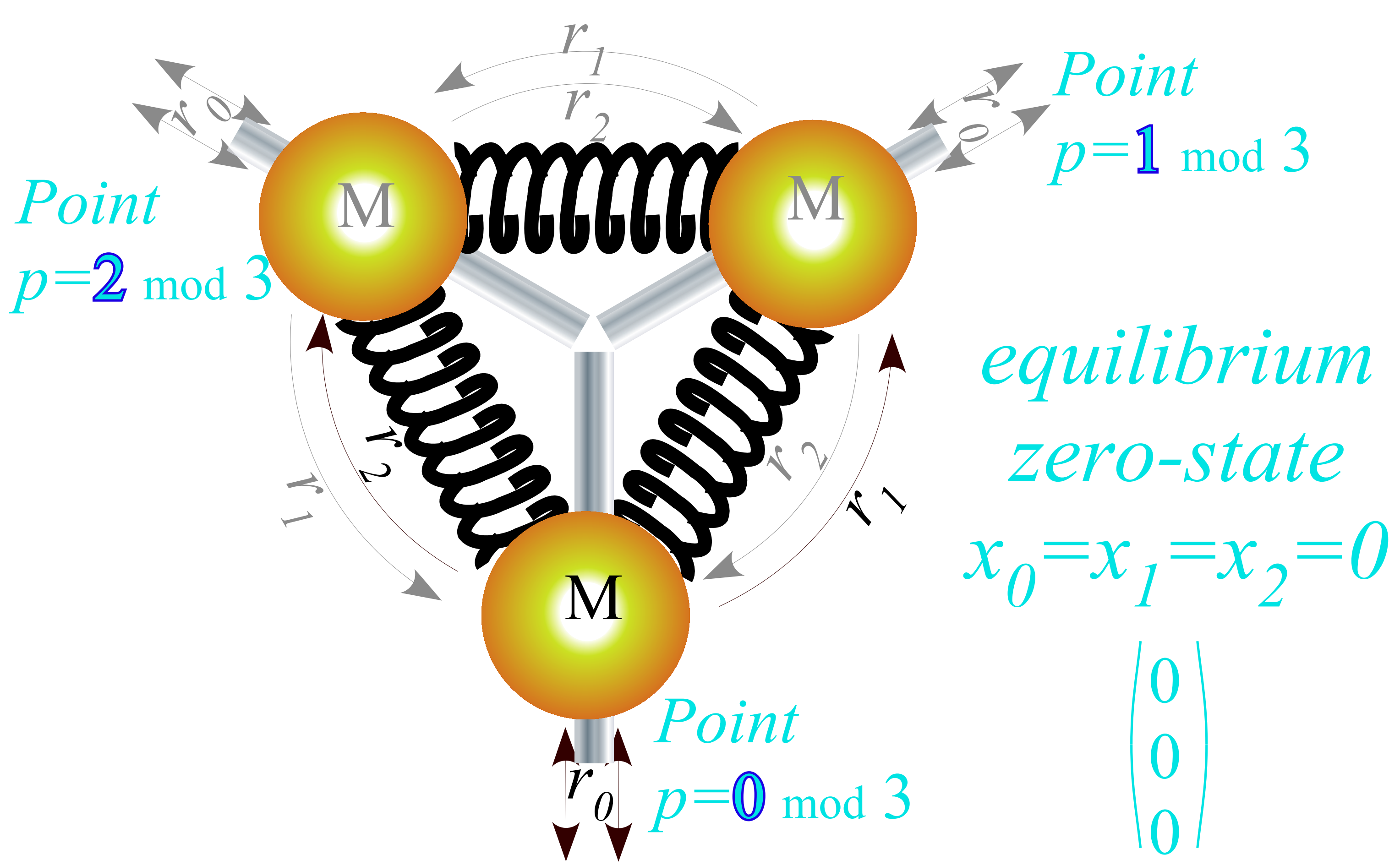
$$\mathbf{H}=\begin{pmatrix} r_0 & r_1 & r_2 \\ r_2 & r_0 & r_1 \\ r_1 & r_2 & r_0 \end{pmatrix} = r_0\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + r_1\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + r_2\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= r_0\cdot\mathbf{1} \qquad + r_1\cdot\mathbf{r}^1 \qquad + r_2\cdot\mathbf{r}^2$$

H-matrix coupling constants $\{r_0, r_1, r_2\}$
relate to particular operators $\{\mathbf{r}^0, \mathbf{r}^1, \mathbf{r}^2\}$
that transmit a particular force or current.



$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

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Constants r_k that are *grayed-out*
may change values
if C_3 symmetry
is broken

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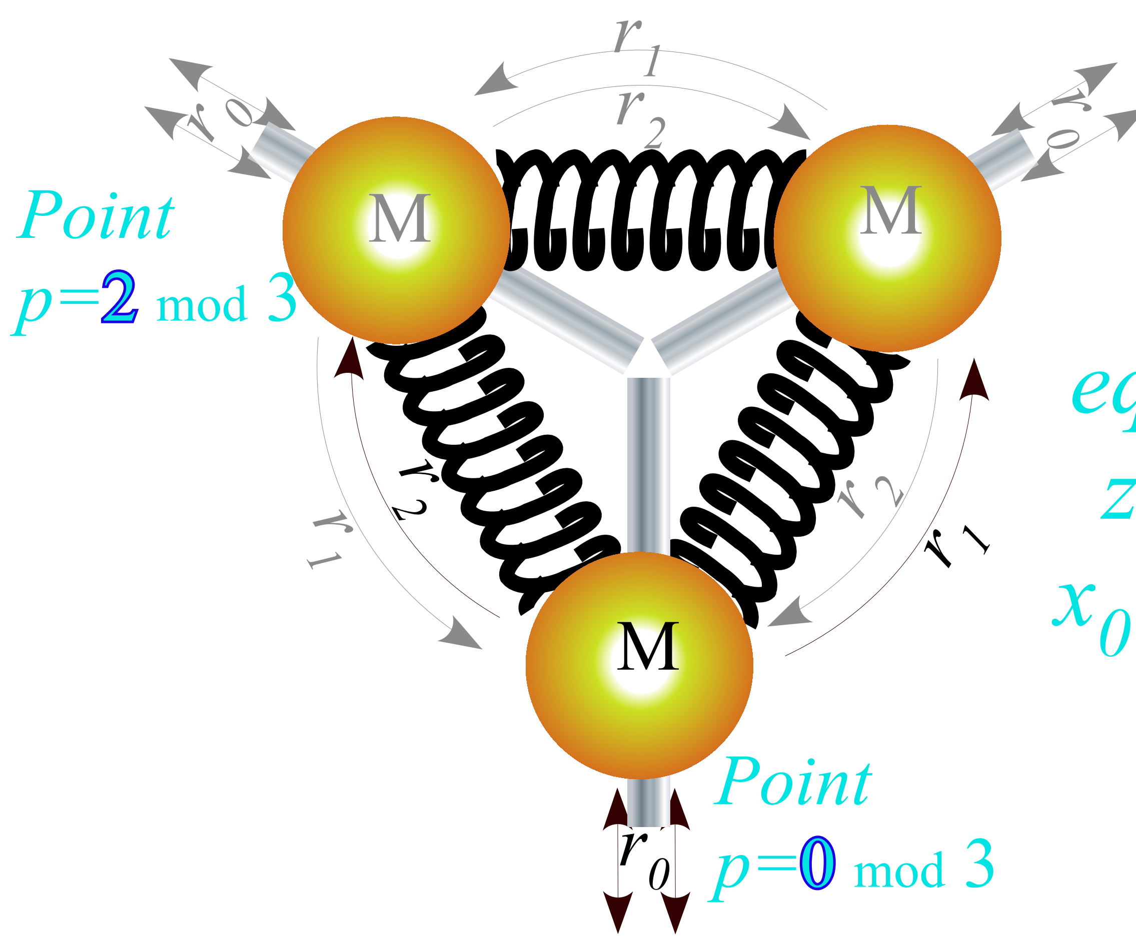

equilibrium zero-state
$$x_0=x_1=x_2=0$$
$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

C_3 $\mathbf{g}^\dagger\mathbf{g}$-*product-table* *and basic group representation theory*

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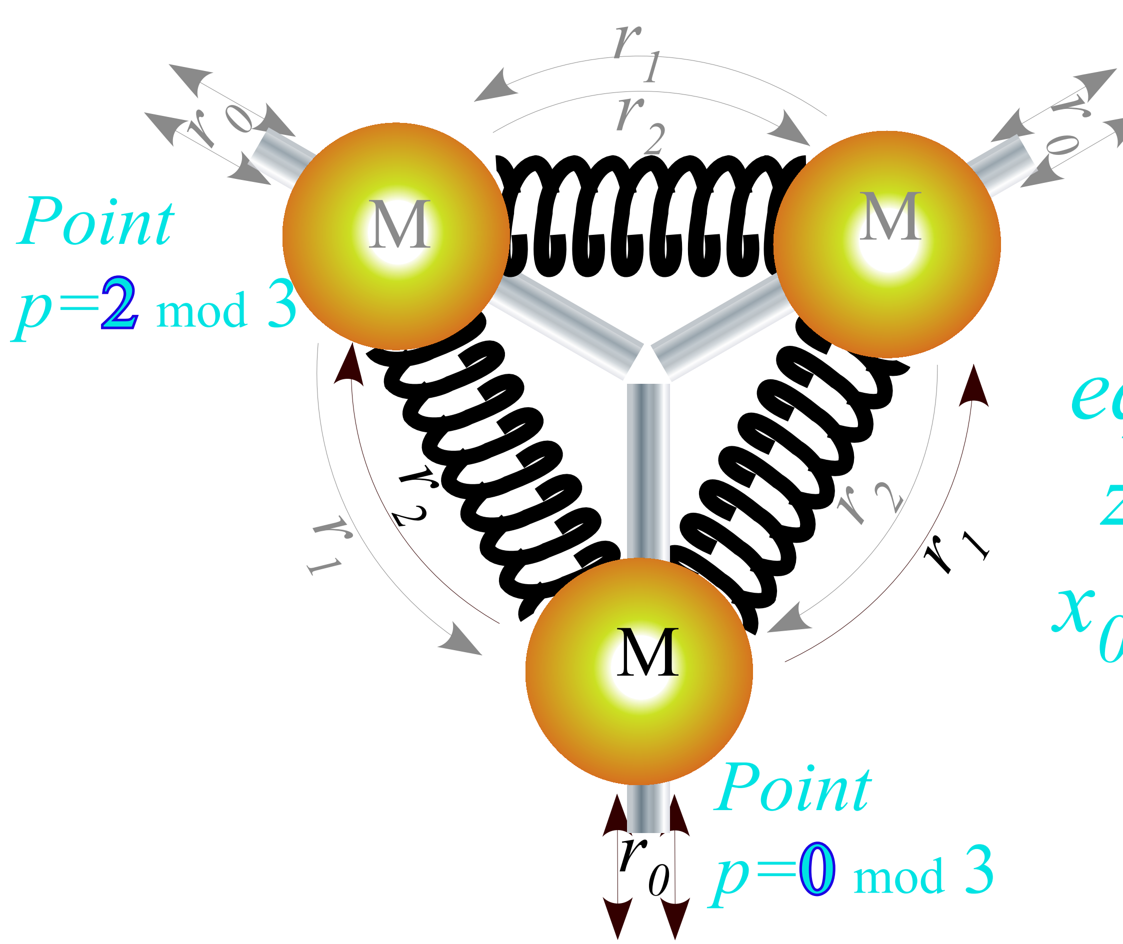
Conjugation symmetry
However, no matter how C_3 is broken,
a Hermitian-symmetric Hamiltonian
$(H^*_{jk}=H_{kj})$ requires that $r_0^*=r_0$ and $r_1^*=r_2$.



equilibrium
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H-matrix coupling constants $\{r_0, r_1, r_2\}$
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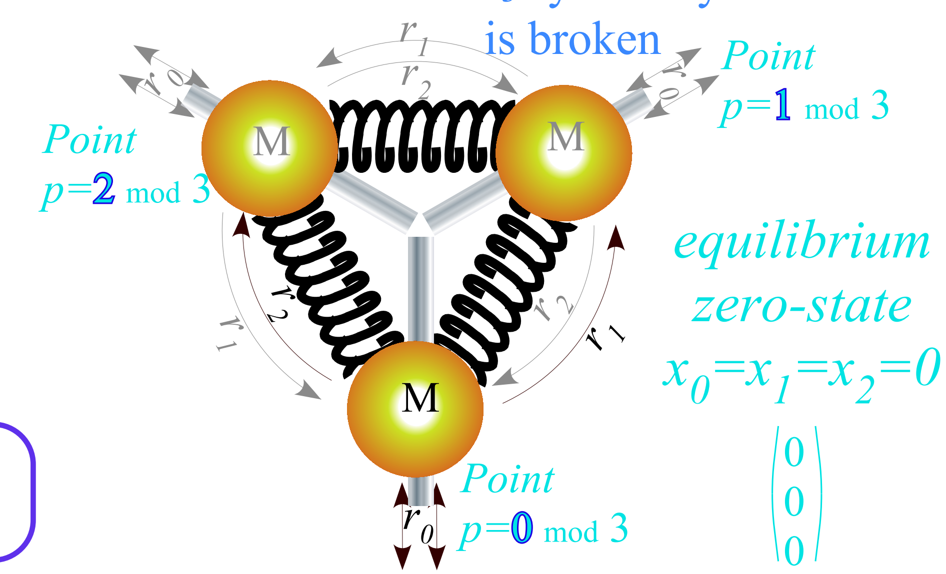
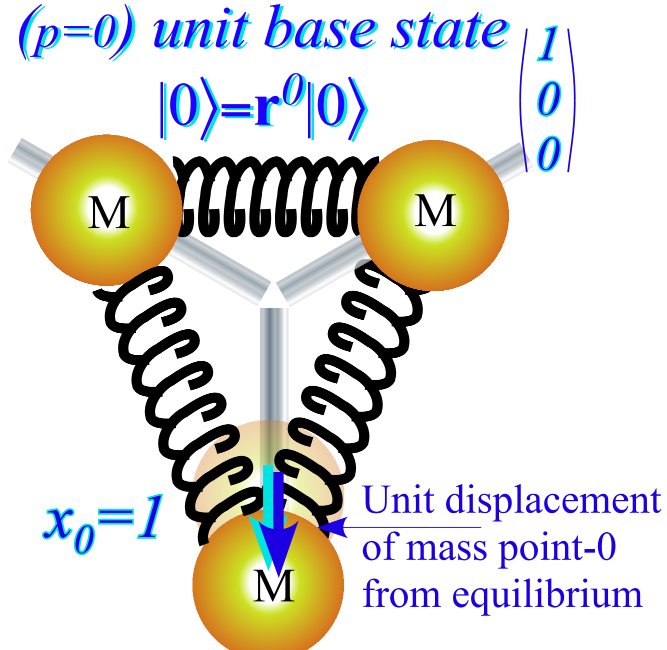
Conjugation symmetry
Hermitian Hamiltonian $(H^*_{jk}=H_{kj})$ requires $r_0^*=r_0$ and $r_1^*=r_2$.

Constants r_k that are *grayed-out* may change values if C_3 symmetry is broken

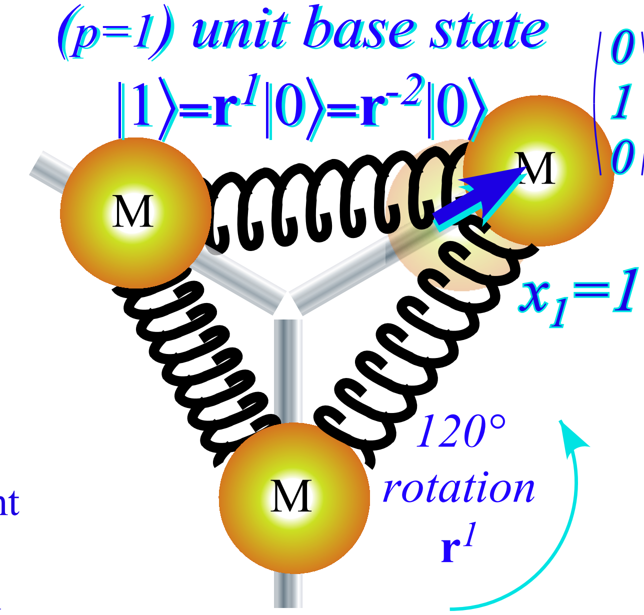


C_3 operators $\{\mathbf{r}^0, \mathbf{r}^1, \mathbf{r}^2\}$
also label unit
base states:
$|0\rangle=\mathbf{r}^0|0\rangle$
$|1\rangle=\mathbf{r}^1|0\rangle$
$|2\rangle=\mathbf{r}^2|0\rangle$
modulo-3

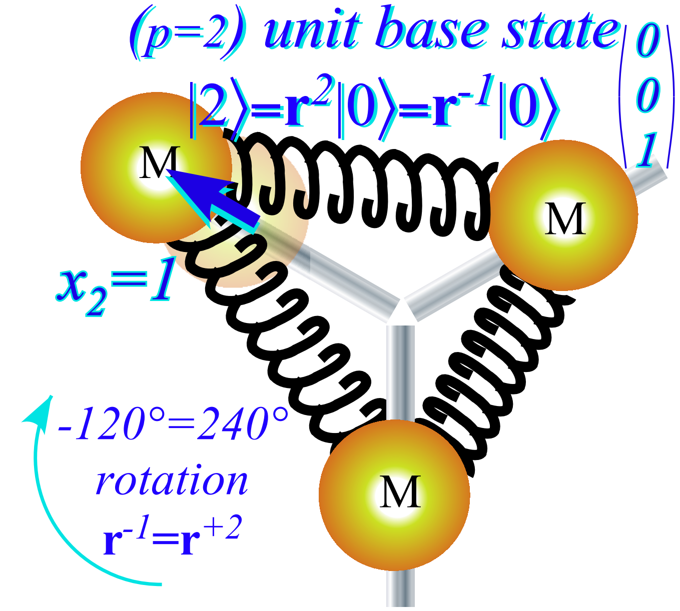
(p=0) unit base state
$|0\rangle=\mathbf{r}^0|0\rangle$



(p=1) unit base state
$|1\rangle=\mathbf{r}^1|0\rangle=\mathbf{r}^{-2}|0\rangle$



(p=2) unit base state
$|2\rangle=\mathbf{r}^2|0\rangle=\mathbf{r}^{-1}|0\rangle$



C_3 $\mathbf{g}^\dagger\mathbf{g}$-*product-table and basic group representation theory*

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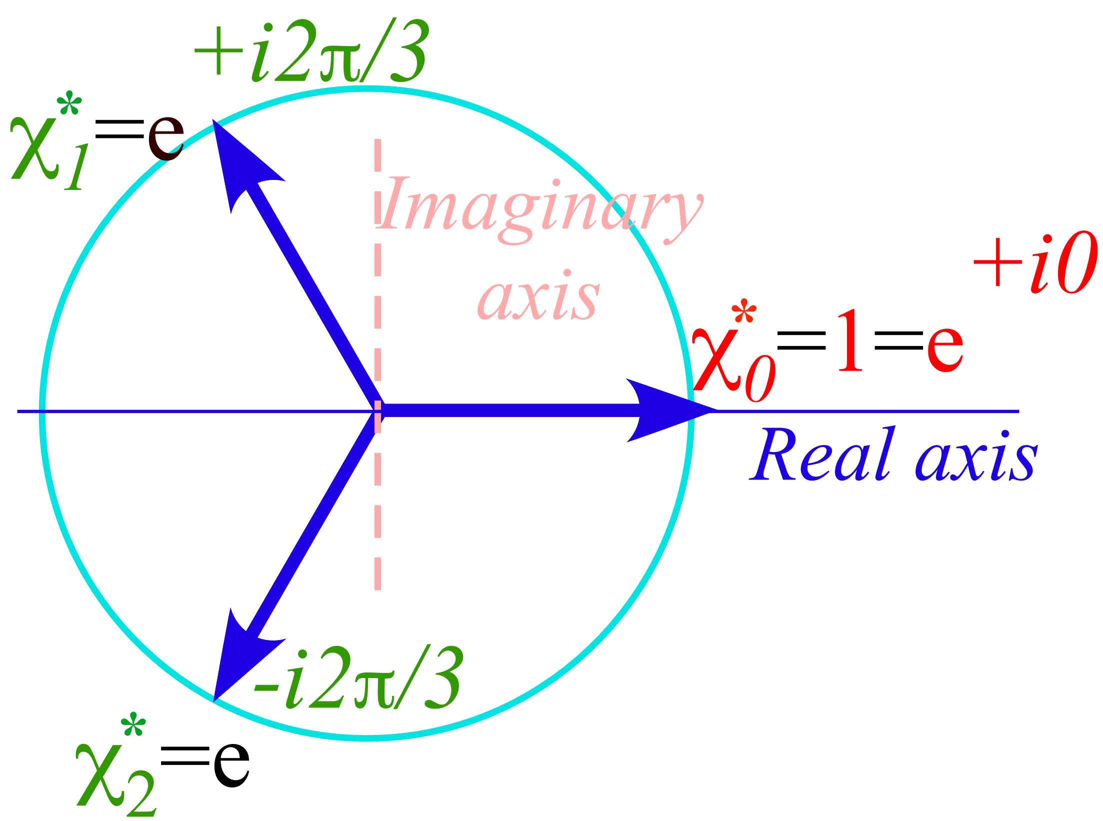

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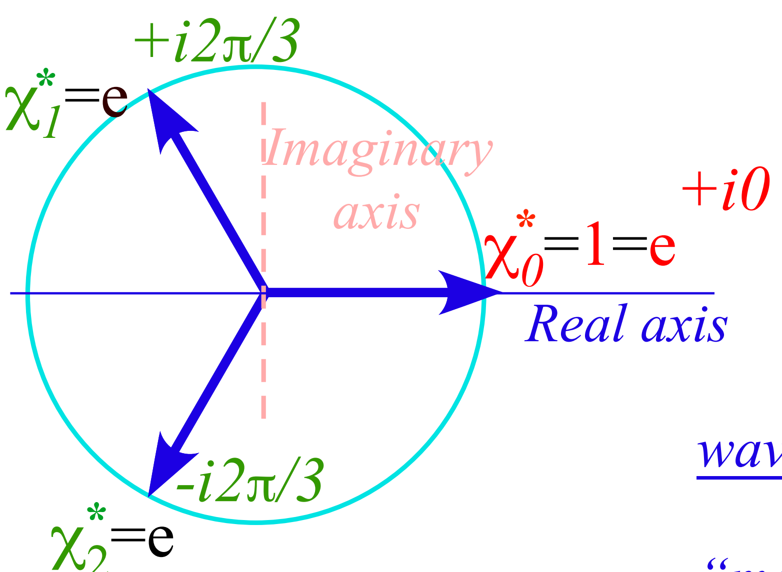

C_3 mode phase character table

<u>*wave-number*</u>
m=
"momentum"

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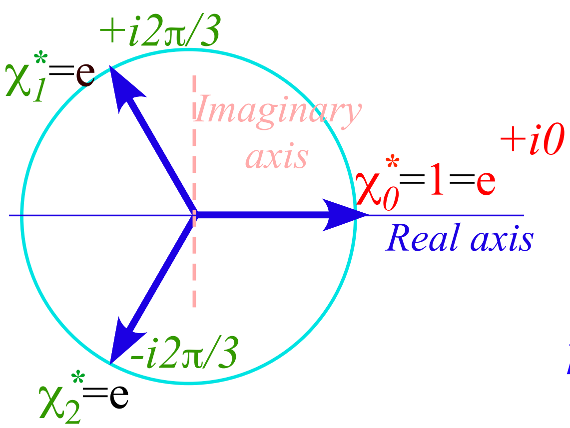

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wave-number
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WaveIt App

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C_3 character conjugate

$$\chi^*_{mp}=e^{imp2\pi/3}$$

is wave function

$$\psi_m(r_p)=\frac{e^{ik_m\cdot r_p}}{\sqrt{3}}$$

C_3 *Spectral resolution: 3rd roots of unity*

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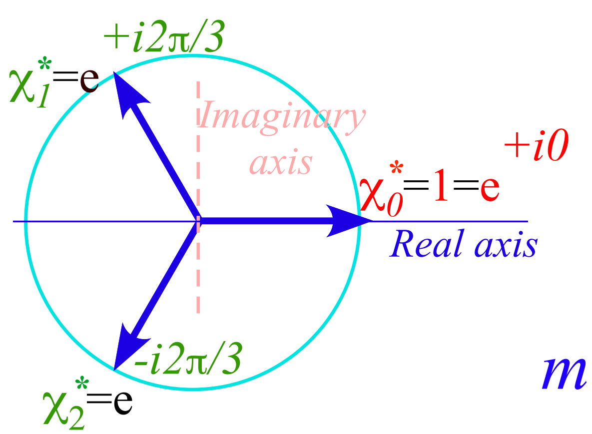

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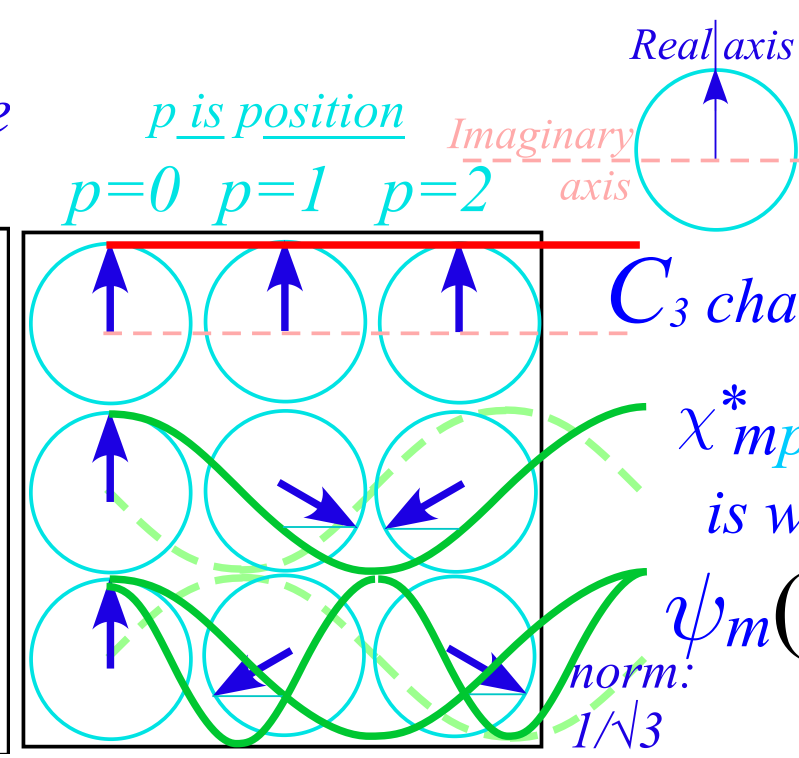

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WaveIt App

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C_3 $\mathbf{g}^\dagger\mathbf{g}$-*product-table* *and basic group representation theory*

C_3 **H**-and-$\mathbf{r}^p$-*matrix representations* *and* *conjugation symmetry*

C_3 Spectral resolution: 3rd roots of unity and ortho-completeness relations

C_3 *character table* *and* *modular labeling*

Ortho-completeness *inversion* *for operators and states*

Comparing wave function operator algebra to bra-ket algebra

Modular quantum number arithmetic

C_3-group jargon and structure of various tables

C_3 Eigenvalues and wave dispersion functions

Standing waves vs Moving waves

C_6 Spectral resolution: 6th roots of unity and higher

Complete sets of coupling parameters and Fourier dispersion

Gauge shifts due to complex coupling

Given unitary *Ortho-Completeness operator* relations:

$$\mathbf{P}^{(0)} + \mathbf{P}^{(1)} + \mathbf{P}^{(1)} = \mathbf{1} = \mathbf{P}^{(0)} + \mathbf{P}^{(1)} + \mathbf{P}^{(2)}$$

$$\chi_0 \mathbf{P}^{(0)} + \chi_1 \mathbf{P}^{(1)} + \chi_2 \mathbf{P}^{(2)} = \mathbf{r}^{1} = 1\ \mathbf{P}^{(0)} + e^{-i2\pi/3}\mathbf{P}^{(1)} + e^{i2\pi/3}\mathbf{P}^{(2)}$$

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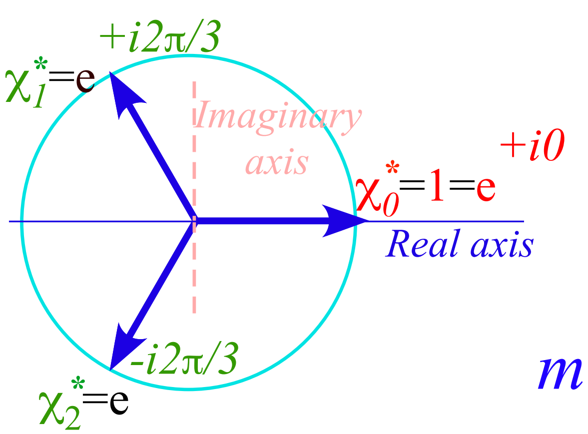

C_3 mode phase character table

	p=0	*p=1*	*p=2*
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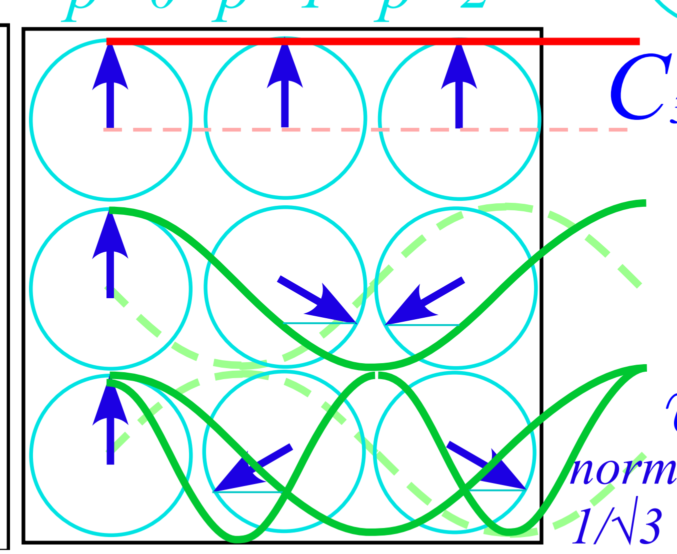

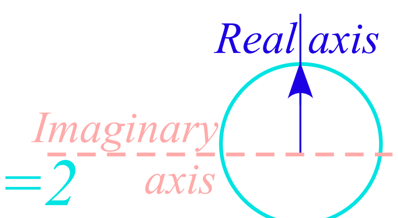

WaveIt App

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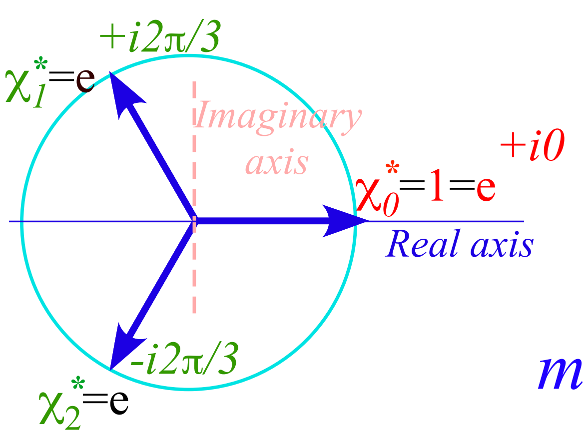
Two distinct types of modular "quantum" numbers:

*p=0,1,*or *2* is *power p* of operator $\mathbf{r}^p$ labeling oscillator *position point p*



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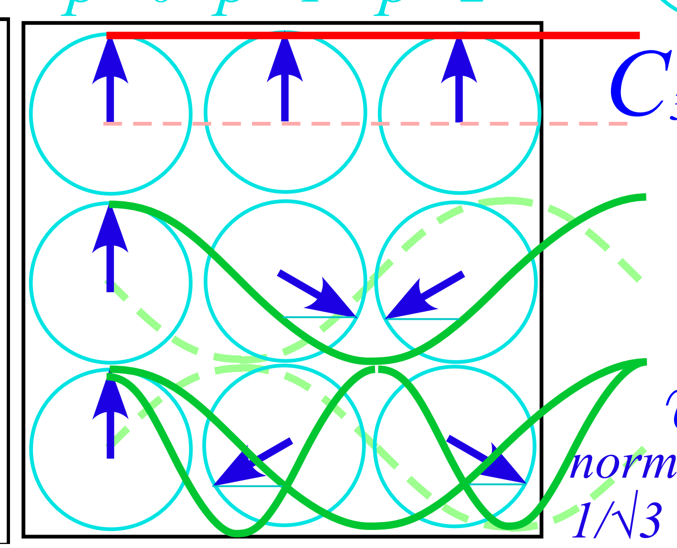
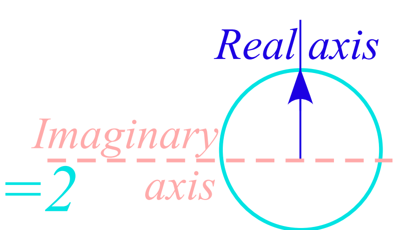


norm: $1/\sqrt{3}$



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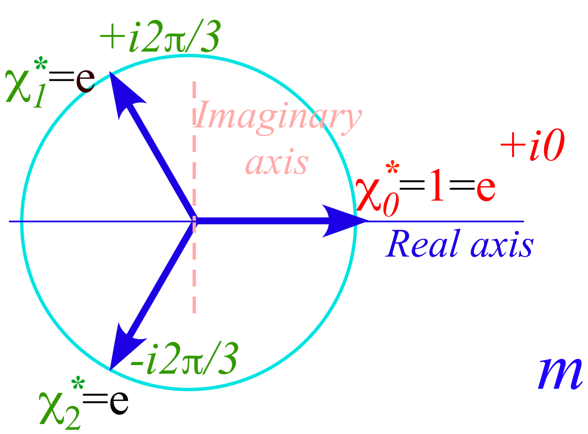
m=0,1,or *2* that is the *mode momentum m* of waves



wave-number

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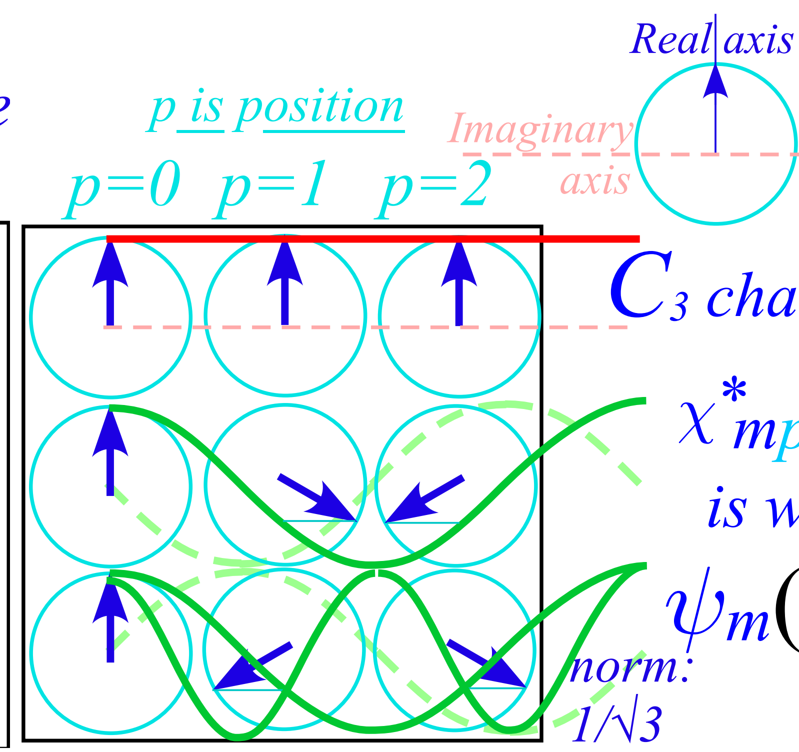

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Inverting *O-C* is easy: just †-*conjugate*! (and norm by $\frac{1}{3}$)

$$\mathbf{P}^{(0)} = \tfrac{1}{3}(\mathbf{r}^0 + \mathbf{r}^1 + \mathbf{r}^2) = \tfrac{1}{3}(\mathbf{1} + \mathbf{r}^1 + \mathbf{r}^2)$$

$$\mathbf{P}^{(1)} = \tfrac{1}{3}(\mathbf{r}^0 + \chi_1^*\mathbf{r}^1 + \chi_2^*\mathbf{r}^2) = \tfrac{1}{3}(\mathbf{1} + e^{+i2\pi/3}\mathbf{r}^1 + e^{-i2\pi/3}\mathbf{r}^2)$$

$$\mathbf{P}^{(2)} = \tfrac{1}{3}(\mathbf{r}^0 + \chi_2^*\mathbf{r}^1 + \chi_1^*\mathbf{r}^2) = \tfrac{1}{3}(\mathbf{1} + e^{-i2\pi/3}\mathbf{r}^1 + e^{+i2\pi/3}\mathbf{r}^2)$$

(or norm by $\sqrt{\frac{1}{3}}$)

$$|0_3\rangle = \mathbf{P}^{(0)}|\mathbf{1}\rangle\sqrt{3} = \frac{|\mathbf{r}^0\rangle + |\mathbf{r}^1\rangle + |\mathbf{r}^2\rangle}{\sqrt{3}}$$

$$|1_3\rangle = \mathbf{P}^{(1)}|\mathbf{1}\rangle\sqrt{3} = \frac{|\mathbf{r}^0\rangle + e^{+i2\pi/3}|\mathbf{r}^1\rangle + e^{-i2\pi/3}|\mathbf{r}^2\rangle}{\sqrt{3}}$$

$$|2_3\rangle = \mathbf{P}^{(2)}|\mathbf{1}\rangle\sqrt{3} = \frac{|\mathbf{r}^0\rangle + e^{-i2\pi/3}|\mathbf{r}^1\rangle + e^{+i2\pi/3}|\mathbf{r}^2\rangle}{\sqrt{3}}$$

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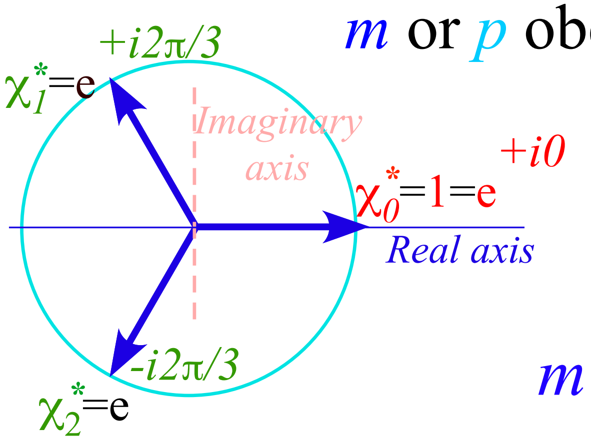
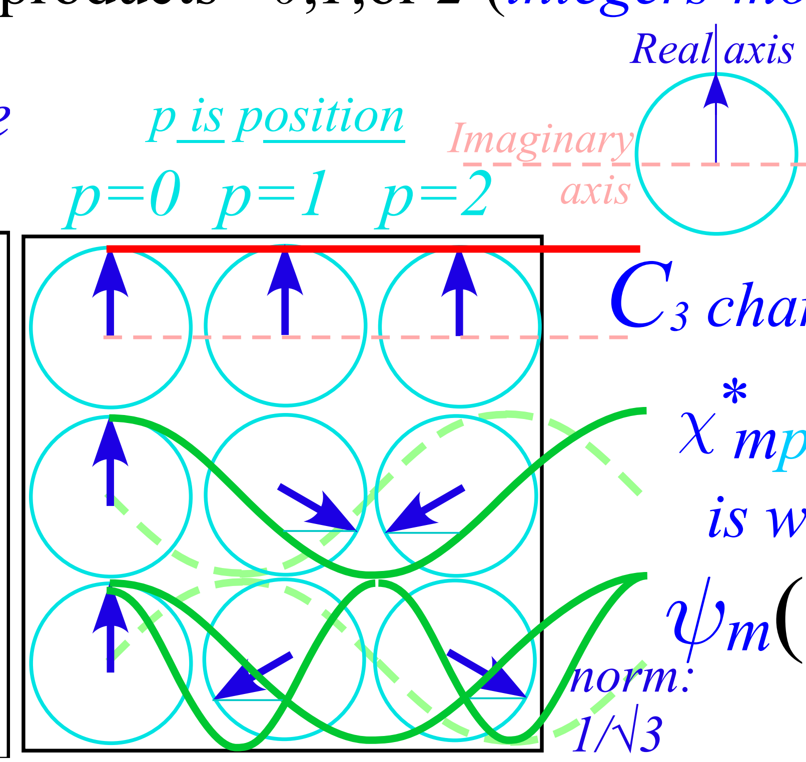


wave-number m= "momentum"

C_3 mode phase character table

	p=0	*p=1*	*p=2*
$m=0_3$	$\chi_{00}=1$	$\chi_{01}=1$	$\chi_{02}=1$
$m=1_3$	$\chi_{10}=1$	$\chi_{11}=e^{-i2\pi/3}$	$\chi_{12}=e^{i2\pi/3}$
$m=2_3$	$\chi_{20}=1$	$\chi_{21}=e^{i2\pi/3}$	$\chi_{22}=e^{-i2\pi/3}$

p is position



WaveIt App

MolVibes

C_3 character conjugate

$$\chi^*_{mp} = e^{imp2\pi/3}$$

is wave function

$$\psi_m(r_p) = \frac{e^{ik_m \cdot r_p}}{\sqrt{3}}$$

C_3 **g†g**-*product-table and basic group representation theory*

C_3 **H**-and-**r**^{p}-*matrix representations and conjugation symmetry*

C_3 Spectral resolution: 3rd roots of unity and ortho-completeness relations

C_3 *character table and modular labeling*

Ortho-completeness inversion for operators and states

Comparing wave function operator algebra to bra-ket algebra

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C_3-group jargon and structure of various tables

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C_3 Plane wave function

$$\psi_m(x_p)=\frac{e^{ik_m\cdot x_p}}{\sqrt{3}}$$

$$=\frac{e^{imp2\pi/3}}{\sqrt{3}}$$

C_3 Lattice position vector

$$x_p=L\cdot p$$

Wavevector

$$k_m=2\pi m/3L=2\pi/\lambda_m$$

Wavelength

$$\lambda_m=2\pi/k_m=3L/m$$

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$$\mathbf{r}^p\,|q\rangle = |q+p\rangle \quad \text{implies:} \quad \langle q|\left(\mathbf{r}^p\right)^\dagger = \langle q|\mathbf{r}^{-p} = \langle q+p| \quad \text{implies:} \quad \langle q|\mathbf{r}^p = \langle q-p|$$

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Action of $\mathbf{r}^p$ on m-ket $|(m)\rangle=|k_m\rangle$ is inverse to action on coordinate bra $\langle x_q|=\langle q|$.

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(Norm factors left out) $\psi_{k_m}(x_q - p \cdot L) = \langle x_q|\mathbf{r}^p|k_m\rangle = e^{ik_m(x_q - p \cdot L)} = e^{ik_m(x_q - x_p)}$

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$$\mathbf{r}^p|(m)\rangle=e^{-ik_m x_p}|(m)\rangle$$

C_3 **g**†**g**-*product-table* *and basic group representation theory*

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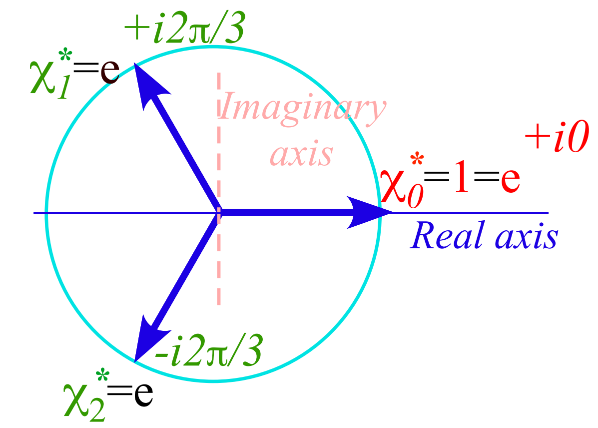


Two distinct types of modular "quantum" numbers:

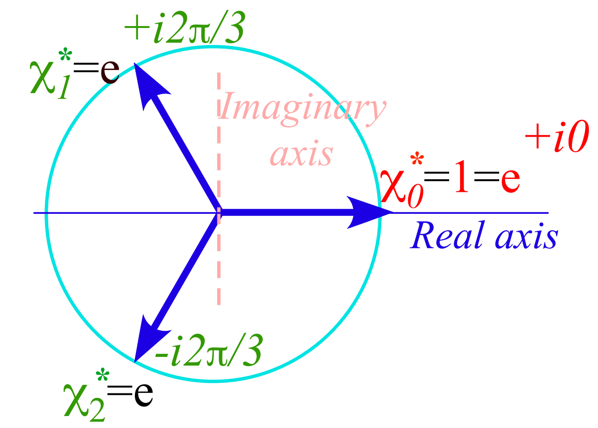
$p=0,1$,or 2 is *power p* of operator $\mathbf{r}^p$ labeling oscillator *position point p*

$m=0,1$,or 2 that is the *mode momentum m* of waves

m or p obey *modular arithmetic* so sums or products $=0,1$,or 2 (*integers-modulo-3)*

For example, for $m=2$ and $p=2$ the number $(\rho_m)^p=(e^{im2\pi/3})^p$ is $e^{imp\cdot 2\pi/3}= e^{i4\cdot 2\pi/3}= e^{i1\cdot 2\pi/3}\ e^{i3\cdot 2\pi/3}= e^{i2\pi/3}=\rho_1$.

Modular quantum number arithmetic



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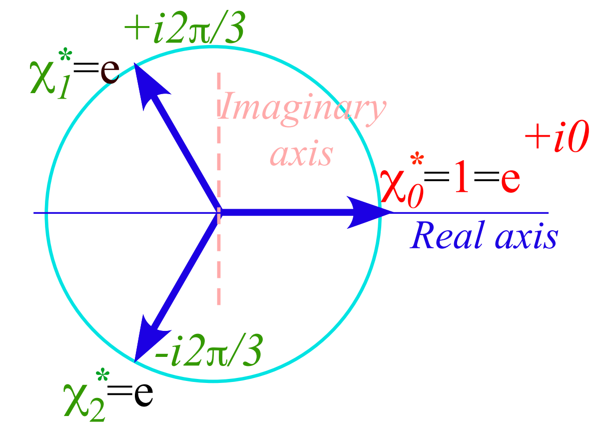

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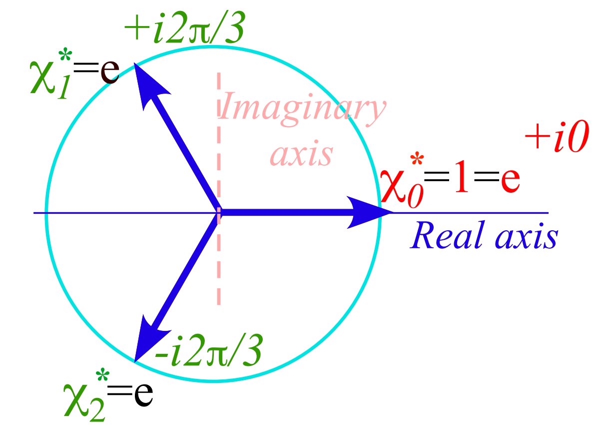
Modular quantum number arithmetic



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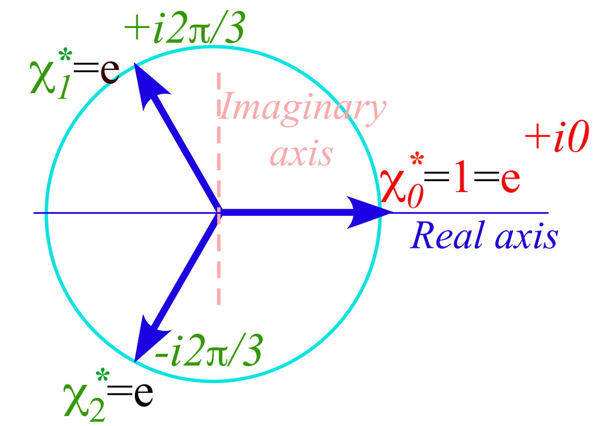

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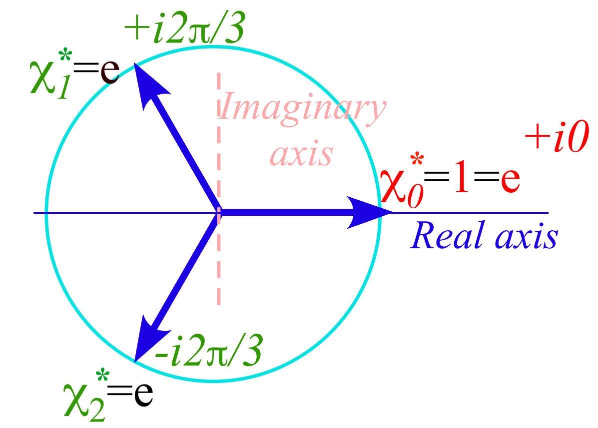
Imagine going around ring reading off address points *p*=... *0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2,*....

..for regular integer points ...*-3,-2,-1, 0, 1, 2, 3, 4, 5, 6, 7, 8,*....

Modular quantum number arithmetic



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Imagine going around ring reading off address points p=... *0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2,....*

..for regular integer points ...*-3,-2,-1, 0, 1, 2, 3, 4, 5, 6, 7, 8,....*

$e^{imp2\pi/3}$ must always equal $e^{i(mp \bmod 3)2\pi/3}$.

$$(\rho_m)^p=(e^{im2\pi/3})^p = e^{imp\cdot 2\pi/3}=\rho_{mp} = e^{i(mp \bmod 3)2\pi/3}=\rho_{mp \bmod 3}$$

C_3 $\mathbf{g}^\dagger\mathbf{g}$-*product-table* *and basic group representation theory*

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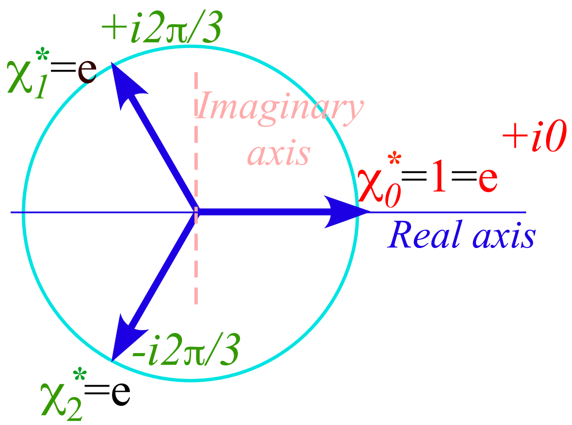

C_3-group $\{\mathbf{r}^0, \mathbf{r}^1, \mathbf{r}^2\}$-table

obeyed by $\{\chi_0 = 1,\ \chi_1 = e^{-i2\pi/3},\ \chi_2 = e^{+i2\pi/3}\}$

C_3	$\mathbf{r}^0 = \mathbf{1}$	$\mathbf{r}^1 = \mathbf{r}^{-2}$	$\mathbf{r}^2 = \mathbf{r}^{-1}$
$\mathbf{r}^0 = \mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{r}^2 = \mathbf{r}^{-1}$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{r}^1 = \mathbf{r}^{-2}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

C_3	$\chi_0 = 1$	$\chi_1 = \chi_2^{-2}$	$\chi_2 = \chi_1^{-1}$
$\chi_0 = 1 = \chi_3$	χ_0	χ_1	χ_2
$\chi_2 = \chi_1^{-1}$	χ_2	χ_0	χ_1
$\chi_1 = \chi_2^{-2}$	χ_1	χ_2	χ_0

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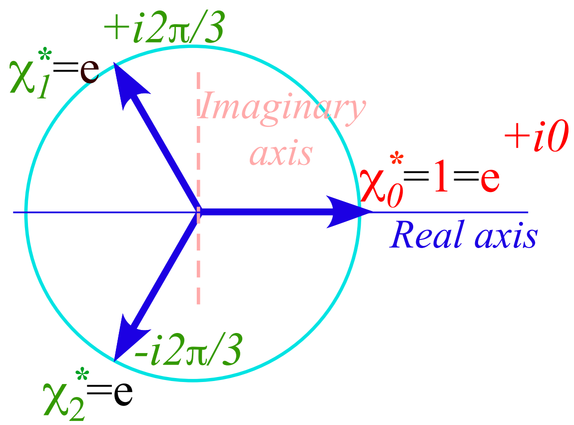

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obeyed by $\{\chi_0=1, \chi_1= e^{-i2\pi/3}, \chi_2= e^{+i2\pi/3}\}$

Set $\{\chi_0, \chi_1, \chi_2\}$ *is an irreducible representation (irrep)* of C_3

$\{D(\mathbf{r}^0)=\chi_0, D(\mathbf{r}^1)=\chi_1, D(\mathbf{r}^2)=\chi_2\}$

C_3	$\mathbf{r}^0=\mathbf{1}$	$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^2=\mathbf{r}^{-1}$
$\mathbf{r}^0=\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{r}^2=\mathbf{r}^{-1}$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

C_3	$\chi_0=1$	$\chi_1=\chi_2^{-2}$	$\chi_2=\chi_1^{-1}$
$\chi_0=1=\chi_3$	χ_0	χ_1	χ_2
$\chi_2=\chi_1^{-1}$	χ_2	χ_0	χ_1
$\chi_1=\chi_2^{-2}$	χ_1	χ_2	χ_0

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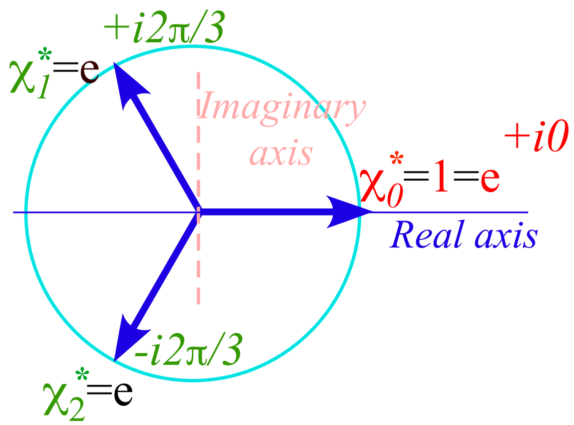

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C_3	$\mathbf{r}^0=\mathbf{1}$	$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^2=\mathbf{r}^{-1}$
$\mathbf{r}^0=\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{r}^2=\mathbf{r}^{-1}$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

C_3	$\chi_0=1$	$\chi_1=\chi_2^{-2}$	$\chi_2=\chi_1^{-1}$
$\chi_0=1=\chi_3$	χ_0	χ_1	χ_2
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$\chi_1=\chi_2^{-2}$	χ_1	χ_2	χ_0

Set $\{\chi_0, \chi_1, \chi_2\}$ *is an*
irreducible representation
(irrep) of C_3
$\{D(\mathbf{r}^0)=\chi_0, D(\mathbf{r}^1)=\chi_1, D(\mathbf{r}^2)=\chi_2\}$

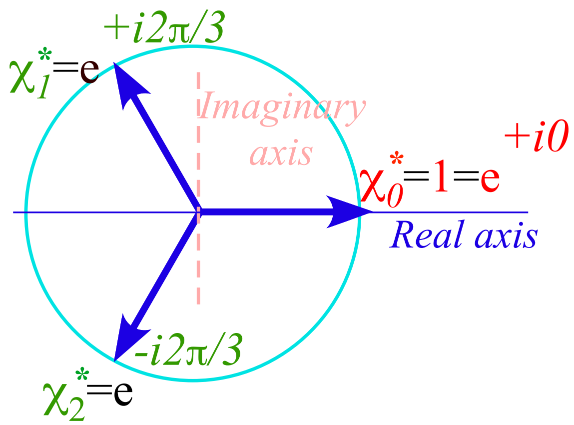
In fact, all ***three*** *irreps* $\{D^{(0)}, D^{(1)}, D^{(2)}\}$ listed in character table obey C_3-group table

$\mathbf{g}=$	$\mathbf{r}^0$	$\mathbf{r}^1$	$\mathbf{r}^2$
$D^{(0)}(\mathbf{g})$	$\chi_0^{(0)}$	$\chi_1^{(0)}$	$\chi_2^{(0)}$
$D^{(1)}(\mathbf{g})$	$\chi_0^{(1)}$	$\chi_1^{(1)}$	$\chi_2^{(1)}$
$D^{(2)}(\mathbf{g})$	$\chi_0^{(2)}$	$\chi_1^{(2)}$	$\chi_2^{(2)}$

=

$\mathbf{g}=$	$\mathbf{r}^0$	$\mathbf{r}^1$	$\mathbf{r}^2$
$D^{(0)}(\mathbf{g})$	1	1	1
$D^{(1)}(\mathbf{g})$	1	$e^{-\frac{2\pi i}{3}}$	$e^{+\frac{2\pi i}{3}}$
$D^{(2)}(\mathbf{g})$	1	$e^{+\frac{2\pi i}{3}}$	$e^{-\frac{2\pi i}{3}}$

C_3-group jargon and structure of various tables



C_3-group $\{\mathbf{r}^0, \mathbf{r}^1, \mathbf{r}^2\}$-table

obeyed by $\{\chi_0=1,\ \chi_1= e^{-i2\pi/3},\ \chi_2= e^{+i2\pi/3}\}$

C_3	$\mathbf{r}^0=1$	$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^2=\mathbf{r}^{-1}$
$\mathbf{r}^0=1$	1	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{r}^2=\mathbf{r}^{-1}$	$\mathbf{r}^2$	1	$\mathbf{r}^1$
$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^1$	$\mathbf{r}^2$	1

Set $\{\chi_0, \chi_1, \chi_2\}$ *is an irreducible representation (irrep)* of C_3

$\{D(\mathbf{r}^0)=\chi_0,\ D(\mathbf{r}^1)=\chi_1,\ D(\mathbf{r}^2)=\chi_2\}$

C_3	$\chi_0=1$	$\chi_1=\chi_2^{-2}$	$\chi_2=\chi_1^{-1}$
$\chi_0=1=\chi_3$	χ_0	χ_1	χ_2
$\chi_2=\chi_1^{-1}$	χ_2	χ_0	χ_1
$\chi_1=\chi_2^{-2}$	χ_1	χ_2	χ_0

In fact, all ***three*** *irreps* $\{D^{(0)}, D^{(1)}, D^{(2)}\}$ listed in character table obey C_3-group table

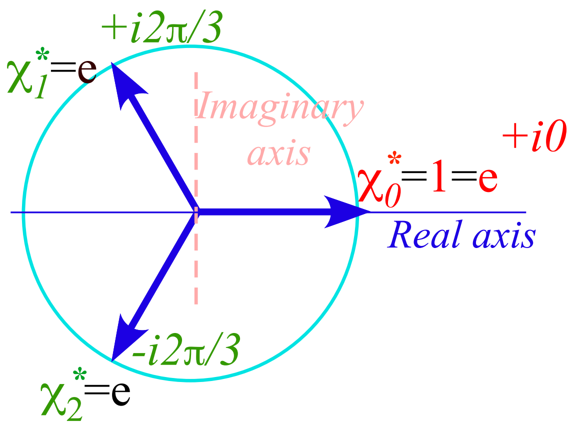
$\mathbf{g}=$	$\mathbf{r}^0$	$\mathbf{r}^1$	$\mathbf{r}^2$
$D^{(0)}(\mathbf{g})$	$\chi_0^{(0)}$	$\chi_1^{(0)}$	$\chi_2^{(0)}$
$D^{(1)}(\mathbf{g})$	$\chi_0^{(1)}$	$\chi_1^{(1)}$	$\chi_2^{(1)}$
$D^{(2)}(\mathbf{g})$	$\chi_0^{(2)}$	$\chi_1^{(2)}$	$\chi_2^{(2)}$

=

$\mathbf{g}=$	$\mathbf{r}^0$	$\mathbf{r}^1$	$\mathbf{r}^2$
$D^{(0)}(\mathbf{g})$	1	1	1
$D^{(1)}(\mathbf{g})$	1	$e^{-\frac{2\pi i}{3}}$	$e^{+\frac{2\pi i}{3}}$
$D^{(2)}(\mathbf{g})$	1	$e^{+\frac{2\pi i}{3}}$	$e^{-\frac{2\pi i}{3}}$

The *identity irrep* $D^{(0)}=\{1,1,1\}$ obeys *any* group table.

C_3-group jargon and structure of various tables



C_3-group $\{\mathbf{r}^0, \mathbf{r}^1, \mathbf{r}^2\}$-table obeyed by $\{\chi_0=1, \chi_1= e^{-i2\pi/3}, \chi_2= e^{+i2\pi/3}\}$

C_3	$\mathbf{r}^0=1$	$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^2=\mathbf{r}^{-1}$
$\mathbf{r}^0=1$	1	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{r}^2=\mathbf{r}^{-1}$	$\mathbf{r}^2$	1	$\mathbf{r}^1$
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C_3	$\chi_0=1$	$\chi_1=\chi_2^{-2}$	$\chi_2=\chi_1^{-1}$
$\chi_0=1=\chi_3$	χ_0	χ_1	χ_2
$\chi_2=\chi_1^{-1}$	χ_2	χ_0	χ_1
$\chi_1=\chi_2^{-2}$	χ_1	χ_2	χ_0

*In fact, all **three** irreps* $\{D^{(0)}, D^{(1)}, D^{(2)}\}$ listed in character table obey C_3-group table

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$D^{(2)}(\mathbf{g})$	$\chi_0^{(2)}$	$\chi_1^{(2)}$	$\chi_2^{(2)}$

=

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The *identity irrep* $D^{(0)}=\{1,1,1\}$ obeys *any* group table.

Irrep $D^{(2)}=\{1, e^{+i2\pi/3}, e^{-i2\pi/3}\}$ is a *conjugate irrep* to $D^{(1)}=\{1, e^{-i2\pi/3}, e^{+i2\pi/3}\}$

$$D^{(2)}= D^{(1)*}$$

C_3 $\mathbf{g}^\dagger\mathbf{g}$-*product-table* *and basic group representation theory*

C_3 **H**-and-$\mathbf{r}^p$-*matrix representations* *and* *conjugation symmetry*

C_3 Spectral resolution: 3^{rd} roots of unity and ***ortho-completeness*** *relations*

C_3 ***character table*** *and* ***modular labeling***

Ortho-completeness ***inversion*** *for operators and states*

Comparing wave function operator algebra to bra-ket algebra

Modular quantum number arithmetic

C_3-group jargon and structure of various tables

→ *C_3 Eigenvalues and wave dispersion functions*

Standing waves vs Moving waves

C_6 Spectral resolution: 6^{th} roots of unity and higher

Complete sets of coupling parameters and Fourier dispersion

Gauge shifts due to complex coupling

Eigenvalues and wave dispersion functions

$$\left\langle m \middle| \mathbf{H} \middle| m \right\rangle = \left\langle m \middle| r_0 \mathbf{r}^0 + r_1 \mathbf{r}^1 + r_2 \mathbf{r}^2 \middle| m \right\rangle = r_0 e^{i0(m)\frac{2\pi}{3}} + r_1 e^{i1(m)\frac{2\pi}{3}} + r_2 e^{i2(m)\frac{2\pi}{3}}$$

Eigenvalues and wave dispersion functions

$$\langle m|\mathbf{H}|m\rangle = \langle m|r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2|m\rangle = r_0 e^{i0(m)\frac{2\pi}{3}} + r_1 e^{i1(m)\frac{2\pi}{3}} + r_2 e^{i2(m)\frac{2\pi}{3}}$$

(*Here we assume* $r_1 = r_2 = r$)
(all-real)

$$= r_0 e^{i0(m)\frac{2\pi}{3}} + r\left(e^{i\frac{2m\pi}{3}} + e^{-i\frac{2m\pi}{3}}\right)$$

Eigenvalues and wave dispersion functions

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$$= r_0 e^{i0(m)\frac{2\pi}{3}} + r(e^{i\frac{2m\pi}{3}} + e^{-i\frac{2m\pi}{3}}) = r_0 + 2r\cos(\tfrac{2m\pi}{3}) = \begin{cases} r_0 + 2r \ (\text{for } m = 0) \\ r_0 - r \ (\text{for } m = \pm 1) \end{cases}$$

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Quantum **H**-values:

$$\begin{pmatrix} r_0 & r & r \\ r & r_0 & r \\ r & r & r_0 \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(r_0 + 2r\cos\left(\tfrac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

Eigenvalues and wave dispersion functions - Moving waves

$$\langle m|\mathbf{H}|m\rangle = \langle m|r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2|m\rangle = r_0 e^{i0(m)\frac{2\pi}{3}} + r_1 e^{i1(m)\frac{2\pi}{3}} + r_2 e^{i2(m)\frac{2\pi}{3}}$$

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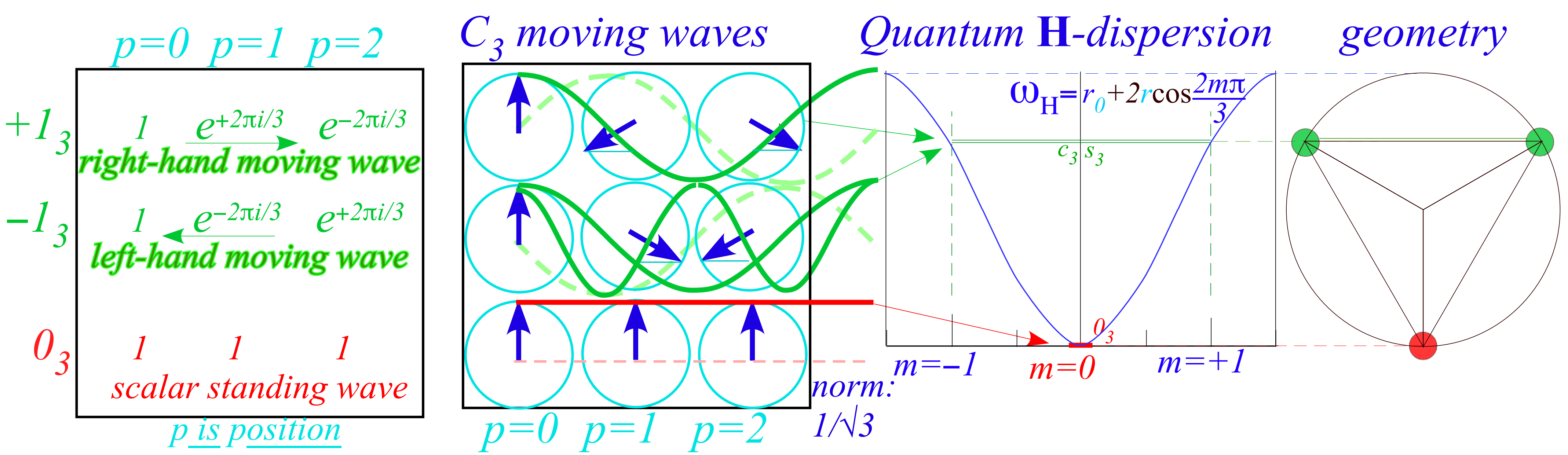

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Eigenvalues and wave dispersion functions - Moving waves

$$\langle m|\mathbf{H}|m\rangle = \langle m|r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2|m\rangle = r_0 e^{i0(m)\frac{2\pi}{3}} + r_1 e^{i1(m)\frac{2\pi}{3}} + r_2 e^{i2(m)\frac{2\pi}{3}}$$

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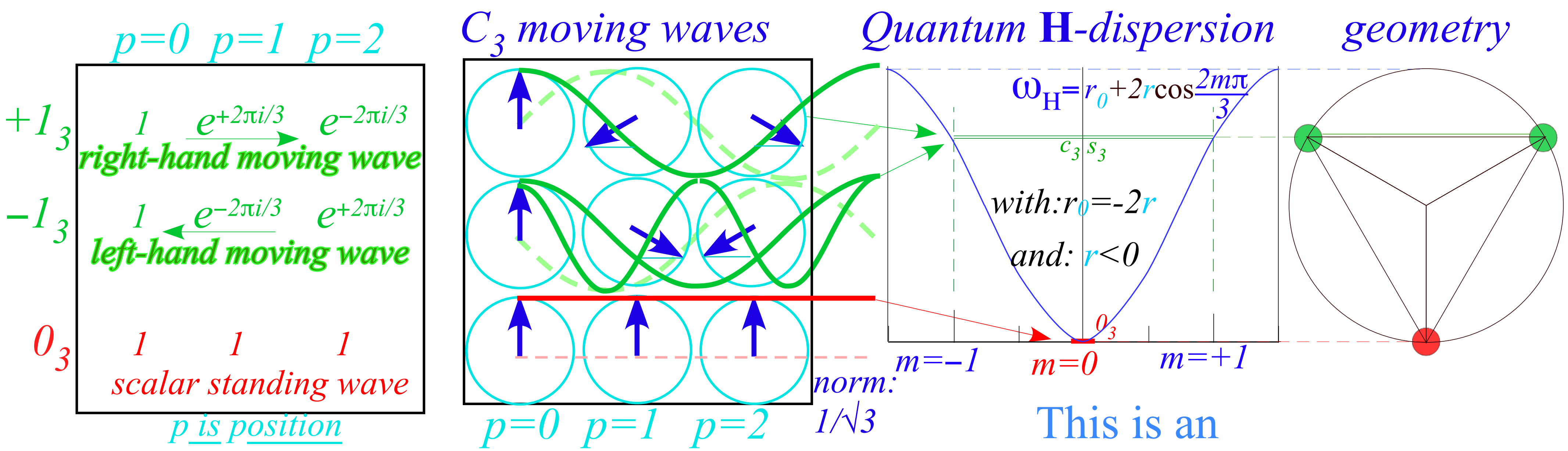

This is an

exciton-like

dispersion function

$$\omega_H(m) = r_0\left(1 - \cos\frac{2m\pi}{3}\right)$$

$$\omega_H(m) \sim 2r_0\left(\frac{m\pi}{3}\right)^2$$

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Eigenvalues and wave dispersion functions - Moving waves

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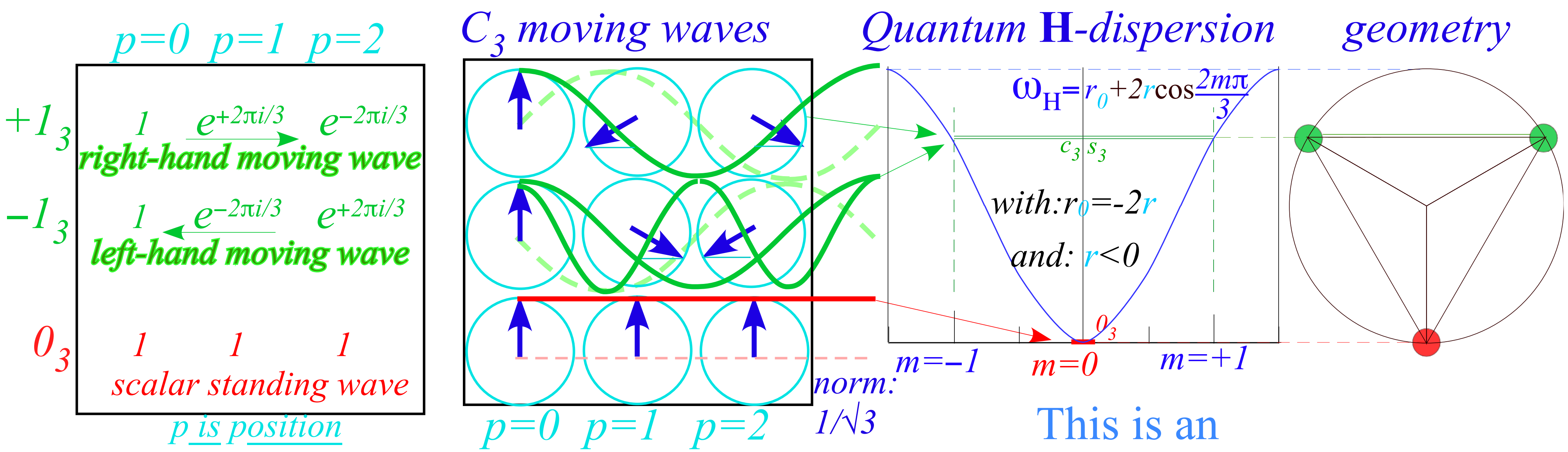
Classical **K**-values:

$$\begin{pmatrix} K & -k & -k \\ -k & K & -k \\ -k & -k & K \end{pmatrix} \begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(K - 2k\cos\left(\frac{2m\pi}{3}\right)\right) \begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$



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Eigenvalues and wave dispersion functions - Moving waves

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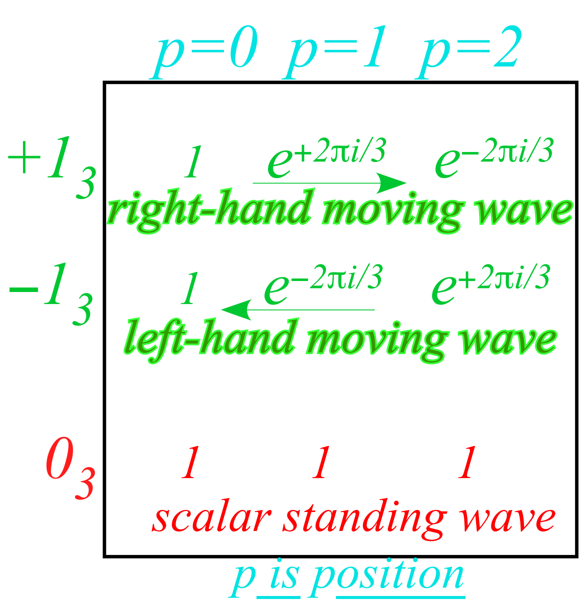
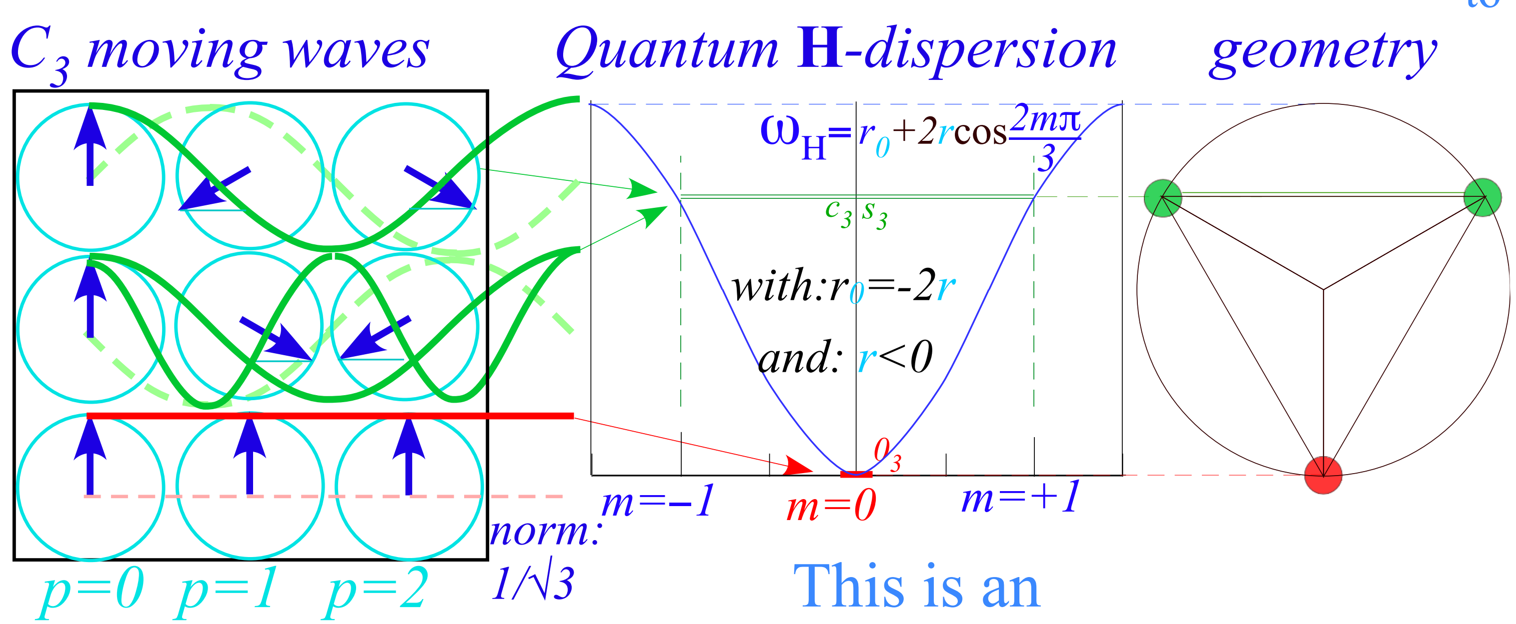
K-eigenvalue...

$$= \left(K - 2k\cos\left(\tfrac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

needs *Square-Root* to be a *frequency*





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Eigenvalues and wave dispersion functions - Moving waves

$$\langle m|\mathbf{H}|m\rangle = \langle m|r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2|m\rangle = r_0 e^{i0(m)\frac{2\pi}{3}} + r_1 e^{i1(m)\frac{2\pi}{3}} + r_2 e^{i2(m)\frac{2\pi}{3}}$$

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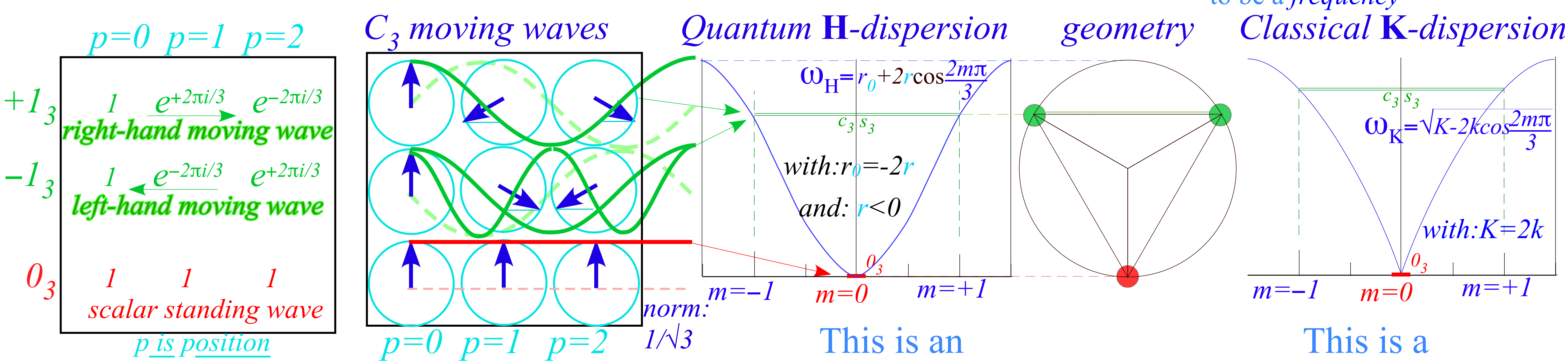
K-eigenvalue... needs *Square-Root* to be a *frequency*



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$$\omega_H(m) \sim 2r_0\left(\tfrac{m\pi}{3}\right)^2$$

$\omega_H(m)$ is *quadratic* for low m

(long wavelength λ)

This is a *phonon-like* dispersion function

$$\omega_K(m) = \sqrt{2k - 2k\cos\tfrac{2m\pi}{3}} = 2\sqrt{k}\,\sin\tfrac{m\pi}{3}$$

$$\omega_K(m) \sim 2\sqrt{k}\left(\tfrac{m\pi}{3}\right)^1$$

$\omega_K(m)$ is *linear* for low m

(long wavelength λ)

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C_3 **g**†**g**-*product-table and basic group representation theory*

C_3 **H**-and-**r**^{p}-*matrix representations and conjugation symmetry*

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➡ *Standing waves vs Moving waves*

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Standing waves possible if **H** *is all-real (No curly C-stuff allowed!)*

Eigenvalues and wave dispersion functions - Standing waves

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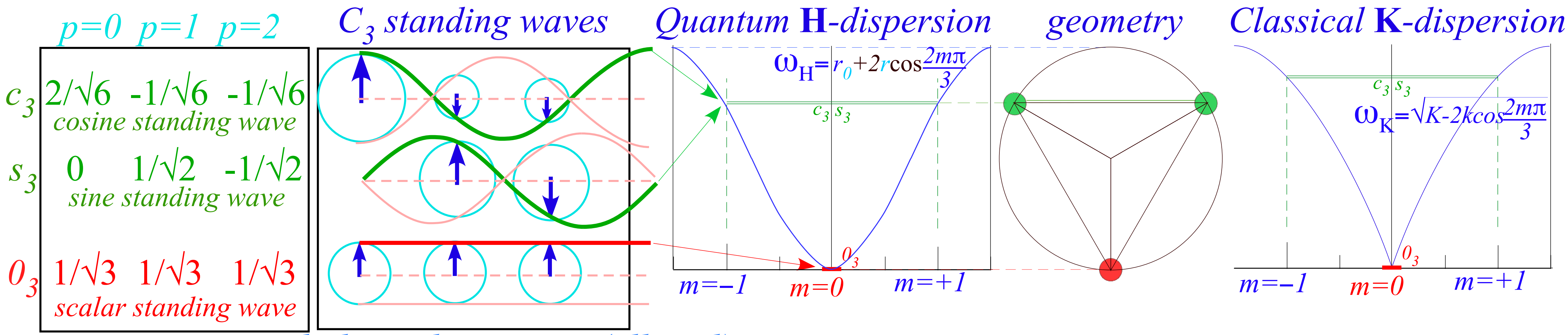
Standing waves possible if **H** *is all-real (No curly C-stuff allowed!)*

Moving eigenwave	*Standing eigenwaves*	**H** – *eigenfrequencies*	**K** – *eigenfrequencies*
$\|(+1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{+i2\pi/3} \\ e^{-i2\pi/3} \end{pmatrix}$	$\|c_3\rangle = \frac{\|(+1)_3\rangle + \|(-1)_3\rangle}{\sqrt{2}} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$	$\omega^{(+1)_3} = r_0 + 2r\cos\left(\frac{+2m\pi}{3}\right)$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos\left(\frac{+2m\pi}{3}\right)}$ $= \sqrt{k_0 + k}$
$\|(-1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{-i2\pi/3} \\ e^{+i2\pi/3} \end{pmatrix}$	$\|s_3\rangle = \frac{\|(+1)_3\rangle - \|(-1)_3\rangle}{i\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ +1 \\ -1 \end{pmatrix}$	$\omega^{(-1)_3} = r_0 + 2r\cos\left(\frac{-2m\pi}{3}\right)$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos\left(\frac{-2m\pi}{3}\right)}$ $= \sqrt{k_0 + k}$
	$\|(0)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$\omega^{(0)_3} = r_0 + 2r$	$\sqrt{k_0 - 2k}$

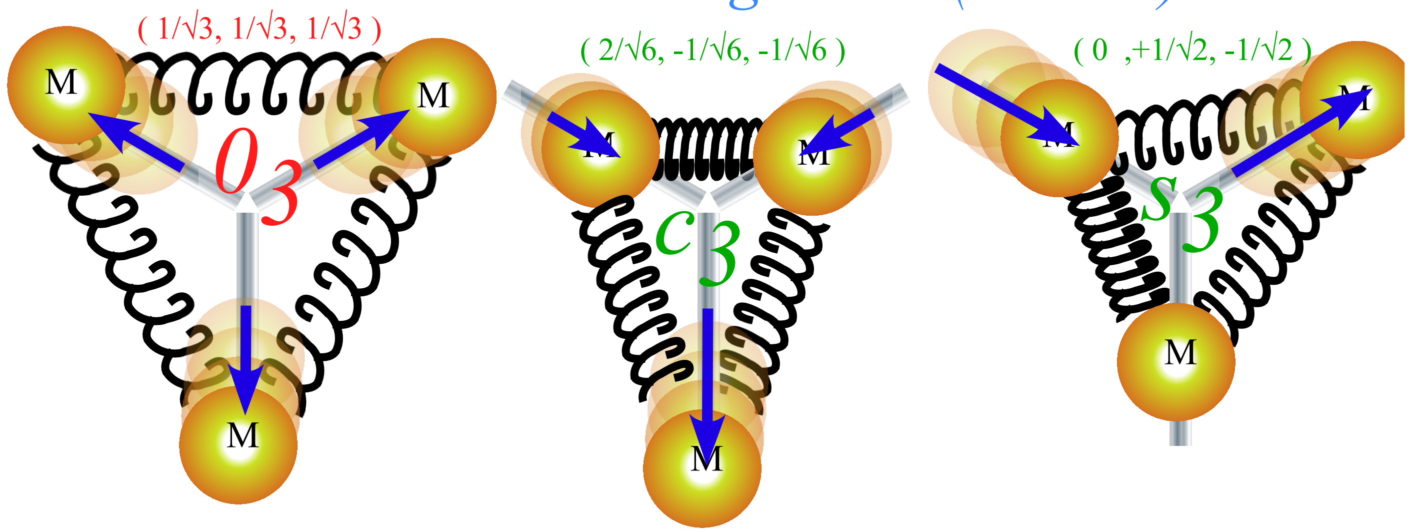
States $|(+)\rangle$ and $|(-)\rangle$ in *any mixtures* are still *stationary* due to $(\pm)$-*degeneracy* ($\cos(+x) = \cos(-x)$)

Eigenvalues and wave dispersion functions - Standing waves

(Possible if **H** *is all-real)*



Radial standing waves (all-real)



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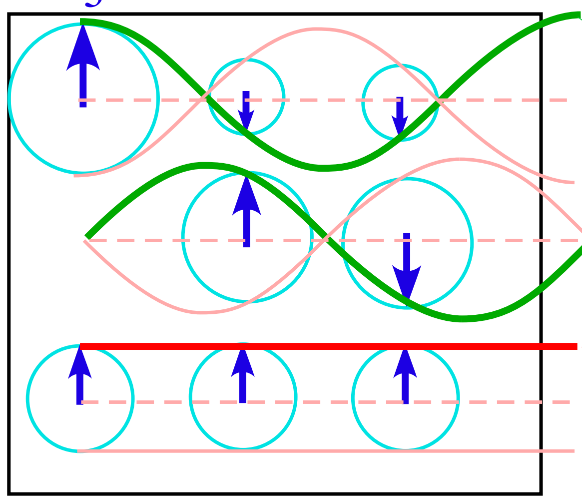
Eigenvalues and wave dispersion functions - Standing waves

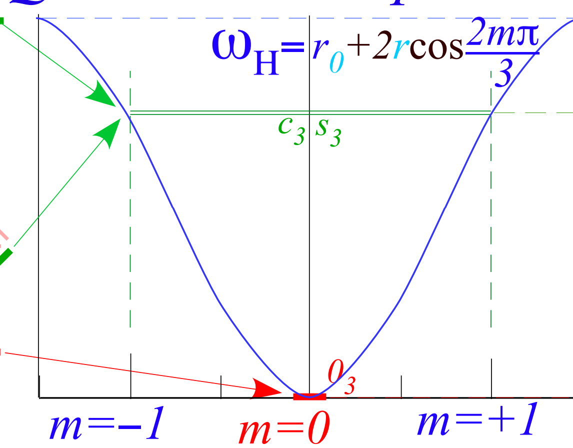
(Possible if **H** *is all-real)*

	$p=0$	$p=1$	$p=2$
c_3	$2/\sqrt{6}$	$-1/\sqrt{6}$	$-1/\sqrt{6}$
	cosine standing wave		
s_3	0	$1/\sqrt{2}$	$-1/\sqrt{2}$
	sine standing wave		
0_3	$1/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{3}$
	scalar standing wave		

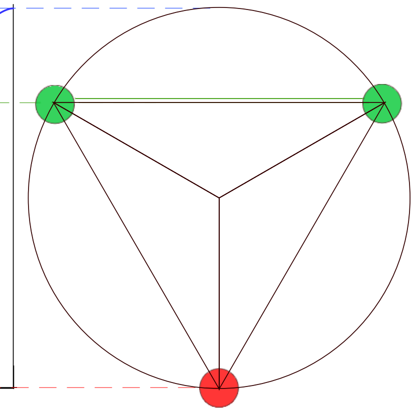
C_3 standing waves



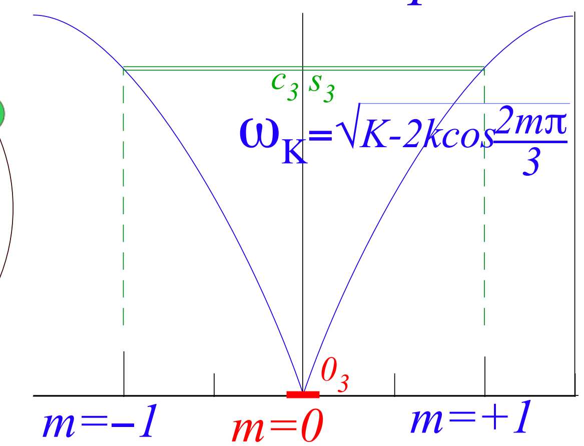
Quantum **H**-*dispersion*



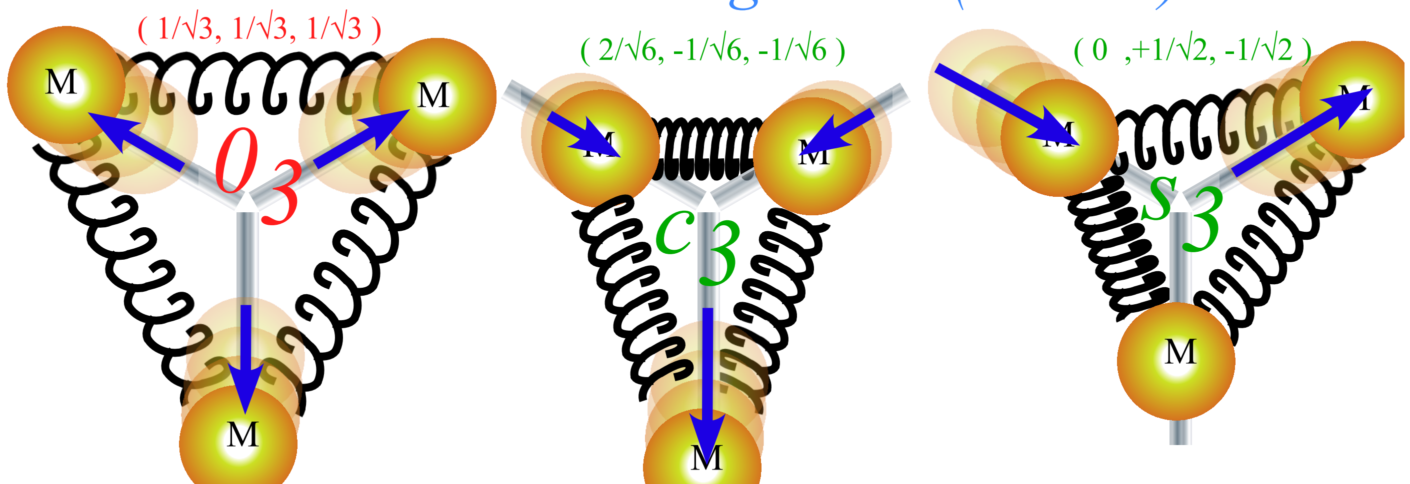
geometry



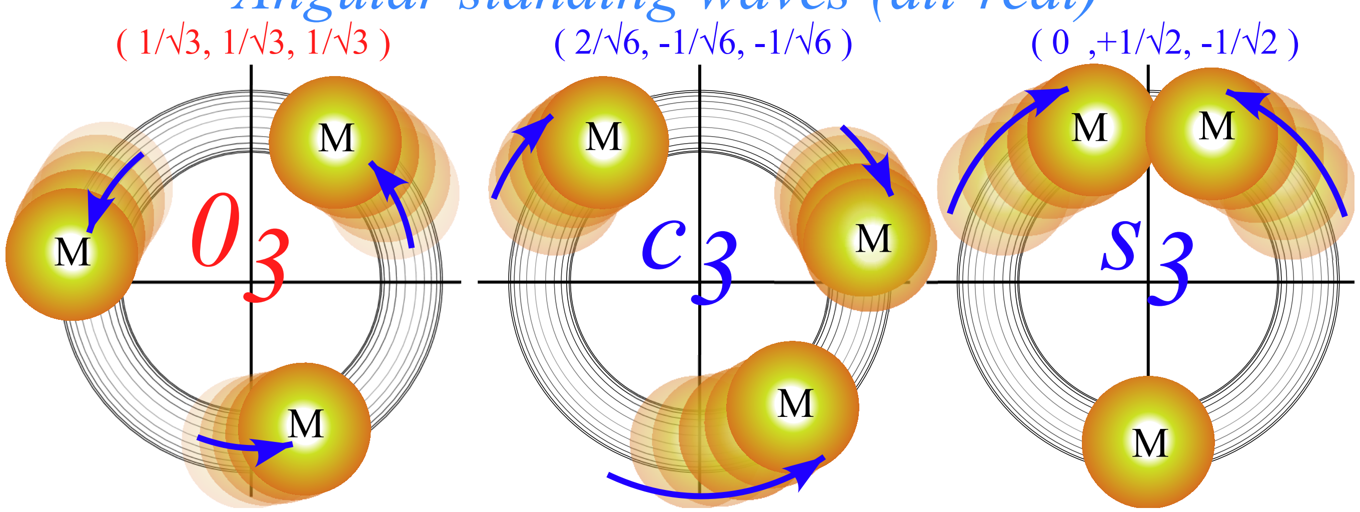
Classical **K**-*dispersion*



Radial standing waves (all-real)



Angular standing waves (all-real)



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