Assignment 2 - **Group Theory and Quantum Physics 5093** Due Th. 2/9/17

Based on: Lectures 3-5 and/or text *QTCA* Unit 1-Ch. 3

Circle-squish-switched

1.3.2 The discussion in Sec. 1.6 showed that a unit circle is mapped onto an ellipse $\langle\mathbf{r}|\mathbf{T}^{-2}|\mathbf{r}\rangle=1$ by matrix $\mathbf{T}=\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$. Consider the same mapping by "switched" matrix $\mathbf{S}=\begin{pmatrix} 1/2 & 1 \\ 1 & 1/2 \end{pmatrix}$.

(a) Find eigenvalues of $\mathbf{S}$ and $\mathbf{S}^{-2}$. Spectrally decompose $\mathbf{S}$ and plot its eigenvectors.

(b) Let $\mathbf{T}^{-1}|\mathbf{r}\rangle=|\mathbf{c}\rangle=\langle\mathbf{r}|\mathbf{T}^{-1}$ or $\mathbf{T}|\mathbf{c}\rangle=|\mathbf{r}\rangle=\langle\mathbf{c}|\mathbf{T}$ so $\langle\mathbf{r}|\mathbf{T}^{-1}|\mathbf{r}\rangle=\langle\mathbf{c}|\mathbf{r}\rangle=\langle\mathbf{c}|\mathbf{T}|\mathbf{c}\rangle$. Suppose all $\mathbf{c}$-vectors lie on a curve $\langle\mathbf{c}|\mathbf{T}|\mathbf{c}\rangle=1$ Discuss curve algebraically and plot this curve and the mapped $\langle\mathbf{r}|\mathbf{T}^{-1}|\mathbf{r}\rangle=1$ curve.

(c) Let $\mathbf{S}^{-1}|\mathbf{r}\rangle=|\mathbf{c}\rangle=\langle\mathbf{r}|\mathbf{S}^{-1}$ or $\mathbf{S}|\mathbf{c}\rangle=|\mathbf{r}\rangle=\langle\mathbf{c}|\mathbf{S}$ so $\langle\mathbf{r}|\mathbf{S}^{-1}|\mathbf{r}\rangle=\langle\mathbf{c}|\mathbf{r}\rangle=\langle\mathbf{c}|\mathbf{S}|\mathbf{c}\rangle$. Suppose all $\mathbf{c}$-vectors lie on a curve $\langle\mathbf{c}|\mathbf{S}|\mathbf{c}\rangle=1$ Discuss curve algebraically and plot this curve and the mapped $\langle\mathbf{r}|\mathbf{S}^{-1}|\mathbf{r}\rangle=1$ curve.

(d) *By conic geometry, derive a map $\mathbf{M}|\mathbf{c}\rangle=|\mathbf{r}\rangle$ of any real vector $|\mathbf{c}\rangle$ by real-symmetric matrix $\mathbf{M}$.

Dagger Your Own Ket

1.3.3 Most quantum matrices have simple relations between eigenvalues ε_m and their conjugates ε_m^*, eigenbras $|\varepsilon_m\rangle$ and kets $\langle\varepsilon_m|$, projectors $\mathbf{P}_m$ and their †-conjugates $(\mathbf{P}_m)^\dagger$, and diagonalizing (*d-tran*) transformations T and their inverses T^{-1}. Let's see what these relations are for...

(a) ...a Hermitian matrix $\mathbf{M} = \mathbf{H}$ such that $\mathbf{H} = \mathbf{H}^\dagger$ by spectrally decomposing and diagonalizing a *2x2* reflection matrix

$$\mathbf{H} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}$$. (Are its eigenvectors meaningful? Discuss.)

(b) ...a Unitary matrix $\mathbf{M} = \mathbf{U}$ such that $\mathbf{U}^{-1}= \mathbf{U}^\dagger$ by spectrally decomposing and diagonalizing a general *2x2* rotation matrix

$$\mathbf{U} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$. (Are its eigenvectors meaningful? Discuss.)

(c) Find <u>*all*</u> the square-roots of $\mathbf{H}$ and of $\mathbf{U}$. (Test them. There are more than two of each!)

(d) A NORMAL MATRIX N obeys $\mathbf{N}^\dagger\mathbf{N}=\mathbf{N}\mathbf{N}^\dagger$. Is a Normal matrix always diagonalizable? Prove or disprove.

Home on Lagrange

1.3.4 Functional spectral decomposition (3.1.17) is related to Lagrange functional interpolation (3.1.18). Use (3.1.18) to approximate *sin x* given only that $\sin 0=0$, $\sin\pi/2=1$, and $\sin\pi=0$. Compare your approximation to order-2 Taylor series approximation of *sin x* around $x=\pi/2$.

Bras-ackwards

1.3.5 See if you can work the spectral decomposition ideas backwards by doing the following "inverse" eigenvalue problems. (Hint: Use ket-bras and ⊗. Normalize first!)

(a) Find a Hermitian 3x3 matrix $\mathbf{H}$ that satisfies.

$$\mathbf{H}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{H}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \quad \mathbf{H}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 9\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

Write down <u>and test</u> at least one square root $\sqrt{\mathbf{H}}$. (How many square roots are there?)

Cures for Nilpotency

1.3.6 Can a *nilpotent* matrix **N** ($\mathbf{N}^m = \mathbf{0}$, $\mathbf{N}^{m-1}$ not zero, integer $m>1$) be Hermitian $\mathbf{N}=\mathbf{N}^\dagger$ ever??
(a) ...for $m=2$? , (b) ...for other m? (Experiment with *2x2* matrices first.)
Use this and exercise *Dagger Your Own Ket* to prove Hermitian matrices must be diagonalizable.

Truly Secular

1.3.7 The coefficients a_k of the general *nxn* secular equation (3.1.5d) and (3.1.5f) of **M** depend on matrix coefficients M_{ij} and on eigenvalues ε_m.
(a) Do they depend on which basis you use to represent **M**? Why or why not?
(b) For a general *4x4* matrix *(n=4)*, compute functions $a_k = a_k(\varepsilon_m)$ in an orderly way that clearly shows how they come out for general n.
(c) For a general *4x4* matrix *(n=4)*, compute functions $a_k = a_k(M_{ij})$ in an orderly way that clearly shows how they come out for general n. Use the ε-expansion in Appendix 1.A and (b) above to help express answer in terms of diagonal minor determinants. (NOTE: This is a "crucial" problem whose solutions belongs in your lab "journal" or equivalent.) May do successively *n= 2, 3,* until a pattern emerges.

Adjunct Junk

1.3.8 Given (1.A.5) or $\mathbf{A}\cdot\mathbf{A}^{ADJ}=\mathbf{1}\,(det|\mathbf{A}|)$ with $\mathbf{A} = \mathbf{M} - \lambda\mathbf{1}$ show that $\mathbf{A}^{ADJ}$ has **M** eigenkets $|\lambda\rangle$ if λ is an eigenvalue of **M**.
Does $\mathbf{A}^{ADJ}$ also harbor **M**'s eigenbras? Use $\mathbf{M}=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$ as one example and $\mathbf{Q}=\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$ as another. (Get **P**-ops of **Q**.)

Pair'em up

1.3.9 An *nxn pairing matrix* Π has 1 for all n^2 matrix elements $\Pi_{ij} = 1$. It has been used in superconductivity theory and theory of nuclear shell structure.
(a) Use 1(c) above to help derive its eigenvalues and spectral decomposition. (Or, you may develop the theory by doing successively *n= 2, 3,* until the pattern emerges.)
(b) Does the matrix $\Pi + (const.)\mathbf{1}$ have the same eigenvectors? eigenvalues? as Π. How about $(const.)\cdot\Pi$? Explain.

All Together Now

1.3.10 Show how to do a <u>simultaneous</u> spectral decomposition using the projector splitting technique.. (a) Spectrally

decompose $\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, *and* $\mathbf{B} = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$.

(b) Calculate the "ridiculous function" $\mathbf{B}^{\mathbf{A}}$ of these two matrices.

Almost Nilpotent

1.3.11 Matrix $\mathbf{N}_{almost}$ is nilpotent for ε=0. Find spectral decomposition and eigen-bra-kets. What happens as ε→0?

$$\mathbf{N}_{almost}(\varepsilon) = \begin{pmatrix} 1 & 1 \\ \varepsilon & 1 \end{pmatrix}$$

Which bras or kets survive ε=0? Plot for ε~0. What ortho-normalization (if any) is left?