Assignment 2 - Group Theory and Quantum Physics 5093  Due Th. 2/9/17
Based on: Lectures 3-5 and/or text QTCA Unit 1-Ch. 3

Circle-squish-switched

1.3.2 The discussion in Sec. 1.6 showed that a unit circle is mapped onto an ellipse \(|r|T^{-2}|r|=1\) by matrix

\[
T = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}.
\]

(a) Find eigenvalues of \(S\) and \(S^{-2}\). Spectrally decompose \(S\) and plot its eigenvectors.

(b) Let \(T^{-1}|r|=|c\) or \(|c|=T|r|\). Suppose all \(c\)-vectors lie on a curve \(|c|T|c|=1\) and plot this curve and the mapped \(|r|T^{-1}|r|=1\) curve.

(c) Let \(S^{-1}|r|=|c\) or \(|c|=S|r|\). Suppose all \(c\)-vectors lie on a curve \(|c|S|c|=1\) and plot this curve and the mapped \(|r|S^{-1}|r|=1\) curve.

(d) By conic geometry, derive a map \(M|c|=|r\) of any real vector \(|c\) by real-symmetric matrix \(M\).

Dagger Your Own Ket

1.3.3 Most quantum matrices have simple relations between eigenvalues \(\varepsilon_m\) and their conjugates \(\varepsilon_m^*\), eigenbras \(|\varepsilon_m\rangle\) and kets \(\langle \varepsilon_m|\), projectors \(P_m\) and their conjugates \((P_m)^\dagger\), and diagonalizing \((d\text{-tran})\) transformations \(T\) and their inverses \(T^{-1}\).

(a) ...a Hermitian matrix \(M = H\) such that \(H = H^\dagger\) by spectrally decomposing and diagonalizing a 2x2 reflection matrix

\[
H = \begin{pmatrix} \cos \varphi & \sin \varphi \\ \sin \varphi & -\cos \varphi \end{pmatrix}.
\]

(b) ...a Unitary matrix \(M = U\) such that \(U^{-1} = U^\dagger\) by spectrally decomposing and diagonalizing a general 2x2 rotation matrix

\[
U = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}.
\]

(c) Find all the square-roots of \(H\) and of \(U\). (Test them. There are more than two of each!)

(d) A NORMAL MATRIX \(N\) obeys \(N^\dagger N = NN^\dagger\). Is a Normal matrix always diagonalizable? Prove or disprove.

Home on Lagrange

1.3.4 Functional spectral decomposition (3.1.17) is related to Lagrange functional interpolation (3.1.18). Use (3.1.18) to approximate \(\sin x\) given only that \(\sin 0=0\), \(\sin \pi/2=1\), and \(\sin \pi=0\). Compare your approximation to order-2 Taylor series approximation of \(\sin x\) around \(x=\pi/2\).

Bras-ackwards

1.3.5 See if you can work the spectral decomposition ideas backwards by doing the following "inverse" eigenvalue problems. (Hint: Use ket-bras and \(\otimes\). Normalize first!)

(a) Find a Hermitian 3x3 matrix \(H\) that satisfies

\[
H \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad H \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \quad H \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.
\]

Write down and test at least one square root \(\sqrt{H}\). (How many square roots are there?)
Cures for Nilpotency

1.3.6 Can a nilpotent matrix \( N (N^m = 0, \ N^{m-1} \) not zero, integer \( m>1 \)) be Hermitian \( N=N^\dagger \) ever??
(a) ...for \( m=2 \), (b) ...for other \( m \)? (Experiment with 2x2 matrices first.)
Use this and exercise Dagger Your Own Ket to prove Hermitian matrices must be diagonalizable.

Truly Secular

1.3.7 The coefficients \( a_k \) of the general \( nxn \) secular equation (3.1.5d) and (3.1.5f) of \( M \) depend on matrix coefficients \( M_{ij} \) and on eigenvalues \( \epsilon_m \).
(a) Do they depend on which basis you use to represent \( M \)? Why or why not?
(b) For a general \( 4x4 \) matrix (\( n=4 \)), compute functions \( a_k = a_k(\epsilon_m) \) in an orderly way that clearly shows how they come out for general \( n \).
(c) For a general \( 4x4 \) matrix (\( n=4 \)), compute functions \( a_k = a_k(M_{ij}) \) in an orderly way that clearly shows how they come out for general \( n \). Use the \( \epsilon \)-expansion in Appendix 1.A and (b) above to help express answer in terms of diagonal minor determinants. (NOTE: This is a "crucial" problem whose solutions belongs in your lab "journal" or equivalent.) May do successively \( n=2,3 \), until a pattern emerges.

Adjunct Junk

1.3.8 Given (1.A.5) or \( A\cdot A^{ADJ}=1 \) (det|\( A \)) with \( A = M - \lambda I \) show that \( A^{ADJ} \) has \( M \) eigenkets \( |\lambda> \) if \( \lambda \) is an eigenvalue of \( M \).
Does \( A^{ADJ} \) also harbor \( M \)'s eigenbras? Use \( M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \) as one example and \( Q=\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} \) as another. (Get \( P-ops \) of \( Q \).)

Pair' em up

1.3.9 An \( nxn \) pairing matrix \( \Pi \) has 1 for all \( n^2 \) matrix elements \( \Pi_{ij}=1 \). It has been used in superconductivity theory and theory of nuclear shell structure.
(a) Use 1(c) above to help derive its eigenvalues and spectral decomposition. (Or, you may develop the theory by doing successively \( n=2,3 \), until the pattern emerges.)
(b) Does the matrix \( \Pi+(\text{const.})I \) have the same eigenvectors? eigenvalues? as \( \Pi \). How about \( (\text{const.})\Pi \)? Explain.

All Together Now

1.3.10 Show how to do a simultaneous spectral decomposition using the projector splitting technique.. (a) Spectrally decompose \( A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \), and \( B = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} \).
(b) Calculate the "ridiculous function" \( BA \) of these two matrices.

Almost Nilpotent

1.3.11 Matrix \( N \) is nilpotent for \( \epsilon=0 \). Find spectral decomposition and eigen-bra-kets. What happens as \( \epsilon \to 0 \)?
\( N_{\text{near}}(\epsilon) = \begin{pmatrix} 1 & 1 \\ \epsilon & 1 \end{pmatrix} \)
Which bras or kets survive \( \epsilon=0 \)? Plot for \( \epsilon \to 0 \). What ortho-normalization (if any) is left?