Euler Canonize

10.A.1 An 2D-oscillator canonical phase state-{$(x_i, p_i, x_j, p_j)$} and a spin-state-{$|\alpha, \beta, \gamma\rangle$} are both defined by the Euler angles {$\{\alpha, \beta, \gamma\}$} through (10.A.1-a) as well as by axis angles {$[\varphi, \theta, \Theta]$} through (10.A.1c). (First, verify all parts of (10.A.1).) If rotation-axis-{$\Theta$} polar angles {$[\varphi, \theta]$} are fixed while rotation angle {$\Theta = Q \Gamma$} varies uniformly with time, Euler angles {$\{\alpha, \beta, \gamma\}$} and phase point-{$(x_i, p_i, x_j, p_j)$} trace spin and oscillator trajectories, respectively. Verify this for the following cases by discussing plots requested below.

(a) {$[\varphi = 0, \theta = 0]$}, (b) {$[\varphi = 0, \theta = \pi/2]$}, (c) {$[\varphi = \pi/2, \theta = \pi/2]$} (option extra credit: (d) {$[\varphi = 0, \theta = \pi/4]$}, (e) {$[\varphi = \pi/2, \theta = \pi/4]$}).

For each case sketch 2D-paths-{$p_1$} vs. {$x_1$} and $x_2$ vs. $x_1$ and sketch $\hat{\Theta} \sin \Theta$ in a 3D ({$p_2$, $x_2$, $p_1$})-space which should also have paths for $-p_2$ vs. $x_2$ and $x_2$ vs. $p_1$ etc. Also, indicate the paths followed by the tip of the $S$-spin-vector (10.5.8c) in 3D-space{$(S_x, S_y, S_z)$} and characterize as A-type, B-type, or C-type motion, etc., in each case.

Invariantipodals (Easy)

10.A.2 When an Euler sphere is rotated from origin-{$|1\rangle$} state-{$\theta = \phi = \beta = \gamma$} to some angles-{$\{\alpha, \beta, \gamma\}$}, there are always points on the sphere which end up exactly where they were before the rotation. Verify this and express the polar-coordinates-{$(\bar{\varphi}, \bar{\theta})$} of all such invariant points in terms of {$\{\alpha, \beta, \gamma\}$}.

Spinor-Vector-Rotor (Deriving 3x3 matrix $e_L \cdot e_L = (\mathbf{R}([\varphi, \theta, \Theta])]_{3x3}$ is a “rite-of-passage” for group theorists.)

10.A.3 Prove and develop the result (10.A.15) or GrpThLect.8 p.47 as described below.

\[
\mathbf{R}([\Phi]) |L\rangle \mathbf{R}([\Phi])^\dagger = \left( \begin{array}{cc} \cos \frac{\Theta}{2} & -i \sin \frac{\Theta}{2} \\ -i \sin \frac{\Theta}{2} & \cos \frac{\Theta}{2} \end{array} \right) |L\rangle \left( \begin{array}{cc} \cos \frac{\Theta}{2} & -i \sin \frac{\Theta}{2} \\ -i \sin \frac{\Theta}{2} & \cos \frac{\Theta}{2} \end{array} \right)^\dagger \\
= |L\rangle^\prime = |L\rangle \cos \Theta - \epsilon_{LMN} \hat{\Theta}_M \sigma_M \sin \Theta + (1 - \cos \Theta) \hat{\Theta}_L (\hat{\Theta}_N \sigma_N)
\]

(a) Using the $\sigma$-product definitions (p. 34-38 of GrpThLect.8) and the Levi-Civita tensor identity $e_{abc}e_{dec} = \delta_{ad}\delta_{be} - \delta_{ae}\delta_{bd}$ (Prove this, too!)

to derive the above result. (In QTCA Equation (10.A.15a) yields Eq. (10.A.15b).)
(b) Above result (Eq. (10.A.15)) applies when $\sigma_2$ are replaced by unit vectors $e_L$. Sketch resulting vectors $\Theta$ and $e_L$ (before rotation) and $e_L^\prime$ (after rotation) for a rotation of $e_2$ by $\Theta = 120^\circ$ around an axis with polar angle $\bar{\varphi} = 54.7^\circ = \arccos (1/\sqrt{3})$ and azimuthal angle $\bar{\theta} = 45^\circ$. (As is conventional, we measure polar angles off the $Z$(or $A$) axis and azimuthal angles from the $X$(or $B$) axis counter clockwise in the $XY$ (or $BC$) plane. Give axis Cartesian coordinates.) Use the above to write down a general 3-3 matrix in terms of axis angles {$[\varphi, \theta, \Theta]$}, and test it using angles in (b).
(c) Derive numerical Euler angles-{$\{\alpha, \beta, \gamma\}$} in degrees for this rotation matrix.
(d) Compare formulas and numerics of 3-3 $R(3)$ matrix with 2-2 $U(2)$ matrix for the same rotations.
(e) Find 3-3 $R(3)$ and 2-2 $U(2)$ matrices for rotation $\mathbf{R}$, by $90^\circ$ around $Y$ (or $C$)-axis.
(f) Do products $\mathbf{R}, \mathbf{R}([\varphi, \theta, \Theta])]$ and $\mathbf{R}([\varphi, \theta, \Theta])\mathbf{R}$, numerically.

You may check products with $U(2)$ product formulaa (10.A.10) or Lect.8 p.42. Compare results to their Hamilton turns.

Spin erection. Does it phase $U(2)$? (Extra credit)

10.B.2. The following general problem may certainly become relevant if the mythical quantum computer materializes. It involves erecting an arbitrary state with spin vector $\mathbf{S}$ to the spin-up $Z$ (or $A$) position with a particular overall phase $\Phi$. In each case make the description of your solution as simple as possible as though you needed to explain it to engineers.

(a) For a state of 0-phase with spin on the $X$ (or $B$), describe a single operator that does the above.
(b) For a state of 0-phase with spin at $\beta$ in the $XZ$ (or $AB$) plane, describe a single operator that does the above.
(c) Of all possible rotations from $\beta$ in the $XZ$ plane to spin-up $Z$, which takes the least energy (that is, least total angle $\Theta$ of rotation) regardless of final phase $\Phi$?
Assignment 4 Solutions  Euler Can Canonize

Solution to 10.A.1 Assuming-\(\Theta\) polar angles \([\varphi, \theta]\) fixed as \(\Theta=\Omega\) varies.

Euler angles \((\alpha=\varphi-\pi/2+\tan^{-1}(\tan(\Omega t/2) \cos\theta), \quad \beta=2\sin^{-1}(\sin(\Omega t/2) \sin\theta), \quad \gamma=\pi/2-\varphi+\tan^{-1}(\tan(\Omega t/2) \cos\theta)\)

Phase point \((x_1=\cos\Omega t/2, \quad p_1=-\hat{\Theta}_Z \sin\Omega t/2=-\sin\Omega t/2)\)

\((x_2=\hat{\Theta}_Y \sin\Omega t/2=\sin\omega\sin\Omega t/2, \quad p_2=-\hat{\Theta}_X \sin\Omega t/2=\cos\omega\sin\Omega t/2)\)

(a) \([\varphi=0, \theta=0]\)

\[(x_1=-\pi/2+\Omega t/2, \quad \beta=0, \quad \gamma=\pi/2+\Omega t/2)\]

\((x_2=0, \quad p_2=0)\)

(b) \([\varphi=0, \theta=\pi/2]\)

\[(x_1=\alpha=-\pi/2=\varphi=\Omega t/2, \quad \beta=2\Omega t/2=\Omega t, \quad \gamma=\pi/2-\varphi=\pi/2)\]

\((x_2=\hat{\Theta}_Y \sin\Omega t/2=0, \quad p_2=-\hat{\Theta}_X \sin\Omega t/2=\sin\Omega t/2)\)

(c) \([\varphi=\pi/2, \theta=\pi/2]\)

\[(x_1=\alpha=\Omega t/2, \quad \beta=2\Omega t/2=\Omega t, \quad \gamma=0)\]

\((x_2=\hat{\Theta}_Y \sin\Omega t/2=\sin\Omega t/2, \quad p_2=-\hat{\Theta}_X \sin\Omega t/2=0)\)

(d) \([\varphi=0, \theta=\pi/4]\)

\[(x_1=\alpha=-\pi/2=\varphi=\pi/2+\tan^{-1}(\tan(\Omega t/2) \sqrt{2}), \quad \beta=2\sin^{-1}(\sin(\Omega t/2) \sqrt{2}), \quad \gamma=\pi/2+\tan^{-1}(\tan(\Omega t/2) \sqrt{2})\]

\((x_2=\hat{\Theta}_Y \sin\Omega t/2=\sqrt{2} \sin\Omega t/2, \quad p_2=-\hat{\Theta}_X \sin\Omega t/2=\sqrt{2} \sin\Omega t/2)\)

\((x_3=\hat{\Theta}_Z \sin\Omega t/2=\sqrt{2} \sin\Omega t/2, \quad p_3=-\hat{\Theta}_X \sin\Omega t/2=0)\)

(e) \([\varphi=\pi/2, \theta=\pi/4]\)

\[(x_1=\alpha=\pi/2=\varphi=\pi/2+\tan^{-1}(\tan(\Omega t/2) \sqrt{2}), \quad \beta=2\sin^{-1}(\sin(\Omega t/2) \sqrt{2}), \quad \gamma=\pi/2+\tan^{-1}(\tan(\Omega t/2) \sqrt{2})\]

\((x_2=\hat{\Theta}_Y \sin\Omega t/2=\sqrt{2} \sin\Omega t/2, \quad p_2=-\hat{\Theta}_X \sin\Omega t/2=0)\)

For each case sketch 2D-paths \(-p_2 \text{ vs. } x_1 \text{ and } x_2 \text{ vs. } x_1\) and sketch \(\hat{\Theta}_Y \sin\Omega t/2\) in a 3D \((p_2, x_2, p_1)\)-space which should also have paths for \(-p_2 \text{ vs. } x_2 \text{ and } x_2 \text{ vs. } p_1\) etc. Also, indicate the paths followed by the tip of the S-spin vector (10.5.8c) in 3D-spin space \((S_x, S_y, S_z)\) and characterize as \(A\)-type, \(B\)-type, or \(C\)-type motion, etc., in each case.

Invariantipodals

Solution to 10.A.2 If Euler sphere is rotated from origin \(|1\rangle\) state \(\{0=\alpha=\beta=\gamma\}\) to some angles \((\alpha, \beta, \gamma)\), there are always points on the sphere that exactly where they were before rotation \(R(\alpha, \beta, \gamma)\). Express the polar-coordinates \((\phi, \theta)\) of such invariant points in terms of \((\alpha, \beta, \gamma)\). \(\pm\)Eigenvectors of \(R(\alpha, \beta, \gamma)\) or \(\pm\)Crank vector \(\Theta\) are points that don’t move and have Darboux angle polar-coordinates \((\phi, \theta) = \{ (\alpha - \gamma + \pi) / 2, \quad \tan^{-1}(\tan(\beta/2) / \sin(\alpha + \gamma)/2) \}\) after a turn by crank-angle: \(\Theta = 2 \cos^{-1}[\cos(\beta/2) \cos(\alpha + \gamma)/2] \) The antipodal axis \(\{\phi, \theta, \pm \pi\}\) has fixed points, too.

Spinor-Rotor

Solution to 10.A.3 Prove and develop the result (10.A.15) as described below.

\[ R[L]\sigma L R[\hat{L}]^+ = \left( \cos \frac{\Theta}{2} - i \sin \frac{\Theta}{2} \hat{\Theta}_K \sigma_K \right) \sigma_L \left( \cos \frac{\Theta}{2} - i \sin \frac{\Theta}{2} \hat{\Theta}_N \sigma_N \right)^+ \]

\[ = \sigma_L^* = \sigma_L \cos \Theta - \varepsilon_{LMK} \hat{\Theta}_K \sigma_M \sin \Theta + (1 - \cos \Theta) \hat{\Theta}_L \left( \hat{\Theta}_N \sigma_N \right) \]

(a) Using the \(\sigma\)-product definitions and the Levi-Civita tensor identity \(\varepsilon_{abc} \varepsilon_{dec} = \delta_{ad} \delta_{be} - \delta_{ae} \delta_{bd}\)

\[ e'_L = e_L \cos \Theta - \varepsilon_{LMK} \hat{\Theta}_K e_M \sin \Theta + (1 - \cos \Theta) \hat{\Theta}_L \left( \hat{\Theta}_N \varepsilon_N \right) = e_L \cos \Theta + \hat{\Theta} \times e_L \sin \Theta + (1 - \cos \Theta) \hat{\Theta} \left( \hat{\Theta} \bullet e_L \right) \]
(b) Rotation of \( e_z \) by \( \Theta = 120^\circ \) around an axis with polar angle \( \vartheta = 54.7^\circ = \arccos(1/\sqrt{3}) \) and azimuthal angle \( \varphi = 45^\circ \). Give
axis Cartesian coordinates and general 3-by-3 matrix in terms of axis angles \([\varphi, \vartheta, \Theta]\). Test it using angles in
(b)
\[
\begin{align*}
\Theta \cdot e_x &= \hat{\Theta}_x = \cos \vartheta \sin \varphi \\
\Theta \cdot e_y &= \hat{\Theta}_y = \sin \varphi \\
\Theta \cdot e_z &= \hat{\Theta}_z = \cos \vartheta \\
\Theta \times e_x &= \hat{\Theta}_x \times e_x + \hat{\Theta}_y \times e_y + \hat{\Theta}_z \times e_z = 0 \\
\Theta \times e_y &= \hat{\Theta}_x \times e_x + \hat{\Theta}_y \times e_y + \hat{\Theta}_z \times e_z = 0 \\
\Theta \times e_z &= \hat{\Theta}_x \times e_x + \hat{\Theta}_y \times e_y + \hat{\Theta}_z \times e_z = 0 \\
\Theta \cdot e_z &= \hat{\Theta}_z \\
\Theta \cdot e_y &= \hat{\Theta}_y \\
\Theta \cdot e_x &= \hat{\Theta}_x \\
(\Theta \cdot e_x) &= (\hat{\Theta}_x \times e_x + \hat{\Theta}_y \times e_y + \hat{\Theta}_z \times e_z) = \hat{\Theta}_x \\
(\Theta \cdot e_y) &= (\hat{\Theta}_x \times e_x + \hat{\Theta}_y \times e_y + \hat{\Theta}_z \times e_z) = \hat{\Theta}_y \\
(\Theta \cdot e_z) &= (\hat{\Theta}_x \times e_x + \hat{\Theta}_y \times e_y + \hat{\Theta}_z \times e_z) = \hat{\Theta}_z \\
e'_x &= e_x \cos \Theta + \hat{\Theta}_x \times e_x \sin \Theta + (1 - \cos \Theta) \hat{\Theta}_x \\
e'_y &= e_y \cos \Theta + \hat{\Theta}_x \times e_y \sin \Theta + (1 - \cos \Theta) \hat{\Theta}_x \\
e'_z &= e_z \cos \Theta + \hat{\Theta}_x \times e_z \sin \Theta + (1 - \cos \Theta) \hat{\Theta}_x \\
\begin{bmatrix}
e_x \cdot e'_x \\
e_y \cdot e'_y \\
e_z \cdot e'_z 
\end{bmatrix} = 
\begin{bmatrix}
\cos \Theta + (1 - \cos \Theta) \hat{\Theta}_x & -\hat{\Theta}_{y} \sin \Theta + (1 - \cos \Theta) \hat{\Theta}_y & \hat{\Theta}_{z} \sin \Theta + (1 - \cos \Theta) \hat{\Theta}_z \\
\hat{\Theta}_{x} \sin \Theta + (1 - \cos \Theta) \hat{\Theta}_x & \cos \Theta + (1 - \cos \Theta) \hat{\Theta}_y & -\hat{\Theta}_{y} \sin \Theta + (1 - \cos \Theta) \hat{\Theta}_z \\
-\hat{\Theta}_{x} \sin \Theta + (1 - \cos \Theta) \hat{\Theta}_x & \hat{\Theta}_{y} \sin \Theta + (1 - \cos \Theta) \hat{\Theta}_y & \cos \Theta + (1 - \cos \Theta) \hat{\Theta}_z 
\end{bmatrix}
\]
Direction cosines \( e_a \cdot e'_b \) expressed in terms of unit crank-vector components \( \hat{\Theta}_x = \cos \varphi \sin \vartheta, \hat{\Theta}_y = \sin \varphi \sin \vartheta, \) and \( \hat{\Theta}_z = \cos \vartheta \)
\[
\begin{bmatrix}
e_x \cdot e'_x \\
e_y \cdot e'_y \\
e_z \cdot e'_z 
\end{bmatrix} = 
\begin{bmatrix}
\cos \Theta + (1 - \cos \Theta) \hat{\Theta}_x & -\hat{\Theta}_{y} \sin \Theta + (1 - \cos \Theta) \hat{\Theta}_y & \hat{\Theta}_{z} \sin \Theta + (1 - \cos \Theta) \hat{\Theta}_z \\
\hat{\Theta}_{x} \sin \Theta + (1 - \cos \Theta) \hat{\Theta}_x & \cos \Theta + (1 - \cos \Theta) \hat{\Theta}_y & -\hat{\Theta}_{y} \sin \Theta + (1 - \cos \Theta) \hat{\Theta}_z \\
-\hat{\Theta}_{x} \sin \Theta + (1 - \cos \Theta) \hat{\Theta}_x & \hat{\Theta}_{y} \sin \Theta + (1 - \cos \Theta) \hat{\Theta}_y & \cos \Theta + (1 - \cos \Theta) \hat{\Theta}_z 
\end{bmatrix}
\]
Direction cosines \( e_a \cdot e'_b \) expressed in terms of unit crank-vector polar angles \( \varphi (\text{azimuth}), \) and \( \vartheta (\text{polar})
\[
\begin{bmatrix}
\cos \Theta + (1 - \cos \Theta) \cos^2 \varphi \sin^2 \vartheta & -\cos \vartheta \sin \Theta + (1 - \cos \Theta) \cos \varphi \sin \varphi \sin^2 \vartheta & \sin \vartheta \sin \Theta + (1 - \cos \Theta) \cos \varphi \sin \varphi \sin \vartheta \\
\cos \Theta \sin \Theta + (1 - \cos \Theta) \cos \varphi \sin \varphi \sin \vartheta & \cos \Theta \sin \Theta + (1 - \cos \Theta) \sin^2 \varphi \sin^2 \vartheta & -\cos \vartheta \sin \Theta + (1 - \cos \Theta) \sin \varphi \sin \vartheta \\
-\sin \vartheta \sin \Theta + (1 - \cos \Theta) \cos \varphi \sin \varphi \sin \vartheta & \sin \vartheta \sin \Theta + (1 - \cos \Theta) \sin \varphi \sin \vartheta & \cos \Theta \sin \Theta + (1 - \cos \Theta) \sin \varphi \sin \vartheta 
\end{bmatrix}
\]
120° rotation around (111) \( \sqrt{3} \) (Polar angle \( \vartheta = 54.7^\circ = \arccos(1/\sqrt{3}) \) and azimuthal angle \( \varphi = 45^\circ \) and \( \hat{\Theta}_x = \hat{\Theta}_y = \hat{\Theta}_z = \sqrt{3}/2 \))
(c) Euler angles \( \alpha = \varphi - \pi/2 + \tan^{-1}(\tan(\Theta/2) \cos \vartheta), \beta = 2\sin^{-1}(\sin(\Theta/2) \sin \vartheta), \gamma = \pi/2 - \varphi + \tan^{-1}(\tan(\Theta/2) \cos \vartheta) \) (10.A.1)
120° rotation Euler angles \( R_1(\alpha \beta \gamma) = (0^\circ, 90^\circ, 90^\circ) \) 90°-Y-Rotation Euler angles \( R_2(\alpha \beta \gamma) = (0^\circ, 90^\circ, 0^\circ) \)
\[
\begin{bmatrix}
e_x \cdot e'_x \\
e_y \cdot e'_y \\
e_z \cdot e'_z 
\end{bmatrix} = 
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 
\end{bmatrix} = R_1 \quad \text{90°-Y-Rotation(} \vartheta = \Theta = 90^\circ \text{)} \) or \( (\hat{\Theta}_x = 0, \hat{\Theta}_y = 1, \hat{\Theta}_z = 0 ) \)
\[
\begin{bmatrix}
e_x \cdot e'_x \\
e_y \cdot e'_y \\
e_z \cdot e'_z 
\end{bmatrix} = 
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 
\end{bmatrix} = R_2 \quad \text{90°-Y-Rotation(} \vartheta = \Theta = 90^\circ \text{)} \) or \( (\hat{\Theta}_x = 0, \hat{\Theta}_y = 1, \hat{\Theta}_z = 0 ) \)
\[
\begin{bmatrix}
e_x \cdot e'_x \\
e_y \cdot e'_y \\
e_z \cdot e'_z 
\end{bmatrix} = 
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 
\end{bmatrix} = R_2 \quad \text{90°-Y-Rotation(} \vartheta = \Theta = 90^\circ \text{)} \) or \( (\hat{\Theta}_x = 0, \hat{\Theta}_y = 1, \hat{\Theta}_z = 0 ) \)
\]
(d) Compare formulas and numerics for 3-by-3 R(3) matrices and corresponding 2-by-2 U(2) matrices.
(e) Find 3-by-3 R(3) and 2-by-2 U(2) matrices for rotation \( R_1 \) by 90° around \( \hat{Y} \) (or \( C \))-axis.
(f) Do products \( R_1 R_2 \) \( R(\varphi, \Theta, \Theta) \) and \( R(\varphi, \Theta, \Theta) R_1 \) numerically and check by product formula (10.A.10). Describe results.
Spin erection. Does it phase \( U(2) \)?

10.B.2. The following general problem may certainly become relevant if the mythical quantum computer materializes. It involves erecting an arbitrary state with spin vector \( S \) to the spin-up \( Z \) (or \( A \)) position with a particular overall phase \( \Phi \). In each case make the description of your solution as simple as possible as though you needed to explain it to engineers.

(a) For a state of 0-phase with spin on the \( X \) (or \( B \)), describe a single operator that does the above.

(b) For a state of 0-phase with spin at \( \beta \) in the \( XZ \) (or \( AB \)) plane, describe a single operator that does the above.

(c) Of all possible rotations from \( \beta \) in the \( XZ \) plane to spin-up \( Z \), which takes the least energy regardless of final phase \( \Phi = -\gamma/2 \)?

Ways to solve a common "spin-erection" problem of finding operations that return a spin vector at initial polar angles \((\alpha, \beta)\) to one of the three main axes such as spin-up the \( Z \) axis.

The "direct" move is done by a \( \beta \)-rotation with crank vector in the \( XY \) plane at azimuth of \( \alpha - \pi/2 \) as shown above.

\[
R[\Theta] = \exp(-i(\Theta \cos(\alpha - \pi/2)J_X + \Theta \sin(\alpha - \pi/2)J_Y)), \text{ where: } \Theta = \beta
\]

The resulting matrix is found from the axis-angle matrix (10.5.15).

\[
R[\beta \text{ direct}] = \exp(-i(\beta \sin \alpha J_X - \Theta \cos \alpha J_Y)) = \begin{pmatrix}
\cos \frac{\beta}{2} e^{-i\alpha} \sin \frac{\beta}{2} \\
-e^{i\alpha} \sin \frac{\beta}{2} \cos \frac{\beta}{2}
\end{pmatrix}
\]

This transformation "erects" the \((\alpha, \beta)\) spin state to spin-up-\( Z \) with no change of overall phase \( \gamma \).

\[
\begin{pmatrix}
\cos \frac{\beta}{2} e^{-i\alpha} \sin \frac{\beta}{2} \\
-e^{i\alpha} \sin \frac{\beta}{2} \cos \frac{\beta}{2}
\end{pmatrix} \begin{pmatrix}
\cos \frac{\beta}{2} e^{-i\alpha/2} \\
\sin \frac{\beta}{2} e^{i\alpha/2}
\end{pmatrix} e^{-i\gamma/2} = \begin{pmatrix}
e^{-i\alpha/2} \\
0
\end{pmatrix} e^{-i\gamma/2}
\]

Any transformation whose crank vector lies in the plane that bisects the \((\alpha, \beta)\)-\( Z \) angle can turn spin-up-\((\alpha, \beta)\) into the erect spin-up-\( Z \), but it will involve crank turn angle \( \Theta \) larger than \( \beta \), a change of phase \( \gamma \), and correspondingly more energy.