Group Theory in Quantum Mechanics

Lecture 5 (1.27.15)

Spectral Decomposition with Repeated Eigenvalues

(Quantum Theory for Computer Age - Ch. 3 of Unit 1)
(Principles of Symmetry, Dynamics, and Spectroscopy - Sec. 1-3 of Ch. 1)

Review: matrix **eigen**states ("**own**states) and **Idempotent projectors** (Non-degeneracy case)
Operator orthonormality, completeness, and spectral decomposition(Non-degenerate e-values)
(Preparing for: Degenerate eigenvalues)

Eigensolutions with degenerate eigenvalues (Possible?... or not?)
Secular→ Hamilton-Cayley→Minimal equations
Diagonalizability criterion

Nilpotents and "Bad degeneracy" examples: $\mathbf{B}=\begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix}$, and: $\mathbf{N}=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
Applications of Nilpotent operators later on

Idempotents and "Good degeneracy" example: $\mathbf{G}=\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$
Secular equation by minor expansion
Example of minimal equation projection

Orthonormalization of degenerate eigensolutions
Projection $\mathbf{P}_j$-matrix anatomy (Gramian matrices)
Gram-Schmidt procedure

Orthonormalization of commuting eigensolutions. Examples: $\mathbf{G}=\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$ and: $\mathbf{H}=\begin{pmatrix} \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & 2 \\ 2 & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \end{pmatrix}$
The old "$\mathbf{1=1\cdot 1}$ trick"-Spectral decomposition by projector splitting
Irreducible projectors and representations (Trace checks)
Minimal equation for projector $\mathbf{P=P}^2$
How symmetry groups become eigen-solvers

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Minimal equation for projector $\mathbf{P}=\mathbf{P}^2$

How symmetry groups become eigen-solvers

Unitary operators and matrices that change state vectors...

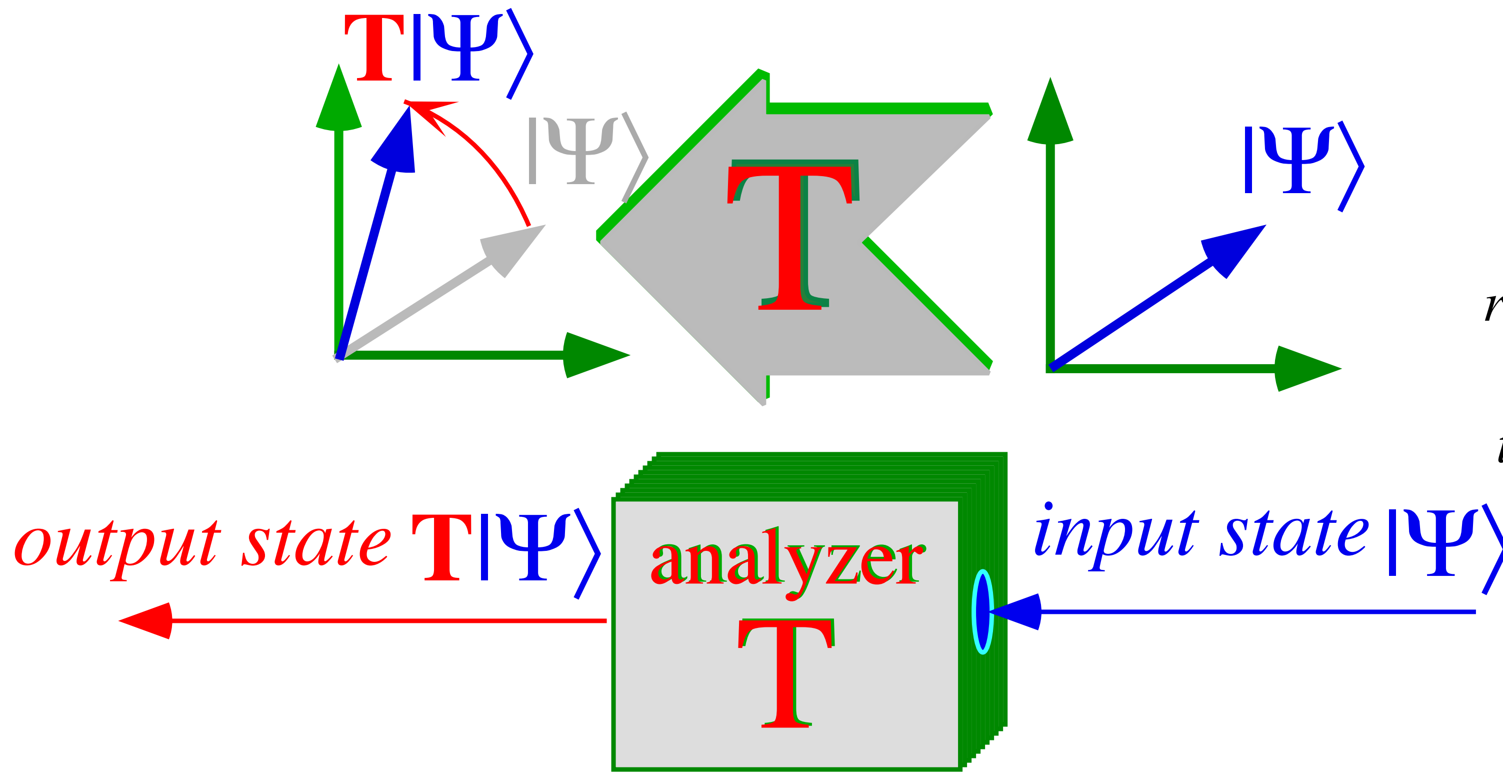

Fig. 3.1.1 Effect of analyzer represented by ket vector transformation of $|\Psi\rangle$ *to new ket vector* $\mathbf{T}|\Psi\rangle$.

*...and **eigen**states (“**own**states) that are mostly immune to* **T***...*

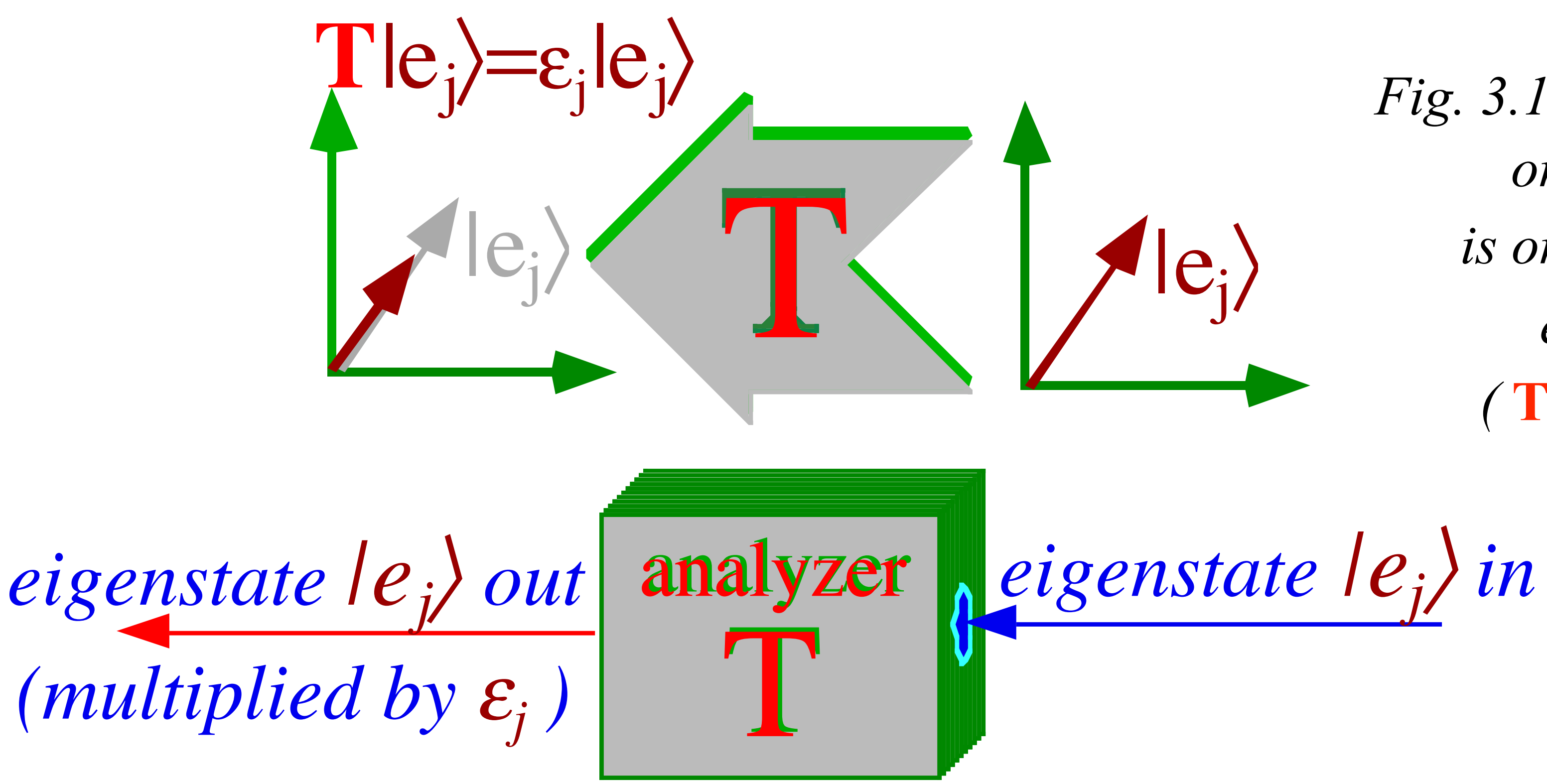

Fig. 3.1.2 Effect of analyzer on eigenket $|\varepsilon_j\rangle$ *is only to multiply by eigenvalue* ε_j *(* $\mathbf{T}|\varepsilon_j\rangle = \varepsilon_j\,|\varepsilon_j\rangle$ *).*

For Unitary operators $\mathbf{T}=\mathbf{U}$*, the eigenvalues must be phase factors* $\varepsilon_k = e^{i\alpha_k}$

Operator ortho-completeness, and spectral decomposition

(For: Non-Degenerate eigenvalues *)*

Eigen-Operator-Projectors $\mathbf{P}_k$:

$$\mathbf{P}_k = \frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$$

$\mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k = \mathbf{P}_k\mathbf{M}$

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Dirac notation form:

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Eigen-Operator-$\mathbf{P}_k$-Orthonormality Relations

$$\mathbf{P}_j \mathbf{P}_k = \delta_{jk} \mathbf{P}_k = \begin{cases} \mathbf{0} & if : j \neq k \\ \mathbf{P}_k & if : j = k \end{cases}$$

Dirac notation form:

$$|\varepsilon_j\rangle\langle\varepsilon_j| \cdot |\varepsilon_k\rangle\langle\varepsilon_k| = \delta_{jk} |\varepsilon_k\rangle\langle\varepsilon_k|$$

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Eigen-Operator-$\mathbf{P}_j$ *-Completeness Relations*

$$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$$

Dirac notation form:

$$\mathbf{1} = |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2| + \ldots + |\varepsilon_n\rangle\langle\varepsilon_n|$$

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Eigen-operators have *Spectral Decomposition*

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To be discussed in this lecture.

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Dirac notation form:

$\mathbf{1} = |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2| + \ldots + |\varepsilon_n\rangle\langle\varepsilon_n|$

$$\mathbf{1} = \mathbf{P}_{\varepsilon_1} + \mathbf{P}_{\varepsilon_2} + \ldots + \mathbf{P}_{\varepsilon_n}$$

(Dirac notation form is more complicated.)
To be discussed in this lecture.

Eigen-operators have *Spectral Decomposition* of operator $\mathbf{M} = \varepsilon_1\mathbf{P}_1 + \varepsilon_2\mathbf{P}_2 + \ldots + \varepsilon_N\mathbf{P}_N$

Dirac notation form:

$\mathbf{M} = \varepsilon_1|\varepsilon_1\rangle\langle\varepsilon_1| + \varepsilon_2|\varepsilon_2\rangle\langle\varepsilon_2| + \ldots + \varepsilon_n|\varepsilon_n\rangle\langle\varepsilon_n|$

$$\mathbf{M} = \varepsilon_1\mathbf{P}_{\varepsilon_1} + \varepsilon_2\mathbf{P}_{\varepsilon_2} + \ldots + \varepsilon_n\mathbf{P}_{\varepsilon_n}$$

(Dirac notation form is more complicated.)

...and operator *Functional Spectral Decomposition* of a function $f(\mathbf{M}) = f(\varepsilon_1)\mathbf{P}_1 + f(\varepsilon_2)\mathbf{P}_2 + \ldots + f(\varepsilon_N)\mathbf{P}_N$

Dirac notation form:

$f(\mathbf{M}) = f(\varepsilon_1)|\varepsilon_1\rangle\langle\varepsilon_1| + f(\varepsilon_2)|\varepsilon_2\rangle\langle\varepsilon_2| + \ldots + f(\varepsilon_n)|\varepsilon_n\rangle\langle\varepsilon_n|$

$$f(\mathbf{M}) = f(\varepsilon_1)\mathbf{P}_{\varepsilon_1} + f(\varepsilon_2)\mathbf{P}_{\varepsilon_2} + \ldots + f(\varepsilon_n)\mathbf{P}_{\varepsilon_n}$$

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(Preparing for: Degenerate eigenvalues)

Review: matrix eigenstates ("ownstates) and Idempotent projectors (Degeneracy case)

Operator orthonormality, completeness, and spectral decomposition(Degenerate e-values)

Eigensolutions with degenerate eigenvalues (Possible?... or not?)

Secular→ Hamilton-Cayley→Minimal equations

Diagonalizability criterion

Nilpotents and "Bad degeneracy" examples: $\mathbf{B}=\begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix}$, *and:* $\mathbf{N}=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

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*The old "***1=1·1** *trick"-Spectral decomposition by projector splitting*

Irreducible projectors and representations (Trace checks)

Minimal equation for projector $\mathbf{P}=\mathbf{P}^2$

How symmetry groups become eigen-solvers

Eigensolutions with degenerate eigenvalues (Possible?... or not?)

What if *secular equation* ($det|\mathbf{M}-\varepsilon_j\mathbf{1}|-0$) of *N-by-N* matrix **H** has ℓ-repeated ε_1-roots $\{\varepsilon_{1_1}, \varepsilon_{1_2} ... \varepsilon_{1_\ell}\}$?

If so, it's possible **H** can't be completely diagonalized, though this is rarely the case.

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$$S(\varepsilon) = 0 = (-1)^N \left(\varepsilon - \varepsilon_1\right)^{\ell_1} \left(\varepsilon - \varepsilon_2\right)^{\ell_2} \ldots \left(\varepsilon - \varepsilon_p\right)^{\ell_p} \quad \text{where:} \quad \ell_1 + \ell_2 + \ldots + \ell_p = N$$

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A *diagonalizability criterion* has just been proved:

In general, matrix $\mathbf{H}$ *can make an ortho-complete set of* $\mathbf{P}_{\varepsilon_j}$ *if and only if, the* $\mathbf{H}$ *minimal equation has no repeated factors. Then and only then is matrix* $\mathbf{H}$ *fully diagonalizable.*

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and *complete*: $\mathbf{1} = \mathbf{P}_{\varepsilon_1} + \mathbf{P}_{\varepsilon_2} + \ldots + \mathbf{P}_{\varepsilon_p}$

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Repeated minimal equation factors means you will not get an ortho-complete set of $\mathbf{P}_j$.
Even *one repeat* is fatal...

(like this ↓)

$$\mathbf{0} = \boxed{\left(\mathbf{H} - \varepsilon_1 \mathbf{1}\right)^2} \left(\mathbf{H} - \varepsilon_2 \mathbf{1}\right)^1 \ldots,$$

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(Preparing for: Degenerate eigenvalues *)*

*Review: matrix **eigen**states (“**own**states) and Idempotent projectors* (*Degeneracy case*)

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The old "$\mathbf{1=1\cdot 1}$ *trick*"-*Spectral decomposition by projector splitting*

Irreducible projectors and representations (Trace checks)

Minimal equation for projector $\mathbf{P=P^2}$

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N and its partners comprise a 4-dimensional *U(2) unit tensor operator space*

U(2) op-space= $\{\mathbf{e}_{11}=|1\rangle\langle 1|,\quad \mathbf{e}_{12}=|1\rangle\langle 2|,\quad \mathbf{e}_{21}=|2\rangle\langle 1|,\quad \mathbf{e}_{22}=|2\rangle\langle 2|\}$

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Their ∞-Dimensional cousins are the *creation-destruction* $\mathbf{a}_i^\dagger \mathbf{a}_j$ operators.

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An example of a 'good' degenerate (but still diagonalizable) matrix
is the anti-diagonal "gamma" matrix **G** (*a Dirac-Lorentz transform generator*)

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$$\varepsilon^4 - (\textstyle\sum \text{1x1 diag of } \mathbf{G})\,\varepsilon^3 + (\textstyle\sum \text{2x2 diag minors of } \mathbf{G})\,\varepsilon^2 - (\textstyle\sum \text{3x3 diag minors of } \mathbf{G})\,\varepsilon^1 + (\text{4x4 determinant of } \mathbf{G})\,\varepsilon^1 = 0$$

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$$\begin{aligned}\det\mathbf{G}&=\\ &=(-1)\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}\\ &=(-1)(1)\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}\\ &=(-1)(1)(-1)\\ &=+1\end{aligned}$$

$$\begin{array}{cccc} + & - & + & - \end{array}$$
$$\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$$

(Preparing for: Degenerate eigenvalues)

Review: matrix eigenstates ("ownstates) and Idempotent projectors (Degeneracy case)

Operator orthonormality, completeness, and spectral decomposition(Degenerate e-values)

Eigensolutions with degenerate eigenvalues (Possible?... or not?)

Secular→ Hamilton-Cayley→Minimal equations

Diagonalizability criterion

Nilpotents and "Bad degeneracy" examples: $\mathbf{B}=\begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix}$, *and:* $\mathbf{N}=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Applications of Nilpotent operators later on

Idempotents and "Good degeneracy" example: $\mathbf{G}=\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$

Secular equation by minor expansion

Example of minimal equation projection

Orthonormalization of degenerate eigensolutions

Projection $\mathbf{P}_j$*-matrix anatomy (Gramian matrices)*

Gram-Schmidt procedure

Orthonormalization of commuting eigensolutions. Examples: $\mathbf{G}=\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$ *and:* $\mathbf{H}=\begin{pmatrix} \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & 2 \\ 2 & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \end{pmatrix}$

*The old "***1=1·1** *trick"-Spectral decomposition by projector splitting*

Irreducible projectors and representations (Trace checks)

Minimal equation for projector $\mathbf{P}=\mathbf{P}^2$

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Each of these projectors contains two linearly independent ket or bra vectors:

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These 4 are more than linearly independent... ...they are *orthogonal*.

An example of a 'good' degenerate (but still diagonalizable) matrix is the anti-diagonal "gamma" matrix **G** (*a Dirac-Lorentz transform generator*)

$$\mathbf{G}=\begin{pmatrix} 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0\\ 1&0&0&0 \end{pmatrix} \qquad \textit{SEq:} \qquad S(\varepsilon)=\det\left|\mathbf{G}-\varepsilon\mathbf{1}\right|=\det\begin{vmatrix} -\varepsilon&0&0&1\\ 0&-\varepsilon&1&0\\ 0&1&-\varepsilon&0\\ 1&0&0&-\varepsilon \end{vmatrix}$$

ε has a 4th degree *Secular Equation (SEq)* with repeat pairs of degenerate roots ($\varepsilon_k=\pm 1$)

$$S(\varepsilon)=0=\varepsilon^4-2\varepsilon^2+1=(\varepsilon-1)^2(\varepsilon+1)^2$$

G has a 4th degree *HC equation (HCeq)* with **G** replacing ε in *SEq:* $S(\varepsilon)\rightarrow S(\mathbf{G})$

$$S(\mathbf{G})=0=\mathbf{G}^4-2\mathbf{G}^2+\mathbf{1}=(\mathbf{G}-\mathbf{1})^2(\mathbf{G}+\mathbf{1})^2$$

Yet **G** satisfies *Minimal Equation (MinEq)* of only 2nd degree with no repeats. So $\mathbf{P}_{\varepsilon_k}$ formulae work!

$$\mathbf{0}=(\mathbf{G}-\mathbf{1})(\mathbf{G}+\mathbf{1})$$

Two ortho-complete projection operators are derived by Projection formula: $\mathbf{P}_{\varepsilon_k}=\dfrac{\prod_{\varepsilon_m\neq\varepsilon_k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{\varepsilon_m\neq\varepsilon_k}(\varepsilon_k-\varepsilon_m)}$

$$\mathbf{P}^{G}_{+1}=\frac{\mathbf{G}-(-1)\mathbf{1}}{+1-(-1)}=\frac{1}{2}\begin{pmatrix} 1&0&0&1\\ 0&1&1&0\\ 0&1&1&0\\ 1&0&0&1 \end{pmatrix} \qquad \mathbf{P}^{G}_{-1}=\frac{\mathbf{G}-(1)\mathbf{1}}{-1-(1)}=\frac{1}{2}\begin{pmatrix} 1&0&0&-1\\ 0&1&-1&0\\ 0&-1&1&0\\ -1&0&0&1 \end{pmatrix}$$

Each of these projectors contains two linearly independent ket or bra vectors:

These 4 are more than linearly independent... ...they are *orthogonal*.

Bra-Ket repeats may need to be made orthogonal. Two methods shown next:
1: *Gram-Schmidt orthogonalization (harder)* **2:** *Commuting projectors (easier)*

(Preparing for: Degenerate eigenvalues)

*Review: matrix **eigen**states (“**own**states) and **Idempotent projectors** (Degeneracy case)*
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Minimal equation for projector $\mathbf{P}=\mathbf{P}^2$

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If projector $\mathbf{P}_j$ is idempotent ($\mathbf{P}_j \mathbf{P}_j = \mathbf{P}_j$), all matrix elements $(\mathbf{P}_j)_{bk}$ are row$_b$-column$_k$-•-products $(j_b|j_k)$

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$$
\begin{array}{ccccc}
(\mathbf{P}_j) & \cdot & (\mathbf{P}_j) & = & (\mathbf{P}_j) \\
\left(\begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)
& \cdot &
\left(\begin{array}{ccc|c|cc}
\cdot & \cdot & \cdot & k_1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & k_2 & \cdot & \cdot \\
\cdot & \cdot & \cdot & k_3 & \cdot & \cdot \\
\cdot & \cdot & \cdot & k_4 & \cdot & \cdot \\
\cdot & \cdot & \cdot & k_5 & \cdot & \cdot \\
\cdot & \cdot & \cdot & k_6 & \cdot & \cdot
\end{array}\right)
& = &
\left(\begin{array}{ccc|c|cc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & (b|k) & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)
\end{array}
$$

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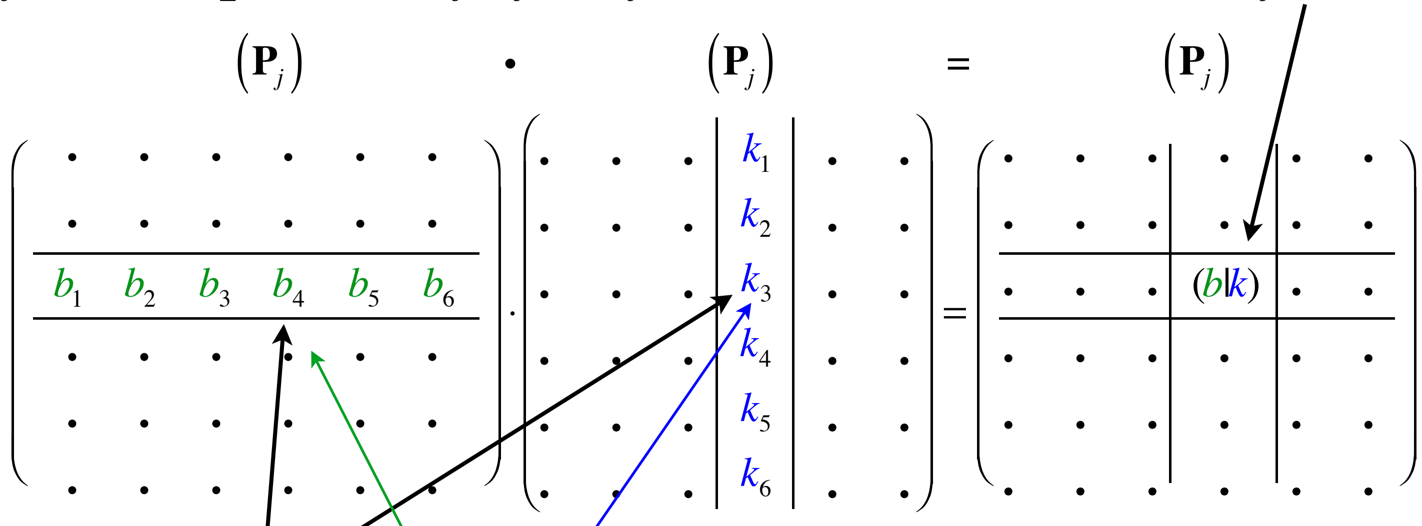

$(\mathbf{P}_j)_{34} = b_4 = k_3 = (j_3|j_4) =$

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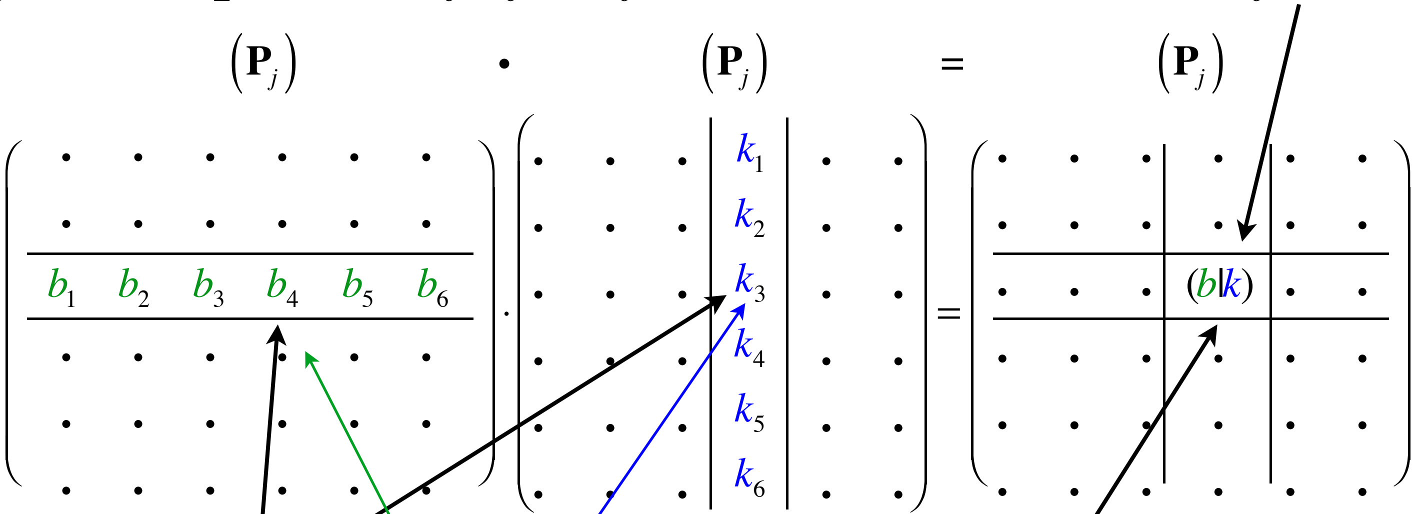

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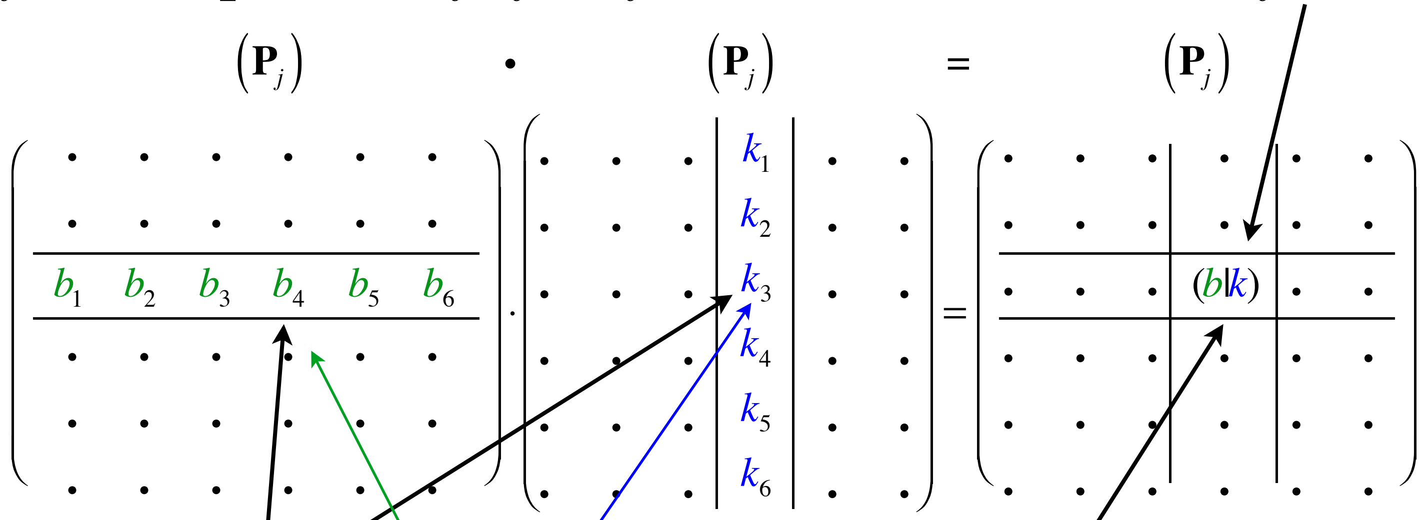

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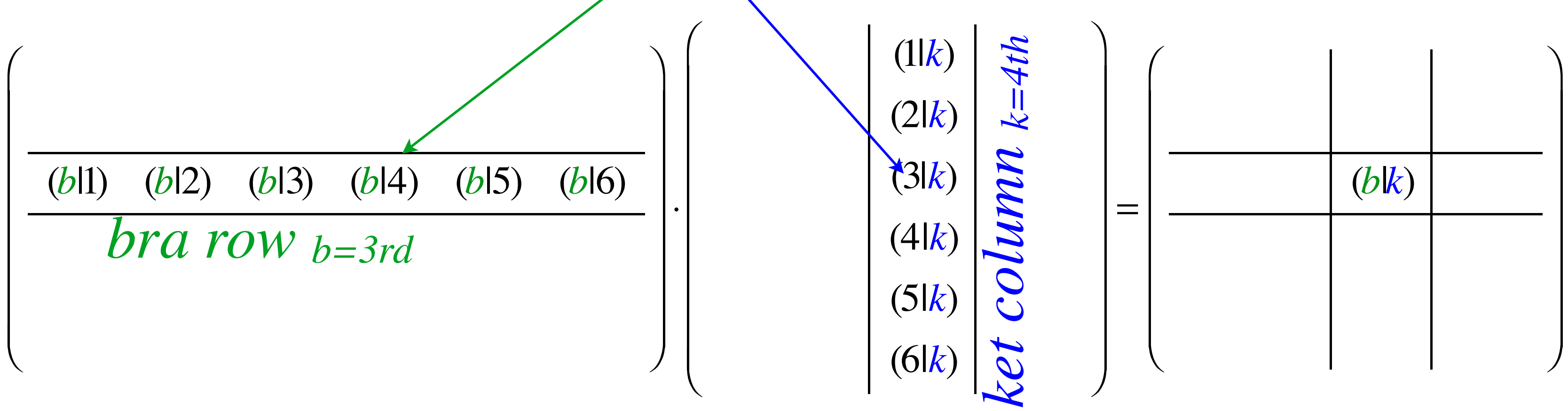

Quasi-Dirac notation shows vector relations

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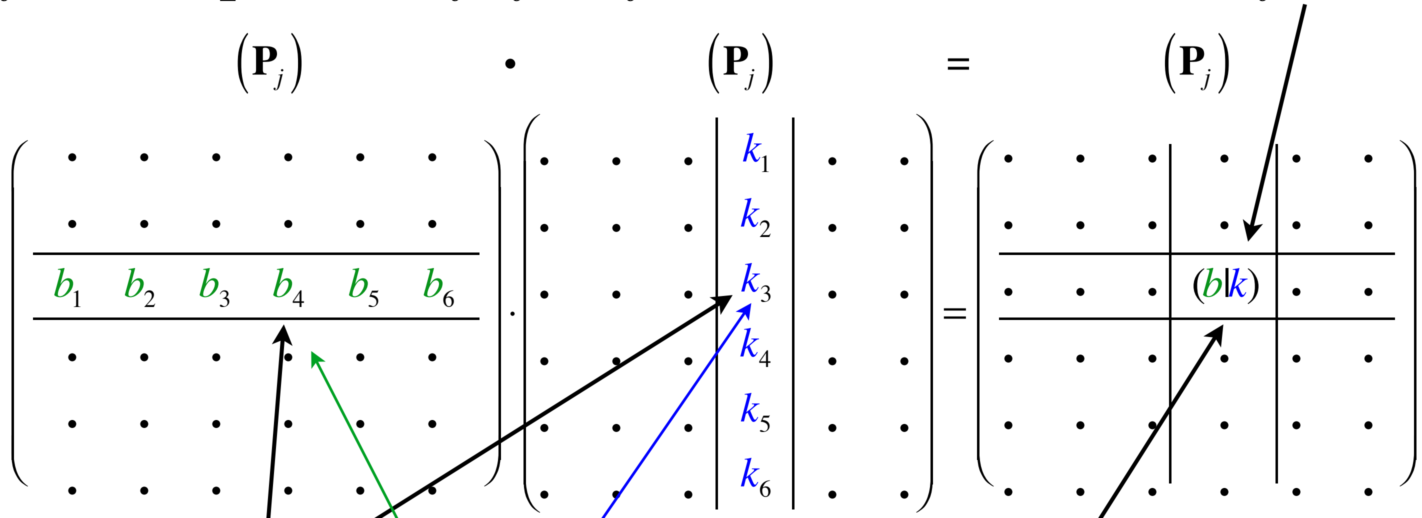

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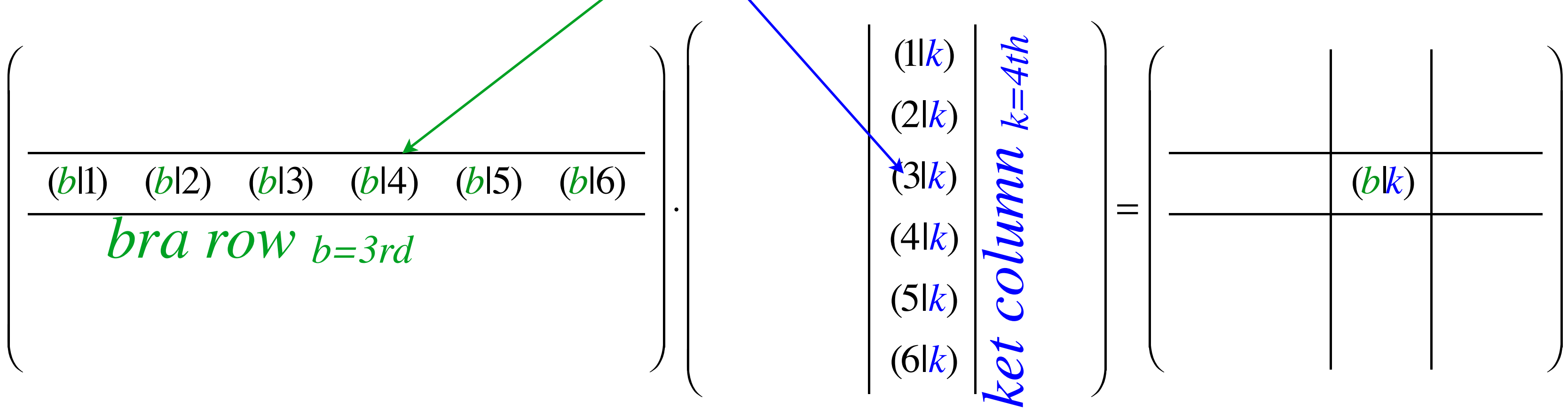

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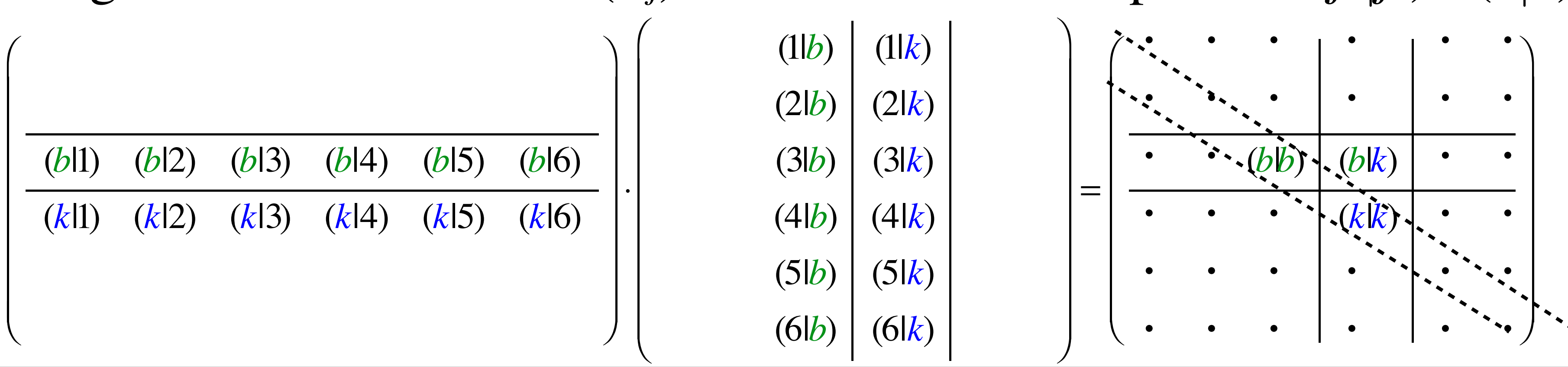
Diagonal matrix elements $(\mathbf{P}_j)_{kk}$ = row$_k$-column$_k$-•-product $(j_k|j_k) = (k|k)$ is k^{th}*-norm value* (usually real)



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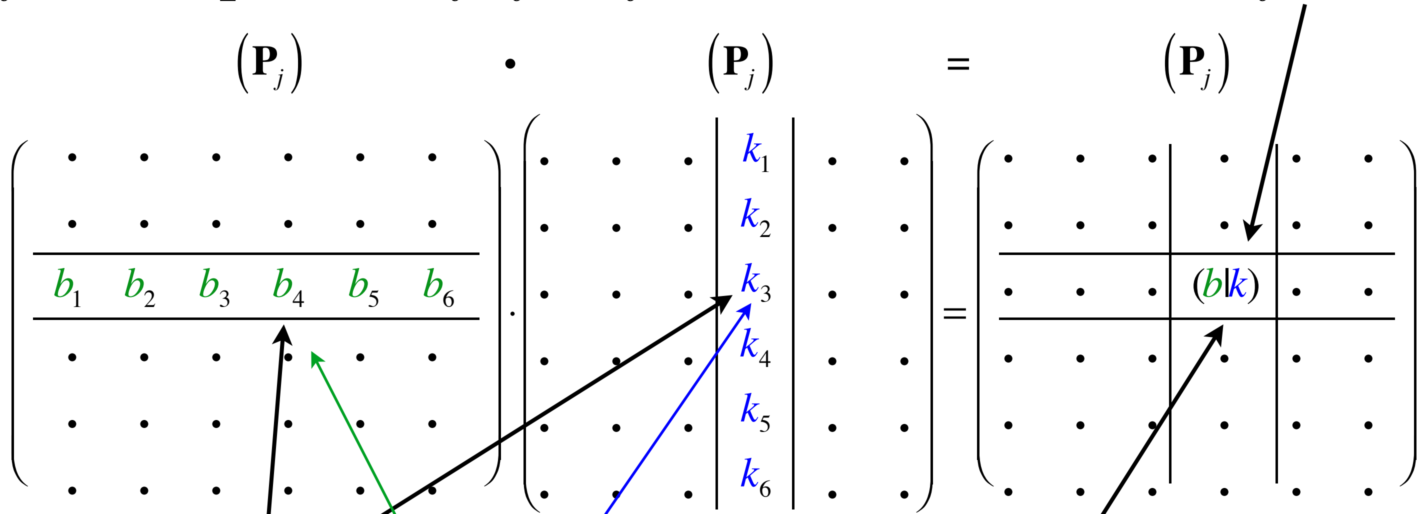

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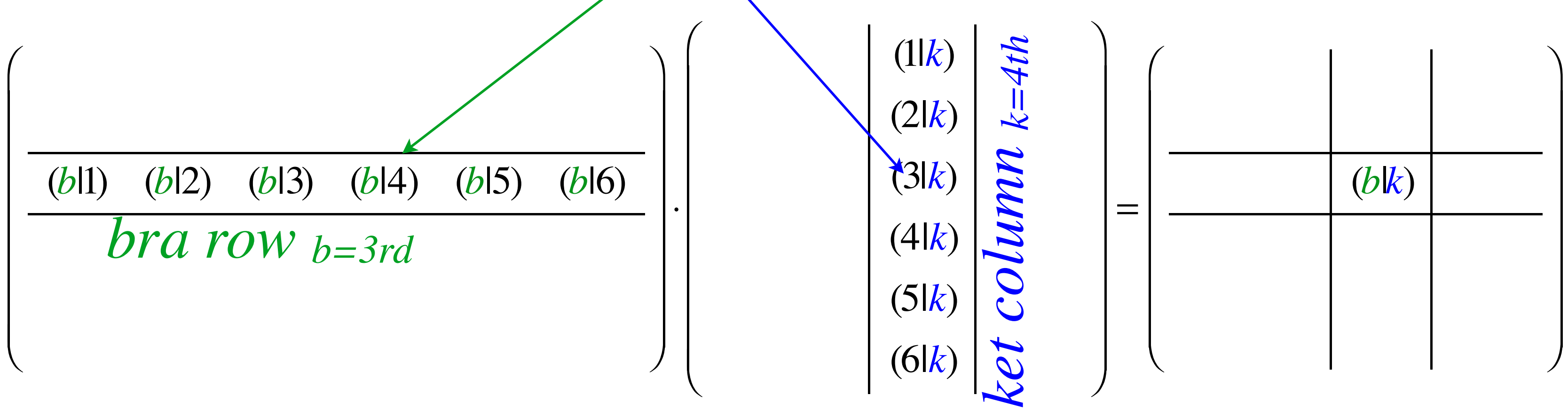

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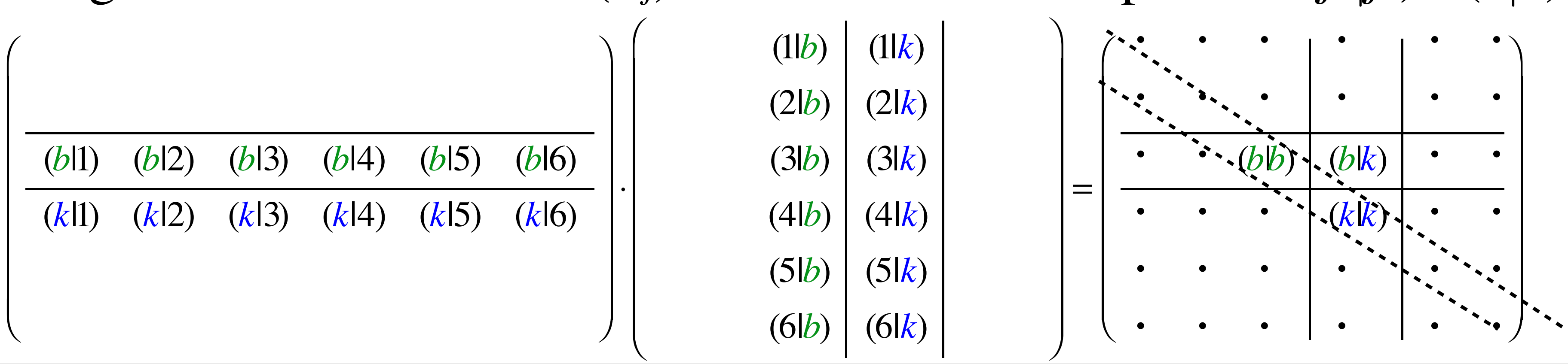

k^{th} <u>*normalized vectors*</u>

ket= $|j_k\rangle = |j_k)/\sqrt{(k|k)}$

bra= $\langle j_k| = (j_k|/\sqrt{(k|k)}$

so: $\langle j_k|j_k\rangle = 1$

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Solve these by substituting: $N_1 =- N_2\,(j_1|j_2)/(j_1|j_1)$

to give: $1= N_2^2\,(j_1|j_2)^2/(j_1|j_1) - N_2^2[(j_1|j_2)+ (j_2|j_1)](j_1|j_2)/(j_1|j_1) + N_2^2(j_2|j_2)$

$1/N_2^2 = (j_2|j_2) + \cancel{(j_1|j_2)^2/(j_1|j_1)} - \cancel{(j_1|j_2)^2/(j_1|j_1)} - (j_2|j_1)(j_1|j_2)/(j_1|j_1)$

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So the new orthonormal pair is:

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$$|j_2\rangle = N_1|j_1) + N_2|j_2) = -\frac{N_2(j_1|j_2)}{(j_1|j_1)}|j_1) + N_2|j_2)$$

$$= N_2\left(|j_2) - \frac{(j_1|j_2)}{(j_1|j_1)}|j_1)\right) = \sqrt{\frac{1}{(j_2|j_2) - \frac{(j_2|j_1)(j_1|j_2)}{(j_1|j_1)}}}\left(|j_2) - \frac{(j_1|j_2)}{(j_1|j_1)}|j_1)\right)$$

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OK. That's for 2 vectors. Like to try for 3?

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Solve these by substituting: $\quad N_1 = - N_2 (j_1|j_2)/(j_1|j_1)$

to give: $\quad 1 = N_2^2 (j_1|j_2)^2/(j_1|j_1) - N_2^2[(j_1|j_2) + (j_2|j_1)](j_1|j_2)/(j_1|j_1) + N_2^2(j_2|j_2)$

$1/N_2^2 = (j_2|j_2) + \cancel{(j_1|j_2)^2/(j_1|j_1)} - \cancel{(j_1|j_2)^2/(j_1|j_1)} - (j_2|j_1)(j_1|j_2)/(j_1|j_1)$

$1/N_2^2 = (j_2|j_2) - (j_2|j_1)(j_1|j_2)/(j_1|j_1)$

So the new orthonormal pair is:
$$|j_1\rangle = \frac{|j_1)}{\sqrt{(j_1|j_1)}}$$

$$|j_2\rangle = N_1|j_1) + N_2|j_2) = -\frac{N_2(j_1|j_2)}{(j_1|j_1)}|j_1) + N_2|j_2)$$

$$= N_2\left(|j_2) - \frac{(j_1|j_2)}{(j_1|j_1)}|j_1)\right) = \sqrt{\frac{1}{(j_2|j_2) - \frac{(j_2|j_1)(j_1|j_2)}{(j_1|j_1)}}}\left(|j_2) - \frac{(j_1|j_2)}{(j_1|j_1)}|j_1)\right)$$

OK. That's for 2 vectors. Like to try for 3?

Instead, let' try another way to "orthogonalize" that might be more *elegante*.

(Preparing for: Degenerate eigenvalues *)*

Review: matrix ***eigen****states ("****own****states) and* ***Idempotent projectors*** *(Degeneracy case)*

*Operator orthonormality, completeness, and spectral decomposition(*Degenerate e-values *)*

Eigensolutions with degenerate eigenvalues (Possible?... or not?)

Secular→ Hamilton-Cayley→Minimal equations

Diagonalizability criterion

Nilpotents and "Bad degeneracy" examples: $\mathbf{B}=\begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix}$, *and:* $\mathbf{N}=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Applications of Nilpotent operators later on

Idempotents and "Good degeneracy" example: $\mathbf{G}=\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$

Secular equation by minor expansion

Example of minimal equation projection

Orthonormalization of degenerate eigensolutions

Projection $\mathbf{P}_j$*-matrix anatomy (Gramian matrices)*

Gram-Schmidt procedure

→ *Orthonormalization of commuting eigensolutions. Examples:* $\mathbf{G}=\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$ *and:* $\mathbf{H}=\begin{pmatrix} \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & 2 \\ 2 & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \end{pmatrix}$

The old "**1=1·1** *trick*"-*Spectral decomposition by projector splitting*

Irreducible projectors and representations (Trace checks)

Minimal equation for projector $\mathbf{P}=\mathbf{P}^2$

How symmetry groups become eigen-solvers

Orthonormalization by commuting projector splitting

The **G** projectors and eigenvectors were derived several pages back: *(And, we got a lucky orthogonality)*

$$\mathbf{P}_{+1}^{G} = \frac{\mathbf{G}-(-1)\mathbf{1}}{+1-(-1)} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{P}_{-1}^{G} = \frac{\mathbf{G}-(1)\mathbf{1}}{-1-(1)} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$\left|1_1\right\rangle = \frac{\left|1_1\right)}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \left|1_2\right\rangle = \frac{\left|1_2\right)}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \left|-1_1\right\rangle = \frac{\left|-1_1\right)}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \quad \left|-1_2\right\rangle = \frac{\left|-1_2\right)}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

Orthonormalization by commuting projector splitting

The **G** projectors and eigenvectors were derived several pages back: *(And, we got a lucky orthogonality)*

$$\mathbf{P}^G_{+1} = \frac{\mathbf{G}-(-1)\mathbf{1}}{+1-(-1)} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{P}^G_{-1} = \frac{\mathbf{G}-(1)\mathbf{1}}{-1-(1)} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$|1_1\rangle = \frac{|1_1)}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad |1_2\rangle = \frac{|1_2)}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad |-1_1\rangle = \frac{|-1_1)}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \quad |-1_2\rangle = \frac{|-1_2)}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

Dirac notation for **G**-*split completeness relation* using eigenvectors is the following:

$$\begin{aligned} 1 = \mathbf{P}^{\mathbf{G}}_{1} + \mathbf{P}^{\mathbf{G}}_{-1} &= |1_1\rangle\langle 1_1| + |1_2\rangle\langle 1_2| + |-1_1\rangle\langle -1_1| + |-1_2\rangle\langle -1_2| \\ &= \mathbf{P}_{1_1} + \mathbf{P}_{1_2} + \mathbf{P}_{-1_1} + \mathbf{P}_{-1_2} \end{aligned}$$

Orthonormalization by commuting projector splitting

The **G** projectors and eigenvectors were derived several pages back: *(And, we got a lucky orthogonality)*

$$\mathbf{P}^{G}_{+1} = \frac{\mathbf{G}-(-1)\mathbf{1}}{+1-(-1)} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{P}^{G}_{-1} = \frac{\mathbf{G}-(1)\mathbf{1}}{-1-(1)} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$\left|1_1\right\rangle = \frac{\left|1_1\right)}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \left|1_2\right\rangle = \frac{\left|1_2\right)}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \left|-1_1\right\rangle = \frac{\left|-1_1\right)}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \quad \left|-1_2\right\rangle = \frac{\left|-1_2\right)}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

Dirac notation for **G**-*split completeness relation* using eigenvectors is the following:

$$\begin{aligned} 1 = \mathbf{P}^{\mathbf{G}}_{1} + \mathbf{P}^{\mathbf{G}}_{-1} &= \left|1_1\right\rangle\left\langle 1_1\right| + \left|1_2\right\rangle\left\langle 1_2\right| + \left|-1_1\right\rangle\left\langle -1_1\right| + \left|-1_2\right\rangle\left\langle -1_2\right| \\ &= \mathbf{P}_{1_1} + \mathbf{P}_{1_2} + \mathbf{P}_{-1_1} + \mathbf{P}_{-1_2} \end{aligned}$$

Each of the original **G** projectors are split in two parts with one ket-bra in each.

$$\mathbf{P}^{\mathbf{G}}_{1} = \mathbf{P}_{1_1} + \mathbf{P}_{1_2} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \left|1_1\right\rangle\left\langle 1_1\right| + \left|1_2\right\rangle\left\langle 1_2\right|$$

$$\mathbf{P}^{\mathbf{G}}_{-1} = \mathbf{P}_{-1_1} + \mathbf{P}_{-1_2} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \left|-1_1\right\rangle\left\langle -1_1\right| + \left|-1_2\right\rangle\left\langle -1_2\right|$$

Orthonormalization by commuting projector splitting

The **G** projectors and eigenvectors were derived several pages back: *(And, we got a lucky orthogonality)*

$$\mathbf{P}^{G}_{+1}=\frac{\mathbf{G}-(-1)\mathbf{1}}{+1-(-1)}=\frac{1}{2}\begin{pmatrix}1&0&0&1\\0&1&1&0\\0&1&1&0\\1&0&0&1\end{pmatrix} \qquad \mathbf{P}^{G}_{-1}=\frac{\mathbf{G}-(1)\mathbf{1}}{-1-(1)}=\frac{1}{2}\begin{pmatrix}1&0&0&-1\\0&1&-1&0\\0&-1&1&0\\-1&0&0&1\end{pmatrix}$$

$$\left|1_1\right\rangle=\frac{\left|1_1\right)}{\sqrt{2}}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\quad \left|1_2\right\rangle=\frac{\left|1_2\right)}{\sqrt{2}}=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}\quad \left|-1_1\right\rangle=\frac{\left|-1_1\right)}{\sqrt{2}}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}\quad \left|-1_2\right\rangle=\frac{\left|-1_2\right)}{\sqrt{2}}=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$$

Dirac notation for **G**-*split completeness relation* using eigenvectors is the following:

$$\begin{aligned}1=\mathbf{P}^{\mathbf{G}}_{1}+\mathbf{P}^{\mathbf{G}}_{-1}=&\left|1_1\right\rangle\left\langle 1_1\right| + \left|1_2\right\rangle\left\langle 1_2\right| + \left|-1_1\right\rangle\left\langle -1_1\right| + \left|-1_2\right\rangle\left\langle -1_2\right|\\ =&\ \mathbf{P}_{1_1}+\mathbf{P}_{1_2}+\mathbf{P}_{-1_1}+\mathbf{P}_{-1_2}\end{aligned}$$

Each of the original **G** projectors are split in two parts with one ket-bra in each.

$$\mathbf{P}^{\mathbf{G}}_{1}=\mathbf{P}_{1_1}+\mathbf{P}_{1_2}=\frac{1}{2}\begin{pmatrix}1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1\end{pmatrix}+\frac{1}{2}\begin{pmatrix}0&0&0&0\\0&1&1&0\\0&1&1&0\\0&0&0&0\end{pmatrix}=\left|1_1\right\rangle\left\langle 1_1\right|+\left|1_2\right\rangle\left\langle 1_2\right|$$

$$\mathbf{P}^{\mathbf{G}}_{-1}=\mathbf{P}_{-1_1}+\mathbf{P}_{-1_2}=\frac{1}{2}\begin{pmatrix}1&0&0&-1\\0&0&0&0\\0&0&0&0\\-1&0&0&1\end{pmatrix}+\frac{1}{2}\begin{pmatrix}0&0&0&0\\0&1&-1&0\\0&-1&1&0\\0&0&0&0\end{pmatrix}=\left|-1_1\right\rangle\left\langle -1_1\right|+\left|-1_2\right\rangle\left\langle -1_2\right|$$

There are ∞-*ly* many ways to split **G** projectors. Now we let another operator **H** do the final splitting.

Orthonormalization of commuting eigensolutions.

Suppose we have two mutually commuting matrix operators: **GH**=**HG**

the **G=** $\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$ from before, and new operator **H=** $\begin{pmatrix} \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & 2 \\ 2 & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \end{pmatrix}$.

Orthonormalization of commuting eigensolutions.

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(First, it is important to verify that they do, in fact, commute.)

$$\mathbf{GH} = \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}\begin{pmatrix} \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & 2 \\ 2 & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & 2 \\ 2 & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \end{pmatrix}\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix} = \mathbf{HG}$$

Orthonormalization of commuting eigensolutions.

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Problem:

Find an ortho-complete projector set that spectrally resolves both **G** and **H**.

Orthonormalization of commuting eigensolutions.

Suppose we have two mutually commuting matrix operators: **GH**=**HG**

the **G**= $\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$ from before, and new operator **H**= $\begin{pmatrix} \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & 2 \\ 2 & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \end{pmatrix}$.

Problem:

Find an ortho-complete projector set that spectrally resolves both **G** and **H**.

Previous completeness for **G**:

$$\mathbf{1} = \mathbf{P}^{\mathbf{G}}_{+1} + \mathbf{P}^{\mathbf{G}}_{-1}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$= \mathbf{P}^{G}_{+1} = \frac{\mathbf{G}-(-1)\mathbf{1}}{+1-(-1)} \quad + \mathbf{P}^{G}_{-1} = \frac{\mathbf{G}-(1)\mathbf{1}}{-1-(1)}$$

Orthonormalization of commuting eigensolutions.

Suppose we have two mutually commuting matrix operators: **GH**=**HG**

the $\mathbf{G}=\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$ from before, and new operator $\mathbf{H}=\begin{pmatrix} \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & 2 \\ 2 & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \end{pmatrix}$.

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Current completeness for **H**:

$$\mathbf{1} = \mathbf{P}^{\mathbf{H}}_{+2} + \mathbf{P}^{\mathbf{H}}_{-2}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

(Left as an exercise)

(*Preparing for:* Degenerate eigenvalues)

Review: matrix eigenstates ("ownstates) and Idempotent projectors (*Degeneracy case*)
Operator orthonormality, completeness, and spectral decomposition(Degenerate e-values)

Eigensolutions with degenerate eigenvalues (Possible?... or not?)
Secular→ Hamilton-Cayley→Minimal equations
Diagonalizability criterion

Nilpotents and "Bad degeneracy" examples: $\mathbf{B}=\begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix}$, *and:* $\mathbf{N}=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
Applications of Nilpotent operators later on

Idempotents and "Good degeneracy" example: $\mathbf{G}=\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$
Secular equation by minor expansion
Example of minimal equation projection
Orthonormalization of degenerate eigensolutions
Projection $\mathbf{P}_j$*-matrix anatomy (Gramian matrices)*
Gram-Schmidt procedure

Orthonormalization of commuting eigensolutions. Examples: $\mathbf{G}=\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$ *and:* $\mathbf{H}=\begin{pmatrix} \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & 2 \\ 2 & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \end{pmatrix}$
The old "**1=1·1** *trick*"-*Spectral decomposition by projector splitting*
Irreducible projectors and representations (Trace checks)
Minimal equation for projector $\mathbf{P}=\mathbf{P}^2$
How symmetry groups become eigen-solvers

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Problem:

Find an ortho-complete projector set that spectrally resolves <u>both</u> **G** <u>and</u> **H**.

Previous completeness for **G**:

$$\mathbf{1} = \mathbf{P}^{\mathbf{G}}_{+1} + \mathbf{P}^{\mathbf{G}}_{-1}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

Current completeness for **H**:

$$\mathbf{1} = \mathbf{P}^{\mathbf{H}}_{+2} + \mathbf{P}^{\mathbf{H}}_{-2}$$

(Left as an exercise)

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

Solution:

The old "**1=1·1** *trick*"-*Spectral decomposition by projector splitting*

Multiplying **G** and **H** completeness relations

$$\mathbf{1}=\mathbf{1}\cdot\mathbf{1} = \left(\mathbf{P}^{\mathbf{G}}_{+1} + \mathbf{P}^{\mathbf{G}}_{-1}\right)\left(\mathbf{P}^{\mathbf{H}}_{+2} + \mathbf{P}^{\mathbf{H}}_{-2}\right) = \mathbf{1} = \left(\mathbf{P}^{\mathbf{G}}_{+1}\mathbf{P}^{\mathbf{H}}_{+2} + \mathbf{P}^{\mathbf{G}}_{+1}\mathbf{P}^{\mathbf{H}}_{-2} + \mathbf{P}^{\mathbf{G}}_{-1}\mathbf{P}^{\mathbf{H}}_{+2} + \mathbf{P}^{\mathbf{G}}_{-1}\mathbf{P}^{\mathbf{H}}_{-2}\right)$$

Orthonormalization of commuting eigensolutions.

Suppose we have two mutually commuting matrix operators: **GH**=**HG**

the $\mathbf{G}=\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$ from before, and new operator $\mathbf{H}=\begin{pmatrix} \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & 2 \\ 2 & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \end{pmatrix}$.

Problem:

Find an ortho-complete projector set that spectrally resolves both **G** and **H**.

Previous completeness for **G**:

$$\mathbf{1} = \mathbf{P}^{\mathbf{G}}_{+1} + \mathbf{P}^{\mathbf{G}}_{-1}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

Current completeness for **H**:

$$\mathbf{1} = \mathbf{P}^{\mathbf{H}}_{+2} + \mathbf{P}^{\mathbf{H}}_{-2}$$

(Left as an exercise)

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

Solution:

The old "**1=1·1** *trick*"-*Spectral decomposition by projector splitting*

Multiplying **G** and **H** completeness relations gives a set of projectors

$$\mathbf{1}=\mathbf{1}\cdot\mathbf{1} = \left(\mathbf{P}^{\mathbf{G}}_{+1} + \mathbf{P}^{\mathbf{G}}_{-1}\right)\left(\mathbf{P}^{\mathbf{H}}_{+2} + \mathbf{P}^{\mathbf{H}}_{-2}\right) = \mathbf{1} = \left(\mathbf{P}^{\mathbf{G}}_{+1}\mathbf{P}^{\mathbf{H}}_{+2} + \mathbf{P}^{\mathbf{G}}_{+1}\mathbf{P}^{\mathbf{H}}_{-2} + \mathbf{P}^{\mathbf{G}}_{-1}\mathbf{P}^{\mathbf{H}}_{+2} + \mathbf{P}^{\mathbf{G}}_{-1}\mathbf{P}^{\mathbf{H}}_{-2}\right)$$

$$\mathbf{P}^{\mathbf{GH}}_{+1,+2} \equiv \mathbf{P}^{\mathbf{G}}_{+1}\mathbf{P}^{\mathbf{H}}_{+2} =$$

$$\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Orthonormalization of commuting eigensolutions.

Suppose we have two mutually commuting matrix operators: **GH**=**HG**

the $\mathbf{G}=\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$ from before, and new operator $\mathbf{H}=\begin{pmatrix} \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & 2 \\ 2 & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \end{pmatrix}$.

Problem:

Find an ortho-complete projector set that spectrally resolves both **G** and **H**.

Previous completeness for **G**:

$$\mathbf{1} = \mathbf{P}^{\mathbf{G}}_{+1} + \mathbf{P}^{\mathbf{G}}_{-1}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

Current completeness for **H**:

$$\mathbf{1} = \mathbf{P}^{\mathbf{H}}_{+2} + \mathbf{P}^{\mathbf{H}}_{-2}$$

(Left as an exercise)

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

Solution:

The old "**1=1·1** *trick*"-*Spectral decomposition by projector splitting*

Multiplying **G** and **H** completeness relations gives a set of projectors

$$\mathbf{1}=\mathbf{1}\cdot\mathbf{1} = \left(\mathbf{P}^{\mathbf{G}}_{+1} + \mathbf{P}^{\mathbf{G}}_{-1}\right)\left(\mathbf{P}^{\mathbf{H}}_{+2} + \mathbf{P}^{\mathbf{H}}_{-2}\right) = \mathbf{1} = \left(\mathbf{P}^{\mathbf{G}}_{+1}\mathbf{P}^{\mathbf{H}}_{+2} + \mathbf{P}^{\mathbf{G}}_{+1}\mathbf{P}^{\mathbf{H}}_{-2} + \mathbf{P}^{\mathbf{G}}_{-1}\mathbf{P}^{\mathbf{H}}_{+2} + \mathbf{P}^{\mathbf{G}}_{-1}\mathbf{P}^{\mathbf{H}}_{-2}\right)$$

$$\mathbf{P}^{\mathbf{GH}}_{+1,+2} \equiv \mathbf{P}^{\mathbf{G}}_{+1}\mathbf{P}^{\mathbf{H}}_{+2} =$$

$$\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Orthonormalization of commuting eigensolutions.

Suppose we have two mutually commuting matrix operators: **GH**=**HG**

the $\mathbf{G}=\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$ from before, and new operator $\mathbf{H}=\begin{pmatrix} \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & 2 \\ 2 & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \end{pmatrix}$.

Problem:

Find an ortho-complete projector set that spectrally resolves <u>both</u> **G** <u>and</u> **H**.

Previous completeness for **G**:

$$\mathbf{1} = \mathbf{P}^{\mathbf{G}}_{+1} + \mathbf{P}^{\mathbf{G}}_{-1}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

Current completeness for **H**: *(Left as an exercise)*

$$\mathbf{1} = \mathbf{P}^{\mathbf{H}}_{+2} + \mathbf{P}^{\mathbf{H}}_{-2}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

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Current completeness for **H**:

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(Left as an exercise)

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

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Current completeness for **H**: *(Left as an exercise)*

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Current completeness for **H**: *(Left as an exercise)*

$$\mathbf{1} = \mathbf{P}^{\mathbf{H}}_{+2} + \mathbf{P}^{\mathbf{H}}_{-2}$$

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Current completeness for $\mathbf{H}$:

$$\mathbf{1} = \mathbf{P}^{\mathbf{H}}_{+2} + \mathbf{P}^{\mathbf{H}}_{-2}$$

(Left as an exercise)

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

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...and a the same $\mathbf{P}^{\mathbf{GH}}_{g,h}$ projectors spectrally resolve <u>both</u> $\mathbf{G}$ <u>and</u> $\mathbf{H}$.

$$\boxed{\mathbf{G} = (+1)\mathbf{P}^{\mathbf{GH}}_{+1,+2} + (+1)\mathbf{P}^{\mathbf{GH}}_{+1,-2} + (-1)\mathbf{P}^{\mathbf{GH}}_{-1,+2} + (-1)\mathbf{P}^{\mathbf{GH}}_{-1,-2}}$$

$$\boxed{\mathbf{H} = (+2)\mathbf{P}^{\mathbf{GH}}_{+1,+2} + (-2)\mathbf{P}^{\mathbf{GH}}_{+1,-2} + (+2)\mathbf{P}^{\mathbf{GH}}_{-1,+2} + (-2)\mathbf{P}^{\mathbf{GH}}_{-1,-2}}$$

(*Preparing for:* Degenerate eigenvalues)

*Review: matrix **eigen**states ("**own**states) and Idempotent projectors* (*Degeneracy case*)

Operator orthonormality, completeness, and spectral decomposition(Degenerate e-values)

Eigensolutions with degenerate eigenvalues (Possible?... or not?)

Secular→ Hamilton-Cayley→Minimal equations

Diagonalizability criterion

Nilpotents and "Bad degeneracy" examples: $\mathbf{B}=\begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix}$, *and:* $\mathbf{N}=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Applications of Nilpotent operators later on

Idempotents and "Good degeneracy" example: $\mathbf{G}=\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$

Secular equation by minor expansion

Example of minimal equation projection

Orthonormalization of degenerate eigensolutions

Projection $\mathbf{P}_j$*-matrix anatomy (Gramian matrices)*

Gram-Schmidt procedure

Orthonormalization of commuting eigensolutions. Examples: $\mathbf{G}=\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$ *and:* $\mathbf{H}=\begin{pmatrix} \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & 2 \\ 2 & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \end{pmatrix}$

The old "**1=1·1** *trick*"-*Spectral decomposition by projector splitting*

→ *Irreducible projectors and representations (Trace checks)*

Minimal equation for projector $\mathbf{P}=\mathbf{P}^2$

How symmetry groups become eigen-solvers

Another Problem: How do you tell when a Projector $\mathbf{P}_g^{\mathbf{G}}$ or $\mathbf{P}_{g,h}^{\mathbf{GH}}$ is 'splittable' (Correct term is *reducible*.)

$$\mathbf{1} = \mathbf{P}_{+1}^{\mathbf{G}} + \mathbf{P}_{-1}^{\mathbf{G}} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

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$$\mathbf{G}\mathbf{P}_{g,h}^{\mathbf{GH}} = \mathbf{G}\mathbf{P}_g^{\mathbf{G}}\mathbf{P}_h^{\mathbf{H}} = \varepsilon_g^{\mathbf{G}}\mathbf{P}_{g,h}^{\mathbf{GH}}$$

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Another Problem: How do you tell when a Projector $\mathbf{P}^{\mathbf{G}}_{g}$ or $\mathbf{P}^{\mathbf{GH}}_{g,h}$ is 'splittable' (Correct term is *reducible*.)

Solution: It's all in the matrix Trace = sum of its diagonal elements.

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$$\mathbf{1} = \mathbf{P}^{\mathbf{H}}_{+2} + \mathbf{P}^{\mathbf{H}}_{-2} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

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$$\mathbf{1}=\mathbf{1}\cdot\mathbf{1}=\left(\mathbf{P}^{\mathbf{G}}_{+1}+\mathbf{P}^{\mathbf{G}}_{-1}\right)\left(\mathbf{P}^{\mathbf{H}}_{+2}+\mathbf{P}^{\mathbf{H}}_{-2}\right)=\mathbf{1}=\left(\mathbf{P}^{\mathbf{G}}_{+1}\mathbf{P}^{\mathbf{H}}_{+2}+\mathbf{P}^{\mathbf{G}}_{+1}\mathbf{P}^{\mathbf{H}}_{-2}+\mathbf{P}^{\mathbf{G}}_{-1}\mathbf{P}^{\mathbf{H}}_{+2}+\mathbf{P}^{\mathbf{G}}_{-1}\mathbf{P}^{\mathbf{H}}_{-2}\right)$$

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(*Preparing for:* Degenerate eigenvalues)

Review: matrix eigenstates ("ownstates) and Idempotent projectors (*Degeneracy case*)

Operator orthonormality, completeness, and spectral decomposition(Degenerate e-values)

Eigensolutions with degenerate eigenvalues (Possible?... or not?)

Secular→ Hamilton-Cayley→Minimal equations

Diagonalizability criterion

Nilpotents and "Bad degeneracy" examples: $\mathbf{B}=\begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix}$, *and:* $\mathbf{N}=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Applications of Nilpotent operators later on

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→ *Minimal equation for projector* $\mathbf{P}=\mathbf{P}^2$

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Minimal equation for an idempotent projector is: $\mathbf{P}^2=\mathbf{P}$ or: $\mathbf{P}^2-\mathbf{P} = (\mathbf{P}-0\cdot\mathbf{1})(\mathbf{P}-1\cdot\mathbf{1}) = \mathbf{0}$
So projector eigenvalues are limited to repeated *0*'s and *1*'s. Trace counts the latter.

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Suppose you need to diagonalize a complicated operator **K** and knew that **K** commutes with some other operators **G** and **H** for which irreducible projectors are more easily found.

$\mathbf{KG} = \mathbf{GK}$ or $\mathbf{G}^\dagger\mathbf{KG} = \mathbf{K}$ or $\mathbf{GKG}^\dagger = \mathbf{K}$

$\mathbf{KH} = \mathbf{HK}$ or $\mathbf{H}^\dagger\mathbf{KH} = \mathbf{K}$ or $\mathbf{HKH}^\dagger = \mathbf{K}$

(Here assuming *unitary* $\mathbf{G}^\dagger=\mathbf{G}^{-1}$ and $\mathbf{H}^\dagger=\mathbf{H}^{-1}$.)

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This means **K** is *invariant* to the transformation by **G** and **H**
and all their products $\mathbf{GH}$, $\mathbf{GH}^2$, $\mathbf{G}^2\mathbf{H}$,.. *etc*. and all their inverses $\mathbf{G}^\dagger$,$\mathbf{H}^\dagger$,.. etc.

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The group $\mathscr{G}_\mathbf{K} = \{\mathbf{1}, \mathbf{G}, \mathbf{H}, .. \}$ so formed by such operators is called a *symmetry group* for **K**.

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and all their products $\mathbf{GH}$, $\mathbf{GH}^2$, $\mathbf{G}^2\mathbf{H}$,.. *etc*. and all their inverses $\mathbf{G}^\dagger$,$\mathbf{H}^\dagger$,.. etc.

The group $\mathscr{G}_{\mathbf{K}} = \{\mathbf{1}, \mathbf{G}, \mathbf{H},.. \}$ so formed by such operators is called a *symmetry group* for **K**.

In certain ideal cases a **K**-matrix $\langle\mathbf{K}\rangle$ is a linear combination of matrices $\langle\mathbf{1}\rangle$,$\langle\mathbf{G}\rangle$,$\langle\mathbf{H}\rangle$,... from $\mathscr{G}_{\mathbf{K}}$.
Then spectral resolution of $\{\langle\mathbf{1}\rangle,\langle\mathbf{G}\rangle,\langle\mathbf{H}\rangle,.. \}$ also resolves $\langle\mathbf{K}\rangle$.

How symmetry groups become eigen-solvers

Suppose you need to diagonalize a complicated operator **K** and knew that **K** commutes with some other operators **G** and **H** for which irreducible projectors are more easily found.

$$\mathbf{KG} = \mathbf{GK} \text{ or } \mathbf{G}^\dagger\mathbf{KG} = \mathbf{K} \text{ or } \mathbf{GKG}^\dagger = \mathbf{K}$$

$$\mathbf{KH} = \mathbf{HK} \text{ or } \mathbf{H}^\dagger\mathbf{KH} = \mathbf{K} \text{ or } \mathbf{HKH}^\dagger = \mathbf{K}$$

(Here assuming *unitary* $\mathbf{G}^\dagger=\mathbf{G}^{-1}$ and $\mathbf{H}^\dagger=\mathbf{H}^{-1}$.)

This means **K** is *invariant* to the transformation by **G** and **H** and all their products $\mathbf{GH}$, $\mathbf{GH}^2$, $\mathbf{G}^2\mathbf{H}$,.. *etc*. and all their inverses $\mathbf{G}^\dagger$,$\mathbf{H}^\dagger$,.. etc.

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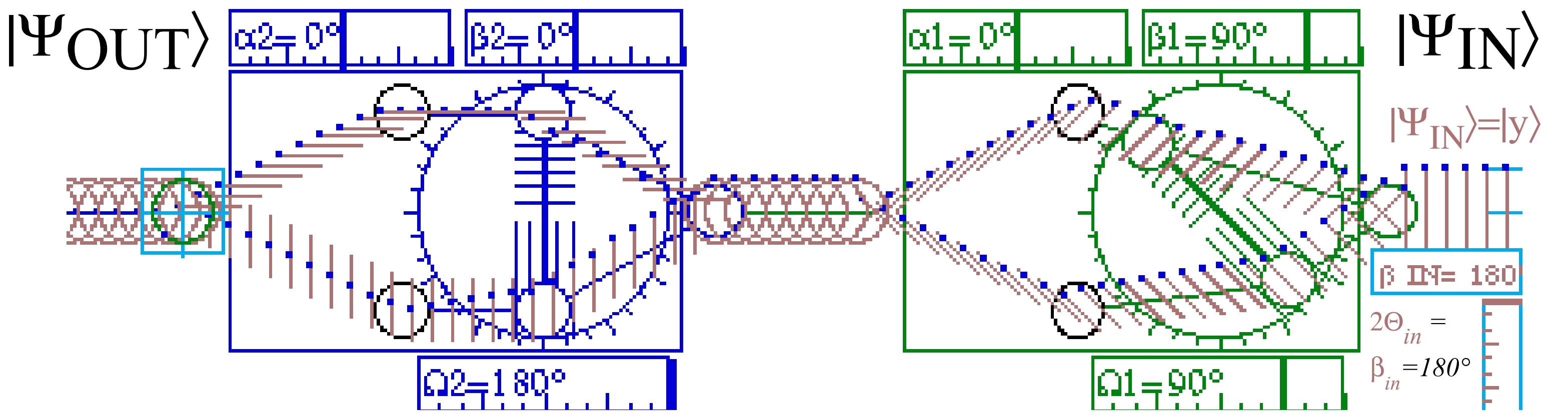
Then spectral resolution of $\{\langle\mathbf{1}\rangle,\langle\mathbf{G}\rangle,\langle\mathbf{H}\rangle,..\}$ also resolves $\langle\mathbf{K}\rangle$.

We will study ideal cases first. More general cases are built from these.

Eigensolutions for active analyzers

Matrix products and eigensolutions for active analyzers

Consider a $45°$ tilted ($\theta_1=\beta_1/2=\pi/4$ or $\beta_1=90°$) analyzer followed by a untilted ($\beta_2=0$) analyzer.
Active analyzers have both paths open and a phase shift $e^{-i\Omega}$ between each path.
Here the first analyzer has $\Omega_1=90°$. The second has $\Omega_2=180°$.



The transfer matrix for each analyzer is a sum of projection operators for each open path multiplied by the phase factor that is active at that path. Apply phase factor $e^{-i\Omega_1}=e^{-i\pi/2}$ to top path in the first analyzer and the factor $e^{-i\Omega_2}=e^{-i\pi}$ to the top path in the second analyzer.

$$T(2)=e^{-i\pi}\left|x\right\rangle\left\langle x\right|+\left|y\right\rangle\left\langle y\right|=\begin{pmatrix} e^{-i\pi} & 0 \\ 0 & 1\end{pmatrix}$$

$$T(1)=e^{-i\pi/2}\left|x'\right\rangle\left\langle x'\right|+\left|y'\right\rangle\left\langle y'\right|=e^{-i\pi/2}\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}+\begin{pmatrix}\frac{1}{2} & \frac{-1}{2}\\ \frac{-1}{2} & \frac{1}{2}\end{pmatrix}=\begin{pmatrix}\frac{1-i}{2} & \frac{-1-i}{2}\\ \frac{-1-i}{2} & \frac{1-i}{2}\end{pmatrix}$$

The matrix product $T(total)=T(2)T(1)$ relates input states $|\Psi_{IN}\rangle$ to output states: $|\Psi_{OUT}\rangle=T(total)|\Psi_{IN}\rangle$

$$T(total)=T(2)T(1)=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1-i}{2} & \frac{-1-i}{2}\\ \frac{-1-i}{2} & \frac{1-i}{2}\end{pmatrix}=\begin{pmatrix}\frac{-1+i}{2} & \frac{1+i}{2}\\ \frac{-1-i}{2} & \frac{1-i}{2}\end{pmatrix}=e^{-i\pi/4}\begin{pmatrix}\frac{-1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\sim\begin{pmatrix}\frac{-1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$$

We drop the overall phase $e^{-i\pi/4}$ since it is unobservable. $T(total)$ yields two eigenvalues and projectors.

$$\lambda^2-0\lambda-1=0,\text{ or: }\lambda=+1,\ -1\text{, gives projectors}\quad P_{+1}=\frac{\begin{pmatrix}\frac{-1}{\sqrt{2}}+1 & \frac{i}{\sqrt{2}}\\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}}+1\end{pmatrix}}{1-(-1)}=\frac{\begin{pmatrix}-1+\sqrt{2} & i\\ -i & 1+\sqrt{2}\end{pmatrix}}{2\sqrt{2}},\quad P_{-1}=\frac{\begin{pmatrix}1+\sqrt{2} & -i\\ i & -1+\sqrt{2}\end{pmatrix}}{2\sqrt{2}}$$