Group Theory in Quantum Mechanics
Lecture 30 *(4.30.15)*
*Symmetry product analysis U(m)*Sn tensors*

(Int.J.Mol.Sci, 14, 714(2013) p.755-774 , QTCA Unit 7 Ch. 23-26)
(PSDS - Ch. 5, 7)

Review : *2-D* $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(2) representations*

Spin-spin $(1/2)^2$ *product states: Hydrogen hyperfine structure*
Kronecker product states and operators
Spin-spin interaction reduces symmetry $U(2)^{proton}\times U(2)^{electron}$ *to* $U(2)^{e+p}$
Clebsch-Gordan Coefficients
Hydrogen hyperfine structure: Fermi-contact interaction
plus B-field gives avoided crossing
Higher-J product states
$(J=1)\otimes(J=1)=2\oplus 1\oplus 0$ *case*
General U(2) case
Multi-spin $(1/2)^N$ *product states*
Magic squares - Intro to Young Tableaus
Tensor operators for spin-1/2 states: Outer products give Hamilton-Pauli-spinors
Tensor operators for spin-1 states: U(3) generalization of Pauli spinors

4.30.15
<u>Multi-Lecture Smorgasbord</u>
Lecture 27 p.5,7,8 Review&more
Asymmetric rotor RES&clusters
Lecture 27 p.36-41 Review&more
O-Symmetry SF_6 *RES&clusters*
Lecture 28 p.36-51
Gyro-rotor RES&levels
Lecture 28 p.54-59
Gyro-rotor REES&levels
Lecture 29 p.36-64 SiF_4 *and* SF_6
Spin Tableau &hyperfine effects
Lecture 29 p.67-80 C_{60} *"Buckyball"*
Superfine &hyperfine effects
Lecture 30 Tensors and CG coeff.??

2^k-pole expansion of an N-by-N matrix $\mathbf{H}$

2-by-2 case: $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

$$= \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{A+D}{2}\,\mathbf{1} + B\,\boldsymbol{\sigma}_x + C\,\boldsymbol{\sigma}_y + \frac{A-D}{2}\,\boldsymbol{\sigma}_z$$

$$= \frac{A+D}{2}\,\mathbf{T}_0 + (B-iC)\,\underset{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}{\mathbf{T}_1} + (B+iC)\,\underset{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}{\mathbf{T}_{-1}} + \frac{A-D}{2}\,\mathbf{T}_0$$

U(2) generators *(spin J=1/2)*

$$\mathbf{u}^1_{+1}=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \mathbf{u}^1_{0}=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{\sqrt{2}} \quad \mathbf{u}^1_{-1}=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

rank-1 (vector)

$$\mathbf{u}^0_{0}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}$$

rank-0 (scalar)

3-by-3 case: $\mathbf{H}=\begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix}$

U(3) generators *(spin J=1)*

$$\mathbf{u}^2_{+2}=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathbf{u}^2_{+1}=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\frac{1}{\sqrt{2}} \quad \mathbf{u}^2_{0}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\frac{1}{\sqrt{6}} \quad \mathbf{u}^2_{-1}=\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}} \quad \mathbf{u}^2_{-2}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

rank-2 (tensor)

$$\mathbf{u}^1_{+1}=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\frac{1}{\sqrt{2}} \quad \mathbf{u}^1_{0}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\frac{1}{\sqrt{2}} \quad \mathbf{u}^1_{-1}=\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}$$

rank-1 (vector)

$$\mathbf{u}^0_{0}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\frac{1}{\sqrt{3}}$$

rank-0 (scalar)

Mutually commuting diagonal operators

Wigner-Clebsch-Gordan expressions for Tensor $\left\langle \mathbf{T}^k_q \right\rangle$

$$\left\langle {J' \atop M'} \middle| \mathbf{T}^k_q \middle| {J \atop M} \right\rangle = \begin{pmatrix} J' & k & J \\ M' & q & -M \end{pmatrix} (J' \| k \| J) = C^{k\,J\,J'}_{q\,M\,M'} \langle J' \| k \| J \rangle$$

Spin-spin $(1/2)^2$ product states: Hydrogen hyperfine structure

electron-proton spin-spin interaction gives a simple example of *hyperfine* spectra

Ket-kets for spin-up and spin-dn states and column matrix representations..

$$\left|\uparrow\right\rangle\left|\uparrow\right\rangle = \left|\begin{matrix}\frac{1}{2}\\ \frac{1}{2}\end{matrix}\right\rangle^{proton}\left|\begin{matrix}\frac{1}{2}\\ \frac{1}{2}\end{matrix}\right\rangle^{electron}, \left|\uparrow\right\rangle\left|\downarrow\right\rangle = \left|\begin{matrix}\frac{1}{2}\\ \frac{1}{2}\end{matrix}\right\rangle^{proton}\left|\begin{matrix}\frac{1}{2}\\ -\frac{1}{2}\end{matrix}\right\rangle^{electron}, \left|\downarrow\right\rangle\left|\uparrow\right\rangle = \left|\begin{matrix}\frac{1}{2}\\ -\frac{1}{2}\end{matrix}\right\rangle^{proton}\left|\begin{matrix}\frac{1}{2}\\ \frac{1}{2}\end{matrix}\right\rangle^{electron}, \left|\downarrow\right\rangle\left|\downarrow\right\rangle = \left|\begin{matrix}\frac{1}{2}\\ -\frac{1}{2}\end{matrix}\right\rangle^{proton}\left|\begin{matrix}\frac{1}{2}\\ -\frac{1}{2}\end{matrix}\right\rangle^{electron}$$

$$\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

Same spin-1/2 representation applies to either proton or electron kets.

$$D^{1/2}(\alpha\beta\gamma) = \begin{pmatrix} D^{1/2}_{+1/2,+1/2} & D^{1/2}_{+1/2,-1/2} \\ D^{1/2}_{-1/2,+1/2} & D^{1/2}_{-1/2,-1/2} \end{pmatrix} = \begin{pmatrix} e^{\frac{-i(\alpha+\gamma)}{2}}\cos\frac{\beta}{2} & -e^{\frac{-i(\alpha-\gamma)}{2}}\sin\frac{\beta}{2} \\ e^{\frac{i(\alpha-\gamma)}{2}}\sin\frac{\beta}{2} & e^{\frac{i(\alpha+\gamma)}{2}}\cos\frac{\beta}{2} \end{pmatrix}$$

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Kronecker product $D^{\frac{1}{2}}\otimes D^{\frac{1}{2}}$

Applies to *outer product symmetry* $U(2)^{proton}\times U(2)^{electron}$ for NO interaction.

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is allowed!

Spin-spin interaction reduces symmetry $U(2)^{proton}\times U(2)^{electron}$ to $U(2)^{e+p}$

$$\begin{pmatrix}1&0&0&0\\0&\frac{1}{\sqrt2}&\frac{1}{\sqrt2}&0\\0&0&0&1\\0&\frac{1}{\sqrt2}&\frac{-1}{\sqrt2}&0\end{pmatrix}\cdot\begin{pmatrix}\cos^2\frac{\beta}{2} & -\sin\frac{\beta}{2}\cos\frac{\beta}{2} & -\sin\frac{\beta}{2}\cos\frac{\beta}{2} & \sin^2\frac{\beta}{2}\\ \sin\frac{\beta}{2}\cos\frac{\beta}{2} & \cos^2\frac{\beta}{2} & -\sin^2\frac{\beta}{2} & -\sin\frac{\beta}{2}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\cos\frac{\beta}{2} & -\sin^2\frac{\beta}{2} & \cos^2\frac{\beta}{2} & -\sin\frac{\beta}{2}\cos\frac{\beta}{2}\\ \sin^2\frac{\beta}{2} & \sin\frac{\beta}{2}\cos\frac{\beta}{2} & \sin\frac{\beta}{2}\cos\frac{\beta}{2} & \cos^2\frac{\beta}{2}\end{pmatrix}\cdot\begin{pmatrix}1&0&0&0\\0&\frac{1}{\sqrt2}&0&\frac{1}{\sqrt2}\\0&\frac{1}{\sqrt2}&0&\frac{-1}{\sqrt2}\\0&0&1&0\end{pmatrix}=\begin{pmatrix}\sin^2\frac{\beta}{2} & \frac{-\sin\beta}{\sqrt2} & \sin^2\frac{\beta}{2} & 0\\ \frac{\sin\beta}{\sqrt2} & \cos\beta & \frac{-\sin\beta}{\sqrt2} & 0\\ \sin^2\frac{\beta}{2} & \frac{\sin\beta}{\sqrt2} & \cos^2\frac{\beta}{2} & 0\\ 0&0&0&1\end{pmatrix}=\left(\begin{array}{c|c}D^{J=1}_{(0\beta0)} & \\ \hline & D^{J=0}\end{array}\right)$$

...and “irreducible” becomes “reducible”...

Spin-spin interaction reduces symmetry $U(2)^{proton}\times U(2)^{electron}$ to $U(2)^{e+p}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix} \cdot \begin{pmatrix} \cos^2\frac{\beta}{2} & -\sin\frac{\beta}{2}\cos\frac{\beta}{2} & -\sin\frac{\beta}{2}\cos\frac{\beta}{2} & \sin^2\frac{\beta}{2} \\ \sin\frac{\beta}{2}\cos\frac{\beta}{2} & \cos^2\frac{\beta}{2} & -\sin^2\frac{\beta}{2} & -\sin\frac{\beta}{2}\cos\frac{\beta}{2} \\ \sin\frac{\beta}{2}\cos\frac{\beta}{2} & -\sin^2\frac{\beta}{2} & \cos^2\frac{\beta}{2} & -\sin\frac{\beta}{2}\cos\frac{\beta}{2} \\ \sin^2\frac{\beta}{2} & \sin\frac{\beta}{2}\cos\frac{\beta}{2} & \sin\frac{\beta}{2}\cos\frac{\beta}{2} & \cos^2\frac{\beta}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \sin^2\frac{\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \sin^2\frac{\beta}{2} & 0 \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} & 0 \\ \sin^2\frac{\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \cos^2\frac{\beta}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} D^{J=1}_{(0\beta 0)} & \\ & D^{J=0} \end{pmatrix}$$

Clebsch-Gordan coefficients(CGC)

$$C^{\frac{1}{2}\ \frac{1}{2}\ J}_{m_p\ m_e\ M} \equiv \left\langle \begin{matrix} \frac{1}{2} & \frac{1}{2} \\ m_p & m_e \end{matrix} \middle| \begin{matrix} J \\ M \end{matrix} \right\rangle$$

reduce $D^{1/2}\otimes D^{1/2}$ to $D^{1}\oplus D^{0}$

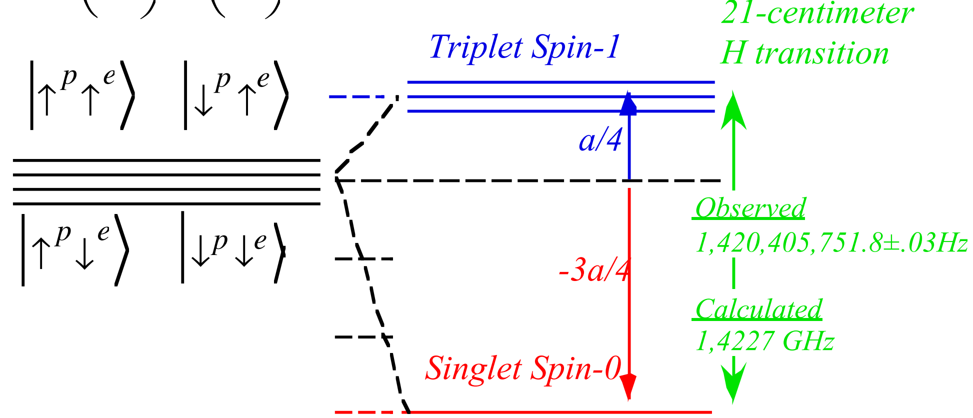
$\frac{1}{2}\otimes\frac{1}{2}$	$J=1$, $M=1$	1, 0	1, -1	0, 0
$\frac{1}{2},\frac{1}{2}$	1	0	0	0
$\frac{1}{2},\frac{-1}{2}$	0	$\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$
$\frac{-1}{2},\frac{1}{2}$	0	$\frac{1}{\sqrt{2}}$	0	$\frac{-1}{\sqrt{2}}$
$\frac{-1}{2},\frac{-1}{2}$	0	0	1	0

$$= \left\langle C^{\frac{1}{2}\ \frac{1}{2}}_{m_p\ m_e} \middle| \begin{matrix} J \\ M \end{matrix} \right\rangle$$

$$\sum_{m_1 m_1'}\sum_{m_2 m_2'} C^{\frac{1}{2}\ \frac{1}{2}\ J}_{m_1\ m_1'\ M} D^{\frac{1}{2}}_{m_1 m_2} D^{\frac{1}{2}}_{m_1' m_2'} C^{\frac{1}{2}\ \frac{1}{2}\ J'}_{m_2\ m_2'\ M'} = \delta^{JJ'} D^{J}_{M\,M'}$$

$$\left(\frac{1}{2}\right)\otimes\left(\frac{1}{2}\right) = (1)\oplus(0)$$

$$\left| \begin{matrix} J \\ M \end{matrix}^{(1/2\otimes 1/2)} \right\rangle = \sum_{m_1, m_2} C^{1/2\ 1/2\ J}_{m_1\ m_2\ M} \left| \begin{matrix} 1/2 \\ m_1 \end{matrix} \right\rangle \left| \begin{matrix} 1/2 \\ m_2 \end{matrix} \right\rangle$$



$$\left|\begin{matrix} 1 \\ +1 \end{matrix}\right\rangle = \left|\uparrow^p\uparrow^e\right\rangle$$

$$\left|\begin{matrix} 1 \\ 0 \end{matrix}\right\rangle = \left(\left|\uparrow^p\downarrow^e\right\rangle + \left|\downarrow^p\uparrow^e\right\rangle\right)/\sqrt{2}$$

$$\left|\begin{matrix} 1 \\ -1 \end{matrix}\right\rangle = \left|\downarrow^p\downarrow^e\right\rangle$$

$$\left|\begin{matrix} 0 \\ 0 \end{matrix}\right\rangle = \left(\left|\uparrow^p\downarrow^e\right\rangle - \left|\downarrow^p\uparrow^e\right\rangle\right)/\sqrt{2}$$

Hydrogen hyperfine structure: Fermi-contact interaction

Racah's trick for energy eigenvalues

$$a_{ep}\mathbf{J}^{proton}\bullet\mathbf{J}^{electron}=\frac{a_{ep}}{2}\left[\left(\mathbf{J}^{proton}+\mathbf{J}^{electron}\right)^2-\left(\mathbf{J}^{proton}\right)^2-\left(\mathbf{J}^{electron}\right)^2\right]$$

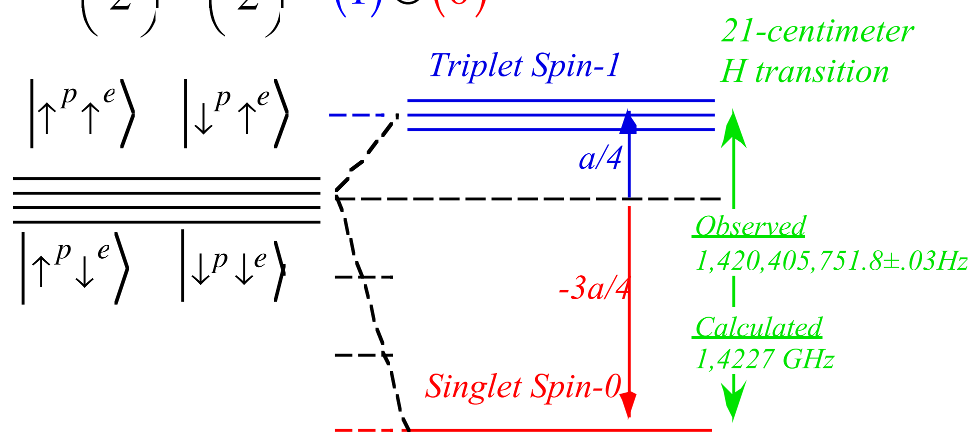
$$=\frac{a_{ep}}{2}\left[\left(\mathbf{J}^{total}\right)^2-\left(\mathbf{J}^{proton}\right)^2-\left(\mathbf{J}^{electron}\right)^2\right].$$

$$\left\langle \begin{matrix} J \\ M \end{matrix}^{(1/2\otimes 1/2)}\right| H_{contact}\left| \begin{matrix} J \\ M \end{matrix}^{(1/2\otimes 1/2)}\right\rangle=\frac{a_{ep}}{2}\left[J(J+1)-\tfrac{1}{2}\left(\tfrac{1}{2}+1\right)-\tfrac{1}{2}\left(\tfrac{1}{2}+1\right)\right]$$

$$=\begin{cases} a_{ep}/4 \; \textit{for the} \; (J=1) & \text{triplet state,} \\ -3a_{ep}/4 \; \textit{for the} \; (J=0) & \text{singlet state.} \end{cases}$$

$$\left(\frac{1}{2}\right)\otimes\left(\frac{1}{2}\right)=(1)\oplus(0)$$

$$\left| \begin{matrix} J \\ M \end{matrix}^{(1/2\otimes 1/2)}\right\rangle=\sum_{m_1,m_2} C^{1/2\;\;1/2\;\;J}_{m_1\;\;m_2\;\;M}\left|\begin{matrix}1/2\\m_1\end{matrix}\right\rangle\left|\begin{matrix}1/2\\m_2\end{matrix}\right\rangle$$



$$\left|\begin{matrix}1\\+1\end{matrix}\right\rangle=\left|\uparrow^p\uparrow^e\right\rangle$$

$$\left|\begin{matrix}1\\0\end{matrix}\right\rangle=\left(\left|\uparrow^p\downarrow^e\right\rangle+\left|\downarrow^p\uparrow^e\right\rangle\right)/\sqrt{2}$$

$$\left|\begin{matrix}1\\-1\end{matrix}\right\rangle=\left|\downarrow^p\downarrow^e\right\rangle$$

$$\left|\begin{matrix}0\\0\end{matrix}\right\rangle=\left(\left|\uparrow^p\downarrow^e\right\rangle-\left|\downarrow^p\uparrow^e\right\rangle\right)/\sqrt{2}$$

Hydrogen hyperfine structure: Fermi-contact interaction + B-field

$$H_{1s-B-field} = -a_p B_z J_z^{proton} + a_e B_z J_z^{electron} + a_{ep} \mathbf{J}^{proton} \bullet \mathbf{J}^{electron}$$

	$g-factor$	$Bohr-magneton$	$gyromagnetic\ factor$
$electron$	$g_e = 2.0023$	$\mu_e = \frac{e\hbar}{2m_e} = 9.27401\cdot 10^{-24}\frac{J}{T}$	$a_e = g_e\mu_e = 1.8570\cdot 10^{-23}\frac{J}{T}$
$proton$	$g_p = 5.585$	$\mu_p = \frac{e\hbar}{2m_p} = 5.05078\cdot 10^{-27}\frac{J}{T}$	$a_p = g_p\mu_p = 2.8209\cdot 10^{-26}\frac{J}{T}$

$Fermi-contact\ factor$
$a_{ep} = \mu_0\frac{2}{3}\frac{1}{\pi a_0^3}a_e a_p = 9.427\cdot 10^{-25}J$
$\mu_0\frac{2}{3}\frac{1}{\pi a_0^3}\frac{a_e a_p}{h} = 1.4227\cdot 10^9 Hz$
$\mu_0\frac{2}{3}\frac{1}{\pi a_0^3}\frac{a_e a_p}{hc} = 4.746 m^{-1} = \frac{1}{21.1}cm^{-1}$

$Magnetic\ constant:\ \mu_0 / 4\pi = 10^{-7} N/A^2$

$$H_{1s-B-field} = -a_p B_z \mathbf{J}_z^{proton} + a_e B_z \mathbf{J}_z^{electron} + a_{ep} \mathbf{J}^{proton} \bullet \mathbf{J}^{electron}$$

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$\mu_0 \frac{2}{3}\frac{1}{\pi a_0^3}\frac{a_e a_p}{hc} = 4.746 m^{-1} = \frac{1}{21.1}cm^{-1}$

$$\left\langle -a_p B_z \mathbf{J}_z^{proton} + a_e B_z \mathbf{J}_z^{electron} \right\rangle =$$

	$\left\|\uparrow^p\uparrow^e\right\rangle$	$\left\|\uparrow^p\downarrow^e\right\rangle$	$\left\|\downarrow^p\uparrow^e\right\rangle$	$\left\|\downarrow^p\downarrow^e\right\rangle$
$\left\langle\uparrow^p\uparrow^e\right\|$	$\frac{1}{2}(a_e - a_p)B_z$	$\cdot$	$\cdot$	$\cdot$
$\left\langle\uparrow^p\downarrow^e\right\|$	$\cdot$	$\frac{-1}{2}(a_e + a_p)B_z$	0	$\cdot$
$\left\langle\downarrow^p\uparrow^e\right\|$	$\cdot$	0	$\frac{1}{2}(a_e + a_p)B_z$	$\cdot$
$\left\langle\downarrow^p\downarrow^e\right\|$	$\cdot$	$\cdot$	$\cdot$	$\frac{-1}{2}(a_e - a_p)B_z$

$$\left\langle a_{ep} \mathbf{J}^{proton} \bullet \mathbf{J}^{electron} \right\rangle =$$

	$\left\|{}^1_1\right\rangle$	$\left\|{}^1_0\right\rangle$	$\left\|{}^0_0\right\rangle$	$\left\|{}^{1}_{-1}\right\rangle$
$\left\langle{}^1_1\right\|$	$\frac{a_{ep}}{4}$	$\cdot$	$\cdot$	$\cdot$
$\left\langle{}^1_0\right\|$	$\cdot$	$\frac{a_{ep}}{4}$	0	$\cdot$
$\left\langle{}^0_0\right\|$	$\cdot$	0	$\frac{-3a_{ep}}{4}$	$\cdot$
$\left\langle{}^{1}_{-1}\right\|$	$\cdot$	$\cdot$	$\cdot$	$\frac{a_{ep}}{4}$

$$H_{1s-B-field} = -a_p B_z J_z^{proton} + a_e B_z J_z^{electron} + a_{ep} \mathbf{J}^{proton} \bullet \mathbf{J}^{electron}$$

	$g-factor$	$Bohr-magneton$	$gyromagnetic\ factor$
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$\mu_0 \frac{2}{3} \frac{1}{\pi a_0^3} \frac{a_e a_p}{hc} = 4.746 m^{-1}$
$= \frac{1}{21.1} cm^{-1}$

$\frac{1}{2} \otimes \frac{1}{2}$	$J=1$, $M=1$	1, 0	1, -1	0, 0
$\frac{1}{2}, \frac{1}{2}$	1	0	0	0
$\frac{1}{2}, \frac{-1}{2}$	0	$\frac{1}{\sqrt{2}}$		$\frac{1}{\sqrt{2}}$
$\frac{-1}{2}, \frac{1}{2}$	0	$\frac{1}{\sqrt{2}}$		$\frac{-1}{\sqrt{2}}$
$\frac{-1}{2}, \frac{-1}{2}$	0	0	1	0

$$= \left\langle C^{\frac{1}{2}\ \frac{1}{2}}_{m_p\ m_e} \middle| {}^{J}_{M} \right\rangle$$

$$\left\langle -a_p B_z J_z^{proton} + a_e B_z J_z^{electron} \right\rangle =$$

	$\lvert \uparrow^p \uparrow^e \rangle$	$\lvert \uparrow^p \downarrow^e \rangle$	$\lvert \downarrow^p \uparrow^e \rangle$	$\lvert \downarrow^p \downarrow^e \rangle$
$\langle \uparrow^p \uparrow^e \rvert$	$\frac{1}{2}(a_e - a_p) B_z$	.	.	.
$\langle \uparrow^p \downarrow^e \rvert$	.	$\frac{-1}{2}(a_e + a_p) B_z$	0	.
$\langle \downarrow^p \uparrow^e \rvert$	.	0	$\frac{1}{2}(a_e + a_p) B_z$	.
$\langle \downarrow^p \downarrow^e \rvert$	.	.	.	$\frac{-1}{2}(a_e - a_p) B_z$

$$\left\langle a_{ep} \mathbf{J}^{proton} \bullet \mathbf{J}^{electron} \right\rangle =$$

	$\lvert {}^{1}_{1} \rangle$	$\lvert {}^{1}_{0} \rangle$	$\lvert {}^{0}_{0} \rangle$	$\lvert {}^{1}_{-1} \rangle$
$\langle {}^{1}_{1} \rvert$	$\frac{a_{ep}}{4}$	.	.	.
$\langle {}^{1}_{0} \rvert$	.	$\frac{a_{ep}}{4}$	0	.
$\langle {}^{0}_{0} \rvert$	.	0	$\frac{-3a_{ep}}{4}$	.
$\langle {}^{1}_{-1} \rvert$	.	.	.	$\frac{a_{ep}}{4}$

$$H_{1s-B-field} = -a_p B_z J_z^{proton} + a_e B_z J_z^{electron} + a_{ep}\mathbf{J}^{proton} \bullet \mathbf{J}^{electron}$$

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Magnetic constant : $\mu_0 / 4\pi = 10^{-7} N/A^2$

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$\mu_0 \frac{2}{3}\frac{1}{\pi a_0^3}\frac{a_e a_p}{hc} = 4.746 m^{-1}$
$= \frac{1}{21.1} cm^{-1}$

$\frac{1}{2}\otimes\frac{1}{2}$	$J=1$, $M=1$	1, 0	1, -1	0, 0
$\frac{1}{2},\frac{1}{2}$	1	0	0	0
$\frac{1}{2},\frac{-1}{2}$	0	$\frac{1}{\sqrt{2}}$		$\frac{1}{\sqrt{2}}$
$\frac{-1}{2},\frac{1}{2}$	0	$\frac{1}{\sqrt{2}}$		$\frac{-1}{\sqrt{2}}$
$\frac{-1}{2},\frac{-1}{2}$	0	0	1	0

$$= \left\langle C^{\frac{1}{2}\ \frac{1}{2}}_{m_p\ m_e} \middle| {J \atop M} \right\rangle$$

$$\left\langle -a_p B_z J_z^{proton} + a_e B_z J_z^{electron} \right\rangle =$$

	$\lvert\uparrow^p\uparrow^e\rangle$	$\lvert\uparrow^p\downarrow^e\rangle$	$\lvert\downarrow^p\uparrow^e\rangle$	$\lvert\downarrow^p\downarrow^e\rangle$
$\langle\uparrow^p\uparrow^e\rvert$	$\frac{1}{2}(a_e - a_p)B_z$	.	.	.
$\langle\uparrow^p\downarrow^e\rvert$	.	$\frac{-1}{2}(a_e + a_p)B_z$	0	.
$\langle\downarrow^p\uparrow^e\rvert$	.	0	$\frac{1}{2}(a_e + a_p)B_z$	.
$\langle\downarrow^p\downarrow^e\rvert$	.	.	.	$\frac{-1}{2}(a_e - a_p)B_z$

$$\left\langle a_{ep}\mathbf{J}^{proton} \bullet \mathbf{J}^{electron} \right\rangle =$$

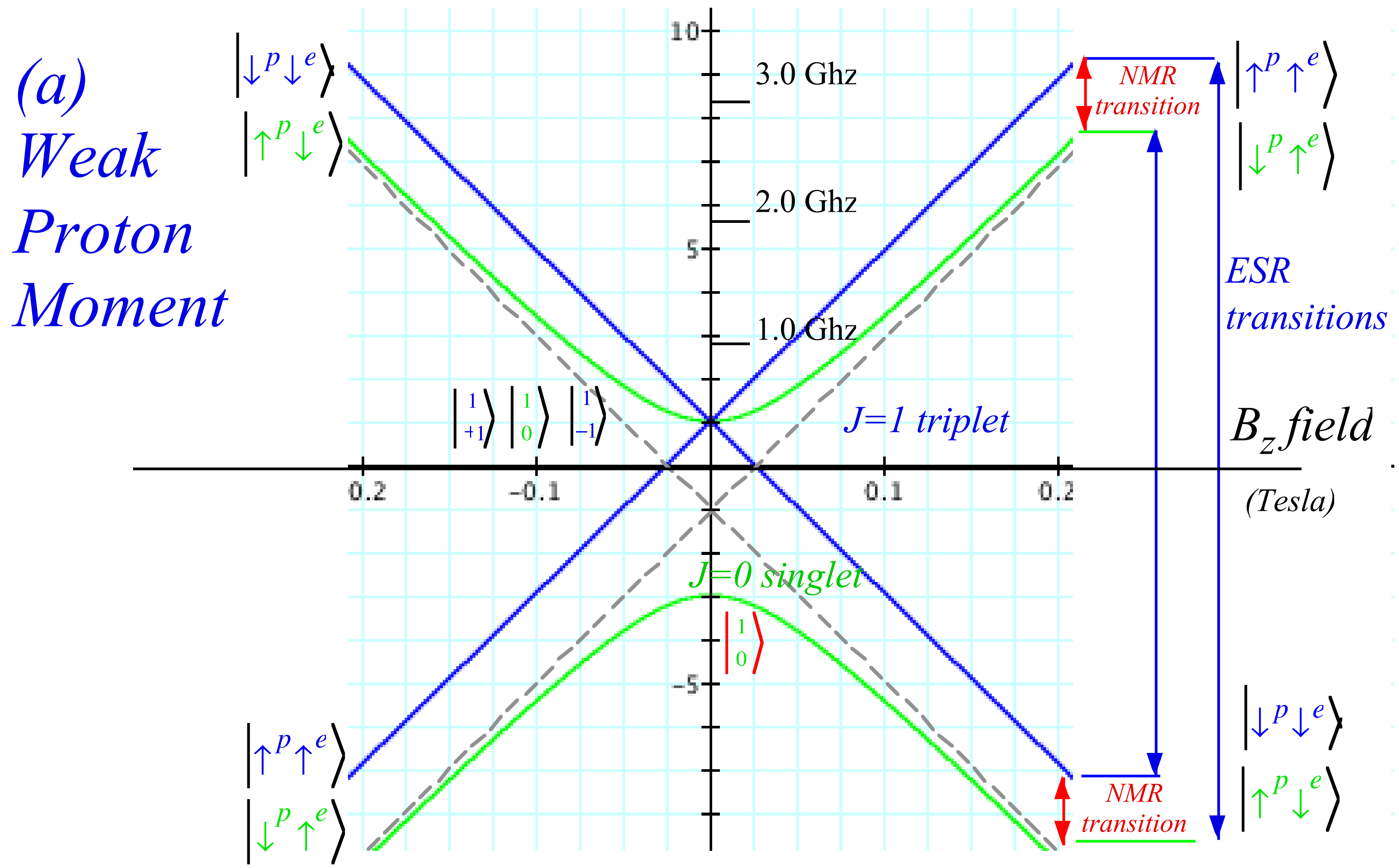
	$\lvert\uparrow^p\uparrow^e\rangle$	$\lvert\uparrow^p\downarrow^e\rangle$	$\lvert\downarrow^p\uparrow^e\rangle$	$\lvert\downarrow^p\downarrow^e\rangle$
$\langle\uparrow^p\uparrow^e\rvert$	$\frac{a_{ep}}{4}$	.	.	.
$\langle\uparrow^p\downarrow^e\rvert$	.	$\frac{-a_{ep}}{4}$	$\frac{a_{ep}}{2}$	.
$\langle\downarrow^p\uparrow^e\rvert$	.	$\frac{a_{ep}}{2}$	$\frac{-a_{ep}}{4}$	.
$\langle\downarrow^p\downarrow^e\rvert$	.	.	.	$\frac{a_{ep}}{4}$

$$\left\langle -a_p B_z J_z^{proton} + a_e B_z J_z^{electron} \right\rangle =$$

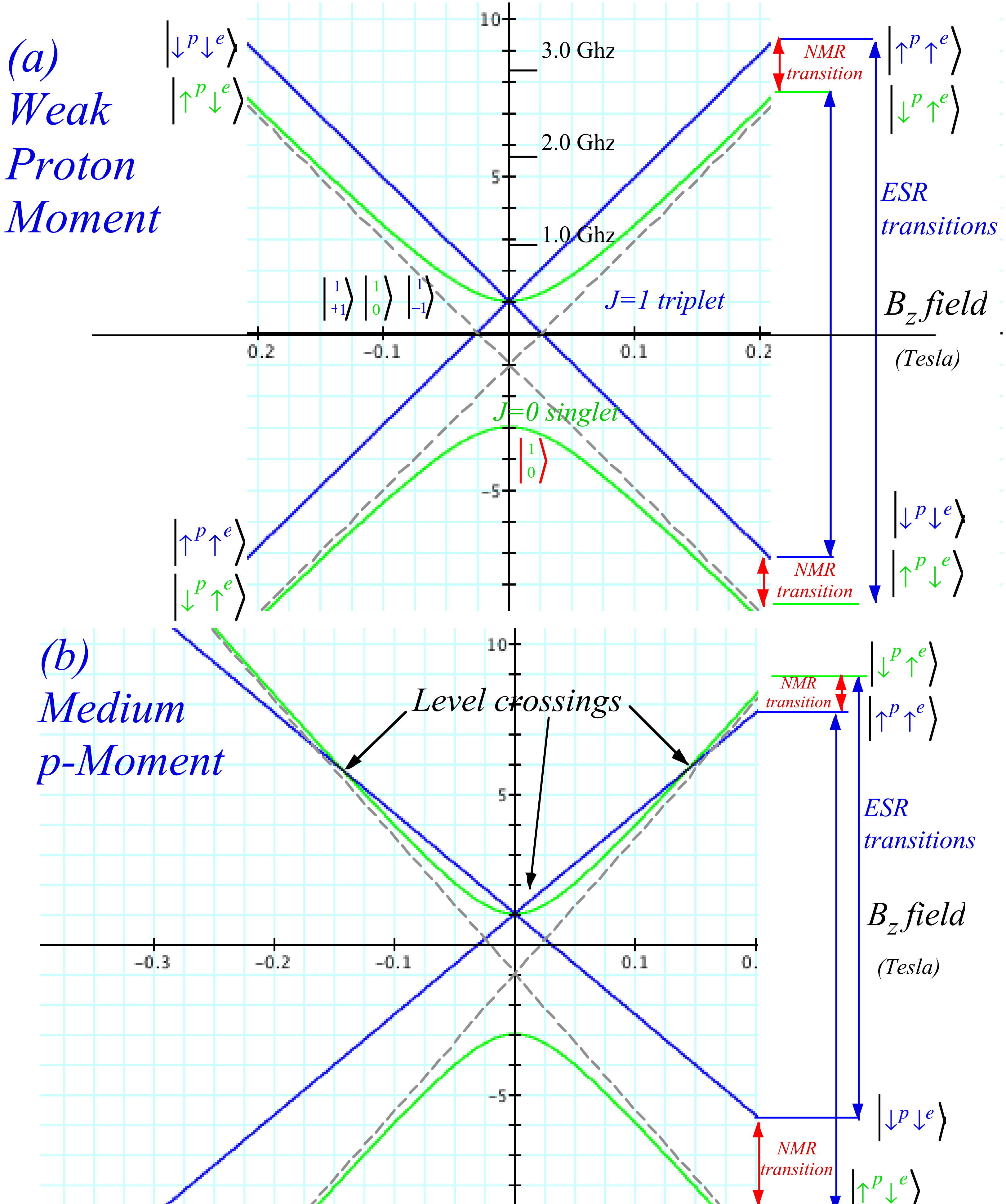
	$\lvert{1 \atop 1}\rangle$	$\lvert{1 \atop 0}\rangle$	$\lvert{0 \atop 0}\rangle$	$\lvert{1 \atop -1}\rangle$
$\langle{1 \atop 1}\rvert$	$\frac{1}{2}(a_e - a_p)B_z$	.	.	.
$\langle{1 \atop 0}\rvert$	.	0	$\frac{-1}{2}(a_e + a_p)B_z$	.
$\langle{0 \atop 0}\rvert$	.	$\frac{-1}{2}(a_e + a_p)B_z$	0	.
$\langle{1 \atop -1}\rvert$	.	.	.	$\frac{-1}{2}(a_e - a_p)B_z$

$$\left\langle a_{ep}\mathbf{J}^{proton} \bullet \mathbf{J}^{electron} \right\rangle =$$

	$\lvert{1 \atop 1}\rangle$	$\lvert{1 \atop 0}\rangle$	$\lvert{0 \atop 0}\rangle$	$\lvert{1 \atop -1}\rangle$
$\langle{1 \atop 1}\rvert$	$\frac{a_{ep}}{4}$	.	.	.
$\langle{1 \atop 0}\rvert$	.	$\frac{a_{ep}}{4}$	0	.
$\langle{0 \atop 0}\rvert$	.	0	$\frac{-3a_{ep}}{4}$	.
$\langle{1 \atop -1}\rvert$	.	.	.	$\frac{a_{ep}}{4}$



(a)
Weak
Proton
Moment
3.0 Ghz
2.0 Ghz
1.0 Ghz
J=1 triplet
J=0 singlet
NMR transition
ESR transitions
NMR transition
B_z field
(Tesla)
$|\downarrow^p\downarrow^e\rangle$
$|\uparrow^p\downarrow^e\rangle$
$|\uparrow^p\uparrow^e\rangle$
$|\downarrow^p\uparrow^e\rangle$
$\left|{1 \atop +1}\right\rangle$ $\left|{1 \atop 0}\right\rangle$ $\left|{1 \atop -1}\right\rangle$
$\left|{1 \atop 0}\right\rangle$
10
5
-5
0.2
-0.1
0.1
0.2


(a)
Weak
Proton
Moment
3.0 Ghz
2.0 Ghz
1.0 Ghz
J=1 triplet
J=0 singlet
NMR
transition
ESR
transitions
B_z field
(Tesla)
NMR
transition
(b)
Medium
p-Moment
Level crossings
NMR
transition
ESR
transitions
B_z field
(Tesla)
NMR
transition

Higher-J product states

(J=1)⊗(J=1)=2⊕1⊕0 case

$\left| C^{1\ \ 1\ \ L}_{m_1\, m_2\, M} \right\rangle =$

$1 \otimes 1$		L: 2	2	2	2	2	1	1	1	0
m_1	m_2	M: 2	1	0	−1	−2	1	0	−1	0
1	1	1	·	·	·	·	·	·	·	·
1	0	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·
1	−1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
0	1	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	·	·
0	0	·	·	$\sqrt{\frac{2}{3}}$	·	·	·	·	·	$-\frac{1}{\sqrt{3}}$
0	−1	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·
−1	1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
−1	0	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·
−1	−1	·	·	·	·	1	·	·	·	·

Higher-J product states

$(J=1)\otimes(J=1)=2\oplus 1\oplus 0$ case

$\left|C^{\,1\;\;1\;\;L}_{m_1 m_2 M}\right\rangle =$

$1 \otimes 1$		2 2	2 1	2 0	2 −1	2 −2	1 1	1 0	1 −1	0 0
1	1	1	·	·	·	·	·	·	·	·
1	0	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·
1	−1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
0	1	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	·	·
0	0	·	·	$\sqrt{\frac{2}{3}}$	·	·	·	·	·	$-\frac{1}{\sqrt{3}}$
0	−1	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·
−1	1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
−1	0	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·
−1	−1	·	·	·	·	1	·	·	·	·

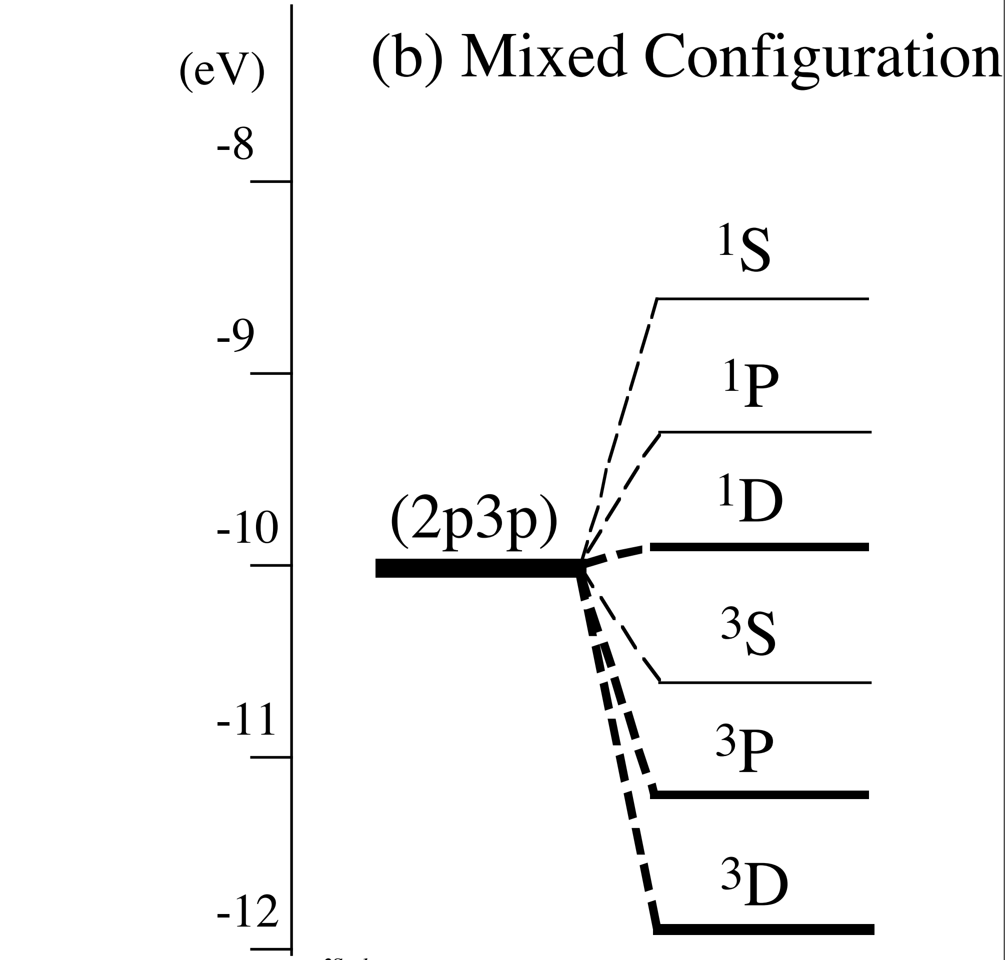


Figure 24.1.3 *Atomic* ^{2S+1}L *multiplet levels for two (ℓ = 1) p electrons.*

Higher product states to Young Tableaus

$(J=1)\otimes(J=1)=2\oplus 1\oplus 0$ case

$\left|C^{\,1\;\;1\;\;L}_{m_1 m_2 M}\right\rangle =$

$1\otimes 1$		2	2	2	2	2	1	1	1	0
		2	1	0	−1	−2	1	0	−1	0
1	1	1	·	·	·	·	·	·	·	·
1	0	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·
1	−1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
0	1	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	·	·
0	0	·	·	$\sqrt{\frac{2}{3}}$	·	·	·	·	·	$-\frac{1}{\sqrt{3}}$
0	−1	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·
−1	1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
−1	0	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·
−1	−1	·	·	·	·	1	·	·	·	·

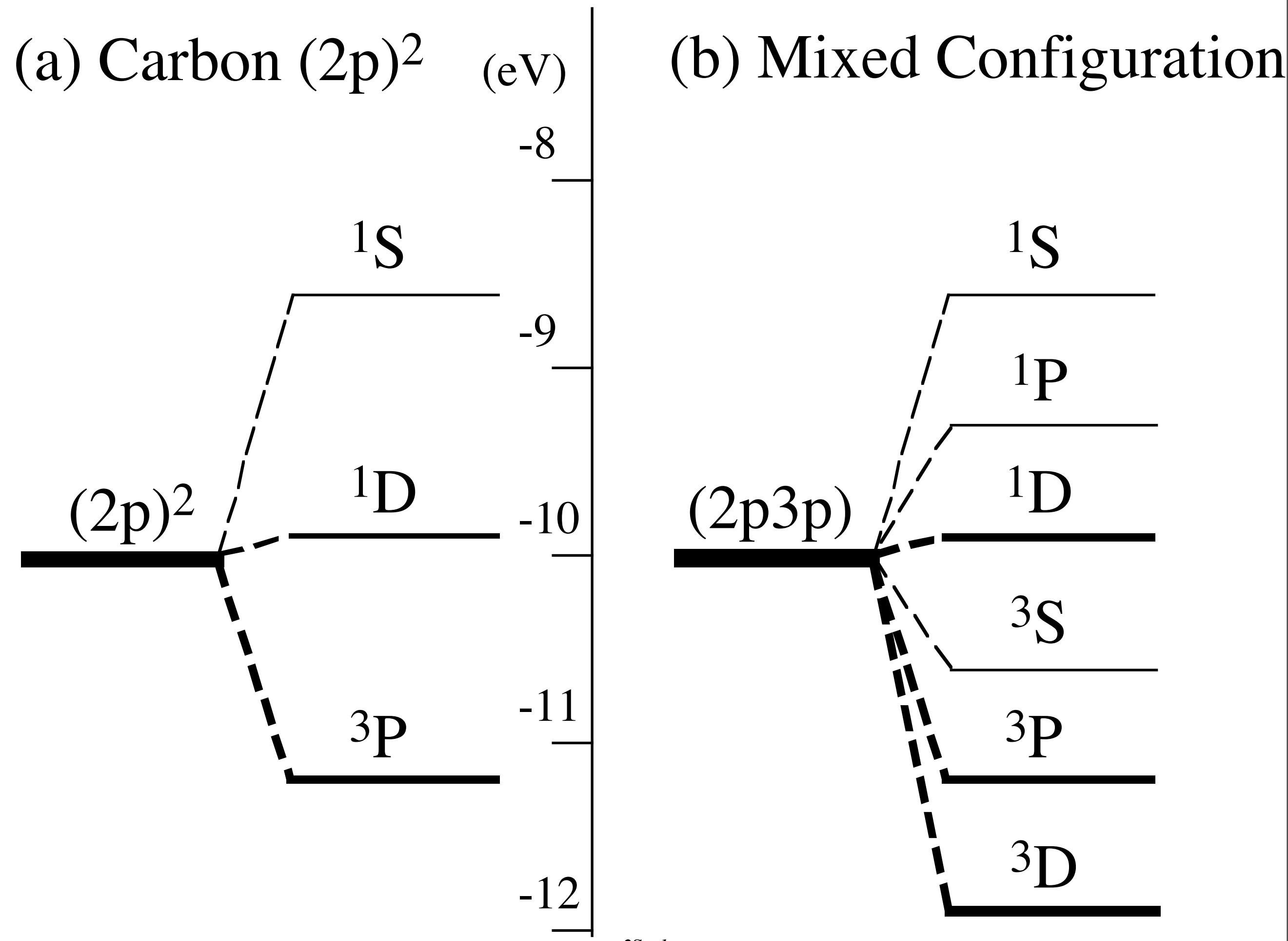


Figure 24.1.3 *Atomic* ^{2S+1}L *multiplet levels for two* ($l = 1$) p *electrons.*

Higher-J product states

$(J=1)\otimes(J=1)=2\oplus 1\oplus 0$ case

$\left|C^{\,1\;\;1\;\;L}_{m_1 m_2 M}\right\rangle =$

$1 \otimes 1$		2	2	2	2	2	1	1	1	0
		2	1	0	−1	−2	1	0	−1	0
1	1	1	·	·	·	·	·	·	·	·
1	0	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·
1	−1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
0	1	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	·	·
0	0	·	·	$\sqrt{\frac{2}{3}}$	·	·	·	·	·	$-\frac{1}{\sqrt{3}}$
0	−1	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·
−1	1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
−1	0	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·
−1	−1	·	·	·	·	1	·	·	·	·

..

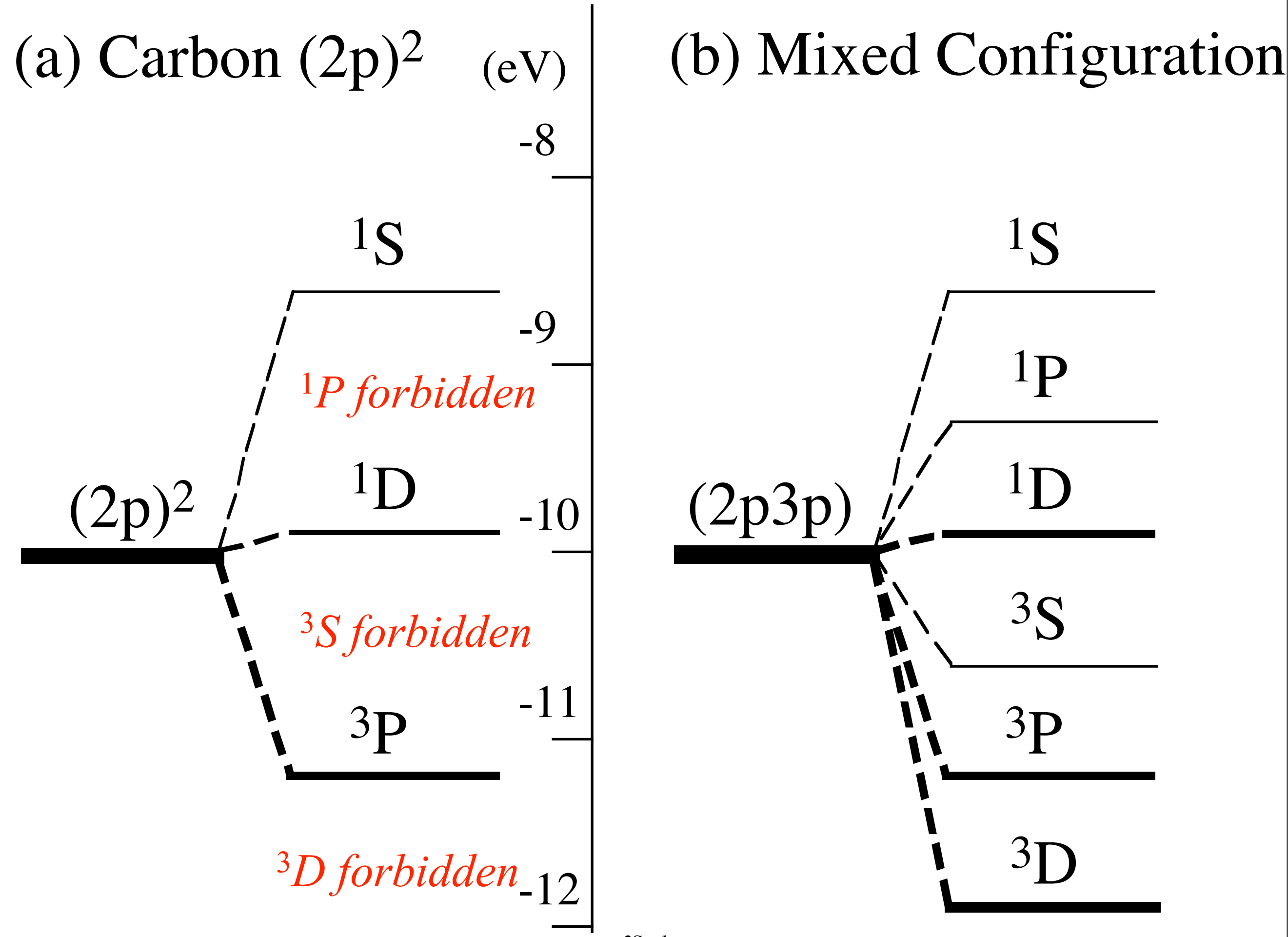


Figure 24.1.3 *Atomic* ^{2S+1}L *multiplet levels for two* ($\ell = 1$) p *electrons.*

Pauli-Fermi *selection rules*
requires total anti-symmetry

Higher-J product states

$(J=1)\otimes(J=1)=2\oplus 1\oplus 0$ case

$\left|C^{\,1\;\;1\;\;L}_{m_1 m_2 M}\right\rangle =$

$1 \otimes 1$		$J=2$	2	2	2	2	1	1	1	0
m_1	m_2	$M=2$	1	0	−1	−2	1	0	−1	0
1	1	1	·	·	·	·	·	·	·	·
1	0	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·
1	−1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
0	1	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	·	·
0	0	·	·	$\sqrt{\frac{2}{3}}$	·	·	·	·	·	$-\frac{1}{\sqrt{3}}$
0	−1	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·
−1	1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
−1	0	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·
−1	−1	·	·	·	·	1	·	·	·	·

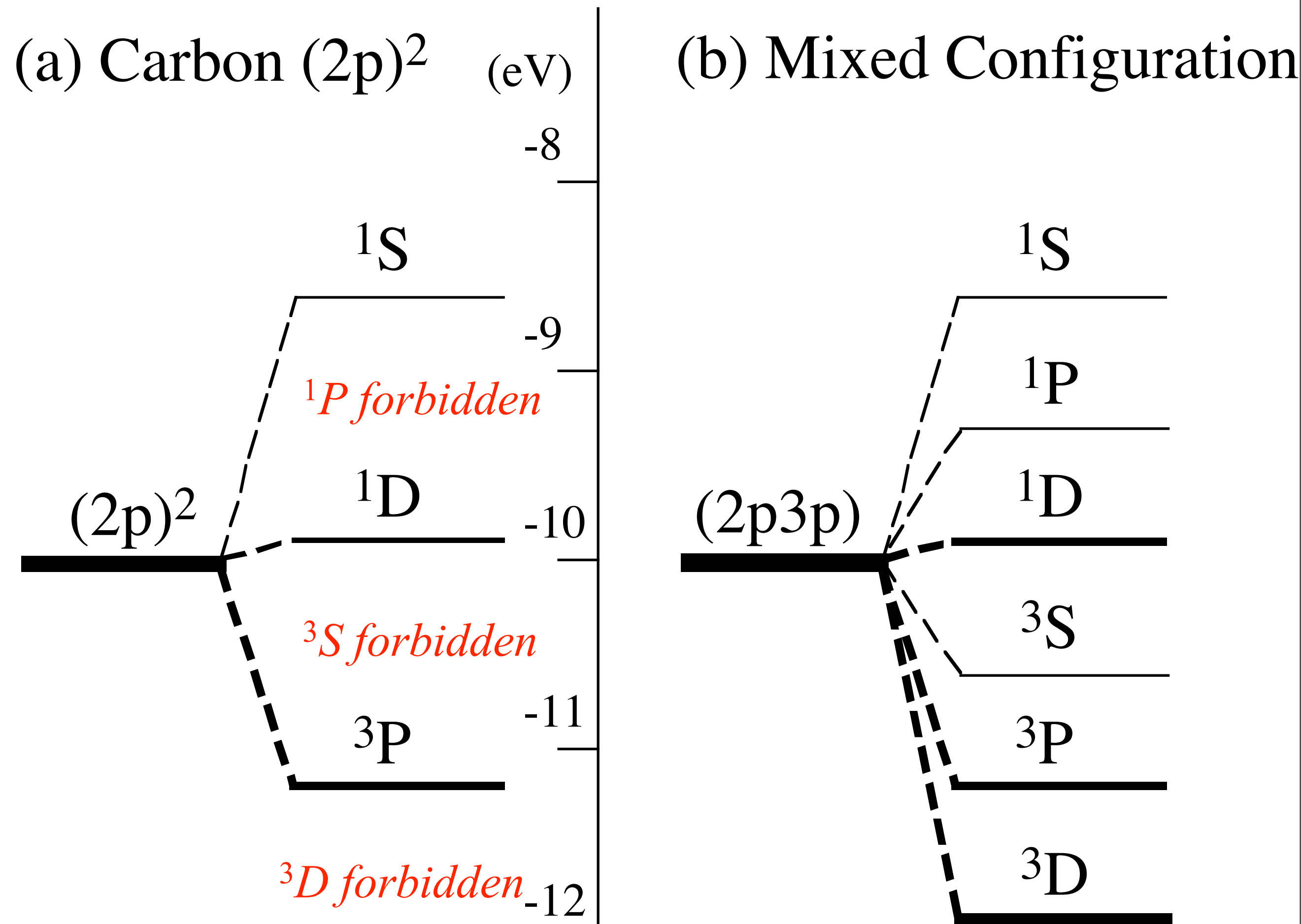
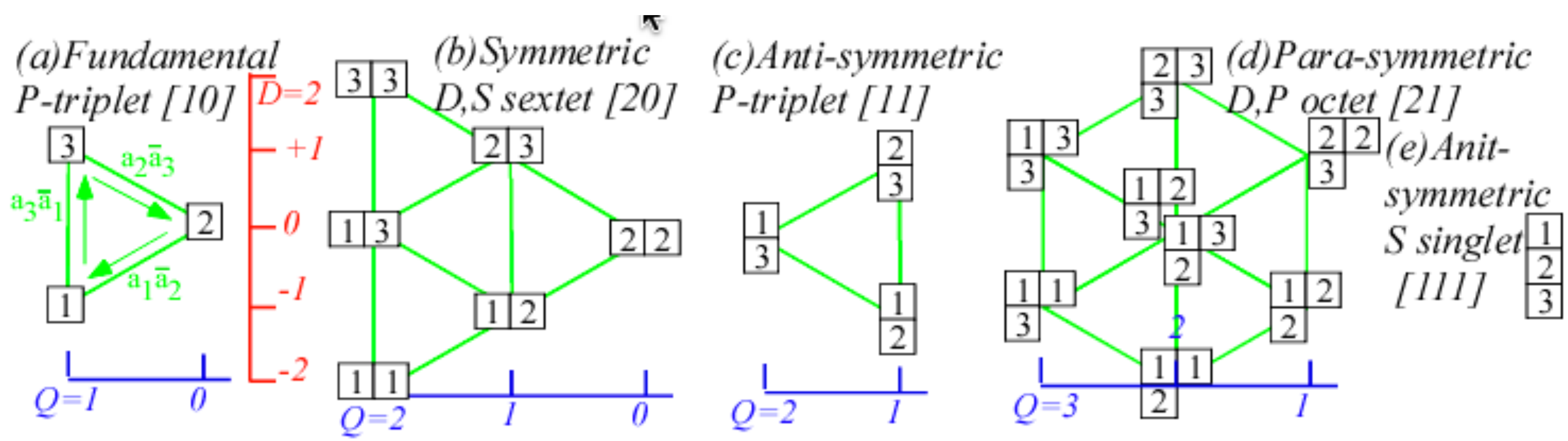


Figure 24.1.3 *Atomic* ^{2S+1}L *multiplet levels for two (*$\ell = 1$*) p electrons.*

Pauli-Fermi *selection rules requires total anti-symmetry*



Higher-J product states

(J=1)⊗(J=1)=2⊕1⊕0 case

$\left|C^{1\ \ 1\ \ L}_{m_1 m_2 M}\right\rangle =$

		2	2	2	2	2	1	1	1	0
1 ⊗	1	2	1	0	−1	−2	1	0	−1	0
1	1	1	·	·	·	·	·	·	·	·
1	0	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·
1	−1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
0	1	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	·	·
0	0	·	·	$\sqrt{\frac{2}{3}}$	·	·	·	·	·	$-\frac{1}{\sqrt{3}}$
0	−1	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·
−1	1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
−1	0	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·
−1	−1	·	·	·	·	1	·	·	·	·

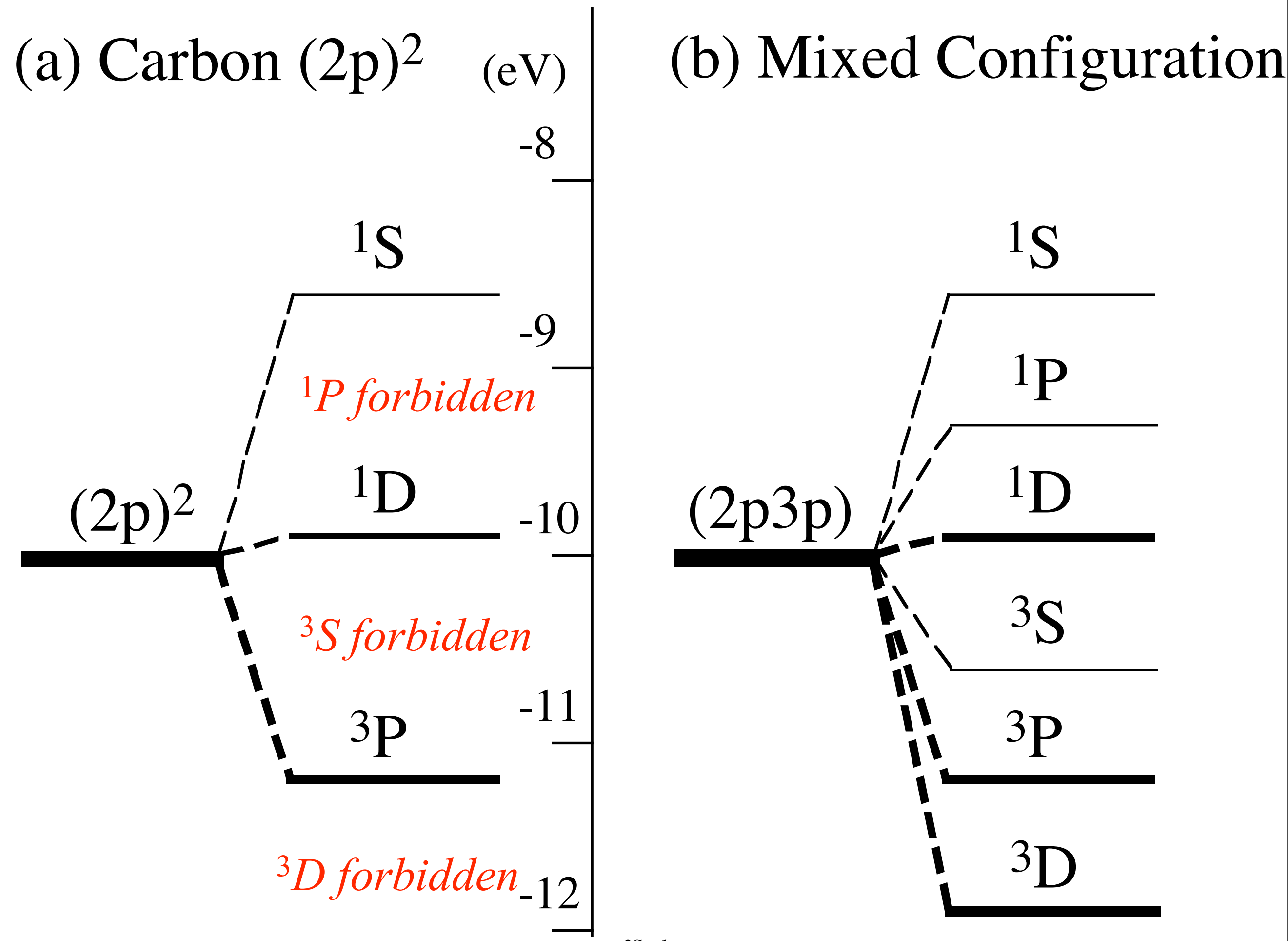


Figure 24.1.3 *Atomic* $^{2S+1}$L *multiplet levels for two* (ℓ = 1) p *electrons.*

Pauli-Fermi *selection rules*
requires total anti-symmetry

General U(2) case

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1-j_2-m_3}\, C^{j_1\ \ j_2\ \ j_3}_{m_1\ m_2\ m_3} \Big/ (2j_3+1)^{\frac{1}{2}}$$

Wigner 3j vs. Clebsch-Gordon (CGC)

Higher-J product states

(J=1)⊗(J=1)=2⊕1⊕0 case

$\left|C^{1\ \ 1\ \ L}_{m_1\,m_2\,M}\right\rangle =$

1 ⊗ 1		2 2	2 1	2 0	2 −1	2 −2	1 1	1 0	1 −1	0 0
1	1	1	·	·	·	·	·	·	·	·
1	0	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·
1	−1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
0	1	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	·	·
0	0	·	·	$\sqrt{\frac{2}{3}}$	·	·	·	·	·	$-\frac{1}{\sqrt{3}}$
0	−1	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·
−1	1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
−1	0	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·
−1	−1	·	·	·	·	1	·	·	·	·

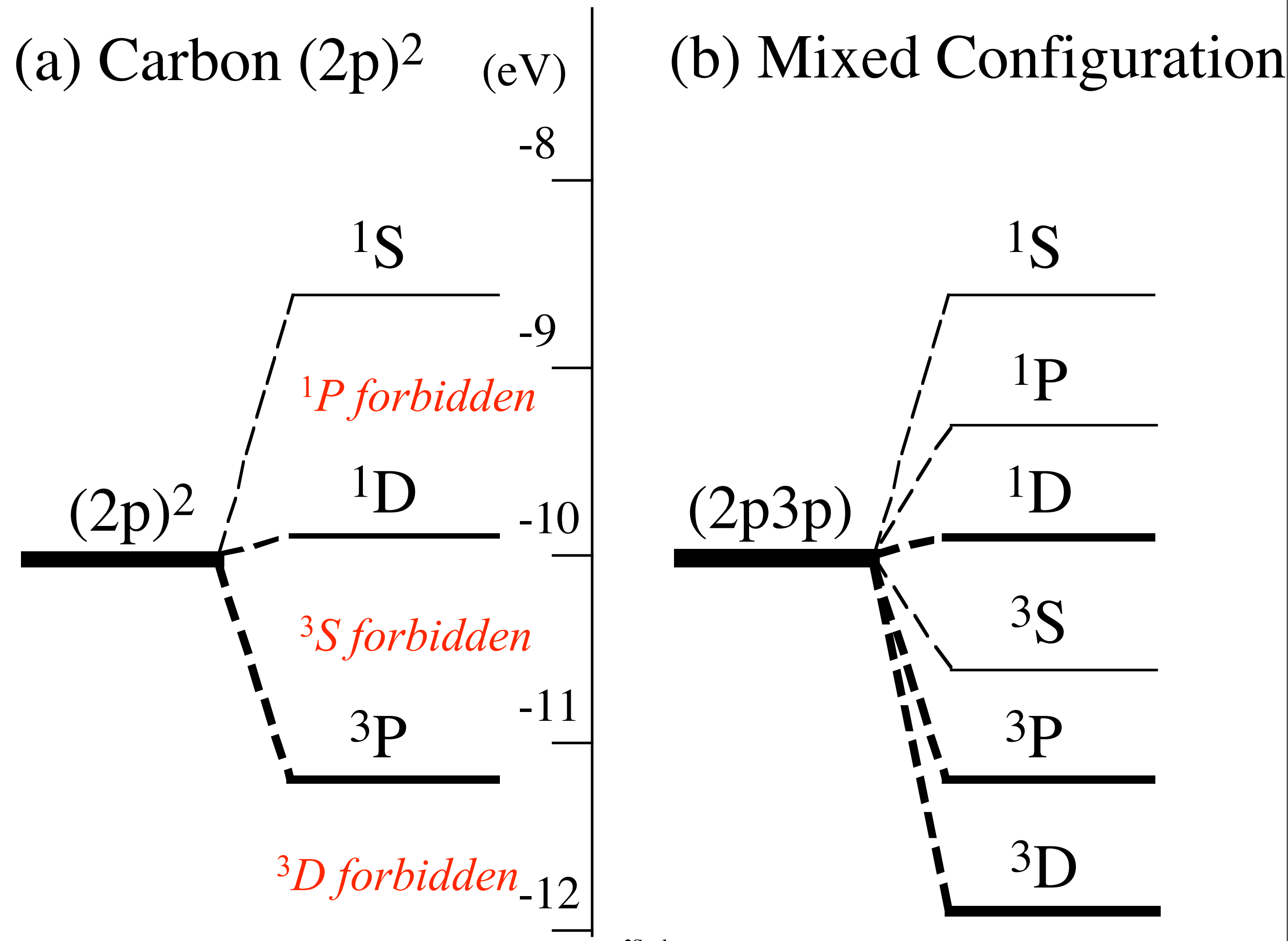


Figure 24.1.3 *Atomic* $^{2S+1}$L *multiplet levels for two (l = 1)* p *electrons.*

Pauli-Fermi *selection rules*
requires total anti-symmetry

General U(2) case

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1-j_2-m_3} C^{j_1\ \ j_2\ \ j_3}_{m_1\ m_2\ m_3} \Big/ (2j_3+1)^{\frac{1}{2}}$$

Wigner 3j vs. Clebsch-Gordon (CGC)

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1-j_2-n_3} \sqrt{\frac{(j_1+j_2-j_3)!(j_1-j_2+j_3)(-j_1+j_2+j_3)}{(j_1+j_2+j_3+1)!}}$$

$$\sum_k \frac{(-1)^k}{k!} \frac{\sqrt{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(j_3+m_3)!(j_3-m_3)!}}{(j_1-m_1-k)!(j_2-m_2-k)!(j_1+j_2-j_3-k)!(j_3-j_2-m_1+k)!(j_3-j_1-m_2+k)!}$$

Higher-J product states

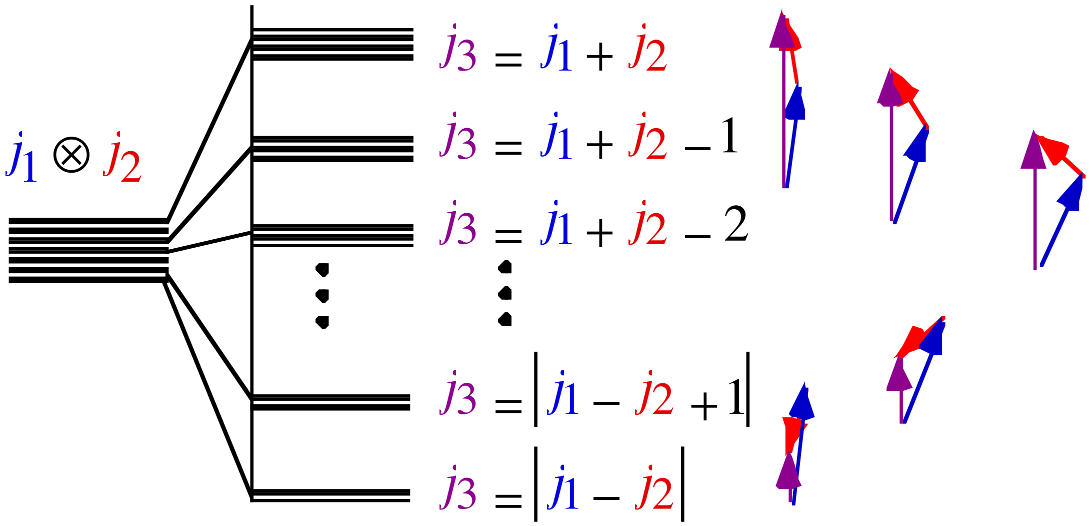


Figure 24.1.6 Level-splitting and vector-addition picture of angular-momentum coupling.

Higher-J product states

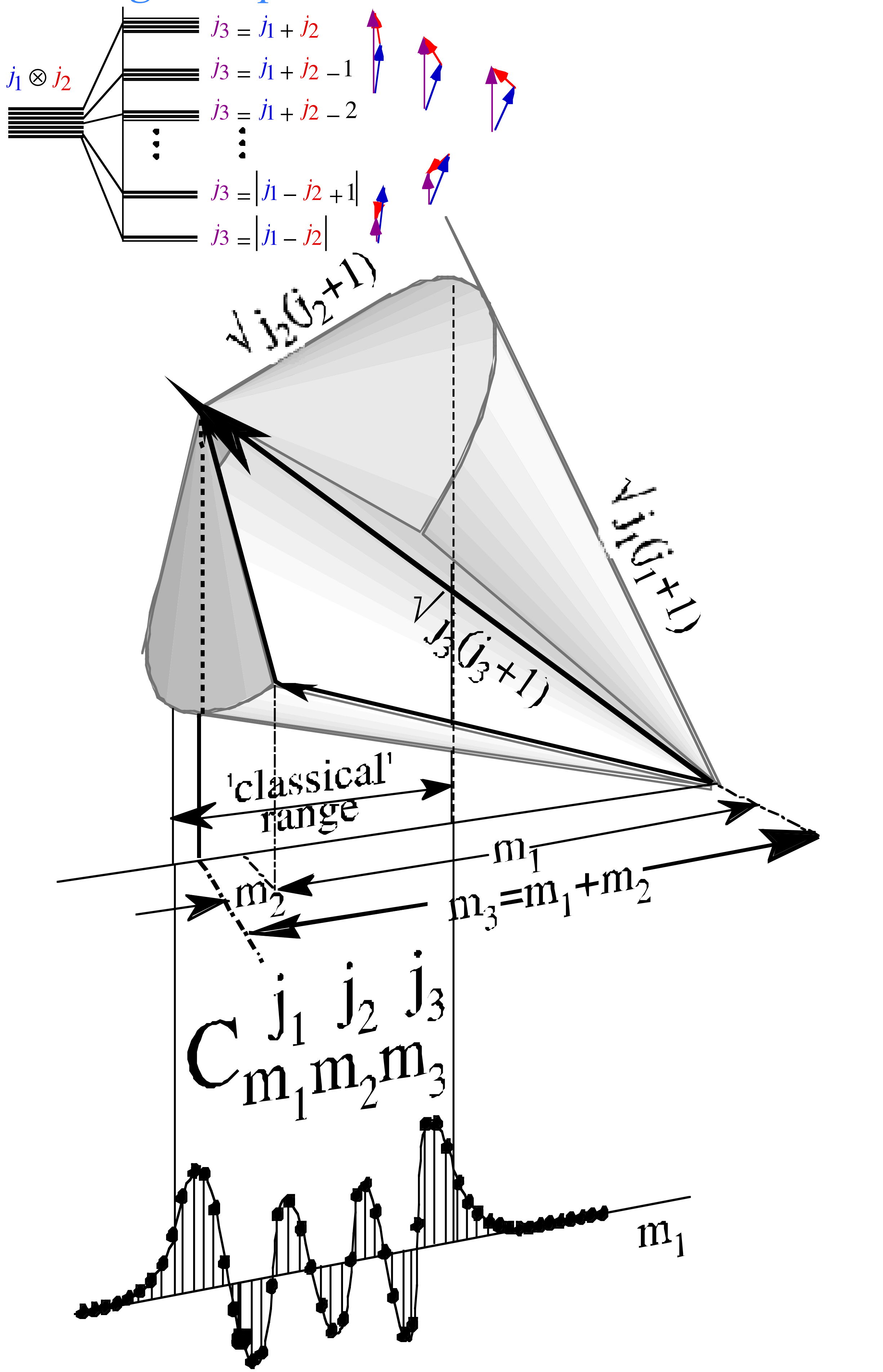


Figure 24.1.7 *Angular-momentum cone picture of Clebsch-Gordan coupling amplitudes.*

Higher-J product states

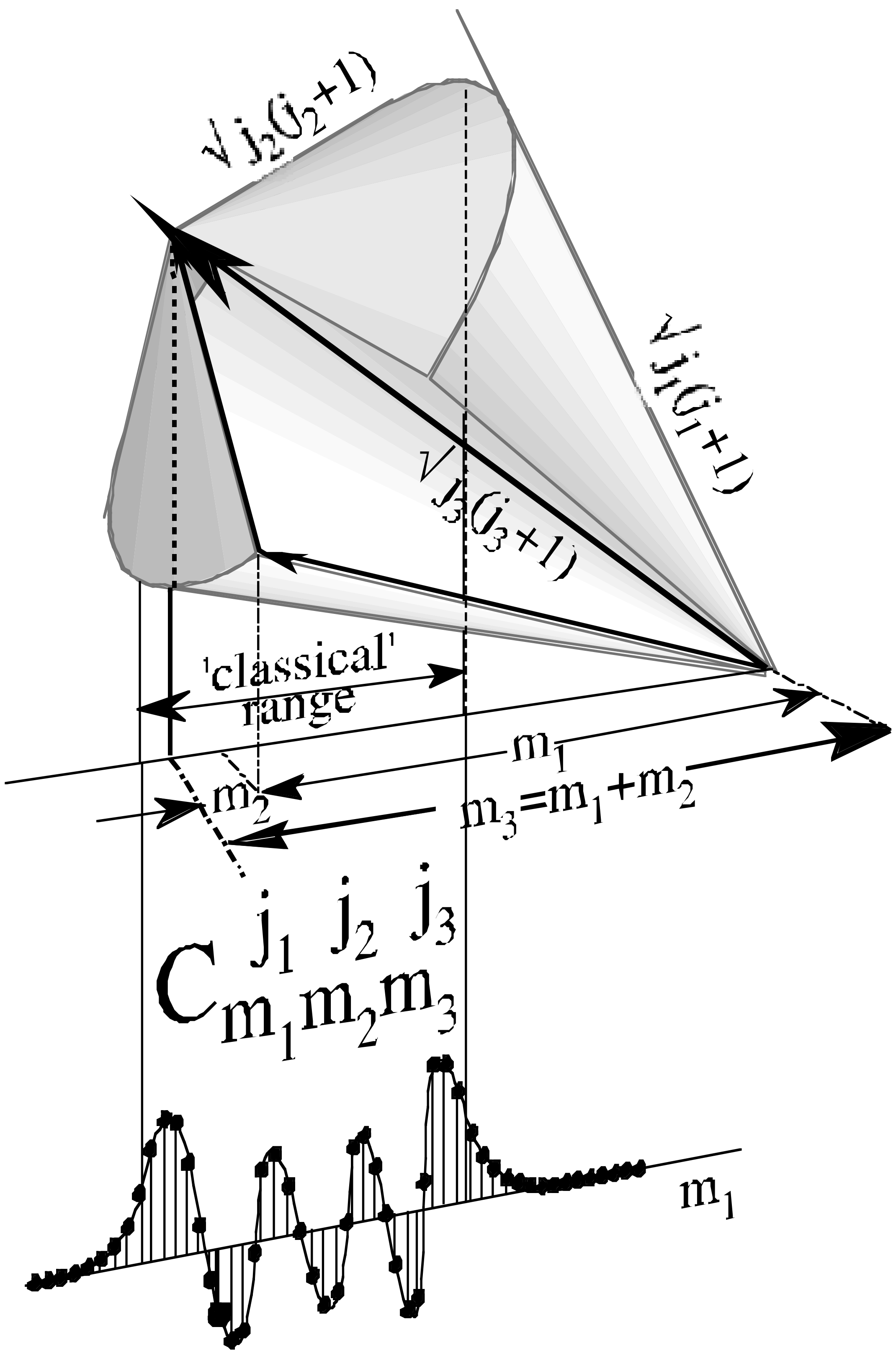


Figure 24.1.7 *Angular-momentum cone picture of Clebsch-Gordan coupling amplitudes.*

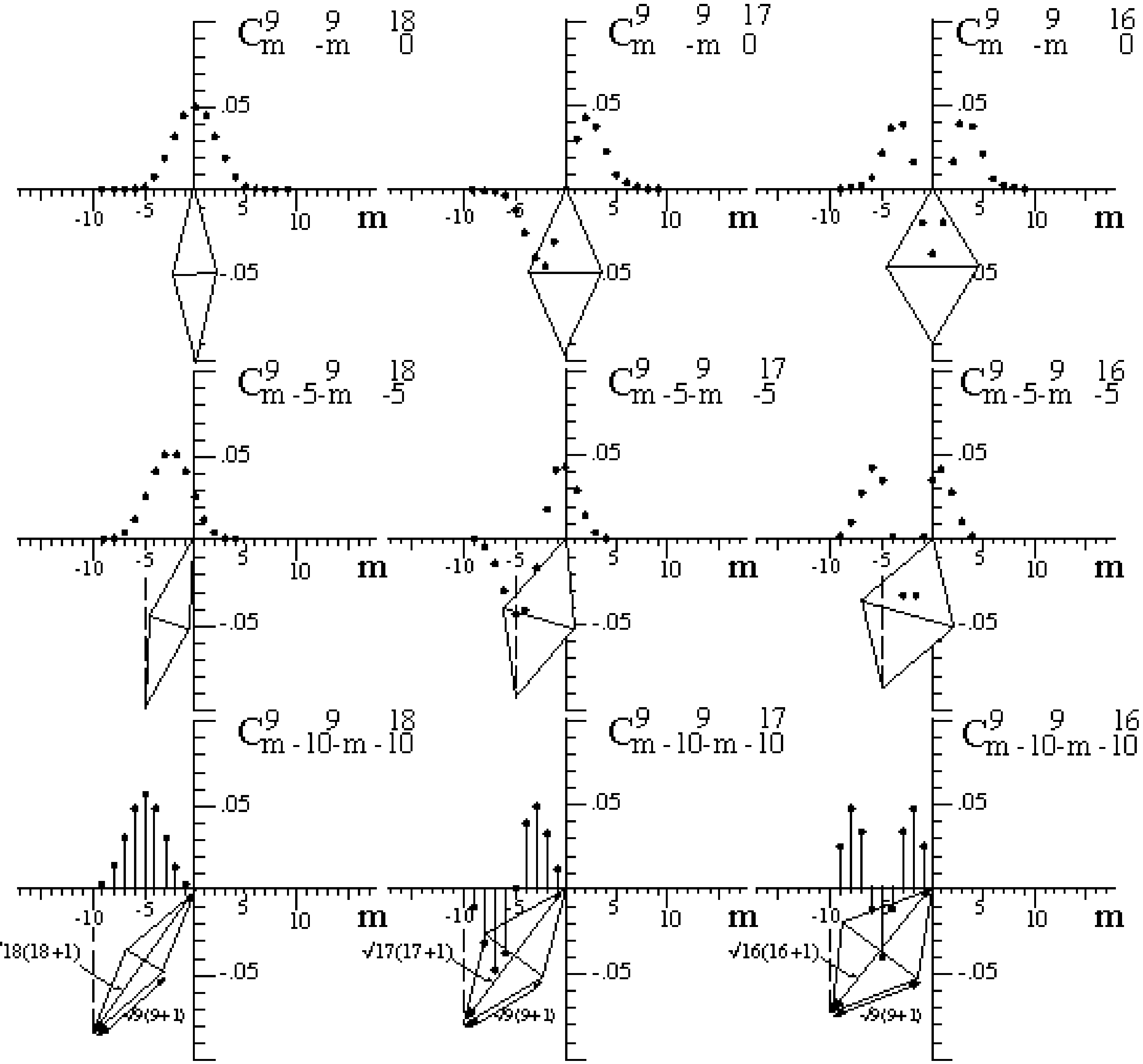
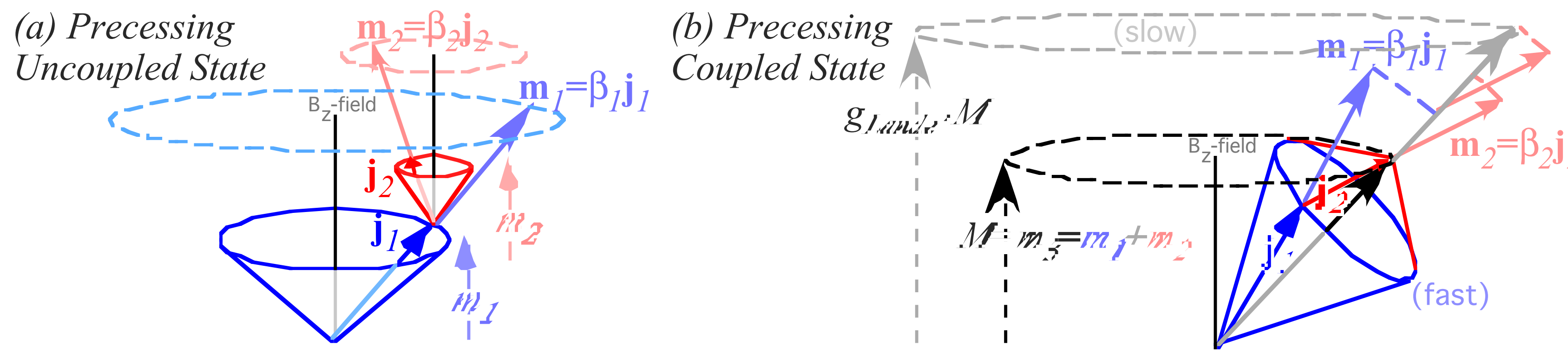


Figure 24.1.8 *Clebsch-Gordan coefficients plotted next to their angular-momentum cones.*

Higher-J product states



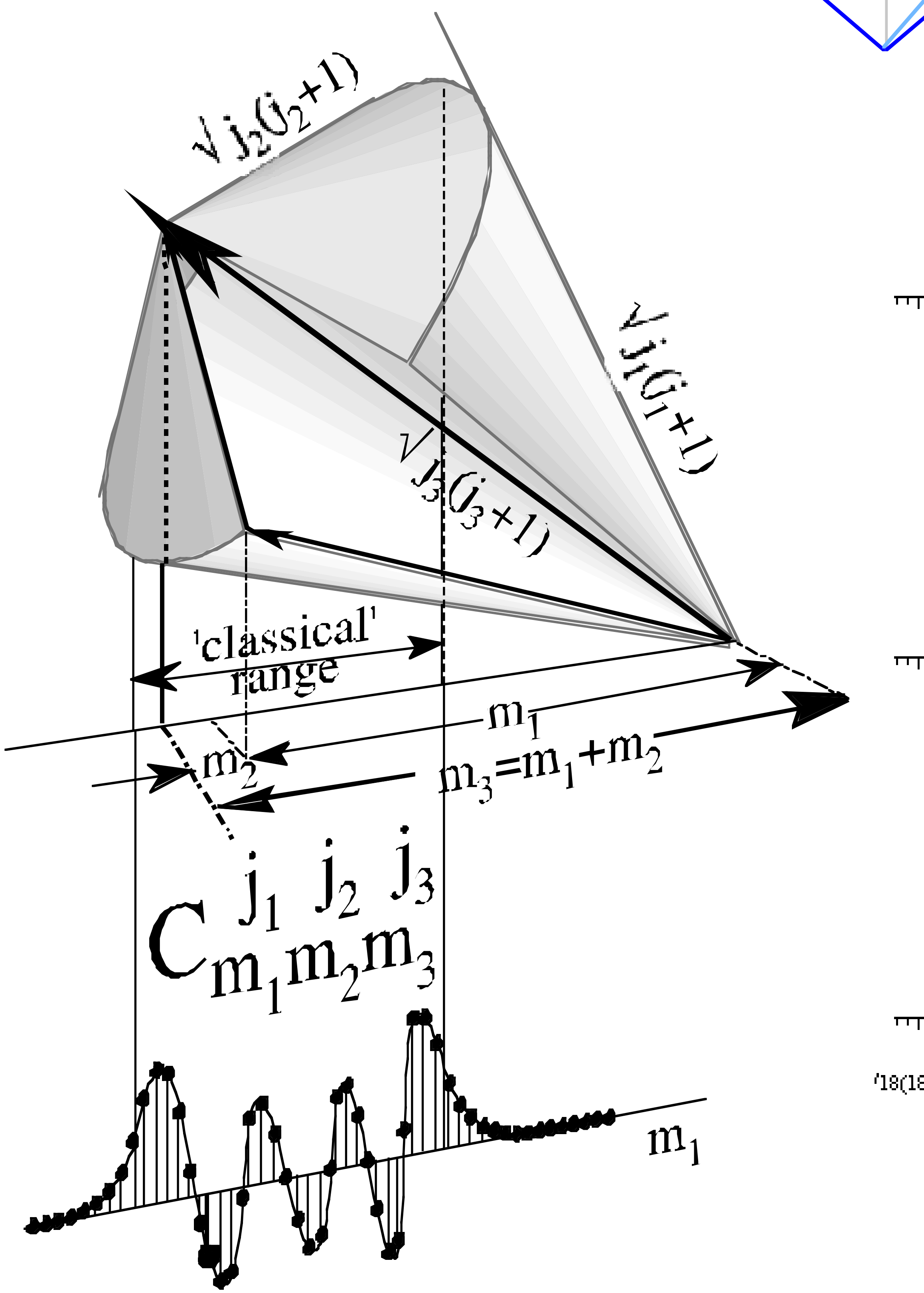


Figure 24.1.7 Angular-momentum cone picture of Clebsch-Gordan coupling amplitudes.

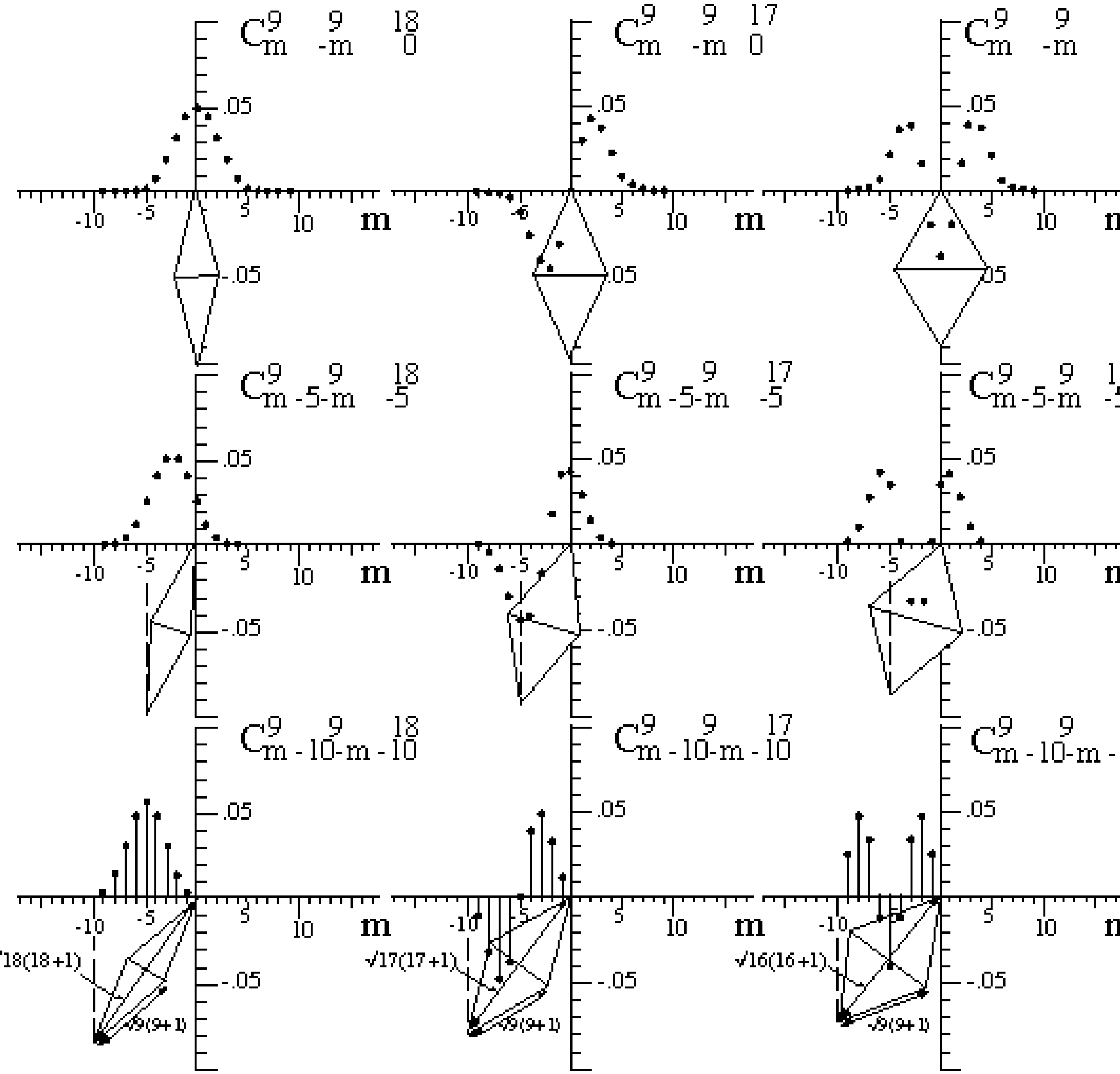


Figure 24.1.8 Clebsch-Gordan coefficients plotted next to their angular-momentum cones.

Multi-spin (1/2)^N product states

$$\left(\frac{1}{2}\otimes\frac{1}{2}\right)\otimes\frac{1}{2}$$

Multi-spin $(1/2)^N$ product states

$$\left(\frac{1}{2}\otimes\frac{1}{2}\right)\otimes\frac{1}{2}=\left(0\oplus 1\right)\otimes\frac{1}{2}$$

Multi-spin (1/2)^N product states

$$\left(\frac{1}{2}\otimes\frac{1}{2}\right)\otimes\frac{1}{2}=(0\oplus 1)\otimes\frac{1}{2}=\left(0\otimes\frac{1}{2}\right)\oplus\quad\left(1\otimes\frac{1}{2}\right)$$

Multi-spin (1/2)^N product states

$$\left(\frac{1}{2}\otimes\frac{1}{2}\right)\otimes\frac{1}{2}=(0\oplus 1)\otimes\frac{1}{2}=\left(0\otimes\frac{1}{2}\right)\oplus\quad\left(1\otimes\frac{1}{2}\right)$$

$$=\quad\left(\frac{1}{2}\right)\quad\oplus\left(\left(\frac{1}{2}\right)\oplus\left(\frac{3}{2}\right)\right)$$

Multi-spin $(1/2)^N$ product states

$$\left(\frac{1}{2}\otimes\frac{1}{2}\right)\otimes\frac{1}{2}=(0\oplus 1)\otimes\frac{1}{2}=\left(0\otimes\frac{1}{2}\right)\oplus\quad\left(1\otimes\frac{1}{2}\right)$$

$$=\quad\left(\frac{1}{2}\right)\quad\oplus\left(\left(\frac{1}{2}\right)\oplus\left(\frac{3}{2}\right)\right)=\frac{1}{2}\oplus\ \frac{1}{2}\oplus\frac{3}{2}$$

Multi-spin $(1/2)^N$ product states

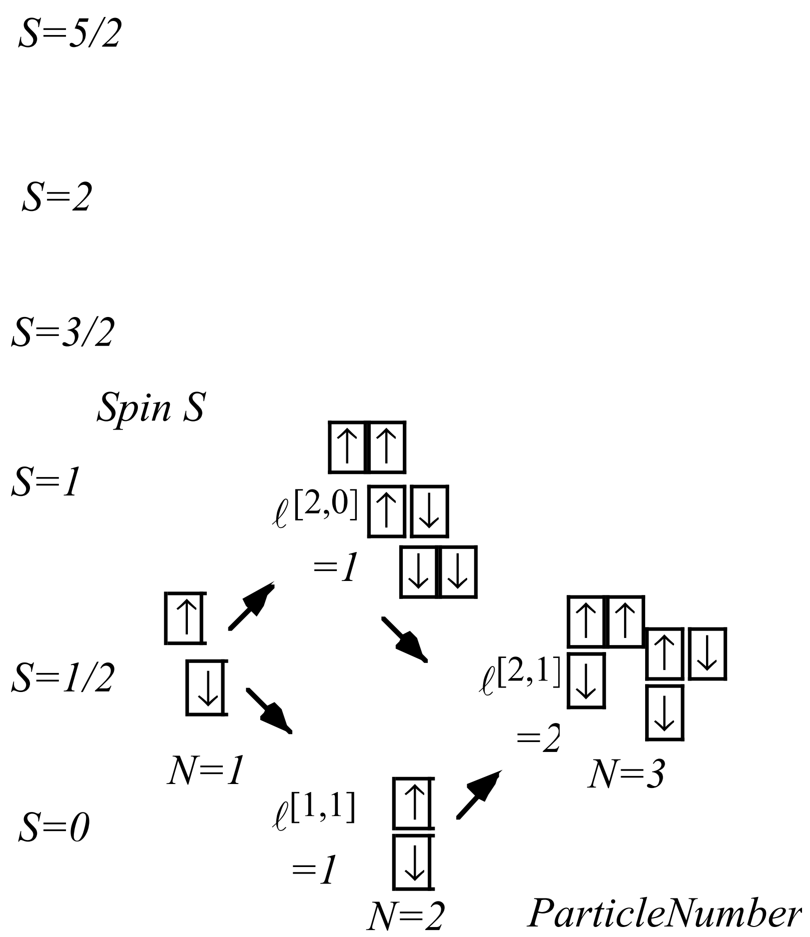
$$\left(\frac{1}{2}\otimes\frac{1}{2}\right)\otimes\frac{1}{2}=\left(0\oplus 1\right)\otimes\frac{1}{2}=\left(0\otimes\frac{1}{2}\right)\oplus\quad\left(1\otimes\frac{1}{2}\right)$$

$$=\quad\left(\frac{1}{2}\right)\quad\oplus\left(\left(\frac{1}{2}\right)\oplus\left(\frac{3}{2}\right)\right)=\frac{1}{2}\oplus\ \frac{1}{2}\oplus\frac{3}{2}=2\left(\frac{1}{2}\right)\oplus 1\left(\frac{3}{2}\right)$$

Multi-spin $(1/2)^N$ product states

$$\left(\frac{1}{2}\otimes\frac{1}{2}\right)\otimes\frac{1}{2}=(0\oplus 1)\otimes\frac{1}{2}=\left(0\otimes\frac{1}{2}\right)\oplus\left(1\otimes\frac{1}{2}\right)$$

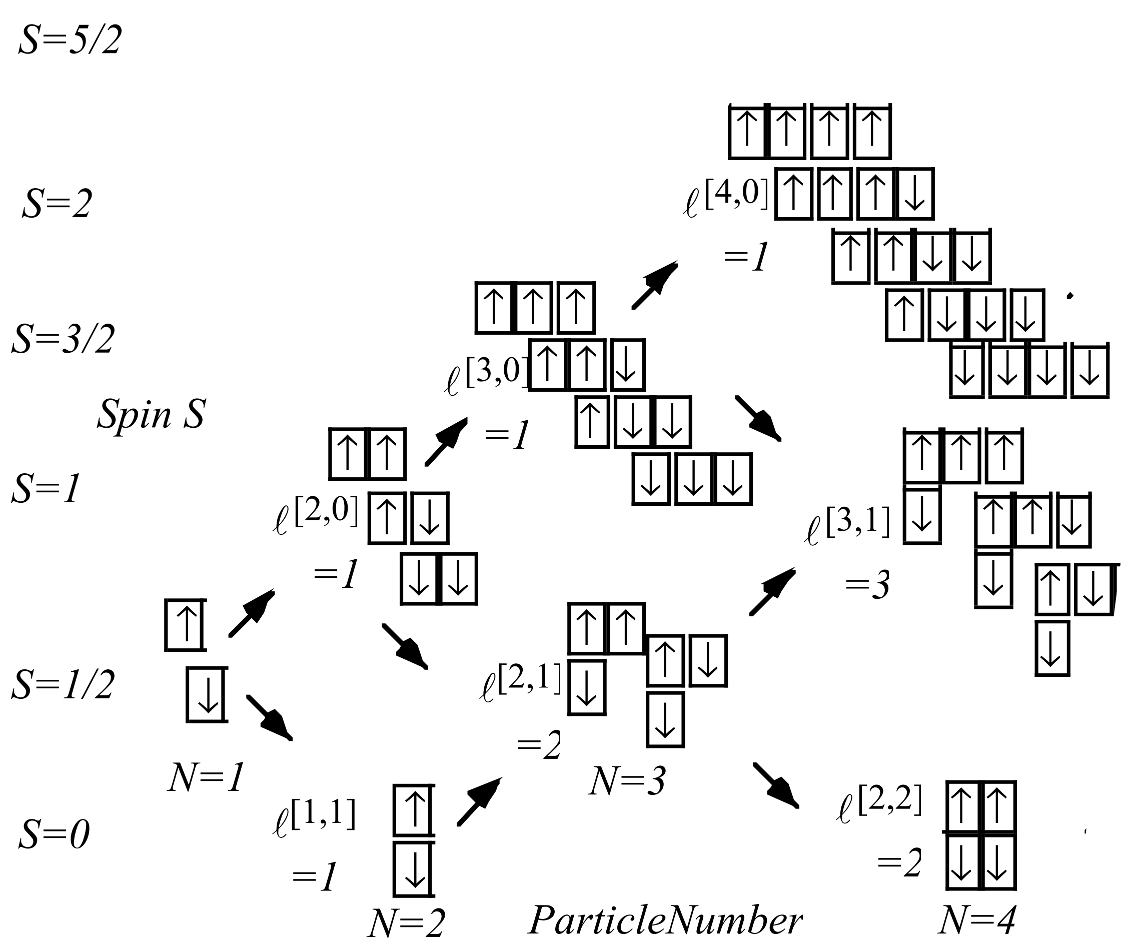
$$=\left(\frac{1}{2}\right)\oplus\left(\left(\frac{1}{2}\right)\oplus\left(\frac{3}{2}\right)\right)=\frac{1}{2}\oplus\frac{1}{2}\oplus\frac{3}{2}=2\left(\frac{1}{2}\right)\oplus 1\left(\frac{3}{2}\right)$$



Multi-spin (1/2)^N product states

$$\left(\frac{1}{2}\otimes\frac{1}{2}\right)\otimes\frac{1}{2}=(0\oplus 1)\otimes\frac{1}{2}=\left(0\otimes\frac{1}{2}\right)\oplus\left(1\otimes\frac{1}{2}\right)$$

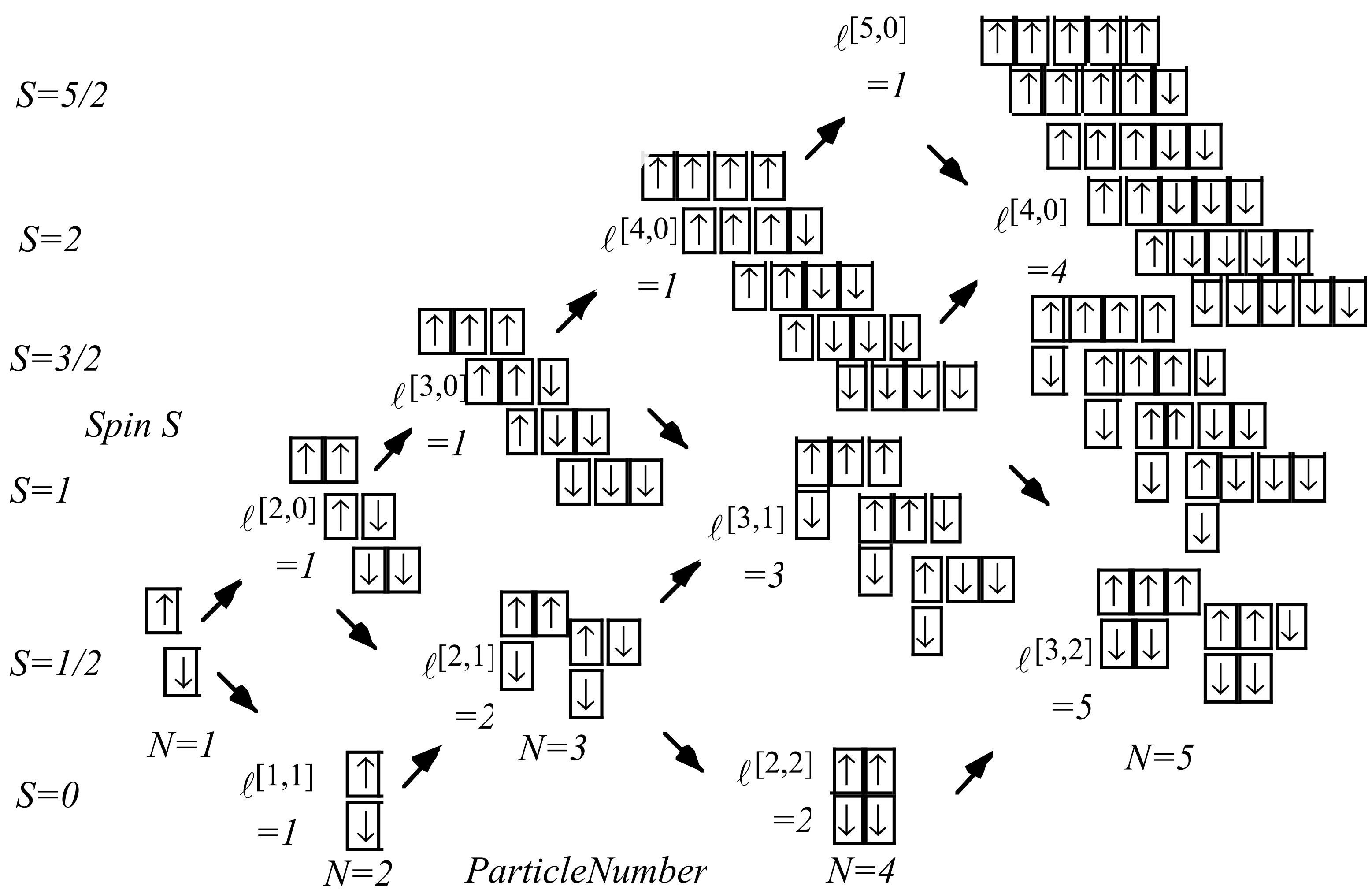
$$=\left(\frac{1}{2}\right)\oplus\left(\left(\frac{1}{2}\right)\oplus\left(\frac{3}{2}\right)\right)=\frac{1}{2}\oplus\frac{1}{2}\oplus\frac{3}{2}=2\left(\frac{1}{2}\right)\oplus 1\left(\frac{3}{2}\right)$$



Multi-spin $(1/2)^N$ product states

$$\left(\frac{1}{2}\otimes\frac{1}{2}\right)\otimes\frac{1}{2}=(0\oplus 1)\otimes\frac{1}{2}=\left(0\otimes\frac{1}{2}\right)\oplus\quad\left(1\otimes\frac{1}{2}\right)$$

$$=\quad\left(\frac{1}{2}\right)\quad\oplus\left(\left(\frac{1}{2}\right)\oplus\left(\frac{3}{2}\right)\right)=\frac{1}{2}\oplus\ \frac{1}{2}\oplus\frac{3}{2}=2\left(\frac{1}{2}\right)\oplus 1\left(\frac{3}{2}\right)$$



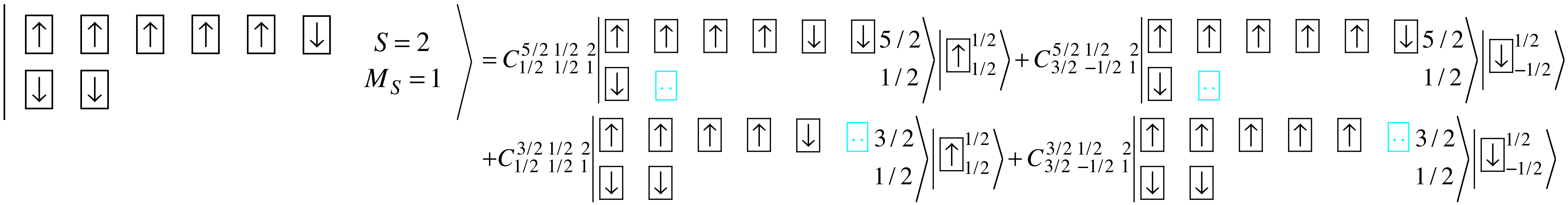
Multi-spin $(1/2)^N$ product states

$$\left|\begin{array}{llllll}\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\downarrow}\\ \boxed{\downarrow}&\boxed{\downarrow}\end{array}\quad\begin{array}{c}S=2\\ M_S=1\end{array}\right\rangle = C^{5/2\ 1/2\ 2}_{1/2\ 1/2\ 1}\left|\begin{array}{llllll}\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\downarrow}&\boxed{\downarrow}\\ \boxed{\downarrow}&\boxed{\cdot\cdot}\end{array}\begin{array}{c}5/2\\ 1/2\end{array}\right\rangle\left|\boxed{\uparrow}^{1/2}_{1/2}\right\rangle + C^{5/2\ 1/2\ 2}_{3/2\ -1/2\ 1}\left|\begin{array}{llllll}\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\downarrow}\\ \boxed{\downarrow}&\boxed{\cdot\cdot}\end{array}\begin{array}{c}5/2\\ 1/2\end{array}\right\rangle\left|\boxed{\downarrow}^{1/2}_{-1/2}\right\rangle$$

$$+C^{3/2\ 1/2\ 2}_{1/2\ 1/2\ 1}\left|\begin{array}{llllll}\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\downarrow}&\boxed{\cdot\cdot}\\ \boxed{\downarrow}&\boxed{\downarrow}\end{array}\begin{array}{c}3/2\\ 1/2\end{array}\right\rangle\left|\boxed{\uparrow}^{1/2}_{1/2}\right\rangle + C^{3/2\ 1/2\ 2}_{3/2\ -1/2\ 1}\left|\begin{array}{llllll}\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\cdot\cdot}\\ \boxed{\downarrow}&\boxed{\downarrow}\end{array}\begin{array}{c}3/2\\ 1/2\end{array}\right\rangle\left|\boxed{\downarrow}^{1/2}_{-1/2}\right\rangle$$

$$\begin{pmatrix} C^{j\ \ 1/2\ \ j+1/2}_{m\ 1/2\ \ m+1/2}=\sqrt{\dfrac{j+m+1}{2j+1}} & C^{j\ \ \ \ 1/2\ \ j+1/2}_{m+1\ -1/2\ \ m+1/2}=\sqrt{\dfrac{j-m}{2j+1}} \\ C^{j+1\ 1/2\ \ j+1/2}_{m\ \ \ 1/2\ \ m+1/2}=-\sqrt{\dfrac{j-m+1}{2j+3}} & C^{j+1\ \ 1/2\ \ j+1/2}_{m+1\ -1/2\ \ m+1/2}=\sqrt{\dfrac{j+m+2}{2j+3}} \end{pmatrix} \quad \text{example:} \quad \begin{pmatrix} C^{3/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=\sqrt{\dfrac{2}{3}} & C^{3/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=\sqrt{\dfrac{1}{3}} \\ C^{5/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=-\sqrt{\dfrac{1}{3}} & C^{5/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=\sqrt{\dfrac{2}{3}} \end{pmatrix}$$

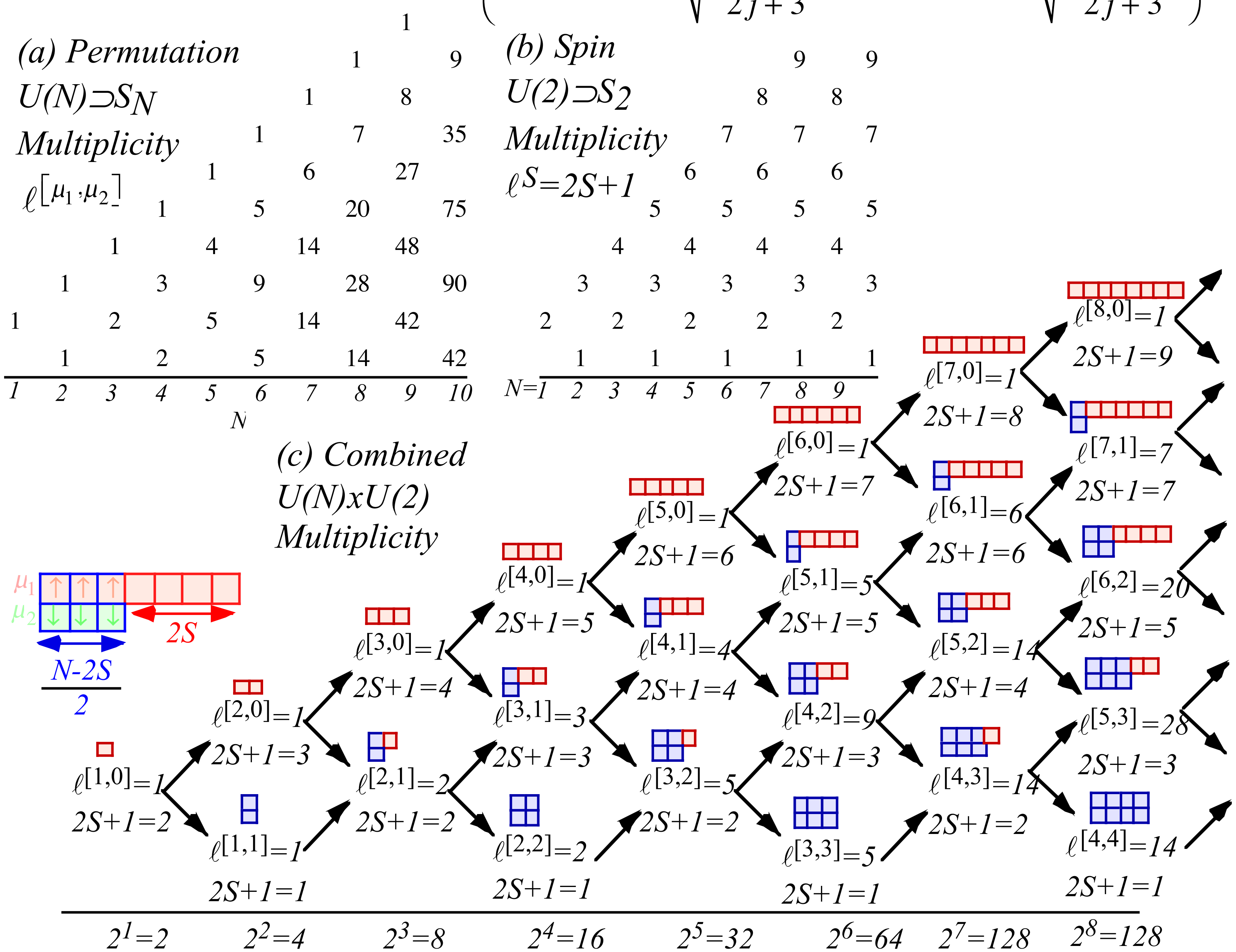
Multi-spin $(1/2)^N$ product states



$$\left(\begin{array}{cc} C^{j\ \ 1/2\ \ j+1/2}_{m\ \ 1/2\ \ m+1/2}=\sqrt{\dfrac{j+m+1}{2j+1}} & C^{j\ \ \ \ 1/2\ \ j+1/2}_{m+1\ -1/2\ \ m+1/2}=\sqrt{\dfrac{j-m}{2j+1}} \\ C^{j+1\ 1/2\ \ j+1/2}_{m\ \ \ \ 1/2\ \ m+1/2}=-\sqrt{\dfrac{j-m+1}{2j+3}} & C^{j+1\ \ \ 1/2\ \ j+1/2}_{m+1\ -1/2\ \ m+1/2}=\sqrt{\dfrac{j+m+2}{2j+3}} \end{array}\right)$$

example:

$$\left(\begin{array}{cc} C^{3/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=\sqrt{\dfrac{2}{3}} & C^{3/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=\sqrt{\dfrac{1}{3}} \\ C^{5/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=-\sqrt{\dfrac{1}{3}} & C^{5/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=\sqrt{\dfrac{2}{3}} \end{array}\right)$$



$$2^N=\sum_S^{N/2}\ell^{[S]}\ell^{[\mu_1,\mu_2]}$$

$$=\sum_S^{N/2}(2S+1)\ell^{\left[\frac{N+2S}{2},\frac{N-2S}{2}\right]}$$

Fig. 23.3.2 *Spin-1/2 and U(2) Tableau branching diagrams*

$$\left|\begin{matrix}\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\downarrow}\\ \boxed{\downarrow}&\boxed{\downarrow}\end{matrix}\quad \begin{matrix}S=2\\ M_S=1\end{matrix}\right\rangle = C^{5/2\ 1/2\ 2}_{1/2\ 1/2\ 1}\left|\begin{matrix}\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\downarrow}&\boxed{\downarrow}\\ \boxed{\downarrow}&\boxed{\cdot\cdot}\end{matrix}\ \begin{matrix}5/2\\ 1/2\end{matrix}\right\rangle\left|\boxed{\uparrow}^{1/2}_{1/2}\right\rangle + C^{5/2\ 1/2\ 2}_{3/2\ -1/2\ 1}\left|\begin{matrix}\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\downarrow}\\ \boxed{\downarrow}&\boxed{\cdot\cdot}\end{matrix}\ \begin{matrix}5/2\\ 1/2\end{matrix}\right\rangle\left|\boxed{\downarrow}^{1/2}_{-1/2}\right\rangle$$

$$+C^{3/2\ 1/2\ 2}_{1/2\ 1/2\ 1}\left|\begin{matrix}\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\downarrow}&\boxed{\cdot\cdot}\\ \boxed{\downarrow}&\boxed{\downarrow}\end{matrix}\ \begin{matrix}3/2\\ 1/2\end{matrix}\right\rangle\left|\boxed{\uparrow}^{1/2}_{1/2}\right\rangle + C^{3/2\ 1/2\ 2}_{3/2\ -1/2\ 1}\left|\begin{matrix}\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\cdot\cdot}\\ \boxed{\downarrow}&\boxed{\downarrow}\end{matrix}\ \begin{matrix}3/2\\ 1/2\end{matrix}\right\rangle\left|\boxed{\downarrow}^{1/2}_{-1/2}\right\rangle$$

$$\begin{pmatrix} C^{j\ \ 1/2\ \ j+1/2}_{m\ 1/2\ m+1/2}=\sqrt{\dfrac{j+m+1}{2j+1}} & C^{j\ \ \ \ 1/2\ \ j+1/2}_{m+1\ -1/2\ m+1/2}=\sqrt{\dfrac{j-m}{2j+1}} \\ C^{j+1\ 1/2\ j+1/2}_{m\ \ \ 1/2\ m+1/2}=-\sqrt{\dfrac{j-m+1}{2j+3}} & C^{j+1\ \ 1/2\ \ j+1/2}_{m+1\ -1/2\ m+1/2}=\sqrt{\dfrac{j+m+2}{2j+3}}\end{pmatrix}\quad \text{example:}\ \begin{pmatrix} C^{3/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=\sqrt{\dfrac{2}{3}} & C^{3/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=\sqrt{\dfrac{1}{3}} \\ C^{5/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=-\sqrt{\dfrac{1}{3}} & C^{5/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=\sqrt{\dfrac{2}{3}}\end{pmatrix}$$

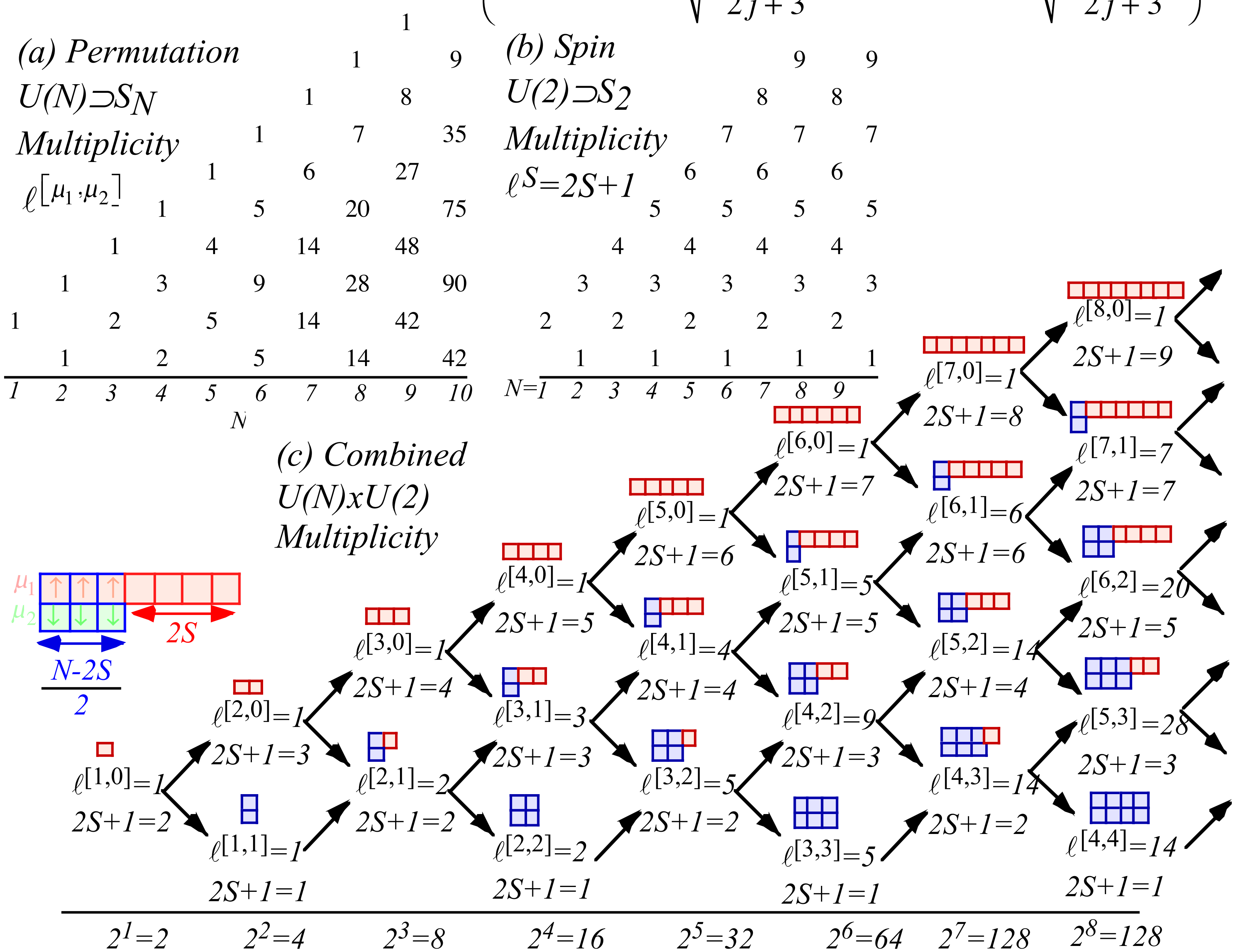


Fig. 23.3.2 *Spin-1/2 and U(2) Tableau branching diagrams*

2	3	4	5
1	2	3	4

5	4	3	2
4	3	2	1

$$2^N = \sum_S^{N/2} \ell^{[S]}\ell^{[\mu_1,\mu_2]}$$

$$= \sum_S^{N/2}(2S+1)\,\ell^{\left[\frac{N+2S}{2},\frac{N-2S}{2}\right]}$$

$8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1$

5	4	3	2
4	3	2	1

$$\left|\begin{array}{cccccc}\uparrow&\uparrow&\uparrow&\uparrow&\uparrow&\downarrow\\ \downarrow&\downarrow&&&&\end{array}\quad\begin{array}{c}S=2\\ M_S=1\end{array}\right\rangle = C^{5/2\ 1/2\ 2}_{1/2\ \ 1/2\ 1}\left|\begin{array}{cccccc}\uparrow&\uparrow&\uparrow&\uparrow&\downarrow&\downarrow\\ \downarrow&\boxed{\cdot\cdot}&&&&\end{array}\begin{array}{c}5/2\\ 1/2\end{array}\right\rangle\left|\boxed{\uparrow}^{1/2}_{1/2}\right\rangle + C^{5/2\ 1/2\ \ 2}_{3/2\ -1/2\ 1}\left|\begin{array}{cccccc}\uparrow&\uparrow&\uparrow&\uparrow&\uparrow&\downarrow\\ \downarrow&\boxed{\cdot\cdot}&&&&\end{array}\begin{array}{c}5/2\\ 1/2\end{array}\right\rangle\left|\boxed{\downarrow}^{1/2}_{-1/2}\right\rangle$$

$$+C^{3/2\ 1/2\ 2}_{1/2\ \ 1/2\ 1}\left|\begin{array}{cccccc}\uparrow&\uparrow&\uparrow&\uparrow&\downarrow&\boxed{\cdot\cdot}\\ \downarrow&\downarrow&&&&\end{array}\begin{array}{c}3/2\\ 1/2\end{array}\right\rangle\left|\boxed{\uparrow}^{1/2}_{1/2}\right\rangle + C^{3/2\ 1/2\ \ 2}_{3/2\ -1/2\ 1}\left|\begin{array}{cccccc}\uparrow&\uparrow&\uparrow&\uparrow&\uparrow&\boxed{\cdot\cdot}\\ \downarrow&\downarrow&&&&\end{array}\begin{array}{c}3/2\\ 1/2\end{array}\right\rangle\left|\boxed{\downarrow}^{1/2}_{-1/2}\right\rangle$$

$$\begin{pmatrix} C^{j\ \ 1/2\ \ j+1/2}_{m\ 1/2\ \ m+1/2}=\sqrt{\dfrac{j+m+1}{2j+1}} & C^{j\ \ \ \ \ 1/2\ \ \ j+1/2}_{m+1\ -1/2\ \ m+1/2}=\sqrt{\dfrac{j-m}{2j+1}} \\ C^{j+1\ 1/2\ \ j+1/2}_{m\ \ \ \ 1/2\ \ m+1/2}=-\sqrt{\dfrac{j-m+1}{2j+3}} & C^{j+1\ \ \ 1/2\ \ j+1/2}_{m+1\ -1/2\ \ m+1/2}=\sqrt{\dfrac{j+m+2}{2j+3}}\end{pmatrix}$$

example:
$$\begin{pmatrix} C^{3/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=\sqrt{\dfrac{2}{3}} & C^{3/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=\sqrt{\dfrac{1}{3}} \\ C^{5/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=-\sqrt{\dfrac{1}{3}} & C^{5/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=\sqrt{\dfrac{2}{3}}\end{pmatrix}$$

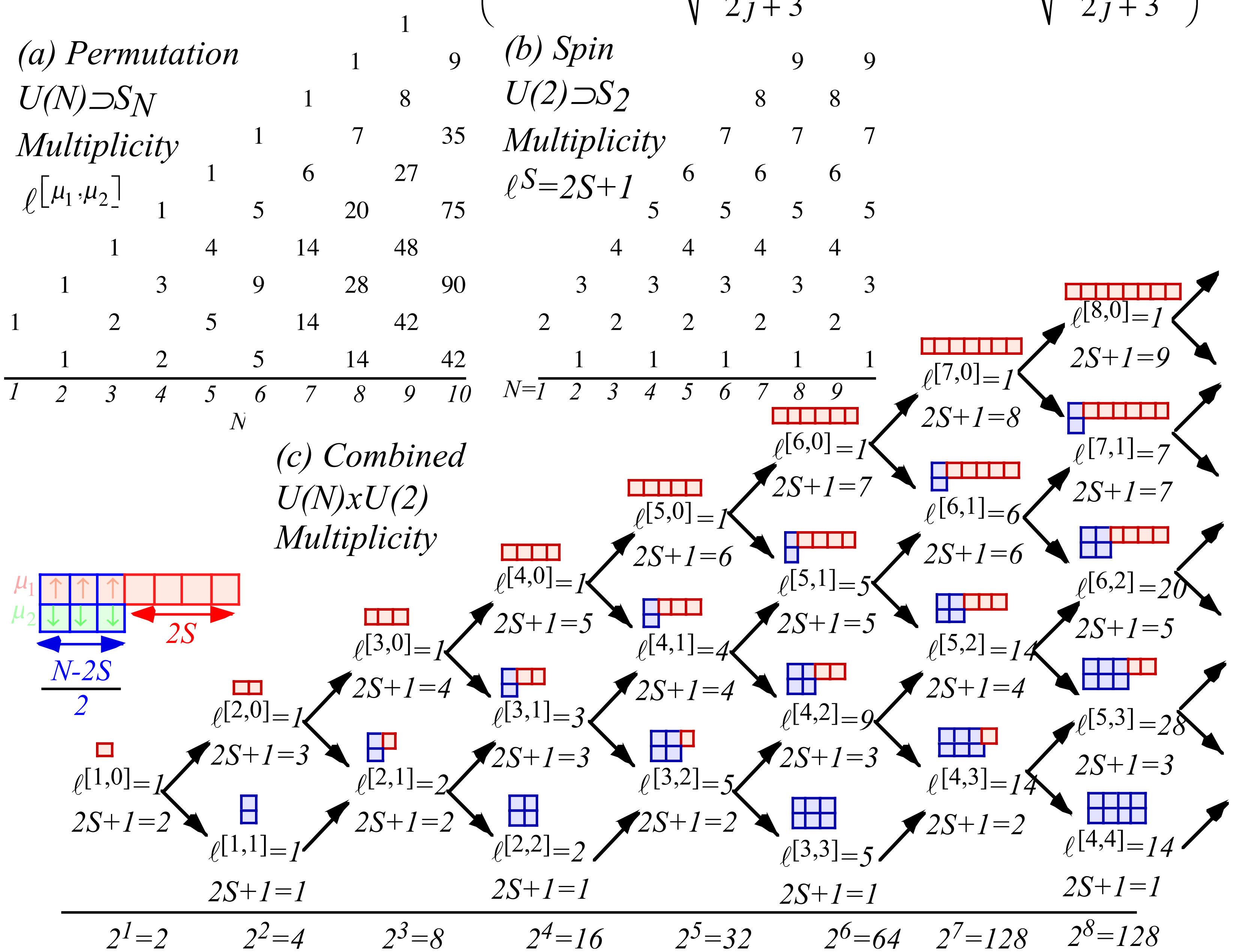


Fig. 23.3.2 Spin-1/2 and U(2) Tableau branching diagrams

2	3	4	5
1	2	3	4

5	4	3	2
4	3	2	1

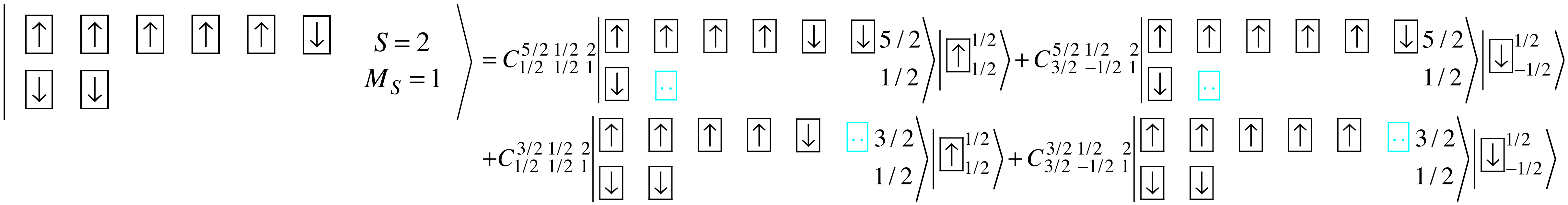
$$2^N = \sum_S^{N/2} \ell^{[S]}\ell^{[\mu_1,\mu_2]}$$

$$= \sum_S^{N/2}(2S+1)\ell^{\left[\frac{N+2S}{2},\frac{N-2S}{2}\right]}$$

$8\cdot7\cdot6\cdot\cancel{5\cdot4\cdot3\cdot2\cdot1}$

~~5~~	~~4~~	~~3~~	~~2~~
4	3	2	1

Magic squares - Intro to Young Tableaus



$$\begin{pmatrix} C^{j\ \ 1/2\ \ j+1/2}_{m\ \ 1/2\ \ m+1/2} = \sqrt{\dfrac{j+m+1}{2j+1}} & C^{j\ \ \ \ 1/2\ \ j+1/2}_{m+1\ \ -1/2\ \ m+1/2} = \sqrt{\dfrac{j-m}{2j+1}} \\ C^{j+1\ \ 1/2\ \ j+1/2}_{m\ \ \ \ 1/2\ \ m+1/2} = -\sqrt{\dfrac{j-m+1}{2j+3}} & C^{j+1\ \ \ 1/2\ \ j+1/2}_{m+1\ \ -1/2\ \ m+1/2} = \sqrt{\dfrac{j+m+2}{2j+3}} \end{pmatrix}$$

example:
$$\begin{pmatrix} C^{3/2\ \ 1/2\ \ 2}_{1/2\ \ 1/2\ \ 1} = \sqrt{\dfrac{2}{3}} & C^{3/2\ \ 1/2\ \ 2}_{1/2\ \ 1/2\ \ 1} = \sqrt{\dfrac{1}{3}} \\ C^{5/2\ \ 1/2\ \ 2}_{1/2\ \ 1/2\ \ 1} = -\sqrt{\dfrac{1}{3}} & C^{5/2\ \ 1/2\ \ 2}_{1/2\ \ 1/2\ \ 1} = \sqrt{\dfrac{2}{3}} \end{pmatrix}$$

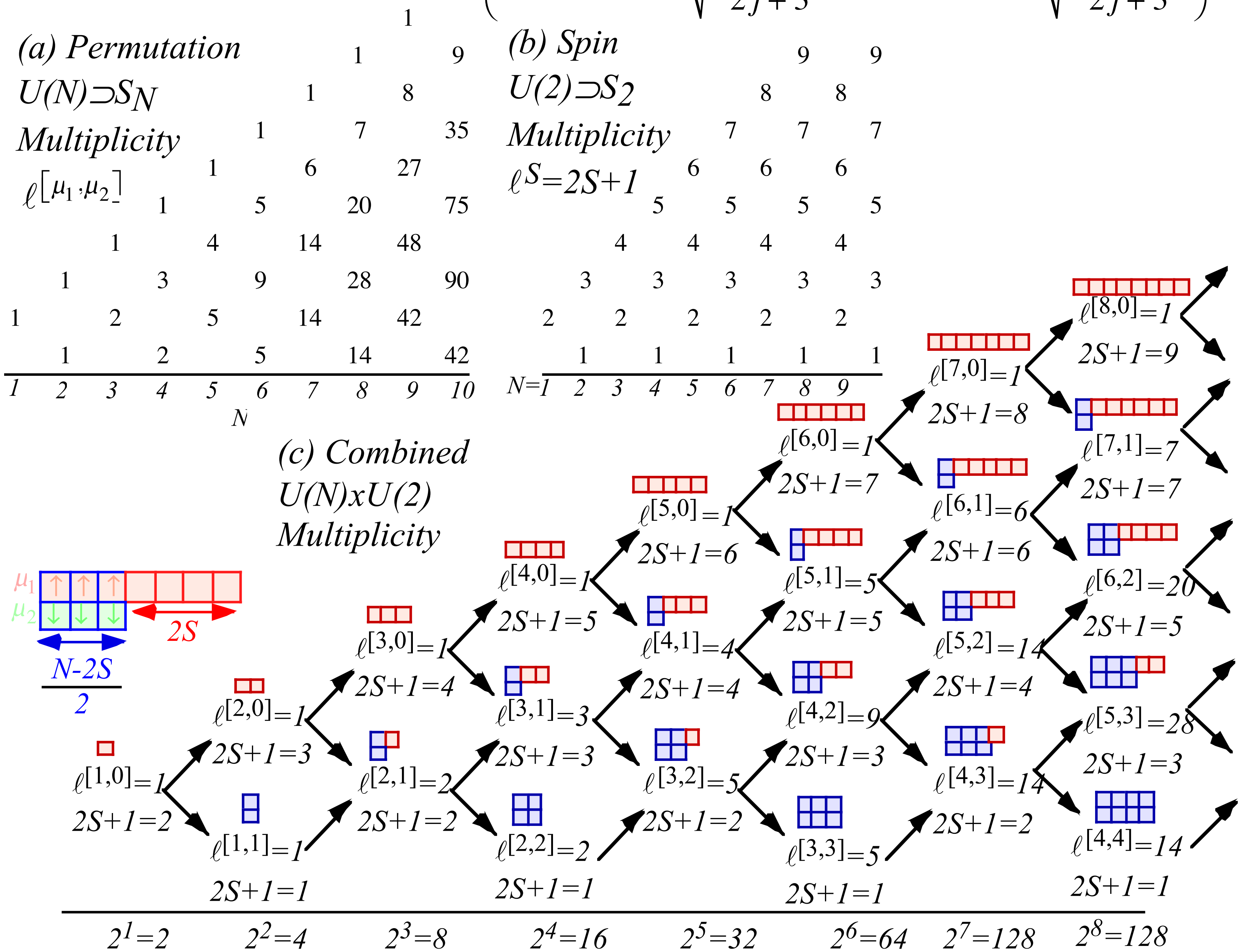


***Fig. 23.3.2** Spin-1/2 and U(2) Tableau branching diagrams*

2	3	4	5
1	2	3	4

5	4	3	2
4	3	2	1

=1

$$2^N = \sum_S^{N/2} \ell^{[S]} \ell^{[\mu_1,\mu_2]}$$

$$= \sum_S^{N/2} (2S+1)\ell^{\left[\frac{N+2S}{2},\frac{N-2S}{2}\right]}$$

2

8·7·6·5·4·3·2·1

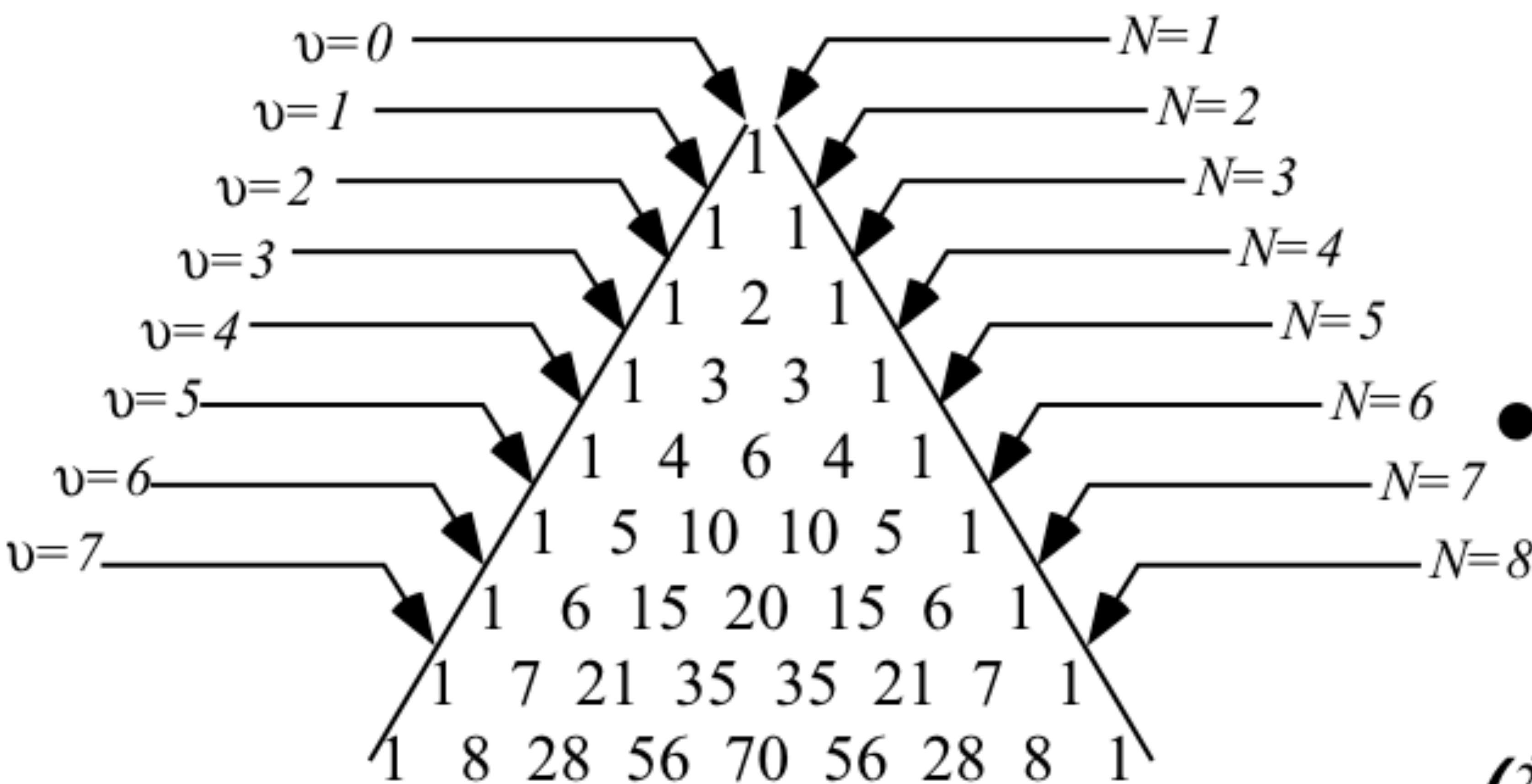
5	4	3	2
4	3	2	1

=14

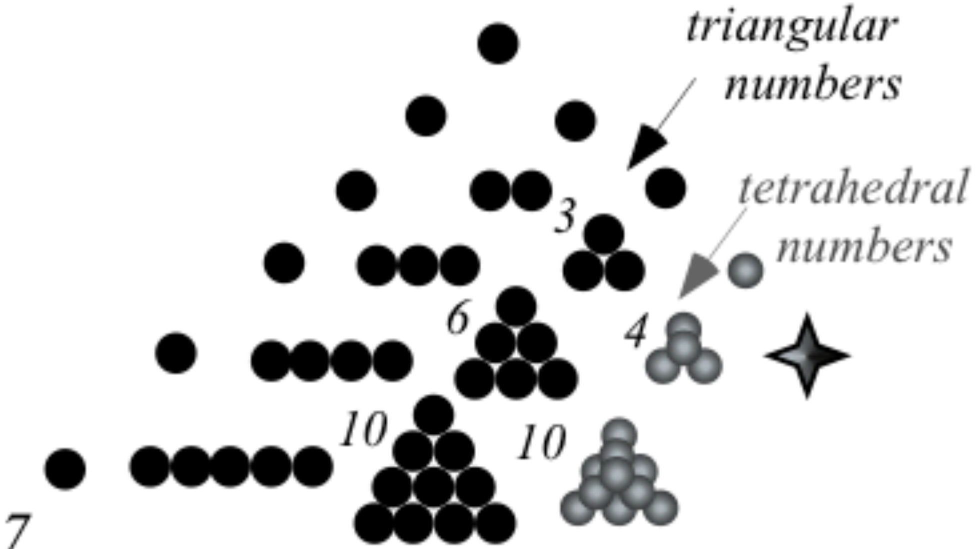
Introducing U(N)

(a) N-D Oscillator Degeneracy ℓ of quamtum level υ

Principal Quantum Number *Dimension of oscillator*

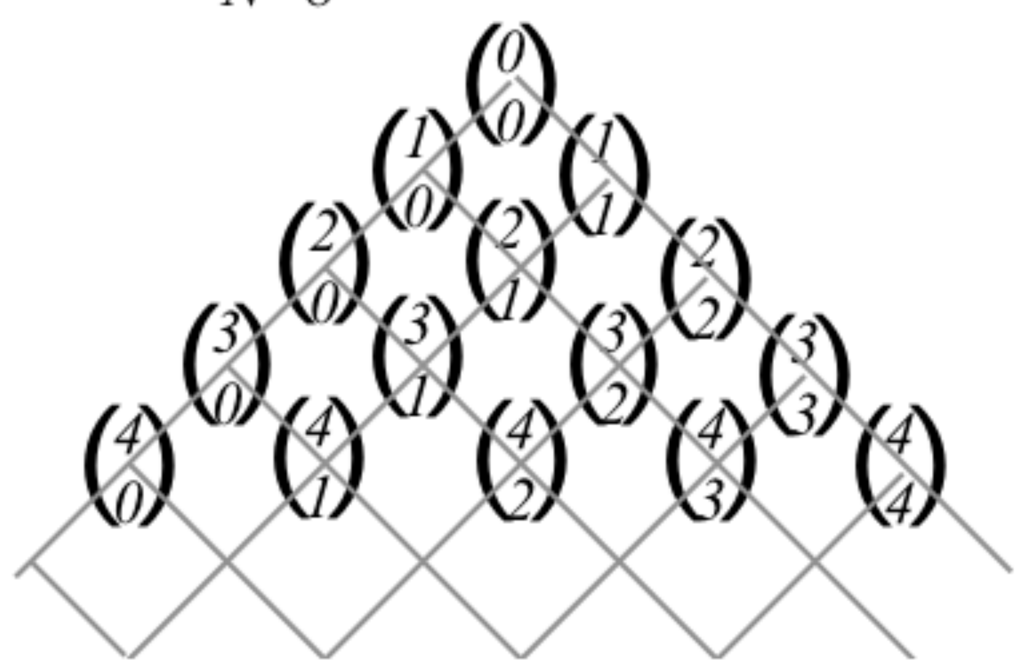


(b) Stacking numbers



(c) Binomial coefficients

$$\frac{(N\text{-}1+\upsilon)!}{(N\text{-}1)!\,\upsilon!} = \binom{N\text{-}1+\upsilon}{\upsilon} = \binom{N\text{-}1+\upsilon}{N\text{-}1}$$



Introducing U(3)

(b) N-particle 3-level states ...or spin-1 states

$\boxed{1} = |1\ 0\ 0\rangle = a_1^\dagger\,|0\ 0\ 0\rangle$

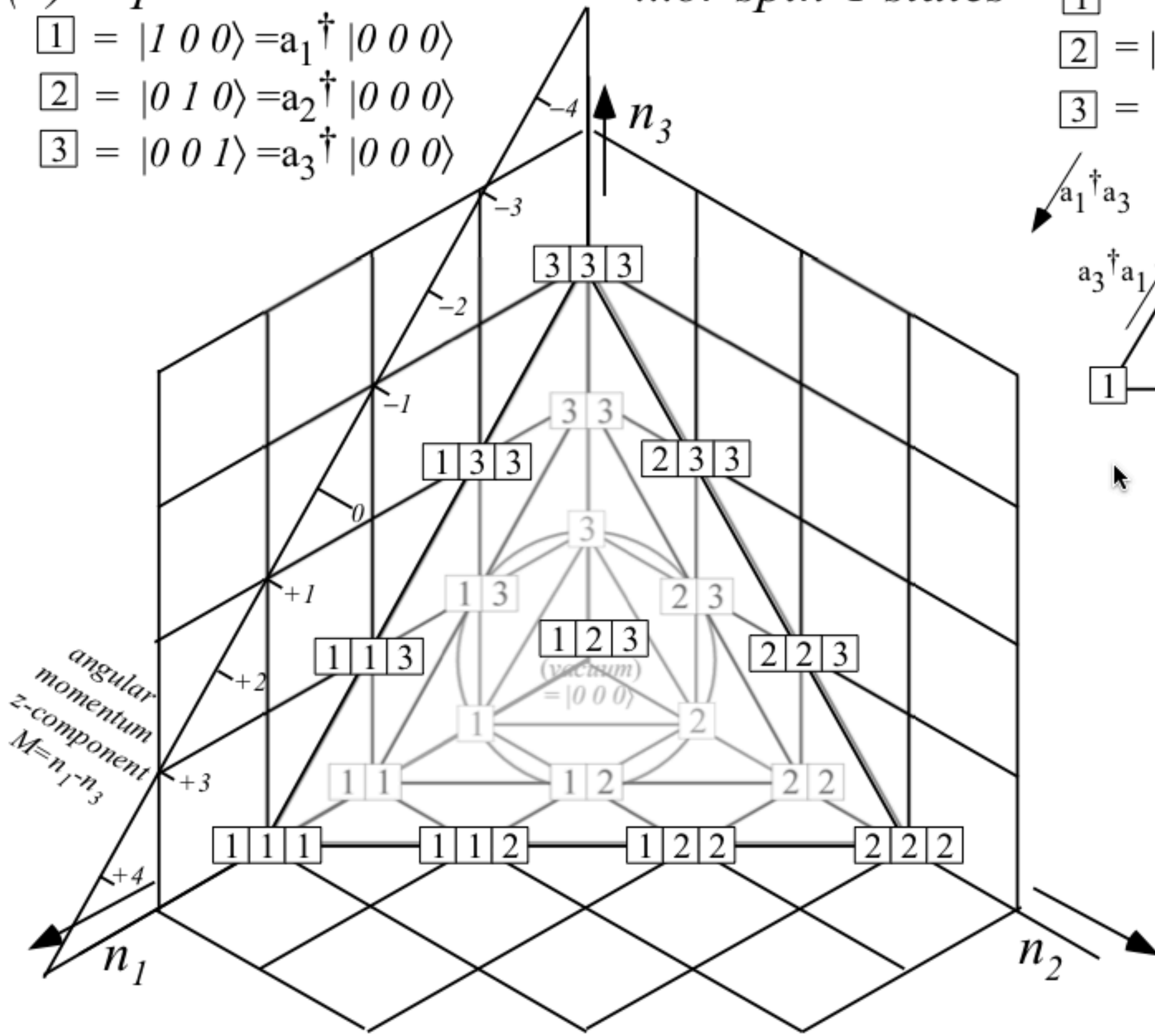
$\boxed{2} = |0\ 1\ 0\rangle = a_2^\dagger\,|0\ 0\ 0\rangle$

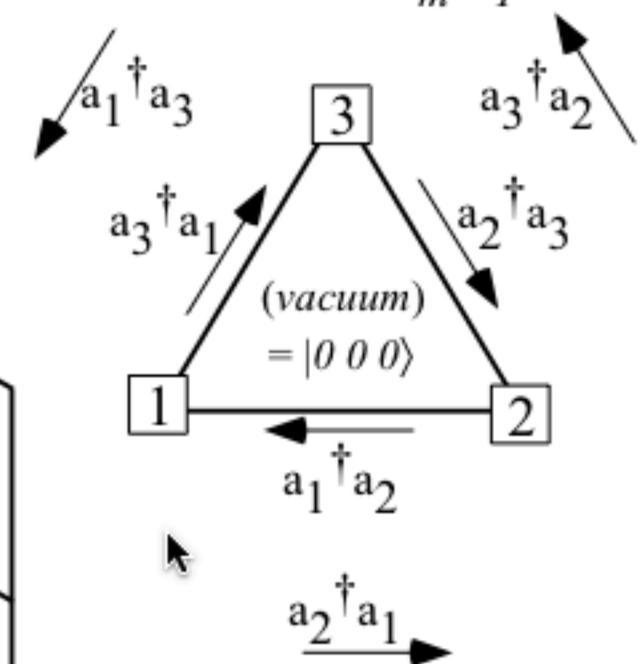
$\boxed{3} = |0\ 0\ 1\rangle = a_3^\dagger\,|0\ 0\ 0\rangle$

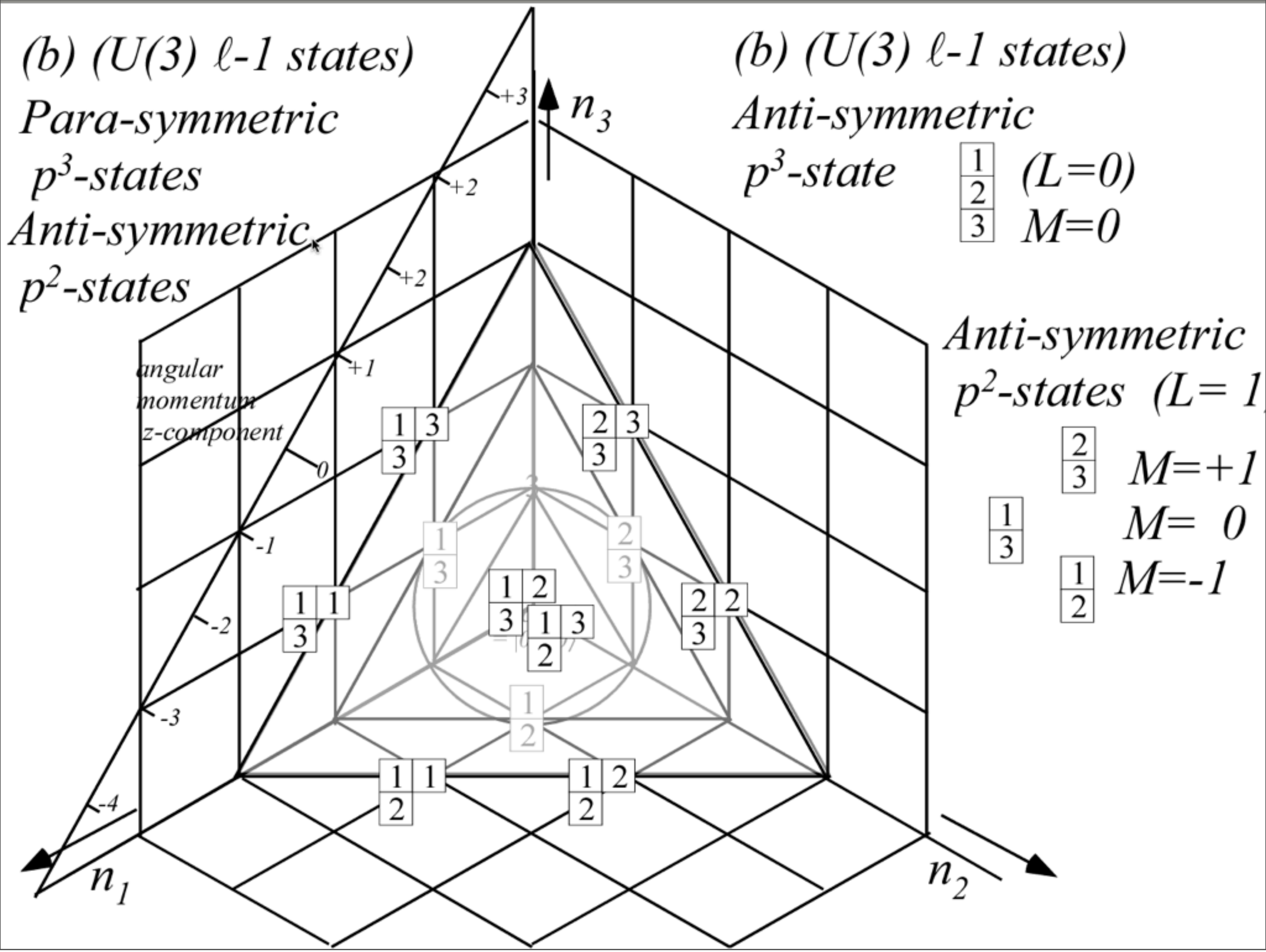
$\boxed{1} = |\uparrow\rangle = \left|{}^{j=1}_{m=+1}\right\rangle$

$\boxed{2} = |\leftrightarrow\rangle = \left|{}^{j=1}_{m=0}\right\rangle$

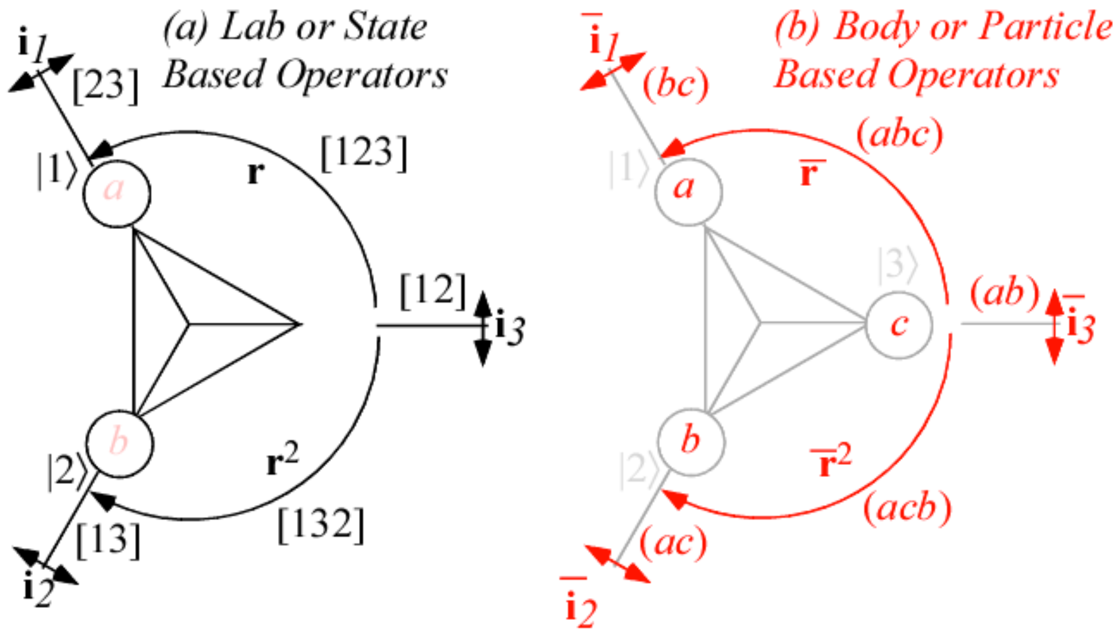
$\boxed{3} = |\downarrow\rangle = \left|{}^{j=1}_{m=-1}\right\rangle$

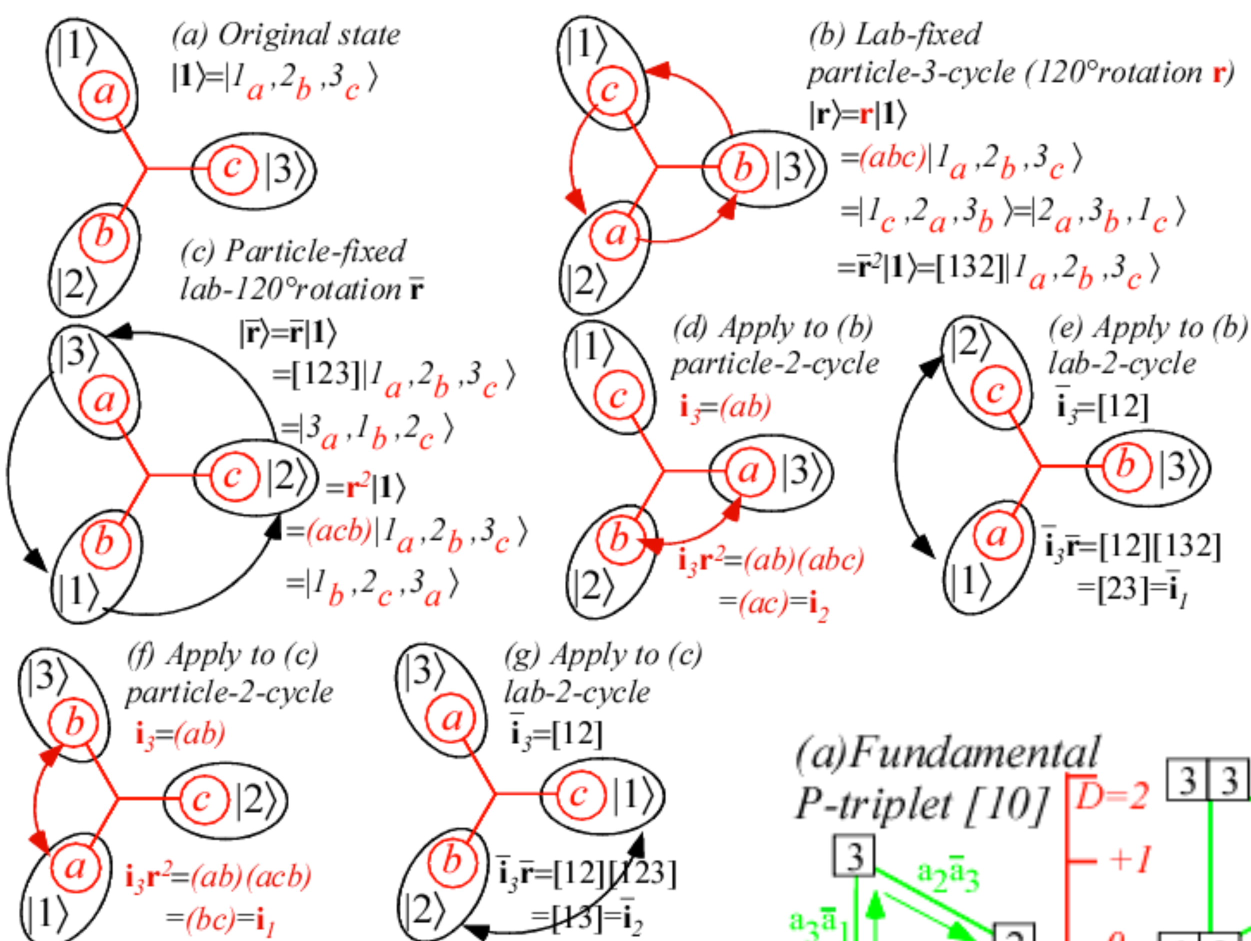


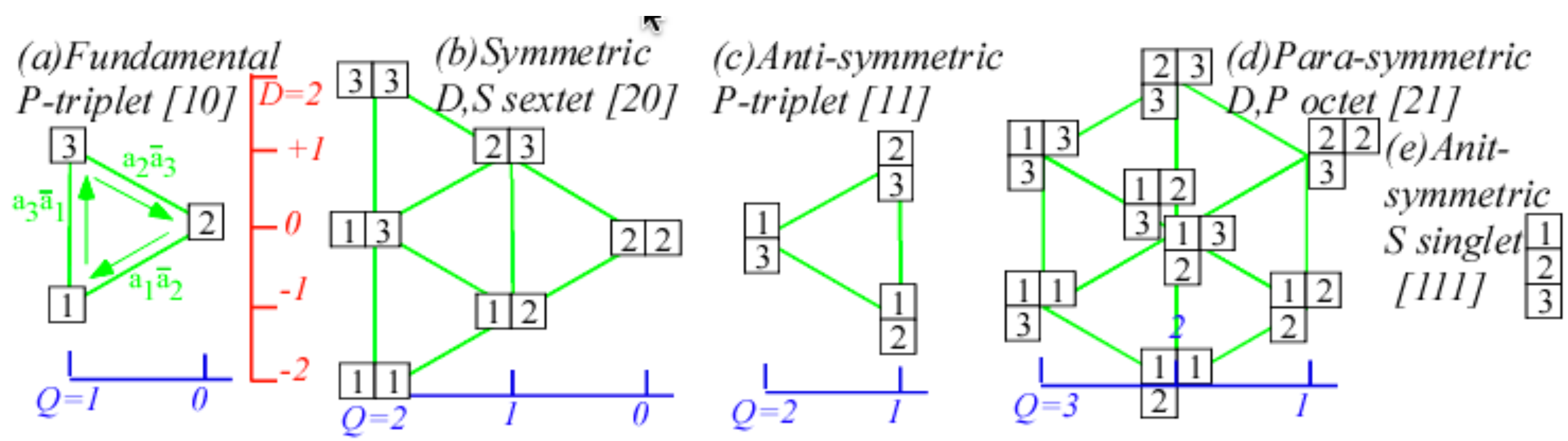




(b) (U(3) ℓ-1 states)
Para-symmetric
p3-states
Anti-symmetric
p2-states
(b) (U(3) ℓ-1 states)
Anti-symmetric
p3-state
(L=0)
M=0
Anti-symmetric
p2-states (L= 1)
M=+1
M= 0
M=-1
angular
momentum
z-component
+3
+2
+2
+1
0
-1
-2
-3
-4
n3
n1
n2


(a) Lab or State
Based Operators
[23]
[123]
[12]
[13]
[132]
(b) Body or Particle
Based Operators
(bc)
(abc)
(ab)
(ac)
(acb)


(a) Original state
|1⟩=|1_a,2_b,3_c⟩
(b) Lab-fixed
particle-3-cycle (120°rotation r)
|r⟩=r|1⟩
=(abc)|1_a,2_b,3_c⟩
=|1_c,2_a,3_b⟩=|2_a,3_b,1_c⟩
=r̄²|1⟩=[132]|1_a,2_b,3_c⟩
(c) Particle-fixed
lab-120°rotation r̄
|r̄⟩=r̄|1⟩
=[123]|1_a,2_b,3_c⟩
=|3_a,1_b,2_c⟩
=r²|1⟩
=(acb)|1_a,2_b,3_c⟩
=|1_b,2_c,3_a⟩
(d) Apply to (b)
particle-2-cycle
i_3=(ab)
i_3r²=(ab)(abc)
=(ac)=i_2
(e) Apply to (b)
lab-2-cycle
ī_3=[12]
ī_3r̄=[12][132]
=[23]=ī_1
(f) Apply to (c)
particle-2-cycle
i_3=(ab)
i_3r²=(ab)(acb)
=(bc)=i_1
(g) Apply to (c)
lab-2-cycle
ī_3=[12]
ī_3r̄=[12][123]
=[13]=ī_2


(a)Fundamental
P-triplet [10]
D=2
+1
0
-1
-2
Q=1
0
(b)Symmetric
D,S sextet [20]
Q=2
1
0
(c)Anti-symmetric
P-triplet [11]
Q=2
1
(d)Para-symmetric
D,P octet [21]
Q=3
1
(e)Anit-
symmetric
S singlet
[111]

Tensor operators for spin-1/2 states: Outer products give Hamilton-Pauli-spinors

CG-Products of spin-$^1/_2$ **ket-bras** $\left\{\left|{1/2 \atop m_1}\right\rangle,\left\langle{1/2 \atop m_2}\right|\right\}$ give scalar/vector operators analogous to: **ket-kets**

$$T^k_q = \sum_{m_1} C^{1/2\;1/2\;k}_{m_1\;m_2\;q}\left|{1/2 \atop m_1}\right\rangle\left\langle{1/2 \atop -m_2}\right|(-1)^{\frac{1}{2}-m_2}$$

analogous to:

$$\left|{J \atop M}(1/2\otimes 1/2)\right\rangle = \sum_{m_1,m_2} C^{1/2\;1/2\;J}_{m_1\;m_2\;M}\left|{1/2 \atop m_1}\right\rangle\left|{1/2 \atop m_2}\right\rangle$$

$$T^1_{-1}=\begin{pmatrix}0&0\\-1&0\end{pmatrix} = -\left|{1/2 \atop -1/2}\right\rangle\left\langle{1/2 \atop 1/2}\right|,\qquad T^1_0=\frac{1}{\sqrt2}\begin{pmatrix}-1&0\\0&1\end{pmatrix} = -\frac{1}{\sqrt2}\left[\left|{1/2 \atop 1/2}\right\rangle\left\langle{1/2 \atop 1/2}\right| - \left|{1/2 \atop -1/2}\right\rangle\left\langle{1/2 \atop -1/2}\right|\right],\qquad T^1_1=\begin{pmatrix}0&1\\0&0\end{pmatrix} = \left|{1/2 \atop 1/2}\right\rangle\left\langle{1/2 \atop -1/2}\right|,$$

analogous to:

$$\left|{1 \atop 1}(1/2\otimes1/2)\right\rangle = \left|{1/2 \atop 1/2}\right\rangle\left|{1/2 \atop 1/2}\right\rangle$$

$$\left|{1 \atop 0}(1/2\otimes1/2)\right\rangle = \frac{1}{\sqrt2}\left|{1/2 \atop 1/2}\right\rangle\left|{1/2 \atop -1/2}\right\rangle + \frac{1}{\sqrt2}\left|{1/2 \atop -1/2}\right\rangle\left|{1/2 \atop 1/2}\right\rangle$$

$$\left|{1 \atop -1}(1/2\otimes1/2)\right\rangle = \left|{1/2 \atop -1/2}\right\rangle\left|{1/2 \atop -1/2}\right\rangle$$

$$T^0_0 = -\frac{1}{\sqrt2}\begin{pmatrix}1&0\\0&1\end{pmatrix} = -\frac{1}{\sqrt2}\left[\left|{1/2 \atop 1/2}\right\rangle\left\langle{1/2 \atop 1/2}\right| + \left|{1/2 \atop -1/2}\right\rangle\left\langle{1/2 \atop -1/2}\right|\right].$$

analogous to:

$$\left|{0 \atop 0}(1/2\otimes1/2)\right\rangle = \frac{1}{\sqrt2}\left|{1/2 \atop 1/2}\right\rangle\left|{1/2 \atop -1/2}\right\rangle + \frac{-1}{\sqrt2}\left|{1/2 \atop -1/2}\right\rangle\left|{1/2 \atop 1/2}\right\rangle$$

1st three operators are a *vector* set with following Cartesian combinations:

$$T_x \equiv -\frac{T^1_{-1}-T^1_1}{\sqrt2} = \frac{1}{\sqrt2}\begin{pmatrix}0&1\\1&0\end{pmatrix} \equiv \frac{1}{\sqrt2}\sigma_x \equiv \sqrt2 J_x$$

$$T_y \equiv -i\frac{T^1_{-1}+T^1_1}{\sqrt2} = \frac{1}{\sqrt2}\begin{pmatrix}0&-i\\i&0\end{pmatrix} \equiv \frac{1}{\sqrt2}\sigma_y \equiv \sqrt2 J_y$$

$$T_z \equiv -T^1_0 = \frac{1}{\sqrt2}\begin{pmatrix}1&0\\0&1\end{pmatrix} \equiv \frac{1}{\sqrt2}\sigma_z \equiv \sqrt2 J_z$$

(Some old friends!)

$$\sigma_X \rightarrow \begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad \sigma_Y \rightarrow \begin{pmatrix}0&-i\\i&0\end{pmatrix},\qquad \sigma_Z \rightarrow \begin{pmatrix}1&0\\0&-1\end{pmatrix},$$

Spherical vs. Cartesian operators

$$T^1_{-1} = J_-/2 = \left(J_x - iJ_y\right)/\sqrt2,\qquad T^1_0 = J_z/\sqrt2,\qquad T^1_{-1} = J_+/2 = \left(J_x + iJ_y\right)/2.$$

Tensor operators for spin-1/2 states: Outer products give Hamilton-Pauli-spinors

CG-Products of spin-1/2 **ket-bras** $\left\{\left|\begin{smallmatrix}1/2\\ m_1\end{smallmatrix}\right\rangle, \left\langle\begin{smallmatrix}1/2\\ m_2\end{smallmatrix}\right|\right\}$ give scalar/vector operators analogous to: **ket-kets**

$$T^k_q = \sum_{m_1} C^{1/2\;1/2\;k}_{m_1\;m_2\;q}\left|\begin{matrix}1/2\\ m_1\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\ -m_2\end{matrix}\right|(-1)^{\frac{1}{2}-m_2}$$

analogous to:

$$\left|\begin{matrix}J\\ M\end{matrix}\;(1/2\otimes 1/2)\right\rangle = \sum_{m_1,m_2} C^{1/2\;1/2\;J}_{m_1\;m_2\;M}\left|\begin{matrix}1/2\\ m_1\end{matrix}\right\rangle\left|\begin{matrix}1/2\\ m_2\end{matrix}\right\rangle$$

$$T^1_{-1} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = -\left|\begin{matrix}1/2\\ -1/2\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\ 1/2\end{matrix}\right|,\quad T^1_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\frac{1}{\sqrt{2}}\left[\left|\begin{matrix}1/2\\ 1/2\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\ 1/2\end{matrix}\right| - \left|\begin{matrix}1/2\\ -1/2\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\ -1/2\end{matrix}\right|\right],\quad T^1_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \left|\begin{matrix}1/2\\ 1/2\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\ -1/2\end{matrix}\right|,$$

analogous to:

$$\left|\begin{matrix}1\\ 1\end{matrix}\;(1/2\otimes 1/2)\right\rangle = \left|\begin{matrix}1/2\\ 1/2\end{matrix}\right\rangle\left|\begin{matrix}1/2\\ 1/2\end{matrix}\right\rangle$$

$$\left|\begin{matrix}1\\ 0\end{matrix}\;(1/2\otimes 1/2)\right\rangle = \frac{1}{\sqrt{2}}\left|\begin{matrix}1/2\\ 1/2\end{matrix}\right\rangle\left|\begin{matrix}1/2\\ -1/2\end{matrix}\right\rangle + \frac{1}{\sqrt{2}}\left|\begin{matrix}1/2\\ -1/2\end{matrix}\right\rangle\left|\begin{matrix}1/2\\ 1/2\end{matrix}\right\rangle$$

$$\left|\begin{matrix}1\\ -1\end{matrix}\;(1/2\otimes 1/2)\right\rangle = \left|\begin{matrix}1/2\\ -1/2\end{matrix}\right\rangle\left|\begin{matrix}1/2\\ -1/2\end{matrix}\right\rangle$$

$$T^0_0 = -\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\frac{1}{\sqrt{2}}\left[\left|\begin{matrix}1/2\\ 1/2\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\ 1/2\end{matrix}\right| + \left|\begin{matrix}1/2\\ -1/2\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\ -1/2\end{matrix}\right|\right].$$

analogous to:

$$\left|\begin{matrix}0\\ 0\end{matrix}\;(1/2\otimes 1/2)\right\rangle = \frac{1}{\sqrt{2}}\left|\begin{matrix}1/2\\ 1/2\end{matrix}\right\rangle\left|\begin{matrix}1/2\\ -1/2\end{matrix}\right\rangle + \frac{-1}{\sqrt{2}}\left|\begin{matrix}1/2\\ -1/2\end{matrix}\right\rangle\left|\begin{matrix}1/2\\ 1/2\end{matrix}\right\rangle$$

1st three operators are a *vector* set that *transform like a vector set*

$$\begin{array}{ccccccc}
R(0\beta 0) & & T^1_0 & & R^\dagger(0\beta 0) & = & T'_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} & & \begin{pmatrix} -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix} & & \begin{pmatrix} \cos\frac{\beta}{2} & \sin\frac{\beta}{2} \\ -\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} & = & -\frac{1}{\sqrt{2}}\begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{pmatrix}
\end{array}$$

$$= D^1_{10}(0\beta 0)T^1_1 \quad + D^1_{00}(0\beta 0)T^1_0 \quad + D^1_{-10}(0\beta 0)T^1_{-1}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$= \frac{-\sin\beta}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \cos\beta\begin{pmatrix} -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix} + \frac{\sin\beta}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

Tensor operators for spin-1/2 states: Outer products give Hamilton-Pauli-spinors

CG-Products of spin-$^1/_2$ **ket-bras** $\left\{\left|{}^{1/2}_{m_1}\right\rangle, \left\langle{}^{1/2}_{m_2}\right|\right\}$ give scalar/vector operators analogous to: **ket-kets**

$$T^k_q = \sum_{m_1} C^{1/2\;1/2\;k}_{m_1\;m_2\;q} \left|{}^{1/2}_{m_1}\right\rangle\left\langle{}^{1/2}_{-m_2}\right| (-1)^{\frac{1}{2}-m_2}$$

analogous to:

$$\left|{}^{J}_{M}\;(1/2\otimes 1/2)\right\rangle = \sum_{m_1,m_2} C^{1/2\;1/2\;J}_{m_1\;m_2\;M}\left|{}^{1/2}_{m_1}\right\rangle\left|{}^{1/2}_{m_2}\right\rangle$$

$$T^1_{-1} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = -\left|{}^{1/2}_{-1/2}\right\rangle\left\langle{}^{1/2}_{1/2}\right|, \quad T^1_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\frac{1}{\sqrt{2}}\left[\left|{}^{1/2}_{1/2}\right\rangle\left\langle{}^{1/2}_{1/2}\right| - \left|{}^{1/2}_{-1/2}\right\rangle\left\langle{}^{1/2}_{-1/2}\right|\right], \quad T^1_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \left|{}^{1/2}_{1/2}\right\rangle\left\langle{}^{1/2}_{-1/2}\right|,$$

analogous to:

$$\left|{}^{1}_{1}\;(1/2\otimes 1/2)\right\rangle = \left|{}^{1/2}_{1/2}\right\rangle\left|{}^{1/2}_{1/2}\right\rangle$$

$$\left|{}^{1}_{0}\;(1/2\otimes 1/2)\right\rangle = \frac{1}{\sqrt{2}}\left|{}^{1/2}_{1/2}\right\rangle\left|{}^{1/2}_{-1/2}\right\rangle + \frac{1}{\sqrt{2}}\left|{}^{1/2}_{-1/2}\right\rangle\left|{}^{1/2}_{1/2}\right\rangle$$

$$\left|{}^{1}_{-1}\;(1/2\otimes 1/2)\right\rangle = \left|{}^{1/2}_{-1/2}\right\rangle\left|{}^{1/2}_{-1/2}\right\rangle$$

$$T^0_0 = -\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\frac{1}{\sqrt{2}}\left[\left|{}^{1/2}_{1/2}\right\rangle\left\langle{}^{1/2}_{1/2}\right| + \left|{}^{1/2}_{-1/2}\right\rangle\left\langle{}^{1/2}_{-1/2}\right|\right].$$

analogous to:

$$\left|{}^{0}_{0}\;(1/2\otimes 1/2)\right\rangle = \frac{1}{\sqrt{2}}\left|{}^{1/2}_{1/2}\right\rangle\left|{}^{1/2}_{-1/2}\right\rangle + \frac{-1}{\sqrt{2}}\left|{}^{1/2}_{-1/2}\right\rangle\left|{}^{1/2}_{1/2}\right\rangle$$

1st three operators are a *vector* set that *transform like a vector set*

so do expectation values

$$R(0\beta 0)\quad T^1_0\quad R^\dagger(0\beta 0) \quad = \quad T'_0$$

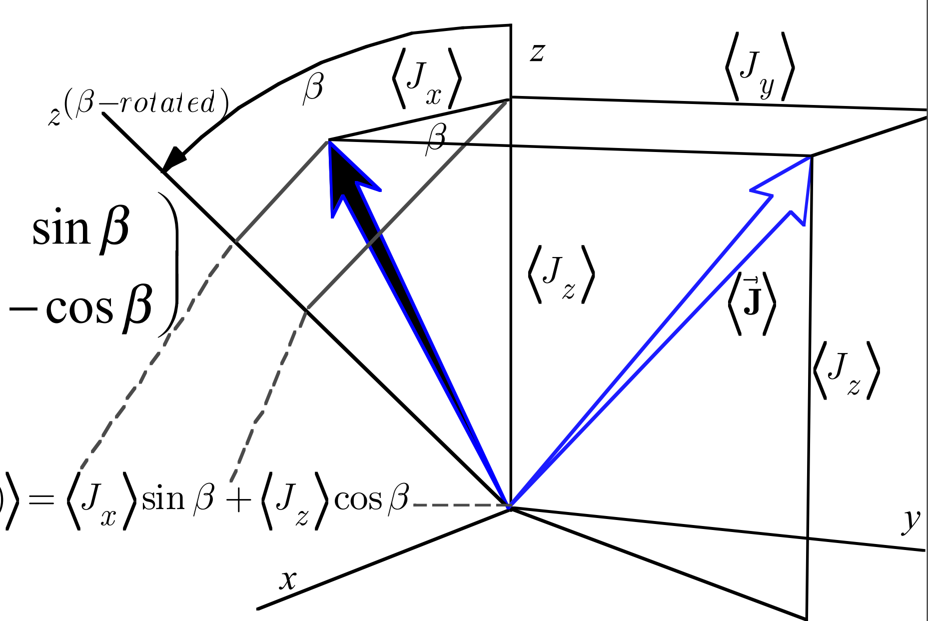
$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & \sin\frac{\beta}{2} \\ -\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} = -\frac{1}{\sqrt{2}}\begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{pmatrix}$$

$$= D^1_{10}(0\beta 0)T^1_1 + D^1_{00}(0\beta 0)T^1_0 + D^1_{-10}(0\beta 0)T^1_{-1}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$= \frac{-\sin\beta}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \cos\beta\begin{pmatrix} -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix} + \frac{\sin\beta}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$



$$\langle J_z(\beta - rotated)\rangle = \langle J_x\rangle\sin\beta + \langle J_z\rangle\cos\beta$$

Tensor operators for spin-1 states: U(1) generalization of Pauli spinors

CGC definition:

$$\mathbf{v}^k_q = \sum_{m,m'} C^{j\ \ j\ \ \ k}_{m\ -m'\ q}(-1)^{j-m'}\left|\begin{matrix}j\\m\end{matrix}\right\rangle\left\langle\begin{matrix}j\\m'\end{matrix}\right| = (-1)^{2j}\,T^k_q .$$

Wigner 3jm definition:

$$\mathbf{v}^k_q = \sum_{m,m'}(-1)^{j-m}\sqrt{2k+1}\begin{pmatrix}k & j & j\\ q & m' & -m\end{pmatrix}\left|\begin{matrix}j\\m\end{matrix}\right\rangle\left\langle\begin{matrix}j\\m'\end{matrix}\right|$$

$$T^2_{-2} = \left|\begin{matrix}1\\-1\end{matrix}\right\rangle\left\langle\begin{matrix}1\\1\end{matrix}\right|,\quad T^2_{-1} = \frac{\left|\begin{matrix}1\\0\end{matrix}\right\rangle\left\langle\begin{matrix}1\\1\end{matrix}\right| - \left|\begin{matrix}1\\-1\end{matrix}\right\rangle\left\langle\begin{matrix}1\\0\end{matrix}\right|}{\sqrt{2}},\quad T^2_0 = \frac{\left|\begin{matrix}1\\1\end{matrix}\right\rangle\left\langle\begin{matrix}1\\1\end{matrix}\right| - 2\left|\begin{matrix}1\\0\end{matrix}\right\rangle\left\langle\begin{matrix}1\\0\end{matrix}\right| + \left|\begin{matrix}1\\-1\end{matrix}\right\rangle\left\langle\begin{matrix}1\\-1\end{matrix}\right|}{\sqrt{6}},\quad T^2_1 = \frac{-\left|\begin{matrix}1\\1\end{matrix}\right\rangle\left\langle\begin{matrix}1\\0\end{matrix}\right| + \left|\begin{matrix}1\\0\end{matrix}\right\rangle\left\langle\begin{matrix}1\\-1\end{matrix}\right|}{\sqrt{2}},\quad T^2_2 = \left|\begin{matrix}1\\1\end{matrix}\right\rangle\left\langle\begin{matrix}1\\-1\end{matrix}\right|$$

$$\rightarrow\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}\quad \rightarrow\begin{pmatrix}0&0&0\\1/\sqrt{2}&0&0\\0&-1/\sqrt{2}&0\end{pmatrix}\quad \rightarrow\begin{pmatrix}1/\sqrt{6}&0&0\\0&-2/\sqrt{6}&0\\0&0&1/\sqrt{6}\end{pmatrix}\quad \rightarrow\begin{pmatrix}0&-1/\sqrt{2}&0\\0&0&1/\sqrt{2}\\0&0&0\end{pmatrix}\quad \rightarrow\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}$$

$$T^1_{-1} = \frac{\left|\begin{matrix}1\\0\end{matrix}\right\rangle\left\langle\begin{matrix}1\\1\end{matrix}\right| + \left|\begin{matrix}1\\-1\end{matrix}\right\rangle\left\langle\begin{matrix}1\\0\end{matrix}\right|}{\sqrt{2}},\quad T^1_0 = \frac{\left|\begin{matrix}1\\1\end{matrix}\right\rangle\left\langle\begin{matrix}1\\1\end{matrix}\right| - \left|\begin{matrix}1\\-1\end{matrix}\right\rangle\left\langle\begin{matrix}1\\-1\end{matrix}\right|}{\sqrt{2}},\quad T^1_1 = \frac{-\left|\begin{matrix}1\\1\end{matrix}\right\rangle\left\langle\begin{matrix}1\\0\end{matrix}\right| - \left|\begin{matrix}1\\0\end{matrix}\right\rangle\left\langle\begin{matrix}1\\-1\end{matrix}\right|}{\sqrt{2}}$$

$$\rightarrow\begin{pmatrix}0&0&0\\1/\sqrt{2}&0&0\\0&1/\sqrt{2}&0\end{pmatrix}\quad \rightarrow\begin{pmatrix}1/\sqrt{2}&0&0\\0&0&0\\0&0&-1/\sqrt{2}\end{pmatrix}\quad \rightarrow\begin{pmatrix}0&-1/\sqrt{2}&0\\0&0&-1/\sqrt{2}\\0&0&0\end{pmatrix}$$

$$T^0_0 = \frac{\left|\begin{matrix}1\\1\end{matrix}\right\rangle\left\langle\begin{matrix}1\\1\end{matrix}\right| + \left|\begin{matrix}1\\0\end{matrix}\right\rangle\left\langle\begin{matrix}1\\0\end{matrix}\right| + \left|\begin{matrix}1\\-1\end{matrix}\right\rangle\left\langle\begin{matrix}1\\-1\end{matrix}\right|}{\sqrt{2}}$$

$$\rightarrow\begin{pmatrix}1/\sqrt{3}&0&0\\0&1/\sqrt{3}&0\\0&0&1/\sqrt{3}\end{pmatrix}$$

Tensor operators for spin-J states: U(2J+1) generalization of Pauli spinors

$$\mathbf{v}^k_q = \sum_{m,m'} (-1)^{j-m}\sqrt{2k+1}\begin{pmatrix} k & j & j \\ q & m' & -m \end{pmatrix} \left| \begin{matrix} j \\ m \end{matrix} \right\rangle \left\langle \begin{matrix} j \\ m' \end{matrix} \right|$$

for $j = 1,2,3.$

$$q = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \qquad (-1)$$

$$v_q^6 = \begin{array}{|ccccccc|}\hline 1 & -\sqrt{2} & 1 & -\sqrt{2} & \sqrt{5} & -1 & 1 \\ \sqrt{2} & -6 & \sqrt{30} & -\sqrt{8} & 3 & -\sqrt{12} & 1 \\ 1 & -\sqrt{30} & 15 & -10 & \sqrt{15} & -3 & \sqrt{5} \\ \sqrt{2} & -\sqrt{8} & 10 & -20 & 10 & -\sqrt{8} & \sqrt{2} \\ \sqrt{5} & -3 & \sqrt{15} & -10 & 15 & -\sqrt{30} & 1 \\ 1 & -\sqrt{12} & 3 & -\sqrt{8} & \sqrt{30} & -6 & \sqrt{2} \\ 1 & -1 & \sqrt{5} & -\sqrt{2} & 1 & -\sqrt{2} & 1 \\ \hline \end{array} \quad \begin{array}{c} 1 \\ \sqrt{2} \\ \sqrt{22} \\ \sqrt{22} \\ \sqrt{33} \\ \sqrt{264} \\ \sqrt{924} \end{array}$$

$$v_q^5 = \begin{array}{|ccccccc|}\hline 1 & -\sqrt{5} & 1 & -\sqrt{2} & 1 & -1 & \cdot \\ \sqrt{5} & -4 & \sqrt{27} & -\sqrt{2} & 1 & 0 & -1 \\ 1 & -\sqrt{27} & 5 & -\sqrt{10} & 0 & 1 & -1 \\ \sqrt{2} & -\sqrt{2} & \sqrt{10} & 0 & -\sqrt{10} & \sqrt{2} & \sqrt{2} \\ 1 & -1 & 0 & \sqrt{10} & -5 & \sqrt{27} & -1 \\ 1 & 0 & -1 & \sqrt{2} & -\sqrt{27} & 4 & -\sqrt{5} \\ \cdot & 1 & -1 & \sqrt{2} & -1 & \sqrt{5} & -1 \\ \hline \end{array} \quad \begin{array}{c} \sqrt{2} \\ \sqrt{2} \\ \sqrt{6} \\ \sqrt{6} \\ \sqrt{84} \\ \sqrt{84} \end{array}$$

$$v_q^4 = \begin{array}{|ccccccc|}\hline 3 & -\sqrt{30} & \sqrt{54} & -3 & \sqrt{3} & \cdot & \cdot \\ \sqrt{30} & -7 & \sqrt{32} & -\sqrt{3} & -\sqrt{2} & \sqrt{5} & \cdot \\ \sqrt{54} & -\sqrt{32} & 1 & \sqrt{15} & -\sqrt{40} & \sqrt{2} & \sqrt{3} \\ 3 & -\sqrt{3} & -\sqrt{15} & 6 & -\sqrt{15} & -\sqrt{3} & 3 \\ \sqrt{3} & \sqrt{2} & -\sqrt{40} & \sqrt{15} & 1 & -\sqrt{32} & \sqrt{54} \\ \cdot & \sqrt{5} & -\sqrt{2} & -\sqrt{3} & \sqrt{32} & -7 & \sqrt{30} \\ \cdot & \cdot & \sqrt{3} & -3 & \sqrt{54} & -\sqrt{30} & 3 \\ \hline \end{array} \quad \begin{array}{c} \sqrt{11} \\ \sqrt{22} \\ \sqrt{154} \\ \sqrt{154} \\ \sqrt{154} \end{array}$$

$$v_q^3 = \begin{array}{|ccccccc|}\hline 1 & -\sqrt{2} & \sqrt{2} & -1 & \cdot & \cdot & \cdot \\ \sqrt{2} & -1 & 0 & 1 & -\sqrt{2} & \cdot & \cdot \\ \sqrt{2} & 0 & -1 & 1 & 0 & -\sqrt{2} & \cdot \\ 1 & 1 & -1 & 0 & 1 & -1 & -1 \\ \cdot & \sqrt{2} & 0 & -1 & 1 & 0 & -\sqrt{2} \\ \cdot & \cdot & \sqrt{2} & -1 & 0 & 1 & -\sqrt{2} \\ \cdot & \cdot & \cdot & 1 & -\sqrt{2} & \sqrt{2} & -1 \\ \hline \end{array} \quad \begin{array}{c} \sqrt{6} \\ \sqrt{6} \\ \sqrt{6} \\ \sqrt{6} \end{array}$$

$$v_q^2 = \begin{array}{|ccccccc|}\hline 5 & -5 & \sqrt{5} & \cdot & \cdot & \cdot & \cdot \\ 5 & 0 & -\sqrt{15} & \sqrt{10} & \cdot & \cdot & \cdot \\ \sqrt{5} & \sqrt{15} & -3 & -\sqrt{2} & \sqrt{12} & \cdot & \cdot \\ \cdot & \sqrt{10} & \sqrt{2} & -4 & \sqrt{2} & \sqrt{10} & \cdot \\ \cdot & \cdot & \sqrt{12} & -\sqrt{2} & -3 & \sqrt{15} & \sqrt{5} \\ \cdot & \cdot & \cdot & \sqrt{10} & -\sqrt{15} & 0 & 5 \\ \cdot & \cdot & \cdot & \cdot & \sqrt{5} & -5 & 5 \\ \hline \end{array} \quad \begin{array}{c} \sqrt{42} \\ \sqrt{84} \\ \sqrt{84} \end{array}$$

$$v_q^1 = \begin{array}{|ccccccc|}\hline 3 & -\sqrt{3} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \sqrt{3} & 2 & -\sqrt{5} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \sqrt{5} & 1 & -\sqrt{6} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \sqrt{6} & 0 & -\sqrt{6} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \sqrt{6} & -1 & -\sqrt{5} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \sqrt{5} & -2 & -\sqrt{3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{3} & -3 \\ \hline \end{array} \quad \begin{array}{c} \sqrt{28} \\ \sqrt{28} \end{array}$$

(f) $l = 3$

$$q = 0 \quad 1 \quad 2 \quad 3 \quad 4$$

$$\begin{array}{|ccccc|}\hline 1 & -1 & \sqrt{3} & -1 & 1 \\ 1 & -4 & \sqrt{6} & -\sqrt{8} & 1 \\ \sqrt{3} & -\sqrt{6} & 6 & -\sqrt{6} & \sqrt{3} \\ 1 & -\sqrt{8} & \sqrt{6} & -4 & 1 \\ 1 & -1 & \sqrt{3} & -1 & 1 \\ \hline \end{array} \quad \begin{array}{c} 1 \\ \sqrt{2} \\ \sqrt{14} \\ \sqrt{14} \\ \sqrt{70} \end{array}$$

$$\begin{array}{|ccccc|}\hline 1 & -\sqrt{3} & 1 & -1 & \cdot \\ \sqrt{3} & -2 & \sqrt{2} & 0 & -1 \\ 1 & -\sqrt{2} & 0 & \sqrt{2} & -1 \\ 1 & 0 & -\sqrt{2} & 2 & -\sqrt{3} \\ \cdot & 1 & -1 & \sqrt{3} & -1 \\ \hline \end{array} \quad \begin{array}{c} \sqrt{2} \\ \sqrt{2} \\ \sqrt{10} \\ \sqrt{10} \end{array}$$

$$\begin{array}{|ccccc|}\hline 2 & -\sqrt{6} & \sqrt{2} & \cdot & \cdot \\ \sqrt{6} & -1 & -1 & \sqrt{3} & \cdot \\ \sqrt{2} & 1 & -2 & 1 & \sqrt{2} \\ \cdot & \sqrt{3} & -1 & -1 & \sqrt{6} \\ \cdot & \cdot & \sqrt{2} & -\sqrt{6} & 2 \\ \hline \end{array} \quad \begin{array}{c} \sqrt{7} \\ \sqrt{14} \\ \sqrt{14} \end{array}$$

$$\begin{array}{|ccccc|}\hline 2 & -\sqrt{2} & \cdot & \cdot & \cdot \\ \sqrt{2} & 1 & -\sqrt{3} & \cdot & \cdot \\ \cdot & \sqrt{3} & 0 & -\sqrt{3} & \cdot \\ \cdot & \cdot & \sqrt{3} & -1 & -\sqrt{2} \\ \cdot & \cdot & \cdot & \sqrt{2} & -2 \\ \hline \end{array} \quad \begin{array}{c} \sqrt{10} \\ \sqrt{10} \end{array}$$

(d) $l = 2$

$$q = 0 \quad 1 \quad 2$$

$$\begin{array}{|ccc|}\hline 1 & -1 & 1 \\ 1 & -2 & 1 \\ 1 & -1 & -1 \\ \hline \end{array}$$

$$\begin{array}{|ccc|}\hline 1 & -1 & \cdot \\ 1 & 0 & -1 \\ \cdot & 1 & -1 \\ \hline \end{array}$$

(p) $l = 1$

Tensor operators for spin-J states: U(2J+1) generalization of Pauli spinors

$$\mathbf{v}^k_q = \sum_{m,m'} (-1)^{j-m}\sqrt{2k+1}\begin{pmatrix} k & j & j \\ q & m' & -m \end{pmatrix} \left| \begin{matrix} j \\ m \end{matrix} \right\rangle\left\langle \begin{matrix} j \\ m' \end{matrix} \right|$$

for $j = 1, 2, 3.$

$q = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$

$$v^6_q = \left[\begin{array}{ccccccc} 1 & -\sqrt{2} & 1 & -\sqrt{2} & \sqrt{5} & -1 & 1 \\ \sqrt{2} & -6 & \sqrt{30} & -\sqrt{8} & 3 & -\sqrt{12} & 1 \\ 1 & -\sqrt{30} & 15 & -10 & \sqrt{15} & -3 & \sqrt{5} \\ \sqrt{2} & -\sqrt{8} & 10 & -20 & 10 & -\sqrt{8} & \sqrt{2} \\ \sqrt{5} & -3 & \sqrt{15} & -10 & 15 & -\sqrt{30} & 1 \\ 1 & -\sqrt{12} & 3 & -\sqrt{8} & \sqrt{30} & -6 & \sqrt{2} \\ 1 & -1 & \sqrt{5} & -\sqrt{2} & 1 & -\sqrt{2} & 1 \end{array}\right] \begin{array}{c} 1 \\ \sqrt{2} \\ \sqrt{22} \\ \sqrt{22} \\ \sqrt{33} \\ \sqrt{264} \\ \sqrt{924} \end{array}$$

$$v^5_q = \left[\begin{array}{ccccccc} 1 & -\sqrt{5} & 1 & -\sqrt{2} & 1 & -1 & \cdot \\ \sqrt{5} & -4 & \sqrt{27} & -\sqrt{2} & 1 & 0 & -1 \\ 1 & -\sqrt{27} & 5 & -\sqrt{10} & 0 & 1 & -1 \\ \sqrt{2} & -\sqrt{2} & \sqrt{10} & 0 & -\sqrt{10} & \sqrt{2} & \sqrt{2} \\ 1 & -1 & 0 & \sqrt{10} & -5 & \sqrt{27} & -1 \\ 1 & 0 & -1 & \sqrt{2} & -\sqrt{27} & 4 & -\sqrt{5} \\ \cdot & 1 & -1 & \sqrt{2} & -1 & \sqrt{5} & -1 \end{array}\right] \begin{array}{c} \sqrt{2} \\ \sqrt{2} \\ \sqrt{6} \\ \sqrt{6} \\ \sqrt{84} \\ \sqrt{84} \end{array}$$

$$v^4_q = \left[\begin{array}{ccccccc} 3 & -\sqrt{30} & \sqrt{54} & -3 & \sqrt{3} & \cdot & \cdot \\ \sqrt{30} & -7 & \sqrt{32} & -\sqrt{3} & -\sqrt{2} & \sqrt{5} & \cdot \\ \sqrt{54} & -\sqrt{32} & 1 & \sqrt{15} & -\sqrt{40} & \sqrt{2} & \sqrt{3} \\ 3 & -\sqrt{3} & -\sqrt{15} & 6 & -\sqrt{15} & -\sqrt{3} & 3 \\ \sqrt{3} & \sqrt{2} & -\sqrt{40} & \sqrt{15} & 1 & -\sqrt{32} & \sqrt{54} \\ \cdot & \sqrt{5} & -\sqrt{2} & -\sqrt{3} & \sqrt{32} & -7 & \sqrt{30} \\ \cdot & \cdot & \sqrt{3} & -3 & \sqrt{54} & -\sqrt{30} & 3 \end{array}\right] \begin{array}{c} \sqrt{11} \\ \sqrt{22} \\ \sqrt{154} \\ \sqrt{154} \\ \sqrt{154} \end{array}$$

$$v^3_q = \left[\begin{array}{ccccccc} 1 & -\sqrt{2} & \sqrt{2} & -1 & \cdot & \cdot & \cdot \\ \sqrt{2} & -1 & 0 & 1 & -\sqrt{2} & \cdot & \cdot \\ \sqrt{2} & 0 & -1 & 1 & 0 & -\sqrt{2} & \cdot \\ 1 & 1 & -1 & 0 & 1 & -1 & -1 \\ \cdot & \sqrt{2} & 0 & -1 & 1 & 0 & -\sqrt{2} \\ \cdot & \cdot & \sqrt{2} & -1 & 0 & 1 & -\sqrt{2} \\ \cdot & \cdot & \cdot & 1 & -\sqrt{2} & \sqrt{2} & -1 \end{array}\right] \begin{array}{c} \sqrt{6} \\ \sqrt{6} \\ \sqrt{6} \\ \sqrt{6} \end{array}$$

$$v^2_q = \left[\begin{array}{ccccccc} 5 & -5 & \sqrt{5} & \cdot & \cdot & \cdot & \cdot \\ 5 & 0 & -\sqrt{15} & \sqrt{10} & \cdot & \cdot & \cdot \\ \sqrt{5} & \sqrt{15} & -3 & -\sqrt{2} & \sqrt{12} & \cdot & \cdot \\ \cdot & \sqrt{10} & \sqrt{2} & -4 & \sqrt{2} & \sqrt{10} & \cdot \\ \cdot & \cdot & \sqrt{12} & -\sqrt{2} & -3 & \sqrt{15} & \sqrt{5} \\ \cdot & \cdot & \cdot & \sqrt{10} & -\sqrt{15} & 0 & 5 \\ \cdot & \cdot & \cdot & \cdot & \sqrt{5} & -5 & 5 \end{array}\right] \begin{array}{c} \sqrt{42} \\ \sqrt{84} \\ \sqrt{84} \end{array}$$

$$v^1_q = \left[\begin{array}{ccccccc} 3 & -\sqrt{3} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \sqrt{3} & 2 & -\sqrt{5} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \sqrt{5} & 1 & -\sqrt{6} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \sqrt{6} & 0 & -\sqrt{6} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \sqrt{6} & -1 & -\sqrt{5} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \sqrt{5} & -2 & -\sqrt{3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{3} & -3 \end{array}\right] \begin{array}{c} \sqrt{28} \\ \sqrt{28} \end{array}$$

(f) $l = 3$

$q = 0 \quad 1 \quad 2 \quad 3 \quad 4$

$$\left[\begin{array}{ccccc} 1 & -1 & \sqrt{3} & -1 & 1 \\ 1 & -4 & \sqrt{6} & -\sqrt{8} & 1 \\ \sqrt{3} & -\sqrt{6} & 6 & -\sqrt{6} & \sqrt{3} \\ 1 & -\sqrt{8} & \sqrt{6} & -4 & 1 \\ 1 & -1 & \sqrt{3} & -1 & 1 \end{array}\right] \begin{array}{c} 1 \\ \sqrt{2} \\ \sqrt{14} \\ \sqrt{14} \\ \sqrt{70} \end{array}$$

$$\left[\begin{array}{ccccc} 1 & -\sqrt{3} & 1 & -1 & \cdot \\ \sqrt{3} & -2 & \sqrt{2} & 0 & -1 \\ 1 & -\sqrt{2} & 0 & \sqrt{2} & -1 \\ 1 & 0 & -\sqrt{2} & 2 & -\sqrt{3} \\ \cdot & 1 & -1 & \sqrt{3} & -1 \end{array}\right] \begin{array}{c} \sqrt{2} \\ \sqrt{2} \\ \sqrt{10} \\ \sqrt{10} \end{array}$$

$$\left[\begin{array}{ccccc} 2 & -\sqrt{6} & \sqrt{2} & \cdot & \cdot \\ \sqrt{6} & -1 & -1 & \sqrt{3} & \cdot \\ \sqrt{2} & 1 & -2 & 1 & \sqrt{2} \\ \cdot & \sqrt{3} & -1 & -1 & \sqrt{6} \\ \cdot & \cdot & \sqrt{2} & -\sqrt{6} & 2 \end{array}\right] \begin{array}{c} \sqrt{7} \\ \sqrt{14} \\ \sqrt{14} \end{array}$$

$$\left[\begin{array}{ccccc} 2 & -\sqrt{2} & \cdot & \cdot & \cdot \\ \sqrt{2} & 1 & -\sqrt{3} & \cdot & \cdot \\ \cdot & \sqrt{3} & 0 & -\sqrt{3} & \cdot \\ \cdot & \cdot & \sqrt{3} & -1 & -\sqrt{2} \\ \cdot & \cdot & \cdot & \sqrt{2} & -2 \end{array}\right] \begin{array}{c} \sqrt{10} \\ \sqrt{10} \end{array}$$

(d) $l = 2$

$q = 0 \quad 1 \quad 2$

$$\left[\begin{array}{ccc} 1 & -1 & 1 \\ 1 & -2 & 1 \\ 1 & -1 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc} 1 & -1 & \cdot \\ 1 & 0 & -1 \\ \cdot & 1 & -1 \end{array}\right]$$

(p) $l = 1$

(a) $j = \frac{1}{2}$

$q = 0 \quad 1$

$$v^1_q = \left[\begin{array}{cc} 1 & -1 \\ 1 & -1 \end{array}\right] \begin{array}{c} 1 \\ \sqrt{2} \end{array}$$

(b) $j = \frac{3}{2}$

$q = 0 \quad 1 \quad 2 \quad 3$

$$v^1_q = \left[\begin{array}{cccc} 3 & -\sqrt{3} & \cdot & \cdot \\ \sqrt{3} & 1 & -2 & \cdot \\ \cdot & 2 & -1 & -\sqrt{3} \\ \cdot & \cdot & \sqrt{3} & -3 \end{array}\right] \begin{array}{c} \sqrt{10} \\ \sqrt{20} \end{array}$$

$$v^2_q = \left[\begin{array}{cccc} 1 & -1 & 1 & \cdot \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ \cdot & 1 & -1 & 1 \end{array}\right] \begin{array}{c} \sqrt{2} \\ \sqrt{2} \\ \sqrt{4} \end{array}$$

$$v^3_q = \left[\begin{array}{cccc} 1 & -1 & 1 & -1 \\ 1 & -3 & \sqrt{3} & -1 \\ 1 & -\sqrt{3} & 3 & -1 \\ 1 & -1 & 1 & -1 \end{array}\right] \begin{array}{c} 1 \\ \sqrt{2} \\ \sqrt{5} \\ \sqrt{20} \end{array}$$

(c) $j = \frac{5}{2}$

$q = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5$

$$v^1q = \left[\begin{array}{cccccc} 5 & -\sqrt{5} & \cdot & \cdot & \cdot & \cdot \\ \sqrt{5} & 3 & -\sqrt{8} & \cdot & \cdot & \cdot \\ \cdot & \sqrt{8} & 1 & -3 & \cdot & \cdot \\ \cdot & \cdot & 3 & -1 & -\sqrt{8} & \cdot \\ \cdot & \cdot & \cdot & \sqrt{8} & -3 & -\sqrt{5} \\ \cdot & \cdot & \cdot & \cdot & \sqrt{5} & -5 \end{array}\right] \begin{array}{c} \sqrt{35} \\ \sqrt{70} \end{array}$$

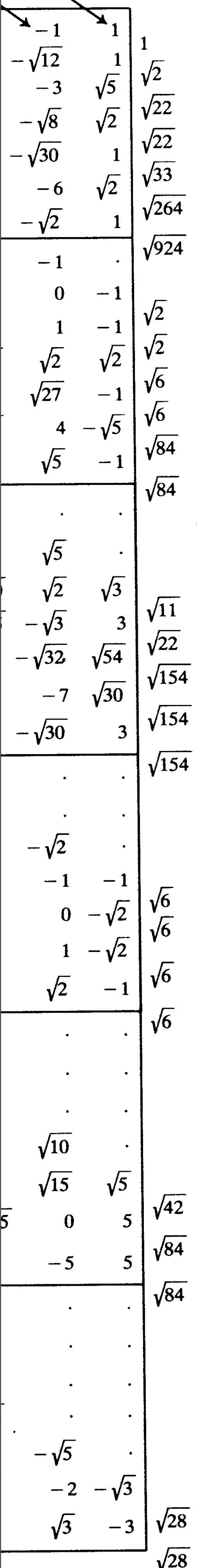
$$v^2_q = \left[\begin{array}{cccccc} 5 & -\sqrt{5} & \sqrt{5} & \cdot & \cdot & \cdot \\ \sqrt{5} & -1 & -\sqrt{2} & 3 & \cdot & \cdot \\ \sqrt{5} & \sqrt{2} & -4 & 0 & 3 & \cdot \\ \cdot & 3 & 0 & -4 & \sqrt{2} & \sqrt{5} \\ \cdot & \cdot & 3 & -\sqrt{2} & -1 & \sqrt{5} \\ \cdot & \cdot & \cdot & \sqrt{5} & -\sqrt{5} & 5 \end{array}\right] \begin{array}{c} \sqrt{28} \\ \sqrt{14} \\ \sqrt{84} \end{array}$$

$$v^3_q = \left[\begin{array}{cccccc} 5 & -\sqrt{10} & \sqrt{5} & -\sqrt{5} & \cdot & \cdot \\ \sqrt{10} & -7 & 1 & 1 & -\sqrt{8} & \cdot \\ \sqrt{5} & -1 & -4 & \sqrt{8} & -1 & -\sqrt{5} \\ \sqrt{5} & 1 & -\sqrt{8} & 4 & 1 & -\sqrt{5} \\ \cdot & \sqrt{8} & -1 & -1 & 7 & -\sqrt{10} \\ \cdot & \cdot & \sqrt{5} & -\sqrt{5} & \sqrt{10} & -5 \end{array}\right] \begin{array}{c} \sqrt{18} \\ \sqrt{12} \\ \sqrt{30} \\ \sqrt{180} \end{array}$$

$$v^4_q = \left[\begin{array}{cccccc} 1 & -\sqrt{2} & 3 & -1 & 1 & \cdot \\ \sqrt{2} & -3 & \sqrt{5} & -\sqrt{5} & 0 & 1 \\ 3 & -\sqrt{5} & 2 & 0 & -\sqrt{5} & 1 \\ 1 & -\sqrt{5} & 0 & 2 & -\sqrt{5} & 3 \\ 1 & 0 & -\sqrt{5} & \sqrt{5} & -3 & \sqrt{2} \\ \cdot & 1 & -1 & 3 & -\sqrt{2} & 1 \end{array}\right] \begin{array}{c} \sqrt{2} \\ \sqrt{2} \\ \sqrt{28} \\ \sqrt{14} \\ \sqrt{28} \end{array}$$

$$v^5_q = \left[\begin{array}{cccccc} 1 & -1 & 1 & -\sqrt{2} & 1 & -1 \\ 1 & -5 & \sqrt{10} & -\sqrt{5} & \sqrt{5} & -1 \\ 1 & -\sqrt{10} & 10 & -\sqrt{20} & \sqrt{5} & -\sqrt{2} \\ \sqrt{2} & -\sqrt{5} & \sqrt{20} & -10 & \sqrt{10} & -1 \\ 1 & -\sqrt{5} & \sqrt{5} & -\sqrt{10} & 5 & -1 \\ 1 & -1 & \sqrt{2} & -1 & 1 & -1 \end{array}\right] \begin{array}{c} 1 \\ \sqrt{2} \\ \sqrt{9} \\ \sqrt{12} \\ \sqrt{42} \\ \sqrt{252} \end{array}$$

Tensor operators for spin-J states: Application to splitting



$$V^{(4)} = D\left[x^4 + y^4 + z^4 - \tfrac{3}{4}r^4\right] = D\left[\frac{2}{\sqrt{70}}\left(X^4_4 + X^4_{-4}\right) + \tfrac{2}{5}X^4_0\right]$$

$$\left\langle V^{(4)}\right\rangle_{j=2} = D\left\langle \frac{2}{\sqrt{70}}\left(v^4_4 + v^4_{-4}\right) + \tfrac{2}{5}v^4_0\right\rangle_{j=2} \frac{\sqrt{5}}{3}\left\langle 2\left|\left|X^4\right|\right|2\right\rangle.$$

$$\left\langle V^{(4)}\right\rangle_{j=2} = \frac{D}{\sqrt{70}}\begin{pmatrix} \frac{2}{5} & \cdot & \cdot & \cdot & 2 \\ \cdot & -\frac{8}{5} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \frac{12}{5} & \cdot & \cdot \\ \cdot & \cdot & \cdot & -\frac{8}{5} & \cdot \\ 2 & \cdot & \cdot & \cdot & \frac{2}{5} \end{pmatrix}\frac{\sqrt{5}}{3}\left\langle 2\left\|X^4\right\|2\right\rangle.$$

$q = 0, 1, 2, 3, 4$

1	−1	$\sqrt{3}$	−1	1	1
1	−4	$\sqrt{6}$	$-\sqrt{8}$	1	$\sqrt{2}$
$\sqrt{3}$	$-\sqrt{6}$	6	$-\sqrt{6}$	$\sqrt{3}$	$\sqrt{14}$
1	$-\sqrt{8}$	$\sqrt{6}$	−4	1	$\sqrt{14}$
1	−1	$\sqrt{3}$	−1	1	$\sqrt{70}$

1	$-\sqrt{3}$	1	−1	·	
$\sqrt{3}$	−2	$\sqrt{2}$	0	−1	$\sqrt{2}$
1	$-\sqrt{2}$	0	$\sqrt{2}$	−1	$\sqrt{2}$
1	0	$-\sqrt{2}$	2	$-\sqrt{3}$	$\sqrt{10}$
·	1	−1	$\sqrt{3}$	−1	$\sqrt{10}$

2	$-\sqrt{6}$	$\sqrt{2}$	·	·	
$\sqrt{6}$	−1	−1	$\sqrt{3}$	·	
$\sqrt{2}$	1	−2	1	$\sqrt{2}$	$\sqrt{7}$
·	$\sqrt{3}$	−1	−1	$\sqrt{6}$	$\sqrt{14}$
·	·	$\sqrt{2}$	$-\sqrt{6}$	2	$\sqrt{14}$

2	$-\sqrt{2}$	·	·	·	
$\sqrt{2}$	1	$-\sqrt{3}$	·	·	
·	$\sqrt{3}$	0	$-\sqrt{3}$	·	
·	·	$\sqrt{3}$	−1	$-\sqrt{2}$	$\sqrt{10}$
·	·	·	$\sqrt{2}$	−2	$\sqrt{10}$

(d) $l = 2$

$q = 0, 1, 2$

1	−1	1
1	−2	1
1	−1	−1

1	−1	·
1	0	−1
·	1	−1

(p) $l = 1$