Group Theory in Quantum Mechanics
Lecture 24 *(4.21.15)*
Rotational symmetry U(2)⊂U(3) and O(3)

(Int.J.Mol.Sci, 14, 714(2013) p.755-774 , QTCA Unit 7 Ch. 21-22)
(PSDS - Ch. 5, 7)

Review : *2-D* $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(2) representations*

Angular momentum generators by U(2) analysis
Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Angular momentum commutation relations
Key Lie theorems

Angular momentum magnitude and uncertainty
Angular momentum uncertainty angle

Generating R(3) rotation and U(2) representations
Applications of R(3) rotation and U(2) representations
Molecular and nuclear wavefunctions
Molecular and nuclear eigenlevels
Generalized Stern-Gerlach and transformation matrices
Angular momentum cones and high J properties

Links as of April 21, 2015
(Apps are being upgraded as time permits)

Links to the current Harter-Soft LearnIt web apps for Physics

(Bold links have default redirect pages. *Italics* are not yet meant for production. Red are in the final stages of testing.

Production Links - *For the students & general public*

BohrIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/BohrItWeb.html"
BounceIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/BounceItWeb.html"
BoxIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/BoxItWeb.html"
CoulIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/CoulItWeb.html"
Cycloidulum - Production; URL is "http://www.uark.edu/ua/modphys/markup/CycloidulumWeb.html"
LearnIt - Production; URL is "**http://www.uark.edu/ua/modphys**" or "http://www.uark.edu/ua/modphys/markup/LearnItWeb.html"
JerkIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/JerkItWeb.html"
Pendulum - Production; URL is "http://www.uark.edu/ua/modphys/markup/PendulumWeb.html"
QuantIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/QuantItWeb.html"
Relativity - Pirelli Entrant - Production; URL is "**http://www.uark.edu/ua/pirelli**" or "http://www.uark.edu/ua/pirelli/html/default.html"
Trebuchet Production; URL is "http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html"

Testing Links - *For internal use and testing by Harter & Heyoka*

BohrIt - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/BohrItWeb.html"
BounceIt - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/BounceItWeb.html"
BounceIt Title Page - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/BounceItTitlePage.html"
BoxIt - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/BoxItWeb.html"
CoulIt - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/CoulItWeb.html"
Cycloidulum - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/CycloidulumWeb.html"
Harter-Soft Web Apps - Quick Reference - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/JerkItWeb.html"
JerkIt - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/JerkItWeb.html"
ModernPhysics - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/IntroCover.html"
Pendulum - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/PendulumWeb.html"
QuantIt - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/QuantItWeb.html"
Trebuchet Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/TrebuchetWeb.html"

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U(2)-2D-HO Hamiltonian and irreducible representations

"Little-Endian" indexing ($\ldots 01, 02, 03 .. 10, 11, 12, 13 \ldots 20, 21, 22, 23, \ldots$)

$$\mathbf{H} = A\left(\mathbf{a}_1^\dagger \mathbf{a}_1 + \mathbf{1}/2\right) + (B - iC)\mathbf{a}_1^\dagger \mathbf{a}_2 + (B + iC)\mathbf{a}_2^\dagger \mathbf{a}_1 + D\left(\mathbf{a}_2^\dagger \mathbf{a}_2 + \mathbf{1}/2\right)$$

$$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$$

	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 02\rangle$	$\cdots$	$\vert 10\rangle$	$\vert 11\rangle$	$\vert 12\rangle$	$\cdots$	$\vert 20\rangle$	$\vert 21\rangle$	$\vert 22\rangle$	$\cdots$
$\langle 00\vert$	0			$\cdots$	$\cdot$			$\cdots$				$\cdots$
$\langle 01\vert$		D		$\cdots$	$B+iC$	$\cdot$		$\cdots$				$\cdots$
$\langle 02\vert$			$2D$	$\cdots$		$\sqrt{2}(B+iC)$	$\cdot$	$\cdots$				$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$				$\cdots$
$\langle 10\vert$	$\cdot$	$B-iC$		$\cdots$	A			$\cdots$	$\cdot$			$\cdots$
$\langle 11\vert$		$\cdot$	$\sqrt{2}(B-iC)$	$\cdots$		$A+D$		$\cdots$	$\sqrt{2}(B+iC)$	$\cdot$		$\cdots$
$\langle 12\vert$		Example:	$\cdot$	$\cdots$			$A+2D$	$\cdots$		$\sqrt{4}(B+iC)$	$\cdot$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\langle 20\vert$					$\cdot$	$\sqrt{2}(B-iC)$		$\cdots$	$2A$			$\cdots$
$\langle 21\vert$						$\cdot$	$\sqrt{4}(B-iC)$	$\cdots$		$2A+D$		$\cdots$
$\langle 22\vert$							$\cdot$	$\cdots$			$2A+2D$	$\cdots$
$\vdots$					$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$$\mathbf{a}_1^\dagger \mathbf{a}_2 \vert 02\rangle = \sqrt{0+1}\sqrt{2}\vert 0{+}1\ 2{-}1\rangle = \sqrt{2}\vert 1\,1\rangle$$

$$\mathbf{a}_1^\dagger \mathbf{a}_2 \vert n_1 n_2\rangle = \sqrt{n_1+1}\sqrt{n_2}\vert n_1{+}1\ n_2{-}1\rangle$$

$$\mathbf{a}_1^\dagger \mathbf{a}_1 \vert n_1 n_2\rangle = n_1 \vert n_1 n_2\rangle$$

$$\mathbf{a}_2^\dagger \mathbf{a}_1 \vert n_1 n_2\rangle = \sqrt{n_1}\sqrt{n_2+1}\vert n_1{-}1\, n_2{+}1\rangle$$

$$\mathbf{a}_1^\dagger \mathbf{a}_2 \vert n_1 n_2\rangle = \sqrt{n_1+1}\sqrt{n_2}\vert n_1{+}1\ n_2{-}1\rangle$$

$$\mathbf{a}_2^\dagger \mathbf{a}_2 \vert n_1 n_2\rangle = n_2 \vert n_1 n_2\rangle$$

Rearrangement of rows and columns brings the matrix to a block-diagonal form.

U(2)-2D-HO Hamiltonian and irreducible representations

"Little-Endian" indexing (...01,02,03..10,11,12,13 ... 20,21,22,23,...)

$$\mathbf{H} = A\left(\mathbf{a}_1^\dagger \mathbf{a}_1 + \mathbf{1}/2\right) + (B - iC)\,\mathbf{a}_1^\dagger \mathbf{a}_2 + (B + iC)\,\mathbf{a}_2^\dagger \mathbf{a}_1 + D\left(\mathbf{a}_2^\dagger \mathbf{a}_2 + \mathbf{1}/2\right)$$

$$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$$

	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 02\rangle$	$\cdots$	$\vert 10\rangle$	$\vert 11\rangle$	$\vert 12\rangle$	$\cdots$	$\vert 20\rangle$	$\vert 21\rangle$	$\vert 22\rangle$	$\cdots$
$\langle 00\vert$	0			$\cdots$	$\cdot$			$\cdots$				$\cdots$
$\langle 01\vert$		D		$\cdots$	$B+iC$	$\cdot$		$\cdots$				$\cdots$
$\langle 02\vert$			$2D$	$\cdots$		$\sqrt{2}(B+iC)$	$\cdot$	$\cdots$				$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$				$\cdots$
$\langle 10\vert$	$\cdot$	$B-iC$		$\cdots$	A			$\cdots$	$\cdot$			$\cdots$
$\langle 11\vert$		$\cdot$	$\sqrt{2}(B-iC)$	$\cdots$		$A+D$		$\cdots$	$\sqrt{2}(B+iC)$	$\cdot$		$\cdots$
$\langle 12\vert$		*Example:*	$\cdot$	$\cdots$			$A+2D$	$\cdots$		$\sqrt{4}(B+iC)$	$\cdot$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\langle 20\vert$	$\mathbf{a}_1^\dagger\mathbf{a}_2\vert 02\rangle = \sqrt{0+1}\sqrt{2}\vert 0+1\ 2-1\rangle = \sqrt{2}\vert 1\,1\rangle$				$\cdot$	$\sqrt{2}(B-iC)$		$\cdots$	$2A$			$\cdots$
$\langle 21\vert$	$\mathbf{a}_1^\dagger\mathbf{a}_2\vert n_1 n_2\rangle = \sqrt{n_1+1}\sqrt{n_2}\vert n_1+1\ n_2-1\rangle$					$\cdot$	$\sqrt{4}(B-iC)$	$\cdots$		$2A+D$		$\cdots$
$\langle 22\vert$							$\cdot$	$\cdots$			$2A+2D$	$\cdots$
$\vdots$					$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$$\mathbf{a}_1^\dagger \mathbf{a}_1 \vert n_1 n_2\rangle = n_1 \vert n_1 n_2\rangle$$

$$\mathbf{a}_2^\dagger \mathbf{a}_1 \vert n_1 n_2\rangle = \sqrt{n_1}\sqrt{n_2+1}\,\vert n_1-1\ n_2+1\rangle$$

$$\mathbf{a}_1^\dagger \mathbf{a}_2 \vert n_1 n_2\rangle = \sqrt{n_1+1}\sqrt{n_2}\,\vert n_1+1\ n_2-1\rangle$$

$$\mathbf{a}_2^\dagger \mathbf{a}_2 \vert n_1 n_2\rangle = n_2 \vert n_1 n_2\rangle$$

Rearrangement of rows and columns brings the matrix to a block-diagonal form.

Base states $\vert n_1\rangle\vert n_2\rangle$ with the same *total quantum number* $\nu = n_1 + n_2$ define each block.

U(2)-2D-HO Hamiltonian and irreducible representations

"Little-Endian" indexing (...01,02,03..10,11,12,13 ... 20,21,22,23,...)

$$\mathbf{H} = A\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{1}/2\right) + (B - iC)\mathbf{a}_1^\dagger\mathbf{a}_2 + (B + iC)\mathbf{a}_2^\dagger\mathbf{a}_1 + D\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + \mathbf{1}/2\right)$$

$$\langle\mathbf{H}\rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$$

	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 02\rangle$	$\cdots$	$\vert 10\rangle$	$\vert 11\rangle$	$\vert 12\rangle$	$\cdots$	$\vert 20\rangle$	$\vert 21\rangle$	$\vert 22\rangle$	$\cdots$
$\langle 00\vert$	0			$\cdots$	$\cdot$			$\cdots$				$\cdots$
$\langle 01\vert$		D		$\cdots$	$B+iC$	$\cdot$		$\cdots$				$\cdots$
$\langle 02\vert$			$2D$	$\cdots$		$\sqrt{2}(B+iC)$	$\cdot$	$\cdots$				$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$				$\cdots$
$\langle 10\vert$	$\cdot$	$B-iC$		$\cdots$	A			$\cdots$	$\cdot$			$\cdots$
$\langle 11\vert$		$\cdot$	$\sqrt{2}(B-iC)$	$\cdots$		$A+D$		$\cdots$	$\sqrt{2}(B+iC)$	$\cdot$		$\cdots$
$\langle 12\vert$				$\cdots$			$A+2D$	$\cdots$		$\sqrt{4}(B+iC)$	$\cdot$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\langle 20\vert$					$\cdot$	$\sqrt{2}(B-iC)$		$\cdots$	$2A$			$\cdots$
$\langle 21\vert$						$\cdot$	$\sqrt{4}(B-iC)$	$\cdots$		$2A+D$		$\cdots$
$\langle 22\vert$							$\cdot$	$\cdots$			$2A+2D$	$\cdots$
$\vdots$					$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Example:

$$\mathbf{a}_1^\dagger\mathbf{a}_2\vert 02\rangle = \sqrt{0+1}\sqrt{2}\vert 0+1\ 2-1\rangle = \sqrt{2}\vert 1\,1\rangle$$

$$\mathbf{a}_1^\dagger\mathbf{a}_2\vert n_1 n_2\rangle = \sqrt{n_1+1}\sqrt{n_2}\vert n_1+1\ n_2-1\rangle$$

$$\mathbf{a}_1^\dagger\mathbf{a}_1\vert n_1 n_2\rangle = n_1\vert n_1 n_2\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1\vert n_1 n_2\rangle = \sqrt{n_1}\sqrt{n_2+1}\vert n_1-1\, n_2+1\rangle$$

$$\mathbf{a}_1^\dagger\mathbf{a}_2\vert n_1 n_2\rangle = \sqrt{n_1+1}\sqrt{n_2}\vert n_1+1\ n_2-1\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_2\vert n_1 n_2\rangle = n_2\vert n_1 n_2\rangle$$

Rearrangement of rows and columns brings the matrix to a block-diagonal form.

Base states $\vert n_1\rangle\vert n_2\rangle$ with the same *total quantum number* $\upsilon = n_1 + n_2$ define each block.

Group reorganized "Little-Endian" indexing (...01,02,03..10,11,12,13 20,21,22,23,...)

$$\langle\mathbf{H}\rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$$

	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 10\rangle$	$\vert 02\rangle$	$\vert 11\rangle$	$\vert 20\rangle$	$\vert 03\rangle$	$\vert 12\rangle$	$\vert 21\rangle$	$\vert 30\rangle$	$\cdots$
$\langle 00\vert$	0	*Vacuum* ($\nu=0$)									
$\langle 01\vert$		D	$B+iC$	*Fundamental* ($\nu=1$)							
$\langle 10\vert$		$B-iC$	A	vibrational sub-space							
$\langle 02\vert$				$2D$	$\sqrt{2}(B+iC)$						
$\langle 11\vert$				$\sqrt{2}(B-iC)$	$A+D$	$\sqrt{2}(B+iC)$	*Overtone* ($\nu=2$)				
$\langle 20\vert$					$\sqrt{2}(B-iC)$	$2A$	vibrational sub-space				
$\langle 03\vert$							$3D$	$\sqrt{3}(B+iC)$			
$\langle 12\vert$							$\sqrt{3}(B-iC)$	$A+2D$	$\sqrt{4}(B+iC)$		*Overtone* ($\nu=3$)
$\langle 21\vert$								$\sqrt{4}(B-iC)$	$2A+D$	$\sqrt{3}(B+iC)$	vibrational sub-space
$\langle 30\vert$									$\sqrt{3}(B-iC)$	$3A$	
$\vdots$											

$$\mathbf{H}^A = A\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{1}/2\right) + D\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + \mathbf{1}/2\right)$$

$$\varepsilon^A_{n_1 n_2} = A\left(n_1 + \frac{1}{2}\right) + D\left(n_2 + \frac{1}{2}\right) = \frac{A+D}{2}(n_1 + n_2 + 1) + \frac{A-D}{2}(n_1 - n_2)$$

Setting $(B=0=C)$ and $(A=\omega_+)$ and $(D=\omega_-)$ gives diagonal block matrices.

$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 10\rangle$	$\vert 02\rangle$	$\vert 11\rangle$	$\vert 20\rangle$	$\vert 03\rangle$	$\vert 12\rangle$	$\vert 21\rangle$	$\vert 30\rangle$	$\cdots$
$\langle 00\vert$	0										
$\langle 01\vert$		ω_-									
$\langle 10\vert$			ω_+								
$\langle 02\vert$				$2\omega_-$							
$\langle 11\vert$					$\omega_+ + \omega_-$						
$\langle 20\vert$						$2\omega_+$					
$\langle 03\vert$							$3\omega_-$				
$\langle 12\vert$								$\omega_+ + 2\omega_-$			
$\langle 21\vert$									$2\omega_+ + \omega_-$		
$\langle 30\vert$										$3\omega_+$	
$\vdots$											

$$\omega_+ - \omega_- = \Omega$$
$$= \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}$$
$$= A - D$$

$$\mathbf{H}^A = A\left(\mathbf{a}_1^\dagger \mathbf{a}_1 + \mathbf{1}/2\right) + D\left(\mathbf{a}_2^\dagger \mathbf{a}_2 + \mathbf{1}/2\right)$$

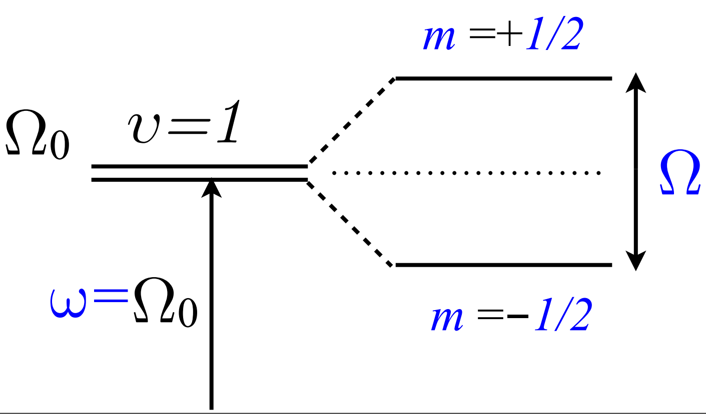
$$\varepsilon^A_{n_1 n_2} = A\left(n_1 + \frac{1}{2}\right) + D\left(n_2 + \frac{1}{2}\right) = \frac{A+D}{2}(n_1 + n_2 + 1) + \frac{A-D}{2}(n_1 - n_2)$$
$$= \Omega_0 (n_1 + n_2 + 1) + \frac{\Omega}{2}(n_1 - n_2) = \Omega_0 (\upsilon + 1) + \Omega\, m$$

Define *total quantum number* $\upsilon=2j$ and half-difference or *asymmetry quantum number m*

$$\upsilon = n_1 + n_2 = 2j \qquad j = \frac{n_1 + n_2}{2} = \frac{\upsilon}{2} \qquad m = \frac{n_1 - n_2}{2}$$

$\upsilon+1=2j+1$ multiplies *base frequency* $\omega=\Omega_0$

m multiplies *beat frequency* Ω



$$\omega_+ = \Omega_0 + \Omega\left(+\tfrac{1}{2}\right)$$

$$\omega_- = \Omega_0 + \Omega\left(-\tfrac{1}{2}\right)$$

Setting $(B=0=C)$ and $(A=\omega_+)$ and $(D=\omega_-)$ gives diagonal block matrices.

$\langle \mathbf{H} \rangle = A(1/2) + D(1/2) +$

	$\|00\rangle$	$\|01\rangle$	$\|10\rangle$	$\|02\rangle$	$\|11\rangle$	$\|20\rangle$	$\|03\rangle$	$\|12\rangle$	$\|21\rangle$	$\|30\rangle$	$\cdots$
$\langle 00\|$	0										
$\langle 01\|$		ω_-									
$\langle 10\|$			ω_+								
$\langle 02\|$				$2\omega_-$							
$\langle 11\|$					$\omega_+ + \omega_-$						
$\langle 20\|$						$2\omega_+$					
$\langle 03\|$							$3\omega_-$				
$\langle 12\|$								$\omega_+ + 2\omega_-$			
$\langle 21\|$									$2\omega_+ + \omega_-$		
$\langle 30\|$										$3\omega_+$	
$\vdots$											

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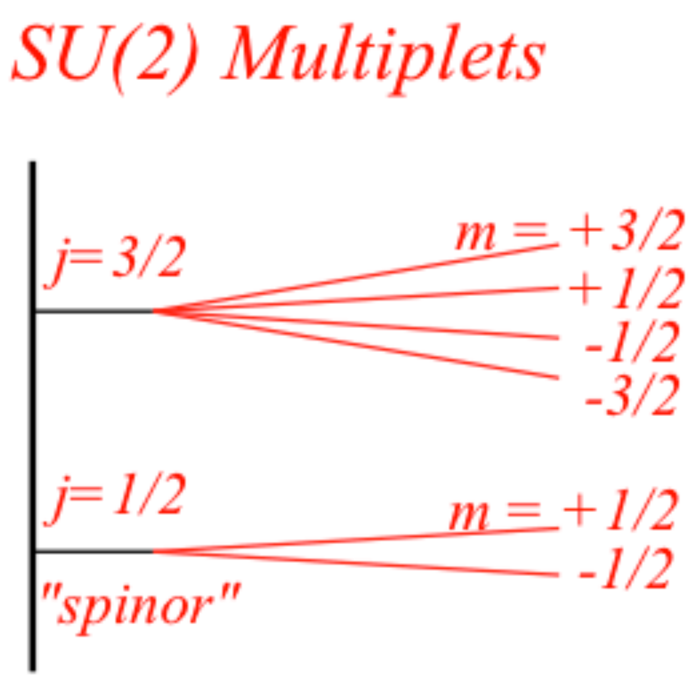

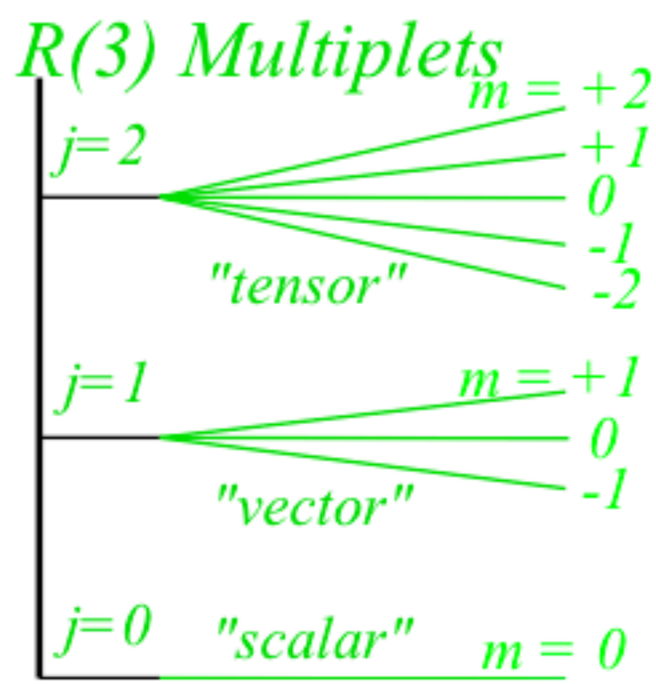

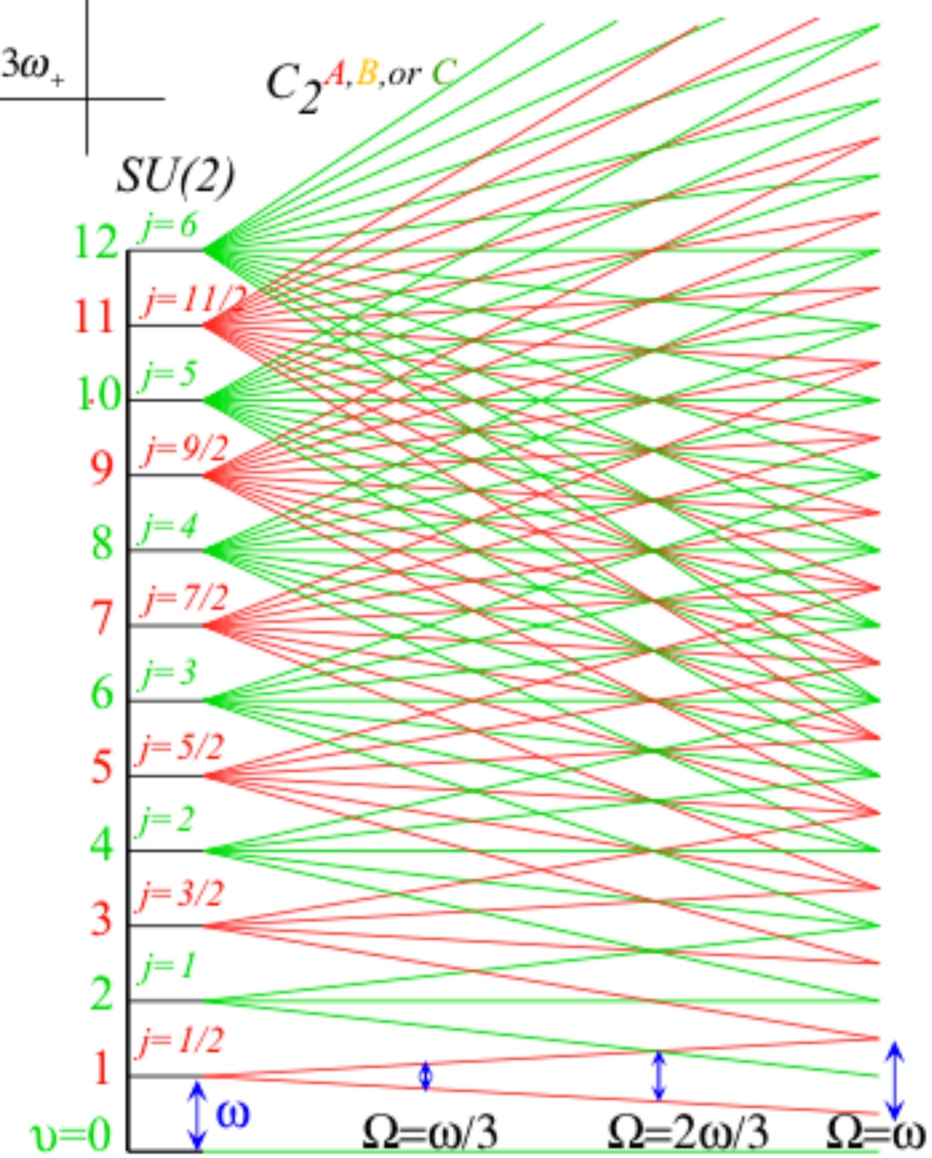

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R(3) Angular momentum generators by U(2) analysis

($\upsilon = 1$) or ($j = 1/2$) block **H** matrices of U(2) oscillator

Use irreps of unit operator $\mathbf{S}_0 = \mathbf{1}$ and spin operators $\{\mathbf{S}_X, \mathbf{S}_Y, \mathbf{S}_Z\}$. (also known as: $\{\mathbf{S}_B, \mathbf{S}_C, \mathbf{S}_A\}$)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + 2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} + (A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

R(3) Angular momentum generators by U(2) analysis

(υ=*1*) or (*j*=*1/2*) block **H** matrices of U(2) oscillator

Use irreps of unit operator $\mathbf{S}_0 = \mathbf{1}$ and spin operators $\{\mathbf{S}_X, \mathbf{S}_Y, \mathbf{S}_Z\}$. (also known as: $\{\mathbf{S}_B, \mathbf{S}_C, \mathbf{S}_A\}$)

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(υ=*2*) or (*j*=*1*) 3-by-3 block uses their vector irreps.

$$\begin{pmatrix} 2A & \sqrt{2}(B-iC) & \cdot \\ \sqrt{2}(B+iC) & A+D & \sqrt{2}(B-iC) \\ \cdot & \sqrt{2}(B+iC) & 2D \end{pmatrix} = (A+D)\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} + 2B\begin{pmatrix} \cdot & \frac{\sqrt{2}}{2} & \cdot \\ \frac{\sqrt{2}}{2} & \cdot & \frac{\sqrt{2}}{2} \\ \cdot & \frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + 2C\begin{pmatrix} \cdot & -i\frac{\sqrt{2}}{2} & \cdot \\ i\frac{\sqrt{2}}{2} & \cdot & -i\frac{\sqrt{2}}{2} \\ \cdot & i\frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + (A-D)\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

R(3) Angular momentum generators by U(2) analysis

($\upsilon=1$) or ($j=1/2$) block **H** matrices of U(2) oscillator

Use irreps of unit operator $\mathbf{S}_0 = \mathbf{1}$ and spin operators $\{\mathbf{S}_X, \mathbf{S}_Y, \mathbf{S}_Z\}$. (also known as: $\{\mathbf{S}_B, \mathbf{S}_C, \mathbf{S}_A\}$)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + 2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} + (A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

($\upsilon=2$) or ($j=1$) 3-by-3 block uses their vector irreps.

$$\begin{pmatrix} 2A & \sqrt{2}(B-iC) & \cdot \\ \sqrt{2}(B+iC) & A+D & \sqrt{2}(B-iC) \\ \cdot & \sqrt{2}(B+iC) & 2D \end{pmatrix} = (A+D)\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} + 2B\begin{pmatrix} \cdot & \frac{\sqrt{2}}{2} & \cdot \\ \frac{\sqrt{2}}{2} & \cdot & \frac{\sqrt{2}}{2} \\ \cdot & \frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + 2C\begin{pmatrix} \cdot & -i\frac{\sqrt{2}}{2} & \cdot \\ i\frac{\sqrt{2}}{2} & \cdot & -i\frac{\sqrt{2}}{2} \\ \cdot & i\frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + (A-D)\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

($\upsilon=3$) or ($j=3/2$) 4-by-4 block uses Dirac spinor irreps.

$$\begin{pmatrix} 3A & \sqrt{3}(B-iC) & & \\ \sqrt{3}(B+iC) & 2A+D & \sqrt{4}(B-iC) & \\ & \sqrt{4}(B+iC) & A+2D & \sqrt{3}(B-iC) \\ & & \sqrt{3}(B+iC) & 3D \end{pmatrix} = \frac{3(A+D)}{2}\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} + 2B\begin{pmatrix} \cdot & \frac{\sqrt{3}}{2} & \cdot & \cdot \\ \frac{\sqrt{3}}{2} & \cdot & \frac{\sqrt{4}}{2} & \cdot \\ \cdot & \frac{\sqrt{4}}{2} & \cdot & \frac{\sqrt{3}}{2} \\ \cdot & \cdot & \frac{\sqrt{3}}{2} & \cdot \end{pmatrix} + 2C\begin{pmatrix} \cdot & -i\frac{\sqrt{3}}{2} & \cdot & \cdot \\ i\frac{\sqrt{3}}{2} & \cdot & -i\frac{\sqrt{4}}{2} & \cdot \\ \cdot & i\frac{\sqrt{4}}{2} & \cdot & -i\frac{\sqrt{3}}{2} \\ \cdot & \cdot & i\frac{\sqrt{3}}{2} & \cdot \end{pmatrix} + (A-D)\begin{pmatrix} \frac{3}{2} & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{2} & \cdot & \cdot \\ \cdot & \cdot & -\frac{1}{2} & \cdot \\ \cdot & \cdot & \cdot & -\frac{3}{2} \end{pmatrix}$$

R(3) Angular momentum generators by U(2) analysis

(υ=*1*) or (*j*=*1/2*) block **H** matrices of U(2) oscillator

Use irreps of unit operator $\mathbf{S}_0 = \mathbf{1}$ and spin operators $\{\mathbf{S}_X, \mathbf{S}_Y, \mathbf{S}_Z\}$. (also known as: $\{\mathbf{S}_B, \mathbf{S}_C, \mathbf{S}_A\}$)

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(υ=*2*) or (*j*=*1*) 3-by-3 block uses their vector irreps.

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(υ=*3*) or (*j*=*3/2*) 4-by-4 block uses Dirac spinor irreps.

$$\begin{pmatrix} 3A & \sqrt{3}(B-iC) & & \\ \sqrt{3}(B+iC) & 2A+D & \sqrt{4}(B-iC) & \\ & \sqrt{4}(B+iC) & A+2D & \sqrt{3}(B-iC) \\ & & \sqrt{3}(B+iC) & 3D \end{pmatrix} = \frac{3(A+D)}{2}\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} + 2B\begin{pmatrix} \cdot & \frac{\sqrt{3}}{2} & \cdot & \cdot \\ \frac{\sqrt{3}}{2} & \cdot & \frac{\sqrt{4}}{2} & \cdot \\ \cdot & \frac{\sqrt{4}}{2} & \cdot & \frac{\sqrt{3}}{2} \\ \cdot & \cdot & \frac{\sqrt{3}}{2} & \cdot \end{pmatrix} + 2C\begin{pmatrix} \cdot & -i\frac{\sqrt{3}}{2} & \cdot & \cdot \\ i\frac{\sqrt{3}}{2} & \cdot & -i\frac{\sqrt{4}}{2} & \cdot \\ \cdot & i\frac{\sqrt{4}}{2} & \cdot & -i\frac{\sqrt{3}}{2} \\ \cdot & \cdot & i\frac{\sqrt{3}}{2} & \cdot \end{pmatrix} + (A-D)\begin{pmatrix} \frac{3}{2} & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{2} & \cdot & \cdot \\ \cdot & \cdot & -\frac{1}{2} & \cdot \\ \cdot & \cdot & \cdot & -\frac{3}{2} \end{pmatrix}$$

(υ=*2j*) or (2*j*+*1*)-by-(2*j*+*1*) block uses $D^{(j)}(\mathbf{s}_\mu)$ irreps of U(2) or R(3).

$$\langle \mathbf{H} \rangle^{j-block} = 2j\Omega_0 \langle \mathbf{1} \rangle^j + \Omega_X \langle \mathbf{s}_X \rangle^j + \Omega_Y \langle \mathbf{s}_Y \rangle^j + \Omega_Z \langle \mathbf{s}_Z \rangle^j$$

R(3) Angular momentum generators by U(2) analysis

($\upsilon=1$) or ($j=1/2$) block **H** matrices of U(2) oscillator

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All j-block matrix operators factor into *raise-n-lower* operators $\mathbf{s}_\pm = \mathbf{s}_X \pm i\mathbf{s}_Y$ plus the diagonal $\mathbf{s}_Z$

$$\langle \mathbf{H} \rangle^{j-block} = 2j\Omega_0\langle \mathbf{1}\rangle^j + \left[(\Omega_X - i\Omega_Y)\langle \mathbf{s}_X + i\mathbf{s}_Y\rangle^j + (\Omega_X + i\Omega_Y)\langle \mathbf{s}_X - i\mathbf{s}_Y\rangle^j\right]/2 + \Omega_Z\langle \mathbf{s}_Z\rangle^j$$

Review : *2-D* **a**†**a** *algebra of U(2) representations*

Angular momentum generators by U(2) analysis
Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Angular momentum commutation relations
Key Lie theorems

Angular momentum magnitude and uncertainty
Angular momentum uncertainty angle

Generating R(3) rotation and U(2) representations
Applications of R(3) rotation and U(2) representations
Molecular and nuclear wavefunctions
Molecular and nuclear eigenstates
Generalized Stern-Gerlach and transformation matrices
Angular momentum cones and high J properties

Angular momentum raise-n-lower operators $\mathbf{S}_+$ and $\mathbf{S}_-$

$$\mathbf{s}_+ = \mathbf{s}_X + i\mathbf{s}_Y \quad \text{and} \quad \mathbf{s}_- = \mathbf{s}_X - i\mathbf{s}_Y = \mathbf{s}_+^\dagger$$

Starting with *j=1/2* we see that $\mathbf{S}+$ is an *elementary projection operator* $\mathbf{e}_{12} = |1\rangle\langle 2| = \mathbf{P}_{12}$

$$\langle \mathbf{s}_+ \rangle^{\frac{1}{2}} = D^{\frac{1}{2}}(\mathbf{s}_+) = D^{\frac{1}{2}}(\mathbf{s}_X + i\mathbf{s}_Y) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + i \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{P}_{12}$$

Such operators can be upgraded to *creation-destruction operator* combinations $\mathbf{a}^\dagger\mathbf{a}$

$$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2 = \mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow \quad , \qquad \mathbf{s}_- = \left(\mathbf{a}_1^\dagger \mathbf{a}_2\right)^\dagger = \mathbf{a}_2^\dagger \mathbf{a}_1 = \mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow$$

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This suggests an $\mathbf{a}^\dagger\mathbf{a}$ form for $\mathbf{s}_Z$.

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Let $\mathbf{a}_1^\dagger = \mathbf{a}_\uparrow^\dagger$ create up-spin $\uparrow$

$$|1\rangle = |\uparrow\rangle = \left| \begin{matrix} 1/2 \\ +1/2 \end{matrix} \right\rangle = \mathbf{a}_1^\dagger |0\rangle = \mathbf{a}_\uparrow^\dagger |0\rangle$$

Angular momentum raise-n-lower operators $\mathbf{S}_+$ and $\mathbf{S}_-$

$$\boxed{\mathbf{s}_+ = \mathbf{s}_X + i\mathbf{s}_Y} \quad \text{and} \quad \boxed{\mathbf{s}_- = \mathbf{s}_X - i\mathbf{s}_Y = \mathbf{s}_+^\dagger}$$

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Let $\mathbf{a}_2^\dagger = \mathbf{a}_\downarrow^\dagger$ create dn-spin $\downarrow$

$$|2\rangle = |\downarrow\rangle = \left| \begin{matrix} 1/2 \\ -1/2 \end{matrix} \right\rangle = \mathbf{a}_2^\dagger|0\rangle = \mathbf{a}_\downarrow^\dagger|0\rangle$$

Angular momentum raise-n-lower operators $\mathbf{S}_+$ and $\mathbf{S}_-$

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$$|2\rangle = |\downarrow\rangle = \left| \begin{matrix} 1/2 \\ -1/2 \end{matrix} \right\rangle = \mathbf{a}_2^\dagger |0\rangle = \mathbf{a}_\downarrow^\dagger |0\rangle$$

$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2 = \mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow$ destroys dn-spin ↓

creates up-spin ↑

to raise angular momentum by one $\hbar$ unit

$$\mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow |\downarrow\rangle = |\uparrow\rangle \quad \text{or:} \quad \mathbf{a}_1^\dagger \mathbf{a}_2 |2\rangle = |1\rangle$$

Angular momentum raise-n-lower operators $\mathbf{S}_+$ and $\mathbf{S}_-$

$$\mathbf{s}_+ = \mathbf{s}_X + i\mathbf{s}_Y \quad \text{and} \quad \mathbf{s}_- = \mathbf{s}_X - i\mathbf{s}_Y = \mathbf{s}_+^\dagger$$

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Such operators can be upgraded to *creation-destruction operator* combinations $\mathbf{a}^\dagger\mathbf{a}$

$$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2 = \mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow \quad , \qquad \mathbf{s}_- = (\mathbf{a}_1^\dagger \mathbf{a}_2)^\dagger = \mathbf{a}_2^\dagger \mathbf{a}_1 = \mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow$$

Hamilton-Pauli-Jordan representation of $\mathbf{s}_Z$ is: $\langle \mathbf{s}_Z \rangle^{\left(\frac{1}{2}\right)} = D^{\left(\frac{1}{2}\right)}(\mathbf{s}_Z) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

This suggests an $\mathbf{a}^\dagger\mathbf{a}$ form for $\mathbf{s}_Z$.

$$\mathbf{s}_Z = \tfrac{1}{2}(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2) = \tfrac{1}{2}(\mathbf{a}_\uparrow^\dagger \mathbf{a}_\uparrow - \mathbf{a}_\downarrow^\dagger \mathbf{a}_\downarrow)$$

Let $\mathbf{a}_1^\dagger = \mathbf{a}_\uparrow^\dagger$ create up-spin $\uparrow$

$$|1\rangle = |\uparrow\rangle = \left| \begin{matrix} 1/2 \\ +1/2 \end{matrix} \right\rangle = \mathbf{a}_1^\dagger|0\rangle = \mathbf{a}_\uparrow^\dagger|0\rangle$$

$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2 = \mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow$ destroys dn-spin $\downarrow$

creates up-spin $\uparrow$

to raise angular momentum by one $\hbar$ unit

$$\mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow |\downarrow\rangle = |\uparrow\rangle \quad \text{or:} \quad \mathbf{a}_1^\dagger \mathbf{a}_2 |2\rangle = |1\rangle$$

Let $\mathbf{a}_2^\dagger = \mathbf{a}_\downarrow^\dagger$ create dn-spin $\downarrow$

$$|2\rangle = |\downarrow\rangle = \left| \begin{matrix} 1/2 \\ -1/2 \end{matrix} \right\rangle = \mathbf{a}_2^\dagger|0\rangle = \mathbf{a}_\downarrow^\dagger|0\rangle$$

$\mathbf{s}_- = \mathbf{a}_2^\dagger \mathbf{a}_1 = \mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow$ destroys up-spin $\uparrow$

creates dn-spin $\downarrow$

to lower angular momentum by one $\hbar$ unit

$$\mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow |\uparrow\rangle = |\downarrow\rangle \quad \text{or:} \quad \mathbf{a}_2^\dagger \mathbf{a}_1 |1\rangle = |2\rangle$$

Review : *2-D* **a**†**a** *algebra of U(2) representations*

Angular momentum generators by U(2) analysis
Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Angular momentum commutation relations
Key Lie theorems

Angular momentum magnitude and uncertainty
Angular momentum uncertainty angle

Generating R(3) rotation and U(2) representations
Applications of R(3) rotation and U(2) representations
Molecular and nuclear wavefunctions
Molecular and nuclear eigenstates
Generalized Stern-Gerlach and transformation matrices
Angular momentum cones and high J properties

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U(2) boson oscillator states $|n_1,n_2\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$

$$|n_1 n_2\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}|0\,0\rangle$$

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U(2) boson oscillator states $|n_1,n_2\rangle$= R(3) spin or rotor states $\left|{j \atop m}\right\rangle$

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$j=1$ vector $\mathbf{s}_+$

$$D^1(\mathbf{s}_+)=D^1(\mathbf{s}_X+i\mathbf{s}_Y)=\begin{pmatrix} \cdot & \frac{\sqrt{2}}{2} & \cdot \\ \frac{\sqrt{2}}{2} & \cdot & \frac{\sqrt{2}}{2} \\ \cdot & \frac{\sqrt{2}}{2} & \cdot \end{pmatrix}+i\begin{pmatrix} \cdot & -i\frac{\sqrt{2}}{2} & \cdot \\ i\frac{\sqrt{2}}{2} & \cdot & -i\frac{\sqrt{2}}{2} \\ \cdot & i\frac{\sqrt{2}}{2} & \cdot \end{pmatrix}=\begin{pmatrix} \cdot & \sqrt{2} & \cdot \\ 0 & \cdot & \sqrt{2} \\ \cdot & 0 & \cdot \end{pmatrix},$$

...and $\mathbf{s}_Z$

$$D^1(\mathbf{s}_Z)=\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

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$$\left|n_\uparrow n_\downarrow\right\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{n_1}\left(\mathbf{a}_\downarrow^\dagger\right)^{n_2}}{\sqrt{n_\uparrow!\,n_\downarrow!}}\left|0\,0\right\rangle = \frac{\left(\mathbf{a}_\uparrow^\dagger\right)^{j+m}\left(\mathbf{a}_\downarrow^\dagger\right)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\left|0\,0\right\rangle = \left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

Oscillator $\mathbf{a}^\dagger\mathbf{a}$ give $\mathbf{s}_+$ and $\mathbf{s}_-$ matrices.

$$\mathbf{a}_1^\dagger\mathbf{a}_2\left|n_1 n_2\right\rangle=\sqrt{n_1+1}\sqrt{n_2}\left|n_1+1\;n_2-1\right\rangle \Rightarrow \mathbf{s}_+\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=\sqrt{j+m+1}\sqrt{j-m}\left|\begin{smallmatrix} j \\ m+1 \end{smallmatrix}\right\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle=\sqrt{n_1}\sqrt{n_2+1}\left|n_1-1\;n_2+1\right\rangle \Rightarrow \mathbf{s}_-\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=\sqrt{j+m}\sqrt{j-m+1}\left|\begin{smallmatrix} j \\ m-1 \end{smallmatrix}\right\rangle$$

1/2-difference of number-ops is $\mathbf{s}_Z$ eigenvalue.

$$\left.\begin{aligned}\mathbf{a}_1^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle&=n_1\left|n_1 n_2\right\rangle\\ \mathbf{a}_2^\dagger\mathbf{a}_2\left|n_1 n_2\right\rangle&=n_2\left|n_1 n_2\right\rangle\end{aligned}\right\}\quad \mathbf{s}_Z\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=\tfrac{1}{2}\left(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2\right)\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=\frac{n_1-n_2}{2}\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=m\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

j=1 vector $\mathbf{s}_+$ ...and $\mathbf{s}_Z$

$$D^1(\mathbf{s}_+)=D^1(\mathbf{s}_X+i\mathbf{s}_Y)=\begin{pmatrix} \cdot & \frac{\sqrt{2}}{2} & \cdot \\ \frac{\sqrt{2}}{2} & \cdot & \frac{\sqrt{2}}{2} \\ \cdot & \frac{\sqrt{2}}{2} & \cdot \end{pmatrix}+i\begin{pmatrix} \cdot & -i\frac{\sqrt{2}}{2} & \cdot \\ i\frac{\sqrt{2}}{2} & \cdot & -i\frac{\sqrt{2}}{2} \\ \cdot & i\frac{\sqrt{2}}{2} & \cdot \end{pmatrix}=\begin{pmatrix} \cdot & \sqrt{2} & \cdot \\ 0 & \cdot & \sqrt{2} \\ \cdot & 0 & \cdot \end{pmatrix},\qquad D^1(\mathbf{s}_Z)=\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

j=3/2 spinor $\mathbf{s}_+$ ...and $\mathbf{s}_Z$

$$D^{\frac{3}{2}}(\mathbf{s}_+)=\begin{pmatrix} \cdot & \sqrt{3} & \cdot & \cdot \\ 0 & \cdot & \sqrt{4} & \cdot \\ \cdot & 0 & \cdot & \sqrt{3} \\ \cdot & \cdot & 0 & \cdot \end{pmatrix}=\left(D^{\frac{3}{2}}(\mathbf{s}_-)\right)^\dagger,\qquad D^{\frac{3}{2}}(\mathbf{s}_Z)=\begin{pmatrix} \frac{3}{2} & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{2} & \cdot & \cdot \\ \cdot & \cdot & -\frac{1}{2} & \cdot \\ \cdot & \cdot & \cdot & -\frac{3}{2} \end{pmatrix}$$

SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

U(2) boson oscillator states $|n_1,n_2\rangle$= R(3) spin or rotor states $\left|{}^{j}_{m}\right\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$ Rotor total momenta: $j=\upsilon/2$ and z-momenta: $m=(n_1-n_2)/2$

$$|n_1 n_2\rangle = \frac{(\mathbf{a}_1^\dagger)^{n_1}(\mathbf{a}_2^\dagger)^{n_2}}{\sqrt{n_1!\,n_2!}}|0\,0\rangle = \frac{(\mathbf{a}_1^\dagger)^{j+m}(\mathbf{a}_2^\dagger)^{j-m}}{\sqrt{(j+m)!(j-m)!}}|0\,0\rangle = \left|{}^{j}_{m}\right\rangle$$

$j=\upsilon/2=(n_1+n_2)/2$

$m=(n_1-n_2)/2$

$n_1=j+m$

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U(2) boson oscillator states = *U(2)* spinor states

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$$\left.\begin{array}{l}\mathbf{a}_1^\dagger\mathbf{a}_1|n_1 n_2\rangle=n_1|n_1 n_2\rangle\\ \mathbf{a}_2^\dagger\mathbf{a}_2|n_1 n_2\rangle=n_2|n_1 n_2\rangle\end{array}\right\}\mathbf{s}_Z\left|{}^{j}_{m}\right\rangle=\tfrac{1}{2}(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2)\left|{}^{j}_{m}\right\rangle=\frac{n_1-n_2}{2}\left|{}^{j}_{m}\right\rangle=m\left|{}^{j}_{m}\right\rangle$$

j=1 vector $\mathbf{s}_+$ ...*and* $\mathbf{s}_Z$

$$D^1(\mathbf{s}_+)=D^1(\mathbf{s}_X+i\mathbf{s}_Y)=\begin{pmatrix}\cdot & \frac{\sqrt2}{2} & \cdot\\ \frac{\sqrt2}{2} & \cdot & \frac{\sqrt2}{2}\\ \cdot & \frac{\sqrt2}{2} & \cdot\end{pmatrix}+i\begin{pmatrix}\cdot & -i\frac{\sqrt2}{2} & \cdot\\ i\frac{\sqrt2}{2} & \cdot & -i\frac{\sqrt2}{2}\\ \cdot & i\frac{\sqrt2}{2} & \cdot\end{pmatrix}=\begin{pmatrix}\cdot & \sqrt2 & \cdot\\ 0 & \cdot & \sqrt2\\ \cdot & 0 & \cdot\end{pmatrix},\quad D^1(\mathbf{s}_Z)=\begin{pmatrix}1 & \cdot & \cdot\\ \cdot & 0 & \cdot\\ \cdot & \cdot & -1\end{pmatrix}$$

j=3/2 spinor $\mathbf{s}_+$ ...*and* $\mathbf{s}_Z$

$$D^{\frac32}(\mathbf{s}_+)=\begin{pmatrix}\cdot & \sqrt3 & \cdot & \cdot\\ 0 & \cdot & \sqrt4 & \cdot\\ \cdot & 0 & \cdot & \sqrt3\\ \cdot & \cdot & 0 & \cdot\end{pmatrix}=\left(D^{\frac32}(\mathbf{s}_-)\right)^\dagger,\quad D^{\frac32}(\mathbf{s}_Z)=\begin{pmatrix}\frac32 & \cdot & \cdot & \cdot\\ \cdot & \frac12 & \cdot & \cdot\\ \cdot & \cdot & -\frac12 & \cdot\\ \cdot & \cdot & \cdot & -\frac32\end{pmatrix}$$

j=2 tensor $\mathbf{s}_+$ ...*and* $\mathbf{s}_Z$

$$D^2(\mathbf{s}_+)=\begin{pmatrix}\cdot & \sqrt4 & \cdot & \cdot & \cdot\\ 0 & \cdot & \sqrt3 & \cdot & \cdot\\ \cdot & 0 & \cdot & \sqrt3 & \cdot\\ \cdot & \cdot & 0 & \cdot & \sqrt4\\ \cdot & \cdot & \cdot & 0 & \cdot\end{pmatrix}=\left(D^2(\mathbf{s}_-)\right)^\dagger,\quad D^2(\mathbf{s}_Z)=\begin{pmatrix}2 & \cdot & \cdot & \cdot & \cdot\\ \cdot & 1 & \cdot & \cdot & \cdot\\ \cdot & \cdot & 0 & \cdot & \cdot\\ \cdot & \cdot & \cdot & -1 & \cdot\\ \cdot & \cdot & \cdot & \cdot & -2\end{pmatrix}$$

Review : *2-D* $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(2) representations*

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QED

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Then there are *up-down commutation relation*: $[\mathbf{s}_+, \mathbf{s}_-] = [\mathbf{e}_{12}, \mathbf{e}_{21}] = \mathbf{e}_{11} - \mathbf{e}_{22} = 2\mathbf{s}_Z$

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General eigen-commutation theorem:

If Hamiltonian $\mathbf{H}$ (or any operator such as $\mathbf{s}_Z$) eigen-commutes with $\mathbf{a}_m$ and $\mathbf{a}^\dagger_n$, that is:

$[\mathbf{H}, \mathbf{a}^\dagger_n] = \omega_n\, \mathbf{a}^\dagger_n$ and $[\mathbf{H}, \mathbf{a}_m] = \omega_m \mathbf{a}_m$, then $\mathbf{H}$ is a combination $\omega_n \mathbf{a}^\dagger_n \mathbf{a}_n$ of number operators.

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$\mathbf{e}_{jk}\, \mathbf{e}_{mn} = \delta_{km}\, \mathbf{e}_{jn}$ gives: $[(\mathbf{e}_{11} - \mathbf{e}_{22})/2, \mathbf{e}_{12}] = +\mathbf{e}_{12}$ and: $[(\mathbf{e}_{11} - \mathbf{e}_{22})/2, \mathbf{e}_{21}] = -\mathbf{e}_{21}$

QED

Then there are *up-down commutation relation*: $[\mathbf{s}_+, \mathbf{s}_-] = [\mathbf{e}_{12}, \mathbf{e}_{21}] = \mathbf{e}_{11} - \mathbf{e}_{22} = 2\mathbf{s}_Z$

General eigen-commutation theorem:

If Hamiltonian **H** (or any operator such as $\mathbf{s}_Z$) eigen-commutes with $\mathbf{a}_m$ and $\mathbf{a}^\dagger_n$, that is:

$[\mathbf{H}, \mathbf{a}^\dagger_n] = \omega_n\, \mathbf{a}^\dagger_n$ and $[\mathbf{H}, \mathbf{a}_m] = \omega_m \mathbf{a}_m$, then **H** is a combination $\omega_n \mathbf{a}^\dagger_n \mathbf{a}_n$ of number operators.

$$\mathbf{H} = \sum_{n=1}^{2} \omega_n \mathbf{a}^\dagger_n \mathbf{a}_n = \omega_1 \mathbf{a}^\dagger_1 \mathbf{a}_1 + \omega_2 \mathbf{a}^\dagger_2 \mathbf{a}_2 \approx \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}$$

Angular momentum commutation relations

Given Hamilton-Jordan-Pauli product relations : $\sigma_\alpha \sigma_\beta = \delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma$ with: $\mathbf{s}_\alpha = \sigma_\alpha / 2$

Commutator formulae for $\mathbf{s}_\alpha$: $\mathbf{s}_\alpha \mathbf{s}_\beta - \mathbf{s}_\beta \mathbf{s}_\alpha = [\mathbf{s}_\alpha, \mathbf{s}_\beta] = i\, \varepsilon_{\alpha\beta\gamma} \mathbf{s}_\gamma$

$\sigma_X \sigma_Y = i\sigma_Z$ implies: $[\mathbf{s}_X, \mathbf{s}_Y] = i\mathbf{s}_Z$

$\sigma_Z \sigma_X = i\sigma_Y$ implies: $[\mathbf{s}_Z, \mathbf{s}_X] = i\mathbf{s}_Y$

$\sigma_Y \sigma_Z = i\sigma_X$ implies: $[\mathbf{s}_Y, \mathbf{s}_Z] = i\mathbf{s}_X$

Key Lie theorem:

$\mathbf{s}_Z$ and $\mathbf{s}_\pm = \mathbf{s}_X \pm i\mathbf{s}_Y$ obey *eigen-commutation relations*. $[\mathbf{s}_Z, \mathbf{s}_+] = (+1)\mathbf{s}_+$ and: $[\mathbf{s}_Z, \mathbf{s}_-] = (-1)\mathbf{s}_-$

Proof using elementary matrix operator multiplication: $\mathbf{e}_{jk}\, \mathbf{e}_{mn} = \delta_{km}\, \mathbf{e}_{jn}$ with: $\mathbf{s}_+ = \mathbf{e}_{12} \approx \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and: $\mathbf{s}_- = \mathbf{e}_{21} \approx \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Also: $\mathbf{s}_Z = (\mathbf{e}_{11} - \mathbf{e}_{22})/2 \approx \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$

$\mathbf{e}_{jk}\, \mathbf{e}_{mn} = \delta_{km}\, \mathbf{e}_{jn}$ gives: $[(\mathbf{e}_{11} - \mathbf{e}_{22})/2, \mathbf{e}_{12}] = +\mathbf{e}_{12}$ and: $[(\mathbf{e}_{11} - \mathbf{e}_{22})/2, \mathbf{e}_{21}] = -\mathbf{e}_{21}$

QED

Then there are *up-down commutation relation*: $[\mathbf{s}_+, \mathbf{s}_-] = [\mathbf{e}_{12}, \mathbf{e}_{21}] = \mathbf{e}_{11} - \mathbf{e}_{22} = 2\mathbf{s}_Z$

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$$\mathbf{H} = \sum_{n=1}^{2} \omega_n \mathbf{a}^\dagger_n \mathbf{a}_n = \omega_1 \mathbf{a}^\dagger_1 \mathbf{a}_1 + \omega_2 \mathbf{a}^\dagger_2 \mathbf{a}_2 \approx \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}$$

U(2) Oscillator eigensolutions:

$$\mathbf{H}|n_1 n_2\rangle = \sum_{n=1}^{2} \omega_n \mathbf{a}^\dagger_n \mathbf{a}_n |n_1 n_2\rangle = (\omega_1 n_1 + \omega_2 n_2)|n_1 n_2\rangle = (\omega_1 (j+m) + \omega_2 (j-m))\left|{}^{j}_{m}\right\rangle$$

Review : *2-D* **a†a** *algebra of U(2) representations*

Angular momentum generators by U(2) analysis
Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Angular momentum commutation relations
Key Lie theorems

➡ *Angular momentum magnitude and uncertainty* ⬅
Angular momentum uncertainty angle

Generating R(3) rotation and U(2) representations
Applications of R(3) rotation and U(2) representations
Molecular and nuclear wavefunctions
Molecular and nuclear eigenstates
Generalized Stern-Gerlach and transformation matrices
Angular momentum cones and high J properties

Angular momentum magnitude and uncertainty

$\mathbf{s}_\pm = \mathbf{s}_X \pm i\mathbf{s}_Y$

$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2$

$\mathbf{s}_- = \mathbf{a}_2^\dagger \mathbf{a}_1$

$\mathbf{s}_Z = \frac{1}{2}\left(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2\right)$

Angular momentum squared $\mathbf{s \bullet s}$ and Z-component $\mathbf{s}_Z$ share eigenstates

$$\mathbf{s \bullet s} = \mathbf{s}_X{}^2 + \mathbf{s}_Y{}^2 + \mathbf{s}_Z{}^2 = (\mathbf{s}_+\mathbf{s}_- + \mathbf{s}_-\mathbf{s}_+)/2 + \mathbf{s}_Z{}^2$$

$\mathbf{s}_{\pm}=\mathbf{s}_X \pm i\mathbf{s}_Y$

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j=1/2 fundamental matrices square up not to *(1/2)*^{2} *=1/4* but to *3/4*.

$$D^{\frac{1}{2}}\left(\mathbf{S}_X^2+\mathbf{S}_Y^2+\mathbf{S}_Z^2\right)=\frac{1}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}+\frac{1}{4}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}+\frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}=\frac{3}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\mathbf{s}_{\pm}=\mathbf{s}_X \pm i\mathbf{s}_Y$

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$$\mathbf{s}\bullet\mathbf{s}=\mathbf{s}_X^2+\mathbf{s}_Y^2+\mathbf{s}_Z^2=(\mathbf{s}_+\mathbf{s}_- + \mathbf{s}_-\mathbf{s}_+)/2+\mathbf{s}_Z^2$$

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$$D^{\frac{1}{2}}\left(\mathbf{S}_X^2+\mathbf{S}_Y^2+\mathbf{S}_Z^2\right)=\frac{1}{4}\begin{pmatrix}0&1\\1&0\end{pmatrix}\cdot\begin{pmatrix}0&1\\1&0\end{pmatrix}+\frac{1}{4}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\cdot\begin{pmatrix}0&-i\\i&0\end{pmatrix}+\frac{1}{4}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\cdot\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\frac{3}{4}\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

In terms of $\mathbf{a}$-operators the squared momentum operator is

$$\mathbf{s}\bullet\mathbf{s}=\frac{1}{4}\left[2\mathbf{a}_1^{\dagger}\mathbf{a}_2\mathbf{a}_2^{\dagger}\mathbf{a}_1+2\mathbf{a}_2^{\dagger}\mathbf{a}_1\mathbf{a}_1^{\dagger}\mathbf{a}_2+\left(\mathbf{a}_1^{\dagger}\mathbf{a}_1-\mathbf{a}_2^{\dagger}\mathbf{a}_2\right)\left(\mathbf{a}_1^{\dagger}\mathbf{a}_1-\mathbf{a}_2^{\dagger}\mathbf{a}_2\right)\right]$$

Angular momentum magnitude and uncertainty

$\mathbf{s}_\pm = \mathbf{s}_X \pm i\mathbf{s}_Y$

$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2$

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$$\mathbf{s}\bullet\mathbf{s} = \mathbf{s}_X^2 + \mathbf{s}_Y^2 + \mathbf{s}_Z^2 = (\mathbf{s}_+\mathbf{s}_- + \mathbf{s}_-\mathbf{s}_+)/2 + \mathbf{s}_Z^2$$

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In terms of **a**-operators the squared momentum operator is

$$\mathbf{s}\bullet\mathbf{s} = \frac{1}{4}\left[2\mathbf{a}_1^\dagger \mathbf{a}_2 \mathbf{a}_2^\dagger \mathbf{a}_1 + 2\mathbf{a}_2^\dagger \mathbf{a}_1 \mathbf{a}_1^\dagger \mathbf{a}_2 + \left(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2\right)\left(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2\right)\right]$$

Using $\mathbf{a}_m \mathbf{a}^\dagger_n = \mathbf{a}^\dagger_n \mathbf{a}_m + \delta_{mn}\mathbf{1}$ gives $\mathbf{s}\bullet\mathbf{s}$ as number operators.

(Normal order: *left←creation*, *destruct→right*.)

Angular momentum magnitude and uncertainty

$\mathbf{s}_\pm = \mathbf{s}_X \pm i\mathbf{s}_Y$

$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2$

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$$D^{\frac{1}{2}}\left(\mathbf{S}_X^2 + \mathbf{S}_Y^2 + \mathbf{S}_Z^2\right) = \frac{1}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{3}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In terms of **a**-operators the squared momentum operator is

$$\mathbf{s}\bullet\mathbf{s} = \frac{1}{4}\left[2\mathbf{a}_1^\dagger\mathbf{a}_2\mathbf{a}_2^\dagger\mathbf{a}_1 + 2\mathbf{a}_2^\dagger\mathbf{a}_1\mathbf{a}_1^\dagger\mathbf{a}_2 + \left(\mathbf{a}_1^\dagger\mathbf{a}_1 - \mathbf{a}_2^\dagger\mathbf{a}_2\right)\left(\mathbf{a}_1^\dagger\mathbf{a}_1 - \mathbf{a}_2^\dagger\mathbf{a}_2\right)\right]$$

Using $\mathbf{a}_m\mathbf{a}^\dagger_n = \mathbf{a}^\dagger_n\,\mathbf{a}_m + \delta_{mn}\mathbf{1}$ gives **s•s** as number operators.
(Normal order: *left←creation*, *destruct→right*.)

$$\mathbf{s}\bullet\mathbf{s} = \frac{1}{4}\left[2\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + \mathbf{1}\right)\mathbf{a}_1^\dagger\mathbf{a}_1 + 2\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{1}\right)\mathbf{a}_2^\dagger\mathbf{a}_2 + \left(\mathbf{a}_1^\dagger\mathbf{a}_1 - \mathbf{a}_2^\dagger\mathbf{a}_2\right)\left(\mathbf{a}_1^\dagger\mathbf{a}_1 - \mathbf{a}_2^\dagger\mathbf{a}_2\right)\right]$$

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Using $\mathbf{a}_m\mathbf{a}^\dagger_n = \mathbf{a}^\dagger_n\,\mathbf{a}_m + \delta_{mn}\mathbf{1}$ gives $\mathbf{s}\bullet\mathbf{s}$ as number operators.
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$$\mathbf{s}\bullet\mathbf{s} = \frac{1}{4}\left[2\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + \mathbf{1}\right)\mathbf{a}_1^\dagger\mathbf{a}_1 + 2\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{1}\right)\mathbf{a}_2^\dagger\mathbf{a}_2 + \left(\mathbf{a}_1^\dagger\mathbf{a}_1 - \mathbf{a}_2^\dagger\mathbf{a}_2\right)\left(\mathbf{a}_1^\dagger\mathbf{a}_1 - \mathbf{a}_2^\dagger\mathbf{a}_2\right)\right]$$

Eigenvalue formula is then found. (Replace number-operator $\mathbf{a}^\dagger_k\mathbf{a}_k$ with its number n_k)

$$\mathbf{S}\bullet\mathbf{S}\left|n_1 n_2\right\rangle = \frac{1}{4}\left[2\left(n_2+1\right)n_1 + 2\left(n_1+1\right)n_2 + \left(n_1 - n_2\right)\left(n_1 - n_2\right)\right]\left|n_1 n_2\right\rangle$$
$$= \frac{1}{4}\left[2n_1 + 2n_2 + 4n_1n_2 + \left(n_1 - n_2\right)\left(n_1 - n_2\right)\right]\left|n_1 n_2\right\rangle$$

Angular momentum magnitude and uncertainty

$\mathbf{s}_\pm = \mathbf{s}_X \pm i\mathbf{s}_Y$

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In terms of $\mathbf{a}$-operators the squared momentum operator is

$$\mathbf{s}\bullet\mathbf{s} = \frac{1}{4}\left[2\mathbf{a}_1^\dagger\mathbf{a}_2\mathbf{a}_2^\dagger\mathbf{a}_1 + 2\mathbf{a}_2^\dagger\mathbf{a}_1\mathbf{a}_1^\dagger\mathbf{a}_2 + \left(\mathbf{a}_1^\dagger\mathbf{a}_1 - \mathbf{a}_2^\dagger\mathbf{a}_2\right)\left(\mathbf{a}_1^\dagger\mathbf{a}_1 - \mathbf{a}_2^\dagger\mathbf{a}_2\right)\right]$$

Using $\mathbf{a}_m\mathbf{a}^\dagger_n = \mathbf{a}^\dagger_n\,\mathbf{a}_m + \delta_{mn}\mathbf{1}$ gives $\mathbf{s}\bullet\mathbf{s}$ as number operators.
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$$\mathbf{s}\bullet\mathbf{s} = \frac{1}{4}\left[2\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + \mathbf{1}\right)\mathbf{a}_1^\dagger\mathbf{a}_1 + 2\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{1}\right)\mathbf{a}_2^\dagger\mathbf{a}_2 + \left(\mathbf{a}_1^\dagger\mathbf{a}_1 - \mathbf{a}_2^\dagger\mathbf{a}_2\right)\left(\mathbf{a}_1^\dagger\mathbf{a}_1 - \mathbf{a}_2^\dagger\mathbf{a}_2\right)\right]$$

Eigenvalue formula is then found. (Replace number-operator $\mathbf{a}^\dagger_k\mathbf{a}_k$ with its number n_k)

$$\mathbf{s}\bullet\mathbf{s}\left|n_1 n_2\right\rangle = \frac{1}{4}\left[2\left(n_2+1\right)n_1 + 2\left(n_1+1\right)n_2 + \left(n_1-n_2\right)\left(n_1-n_2\right)\right]\left|n_1 n_2\right\rangle$$

$$= \frac{1}{4}\left[2n_1 + 2n_2 + 4n_1 n_2 + \left(n_1-n_2\right)\left(n_1-n_2\right)\right]\left|n_1 n_2\right\rangle$$

R(3) angular quanta in $n_1 = j+m$ and $n_2 = j-m$ give R(3) eigenvalue formula.

$$\mathbf{s}\bullet\mathbf{s}\left|{}^{j}_{m}\right\rangle = \frac{1}{4}\left[2\left(j+m+1\right)\left(j-m\right) + 2\left(j-m+1\right)\left(j+m\right) + 4m^2\right]\left|{}^{j}_{m}\right\rangle = j\left(j+1\right)\left|{}^{j}_{m}\right\rangle$$

Angular momentum magnitude and uncertainty

$$\mathbf{s}_\pm=\mathbf{s}_X \pm i\mathbf{s}_Y$$
$$\mathbf{s}_+=\mathbf{a}_1^\dagger\mathbf{a}_2$$
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$$\mathbf{s}\bullet\mathbf{s}=\mathbf{s}_X{}^2+\mathbf{s}_Y{}^2+\mathbf{s}_Z{}^2=(\mathbf{s}_+\mathbf{s}_-+\mathbf{s}_-\mathbf{s}_+)/2+\mathbf{s}_Z{}^2$$

j=1/2 fundamental matrices square up not to $(1/2)^2=1/4$ but to *3/4*.

$$D^{\frac{1}{2}}\left(\mathbf{S}_X^2+\mathbf{S}_Y^2+\mathbf{S}_Z^2\right)=\frac{1}{4}\begin{pmatrix}0&1\\1&0\end{pmatrix}\cdot\begin{pmatrix}0&1\\1&0\end{pmatrix}+\frac{1}{4}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\cdot\begin{pmatrix}0&-i\\i&0\end{pmatrix}+\frac{1}{4}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\cdot\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\frac{3}{4}\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

In terms of **a**-operators the squared momentum operator is

$$\mathbf{s}\bullet\mathbf{s}=\frac{1}{4}\left[2\mathbf{a}_1^\dagger\mathbf{a}_2\mathbf{a}_2^\dagger\mathbf{a}_1+2\mathbf{a}_2^\dagger\mathbf{a}_1\mathbf{a}_1^\dagger\mathbf{a}_2+\left(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2\right)\left(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2\right)\right]$$

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(Normal order: *left←creation*, *destruct→right*.)

$$\mathbf{s}\bullet\mathbf{s}=\frac{1}{4}\left[2\left(\mathbf{a}_2^\dagger\mathbf{a}_2+\mathbf{1}\right)\mathbf{a}_1^\dagger\mathbf{a}_1+2\left(\mathbf{a}_1^\dagger\mathbf{a}_1+\mathbf{1}\right)\mathbf{a}_2^\dagger\mathbf{a}_2+\left(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2\right)\left(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2\right)\right]$$

U(2) eigenvalue formula is then found.

$$\mathbf{s}\bullet\mathbf{s}\left|n_1n_2\right\rangle=\frac{1}{4}\left[2\left(n_2+1\right)n_1+2\left(n_1+1\right)n_2+\left(n_1-n_2\right)\left(n_1-n_2\right)\right]\left|n_1n_2\right\rangle$$
$$=\frac{1}{4}\left[2n_1+2n_2+4n_1n_2+\left(n_1-n_2\right)\left(n_1-n_2\right)\right]\left|n_1n_2\right\rangle$$

R(3) angular quanta in $n_1=j+m$ and $n_2=j-m$ give R(3) eigenvalue formula.

$$\mathbf{s}\bullet\mathbf{s}\left|{}^{j}_{m}\right\rangle=\frac{1}{4}\left[2\left(j+m+1\right)\left(j-m\right)+2\left(j-m+1\right)\left(j+m\right)+4m^2\right]\left|{}^{j}_{m}\right\rangle=j\left(j+1\right)\left|{}^{j}_{m}\right\rangle$$

For Large j:

Magnitude of angular momentum |**s**| approaches *j+1/2*: $\left|\mathbf{s}\right|\left|{}^{j}_{m}\right\rangle=\sqrt{\mathbf{s}\bullet\mathbf{s}}\left|{}^{j}_{m}\right\rangle=\sqrt{j\left(j+1\right)}\left|{}^{j}_{m}\right\rangle\cong\left(j+\frac{1}{2}\right)\left|{}^{j}_{m}\right\rangle$

Review : *2-D* **a†a** *algebra of U(2) representations*

Angular momentum generators by U(2) analysis
Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Angular momentum commutation relations
Key Lie theorems

Angular momentum magnitude and uncertainty
Angular momentum uncertainty angle

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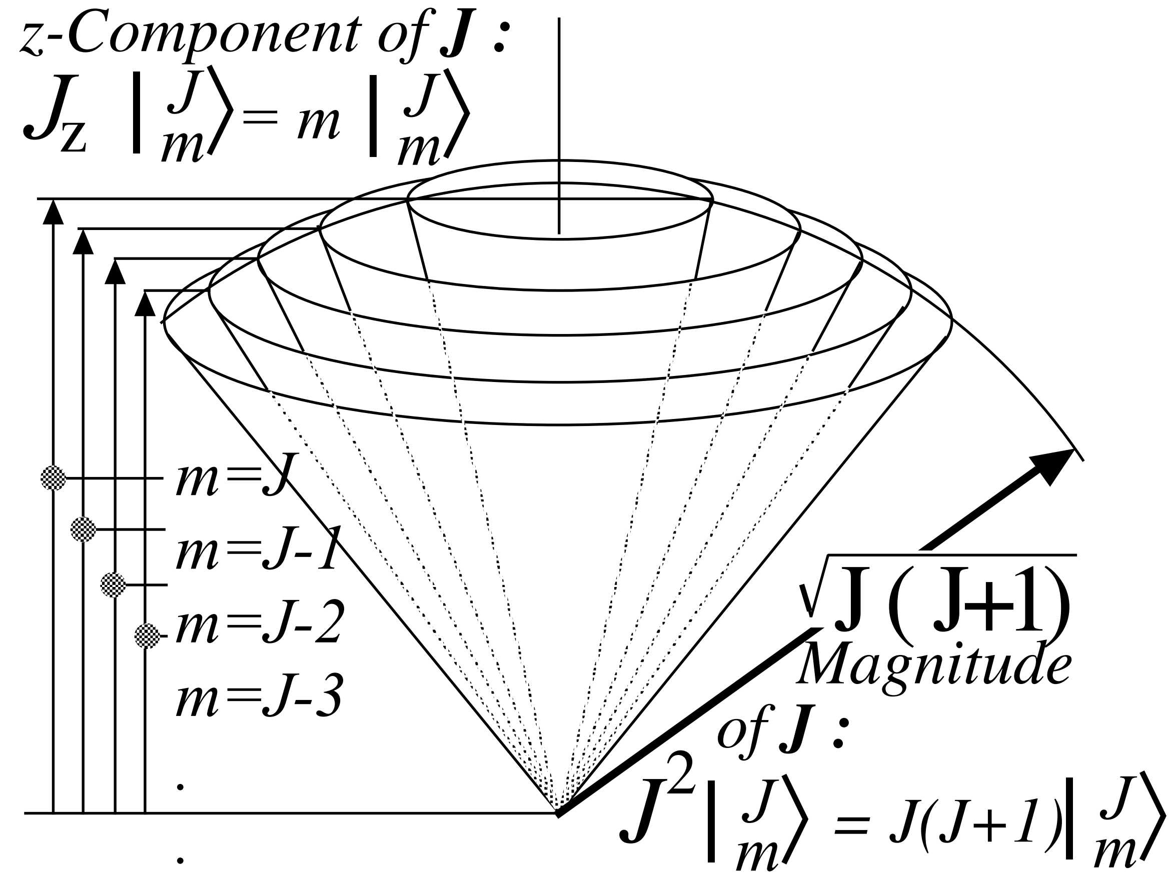

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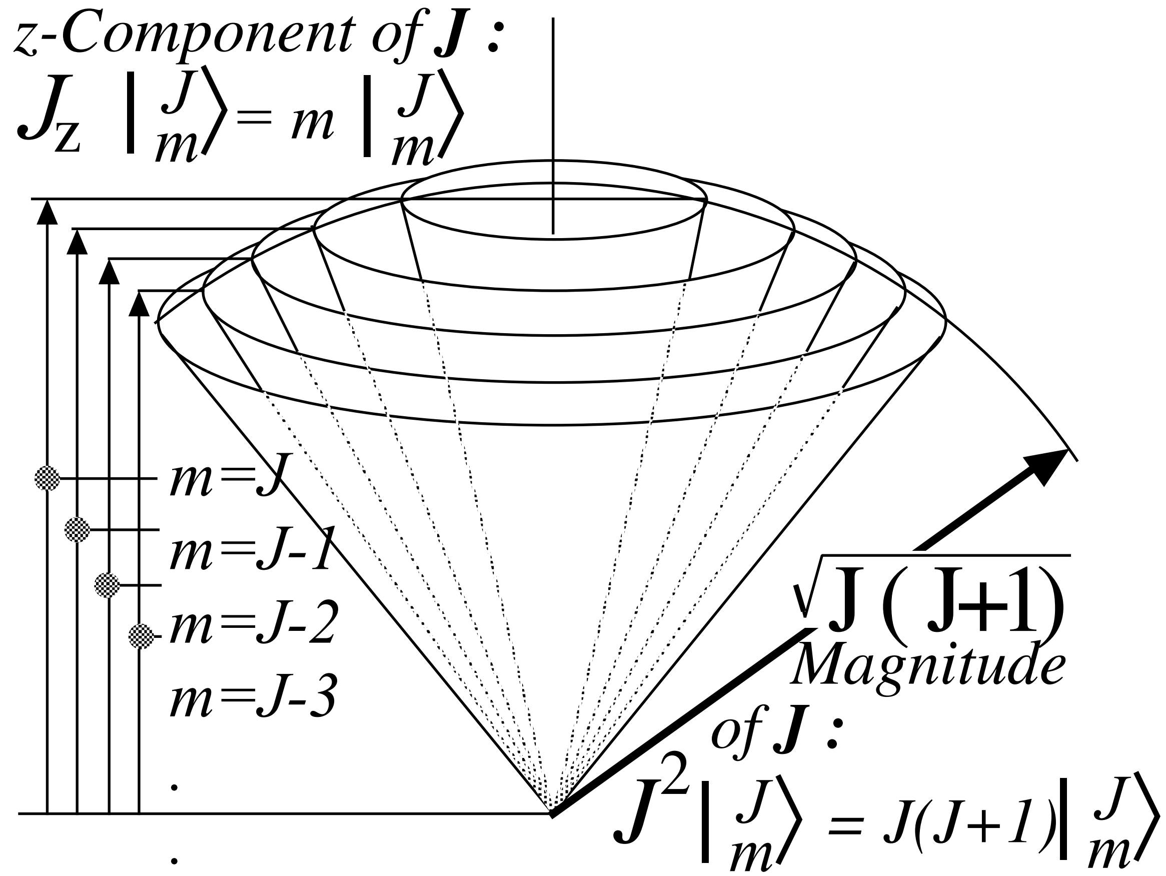

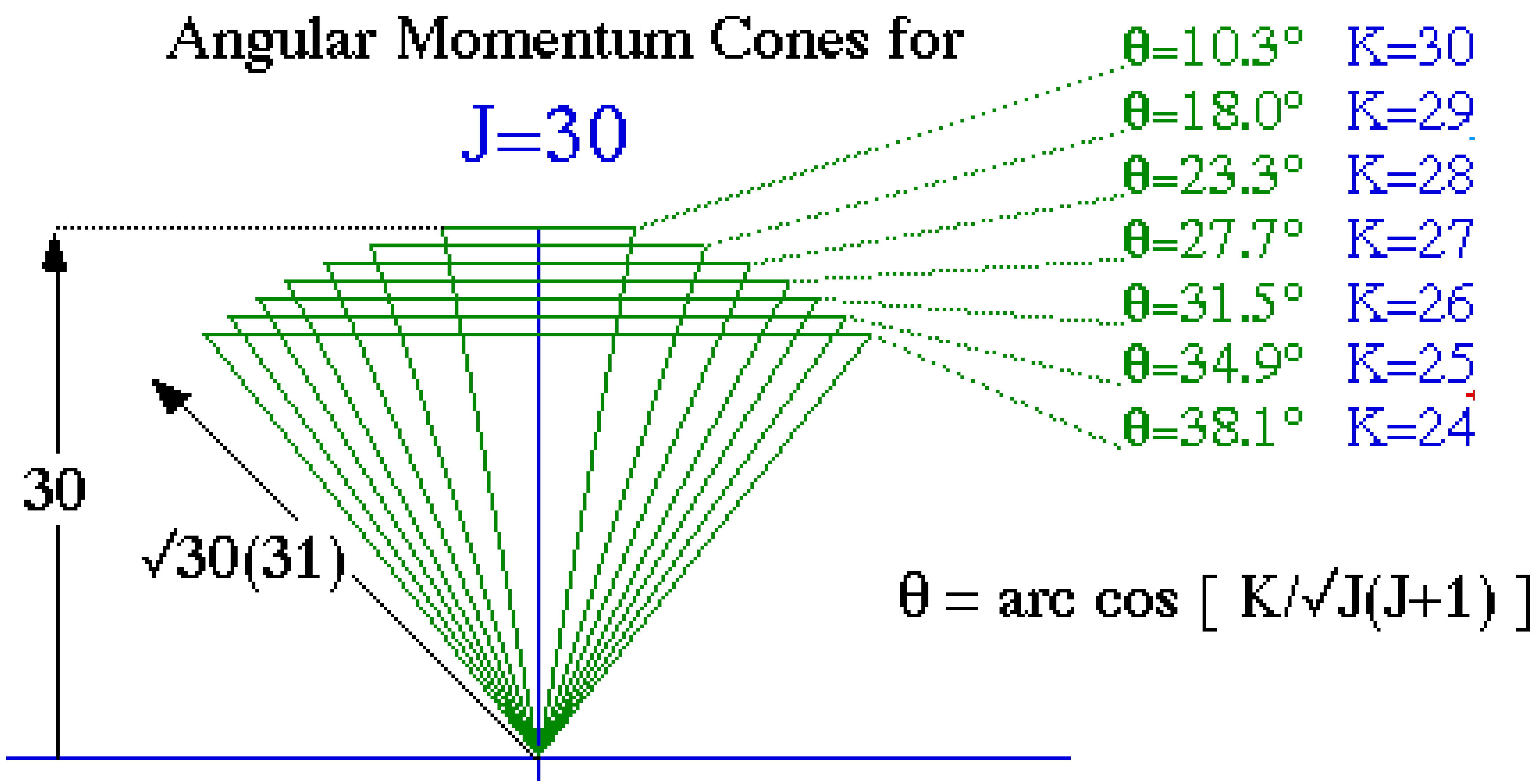

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A fundamental (spin-1/2) Euler transformation $\mathbf{R}(\alpha\beta\gamma)$ given in three notations.

$$\begin{pmatrix} \langle 1|1'\rangle & \langle 1|2'\rangle \\ \langle 2|1'\rangle & \langle 2|2'\rangle \end{pmatrix} = \begin{pmatrix} \langle 1|\mathbf{R}(\alpha\beta\gamma)|1\rangle & \langle 1|\mathbf{R}(\alpha\beta\gamma)|2\rangle \\ \langle 2|\mathbf{R}(\alpha\beta\gamma)|1\rangle & \langle 2|\mathbf{R}(\alpha\beta\gamma)|2\rangle \end{pmatrix} = \begin{pmatrix} D^{1/2}_{11}(\alpha\beta\gamma) & D^{1/2}_{12}(\alpha\beta\gamma) \\ D^{1/2}_{21}(\alpha\beta\gamma) & D^{1/2}_{22}(\alpha\beta\gamma) \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}$$

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Corresponding transformation of the fundamental creation operators

$$\mathbf{a}^\dagger_{1'} = D^{1/2}_{11}(\alpha\beta\gamma)\mathbf{a}^\dagger_1 + D^{1/2}_{21}(\alpha\beta\gamma)\mathbf{a}^\dagger_2 = e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\mathbf{a}^\dagger_1 + e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\mathbf{a}^\dagger_2$$

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Problem: Find corresponding transformation $D^{(j)}(\alpha\beta\gamma)$ matrix for a ($\upsilon=2j$)-oscillator state

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$$\mathbf{R}(\alpha\beta\gamma)\left|{}^{j}_{n}\right\rangle = \frac{\left(\mathbf{a}^\dagger_{1'}\right)^{j+n}\left(\mathbf{a}^\dagger_{2'}\right)^{j-n}}{\sqrt{(j+n)!(j-n)!}}\left|0\,0\right\rangle = \frac{\left(D_{11}\mathbf{a}^\dagger_1 + D_{21}\mathbf{a}^\dagger_2\right)^{j+n}\left(D_{21}\mathbf{a}^\dagger_1 + D_{22}\mathbf{a}^\dagger_2\right)^{j-n}}{\sqrt{(j+n)!(j-n)!}}\left|0\,0\right\rangle$$

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Binomial expansion is a double sum over binomial coefficients: $\binom{n}{k} = n!/k!(n-k)!$

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$$= \sqrt{(j+n)!(j-n)!}\frac{\sum_\ell\sum_k \left(D_{11}\mathbf{a}^\dagger_1\right)^\ell\left(D_{21}\mathbf{a}^\dagger_2\right)^{j+n-\ell}\left(D_{12}\mathbf{a}^\dagger_1\right)^k\left(D_{22}\mathbf{a}^\dagger_2\right)^{j-n-k}}{\ell!(j+n-\ell)!k!(j-n-k)!}|0\,0\rangle$$

Let $\mathbf{a}^\dagger$-operator powers be $j\pm m$ forms : $j+m=\ell+k$, $j-m=2j-\ell-k$ so $\ell=j+m-k$ and $j+n-\ell=n-m+k$

$$= \sqrt{(j+n)!(j-n)!}\frac{\sum_\ell\sum_k \left(D_{11}\right)^\ell\left(D_{21}\right)^{j+n-\ell}\left(D_{12}\right)^k\left(D_{22}\right)^{j-n-k}}{\ell!(j+n-\ell)!k!(j-n-k)!}\left(\mathbf{a}^\dagger_1\right)^{\ell+k}\left(\mathbf{a}^\dagger_2\right)^{2j-\ell-k}|0\,0\rangle = \sqrt{(j+n)!(j-n)!}\frac{\sum_m\sum_k \left(D_{11}\right)^{j+m-k}\left(D_{21}\right)^{n-m+k}\left(D_{12}\right)^k\left(D_{22}\right)^{j-n-k}}{(j+m-k)!(n-m+k)!k!(j-n-k)!}\left(\mathbf{a}^\dagger_1\right)^{j+m}\left(\mathbf{a}^\dagger_2\right)^{j-m}|0\,0\rangle$$

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This gives general *irreducible representation of U(2)* :

$$\left\langle {}^{j}_{m}\right|\mathbf{R}(\alpha\beta\gamma)\left|{}^{j}_{n}\right\rangle = D^{j}_{m,n}(\alpha\beta\gamma) = \sqrt{(j+n)!(j-n)!}\sqrt{(j+m)!(j-m)!}\frac{\sum_k (D_{11})^{j+m-k}(D_{21})^{n-m+k}(D_{12})^{k}(D_{22})^{j-n-k}}{(j+m-k)!(n-m+k)!k!(j-n-k)!}$$

And general *SU(2) irreducible representation for Euler angles (αβγ)*.

$$\left\langle {}^{j}_{m}\right|\mathbf{R}(\alpha\beta\gamma)\left|{}^{j}_{n}\right\rangle = D^{j}_{m,n}(\alpha\beta\gamma) = \sqrt{(j+n)!(j-n)!}\sqrt{(j+m)!(j-m)!}\frac{\sum_k (-1)^k\left(\cos\frac{\beta}{2}\right)^{2j+m-n-2k}\left(\sin\frac{\beta}{2}\right)^{n-m+2k} e^{-i(m\alpha+n\gamma)}}{(j+m-k)!(n-m+k)!k!(j-n-k)!}$$

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Vector (j=1) representation

$$D^1(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha} \end{pmatrix}\begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\gamma} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\gamma} \end{pmatrix}=\begin{pmatrix} e^{-i\alpha}\frac{1+\cos\beta}{2}e^{-i\gamma} & e^{-i\alpha}\frac{-\sin\beta}{\sqrt{2}} & e^{-i\alpha}\frac{1-\cos\beta}{2}e^{i\gamma} \\ \frac{\sin\beta}{\sqrt{2}}e^{-i\gamma} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}}e^{i\gamma} \\ e^{i\alpha}\frac{1-\cos\beta}{2}e^{-i\gamma} & e^{i\alpha}\frac{\sin\beta}{\sqrt{2}} & e^{i\alpha}\frac{1+\cos\beta}{2}e^{i\gamma} \end{pmatrix}$$

Here half-angle identities were used. $\cos^2\frac{\beta}{2}=\frac{1+\cos\beta}{2},\ \sin^2\frac{\beta}{2}=\frac{1-\cos\beta}{2},\ \sin\frac{\beta}{2}\cos\frac{\beta}{2}=\frac{\sin\beta}{2},$

Applications of R(3) rotation and U(2) representations

Vector (j=ℓ=1) representation

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Center ($n=0$) column with the factor $\sqrt{\frac{2\ell+1}{4\pi}}$ gives set of *spherical harmonics* Y^ℓ_m.

$$Y^\ell_m(\phi\theta)=D^{\ell *}_{m,n=0}(\phi\theta 0)\sqrt{\frac{2\ell+1}{4\pi}}$$

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$$D^1(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha}\end{pmatrix}\begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\gamma} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\gamma}\end{pmatrix}=\begin{pmatrix} e^{-i\alpha}\frac{1+\cos\beta}{2}e^{-i\gamma} & e^{-i\alpha}\frac{-\sin\beta}{\sqrt{2}} & e^{-i\alpha}\frac{1-\cos\beta}{2}e^{i\gamma} \\ \frac{\sin\beta}{\sqrt{2}}e^{-i\gamma} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}}e^{i\gamma} \\ e^{i\alpha}\frac{1-\cos\beta}{2}e^{-i\gamma} & e^{i\alpha}\frac{\sin\beta}{\sqrt{2}} & e^{i\alpha}\frac{1+\cos\beta}{2}e^{i\gamma}\end{pmatrix}$$

Here half-angle identities were used. $\cos^2\frac{\beta}{2}=\frac{1+\cos\beta}{2},\ \sin^2\frac{\beta}{2}=\frac{1-\cos\beta}{2},\ \sin\frac{\beta}{2}\cos\frac{\beta}{2}=\frac{\sin\beta}{2},$

$$Y_1^{1*}(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\,e^{-i\phi}\frac{-\sin\theta}{\sqrt{2}}=D^1_{1,0}(\phi,\theta)$$

$$Y_0^{1*}(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\,\cos\beta=D^1_{0,0}(\phi,\theta)$$

$$Y_{-1}^{1*}(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\,e^{+i\phi}\frac{\sin\theta}{\sqrt{2}}=D^1_{-1,0}(\phi,\theta)$$

Center (n=*0*) column with the factor $\sqrt{\frac{2\ell+1}{4\pi}}$ gives set of *spherical harmonics* Y^ℓ_m.

$$Y^\ell_m(\phi\theta)=D^{\ell*}_{m,n=0}(\phi\theta 0)\sqrt{\frac{2\ell+1}{4\pi}}$$

Applications of R(3) rotation and U(2) representations

Vector (j=ℓ=1) representation

$$D^1(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha}\end{pmatrix}\begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\gamma} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\gamma}\end{pmatrix}=\begin{pmatrix} e^{-i\alpha}\frac{1+\cos\beta}{2}e^{-i\gamma} & e^{-i\alpha}\frac{-\sin\beta}{\sqrt{2}} & e^{-i\alpha}\frac{1-\cos\beta}{2}e^{i\gamma} \\ \frac{\sin\beta}{\sqrt{2}}e^{-i\gamma} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}}e^{i\gamma} \\ e^{i\alpha}\frac{1-\cos\beta}{2}e^{-i\gamma} & e^{i\alpha}\frac{\sin\beta}{\sqrt{2}} & e^{i\alpha}\frac{1+\cos\beta}{2}e^{i\gamma}\end{pmatrix}$$

Here half-angle identities were used. $\cos^2\frac{\beta}{2}=\frac{1+\cos\beta}{2},\ \sin^2\frac{\beta}{2}=\frac{1-\cos\beta}{2},\ \sin\frac{\beta}{2}\cos\frac{\beta}{2}=\frac{\sin\beta}{2},$

$$Y_1^{1*}(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}e^{-i\phi}\frac{-\sin\theta}{\sqrt{2}}=D^1_{1,0}(\phi,\theta)$$

$$Y_0^{1*}(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\cos\beta=D^1_{0,0}(\phi,\theta)$$

$$Y_{-1}^{1*}(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}e^{+i\phi}\frac{\sin\theta}{\sqrt{2}}=D^1_{-1,0}(\phi,\theta)$$

Center (n=0) column with the factor $\sqrt{\frac{2\ell+1}{4\pi}}$ gives set of *spherical harmonics* Y^ℓ_m.

$$Y^\ell_m(\phi\theta)=D^{\ell*}_{m,n=0}(\phi\theta 0)\sqrt{\frac{2\ell+1}{4\pi}}$$

Dipole (j=ℓ=1) wavefunctions

$$D^{1*}_{1,0}(\phi\theta 0)=-e^{i\phi}\frac{\sin\theta}{\sqrt{2}}=-\frac{\cos\phi\sin\theta+i\cos\phi\sin\theta}{\sqrt{2}}=-\frac{x+iy}{r\sqrt{2}}$$

$$D^{1*}_{0,0}(\phi\theta 0)=\cos\theta=\cos\theta=z/r$$

$$D^{1*}_{-1,0}(\phi\theta 0)=e^{-i\phi}\frac{\sin\theta}{\sqrt{2}}=\frac{\cos\phi\sin\theta-i\cos\phi\sin\theta}{\sqrt{2}}=\frac{x-iy}{r\sqrt{2}}$$

Applications of R(3) rotation and U(2) representations

Vector (j=ℓ=1) representation

$$D^1(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha}\end{pmatrix}\begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\gamma} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\gamma}\end{pmatrix}=\begin{pmatrix} e^{-i\alpha}\frac{1+\cos\beta}{2}e^{-i\gamma} & e^{-i\alpha}\frac{-\sin\beta}{\sqrt{2}} & e^{-i\alpha}\frac{1-\cos\beta}{2}e^{i\gamma} \\ \frac{\sin\beta}{\sqrt{2}}e^{-i\gamma} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}}e^{i\gamma} \\ e^{i\alpha}\frac{1-\cos\beta}{2}e^{-i\gamma} & e^{i\alpha}\frac{\sin\beta}{\sqrt{2}} & e^{i\alpha}\frac{1+\cos\beta}{2}e^{i\gamma}\end{pmatrix}$$

Here half-angle identities were used. $\cos^2\frac{\beta}{2}=\frac{1+\cos\beta}{2},\ \sin^2\frac{\beta}{2}=\frac{1-\cos\beta}{2},\ \sin\frac{\beta}{2}\cos\frac{\beta}{2}=\frac{\sin\beta}{2},$

$$Y_1^{1*}(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\,e^{-i\phi}\frac{-\sin\theta}{\sqrt{2}}=D^1_{1,0}(\phi,\theta)$$

$$Y_0^{1*}(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\qquad\cos\beta\ =D^1_{0,0}(\phi,\theta)$$

$$Y_{-1}^{1*}(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\,e^{+i\phi}\frac{\sin\theta}{\sqrt{2}}\ =D^1_{-1,0}(\phi,\theta)$$

Center ($n=0$) column with the factor $\sqrt{\frac{2\ell+1}{4\pi}}$ gives set of *spherical harmonics* Y^ℓ_m.

$$Y^\ell_m(\phi\theta)=D^{\ell*}_{m,n=0}(\phi\theta 0)\sqrt{\frac{2\ell+1}{4\pi}}$$

Dipole (j=ℓ=1) wavefunctions

$$D^{1*}_{1,0}(\phi\theta 0)=-e^{i\phi}\frac{\sin\theta}{\sqrt{2}}=-\frac{\cos\phi\sin\theta+i\cos\phi\sin\theta}{\sqrt{2}}=-\frac{x+iy}{r\sqrt{2}}$$

$$D^{1*}_{0,0}(\phi\theta 0)=\cos\theta\ =\ \cos\theta\ =z/r$$

$$D^{1*}_{-1,0}(\phi\theta 0)=e^{-i\phi}\frac{\sin\theta}{\sqrt{2}}=\frac{\cos\phi\sin\theta-i\cos\phi\sin\theta}{\sqrt{2}}=\frac{x-iy}{r\sqrt{2}}$$

3-D linear-circular polarization T-matrix.

$$\begin{pmatrix} \left\langle \begin{smallmatrix}1\\1\end{smallmatrix}\middle|\begin{smallmatrix}1\\x\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\1\end{smallmatrix}\middle|\begin{smallmatrix}1\\y\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\1\end{smallmatrix}\middle|\begin{smallmatrix}1\\z\end{smallmatrix}\right\rangle \\ \left\langle \begin{smallmatrix}1\\0\end{smallmatrix}\middle|\begin{smallmatrix}1\\x\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\0\end{smallmatrix}\middle|\begin{smallmatrix}1\\y\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\0\end{smallmatrix}\middle|\begin{smallmatrix}1\\z\end{smallmatrix}\right\rangle \\ \left\langle \begin{smallmatrix}1\\-1\end{smallmatrix}\middle|\begin{smallmatrix}1\\x\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\-1\end{smallmatrix}\middle|\begin{smallmatrix}1\\y\end{smallmatrix}\right\rangle & \left\langle \begin{smallmatrix}1\\-1\end{smallmatrix}\middle|\begin{smallmatrix}1\\z\end{smallmatrix}\right\rangle \end{pmatrix}=\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \end{pmatrix}$$

Applications of R(3) rotation and U(2) representations

Vector ($j=\ell=1$) representation

$$D^1(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha}\end{pmatrix}\begin{pmatrix}\frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2}\\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}}\\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\gamma} & \cdot & \cdot\\ \cdot & 1 & \cdot\\ \cdot & \cdot & e^{i\gamma}\end{pmatrix}=\begin{pmatrix} e^{-i\alpha}\frac{1+\cos\beta}{2}e^{-i\gamma} & e^{-i\alpha}\frac{-\sin\beta}{\sqrt{2}} & e^{-i\alpha}\frac{1-\cos\beta}{2}e^{i\gamma}\\ \frac{\sin\beta}{\sqrt{2}}e^{-i\gamma} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}}e^{i\gamma}\\ e^{i\alpha}\frac{1-\cos\beta}{2}e^{-i\gamma} & e^{i\alpha}\frac{\sin\beta}{\sqrt{2}} & e^{i\alpha}\frac{1+\cos\beta}{2}e^{i\gamma}\end{pmatrix}$$

Here half-angle identities were used. $\cos^2\frac{\beta}{2}=\frac{1+\cos\beta}{2},\quad \sin^2\frac{\beta}{2}=\frac{1-\cos\beta}{2},\quad \sin\frac{\beta}{2}\cos\frac{\beta}{2}=\frac{\sin\beta}{2},$

$$Y_1^{1*}(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\,e^{-i\phi}\frac{-\sin\theta}{\sqrt{2}}=D^1_{1,0}(\phi,\theta)$$

$$Y_0^{1*}(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\quad\cos\beta\;=D^1_{0,0}(\phi,\theta)$$

$$Y_{-1}^{1*}(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\,e^{+i\phi}\frac{\sin\theta}{\sqrt{2}}\;=D^1_{-1,0}(\phi,\theta)$$

Center ($n=0$) column with the factor $\sqrt{\frac{2\ell+1}{4\pi}}$ gives set of *spherical harmonics* Y^ℓ_m.

$$Y^\ell_m(\phi\theta)=D^{\ell*}_{m,n=0}(\phi\theta 0)\sqrt{\frac{2\ell+1}{4\pi}}$$

Applying T-matrix:

$$\begin{pmatrix}\frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}}\\ 0 & 1 & 0\end{pmatrix}\cdot\begin{pmatrix}\frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2}\\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}}\\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2}\end{pmatrix}\cdot\begin{pmatrix}\frac{-1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0\\ 0 & 0 & 1\\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0\end{pmatrix}=\begin{pmatrix}\cos\beta & 0 & \sin\beta\\ 0 & 1 & 0\\ -\sin\beta & 0 & \sin\beta\end{pmatrix}$$

Dipole ($j=\ell=1$) wavefunctions

$$D^{1*}_{1,0}(\phi\theta 0)=-e^{i\phi}\frac{\sin\theta}{\sqrt{2}}=-\frac{\cos\phi\sin\theta+i\cos\phi\sin\theta}{\sqrt{2}}=-\frac{x+iy}{r\sqrt{2}}$$

$$D^{1*}_{0,0}(\phi\theta 0)=\cos\theta\quad=\quad\cos\theta\quad=z/r$$

$$D^{1*}_{-1,0}(\phi\theta 0)=e^{-i\phi}\frac{\sin\theta}{\sqrt{2}}=\frac{\cos\phi\sin\theta-i\cos\phi\sin\theta}{\sqrt{2}}=\frac{x-iy}{r\sqrt{2}}$$

3-D linear-circular polarization T-matrix:

$$\begin{pmatrix}\left\langle\begin{matrix}1\\1\end{matrix}\middle|\begin{matrix}1\\x\end{matrix}\right\rangle & \left\langle\begin{matrix}1\\1\end{matrix}\middle|\begin{matrix}1\\y\end{matrix}\right\rangle & \left\langle\begin{matrix}1\\1\end{matrix}\middle|\begin{matrix}1\\z\end{matrix}\right\rangle\\ \left\langle\begin{matrix}1\\0\end{matrix}\middle|\begin{matrix}1\\x\end{matrix}\right\rangle & \left\langle\begin{matrix}1\\0\end{matrix}\middle|\begin{matrix}1\\y\end{matrix}\right\rangle & \left\langle\begin{matrix}1\\0\end{matrix}\middle|\begin{matrix}1\\z\end{matrix}\right\rangle\\ \left\langle\begin{matrix}1\\-1\end{matrix}\middle|\begin{matrix}1\\x\end{matrix}\right\rangle & \left\langle\begin{matrix}1\\-1\end{matrix}\middle|\begin{matrix}1\\y\end{matrix}\right\rangle & \left\langle\begin{matrix}1\\-1\end{matrix}\middle|\begin{matrix}1\\z\end{matrix}\right\rangle\end{pmatrix}=\begin{pmatrix}\frac{-1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0\\ 0 & 0 & 1\\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0\end{pmatrix}$$

Applications of R(3) rotation and U(2) representations

Vector (j=ℓ=1) representation

$$D^1(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha}\end{pmatrix}\begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\gamma} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\gamma}\end{pmatrix}=\begin{pmatrix} e^{-i\alpha}\frac{1+\cos\beta}{2}e^{-i\gamma} & e^{-i\alpha}\frac{-\sin\beta}{\sqrt{2}} & e^{-i\alpha}\frac{1-\cos\beta}{2}e^{i\gamma} \\ \frac{\sin\beta}{\sqrt{2}}e^{-i\gamma} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}}e^{i\gamma} \\ e^{i\alpha}\frac{1-\cos\beta}{2}e^{-i\gamma} & e^{i\alpha}\frac{\sin\beta}{\sqrt{2}} & e^{i\alpha}\frac{1+\cos\beta}{2}e^{i\gamma}\end{pmatrix}$$

Here half-angle identities were used. $\cos^2\frac{\beta}{2}=\frac{1+\cos\beta}{2},\ \sin^2\frac{\beta}{2}=\frac{1-\cos\beta}{2},\ \sin\frac{\beta}{2}\cos\frac{\beta}{2}=\frac{\sin\beta}{2},$

$$Y^{1*}_1(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\,e^{-i\phi}\frac{-\sin\theta}{\sqrt{2}}=D^1_{1,0}(\phi,\theta)$$

$$Y^{1*}_0(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\,\cos\beta=D^1_{0,0}(\phi,\theta)$$

$$Y^{1*}_{-1}(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\,e^{+i\phi}\frac{\sin\theta}{\sqrt{2}}=D^1_{-1,0}(\phi,\theta)$$

Center (n=0) column with the factor $\sqrt{\frac{2\ell+1}{4\pi}}$ gives set of *spherical harmonics* Y^ℓ_m.

$$Y^\ell_m(\phi\theta)=D^{\ell*}_{m,n=0}(\phi\theta 0)\sqrt{\frac{2\ell+1}{4\pi}}$$

Dipole (j=ℓ=1) wavefunctions

$$D^{1*}_{1,0}(\phi\theta 0)=-e^{i\phi}\frac{\sin\theta}{\sqrt{2}}=-\frac{\cos\phi\sin\theta+i\cos\phi\sin\theta}{\sqrt{2}}=-\frac{x+iy}{r\sqrt{2}}$$

$$D^{1*}_{0,0}(\phi\theta 0)=\cos\theta\quad=\quad\cos\theta\quad=z/r$$

$$D^{1*}_{-1,0}(\phi\theta 0)=e^{-i\phi}\frac{\sin\theta}{\sqrt{2}}=\frac{\cos\phi\sin\theta-i\cos\phi\sin\theta}{\sqrt{2}}=\frac{x-iy}{r\sqrt{2}}$$

3-D linear-circular polarization T-matrix:

$$\begin{pmatrix} \left\langle\begin{smallmatrix}1\\1\end{smallmatrix}\middle|\begin{smallmatrix}1\\x\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}1\\1\end{smallmatrix}\middle|\begin{smallmatrix}1\\y\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}1\\1\end{smallmatrix}\middle|\begin{smallmatrix}1\\z\end{smallmatrix}\right\rangle \\ \left\langle\begin{smallmatrix}1\\0\end{smallmatrix}\middle|\begin{smallmatrix}1\\x\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}1\\0\end{smallmatrix}\middle|\begin{smallmatrix}1\\y\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}1\\0\end{smallmatrix}\middle|\begin{smallmatrix}1\\z\end{smallmatrix}\right\rangle \\ \left\langle\begin{smallmatrix}1\\-1\end{smallmatrix}\middle|\begin{smallmatrix}1\\x\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}1\\-1\end{smallmatrix}\middle|\begin{smallmatrix}1\\y\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}1\\-1\end{smallmatrix}\middle|\begin{smallmatrix}1\\z\end{smallmatrix}\right\rangle\end{pmatrix}=\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0\end{pmatrix}$$

Applying T-matrix:

$$\begin{pmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & 1 & 0\end{pmatrix}\cdot\begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2}\end{pmatrix}\cdot\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0\end{pmatrix}=\begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \sin\beta\end{pmatrix}$$

$$\begin{pmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & 1 & 0\end{pmatrix}\cdot\begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha}\end{pmatrix}\cdot\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0\end{pmatrix}=\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1\end{pmatrix}$$

Applications of R(3) rotation and U(2) representations

Vector (j=ℓ=1) representation

$$D^1(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha}\end{pmatrix}\begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\gamma} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\gamma}\end{pmatrix}=\begin{pmatrix} e^{-i\alpha}\frac{1+\cos\beta}{2}e^{-i\gamma} & e^{-i\alpha}\frac{-\sin\beta}{\sqrt{2}} & e^{-i\alpha}\frac{1-\cos\beta}{2}e^{i\gamma} \\ \frac{\sin\beta}{\sqrt{2}}e^{-i\gamma} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}}e^{i\gamma} \\ e^{i\alpha}\frac{1-\cos\beta}{2}e^{-i\gamma} & e^{i\alpha}\frac{\sin\beta}{\sqrt{2}} & e^{i\alpha}\frac{1+\cos\beta}{2}e^{i\gamma}\end{pmatrix}$$

Here half-angle identities were used. $\cos^2\frac{\beta}{2}=\frac{1+\cos\beta}{2},\ \sin^2\frac{\beta}{2}=\frac{1-\cos\beta}{2},\ \sin\frac{\beta}{2}\cos\frac{\beta}{2}=\frac{\sin\beta}{2},$

$$Y_1^{1*}(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\,e^{-i\phi}\frac{-\sin\theta}{\sqrt{2}}=D^1_{1,0}(\phi,\theta)$$

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$$Y_{-1}^{1*}(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\,e^{+i\phi}\frac{\sin\theta}{\sqrt{2}}=D^1_{-1,0}(\phi,\theta)$$

Center ($n=0$) column with the factor $\sqrt{\frac{2\ell+1}{4\pi}}$ gives set of *spherical harmonics* Y^{ℓ}_{m}.

$$Y^{\ell}_{m}(\phi\theta)=D^{\ell *}_{m,n=0}(\phi\theta 0)\sqrt{\frac{2\ell+1}{4\pi}}$$

Dipole (j=ℓ=1) wavefunctions

$$D^{1*}_{1,0}(\phi\theta 0)=-e^{i\phi}\frac{\sin\theta}{\sqrt{2}}=-\frac{\cos\phi\sin\theta+i\cos\phi\sin\theta}{\sqrt{2}}=-\frac{x+iy}{r\sqrt{2}}$$

$$D^{1*}_{0,0}(\phi\theta 0)=\cos\theta=\cos\theta=z/r$$

$$D^{1*}_{-1,0}(\phi\theta 0)=e^{-i\phi}\frac{\sin\theta}{\sqrt{2}}=\frac{\cos\phi\sin\theta-i\cos\phi\sin\theta}{\sqrt{2}}=\frac{x-iy}{r\sqrt{2}}$$

3-D linear-circular polarization T-matrix:

$$\begin{pmatrix} \left\langle {1 \atop 1}\middle| {1 \atop x}\right\rangle & \left\langle {1 \atop 1}\middle| {1 \atop y}\right\rangle & \left\langle {1 \atop 1}\middle| {1 \atop z}\right\rangle \\ \left\langle {1 \atop 0}\middle| {1 \atop x}\right\rangle & \left\langle {1 \atop 0}\middle| {1 \atop y}\right\rangle & \left\langle {1 \atop 0}\middle| {1 \atop z}\right\rangle \\ \left\langle {1 \atop -1}\middle| {1 \atop x}\right\rangle & \left\langle {1 \atop -1}\middle| {1 \atop y}\right\rangle & \left\langle {1 \atop -1}\middle| {1 \atop z}\right\rangle \end{pmatrix}=\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0\end{pmatrix}$$

Applying T-matrix:

$$\begin{pmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & 1 & 0\end{pmatrix}\cdot\begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2}\end{pmatrix}\cdot\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0\end{pmatrix}=\begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \sin\beta\end{pmatrix}$$

$$\begin{pmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & 1 & 0\end{pmatrix}\cdot\begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha}\end{pmatrix}\cdot\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0\end{pmatrix}=\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1\end{pmatrix}$$

$$\begin{pmatrix} D^1_{x,x}(\alpha\beta\gamma) & D^1_{x,y} & D^1_{x,z} \\ D^1_{y,x} & D^1_{y,y} & D^1_{y,z} \\ D^1_{z,x} & D^1_{z,y} & D^1_{z,z}\end{pmatrix}=$$

$$\begin{pmatrix} \cos\alpha\cos\beta\cos\gamma-\sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma-\sin\alpha\cos\gamma & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta\cos\gamma+\cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma+\cos\alpha\cos\gamma & \sin\alpha\sin\beta \\ -\cos\gamma\sin\beta & \sin\gamma\sin\beta & \cos\beta\end{pmatrix}$$

Applications of R(3) rotation and U(2) representations

Vector (j=ℓ=1) representation

$$D^1(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\alpha} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{i\alpha}\end{pmatrix}\begin{pmatrix}\frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2}\\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}}\\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2}\end{pmatrix}\begin{pmatrix}e^{-i\gamma}&\cdot&\cdot\\ \cdot&1&\cdot\\ \cdot&\cdot&e^{i\gamma}\end{pmatrix}=\begin{pmatrix} e^{-i\alpha}\frac{1+\cos\beta}{2}e^{-i\gamma} & e^{-i\alpha}\frac{-\sin\beta}{\sqrt{2}} & e^{-i\alpha}\frac{1-\cos\beta}{2}e^{i\gamma}\\ \frac{\sin\beta}{\sqrt{2}}e^{-i\gamma} & \cos\beta & \frac{-\sin\beta}{\sqrt{2}}e^{i\gamma}\\ e^{i\alpha}\frac{1-\cos\beta}{2}e^{-i\gamma} & e^{i\alpha}\frac{\sin\beta}{\sqrt{2}} & e^{i\alpha}\frac{1+\cos\beta}{2}e^{i\gamma}\end{pmatrix}$$

Here half-angle identities were used. $\cos^2\frac{\beta}{2}=\frac{1+\cos\beta}{2},\ \sin^2\frac{\beta}{2}=\frac{1-\cos\beta}{2},\ \sin\frac{\beta}{2}\cos\frac{\beta}{2}=\frac{\sin\beta}{2},$

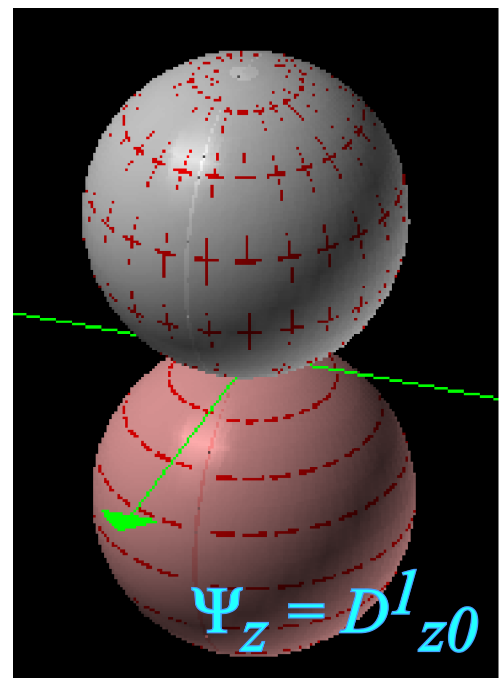
$$Y_1^1(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\,e^{-i\phi}\frac{-\sin\theta}{\sqrt{2}}$$
$$Y_0^1(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\,\cos\beta$$
$$Y_{-1}^1(\phi,\theta)=\sqrt{\tfrac{3}{4\pi}}\,e^{+i\phi}\frac{\sin\theta}{\sqrt{2}}$$

Center ($n=0$) column with the factor $\sqrt{\frac{2\ell+1}{4\pi}}$ gives set of *spherical harmonics* Y^ℓ_m.

$$Y^\ell_m(\phi\theta)=D^{\ell *}_{m,n=0}(\phi\theta 0)\sqrt{\frac{2\ell+1}{4\pi}}$$

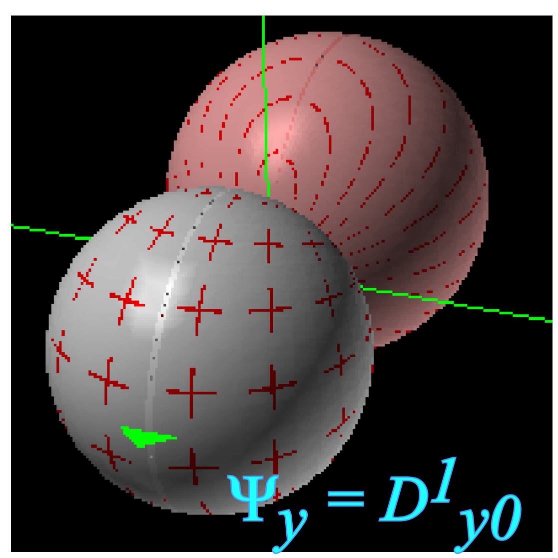
Dipole (j=ℓ=1) wavefunctions

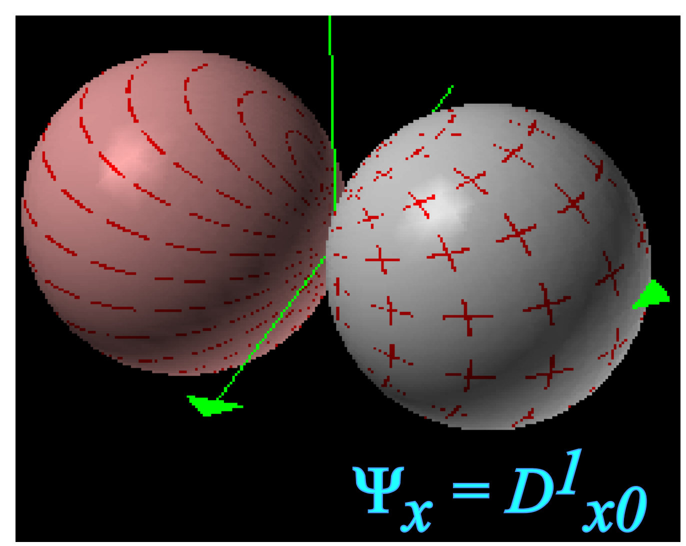
$$D^{1*}_{1,0}(\phi\theta 0)=-e^{i\phi}\frac{\sin\theta}{\sqrt{2}}=-\frac{\cos\phi\sin\theta+i\cos\phi\sin\theta}{\sqrt{2}}=-\frac{x+iy}{r\sqrt{2}}$$
$$D^{1*}_{0,0}(\phi\theta 0)=\cos\theta\quad=\quad\cos\theta\quad=z/r$$
$$D^{1*}_{-1,0}(\phi\theta 0)=e^{-i\phi}\frac{\sin\theta}{\sqrt{2}}=\frac{\cos\phi\sin\theta-i\cos\phi\sin\theta}{\sqrt{2}}=\frac{x-iy}{r\sqrt{2}}$$

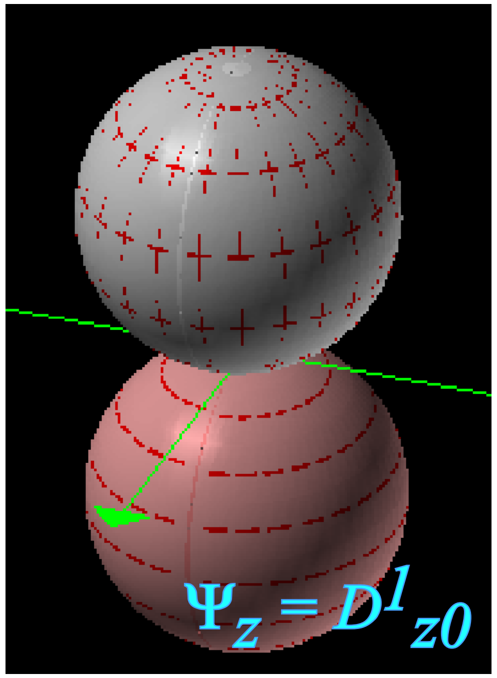


j = 1
Standing p-Waves

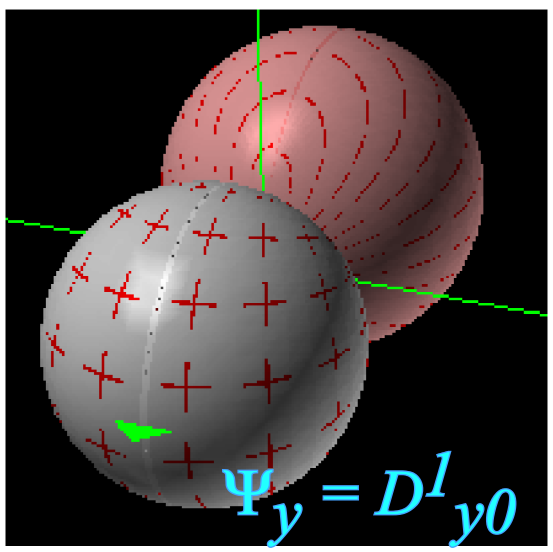
$$\Psi^1_x(\phi,\theta)=D^1_{x,z}(\phi,\theta,0)=\cos\phi\sin\theta$$
$$\Psi^1_y(\phi,\theta)=D^1_{y,z}(\phi,\theta,0)=\sin\phi\sin\theta$$
$$\Psi^1_z(\phi,\theta)=D^1_{z,z}(\phi,\theta,0)=\cos\theta$$

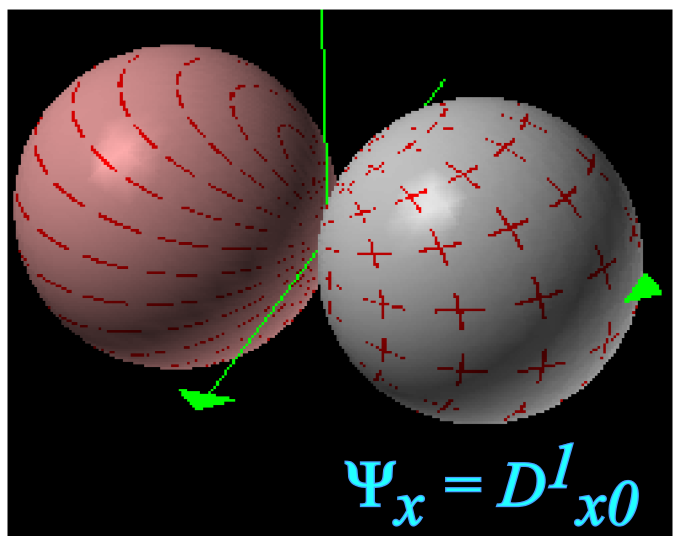




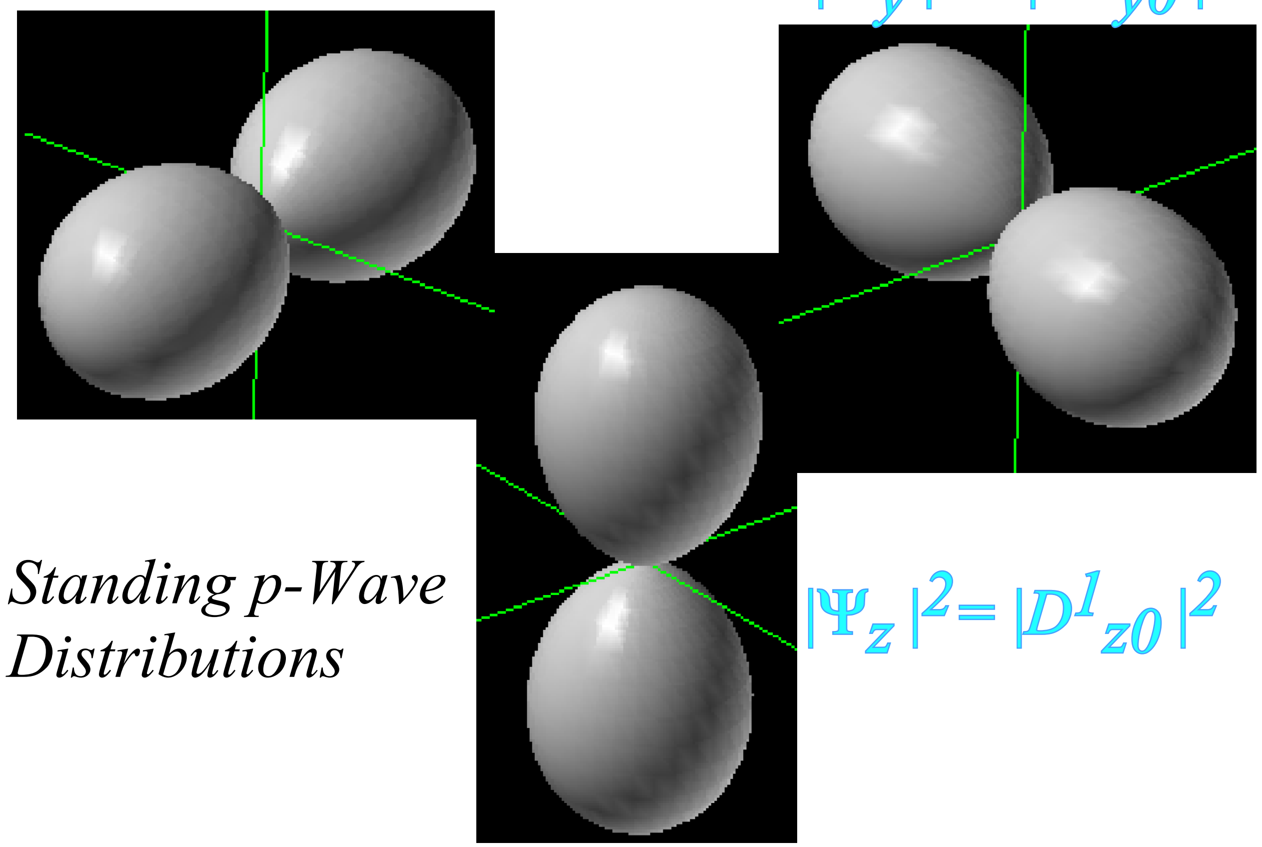


j = 1
Standing
p-Waves





$|\Psi_x|^2 = |D^1{}_{x0}|^2$ $|\Psi_y|^2 = |D^1{}_{y0}|^2$

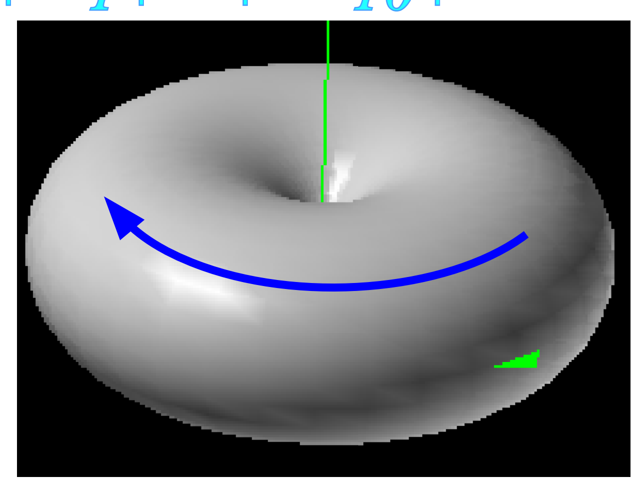


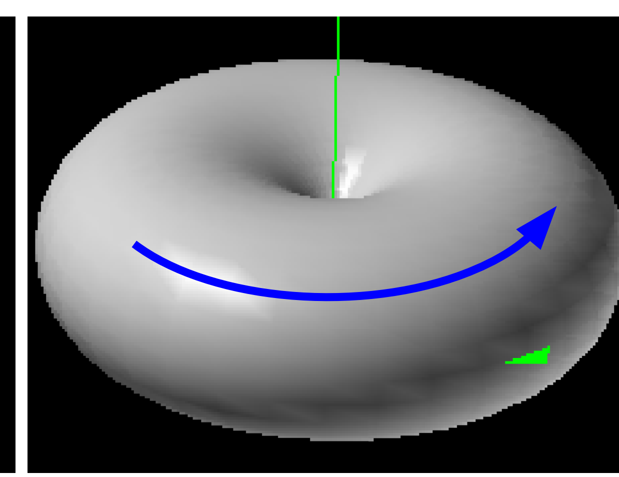
Standing p-Wave Distributions

$|\Psi_z|^2 = |D^1{}_{z0}|^2$

Moving p-Wave Distributions

$|\Psi_{-1}|^2 = |D^1{}_{-10}|^2$ $|\Psi_1|^2 = |D^1{}_{10}|^2$





$$\Psi^1_x(\phi,\theta) = D^1_{x,z}(\phi,\theta,0) = \cos\phi\sin\theta$$

$$\Psi^1_y(\phi,\theta) = D^1_{y,z}(\phi,\theta,0) = \sin\phi\sin\theta$$

$$\Psi^1_z(\phi,\theta) = D^1_{z,z}(\phi,\theta,0) = \cos\theta$$

Applications of R(3) rotation and U(2) representations

Tensor (j=ℓ=2) representation

$$D^2(\alpha\beta 0)=\left(\begin{array}{cc|c|cc}
e^{-i2\alpha}\left(\frac{1+\cos\beta}{2}\right)^2 & e^{-i2\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & \sqrt{\frac{3}{8}}e^{-i2\alpha}\sin^2\beta & e^{-i2\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{-i2\alpha}\left(\frac{1-\cos\beta}{2}\right)^2 \\
e^{-i\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{-i\alpha}\left(\frac{1+\cos\beta}{2}\right)(2\cos\beta-1) & -\sqrt{\frac{3}{2}}e^{-i\alpha}\sin\beta\cos\beta & e^{-i\alpha}\left(\frac{1-\cos\beta}{2}\right)(2\cos\beta+1) & -e^{-i\alpha}\left(\frac{1-\cos\beta}{2}\right)\sin\beta \\
\sqrt{\frac{3}{8}}\sin^2\beta & \sqrt{\frac{3}{2}}\sin\beta\cos\beta & \frac{3\cos^2\beta-1}{2} & \sqrt{\frac{3}{2}}\sin\beta\cos\beta & \sqrt{\frac{3}{8}}\sin^2\beta \\
e^{i\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{i\alpha}\left(\frac{1-\cos\beta}{2}\right)(2\cos\beta+1) & \sqrt{\frac{3}{2}}e^{i\alpha}\sin\beta\cos\beta & e^{i\alpha}\left(\frac{1+\cos\beta}{2}\right)(2\cos\beta-1) & -e^{i\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta \\
e^{i2\alpha}\left(\frac{1-\cos\beta}{2}\right)^2 & e^{i2\alpha}\left(\frac{1-\cos\beta}{2}\right)\sin\beta & \sqrt{\frac{3}{8}}e^{i2\alpha}\sin^2\beta & e^{i2\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{i2\alpha}\left(\frac{1+\cos\beta}{2}\right)^2
\end{array}\right)$$

Applications of R(3) rotation and U(2) representations

Tensor (j=ℓ=2) representation

$$D^2(\alpha\beta 0)=\left(\begin{array}{cc|c|cc}
e^{-i2\alpha}\left(\frac{1+\cos\beta}{2}\right)^2 & e^{-i2\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & \sqrt{\frac{3}{8}}e^{-i2\alpha}\sin^2\beta & e^{-i2\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{-i2\alpha}\left(\frac{1-\cos\beta}{2}\right)^2 \\
e^{-i\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{-i\alpha}\left(\frac{1+\cos\beta}{2}\right)(2\cos\beta-1) & -\sqrt{\frac{3}{2}}e^{-i\alpha}\sin\beta\cos\beta & e^{-i\alpha}\left(\frac{1-\cos\beta}{2}\right)(2\cos\beta+1) & -e^{-i\alpha}\left(\frac{1-\cos\beta}{2}\right)\sin\beta \\
\sqrt{\frac{3}{8}}\sin^2\beta & \sqrt{\frac{3}{2}}\sin\beta\cos\beta & \frac{3\cos^2\beta-1}{2} & \sqrt{\frac{3}{2}}\sin\beta\cos\beta & \sqrt{\frac{3}{8}}\sin^2\beta \\
e^{i\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{i\alpha}\left(\frac{1-\cos\beta}{2}\right)(2\cos\beta+1) & \sqrt{\frac{3}{2}}e^{i\alpha}\sin\beta\cos\beta & e^{i\alpha}\left(\frac{1+\cos\beta}{2}\right)(2\cos\beta-1) & -e^{i\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta \\
e^{i2\alpha}\left(\frac{1-\cos\beta}{2}\right)^2 & e^{i2\alpha}\left(\frac{1-\cos\beta}{2}\right)\sin\beta & \sqrt{\frac{3}{8}}e^{i2\alpha}\sin^2\beta & e^{i2\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{i2\alpha}\left(\frac{1+\cos\beta}{2}\right)^2
\end{array}\right)$$

Spherical *2^k-multipole functions X^k_q* or *X-functions* are *D**-functions times the k^{th} power of radius *(r^k)*.

$$\sqrt{4\pi/5}\,Y^{\ell=2}_{m=2}(\phi\theta)=D^{2*}_{2,0}(\phi\theta 0)=\sqrt{\frac{3}{8}}e^{i2\phi}\sin^2\theta=\sqrt{\frac{3}{8}}\frac{(x+iy)^2}{r^2}$$

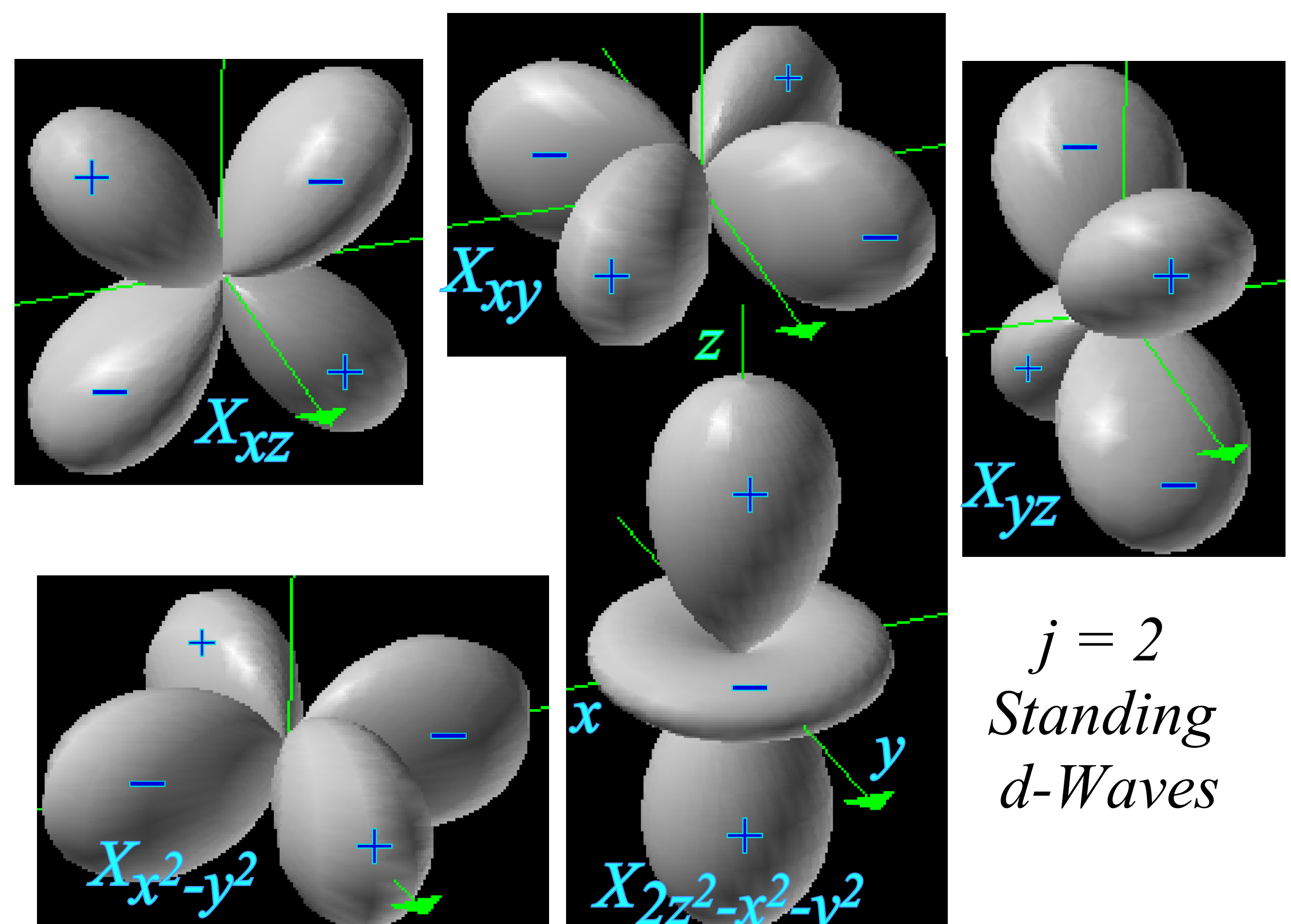
$$\sqrt{4\pi/5}\,Y^{\ell=2}_{m=1}(\phi\theta)=D^{2*}_{1,0}(\phi\theta 0)=-\sqrt{\frac{3}{2}}e^{i\phi}\sin\theta\cos\theta=-\sqrt{\frac{3}{2}}\frac{(x+iy)z}{r^2}$$

$$\sqrt{4\pi/5}\,Y^{\ell=2}_{m=0}(\phi\theta)=D^{2*}_{0,0}(\phi\theta 0)=\frac{3\cos^2\theta-1}{2}=\frac{3z^2-r^2}{2r^2}$$

$$\sqrt{4\pi/5}\,Y^{\ell=2}_{m=-1}(\phi\theta)=D^{2*}_{-1,0}(\phi\theta 0)=\sqrt{\frac{3}{2}}e^{-i\phi}\sin\theta\cos\theta=\sqrt{\frac{3}{2}}\frac{(x-iy)z}{r^2}$$

$$\sqrt{4\pi/5}\,Y^{\ell=2}_{m=-2}(\phi\theta)=D^{2*}_{-2,0}(\phi\theta 0)=\sqrt{\frac{3}{8}}e^{-i2\phi}\sin^2\theta=\sqrt{\frac{3}{8}}\frac{(x-iy)^2}{r^2}$$

$$X^k_q=r^kD^{k*}_{q,0}=\sqrt{\frac{4\pi}{2k+1}}r^kY^k_q$$



j = 2
Standing
d-Waves

Tensor (j=ℓ=2) representation

Spherical 2^k-*multipole functions* X^k_q or *X-functions* are D^*-functions times the k^{th} power of radius (r^k).

$$\sqrt{4\pi/5}\,Y^{\ell=2}_{m=2}(\phi\theta) = D^{2*}_{2,0}(\phi\theta 0) = \sqrt{\frac{3}{8}}e^{i2\phi}\sin^2\theta = \sqrt{\frac{3}{8}}\frac{(x+iy)^2}{r^2}$$

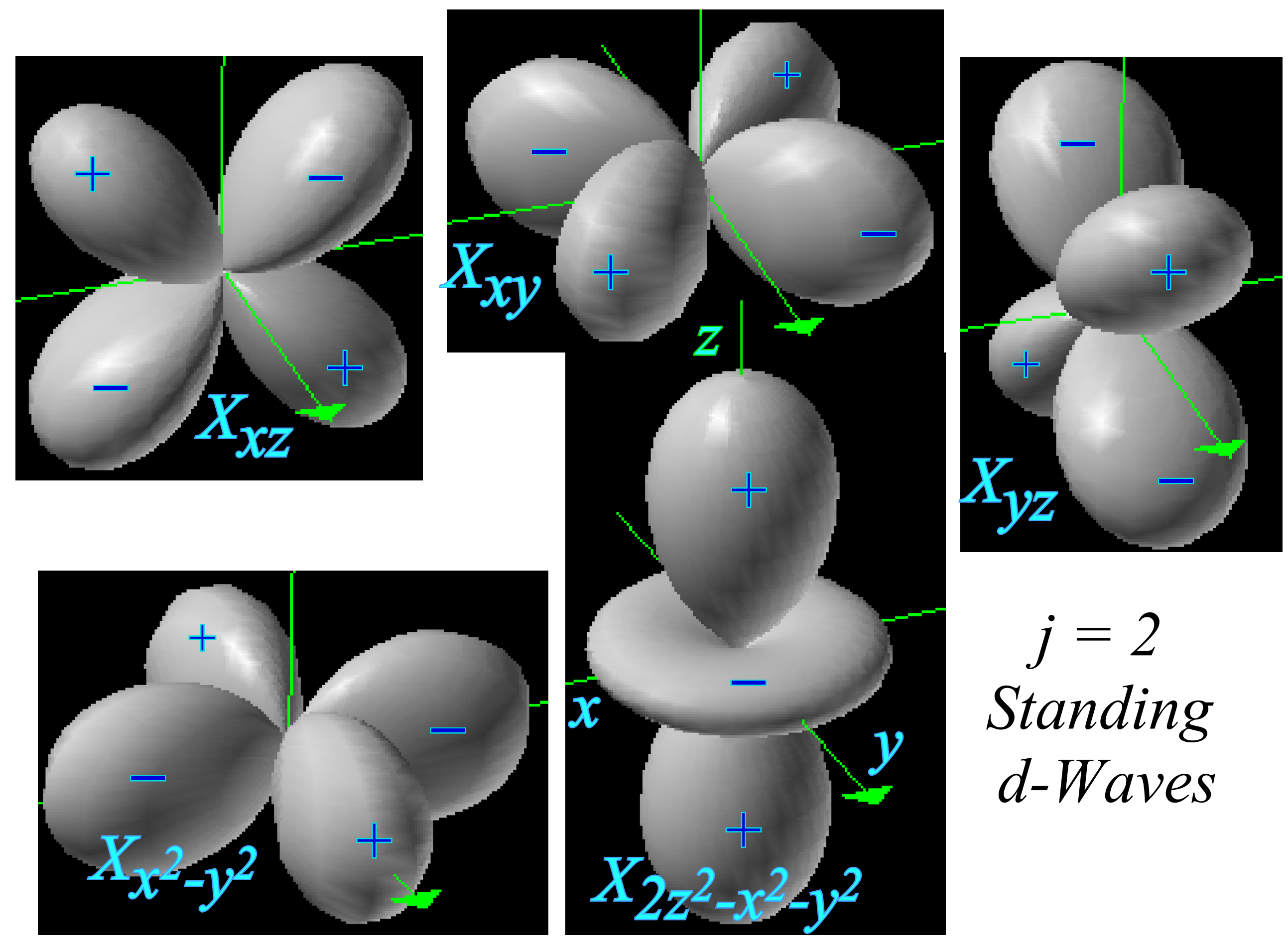
$$\sqrt{4\pi/5}\,Y^{\ell=2}_{m=1}(\phi\theta) = D^{2*}_{1,0}(\phi\theta 0) = -\sqrt{\frac{3}{2}}e^{i\phi}\sin\theta\cos\theta = -\sqrt{\frac{3}{2}}\frac{(x+iy)z}{r^2}$$

$$\sqrt{4\pi/5}\,Y^{\ell=2}_{m=0}(\phi\theta) = D^{2*}_{0,0}(\phi\theta 0) = \frac{3\cos^2\theta-1}{2} = \frac{3z^2-r^2}{2r^2}$$

$$\sqrt{4\pi/5}\,Y^{\ell=2}_{m=-1}(\phi\theta) = D^{2*}_{-1,0}(\phi\theta 0) = \sqrt{\frac{3}{2}}e^{-i\phi}\sin\theta\cos\theta = \sqrt{\frac{3}{2}}\frac{(x-iy)z}{r^2}$$

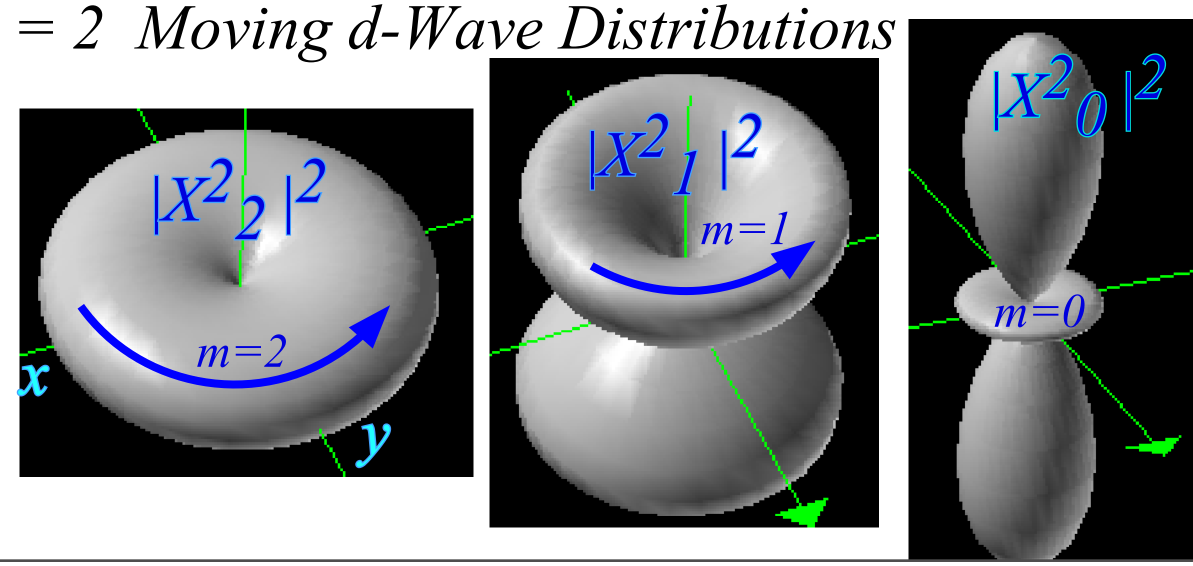
$$\sqrt{4\pi/5}\,Y^{\ell=2}_{m=-2}(\phi\theta) = D^{2*}_{-2,0}(\phi\theta 0) = \sqrt{\frac{3}{8}}e^{-i2\phi}\sin^2\theta = \sqrt{\frac{3}{8}}\frac{(x-iy)^2}{r^2}$$

$$X^k_q = r^k D^{k*}_{q,0} = \sqrt{\frac{4\pi}{2k+1}}\,r^k Y^k_q$$



j = 2
Standing
d-Waves

j = 2 Moving d-Wave Distributions



Review : *2-D* **a†a** *algebra of U(2) representations*

Angular momentum generators by U(2) analysis
Angular momentum raise-n-lower operators $\mathbf{s}_+$ and $\mathbf{s}_-$
SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors

Angular momentum commutation relations
Key Lie theorems

Angular momentum magnitude and uncertainty
Angular momentum uncertainty angle

Generating R(3) rotation and U(2) representations
Applications of R(3) rotation and U(2) representations
→ *Molecular and nuclear wavefunctions* ←
Molecular and nuclear eigenlevels
Generalized Stern-Gerlach and transformation matrices
Angular momentum cones and high J properties

For *SU(2)* and *R(3)*, sum over rotations is an integral over Euler angles *(αβγ)*.

For integral-*j=0, 1, 2,..* the *R(3)* integral over polar angle β ranges from 0 to π.

$$\text{for } R(3): \frac{\ell^j}{N}\int d(\alpha\beta\gamma) = \frac{2j+1}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma \;= 2j+1$$

Applications of R(3) rotation and U(2) representations

Molecular and nuclear wavefunctions

For *SU(2)* and *R(3)*, sum over rotations is an integral over Euler angles *(αβγ)*.

For integral-*j=0, 1, 2,..* the *R(3)* integral over polar angle β ranges from 0 to π.

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For ½-integral-*j=1/2, 3/2,..* the *U(2)* integral over polar angle β ranges from -π to π.

$$for\ SU(2): \frac{\ell^j}{N}\int d(\alpha\beta\gamma) = \frac{2j+1}{16\pi^2}\int_0^{2\pi} d\alpha \int_{-\pi}^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma\ = 2j+1 = \ell^j$$

Molecular and nuclear wavefunctions

For *SU(2)* and *R(3)*, sum over rotations is an integral over Euler angles *(αβγ)*.

For integral-*j=0, 1, 2,..* the *R(3)* integral over polar angle β ranges from 0 to π.

$$\text{for } R(3): \frac{\ell^j}{N}\int d(\alpha\beta\gamma) = \frac{2j+1}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma = 2j+1 = \ell^j$$

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Eigenstates of angular momentum are built from projected initial position states |000⟩.

$$\left|{}^{j}_{m,n}\right\rangle = \frac{\mathbf{P}^j_{m,n}|000\rangle}{\sqrt{\ell^j}} = \frac{1}{N}\int d(\alpha\beta\gamma)\, D^{j*}_{m,n}(\alpha\beta\gamma)\mathbf{R}(\alpha\beta\gamma)|000\rangle\sqrt{\ell^j} = \frac{1}{N}\int d(\alpha\beta\gamma)\, D^{j*}_{m,n}(\alpha\beta\gamma)\sqrt{\ell^j}\,|\alpha\beta\gamma\rangle$$

Applications of R(3) rotation and U(2) representations

Molecular and nuclear wavefunctions

For *SU(2)* and *R(3)*, sum over rotations is an integral over Euler angles *(αβγ)*.

For integral-*j=0, 1, 2,..* the *R(3)* integral over polar angle β ranges from 0 to π.

$$\text{for } R(3): \frac{\ell^j}{N}\int d(\alpha\beta\gamma) = \frac{2j+1}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma \ = 2j+1 = \ell^j$$

For ½-integral-*j=1/2, 3/2,..* the *U(2)* integral over polar angle β ranges from -π to π.

$$\text{for } SU(2): \frac{\ell^j}{N}\int d(\alpha\beta\gamma) = \frac{2j+1}{16\pi^2}\int_0^{2\pi} d\alpha \int_{-\pi}^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma \ = 2j+1 = \ell^j$$

Eigenstates of angular momentum are built from projected initial position states |000⟩.

$$\left|{}^{j}_{m,n}\right\rangle = \frac{\mathbf{P}^j_{m,n}|000\rangle}{\sqrt{\ell^j}} = \frac{1}{N}\int d(\alpha\beta\gamma)\, D^{j*}_{m,n}(\alpha\beta\gamma)\mathbf{R}(\alpha\beta\gamma)|000\rangle\sqrt{\ell^j} = \frac{1}{N}\int d(\alpha\beta\gamma)\, D^{j*}_{m,n}(\alpha\beta\gamma)\sqrt{\ell^j}\,|\alpha\beta\gamma\rangle$$

Angular position is defined by a *rotational duality relativity relations* or "Mock-Mach" principle

$$\mathbf{R}(\alpha\beta\gamma)|000\rangle = |\alpha\beta\gamma\rangle = \bar{\mathbf{R}}^\dagger(\alpha\beta\gamma)|000\rangle$$

Applications of R(3) rotation and U(2) representations

Molecular and nuclear wavefunctions

For *SU(2)* and *R(3)*, sum over rotations is an integral over Euler angles *(αβγ)*.

For integral-$j=0, 1, 2,..$ the *R(3)* integral over polar angle β ranges from 0 to π.

$$\text{for } R(3): \frac{\ell^j}{N}\int d(\alpha\beta\gamma) = \frac{2j+1}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma \;= 2j+1 = \ell^j$$

For ½-integral-$j=1/2, 3/2,..$ the *U(2)* integral over polar angle β ranges from -π to π.

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$$\mathbf{R}(\alpha\beta\gamma)|000\rangle = |\alpha\beta\gamma\rangle = \bar{\mathbf{R}}^{\dagger}(\alpha\beta\gamma)|000\rangle$$

$$\mathbf{R}(\alpha\beta\gamma)\bar{\mathbf{R}}(\alpha'\beta'\gamma') = \bar{\mathbf{R}}(\alpha'\beta'\gamma')\mathbf{R}(\alpha\beta\gamma)$$

$$\text{for all}(\alpha\beta\gamma)\text{and}(\alpha'\beta'\gamma')$$

Applications of R(3) rotation and U(2) representations

Molecular and nuclear wavefunctions

For *SU(2)* and *R(3)*, sum over rotations is an integral over Euler angles *(αβγ)*.

For integral-$j=0, 1, 2,..$ the *R(3)* integral over polar angle β ranges from 0 to π.

$$\textit{for } R(3): \frac{\ell^j}{N}\int d(\alpha\beta\gamma) = \frac{2j+1}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma \;= 2j+1 = \ell^j$$

For $^1/_2$-integral-$j=1/2, 3/2,..$ the *U(2)* integral over polar angle β ranges from -π to π.

$$\textit{for } SU(2): \frac{\ell^j}{N}\int d(\alpha\beta\gamma) = \frac{2j+1}{16\pi^2}\int_0^{2\pi} d\alpha \int_{-\pi}^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma \;= 2j+1 = \ell^j$$

Eigenstates of angular momentum are built from projected initial position states |000⟩.

$$\left|{}^{j}_{m,n}\right\rangle = \frac{\mathbf{P}^j_{m,n}|000\rangle}{\sqrt{\ell^j}} = \frac{1}{N}\int d(\alpha\beta\gamma)\, D^{j*}_{m,n}(\alpha\beta\gamma)\mathbf{R}(\alpha\beta\gamma)|000\rangle\sqrt{\ell^j} = \frac{1}{N}\int d(\alpha\beta\gamma)\, D^{j*}_{m,n}(\alpha\beta\gamma)\sqrt{\ell^j}\,|\alpha\beta\gamma\rangle$$

Angular position is defined by a *rotational duality relativity relation* or "Mock-Mach" principle

$$\mathbf{R}(\alpha\beta\gamma)|000\rangle = |\alpha\beta\gamma\rangle = \bar{\mathbf{R}}^\dagger(\alpha\beta\gamma)|000\rangle$$

$$\mathbf{R}(\alpha\beta\gamma)\bar{\mathbf{R}}(\alpha'\beta'\gamma') = \bar{\mathbf{R}}(\alpha'\beta'\gamma')\mathbf{R}(\alpha\beta\gamma)$$

$$\text{for all}(\alpha\beta\gamma)\text{and}(\alpha'\beta'\gamma')$$

Left hand (lab-*m*) and right hand (body-*n*) quantum numbers apply.

$$\mathbf{R}(\alpha\beta\gamma)\left|{}^{j}_{m,n}\right\rangle = \sum_{m'=-j}^{j} D^{j}_{m',m}(\alpha\beta\gamma)\left|{}^{j}_{m',n}\right\rangle \qquad \bar{\mathbf{R}}(\alpha\beta\gamma)\left|{}^{j}_{m,n}\right\rangle = \sum_{n'=-j}^{j} D^{j*}_{n',n}(\alpha\beta\gamma)\left|{}^{j}_{m,n'}\right\rangle$$

Applications of R(3) rotation and U(2) representations

Molecular and nuclear wavefunctions

For *SU(2)* and *R(3)*, sum over rotations is an integral over Euler angles *(αβγ)*.

For integral-*j=0, 1, 2,..* the *R(3)* integral over polar angle β ranges from 0 to π.

$$\textit{for } R(3): \frac{\ell^j}{N}\int d(\alpha\beta\gamma) = \frac{2j+1}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma = 2j+1 = \ell^j$$

For ½-integral-*j=1/2, 3/2,..* the *U(2)* integral over polar angle β ranges from -π to π.

$$\textit{for } SU(2): \frac{\ell^j}{N}\int d(\alpha\beta\gamma) = \frac{2j+1}{16\pi^2}\int_0^{2\pi} d\alpha \int_{-\pi}^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma = 2j+1 = \ell^j$$

Eigenstates of angular momentum are built from projected initial position states |000⟩.

$$\left|{}^{j}_{m,n}\right\rangle = \frac{\mathbf{P}^j_{m,n}|000\rangle}{\sqrt{\ell^j}} = \frac{1}{N}\int d(\alpha\beta\gamma) D^{j*}_{m,n}(\alpha\beta\gamma)\mathbf{R}(\alpha\beta\gamma)|000\rangle\sqrt{\ell^j} = \frac{1}{N}\int d(\alpha\beta\gamma) D^{j*}_{m,n}(\alpha\beta\gamma)\sqrt{\ell^j}\,|\alpha\beta\gamma\rangle$$

Angular position is defined by a *rotational duality relativity relation* or "Mock-Mach" principle

$$\mathbf{R}(\alpha\beta\gamma)|000\rangle = |\alpha\beta\gamma\rangle = \bar{\mathbf{R}}^{\dagger}(\alpha\beta\gamma)|000\rangle$$

$$\mathbf{R}(\alpha\beta\gamma)\bar{\mathbf{R}}(\alpha'\beta'\gamma') = \bar{\mathbf{R}}(\alpha'\beta'\gamma')\mathbf{R}(\alpha\beta\gamma)$$

$$\text{for all}(\alpha\beta\gamma)\text{and}(\alpha'\beta'\gamma')$$

Left hand (lab-*m*) and right hand (body-*n*) quantum numbers apply.

$$\mathbf{R}(\alpha\beta\gamma)\left|{}^{j}_{m,n}\right\rangle = \sum_{m'=-j}^{j} D^j_{m',m}(\alpha\beta\gamma)\left|{}^{j}_{m',n}\right\rangle \qquad \bar{\mathbf{R}}(\alpha\beta\gamma)\left|{}^{j}_{m,n}\right\rangle = \sum_{n'=-j}^{j} D^{j*}_{n',n}(\alpha\beta\gamma)\left|{}^{j}_{m,n'}\right\rangle$$

Same applies to the generators $\mathbf{s}_Z$ or $\mathbf{J}_Z$ of *SU(2)* or *R(3)*.

$$\mathbf{s}_Z\left|{}^{j}_{m,n}\right\rangle = m\left|{}^{j}_{m,n}\right\rangle \qquad \bar{\mathbf{s}}_Z\left|{}^{j}_{m,n}\right\rangle = -n\left|{}^{j}_{m,n}\right\rangle$$

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The reversed sign is a nuisance, so let us define ***reversed momentum operators*** that give a positive sign.

$$\mathbf{s}_{\bar{Z}}\left|\begin{matrix} j \\ m,n \end{matrix}\right\rangle = +n\left|\begin{matrix} j \\ m,n \end{matrix}\right\rangle \qquad \mathbf{s}_{\bar{Z}} = -\bar{\mathbf{s}}_{Z}$$

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Example of a rotor spectrum and Hamiltonian of a symmetric top molecule.

$$\mathbf{H}_{symmetric\ top} = B\mathbf{J}_{\bar{X}}^{2} + B\mathbf{J}_{\bar{Y}}^{2} + A\mathbf{J}_{\bar{Z}}^{2}$$

(Molecular spin is labeled **J** instead of **s**)

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Eigensolution equations:

$$\mathbf{H}_{symmetric\ top}\left|\begin{matrix} j \\ m,n \end{matrix}\right\rangle$$

$$= B\mathbf{J}\bullet\mathbf{J} + (A-B)\mathbf{J}_{\bar{Z}}^2\left|\begin{matrix} j \\ m,n \end{matrix}\right\rangle$$

$$= \left[BJ(J+1) + (A-B)n^2\right]\left|\begin{matrix} j \\ m,n \end{matrix}\right\rangle$$

Eigenvalue and energy level spectrum is shown next.

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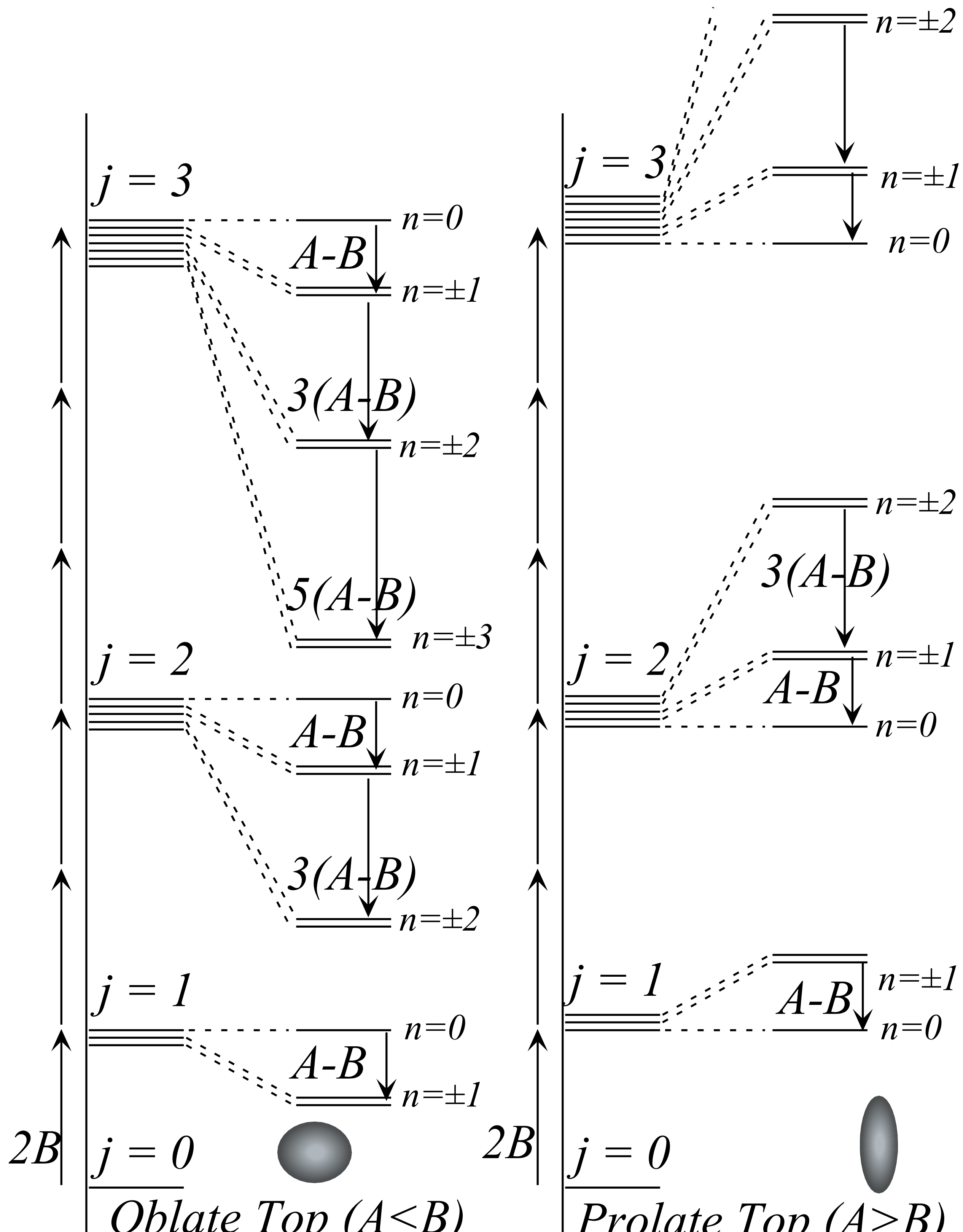

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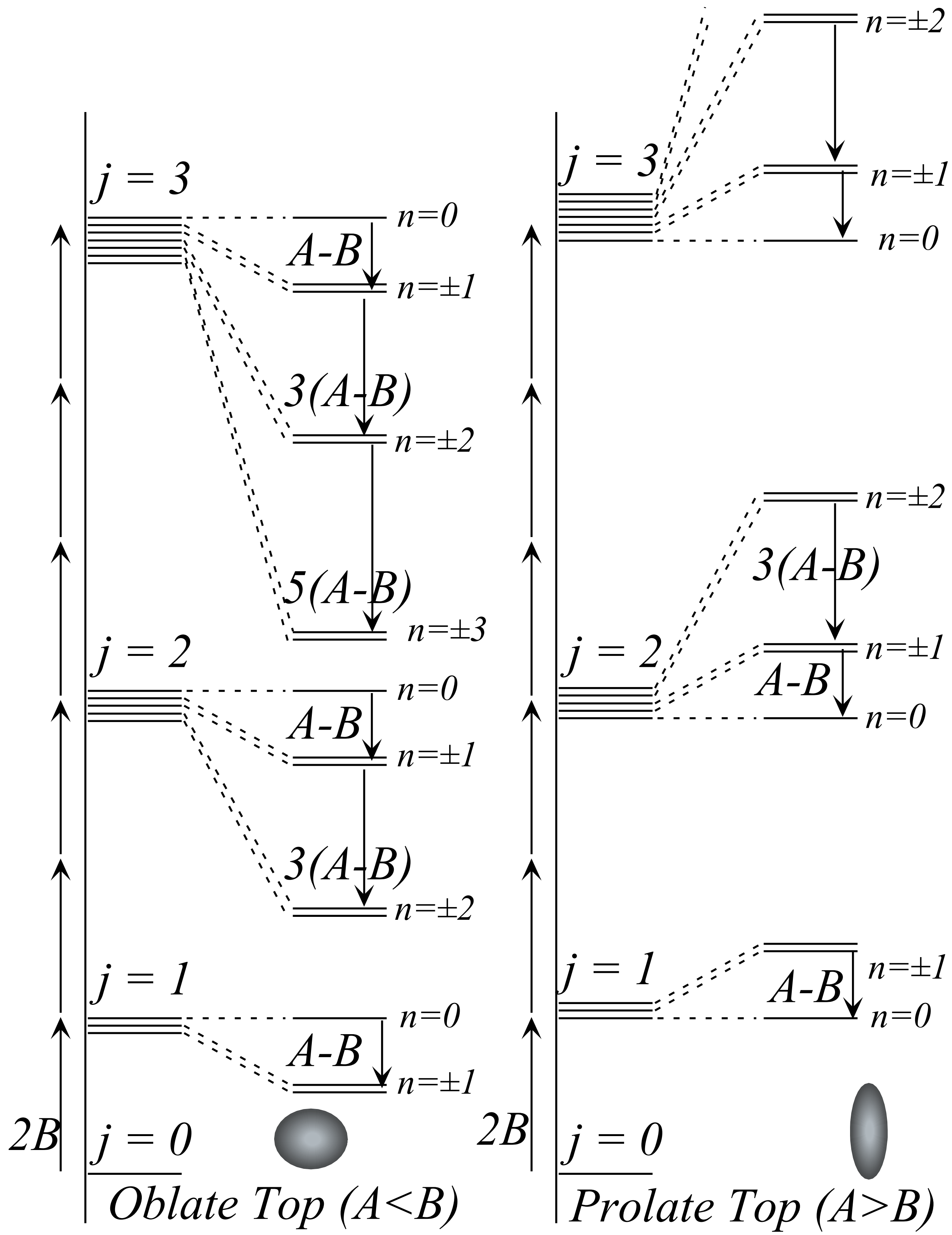
Mock-Mach-Multiplicity is $(2j+1)^2$ for each j



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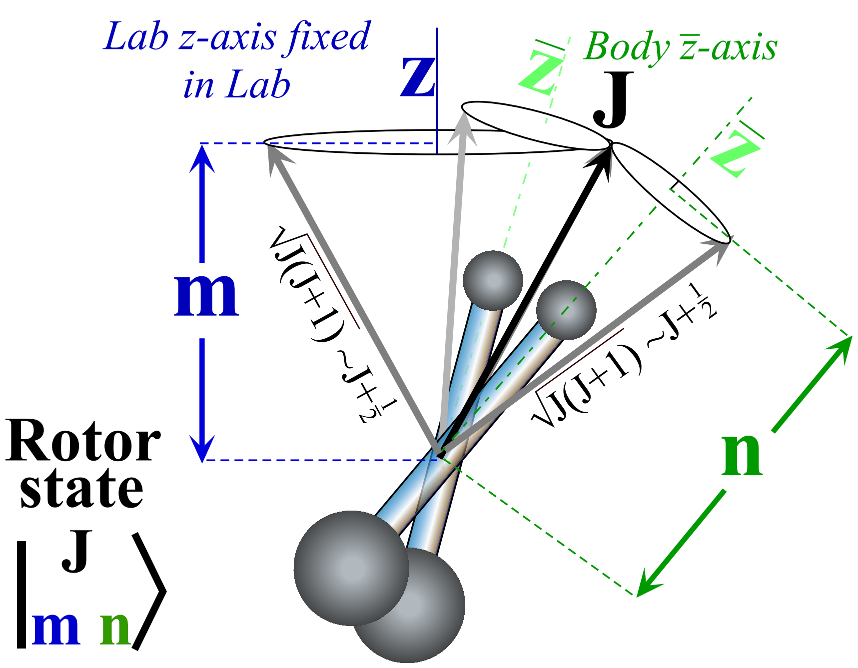

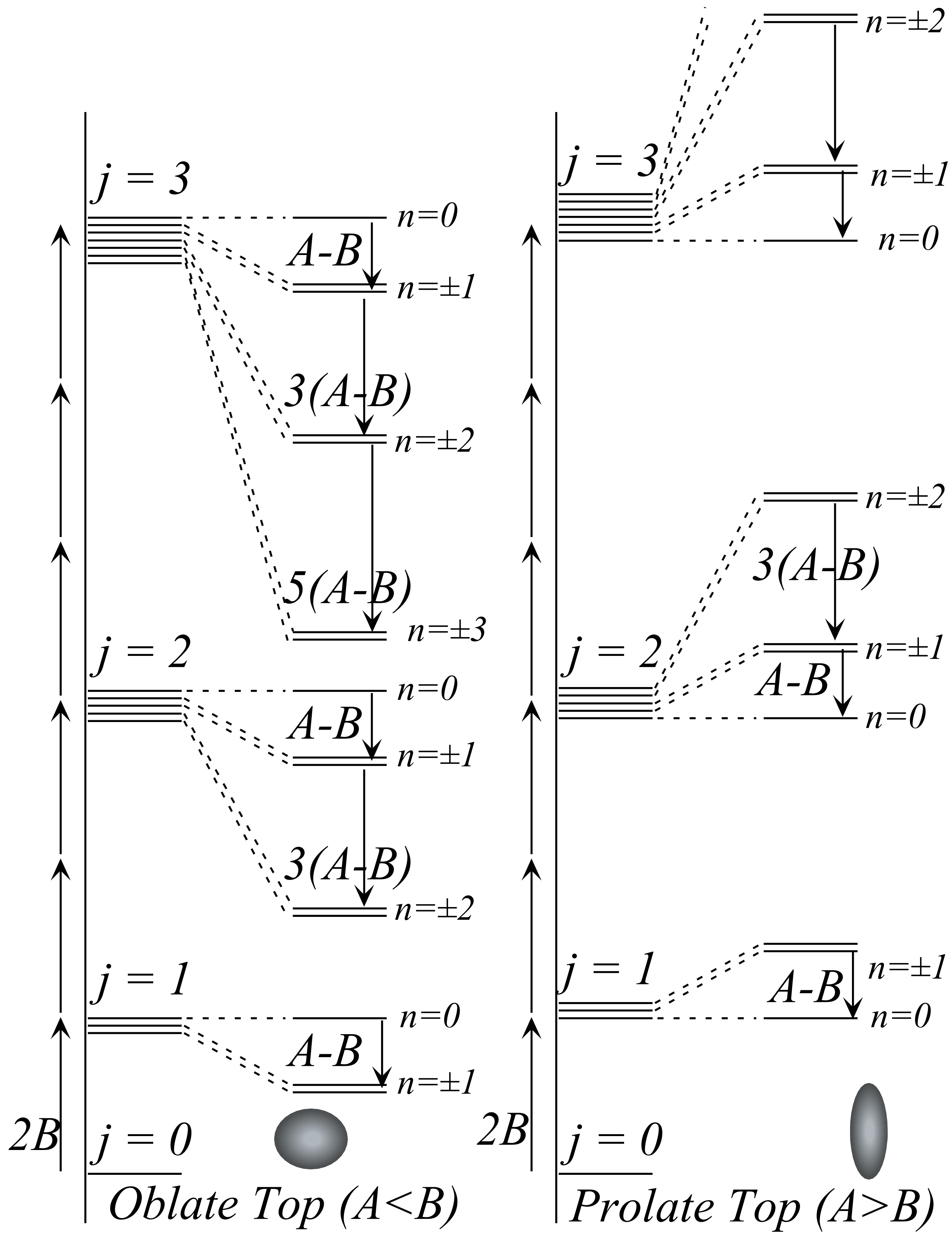

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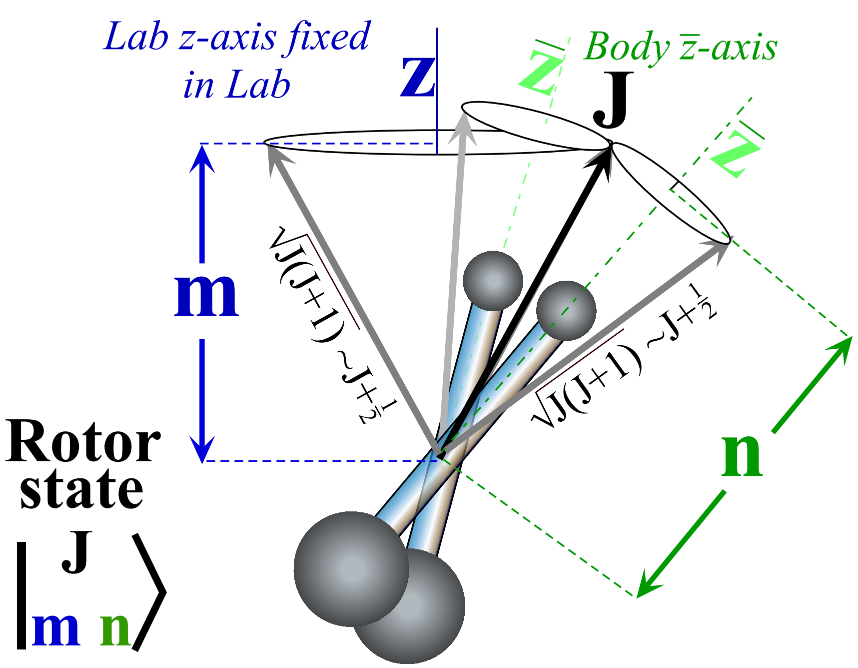
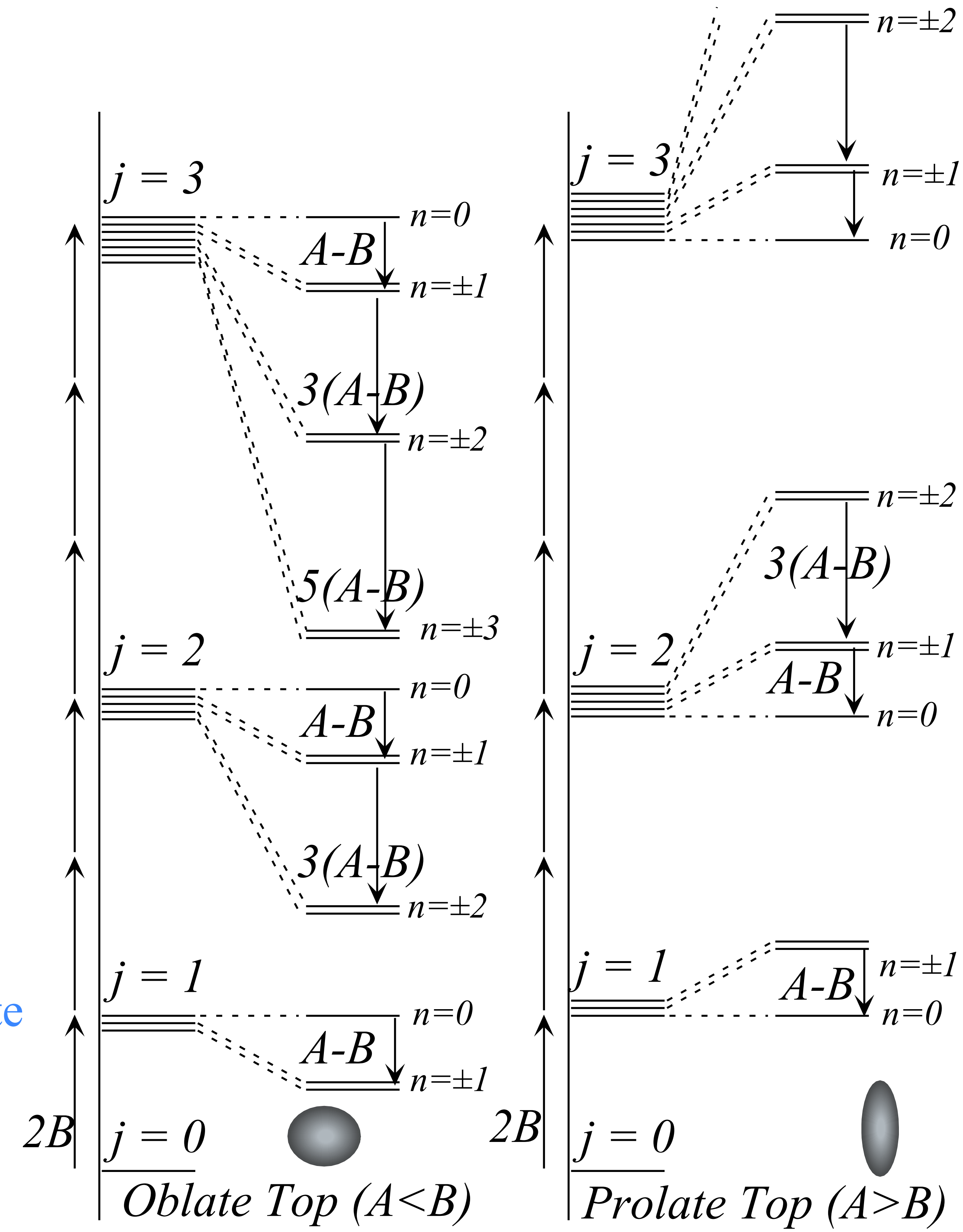


The $n=0$ levels are $2j+1$-fold degenerate

$n\neq 0$ level degeneracy is $4j+2$.



Applications of R(3) rotation and U(2) representations

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Introducing Racah tensor notation

Eigensolution equations:

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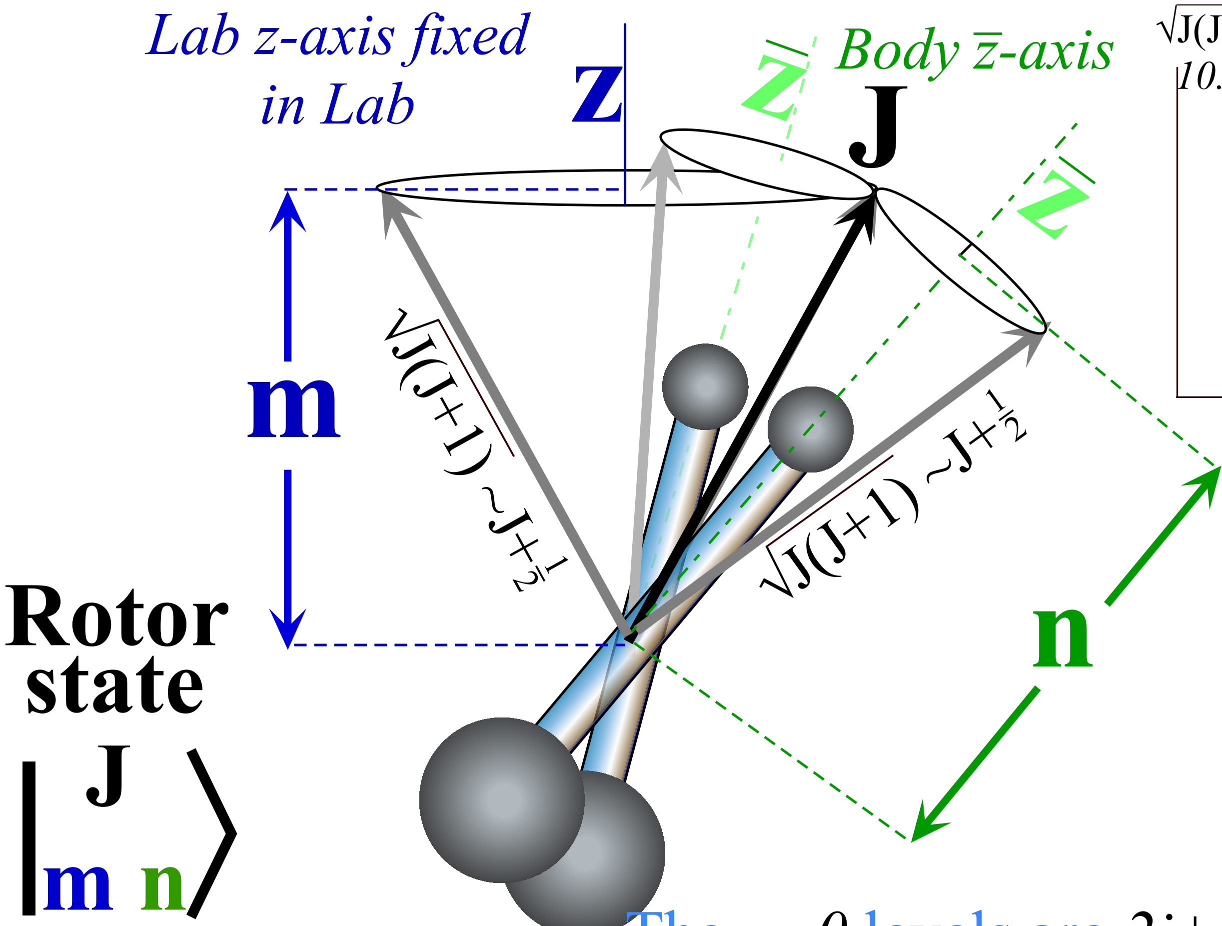
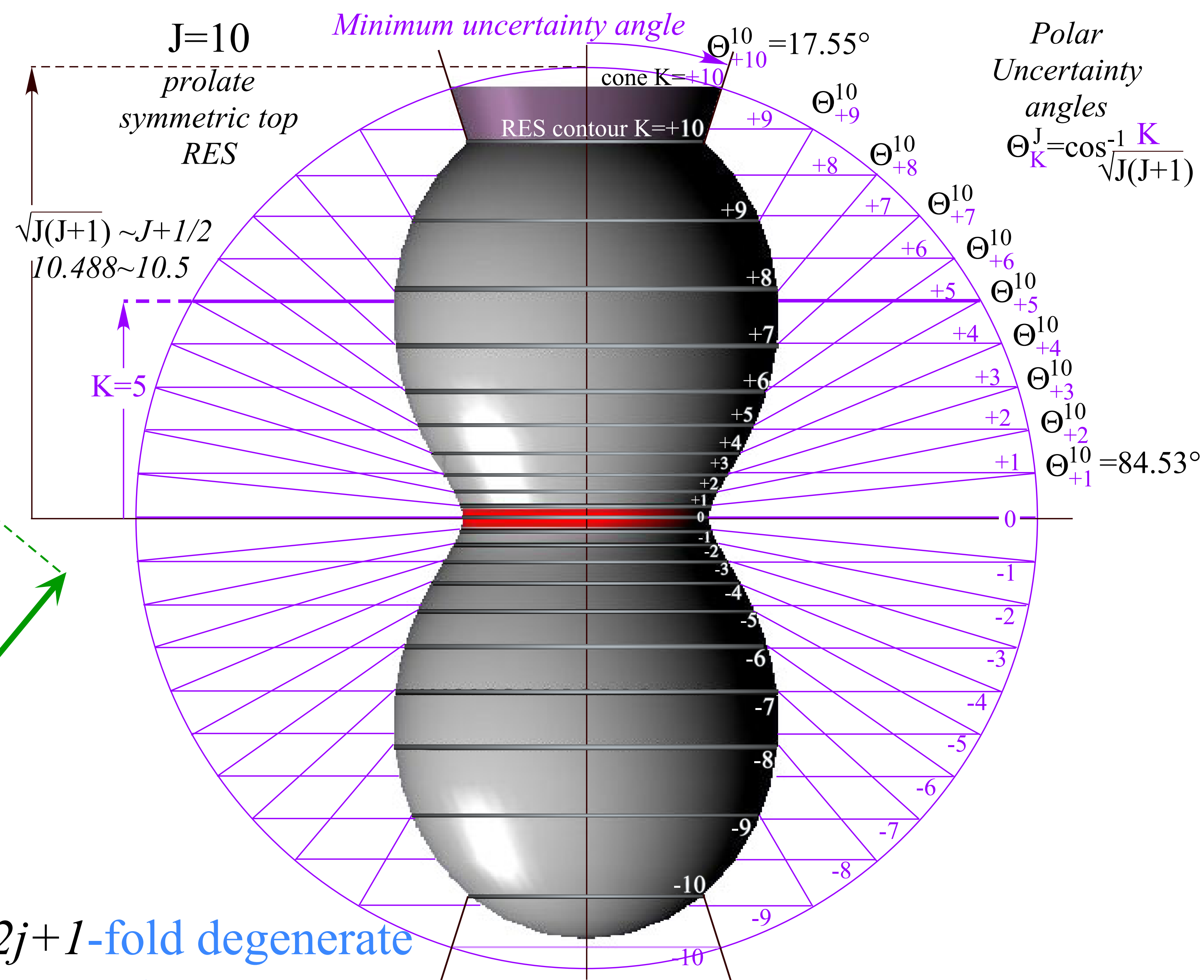
$$T_0^0 = \mathrm{J}\cdot\mathrm{J} = \langle J\rangle^2 \quad = \quad \left(J_x^2 + J_y^2 + J_y^2\right),$$

$$T_0^2 = \tfrac{1}{2}\langle J\rangle^2\left(3\cos^2\beta - 1\right) = \tfrac{1}{2}\left(2J_z^2 - J_x^2 - J_y^2\right),$$

$$H = B\ T_0^0 + \frac{2}{3}(A-B)\ T_0^2$$

Mock-Mach-Multiplicity is $(2j+1)^2$ for each j





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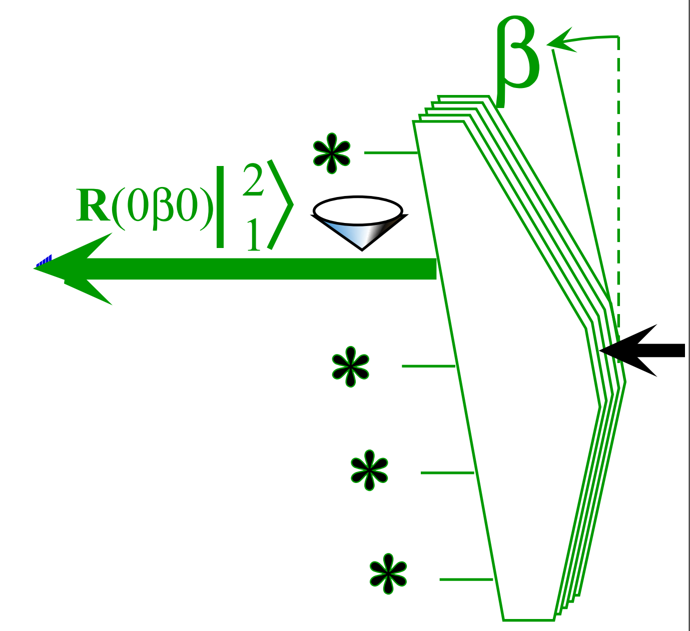
Generalized Stern-Gerlach and transformation matrices

Polarization analysis: Suppose a spin-j state $\mathbf{R}(0\beta 0)\left|{}^{j=2}_{m=1}\right\rangle$ exits an analyzer rotated by β

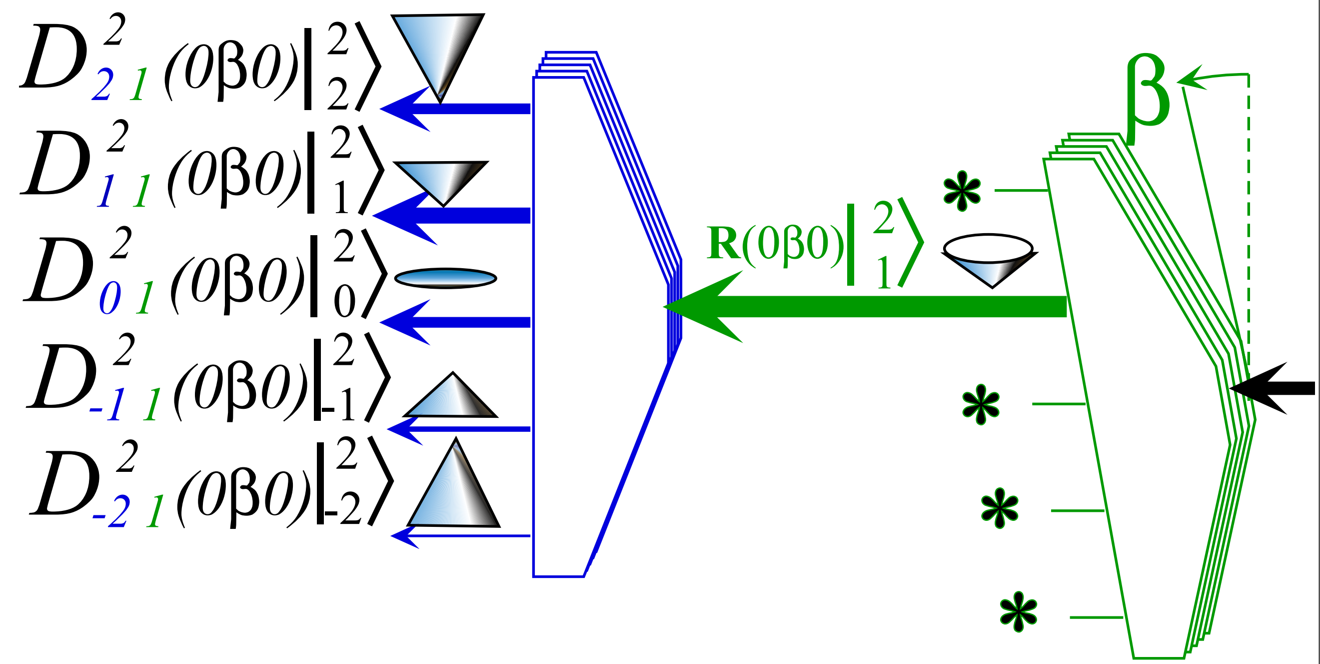


Generalized Stern-Gerlach and transformation matrices

Polarization analysis: Suppose a spin-j state $\mathbf{R}(0\beta 0)\left|{}^{j=2}_{m=1}\right\rangle$ exits an analyzer rotated by β and then enters a vertical ($\beta=0$) analyzer that forces it to choose from unrotated states $\left|{}^{j=2}_{m'}\right\rangle$

$$\mathbf{R}(0\beta 0)\left|{}^{j}_{m}\right\rangle = \sum_{m'=-j}^{j}\left|{}^{j}_{m'}\right\rangle\left\langle{}^{j}_{m'}\right|\mathbf{R}(0\beta 0)\left|{}^{j}_{m}\right\rangle$$

$$= \sum_{m'=-j}^{j}\left|{}^{j}_{m'}\right\rangle D^{j}_{m'm}(0\beta 0)$$



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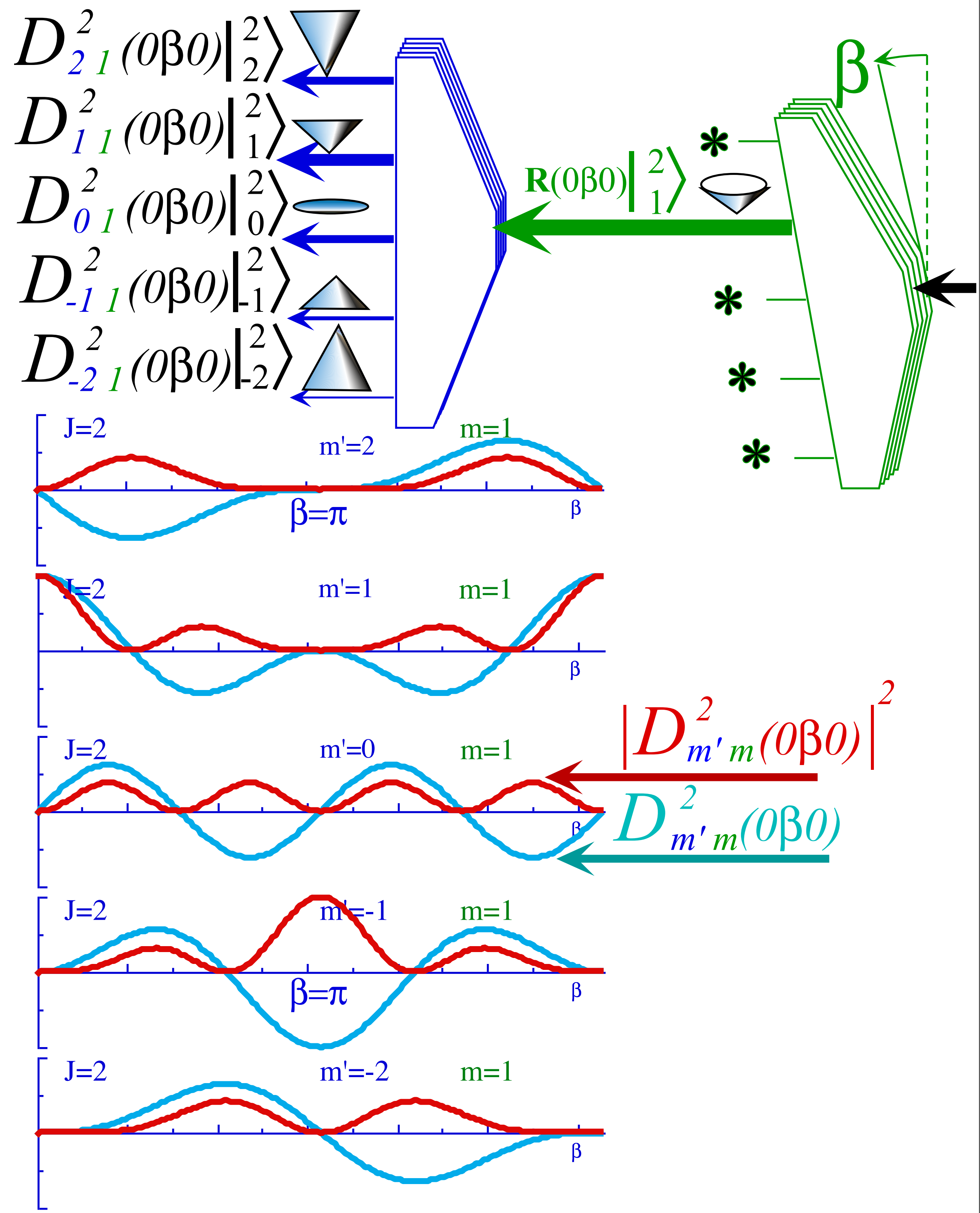
Overlap of state $\mathbf{R}(\alpha\beta\gamma)\,|{}^{2}_{1}\rangle$ with unrotated $|{}^{j=2}_{m'}\rangle$ is the corresponding D-matrix element.

$$\left\langle{}^{j'}_{m'}\right|\mathbf{R}(\alpha\beta\gamma)\left|{}^{2}_{1}\right\rangle = \delta^{j'2}D^{2}_{m'1}(\alpha\beta\gamma) = \left\langle{}^{j'}_{m'}\middle|{}^{2}_{1}\right\rangle_R$$

$D^{j}_{m'n}(0\beta 0)$ plotted vs. β for fixed j,m',n



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$$D^2(\alpha\beta 0)=\left(\begin{array}{cc|c|cc}
e^{-i2\alpha}\left(\frac{1+\cos\beta}{2}\right)^2 & e^{-i2\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & \sqrt{\frac{3}{8}}e^{-i2\alpha}\sin^2\beta & e^{-i2\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{-i2\alpha}\left(\frac{1-\cos\beta}{2}\right)^2 \\
e^{-i\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{-i\alpha}\left(\frac{1+\cos\beta}{2}\right)(2\cos\beta-1) & -\sqrt{\frac{3}{2}}e^{-i\alpha}\sin\beta\cos\beta & e^{-i\alpha}\left(\frac{1-\cos\beta}{2}\right)(2\cos\beta+1) & -e^{-i\alpha}\left(\frac{1-\cos\beta}{2}\right)\sin\beta \\
\sqrt{\frac{3}{8}}\sin^2\beta & \sqrt{\frac{3}{2}}\sin\beta\cos\beta & \frac{3\cos^2\beta-1}{2} & \sqrt{\frac{3}{2}}\sin\beta\cos\beta & \sqrt{\frac{3}{8}}\sin^2\beta \\
e^{i\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{i\alpha}\left(\frac{1-\cos\beta}{2}\right)(2\cos\beta+1) & \sqrt{\frac{3}{2}}e^{i\alpha}\sin\beta\cos\beta & e^{i\alpha}\left(\frac{1+\cos\beta}{2}\right)(2\cos\beta-1) & -e^{i\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta \\
e^{i2\alpha}\left(\frac{1-\cos\beta}{2}\right)^2 & e^{i2\alpha}\left(\frac{1-\cos\beta}{2}\right)\sin\beta & \sqrt{\frac{3}{8}}e^{i2\alpha}\sin^2\beta & e^{i2\alpha}\left(\frac{1+\cos\beta}{2}\right)\sin\beta & e^{i2\alpha}\left(\frac{1+\cos\beta}{2}\right)^2
\end{array}\right)$$

$$\mathbf{R}(0\beta 0)\left|\begin{matrix} j \\ m\end{matrix}\right\rangle = \sum_{m'=-j}^{j}\left|\begin{matrix} j \\ m'\end{matrix}\right\rangle\left\langle\begin{matrix} j \\ m'\end{matrix}\right|\mathbf{R}(0\beta 0)\left|\begin{matrix} j \\ m\end{matrix}\right\rangle$$

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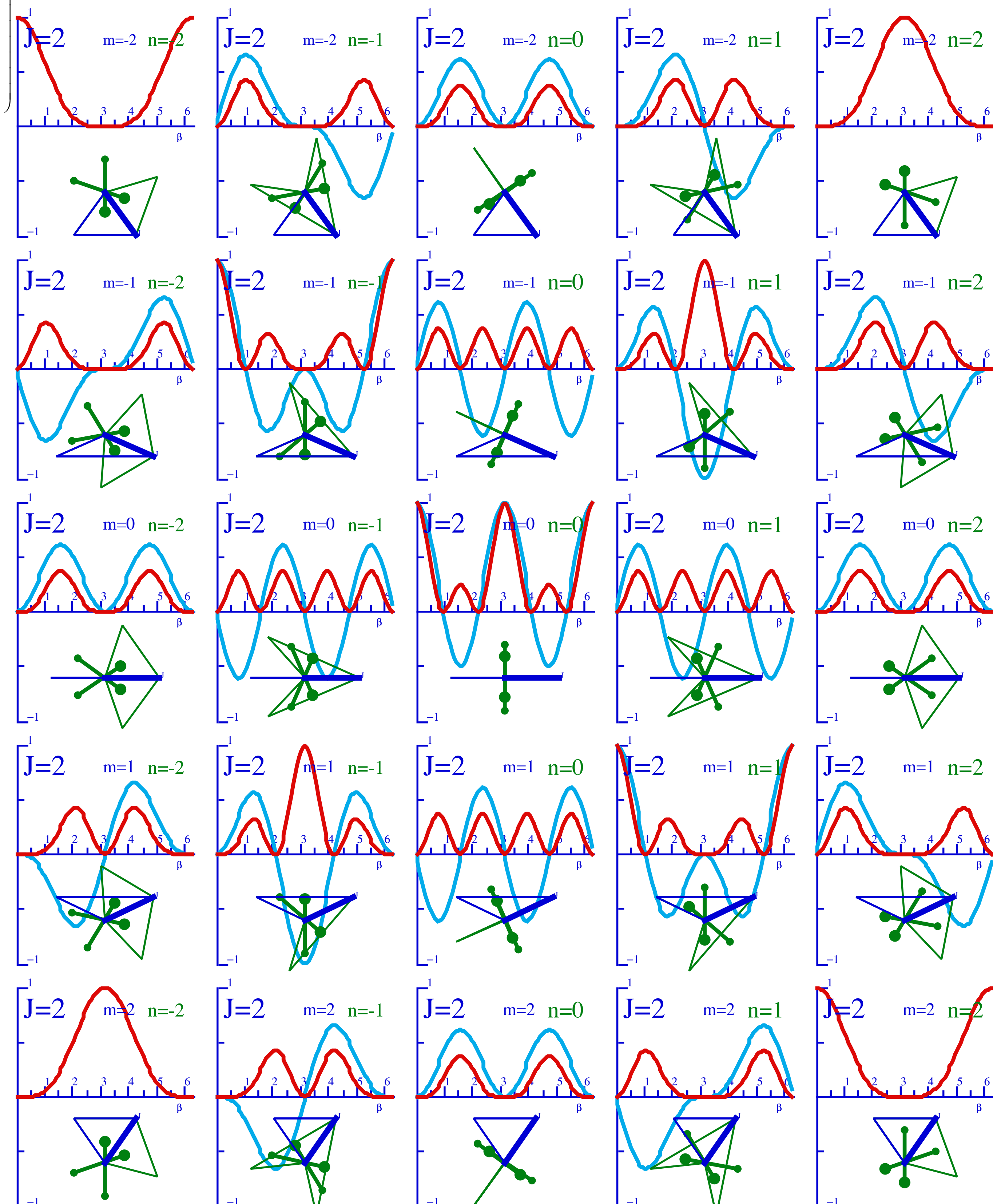

$D^j_{m'n}(0\beta 0)$ plotted vs. β for fixed j, m', n

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$D^{j}_{m'n}(0\beta 0)$
plotted
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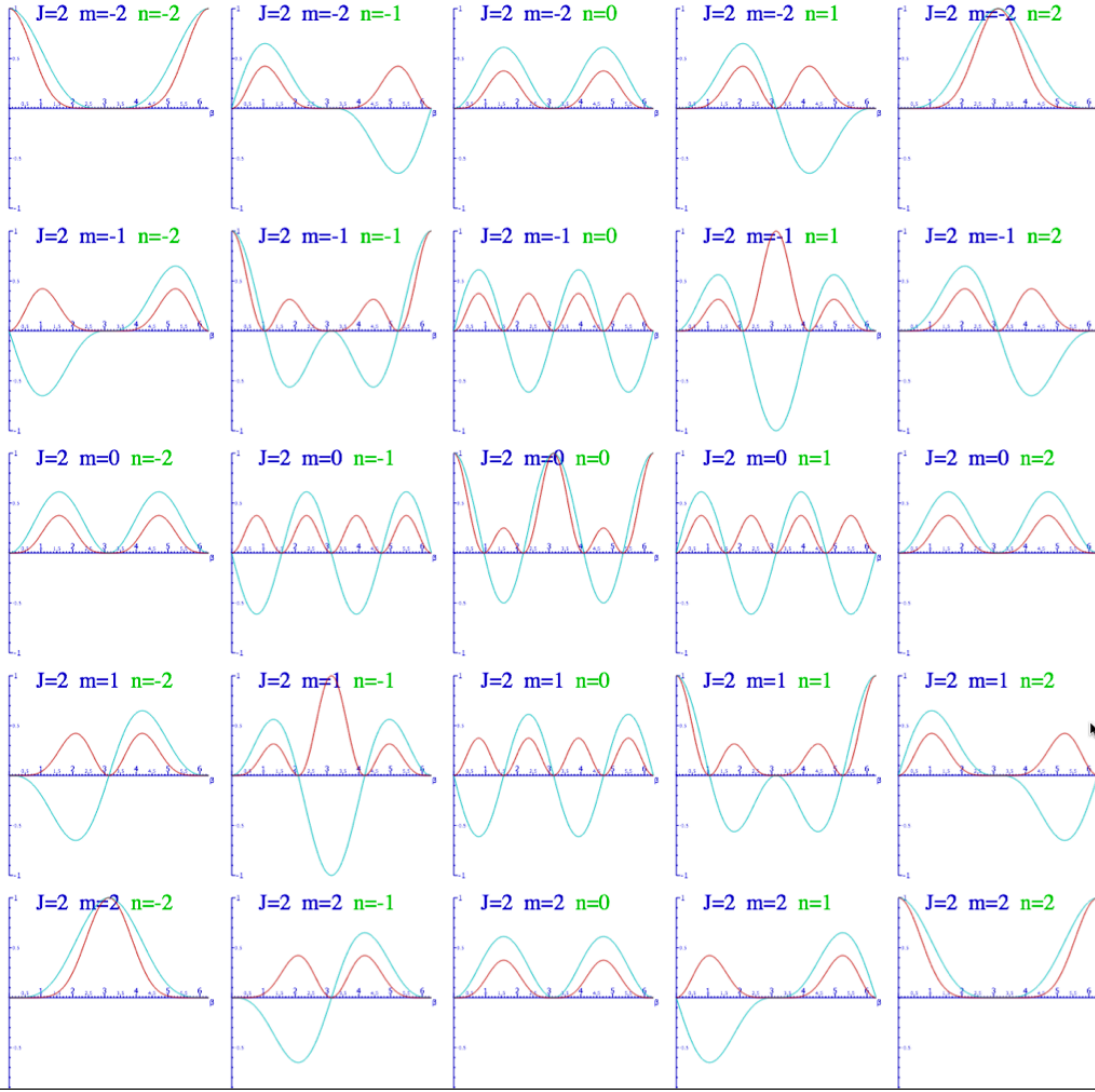

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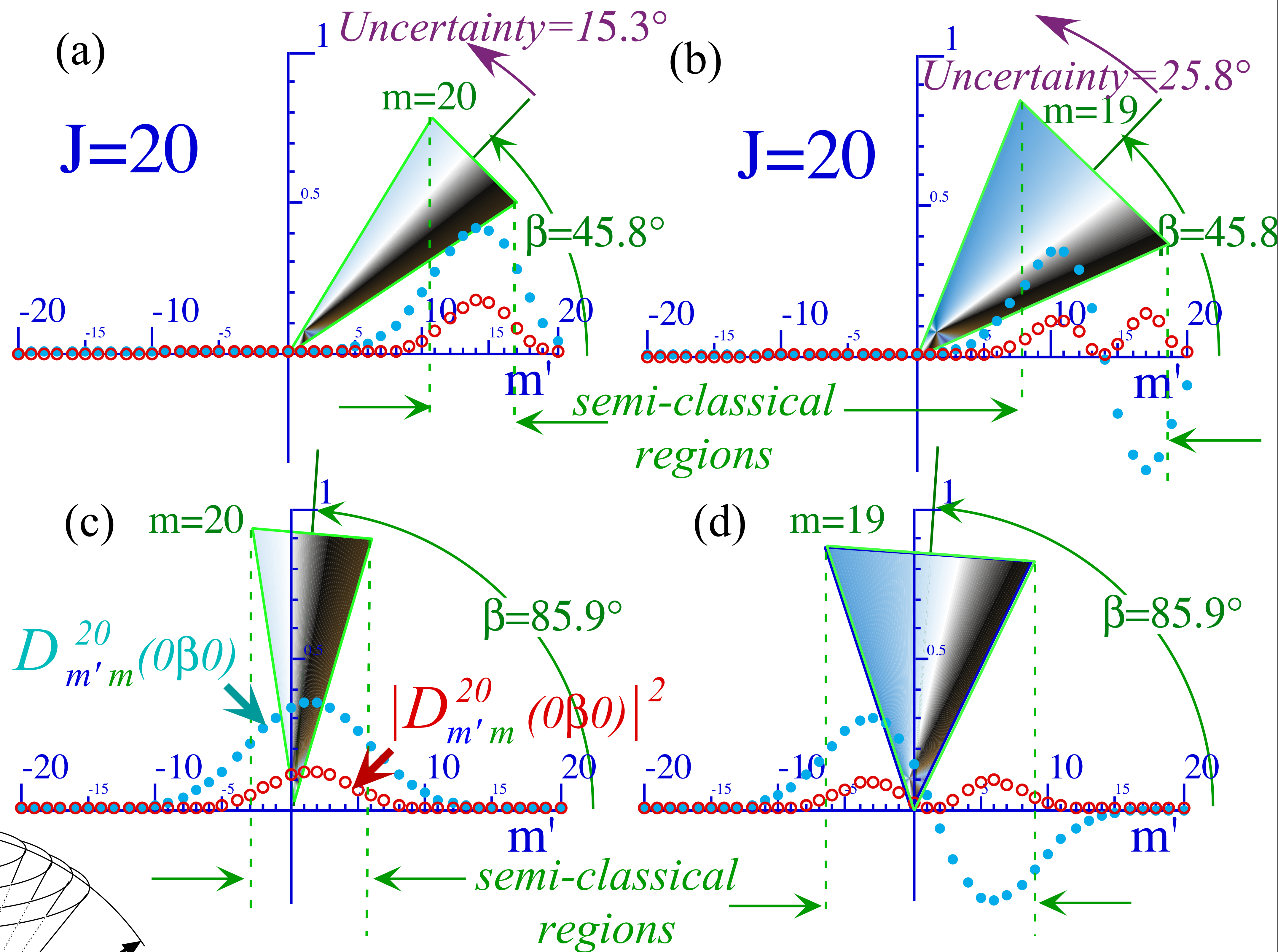
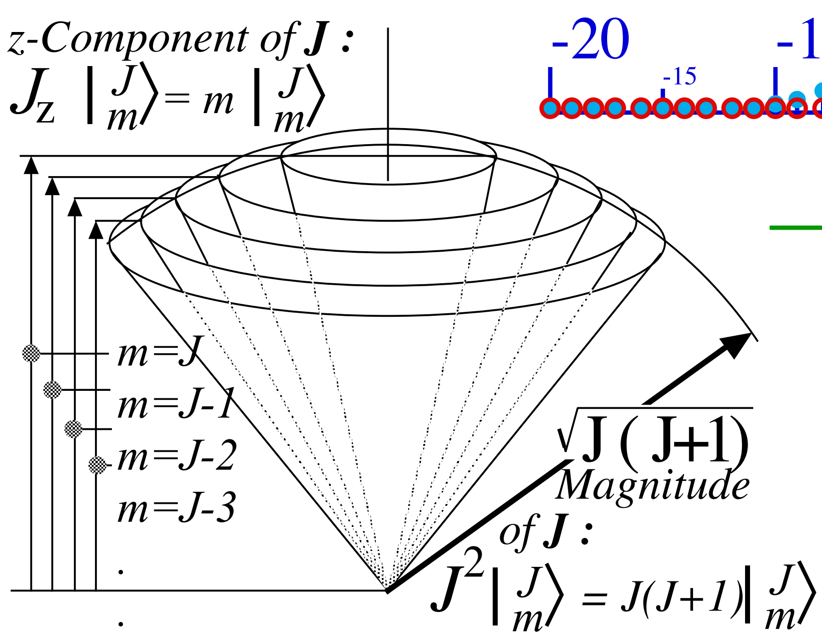
Angular momentum cones and high J properties

$D^J_{m'm}(0\beta 0)$ plotted vs. m' for fixed J,β,m





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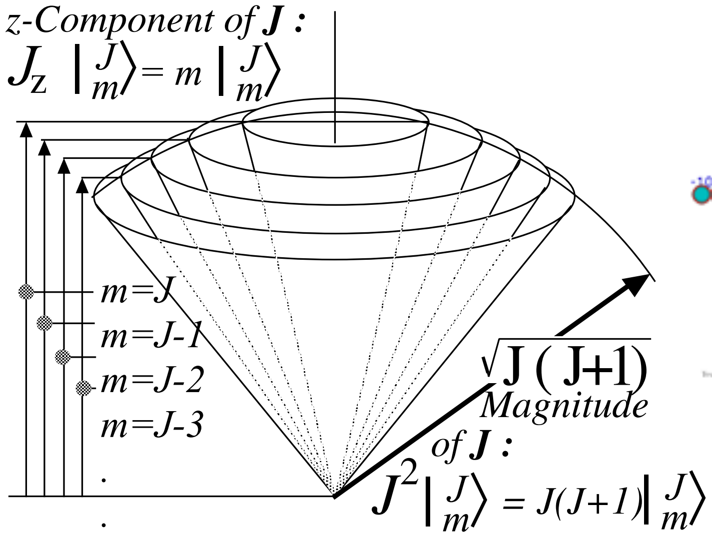

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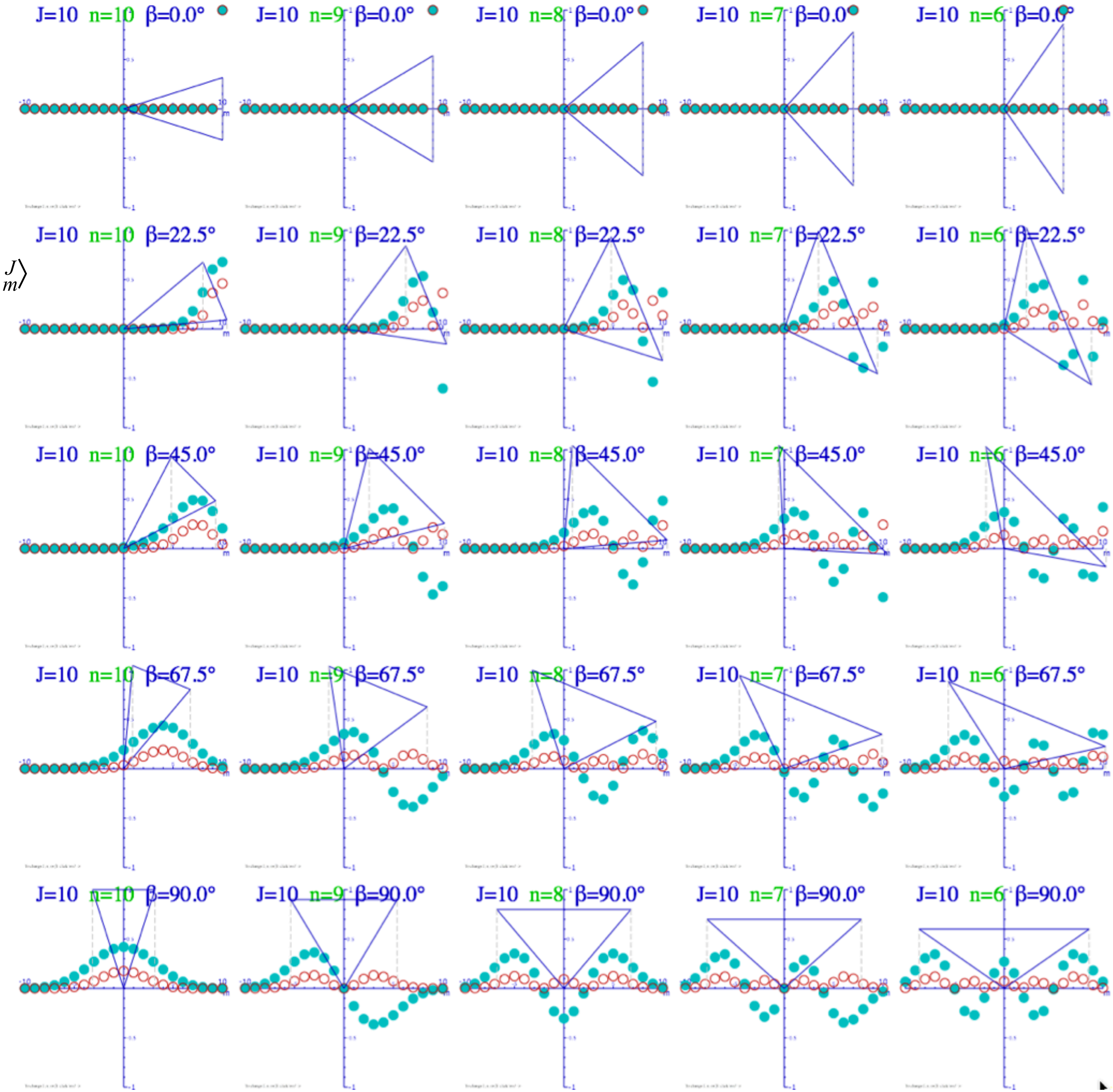
$D^{J=10}_{m,n}(0\beta 0)$

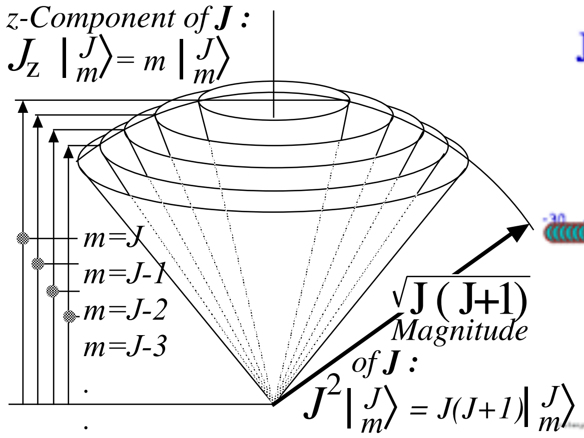
plotted

vs. m

for fixed

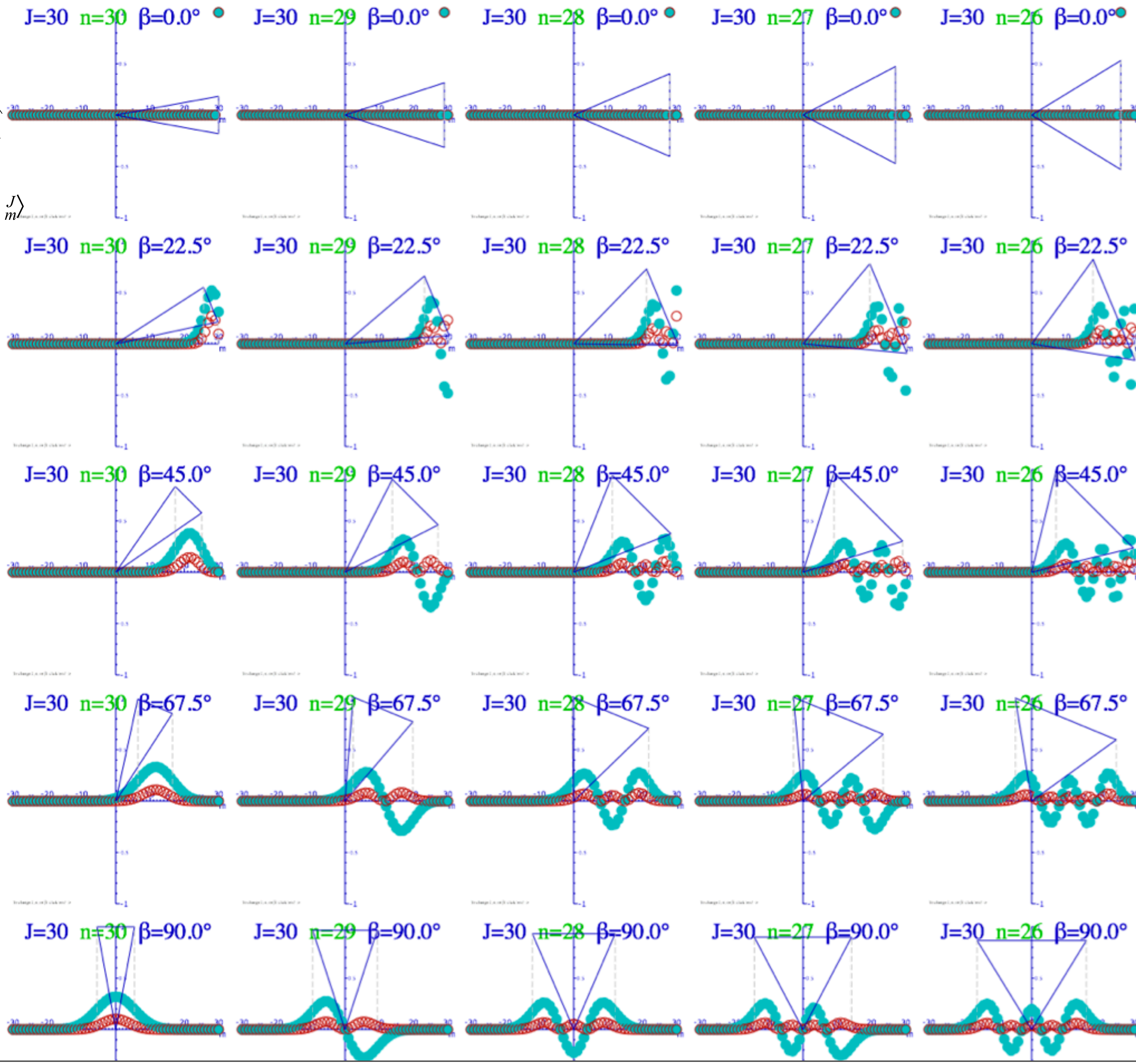
J=10,β,n





$D^{J=30}_{m,n}(0\beta 0)$

plotted
vs. m
for fixed
J=30,β,n



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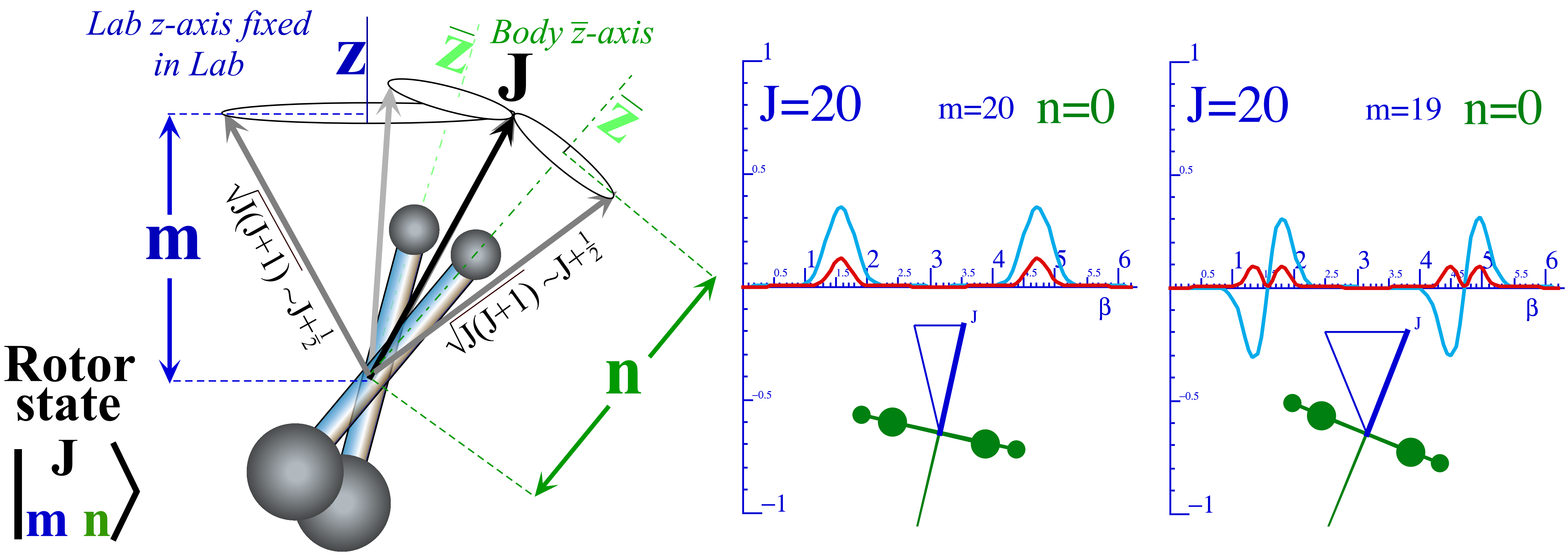
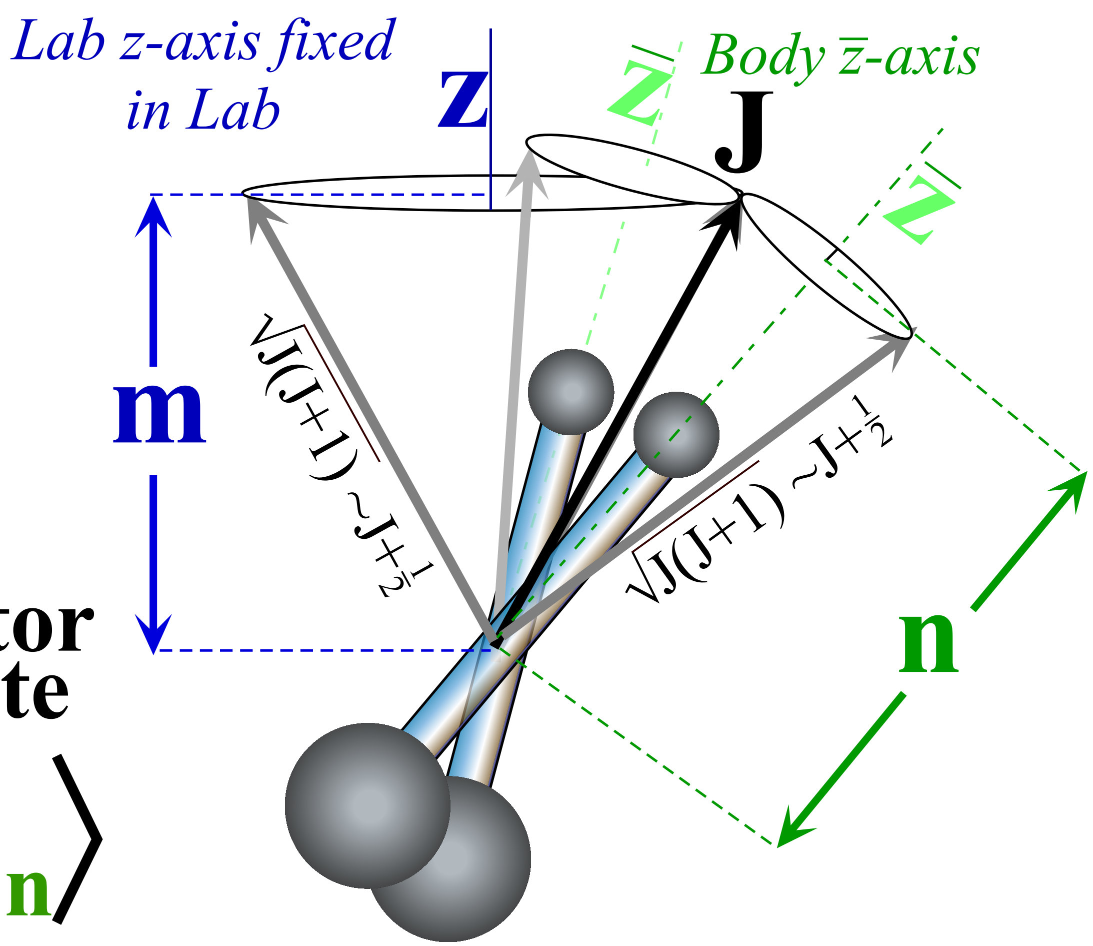
$D^{J=20}_{m,n}(0\beta 0)$
plotted
vs. β
for fixed
J=20,m,n



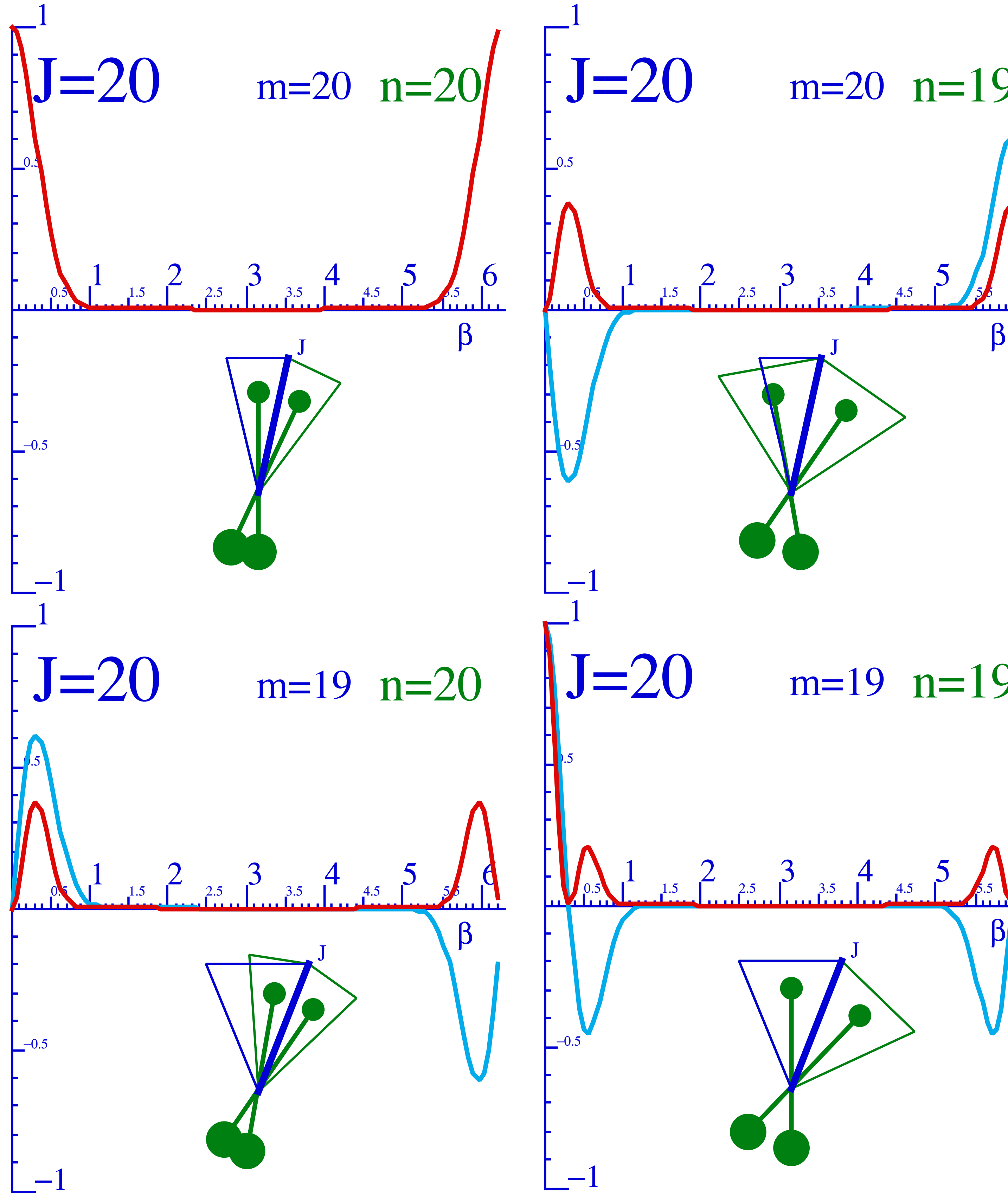
$D^{J}_{m,n}(0\beta 0)$ plotted vs. β for fixed J,m,n

QuantIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/QuantItWeb.html"

$D^{J=20}_{m,n}(0\beta 0)$
plotted
vs. β
for fixed
J=20,m,n

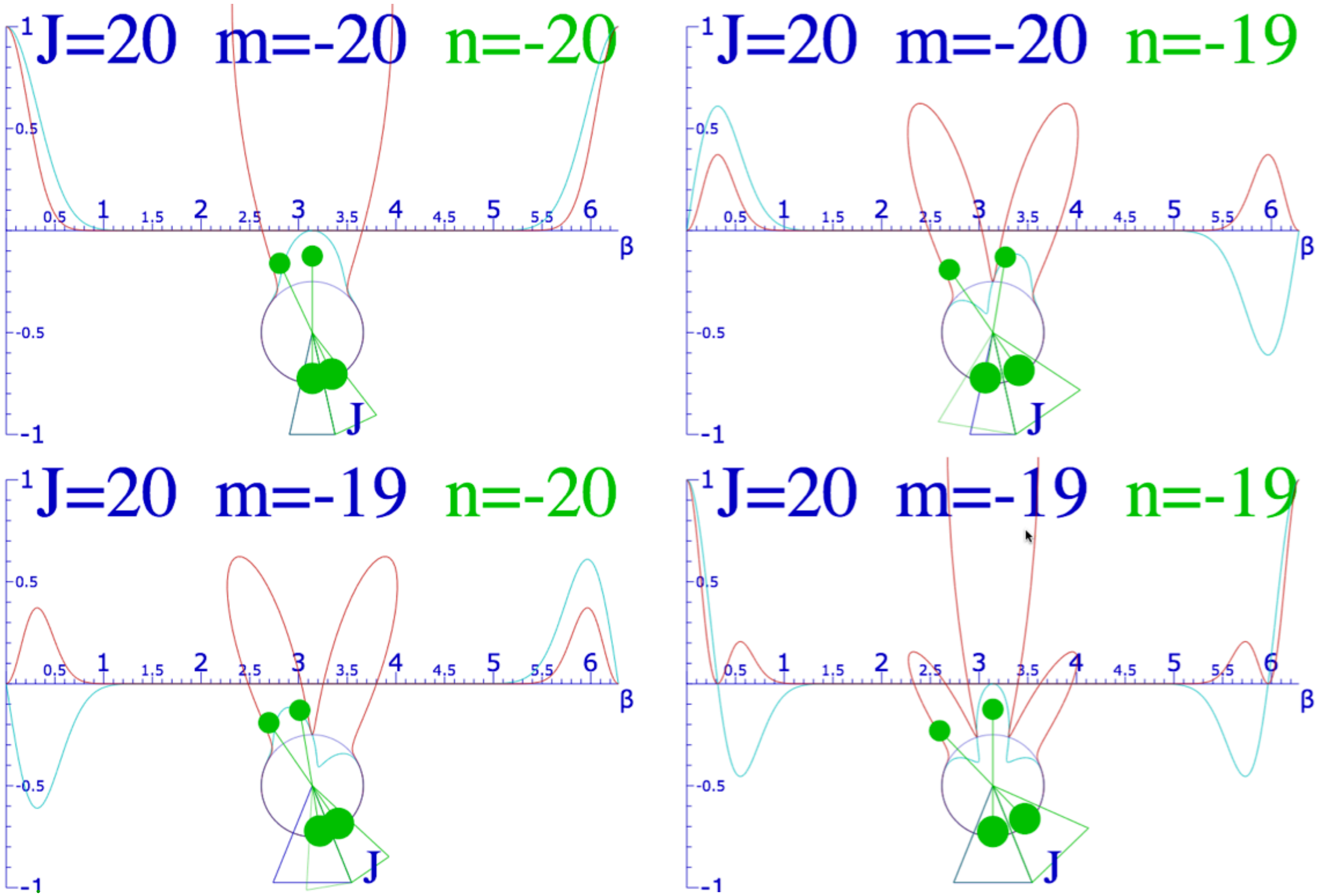


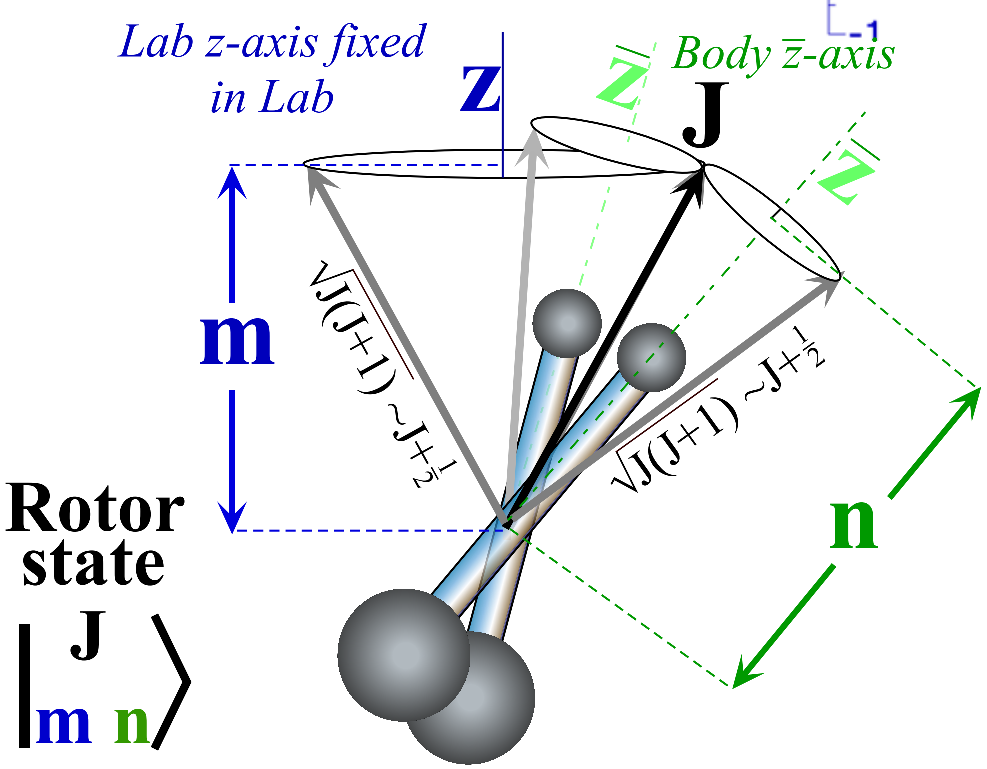
Rotor state
$\left|{}^{J}_{m\ n}\right\rangle$



QuantIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/QuantItWeb.html"

$D^{J=20}_{m,n}(0\beta 0)$
plotted
vs. β
for fixed
J=20,m,n





QuantIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/QuantItWeb.html"

Links as of April 21, 2015
(Apps are being upgraded as time permits)

Links to the current Harter-Soft LearnIt web apps for Physics

(Bold links have default redirect pages. ***Italics*** **are not yet meant for production. Red are in the final stages of testing.**

Production Links - *For the students & general public*

BohrIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/BohrItWeb.html"
BounceIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/BounceItWeb.html"
BoxIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/BoxItWeb.html"
CoulIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/CoulItWeb.html"
Cycloidulum - Production; URL is "http://www.uark.edu/ua/modphys/markup/CycloidulumWeb.html"
LearnIt - Production; URL is "**http://www.uark.edu/ua/modphys**" or "http://www.uark.edu/ua/modphys/markup/LearnItWeb.html"
JerkIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/JerkItWeb.html"
Pendulum - Production; URL is "http://www.uark.edu/ua/modphys/markup/PendulumWeb.html"
QuantIt - Production; URL is "http://www.uark.edu/ua/modphys/markup/QuantItWeb.html"
Relativity - Pirelli Entrant - Production; URL is "**http://www.uark.edu/ua/pirelli**" or "http://www.uark.edu/ua/pirelli/html/default.html"
Trebuchet Production; URL is "http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html"

Testing Links - *For internal use and testing by Harter & Heyoka*

BohrIt - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/BohrItWeb.html"
BounceIt - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/BounceItWeb.html"
BounceIt Title Page - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/BounceItTitlePage.html"
BoxIt - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/BoxItWeb.html"
CoulIt - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/CoulItWeb.html"
Cycloidulum - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/CycloidulumWeb.html"
Harter-Soft Web Apps - Quick Reference - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/JerkItWeb.html"
JerkIt - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/JerkItWeb.html"
ModernPhysics - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/IntroCover.html"
Pendulum - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/PendulumWeb.html"
QuantIt - Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/QuantItWeb.html"
Trebuchet Testing; URL is "http://www.uark.edu/ua/modphys/testing/markup/TrebuchetWeb.html"