# Group Theory in Quantum Mechanics

## Lecture 9 (2.12.13)

# Applications of U(2) and R(3) representations

(Quantum Theory for Computer Age - Ch. 10A-B of Unit 3 )
(Principles of Symmetry, Dynamics, and Spectroscopy - Sec. 1-3 of Ch. 5 and Ch. 7 )

Review: Fundamental Euler **R**$(\alpha\beta\gamma)$ and Darboux **R**$[\varphi\vartheta\Theta]$ representations of U(2) and R(3)

- Euler **R**$(\alpha\beta\gamma)$ derived from Darboux **R**$[\varphi\vartheta\Theta]$ and vice versa
- Euler **R**$(\alpha\beta\gamma)$ rotation $\Theta=0-4\pi$-sequence $[\varphi\vartheta]$ fixed
- R(3)-U(2) slide rule for converting **R**$(\alpha\beta\gamma)$ ↔ **R**$[\varphi\vartheta\Theta]$ and Sundial

U(2) density operator approach to symmetry dynamics

- Bloch equation for density operator

The *ABC*'s of U(2) dynamics-Archetypes

- Asymmetric-Diagonal *A*-Type motion
- Bilateral-Balanced *B*-Type motion
- Circular-Coriolis… *C*-Type motion

The *ABC*'s of U(2) dynamics-Mixed modes

- *AB*-Type motion and Wigner's Avoided-Symmetry-Crossings
- *ABC*-Type elliptical polarized motion

Ellipsometry using U(2) symmetry coordinates

- Conventional amp-phase ellipse coordinates
- Euler Angle $(\alpha\beta\gamma)$ ellipse coordinates

*Review: Fundamental Euler* **R***(αβγ) and Darboux* **R**[φϑΘ] *representations of U(2) and R(3)*

- *Euler* **R***(αβγ) derived from Darboux* **R**[φϑΘ] *and vice versa*
- *Euler* **R***(αβγ) rotation* Θ=0−4π*-sequence* [φϑ] *fixed*
- *R(3)-U(2) slide rule for converting* **R***(αβγ)* ↔ **R**[φϑΘ] *and Sundial*

*U(2) density operator approach to symmetry dynamics*

- *Bloch equation for density operator*

*The ABC's of U(2) dynamics-Archetypes*

- *Asymmetric-Diagonal A-Type motion*
- *Bilateral-Balanced B-Type motion*
- *Circular-Coriolis… C-Type motion*

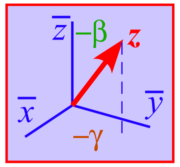
*The ABC's of U(2) dynamics-Mixed modes*

- *AB-Type motion and Wigner's Avoided-Symmetry-Crossings*
- *ABC-Type elliptical polarized motion*

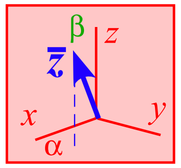
*Ellipsometry using U(2) symmetry coordinates*

- *Conventional amp-phase ellipse coordinates*
- *Euler Angle (αβγ) ellipse coordinates*

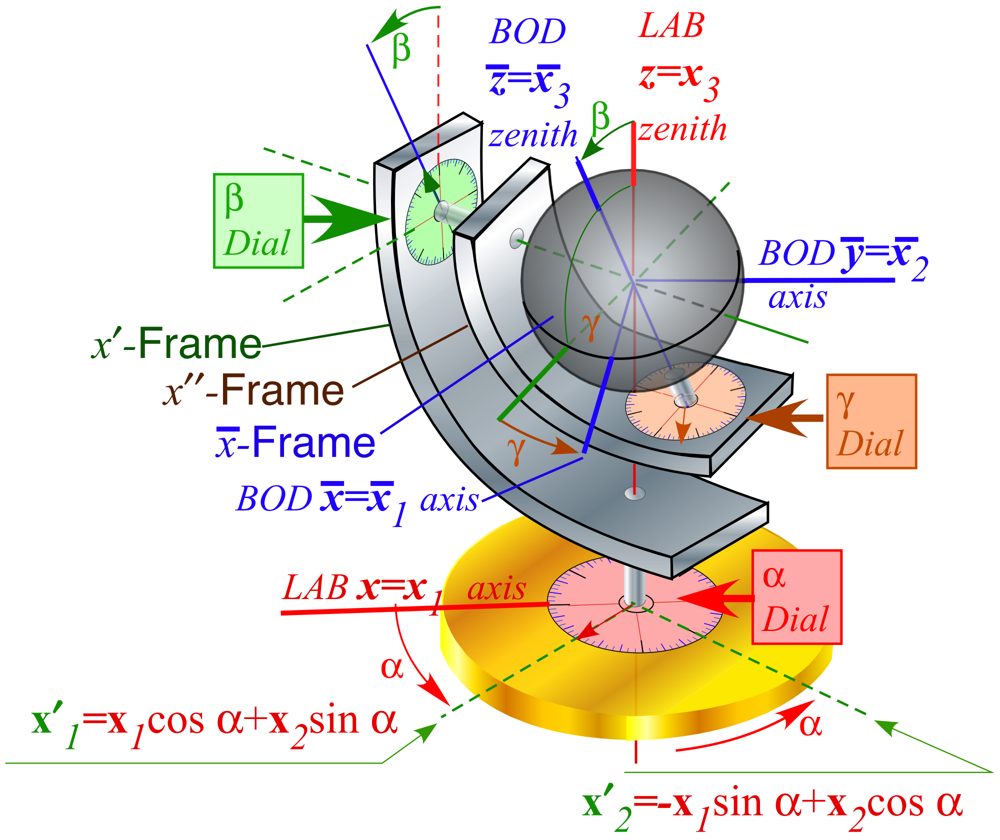
# *Here spin-rotor* **S***-polar coordinates are Euler angles*

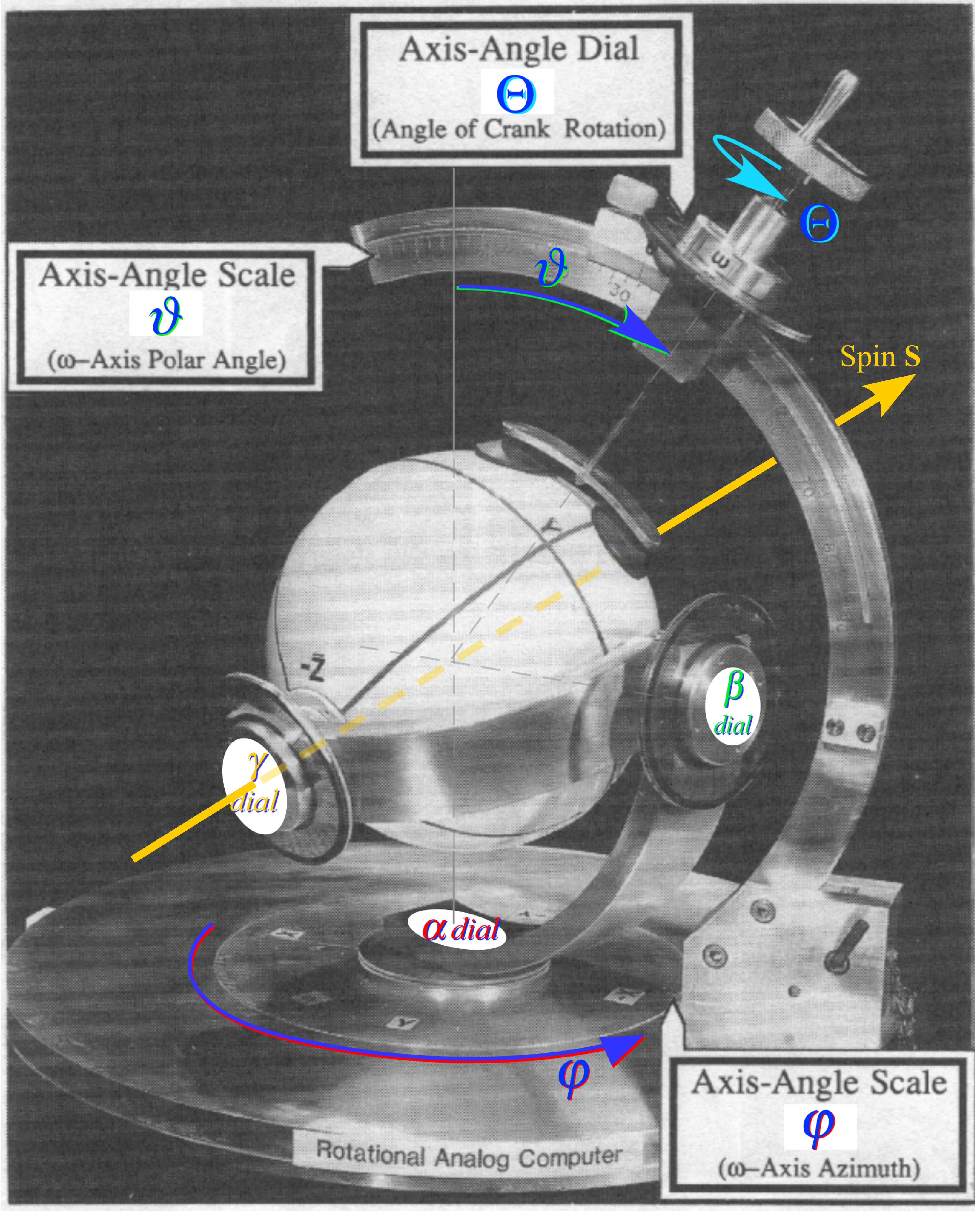


*BOD frame view*
*Polar angles of LAB zenith $z=x_3$ are*
*(azimuth angle=−γ,*
*polar angle=−β )*



*LAB frame view*
*Polar angles of BOD zenith $\bar{z}=\bar{x}_3$ are*
*(azimuth angle=α,*
*polar angle=β )*

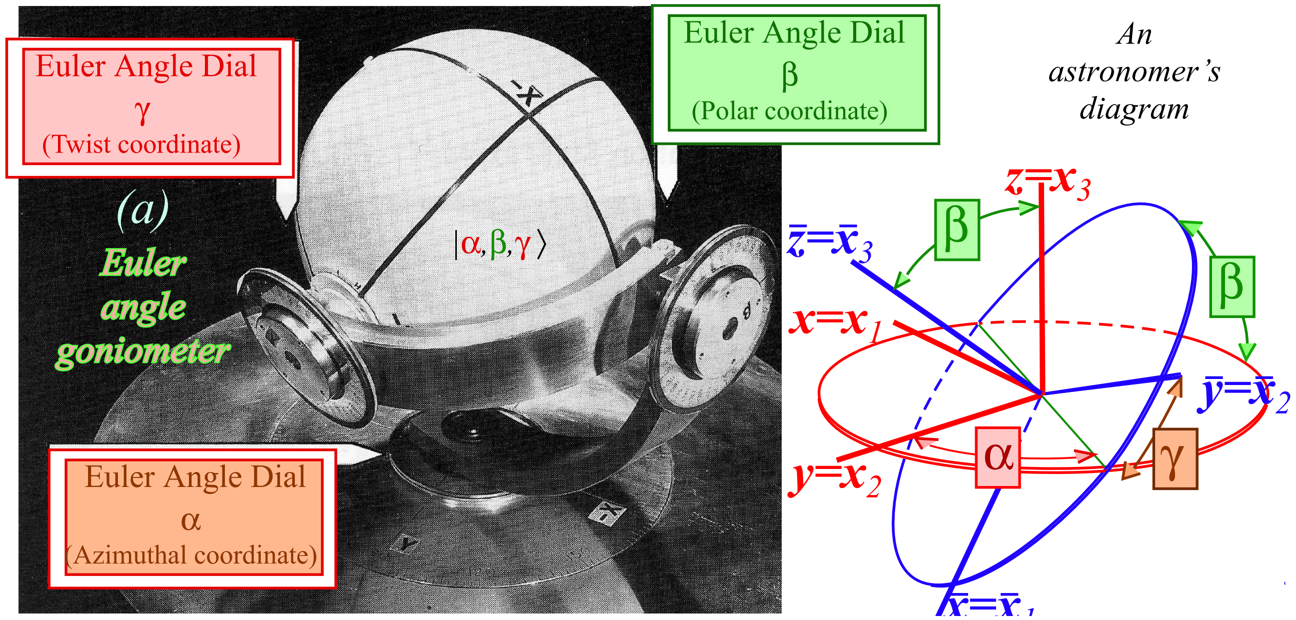




## *Polar coordinates for unit axis vector* $\hat{\boldsymbol{\Theta}}$

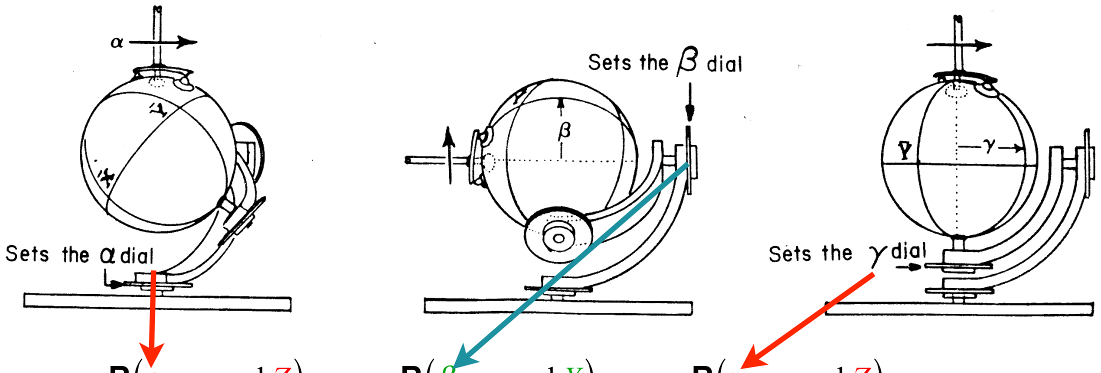
$$\hat{\Theta}_X = \cos\varphi \ \sin\vartheta$$
$$\hat{\Theta}_Y = \sin\varphi \ \sin\vartheta$$
$$\hat{\Theta}_Z = \cos\vartheta$$

# *Euler* **R**(αβγ) versus *Darboux* **R**[φϑΘ]



*An astronomer's diagram*

Third rotation **R**(α00) Second rotation **R**(0β0) First rotation **R**(00γ)

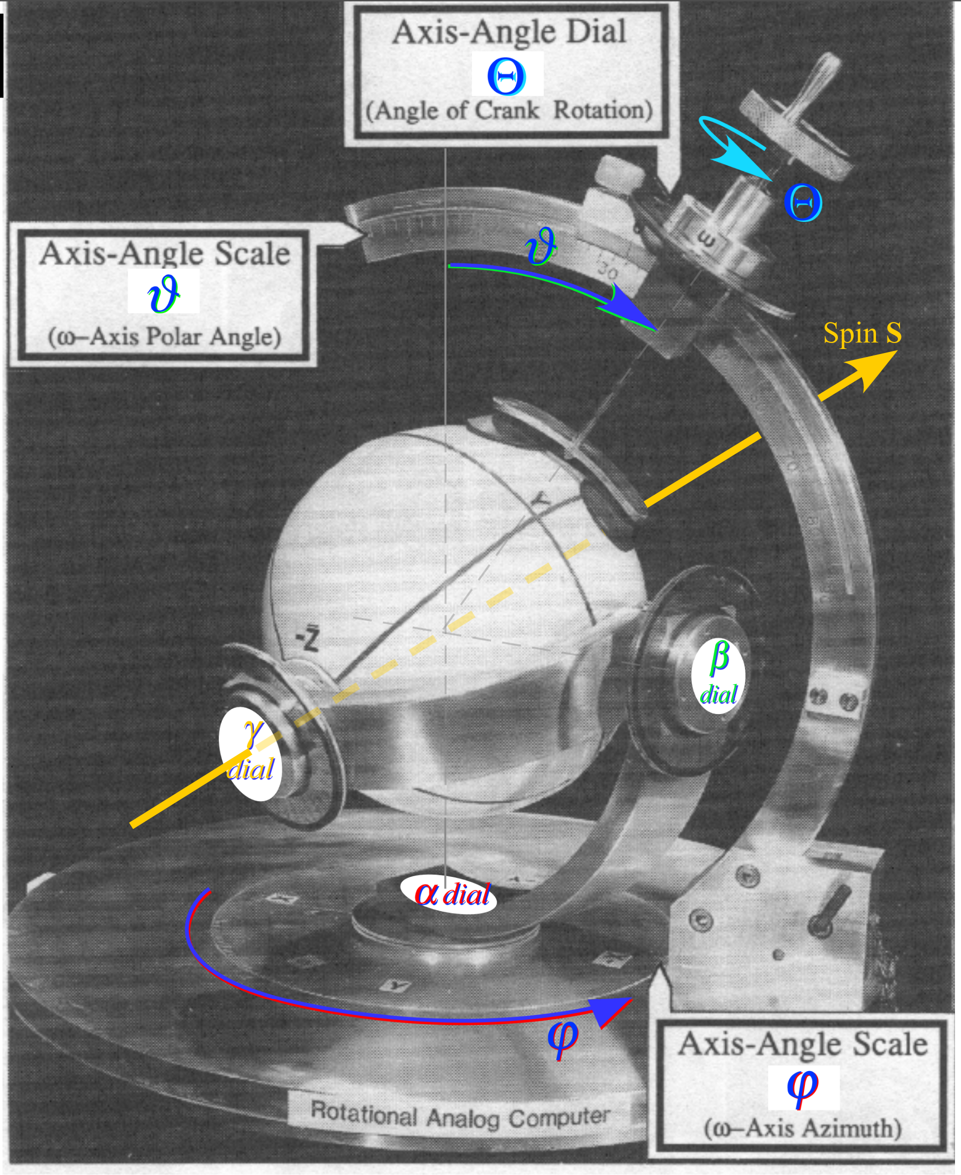


**R**(α around Z) **R**(β around Y) **R**(γ around Z)

$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

*Euler* **R**(αβγ) is simpler to form than Θ-axis *Darboux* **R**[φϑΘ].

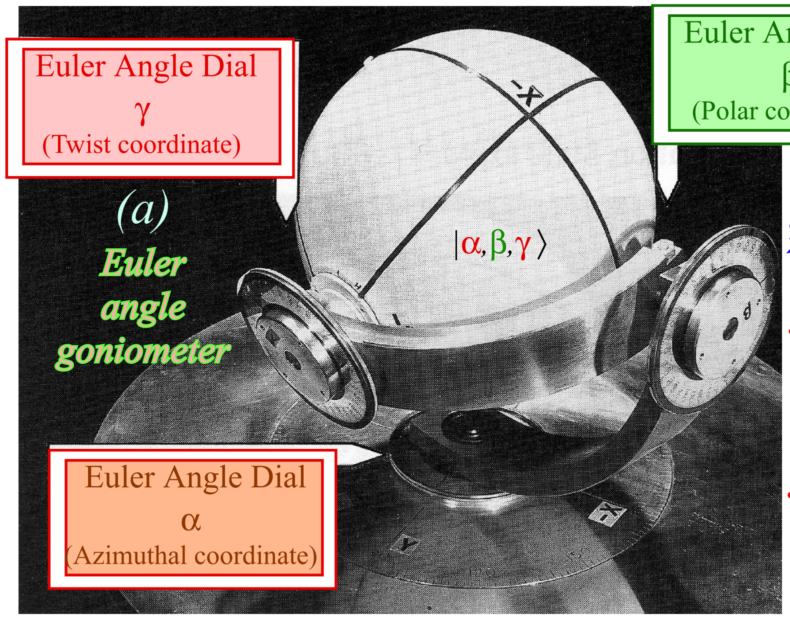


$$\mathbf{R}\left[\vec{\Theta}\right]=\begin{pmatrix} \cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X-i\hat{\Theta}_Y\right) \\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X+i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2} \end{pmatrix}=\mathbf{R}\left[\varphi\vartheta\Theta\right]=e^{-i\mathbf{H}t}$$

$$=\cos\frac{\Theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

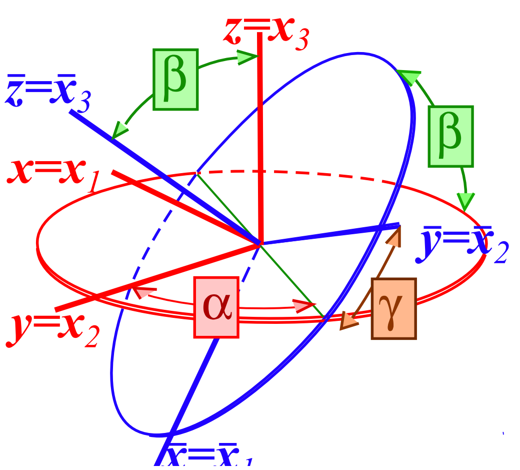
$$=\begin{pmatrix} \cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta\right) \\ \sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta\right) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2} \end{pmatrix}$$

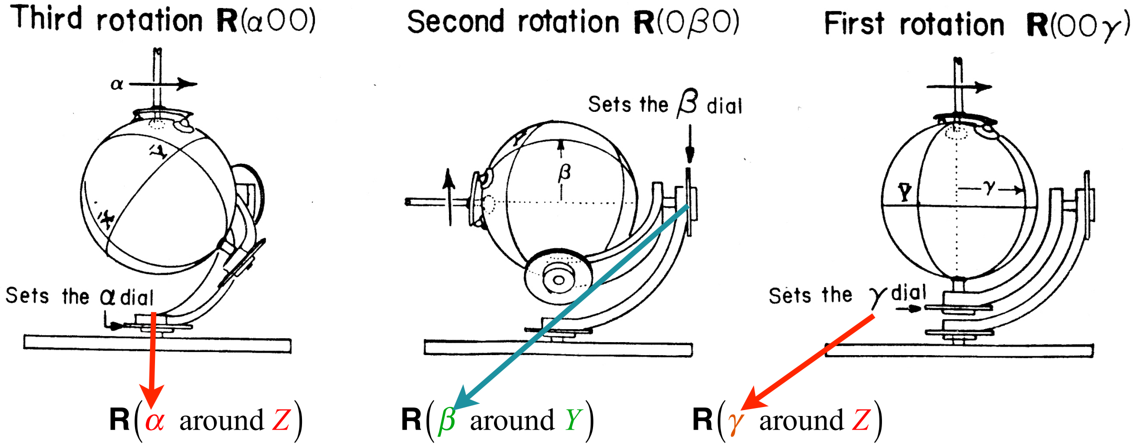
# *Euler* R(αβγ) versus *Darboux* R[φϑΘ]

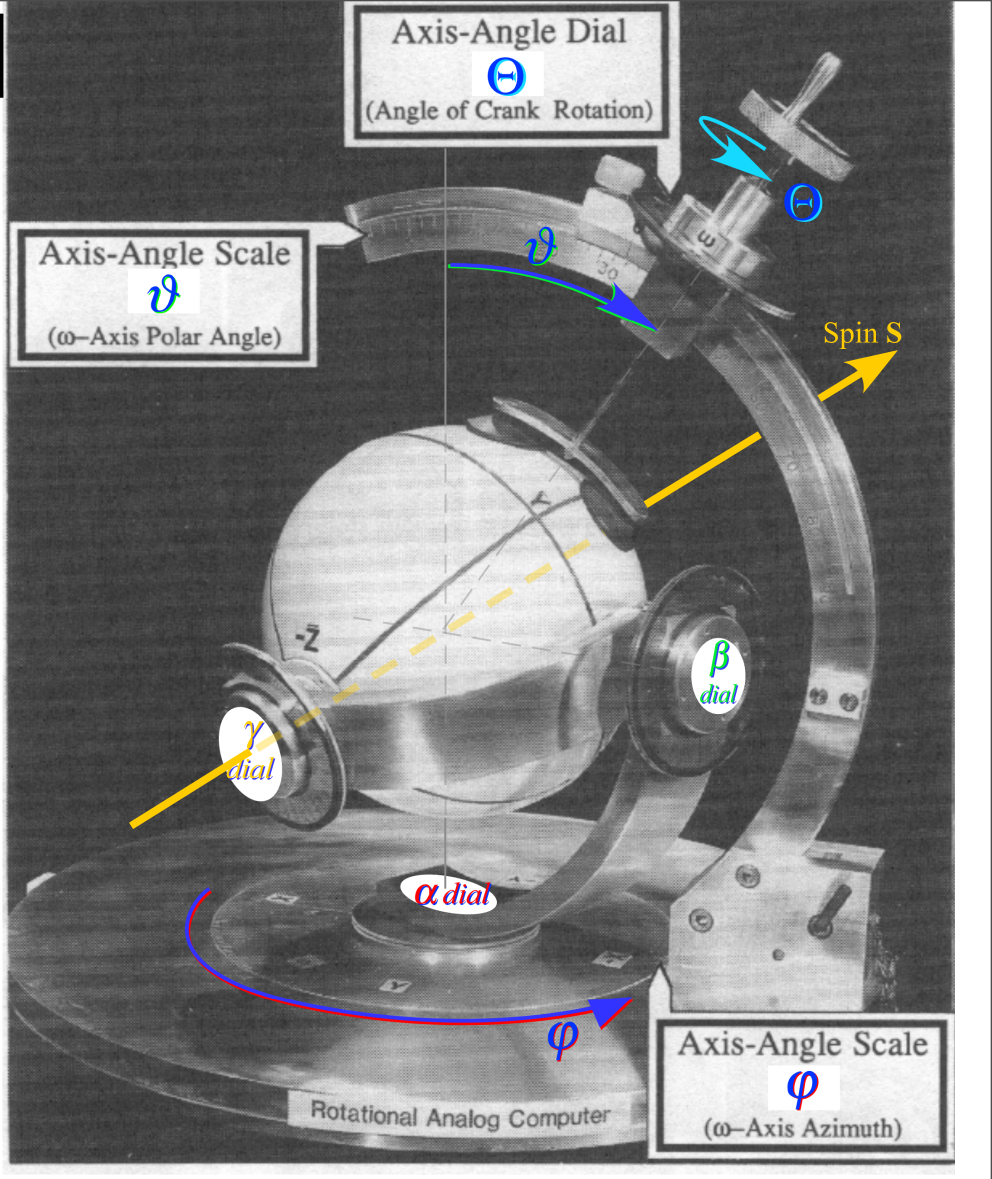


Euler Angle Dial
β
(Polar coordinate)

*An astronomer's diagram*







$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}}\end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}}\end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

$$\mathbf{R}[\vec{\Theta}]=\begin{pmatrix}\cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}(\hat{\Theta}_X-i\hat{\Theta}_Y) \\ -i\sin\frac{\Theta}{2}(\hat{\Theta}_X+i\hat{\Theta}_Y) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2}\end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

$$=\cos\frac{\Theta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$=\begin{pmatrix}\cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta) \\ \sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2}\end{pmatrix}$$

*Euler* **R**(αβγ) is simpler to form than Θ-axis *Darboux* **R**[φϑΘ].

Euler *state definition* lets us relate **R**(αβγ) to **R**[φϑΘ] ...

|αβγ⟩= **R**(αβγ)|000⟩ (αβγ *make better coordinates)*

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}x_1+ip_1\\x_2+ip_2\end{pmatrix}$$

$x_1$ = cos[(γ+α)/2] cosβ/2 = cosΘ/2

$-p_2$ = sin[(γ−α)/2] sinβ/2 = $\hat{\Theta}_X$ sinΘ/2 = cosφ sinϑ sinΘ/2

$x_2$ = cos[(γ−α)/2] sinβ/2 = $\hat{\Theta}_Y$ sinΘ/2 = sinφ sinϑ sinΘ/2

$-p_1$ = sin[(γ+α)/2] cosβ/2 = $\hat{\Theta}_Z$ sinΘ/2 = cosϑ sinΘ/2

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# *Euler* **R**(αβγ) *related to Darboux* **R**[φϑΘ]

Euler *state definition* lets us relate **R**(αβγ) to **R**[φϑΘ] ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ αβγ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$\begin{aligned} x_1 &= \cos[(\gamma+\alpha)/2]\cos\beta/2 = && \cos\Theta/2 \\ -p_2 &= \sin[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_X \sin\Theta/2 = \cos\varphi\sin\vartheta && \sin\Theta/2 \\ x_2 &= \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi\sin\vartheta && \sin\Theta/2 \\ -p_1 &= \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z \sin\Theta/2 = \cos\vartheta && \sin\Theta/2 \end{aligned}$$

# *Euler* **R**$(\alpha\beta\gamma)$ *related to Darboux* **R**$[\varphi\vartheta\Theta]$

Euler *state definition* lets us relate **R**$(\alpha\beta\gamma)$ to **R**$[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ $\alpha\beta\gamma$ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$$

$$-p_2 = \sin[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_X\sin\Theta/2 = \cos\varphi\sin\vartheta\ \sin\Theta/2$$

$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y\sin\Theta/2 = \sin\varphi\sin\vartheta\ \sin\Theta/2$$

$$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z\sin\Theta/2 = \cos\vartheta\ \sin\Theta/2$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2$$

# *Euler* **R**(αβγ) *related to Darboux* **R**[φϑΘ]

Euler *state definition* lets us relate **R**(αβγ) to **R**[φϑΘ] ...

|αβγ⟩= **R**(αβγ)|000⟩    αβγ *make better coordinates but:* **R**(αβγ)|000⟩ = **R**(αβγ) |**1**⟩ = **R**[φϑΘ]|**1**⟩

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$$

$$-p_2 = \sin[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_X \sin\Theta/2 = \cos\varphi\sin\vartheta\,\sin\Theta/2$$

$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi\sin\vartheta\,\sin\Theta/2$$

$$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z \sin\Theta/2 = \cos\vartheta\,\sin\Theta/2$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2$$

$$\tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2}-\varphi]$$

# *Euler* $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ $\alpha\beta\gamma$ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\ |\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \qquad \cos\Theta/2$$

$$-p_2 = \sin[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_X \sin\Theta/2 = \cos\varphi\sin\vartheta\ \sin\Theta/2$$

$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi\sin\vartheta\ \sin\Theta/2$$

$$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z \sin\Theta/2 = \qquad \cos\vartheta\ \sin\Theta/2$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2$$

$$\tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2}-\varphi]$$

$$(\gamma+\alpha)/2 = \tan^{-1}[\cos\vartheta\tan\Theta/2]$$

$$(\gamma-\alpha)/2 = \frac{\pi}{2}-\varphi$$

# *Euler* **R**(αβγ) *related to Darboux* **R**[φϑΘ]

Euler *state definition* lets us relate **R**(αβγ) to **R**[φϑΘ] ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ αβγ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

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$$-p_2 = \sin[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_X \sin\Theta/2 = \cos\varphi\sin\vartheta\ \sin\Theta/2$$

$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi\sin\vartheta\ \sin\Theta/2$$

$$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z \sin\Theta/2 = \cos\vartheta\ \sin\Theta/2$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2$$

$$\tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2}-\varphi]$$

$$(\gamma+\alpha)/2 = \tan^{-1}[\cos\vartheta\tan\Theta/2]$$

$$(\gamma-\alpha)/2 = \frac{\pi}{2}-\varphi$$

$$\sin[(\gamma-\alpha)/2] = \sin[\frac{\pi}{2}-\varphi] = \cos\varphi$$

This gives *Euler angles* (αβγ) in terms of *Darboux angles* [φϑΘ]

# *Euler* **R**$(\alpha\beta\gamma)$ *related to Darboux* **R**$[\varphi\vartheta\Theta]$

Euler *state definition* lets us relate **R**$(\alpha\beta\gamma)$ to **R**$[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ $\alpha\beta\gamma$ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$$

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$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi\sin\vartheta\ \sin\Theta/2$$

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$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2 \qquad \tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2}-\varphi]$$

$$(\gamma+\alpha)/2 = \tan^{-1}[\cos\vartheta\tan\Theta/2] \qquad (\gamma-\alpha)/2 = \frac{\pi}{2}-\varphi$$

$$\sin[(\gamma-\alpha)/2] = \sin[\frac{\pi}{2}-\varphi] = \cos\varphi$$

This gives *Euler angles* $(\alpha\beta\gamma)$ in terms of *Darboux angles* $[\varphi\vartheta\Theta]$

$$\alpha = \varphi - \pi/2 + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

$$\gamma = \pi/2 - \varphi + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

# *Euler* **R**(αβγ) *related to Darboux* **R**[φϑΘ]

Euler *state definition* lets us relate **R**(αβγ) to **R**[φϑΘ] ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ *αβγ make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$$

$$-p_2 = \sin[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_X \sin\Theta/2 = \cos\varphi\sin\vartheta\ \sin\Theta/2$$

$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi\sin\vartheta\ \sin\Theta/2$$

$$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z \sin\Theta/2 = \cos\vartheta\ \sin\Theta/2$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2$$

$$\tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2}-\varphi]$$

$$(\gamma+\alpha)/2 = \tan^{-1}[\cos\vartheta\tan\Theta/2]$$

$$(\gamma-\alpha)/2 = \frac{\pi}{2}-\varphi$$

$$\sin[(\gamma-\alpha)/2] = \sin[\frac{\pi}{2}-\varphi] = \cos\varphi$$

$$\sin\beta/2 = \sin\vartheta\ \sin\Theta/2$$

This gives *Euler angles* (αβγ) in terms of *Darboux angles* [φϑΘ]

$$\alpha = \varphi - \pi/2 + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

$$\beta = 2\sin^{-1}(\sin\Theta/2\ \sin\vartheta)$$

$$\gamma = \pi/2 - \varphi + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

# *Euler* $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ αβγ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \qquad \cos\Theta/2$

$-p_2 = \sin[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_X\,\sin\Theta/2 = \cos\varphi\sin\vartheta\;\sin\Theta/2$

$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y\,\sin\Theta/2 = \sin\varphi\,\sin\vartheta\;\sin\Theta/2$

$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z\,\sin\Theta/2 = \qquad \cos\vartheta\;\sin\Theta/2$

$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2$

$\tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2}-\varphi]$

$(\gamma+\alpha)/2 = \tan^{-1}[\cos\vartheta\tan\Theta/2]$

$(\gamma-\alpha)/2 = \frac{\pi}{2}-\varphi$

$\sin[(\gamma-\alpha)/2] = \sin[\frac{\pi}{2}-\varphi] = \cos\varphi$

$\sin\beta/2 = \sin\vartheta\;\sin\Theta/2$

This gives *Euler angles* (αβγ) in terms of *Darboux angles* [φϑΘ]

$$\alpha = \varphi-\pi/2+\tan^{-1}(\cos\vartheta\tan\Theta/2)$$

$$\beta = 2\sin^{-1}(\sin\Theta/2\;\sin\vartheta)$$

$$\gamma = \pi/2-\varphi+\tan^{-1}(\cos\vartheta\tan\Theta/2)$$

Inverse relations have *Darboux axis angles* [φϑΘ] in terms of *Euler angles* (αβγ)

$$\varphi = (\alpha-\gamma+\pi)/2$$

$\cos[(\gamma-\alpha)/2] = \cos[\frac{\pi}{2}-\varphi] = \sin\varphi$

# *Euler* **R(αβγ)** *related to Darboux* **R[φϑΘ]**

Euler *state definition* lets us relate **R**(αβγ) to **R**[φϑΘ] ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ αβγ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$$

$$-p_2 = \sin[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_X \sin\Theta/2 = \cos\varphi\sin\vartheta\ \sin\Theta/2$$

$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi\sin\vartheta\ \sin\Theta/2$$

$$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z \sin\Theta/2 = \cos\vartheta\ \sin\Theta/2$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2$$

$$\tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2}-\varphi]$$

$$(\gamma+\alpha)/2 = \tan^{-1}[\cos\vartheta\tan\Theta/2]$$

$$(\gamma-\alpha)/2 = \frac{\pi}{2}-\varphi$$

$$\sin[(\gamma-\alpha)/2] = \sin[\frac{\pi}{2}-\varphi] = \cos\varphi$$

$$\sin\beta/2 = \sin\vartheta\ \sin\Theta/2$$

This gives *Euler angles* (αβγ) in terms of *Darboux angles* [φϑΘ]

$$\alpha = \varphi - \pi/2 + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

$$\beta = 2\sin^{-1}(\sin\Theta/2\ \sin\vartheta)$$

$$\gamma = \pi/2 - \varphi + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

Inverse relations have *Darboux axis angles* [φϑΘ] in terms of *Euler angles* (αβγ)

$$\varphi = (\alpha-\gamma+\pi)/2$$

$$\vartheta = \tan^{-1}[\tan\beta/2/\sin(\alpha+\gamma)/2]$$

$$\cos[(\gamma-\alpha)/2] = \cos[\frac{\pi}{2}-\varphi] = \sin\varphi$$

$$\frac{\cos[(\gamma-\alpha)/2]\sin\beta/2}{\sin[(\gamma+\alpha)/2]\cos\beta/2} = \sin\varphi\tan\vartheta \Rightarrow \frac{\tan\beta/2}{\sin[(\gamma+\alpha)/2]} = \tan\vartheta$$

# *Euler* **R**(αβγ) *related to Darboux* **R**[φϑΘ]

Euler *state definition* lets us relate **R**(αβγ) to **R**[φϑΘ] ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ αβγ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \qquad \cos\Theta/2$$

$$-p_2 = \sin[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_X \sin\Theta/2 = \cos\varphi\sin\vartheta\ \sin\Theta/2$$

$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi\sin\vartheta\ \sin\Theta/2$$

$$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z \sin\Theta/2 = \qquad \cos\vartheta\ \sin\Theta/2$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2$$

$$\tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2}-\varphi]$$

$$(\gamma+\alpha)/2 = \tan^{-1}[\cos\vartheta\tan\Theta/2]$$

$$(\gamma-\alpha)/2 = \frac{\pi}{2}-\varphi$$

$$\sin[(\gamma-\alpha)/2] = \sin[\frac{\pi}{2}-\varphi] = \cos\varphi$$

$$\sin\beta/2 = \sin\vartheta\ \sin\Theta/2$$

This gives *Euler angles* (αβγ) in terms of *Darboux angles* [φϑΘ]

$$\alpha = \varphi-\pi/2+\tan^{-1}(\cos\vartheta\tan\Theta/2)$$

$$\beta = 2\sin^{-1}(\sin\Theta/2\ \sin\vartheta)$$

$$\gamma = \pi/2-\varphi+\tan^{-1}(\cos\vartheta\tan\Theta/2)$$

Inverse relations have *Darboux axis angles* [φϑΘ] in terms of *Euler angles* (αβγ)

$$\varphi = (\alpha-\gamma+\pi)/2$$

$$\vartheta = \tan^{-1}[\tan\beta/2/\ \sin(\alpha+\gamma)/2]$$

$$\Theta = 2\cos^{-1}[\cos\beta/2\ \cos(\alpha+\gamma)/2]$$

$$\cos[(\gamma-\alpha)/2] = \cos[\frac{\pi}{2}-\varphi] = \sin\varphi$$

$$\frac{\cos[(\gamma-\alpha)/2]\sin\beta/2}{\sin[(\gamma+\alpha)/2]\cos\beta/2} = \sin\varphi\tan\vartheta \Rightarrow \frac{\tan\beta/2}{\sin[(\gamma+\alpha)/2]} = \tan\vartheta$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \qquad \cos\Theta/2$$

# *Euler* $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ $\alpha\beta\gamma$ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix}1\\ \\ 0\end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ \\ x_2+ip_2\end{pmatrix}$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \qquad \cos\Theta/2$$

$$-p_2 = \sin[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_X\sin\Theta/2 = \cos\varphi\sin\vartheta\ \sin\Theta/2$$

$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y\sin\Theta/2 = \sin\varphi\sin\vartheta\ \sin\Theta/2$$

$$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z\sin\Theta/2 = \qquad \cos\vartheta\ \sin\Theta/2$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2$$

$$\tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2}-\varphi]$$

$$(\gamma+\alpha)/2 = \tan^{-1}[\cos\vartheta\tan\Theta/2]$$

$$(\gamma-\alpha)/2 = \frac{\pi}{2}-\varphi$$

$$\sin[(\gamma-\alpha)/2] = \sin[\frac{\pi}{2}-\varphi] = \cos\varphi$$

$$\sin\beta/2 = \sin\vartheta\ \sin\Theta/2$$

This gives *Euler angles* $(\alpha\beta\gamma)$ in terms of *Darboux angles* $[\varphi\vartheta\Theta]$

$$\alpha = \varphi-\pi/2+\tan^{-1}(\cos\vartheta\tan\Theta/2)$$

$$\beta = 2\sin^{-1}(\sin\Theta/2\ \sin\vartheta)$$

$$\gamma = \pi/2-\varphi+\tan^{-1}(\cos\vartheta\tan\Theta/2)$$

Inverse relations have *Darboux axis angles* $[\varphi\vartheta\Theta]$ in terms of *Euler angles* $(\alpha\beta\gamma)$

$$\varphi = (\alpha-\gamma+\pi)/2$$

$$\vartheta = \tan^{-1}[\tan\beta/2/\ \sin(\alpha+\gamma)/2]$$

$$\Theta = 2\cos^{-1}[\cos\beta/2\ \cos(\alpha+\gamma)/2]$$

$$\cos[(\gamma-\alpha)/2] = \cos[\frac{\pi}{2}-\varphi] = \sin\varphi$$

$$\frac{\cos[(\gamma-\alpha)/2]\sin\beta/2}{\sin[(\gamma+\alpha)/2]\cos\beta/2} = \sin\varphi\tan\vartheta \Rightarrow \frac{\tan\beta/2}{\sin[(\gamma+\alpha)/2]} = \tan\vartheta$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \qquad \cos\Theta/2$$

Example: *Euler angles* ($\alpha$=50° $\beta$=60° $\gamma$=70°)

$$\varphi = \quad (50°-70° +180°)/2 \qquad = 80°$$

$$\vartheta = \quad \tan^{-1}[\tan 60°/2/\ \sin(50°+\gamma)/2] \qquad = 33.7°$$

$$\Theta = 2\cos^{-1}[\cos 60°/2\ \cos(50°+\gamma)/2] \qquad = 128.7°$$

# *Euler* **R**(αβγ) *related to Darboux* **R**[φϑΘ]

Euler *state definition* lets us relate **R**(αβγ) to **R**[φϑΘ] ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ αβγ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$$

$$-p_2 = \sin[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_X \sin\Theta/2 = \cos\varphi\sin\vartheta\ \sin\Theta/2$$

$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi\sin\vartheta\ \sin\Theta/2$$

$$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z \sin\Theta/2 = \cos\vartheta\ \sin\Theta/2$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2 \qquad \tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2}-\varphi]$$

$$(\gamma+\alpha)/2 = \tan^{-1}[\cos\vartheta\tan\Theta/2] \qquad (\gamma-\alpha)/2 = \frac{\pi}{2}-\varphi$$

$$\sin[(\gamma-\alpha)/2] = \sin[\frac{\pi}{2}-\varphi] = \cos\varphi$$

$$\sin\beta/2 = \sin\vartheta\ \sin\Theta/2$$

This gives *Euler angles* (αβγ) in terms of *Darboux angles* [φϑΘ]

$$\alpha = \varphi - \pi/2 + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

$$\beta = 2\sin^{-1}(\sin\Theta/2\ \sin\vartheta)$$

$$\gamma = \pi/2 - \varphi + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

Inverse relations have *Darboux axis angles* [φϑΘ] in terms of *Euler angles* (αβγ)

$$\varphi = (\alpha - \gamma + \pi)/2$$

$$\vartheta = \tan^{-1}[\tan\beta/2/\ \sin(\alpha+\gamma)/2]$$

$$\Theta = 2\cos^{-1}[\cos\beta/2\cos(\alpha+\gamma)/2]$$

$$\cos[(\gamma-\alpha)/2] = \cos[\frac{\pi}{2}-\varphi] = \sin\varphi$$

$$\frac{\cos[(\gamma-\alpha)/2]\sin\beta/2}{\sin[(\gamma+\alpha)/2]\cos\beta/2} = \sin\varphi\tan\vartheta \Rightarrow \frac{\tan\beta/2}{\sin[(\gamma+\alpha)/2]} = \tan\vartheta$$

Example: *Euler angles* (α=50° β=60° γ=70°)

$$\varphi = (50° - 70° + 180°)/2 = 80°$$

$$\vartheta = \tan^{-1}[\tan 60°/2/\ \sin(50°+\gamma)/2] = 33.7°$$

$$\Theta = 2\cos^{-1}[\cos 60°/2\cos(50°+\gamma)/2] = 128.7°$$

Reverse check: (αβγ) in terms of [φϑΘ]

$$\alpha = 80° - 90° + \tan^{-1}(\tan(128.7°/2)\cos 33.7°) = 50.007°$$

$$\beta = 2\sin^{-1}(\sin 128.7°/2\ \sin 33.7°) = 60.022°$$

$$\gamma = \pi/2 - 128.7° + \tan^{-1}(\tan(128.7°/2) = 70.007°$$

*Review: Fundamental Euler* $\mathbf{R}(\alpha\beta\gamma)$ *and Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

- *Euler* $\mathbf{R}(\alpha\beta\gamma)$ *derived from Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *and vice versa*
- *Euler* $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*
- *R(3)-U(2) slide rule for converting* $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$ *and Sundial*

*U(2) density operator approach to symmetry dynamics*

- *Bloch equation for density operator*

*The ABC's of U(2) dynamics-Archetypes*

- *Asymmetric-Diagonal A-Type motion*
- *Bilateral-Balanced B-Type motion*
- *Circular-Coriolis… C-Type motion*

*The ABC's of U(2) dynamics-Mixed modes*

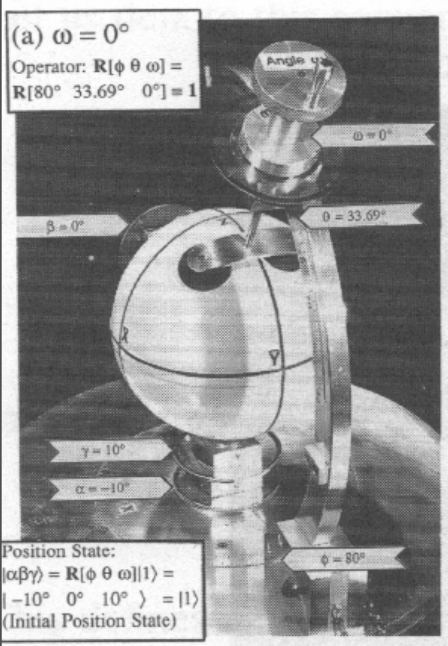
- *AB-Type motion and Wigner's Avoided-Symmetry-Crossings*
- *ABC-Type elliptical polarized motion*

*Ellipsometry using U(2) symmetry coordinates*

- *Conventional amp-phase ellipse coordinates*
- *Euler Angle* $(\alpha\beta\gamma)$ *ellipse coordinates*

# *Euler* $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*

$\Theta=0^\circ$



(a) $\omega = 0^\circ$

Operator: $\mathbf{R}[\phi\ \theta\ \omega] = \mathbf{R}[80^\circ\ \ 33.69^\circ\ \ 0^\circ] = 1$

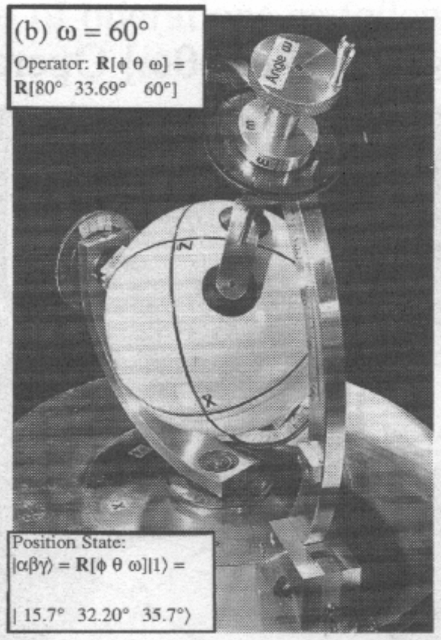
Position State:

$|\alpha\beta\gamma\rangle = \mathbf{R}[\phi\ \theta\ \omega]|1\rangle =$

$|-10^\circ\ \ 0^\circ\ \ 10^\circ\ \rangle\ = |1\rangle$

(Initial Position State)

$\Theta=60^\circ$



(b) $\omega = 60^\circ$

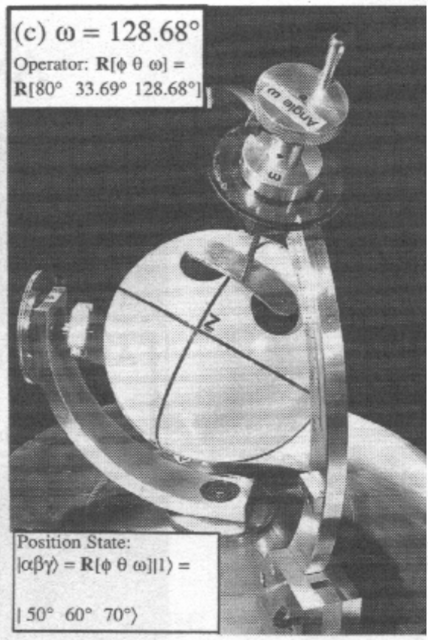
Operator: $\mathbf{R}[\phi\ \theta\ \omega] = \mathbf{R}[80^\circ\ \ 33.69^\circ\ \ 60^\circ]$

Position State:

$|\alpha\beta\gamma\rangle = \mathbf{R}[\phi\ \theta\ \omega]|1\rangle =$

$|\ 15.7^\circ\ \ 32.20^\circ\ \ 35.7^\circ\rangle$

$\Theta=128.7^\circ$



(c) $\omega = 128.68^\circ$

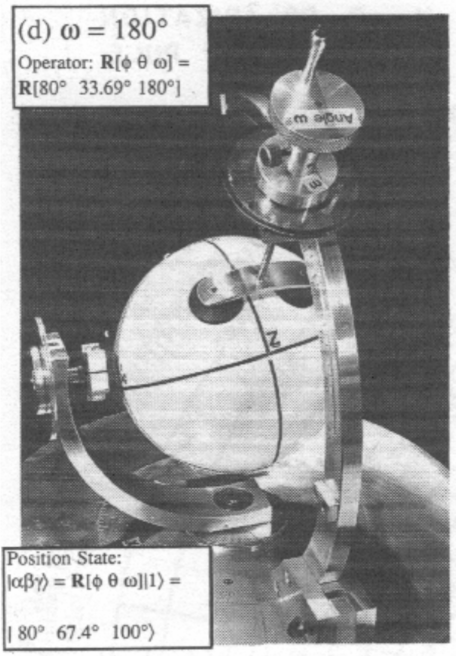
Operator: $\mathbf{R}[\phi\ \theta\ \omega] = \mathbf{R}[80^\circ\ \ 33.69^\circ\ 128.68^\circ]$

Position State:

$|\alpha\beta\gamma\rangle = \mathbf{R}[\phi\ \theta\ \omega]|1\rangle =$

$|\ 50^\circ\ \ 60^\circ\ \ 70^\circ\rangle$

$\Theta=180^\circ$



(d) $\omega = 180^\circ$

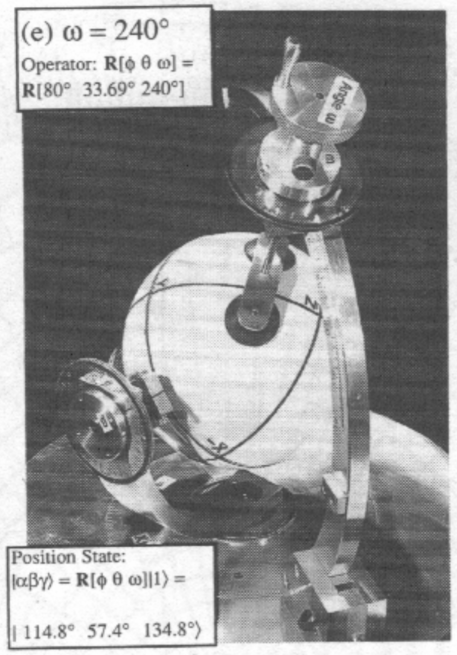
Operator: $\mathbf{R}[\phi\ \theta\ \omega] = \mathbf{R}[80^\circ\ \ 33.69^\circ\ 180^\circ]$

Position State:

$|\alpha\beta\gamma\rangle = \mathbf{R}[\phi\ \theta\ \omega]|1\rangle =$

$|\ 80^\circ\ \ 67.4^\circ\ \ 100^\circ\rangle$

$\Theta=240^\circ$



(e) $\omega = 240^\circ$

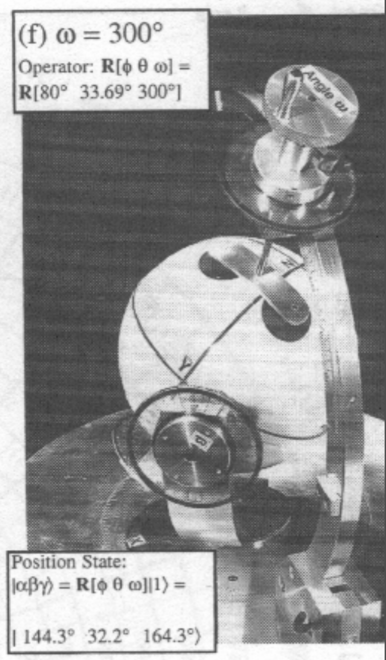
Operator: $\mathbf{R}[\phi\ \theta\ \omega] = \mathbf{R}[80^\circ\ \ 33.69^\circ\ 240^\circ]$

Position State:

$|\alpha\beta\gamma\rangle = \mathbf{R}[\phi\ \theta\ \omega]|1\rangle =$

$|\ 114.8^\circ\ \ 57.4^\circ\ \ 134.8^\circ\rangle$

$\Theta=300^\circ$



(f) $\omega = 300^\circ$

Operator: $\mathbf{R}[\phi\ \theta\ \omega] = \mathbf{R}[80^\circ\ \ 33.69^\circ\ 300^\circ]$

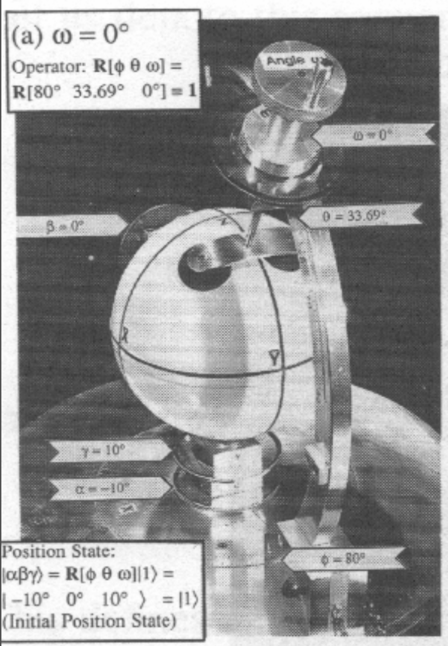
Position State:

$|\alpha\beta\gamma\rangle = \mathbf{R}[\phi\ \theta\ \omega]|1\rangle =$

$|\ 144.3^\circ\ \ 32.2^\circ\ \ 164.3^\circ\rangle$

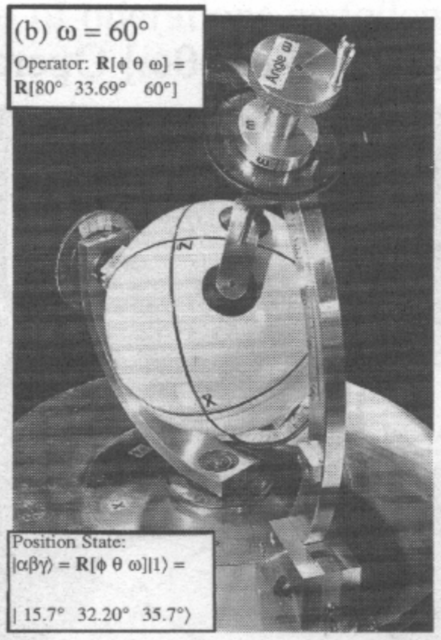
# Euler $\mathbf{R}(\alpha\beta\gamma)$ rotation $\Theta=0-4\pi$-sequence $[\varphi\vartheta]$ fixed

$\Theta=0°$


(a) $\omega = 0°$
Operator: $\mathbf{R}[\phi\ \theta\ \omega] = \mathbf{R}[80°\ \ 33.69°\ \ 0°] = \mathbf{1}$

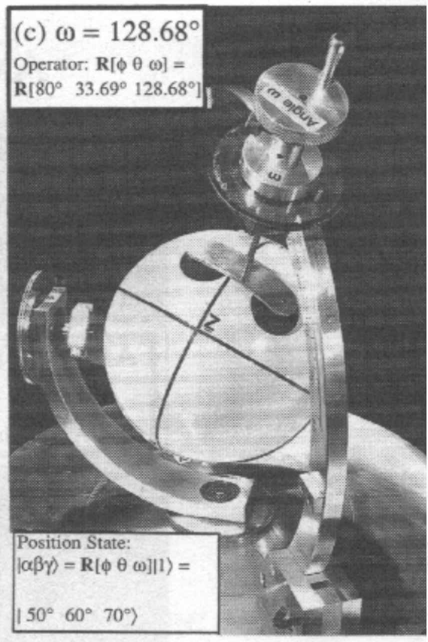
Position State:
$|\alpha\beta\gamma\rangle = \mathbf{R}[\phi\ \theta\ \omega]|1\rangle =$
$|-10°\ \ 0°\ \ 10°\ \rangle = |1\rangle$
(Initial Position State)

$\Theta=60°$


(b) $\omega = 60°$
Operator: $\mathbf{R}[\phi\ \theta\ \omega] = \mathbf{R}[80°\ \ 33.69°\ \ 60°]$

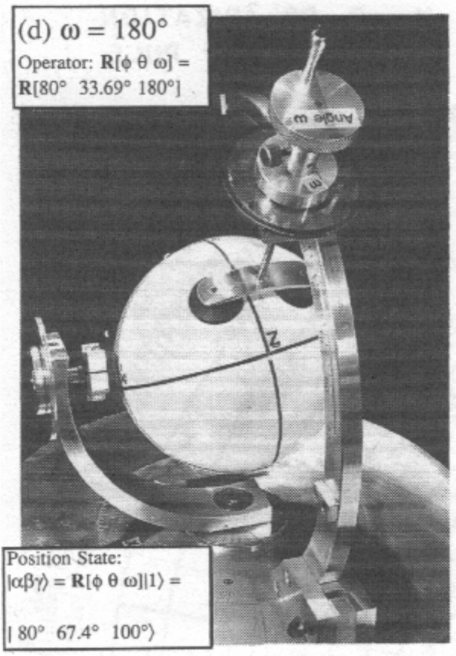
Position State:
$|\alpha\beta\gamma\rangle = \mathbf{R}[\phi\ \theta\ \omega]|1\rangle =$
$|\ 15.7°\ \ 32.20°\ \ 35.7°\rangle$

$\Theta=128.7°$


(c) $\omega = 128.68°$
Operator: $\mathbf{R}[\phi\ \theta\ \omega] = \mathbf{R}[80°\ \ 33.69°\ 128.68°]$

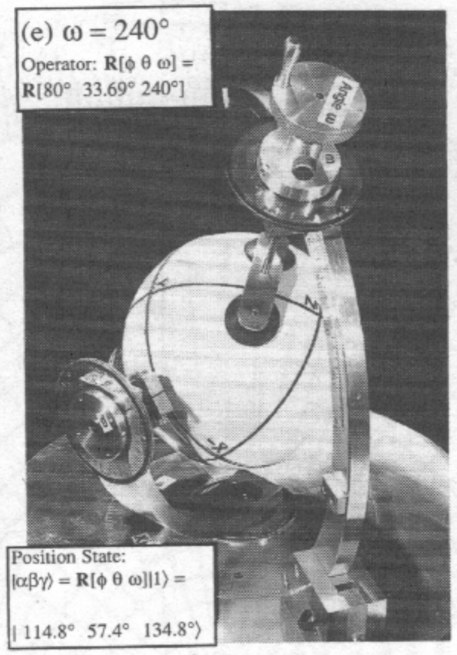
Position State:
$|\alpha\beta\gamma\rangle = \mathbf{R}[\phi\ \theta\ \omega]|1\rangle =$
$|\ 50°\ \ 60°\ \ 70°\rangle$

$\Theta=180°$


(d) $\omega = 180°$
Operator: $\mathbf{R}[\phi\ \theta\ \omega] = \mathbf{R}[80°\ \ 33.69°\ 180°]$

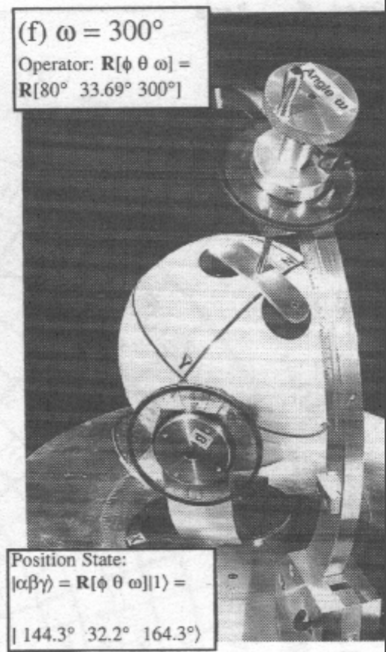
Position State:
$|\alpha\beta\gamma\rangle = \mathbf{R}[\phi\ \theta\ \omega]|1\rangle =$
$|\ 80°\ \ 67.4°\ \ 100°\rangle$

$\Theta=240°$

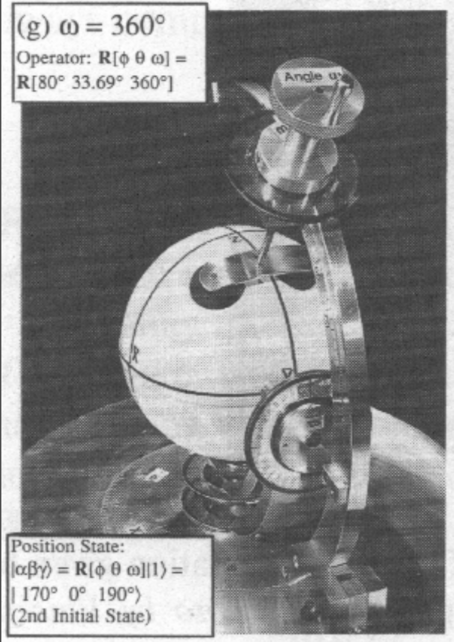

(e) $\omega = 240°$
Operator: $\mathbf{R}[\phi\ \theta\ \omega] = \mathbf{R}[80°\ \ 33.69°\ 240°]$

Position State:
$|\alpha\beta\gamma\rangle = \mathbf{R}[\phi\ \theta\ \omega]|1\rangle =$
$|\ 114.8°\ \ 57.4°\ \ 134.8°\rangle$

$\Theta=300°$

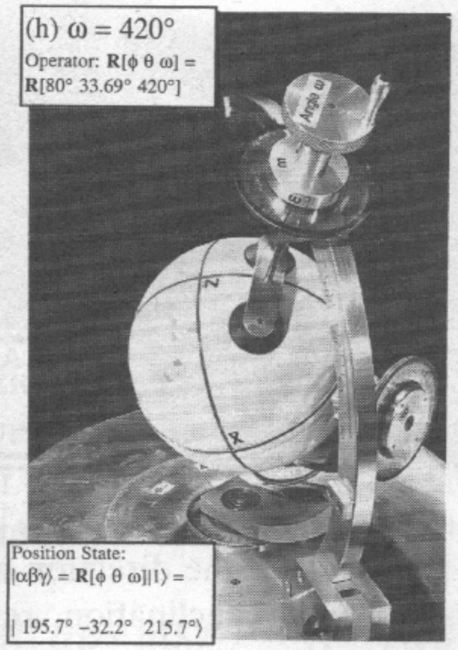

(f) $\omega = 300°$
Operator: $\mathbf{R}[\phi\ \theta\ \omega] = \mathbf{R}[80°\ \ 33.69°\ 300°]$

Position State:
$|\alpha\beta\gamma\rangle = \mathbf{R}[\phi\ \theta\ \omega]|1\rangle =$
$|\ 144.3°\ \ 32.2°\ \ 164.3°\rangle$


(g) $\omega = 360°$
Operator: $\mathbf{R}[\phi\ \theta\ \omega] = \mathbf{R}[80°\ 33.69°\ 360°]$

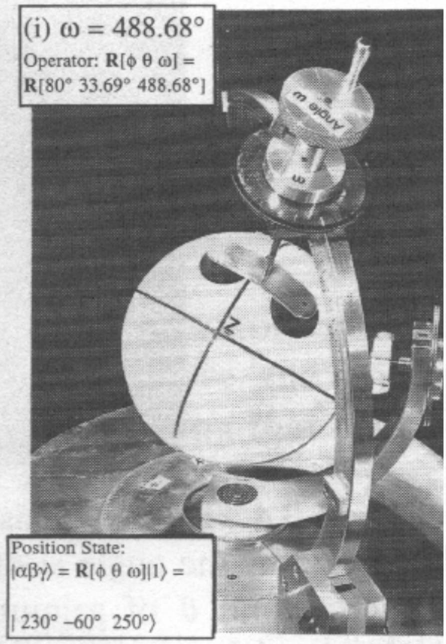
Position State:
$|\alpha\beta\gamma\rangle = \mathbf{R}[\phi\ \theta\ \omega]|1\rangle =$
$|\ 170°\ \ 0°\ \ 190°\rangle$
(2nd Initial State)

$\Theta=360°$


(h) $\omega = 420°$
Operator: $\mathbf{R}[\phi\ \theta\ \omega] = \mathbf{R}[80°\ 33.69°\ 420°]$

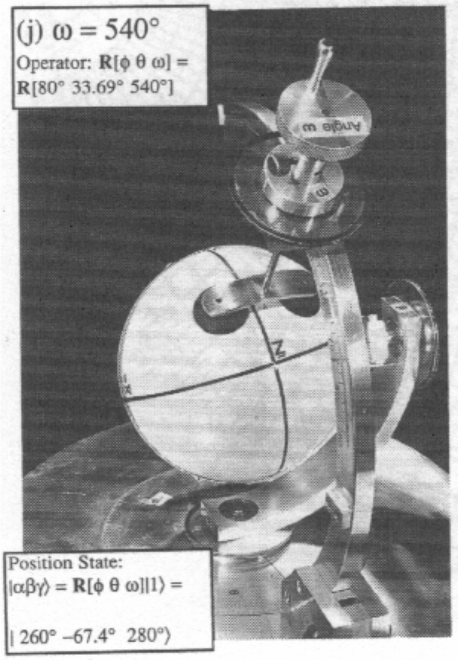
Position State:
$|\alpha\beta\gamma\rangle = \mathbf{R}[\phi\ \theta\ \omega]|1\rangle =$
$|\ 195.7°\ -32.2°\ \ 215.7°\rangle$

$\Theta=420°$


(i) $\omega = 488.68°$
Operator: $\mathbf{R}[\phi\ \theta\ \omega] = \mathbf{R}[80°\ 33.69°\ 488.68°]$

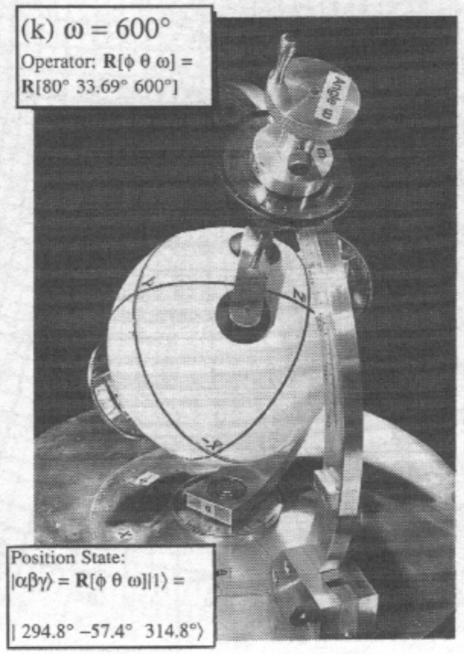
Position State:
$|\alpha\beta\gamma\rangle = \mathbf{R}[\phi\ \theta\ \omega]|1\rangle =$
$|\ 230°\ -60°\ \ 250°\rangle$

$\Theta=488.7°$


(j) $\omega = 540°$
Operator: $\mathbf{R}[\phi\ \theta\ \omega] = \mathbf{R}[80°\ 33.69°\ 540°]$

Position State:
$|\alpha\beta\gamma\rangle = \mathbf{R}[\phi\ \theta\ \omega]|1\rangle =$
$|\ 260°\ -67.4°\ \ 280°\rangle$

$\Theta=540°$


(k) $\omega = 600°$
Operator: $\mathbf{R}[\phi\ \theta\ \omega] = \mathbf{R}[80°\ 33.69°\ 600°]$

Position State:
$|\alpha\beta\gamma\rangle = \mathbf{R}[\phi\ \theta\ \omega]|1\rangle =$
$|\ 294.8°\ -57.4°\ \ 314.8°\rangle$

$\Theta=600°$


(l) $\omega = 660°$
Operator: $\mathbf{R}[\phi\ \theta\ \omega] = \mathbf{R}[80°\ 33.69°\ 660°]$

Position State:
$|\alpha\beta\gamma\rangle = \mathbf{R}[\phi\ \theta\ \omega]|1\rangle =$
$|\ 324.3°\ -32.2°\ \ 344.3°\rangle$

$\Theta=660°$

*Review: Fundamental Euler* **R***(αβγ) and Darboux* **R**[φϑΘ] *representations of U(2) and R(3)*

- *Euler* **R***(αβγ) derived from Darboux* **R**[φϑΘ] *and vice versa*
- *Euler* **R***(αβγ) rotation* Θ=0−4π*-sequence* [φϑ] *fixed*
- ➡ *R(3)-U(2) slide rule for converting* **R***(αβγ)* ↔ **R**[φϑΘ] *and Sundial*

*U(2) density operator approach to symmetry dynamics*

- *Bloch equation for density operator*

*The ABC's of U(2) dynamics-Archetypes*

- *Asymmetric-Diagonal A-Type motion*
- *Bilateral-Balanced B-Type motion*
- *Circular-Coriolis… C-Type motion*

*The ABC's of U(2) dynamics-Mixed modes*

- *AB-Type motion and Wigner's Avoided-Symmetry-Crossings*
- *ABC-Type elliptical polarized motion*

*Ellipsometry using U(2) symmetry coordinates*

- *Conventional amp-phase ellipse coordinates*
- *Euler Angle (αβγ) ellipse coordinates*

# *R(3)-U(2) slide rule for converting* **R**$(\alpha\beta\gamma)$ ↔ **R**$[\varphi\vartheta\Theta]$

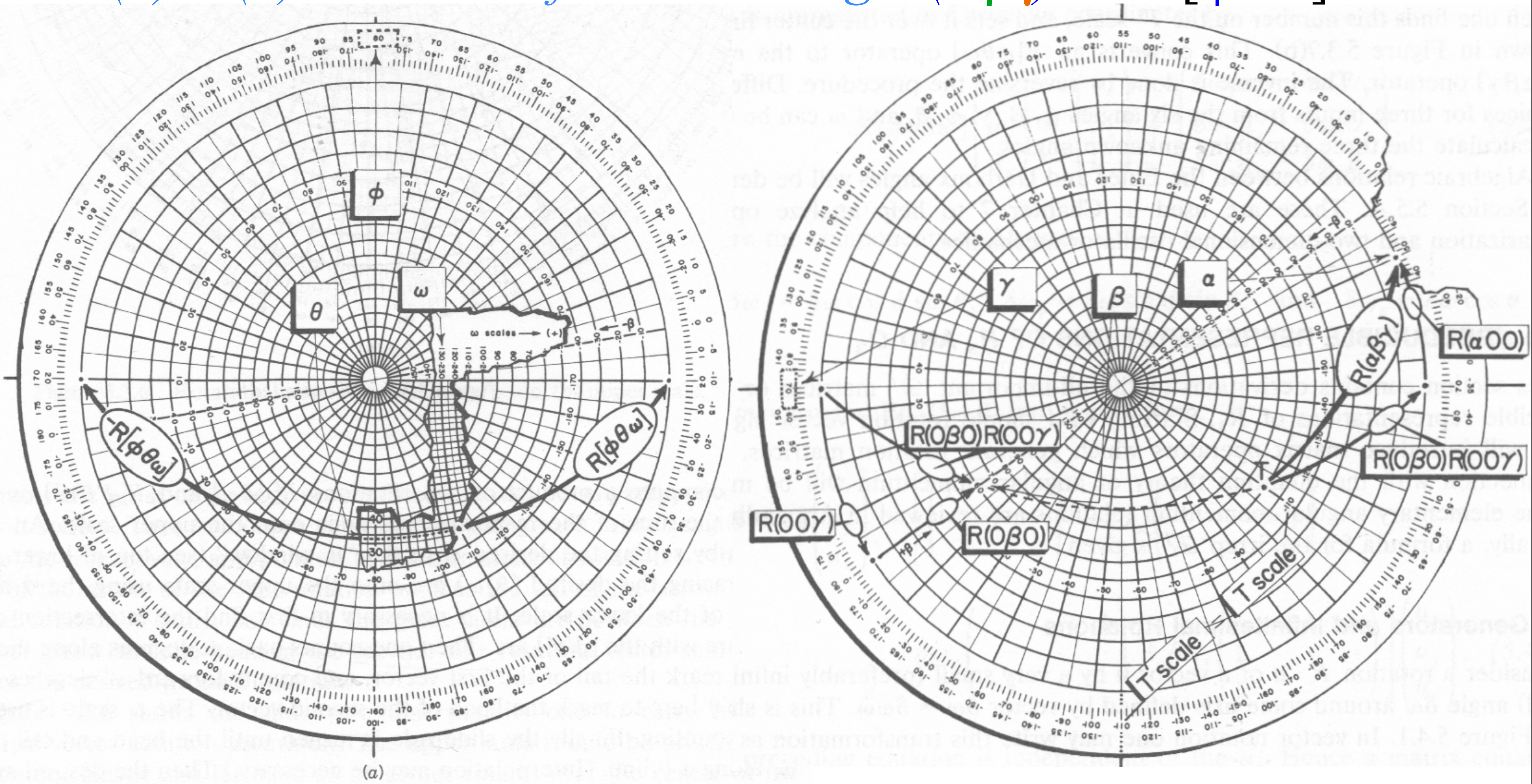
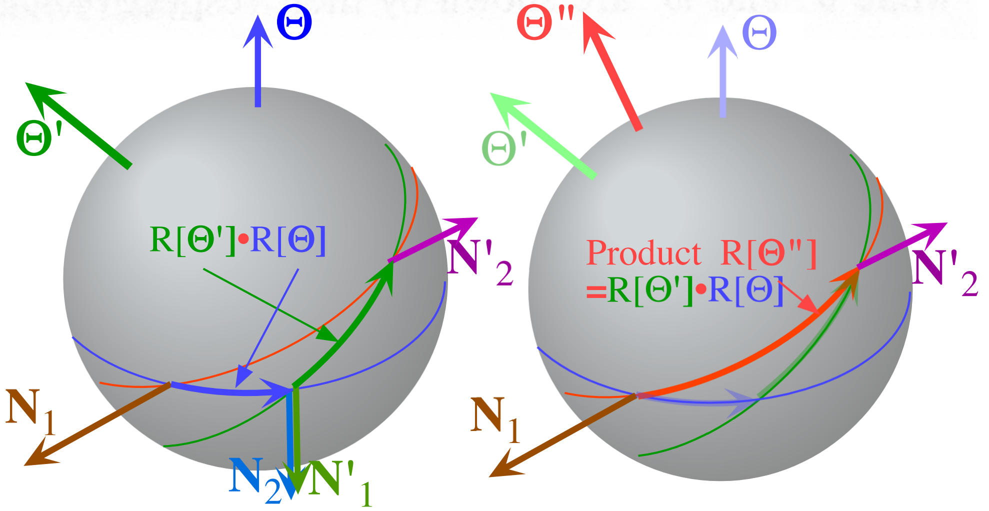
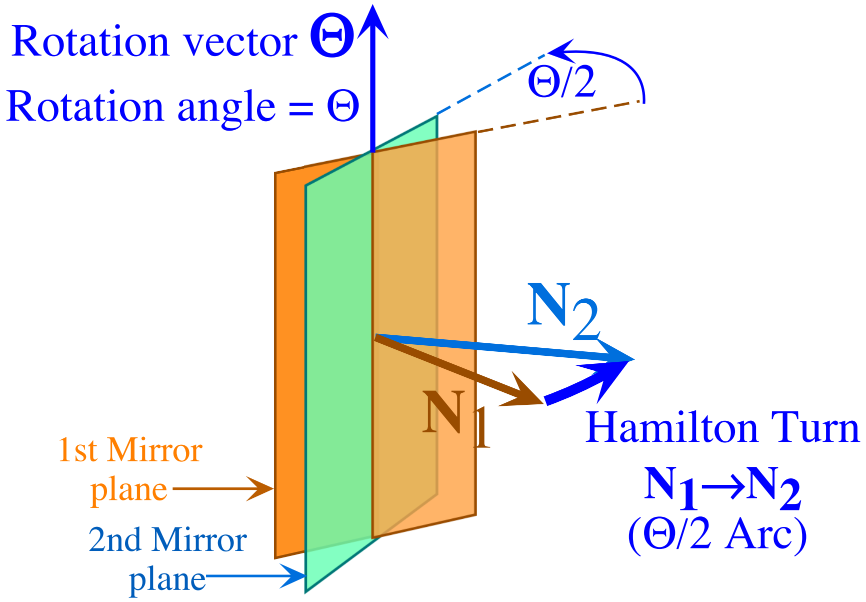
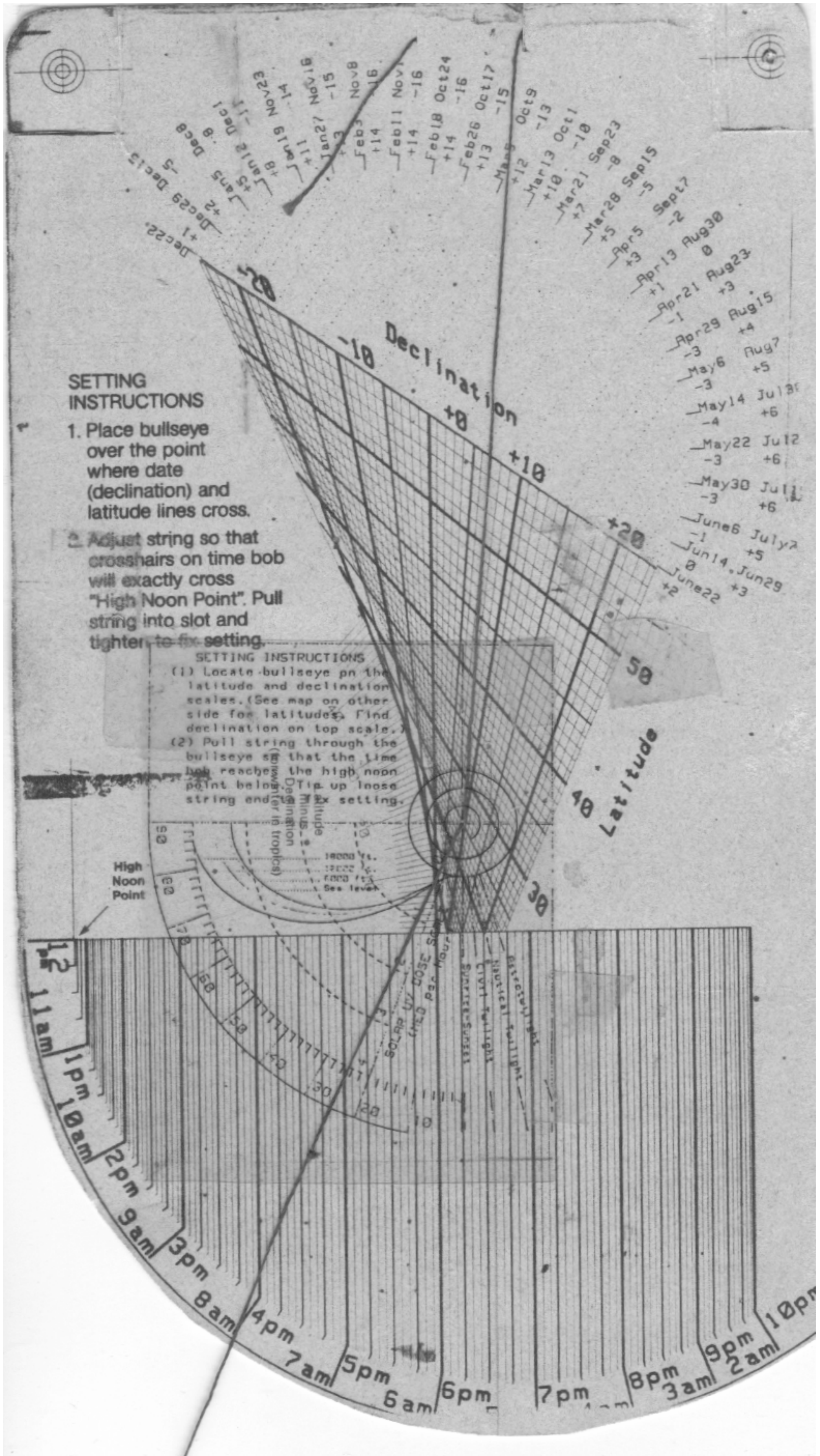


**Figure 5.3.7** Setting the rotational slide rule. (a) Darboux or axis angles. (b) Euler angles.







SETTING INSTRUCTIONS

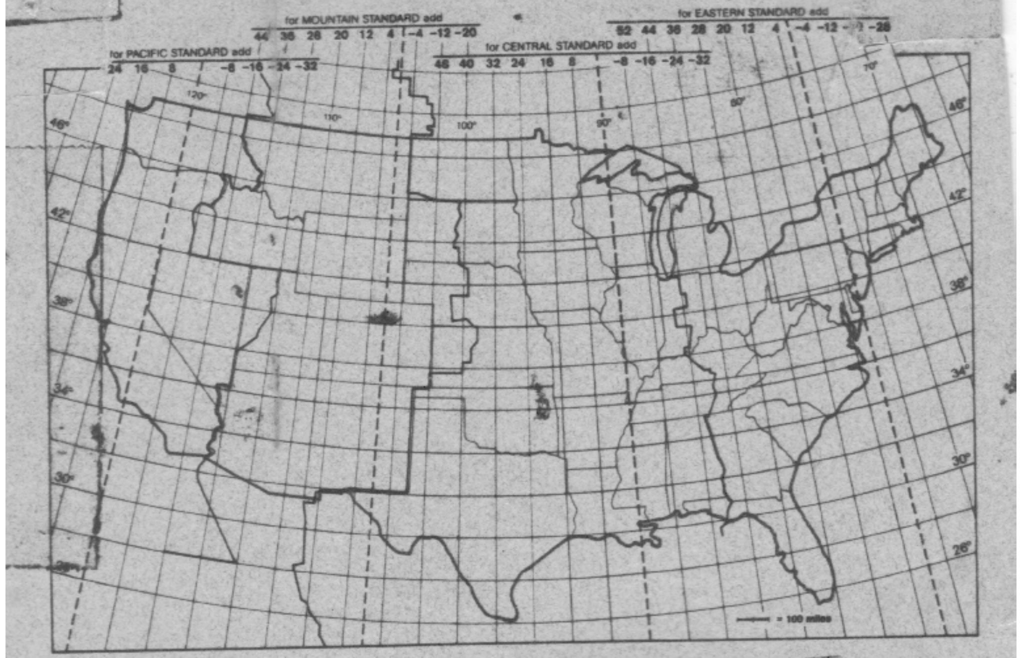
1. Place bullseye over the point where date (declination) and latitude lines cross.
2. Adjust string so that crosshairs on time bob will exactly cross "High Noon Point". Pull string into slot and tighten to fix setting.

SETTING INSTRUCTIONS

(1) Locate bullseye on the latitude and declination scales. (See map on other side for latitudes. Find declination on top scale.)

(2) Pull string through the bullseye so that the time bob reaches the high noon point below. Tie up loose string end to fix setting.

# *Euler* **R**(αβγ) *Sundial*



FYV +16

## INSTRUCTIONS

1. Follow "Setting Instructions" on other side.
2. Fold aiming tabs into place.
3. Holding card vertical, tilt card so that sunlight passes through hole in tab and strikes target on opposite tab.
4. Allow time bob to come to rest.
5. Gently tilt card or hold time bob to keep it in position. Read SOLAR time under crosshairs.
6. To convert SOLAR time to CIVIL (standard) or DAYLIGHT time, use the following formula:

   CIVIL time = SOLAR time + date correction (see calendar) + map correction (see map)

   DAYLIGHT time = CIVIL time + 1 hour

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**SOLAR COMPUTER**™

115 N. Rocky River Drive
Berea, OH 44017

- *Review: Fundamental Euler* $\mathbf{R}(\alpha\beta\gamma)$ *and Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*
  - *Euler* $\mathbf{R}(\alpha\beta\gamma)$ *derived from Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *and vice versa*
  - *Euler* $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*
  - *R(3)-U(2) slide rule for converting* $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$ *and Sundial*

- *U(2) density operator approach to symmetry dynamics*
  - *Bloch equation for density operator*

- *The ABC's of U(2) dynamics-Archetypes*
  - *Asymmetric-Diagonal A-Type motion*
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- *Ellipsometry using U(2) symmetry coordinates*
  - *Conventional amp-phase ellipse coordinates*
  - *Euler Angle* $(\alpha\beta\gamma)$ *ellipse coordinates*

# *U(2) density operator approach to symmetry dynamics*

*Euler phase-angle coordinates* $(\alpha,\beta,\gamma)$
*and norm N of quantum state* $|\Psi\rangle$

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\begin{pmatrix}\langle 1|\Psi\rangle\\ \langle 2|\Psi\rangle\end{pmatrix}=\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\frac{\beta}{2}\\ e^{i\alpha/2}\sin\frac{\beta}{2}\end{pmatrix}e^{-i\gamma/2}$$

Spin **S**-vector components are *one-half* the Pauli *spinor operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ .

$$\langle\Psi|\sigma_Z|\Psi\rangle=2S_A=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\left(p_1^2+x_1^2-p_2^2-x_2^2\right)=|\Psi_1|^2-|\Psi_2|^2$$

$$\langle\Psi|\sigma_X|\Psi\rangle=2S_B=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1x_2+p_1p_2\right)\qquad=2\,\mathrm{Re}\,\Psi_1^*\Psi_2$$

$$\langle\Psi|\sigma_Y|\Psi\rangle=2S_C=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1p_2-x_2p_1\right)\qquad=2\,\mathrm{Im}\,\Psi_1^*\Psi_2$$

# *U(2) density operator approach to symmetry dynamics*

*Euler phase-angle coordinates* $(\alpha,\beta,\gamma)$
*and norm* $N$ *of quantum state* $|\Psi\rangle$

$$|\Psi\rangle = \begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = \begin{pmatrix}\langle 1|\Psi\rangle\\ \langle 2|\Psi\rangle\end{pmatrix} = \begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix} = \sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\frac{\beta}{2}\\ e^{i\alpha/2}\sin\frac{\beta}{2}\end{pmatrix} e^{-i\gamma/2}$$

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$$\langle\Psi|\sigma_X|\Psi\rangle = 2S_B = \begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = 2N\left(x_1x_2+p_1p_2\right)$$

$$\langle\Psi|\sigma_Y|\Psi\rangle = 2S_C = \begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = 2N\left(x_1p_2-x_2p_1\right)$$

$$S_Z = S_A = \frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right) = \frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right) = \frac{N}{2}\cos\beta$$

$$S_X = S_B = \mathrm{Re}\,\Psi_1^*\Psi_2 \qquad = N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} \quad = \frac{N}{2}\cos\alpha\,\sin\beta$$

$$S_Y = S_C = \mathrm{Im}\,\Psi_1^*\Psi_2 \qquad = N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} \quad = \frac{N}{2}\sin\alpha\,\sin\beta$$

# *U(2) density operator approach to symmetry dynamics*

*Euler phase-angle coordinates* $(\alpha,\beta,\gamma)$

*and norm N of quantum state* $|\Psi\rangle$

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\begin{pmatrix}\langle 1|\Psi\rangle\\ \langle 2|\Psi\rangle\end{pmatrix}=\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\frac{\beta}{2}\\ e^{i\alpha/2}\sin\frac{\beta}{2}\end{pmatrix}e^{-i\gamma/2}$$

Spin **S**-vector components are *one-half* the Pauli *spinor operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ .

$$\langle\Psi|\boldsymbol{\sigma}_Z|\Psi\rangle=2S_A=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\left(p_1^2+x_1^2-p_2^2-x_2^2\right)$$

$$S_Z=S_A=\frac{1}{2}\left(\left|\Psi_1\right|^2-\left|\Psi_2\right|^2\right)=\frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)=\frac{N}{2}\cos\beta$$

$$\langle\Psi|\boldsymbol{\sigma}_X|\Psi\rangle=2S_B=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1x_2+p_1p_2\right)$$

$$S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2=N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\cos\alpha\,\sin\beta$$

$$\langle\Psi|\boldsymbol{\sigma}_Y|\Psi\rangle=2S_C=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1p_2-x_2p_1\right)$$

$$S_Y=S_C=\mathrm{Im}\,\Psi_1^*\Psi_2=N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\sin\alpha\,\sin\beta$$

The *density operator* $\rho=|\Psi\rangle\langle\Psi|=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}\otimes\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}=\begin{pmatrix}\Psi_1\Psi_1^* & \Psi_1\Psi_2^*\\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^*\end{pmatrix}=\begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix}=\begin{pmatrix}\Psi_1^*\Psi_1 & \Psi_2^*\Psi_1\\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2\end{pmatrix}$

# U(2) density operator approach to symmetry dynamics

*Euler phase-angle coordinates* $(\alpha, \beta, \gamma)$
*and norm $N$ of quantum state* $|\Psi\rangle$

$$|\Psi\rangle = \begin{pmatrix}\Psi_1 \\ \Psi_2\end{pmatrix} = \begin{pmatrix}\langle 1|\Psi\rangle \\ \langle 2|\Psi\rangle\end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2\end{pmatrix} = \sqrt{N}\begin{pmatrix} e^{-i\alpha/2}\cos\frac{\beta}{2} \\ e^{i\alpha/2}\sin\frac{\beta}{2}\end{pmatrix} e^{-i\gamma/2}$$

Spin **S**-vector components are *one-half* the Pauli *spinor operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ .

$$\langle\Psi|\sigma_Z|\Psi\rangle = 2S_A = \begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}\Psi_1 \\ \Psi_2\end{pmatrix} = N\left(p_1^2 + x_1^2 - p_2^2 - x_2^2\right)$$

$$\langle\Psi|\sigma_X|\Psi\rangle = 2S_B = \begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}\Psi_1 \\ \Psi_2\end{pmatrix} = 2N\left(x_1x_2 + p_1p_2\right)$$

$$\langle\Psi|\sigma_Y|\Psi\rangle = 2S_C = \begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0 & -i \\ i & 0\end{pmatrix}\begin{pmatrix}\Psi_1 \\ \Psi_2\end{pmatrix} = 2N\left(x_1p_2 - x_2p_1\right)$$

$$S_Z = S_A = \frac{1}{2}\left(|\Psi_1|^2 - |\Psi_2|^2\right) = \frac{N}{2}\left(\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right) = \frac{N}{2}\cos\beta$$

$$S_X = S_B = \mathrm{Re}\,\Psi_1^*\Psi_2 \quad = N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} \quad = \frac{N}{2}\cos\alpha\,\sin\beta$$

$$S_Y = S_C = \mathrm{Im}\,\Psi_1^*\Psi_2 \quad = N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} \quad = \frac{N}{2}\sin\alpha\,\sin\beta$$

The *density operator* $\rho = |\Psi\rangle\langle\Psi| =$ $\begin{pmatrix}\Psi_1 \\ \Psi_2\end{pmatrix}\otimes\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix} = \begin{pmatrix}\Psi_1\Psi_1^* & \Psi_1\Psi_2^* \\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^*\end{pmatrix} = \begin{pmatrix}\rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22}\end{pmatrix} = \begin{pmatrix}\Psi_1^*\Psi_1 & \Psi_2^*\Psi_1 \\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2\end{pmatrix}$

$$= \begin{pmatrix}\frac{1}{2}N + S_Z & S_X - iS_Y \\ S_X + iS_Y & \frac{1}{2}N - S_Z\end{pmatrix} = \frac{1}{2}N\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix} + S_X\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix} + S_Y\begin{pmatrix}0 & -i \\ i & 0\end{pmatrix} + S_Z\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$$

# U(2) density operator approach to symmetry dynamics

*Euler phase-angle coordinates* $(\alpha,\beta,\gamma)$
*and norm N of quantum state* $|\Psi\rangle$

$$|\Psi\rangle = \begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = \begin{pmatrix}\langle 1|\Psi\rangle\\ \langle 2|\Psi\rangle\end{pmatrix} = \begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix} = \sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\frac{\beta}{2}\\ e^{i\alpha/2}\sin\frac{\beta}{2}\end{pmatrix}e^{-i\gamma/2}$$

Spin **S**-vector components are *one-half* the Pauli *spinor operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ .

$$\langle\Psi|\sigma_Z|\Psi\rangle = 2S_A = \begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = N\left(p_1^2+x_1^2-p_2^2-x_2^2\right)$$

$$\langle\Psi|\sigma_X|\Psi\rangle = 2S_B = \begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = 2N\left(x_1x_2+p_1p_2\right)$$

$$\langle\Psi|\sigma_Y|\Psi\rangle = 2S_C = \begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = 2N\left(x_1p_2-x_2p_1\right)$$

$$S_Z = S_A = \frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right) = \frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right) = \frac{N}{2}\cos\beta$$

$$S_X = S_B = \mathrm{Re}\,\Psi_1^*\Psi_2 = N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{N}{2}\cos\alpha\,\sin\beta$$

$$S_Y = S_C = \mathrm{Im}\,\Psi_1^*\Psi_2 = N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{N}{2}\sin\alpha\,\sin\beta$$

The *density operator* $\rho = |\Psi\rangle\langle\Psi| = \begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}\otimes\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix} = \begin{pmatrix}\Psi_1\Psi_1^* & \Psi_1\Psi_2^*\\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^*\end{pmatrix} = \begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix} = \begin{pmatrix}\Psi_1^*\Psi_1 & \Psi_2^*\Psi_1\\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2\end{pmatrix}$

| | |
|---|---|
| $\rho_{11} = \Psi_1^*\Psi_1$ <br> $= \frac{1}{2}N + S_Z$ | $\rho_{12} = \Psi_2^*\Psi_1$ <br> $= S_X - iS_Y,$ |
| $\rho_{21} = \Psi_1^*\Psi_2$ <br> $= S_X + iS_Y$ | $\rho_{22} = \Psi_2^*\Psi_2$ <br> $= \frac{1}{2}N - S_Z$ |

*Norm:* $N = \Psi_1^*\Psi_1 + \Psi_2^*\Psi_2$

$$= \begin{pmatrix}\frac{1}{2}N + S_Z & S_X - iS_Y\\ S_X + iS_Y & \frac{1}{2}N - S_Z\end{pmatrix} = \frac{1}{2}N\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + S_X\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + S_Y\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + S_Z\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

← *2-by-2 density matrix*

# *U(2) density operator approach to symmetry dynamics*

*Euler phase-angle coordinates* $(\alpha, \beta, \gamma)$
*and norm $N$ of quantum state* $|\Psi\rangle$

$$|\Psi\rangle = \begin{pmatrix}\Psi_1 \\ \Psi_2\end{pmatrix} = \begin{pmatrix}\langle 1|\Psi\rangle \\ \langle 2|\Psi\rangle\end{pmatrix} = \begin{pmatrix}x_1 + ip_1 \\ x_2 + ip_2\end{pmatrix} = \sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\frac{\beta}{2} \\ e^{i\alpha/2}\sin\frac{\beta}{2}\end{pmatrix} e^{-i\gamma/2}$$

Spin **S**-vector components are *one-half* the Pauli *spinor operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ .

$$\langle\Psi|\boldsymbol{\sigma}_Z|\Psi\rangle = 2S_A = \begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}\Psi_1 \\ \Psi_2\end{pmatrix} = N\left(p_1^2 + x_1^2 - p_2^2 - x_2^2\right)$$

$$S_Z = S_A = \frac{1}{2}\left(|\Psi_1|^2 - |\Psi_2|^2\right) = \frac{N}{2}\left(\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right) = \frac{N}{2}\cos\beta$$

$$\langle\Psi|\boldsymbol{\sigma}_X|\Psi\rangle = 2S_B = \begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}\Psi_1 \\ \Psi_2\end{pmatrix} = 2N\left(x_1x_2 + p_1p_2\right)$$

$$S_X = S_B = \mathrm{Re}\,\Psi_1^*\Psi_2 \qquad = N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} \quad = \frac{N}{2}\cos\alpha\,\sin\beta$$

$$\langle\Psi|\boldsymbol{\sigma}_Y|\Psi\rangle = 2S_C = \begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0 & -i \\ i & 0\end{pmatrix}\begin{pmatrix}\Psi_1 \\ \Psi_2\end{pmatrix} = 2N\left(x_1p_2 - x_2p_1\right)$$

$$S_Y = S_C = \mathrm{Im}\,\Psi_1^*\Psi_2 \qquad = N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} \quad = \frac{N}{2}\sin\alpha\,\sin\beta$$

The *density operator* $\rho = |\Psi\rangle\langle\Psi| = \begin{pmatrix}\Psi_1 \\ \Psi_2\end{pmatrix}\otimes\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix} = \begin{pmatrix}\Psi_1\Psi_1^* & \Psi_1\Psi_2^* \\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^*\end{pmatrix} = \begin{pmatrix}\rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22}\end{pmatrix} = \begin{pmatrix}\Psi_1^*\Psi_1 & \Psi_2^*\Psi_1 \\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2\end{pmatrix}$

| | |
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*Norm: $N = \Psi_1^*\Psi_1 + \Psi_2^*\Psi_2$*

$$= \begin{pmatrix}\frac{1}{2}N + S_Z & S_X - iS_Y \\ S_X + iS_Y & \frac{1}{2}N - S_Z\end{pmatrix} = \frac{1}{2}N\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix} + S_X\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix} + S_Y\begin{pmatrix}0 & -i \\ i & 0\end{pmatrix} + S_Z\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$$

$$\rho \quad = \frac{1}{2}N\;\mathbf{1} \;+ S_X\;\boldsymbol{\sigma}_X \;+ S_Y\;\boldsymbol{\sigma}_Y \;+ S_Z\;\boldsymbol{\sigma}_Z \;= \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

*2-by-2 density matrix*

# *U(2) density operator approach to symmetry dynamics*

*Euler phase-angle coordinates* (α, β, γ)
*and norm N of quantum state* $|\Psi\rangle$

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} \langle 1|\Psi\rangle \\ \langle 2|\Psi\rangle \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \sqrt{N} \begin{pmatrix} e^{-i\alpha/2}\cos\frac{\beta}{2} \\ e^{i\alpha/2}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\gamma/2}$$

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$$\langle\Psi|\boldsymbol{\sigma}_Z|\Psi\rangle = 2S_A = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N\left(p_1^2 + x_1^2 - p_2^2 - x_2^2\right)$$

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$$\langle\Psi|\boldsymbol{\sigma}_Y|\Psi\rangle = 2S_C = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = 2N\left(x_1 p_2 - x_2 p_1\right)$$

$$S_Z = S_A = \frac{1}{2}\left(|\Psi_1|^2 - |\Psi_2|^2\right) = \frac{N}{2}\left(\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right) = \frac{N}{2}\cos\beta$$

$$S_X = S_B = \mathrm{Re}\,\Psi_1^*\Psi_2 = N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{N}{2}\cos\alpha\,\sin\beta$$

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The *density operator* ρ = |Ψ⟩⟨Ψ|=

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \otimes \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix} = \begin{pmatrix} \Psi_1\Psi_1^* & \Psi_1\Psi_2^* \\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^* \end{pmatrix} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \begin{pmatrix} \Psi_1^*\Psi_1 & \Psi_2^*\Psi_1 \\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2 \end{pmatrix}$$

| | |
|---|---|
| $\rho_{11} = \Psi_1^*\Psi_1$<br>$= \frac{1}{2}N + S_Z$ | $\rho_{12} = \Psi_2^*\Psi_1$<br>$= S_X - iS_Y$, |
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*Norm: N* = Ψ1*Ψ1 + Ψ2*Ψ2

$$= \begin{pmatrix} \frac{1}{2}N + S_Z & S_X - iS_Y \\ S_X + iS_Y & \frac{1}{2}N - S_Z \end{pmatrix} = \frac{1}{2}N\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + S_X\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + S_Y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + S_Z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\uparrow\ \boldsymbol{\rho} \quad = \frac{1}{2}N\ \mathbf{1} + S_X\ \boldsymbol{\sigma}_X + S_Y\ \boldsymbol{\sigma}_Y + S_Z\ \boldsymbol{\sigma}_Z = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

...so state *density operator* ρ has σ-expansion like *Hamiltonian operator* **H**

$$\begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{H} = \omega_0\ \ \sigma_0 + \frac{\Omega_A}{2}\ \sigma_A + \frac{\Omega_B}{2}\ \sigma_B + \frac{\Omega_C}{2}\sigma_C = \omega_0\sigma_0 + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

# *U(2) density operator approach to symmetry dynamics*

*Euler phase-angle coordinates* (α, β, γ)
*and norm N of quantum state* |Ψ⟩

$$|\Psi\rangle = \begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = \begin{pmatrix}\langle 1|\Psi\rangle\\ \langle 2|\Psi\rangle\end{pmatrix} = \begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix} = \sqrt{N}\begin{pmatrix} e^{-i\alpha/2}\cos\frac{\beta}{2}\\ e^{i\alpha/2}\sin\frac{\beta}{2}\end{pmatrix} e^{-i\gamma/2}$$

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*Norm:* $N = \Psi_1^*\Psi_1 + \Psi_2^*\Psi_2$

$$= \begin{pmatrix}\frac{1}{2}N+S_Z & S_X-iS_Y\\ S_X+iS_Y & \frac{1}{2}N-S_Z\end{pmatrix} = \frac{1}{2}N\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + S_X\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + S_Y\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + S_Z\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$\uparrow\ \rho \quad = \frac{1}{2}N\ \mathbf{1} + S_X\ \boldsymbol{\sigma}_X + S_Y\ \boldsymbol{\sigma}_Y + S_Z\ \boldsymbol{\sigma}_Z = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

←

...so state *density operator* ρ has σ-expansion like *Hamiltonian operator* **H**

$$\begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{A-D}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + B\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + C\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

$$\boxed{\begin{aligned}\rho &= \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\\ \mathbf{H} &= \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}\end{aligned}}$$

$$\mathbf{H} = \omega_0\ \sigma_0 + \frac{\Omega_A}{2}\ \sigma_A + \frac{\Omega_B}{2}\ \sigma_B + \frac{\Omega_C}{2}\sigma_C = \omega_0\sigma_0 + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\ \mathbf{1} + \Omega_A\ \mathbf{S}_A + \Omega_B\ \mathbf{S}_B + \Omega_C\mathbf{S}_C = \Omega_0\mathbf{1} + \vec{\Omega}\bullet\mathbf{S}$$

*Review: Fundamental Euler* **R**$(\alpha\beta\gamma)$ *and Darboux* **R**$[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

- *Euler* **R**$(\alpha\beta\gamma)$ *derived from Darboux* **R**$[\varphi\vartheta\Theta]$ *and vice versa*
- *Euler* **R**$(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*
- *R(3)-U(2) slide rule for converting* **R**$(\alpha\beta\gamma)$ ↔ **R**$[\varphi\vartheta\Theta]$ *and Sundial*

*U(2) density operator approach to symmetry dynamics*

- *Bloch equation for density operator*

*The ABC's of U(2) dynamics-Archetypes*

- *Asymmetric-Diagonal A-Type motion*
- *Bilateral-Balanced B-Type motion*
- *Circular-Coriolis… C-Type motion*

*The ABC's of U(2) dynamics-Mixed modes*

- *AB-Type motion and Wigner's Avoided-Symmetry-Crossings*
- *ABC-Type elliptical polarized motion*

*Ellipsometry using U(2) symmetry coordinates*

- *Conventional amp-phase ellipse coordinates*
- *Euler Angle* $(\alpha\beta\gamma)$ *ellipse coordinates*

# U(2) density operator approach to symmetry dynamics

## Bloch equation for density operator

$$\rho = \frac{1}{2} N\mathbf{1} + \vec{\mathbf{S}} \bullet \boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0 \mathbf{1} + \frac{\vec{\Omega}}{2} \bullet \boldsymbol{\sigma}$$

Ket equation (time *forward*) and "daggered" bra-equation (time *reversed*).

$$i\hbar\left|\dot{\Psi}\right\rangle = \mathbf{H}\left|\Psi\right\rangle, \qquad \Leftarrow Daggar^{\dagger} \Rightarrow \qquad -i\hbar\left\langle\dot{\Psi}\right| = \left\langle\Psi\right|\mathbf{H}$$

Note: $\mathbf{H}^{\dagger} = \mathbf{H}$.

# U(2) density operator approach to symmetry dynamics

## Bloch equation for density operator

$$\boldsymbol{\rho} = \frac{1}{2} N\mathbf{1} + \vec{\mathbf{S}} \bullet \boldsymbol{\sigma}$$

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Note: $\mathbf{H}^{\dagger} = \mathbf{H}$.

Combining these gives a time derivative of the density operator $\boldsymbol{\rho} = |\Psi\rangle\langle\Psi|$

$$\boldsymbol{\rho}^{\dagger} = \boldsymbol{\rho}$$

$$i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho} = i\hbar\dot{\boldsymbol{\rho}} = i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right| + i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right| = \mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

# U(2) density operator approach to symmetry dynamics

## Bloch equation for density operator

$$\boldsymbol{\rho} = \frac{1}{2}\left(N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)$$

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$\boldsymbol{\rho}^{\dagger} = \boldsymbol{\rho}$

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$$i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho} = i\hbar\dot{\boldsymbol{\rho}} = i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right| + i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right| = \mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

The result is called a *Bloch equation*.

$$i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho} = i\hbar\dot{\boldsymbol{\rho}} = \mathbf{H}\boldsymbol{\rho} - \boldsymbol{\rho}\mathbf{H} = \left[\mathbf{H},\boldsymbol{\rho}\right]$$

# *U(2) density operator approach to symmetry dynamics*

## *Bloch equation for density operator*

$$\rho = \frac{1}{2}N\mathbf{1}+\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}$$

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Ket equation (time *forward*) and "daggered" bra-equation (time *reversed*).

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Combining these gives a time derivative of the density operator $\rho = |\Psi\rangle\langle\Psi|$

$\rho^{\dagger} = \rho$

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right| + i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right| = \mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

The result is called a *Bloch equation*.

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = \mathbf{H}\rho - \rho\mathbf{H} = \left[\mathbf{H},\rho\right]$$

Given ρ and **H** in terms *spin* S-vector and *crank* **Ω**-vector:

$$\mathbf{H}\rho = \left(\hbar\Omega_0\mathbf{1}+\frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\frac{N}{2}\mathbf{1}+\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1}+\frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}+\hbar\Omega_0\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}+\frac{\hbar}{2}\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right)$$

$$-\rho\mathbf{H} = \left(\frac{N}{2}\mathbf{1}+\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right)\left(\hbar\Omega_0\mathbf{1}+\frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1}+\frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}+\hbar\Omega_0\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}+\frac{\hbar}{2}\left(\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)$$

# *U(2) density operator approach to symmetry dynamics*

## *Bloch equation for density operator*

$$\rho = \frac{1}{2} N\mathbf{1} + \vec{\mathrm{S}}\bullet\boldsymbol{\sigma}$$
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Ket equation (time *forward*) and "daggered" bra-equation (time *reversed*).

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Combining these gives a time derivative of the density operator $\rho = |\Psi\rangle\langle\Psi|$

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The result is called a *Bloch equation*.

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = \mathbf{H}\rho - \rho\mathbf{H} = \left[\mathbf{H},\rho\right]$$

Given ρ and **H** in terms *spin* S-vector and *crank* **Ω**-vector:

$$\mathbf{H}\rho = \left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\Omega}\bullet\boldsymbol{\sigma}\right)\left(\frac{N}{2}\mathbf{1} + \vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right) = \cancel{\hbar\Omega_0\frac{N}{2}\mathbf{1}} + \cancel{\frac{N}{4}\hbar\vec{\Omega}\bullet\boldsymbol{\sigma}} + \cancel{\hbar\Omega_0\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}} + \frac{\hbar}{2}\left(\vec{\Omega}\bullet\boldsymbol{\sigma}\right)\left(\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right)$$

$$-\rho\mathbf{H} = \left(\frac{N}{2}\mathbf{1} + \vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right)\left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\Omega}\bullet\boldsymbol{\sigma}\right) = \cancel{\hbar\Omega_0\frac{N}{2}\mathbf{1}} + \cancel{\frac{N}{4}\hbar\vec{\Omega}\bullet\boldsymbol{\sigma}} + \cancel{\hbar\Omega_0\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}} + \frac{\hbar}{2}\left(\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\Omega}\bullet\boldsymbol{\sigma}\right)$$

Last terms don't cancel if the *spin* S and *crank* **Ω** *point in different directions*.

# *U(2) density operator approach to symmetry dynamics*

## *Bloch equation for density operator*

$$\boxed{\begin{aligned}\boldsymbol{\rho} &= \frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\\ \mathbf{H} &= \Omega_0\mathbf{1}+\frac{\vec{\boldsymbol{\Omega}}}{2}\bullet\boldsymbol{\sigma}\end{aligned}}$$

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Combining these gives a time derivative of the density operator $\boldsymbol{\rho} = |\Psi\rangle\langle\Psi|$

$\boldsymbol{\rho}^{\dagger} = \boldsymbol{\rho}$

$$i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho} = i\hbar\dot{\boldsymbol{\rho}} = i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right| + i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right| = \mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

The result is called a *Bloch equation*.

$$\boxed{i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho} = i\hbar\dot{\boldsymbol{\rho}} = \mathbf{H}\boldsymbol{\rho} - \boldsymbol{\rho}\mathbf{H} = \left[\mathbf{H},\boldsymbol{\rho}\right]}$$

$$(\mathbf{A}\bullet\boldsymbol{\sigma})(\mathbf{B}\bullet\boldsymbol{\sigma}) = A_\alpha B_\beta \sigma_\alpha\sigma_\beta = A_\alpha B_\beta\left(\delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma\right)$$

$$= A_\alpha B_\alpha + i\varepsilon_{\alpha\beta\gamma}A_\alpha B_\beta\sigma_\gamma$$

$$= \mathbf{A}\bullet\mathbf{B} + i\left(\mathbf{A}\times\mathbf{B}\right)\cdot\boldsymbol{\sigma}$$

*This cancels* *This remains*

Given $\boldsymbol{\rho}$ and $\mathbf{H}$ in terms *spin* $\mathbf{S}$-vector and *crank* $\boldsymbol{\Omega}$-vector:

$$\mathbf{H}\boldsymbol{\rho} = \left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\frac{N}{2}\mathbf{1} + \vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right) = \cancel{\hbar\Omega_0\frac{N}{2}\mathbf{1}} + \cancel{\frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}} + \cancel{\hbar\Omega_0\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}} + \frac{\hbar}{2}\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right)$$

$$-\boldsymbol{\rho}\mathbf{H} = \left(\frac{N}{2}\mathbf{1} + \vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right)\left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right) = \cancel{\hbar\Omega_0\frac{N}{2}\mathbf{1}} + \cancel{\frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}} + \cancel{\hbar\Omega_0\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}} + \frac{\hbar}{2}\left(\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)$$

Last terms don't cancel if the *spin* $\mathbf{S}$ and *crank* $\boldsymbol{\Omega}$ *point in different directions*.

$$\mathbf{H}\boldsymbol{\rho} - \boldsymbol{\rho}\mathbf{H} = \frac{\hbar}{2}\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right) - \frac{\hbar}{2}\left(\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)$$

# U(2) density operator approach to symmetry dynamics

## Bloch equation for density operator

$$\boxed{\begin{aligned}\boldsymbol{\rho} &= \frac{1}{2}N\mathbf{1} + \vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\\ \mathbf{H} &= \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}\end{aligned}}$$

Ket equation (time forward) and "daggered" bra-equation (time reversed).

$$i\hbar\left|\dot{\Psi}\right\rangle = \mathbf{H}\left|\Psi\right\rangle, \qquad \Leftarrow \mathit{Daggar}^{\dagger} \Rightarrow \qquad -i\hbar\left\langle\dot{\Psi}\right| = \left\langle\Psi\right|\mathbf{H}$$

Note: $\mathbf{H}^{\dagger} = \mathbf{H}$.

$\boldsymbol{\rho}^{\dagger} = \boldsymbol{\rho}$

Combining these gives a time derivative of the density operator $\boldsymbol{\rho} = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho} = i\hbar\dot{\boldsymbol{\rho}} = i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right| + i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right| = \mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

The result is called a *Bloch equation*.

$$\boxed{i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho} = i\hbar\dot{\boldsymbol{\rho}} = \mathbf{H}\boldsymbol{\rho} - \boldsymbol{\rho}\mathbf{H} = \left[\mathbf{H},\boldsymbol{\rho}\right]}$$

$$(\mathbf{A}\bullet\boldsymbol{\sigma})(\mathbf{B}\bullet\boldsymbol{\sigma}) = A_\alpha B_\beta\sigma_\alpha\sigma_\beta = A_\alpha B_\beta\left(\delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma\right)$$

$$= A_\alpha B_\alpha + i\varepsilon_{\alpha\beta\gamma}A_\alpha B_\beta\sigma_\gamma$$

$$= \mathbf{A}\bullet\mathbf{B} + i\left(\mathbf{A}\times\mathbf{B}\right)\cdot\boldsymbol{\sigma}$$

*This cancels*

*This remains*

Given ρ and **H** in terms *spin* S-vector and *crank* **Ω**-vector:

$$\mathbf{H}\boldsymbol{\rho} = \left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\frac{N}{2}\mathbf{1} + \vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma} + \hbar\Omega_0\vec{\mathrm{S}}\bullet\boldsymbol{\sigma} + \frac{\hbar}{2}\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right)$$

$$\boldsymbol{\rho}\mathbf{H} = \left(\frac{N}{2}\mathbf{1} + \vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right)\left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma} + \hbar\Omega_0\vec{\mathrm{S}}\bullet\boldsymbol{\sigma} + \frac{\hbar}{2}\left(\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)$$

Last terms don't cancel if the *spin* S and *crank* **Ω** *point in different directions*.

$$\mathbf{H}\boldsymbol{\rho} - \boldsymbol{\rho}\mathbf{H} = \frac{\hbar}{2}\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right) - \frac{\hbar}{2}\left(\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)$$

$$i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho} = i\hbar\dot{\boldsymbol{\rho}} = \frac{i\hbar}{2}\left(\vec{\boldsymbol{\Omega}}\times\vec{\mathrm{S}}\right)\bullet\boldsymbol{\sigma} - \frac{i\hbar}{2}\left(\vec{\mathrm{S}}\times\vec{\boldsymbol{\Omega}}\right)\bullet\boldsymbol{\sigma}$$

# U(2) density operator approach to symmetry dynamics

## Bloch equation for density operator

$$\boxed{\begin{aligned}\boldsymbol{\rho} &= \frac{1}{2}N\mathbf{1}+\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\\ \mathbf{H} &= \Omega_0\mathbf{1}+\frac{\vec{\boldsymbol{\Omega}}}{2}\bullet\boldsymbol{\sigma}\end{aligned}}$$

Ket equation (time forward) and "daggered" bra-equation (time reversed).

$$i\hbar\left|\dot{\Psi}\right\rangle=\mathbf{H}\left|\Psi\right\rangle, \qquad \Leftarrow Daggar^{\dagger}\Rightarrow \qquad -i\hbar\left\langle\dot{\Psi}\right|=\left\langle\Psi\right|\mathbf{H}$$

Note: $\mathbf{H}^{\dagger}=\mathbf{H}$.

Combining these gives a time derivative of the density operator $\boldsymbol{\rho}=|\Psi\rangle\langle\Psi|$

$\boldsymbol{\rho}^{\dagger}=\boldsymbol{\rho}$

$$i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho}=i\hbar\dot{\boldsymbol{\rho}}=i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right|+i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right|=\mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right|-\left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

The result is called a *Bloch equation*.

$$\boxed{i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho}=i\hbar\dot{\boldsymbol{\rho}}=\mathbf{H}\boldsymbol{\rho}-\boldsymbol{\rho}\mathbf{H}=\left[\mathbf{H},\boldsymbol{\rho}\right]}$$

$$\left(\mathbf{A}\bullet\boldsymbol{\sigma}\right)\left(\mathbf{B}\bullet\boldsymbol{\sigma}\right)=A_{\alpha}B_{\beta}\sigma_{\alpha}\sigma_{\beta}=A_{\alpha}B_{\beta}\left(\delta_{\alpha\beta}+i\varepsilon_{\alpha\beta\gamma}\sigma_{\gamma}\right)$$
$$=A_{\alpha}B_{\alpha}+i\varepsilon_{\alpha\beta\gamma}A_{\alpha}B_{\beta}\sigma_{\gamma}$$
$$=\mathbf{A}\bullet\mathbf{B}+i\left(\mathbf{A}\times\mathbf{B}\right)\cdot\boldsymbol{\sigma}$$

*This cancels*

*This remains*

Given $\boldsymbol{\rho}$ and $\mathbf{H}$ in terms *spin* $\mathrm{S}$-vector and *crank* $\boldsymbol{\Omega}$-vector:

$$\mathbf{H}\boldsymbol{\rho}=\left(\hbar\Omega_0\mathbf{1}+\frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\frac{N}{2}\mathbf{1}+\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right)=\hbar\Omega_0\frac{N}{2}\mathbf{1}+\frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}+\hbar\Omega_0\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}+\frac{\hbar}{2}\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right)$$

$$\boldsymbol{\rho}\mathbf{H}=\left(\frac{N}{2}\mathbf{1}+\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right)\left(\hbar\Omega_0\mathbf{1}+\frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)=\hbar\Omega_0\frac{N}{2}\mathbf{1}+\frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}+\hbar\Omega_0\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}+\frac{\hbar}{2}\left(\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)$$

Last terms don't cancel if the *spin* $\mathrm{S}$ and *crank* $\boldsymbol{\Omega}$ *point in different directions*.

$$\mathbf{H}\boldsymbol{\rho}-\boldsymbol{\rho}\mathbf{H}=\frac{\hbar}{2}\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right)-\frac{\hbar}{2}\left(\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)$$

$$i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho}=i\hbar\dot{\boldsymbol{\rho}}=\frac{i\hbar}{2}\left(\vec{\boldsymbol{\Omega}}\times\vec{\mathrm{S}}\right)\bullet\boldsymbol{\sigma}-\frac{i\hbar}{2}\left(\vec{\mathrm{S}}\times\vec{\boldsymbol{\Omega}}\right)\bullet\boldsymbol{\sigma}$$

$$i\hbar\frac{\partial}{\partial t}\left(\frac{N}{2}\mathbf{1}+\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right)=i\hbar\dot{\vec{\mathrm{S}}}\bullet\boldsymbol{\sigma}=\ i\hbar\left(\vec{\boldsymbol{\Omega}}\times\mathbf{S}\right)\bullet\boldsymbol{\sigma}$$

# *U(2) density operator approach to symmetry dynamics*

## *Bloch equation for density operator*

$$\boldsymbol{\rho} = \frac{1}{2}N\mathbf{1} + \vec{\mathrm{S}}\bullet\boldsymbol{\sigma}$$
$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

Ket equation (time forward) and "daggered" bra-equation (time reversed).

$$i\hbar\left|\dot{\Psi}\right\rangle = \mathbf{H}\left|\Psi\right\rangle, \qquad \Leftarrow \textit{Daggar}^{\dagger} \Rightarrow \quad \text{-}i\hbar\left\langle\dot{\Psi}\right| = \left\langle\Psi\right|\mathbf{H}$$

Note: $\mathbf{H}^{\dagger} = \mathbf{H}$

$\boldsymbol{\rho}^{\dagger} = \boldsymbol{\rho}$

Combining these gives a time derivative of the density operator $\boldsymbol{\rho} = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho} = i\hbar\dot{\boldsymbol{\rho}} = i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right| + i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right| = \mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

The result is called a *Bloch equation*.

$$i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho} = i\hbar\dot{\boldsymbol{\rho}} = \mathbf{H}\boldsymbol{\rho} - \boldsymbol{\rho}\mathbf{H} = \left[\mathbf{H},\boldsymbol{\rho}\right]$$

$$(\mathbf{A}\bullet\boldsymbol{\sigma})(\mathbf{B}\bullet\boldsymbol{\sigma}) = A_\alpha B_\beta\sigma_\alpha\sigma_\beta = A_\alpha B_\beta\left(\delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma\right)$$
$$= A_\alpha B_\alpha + i\varepsilon_{\alpha\beta\gamma}A_\alpha B_\beta\sigma_\gamma$$
$$= \mathbf{A}\bullet\mathbf{B} + i(\mathbf{A}\times\mathbf{B})\cdot\boldsymbol{\sigma}$$

Given $\boldsymbol{\rho}$ and **H** in terms *spin* **S**-vector and *crank* **Ω**-vector:

$$\mathbf{H}\boldsymbol{\rho} = \left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\Omega}\bullet\boldsymbol{\sigma}\right)\left(\frac{N}{2}\mathbf{1} + \vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\Omega}\bullet\boldsymbol{\sigma} + \hbar\Omega_0\vec{\mathrm{S}}\bullet\boldsymbol{\sigma} + \frac{\hbar}{2}(\vec{\Omega}\bullet\boldsymbol{\sigma})(\vec{\mathrm{S}}\bullet\boldsymbol{\sigma})$$

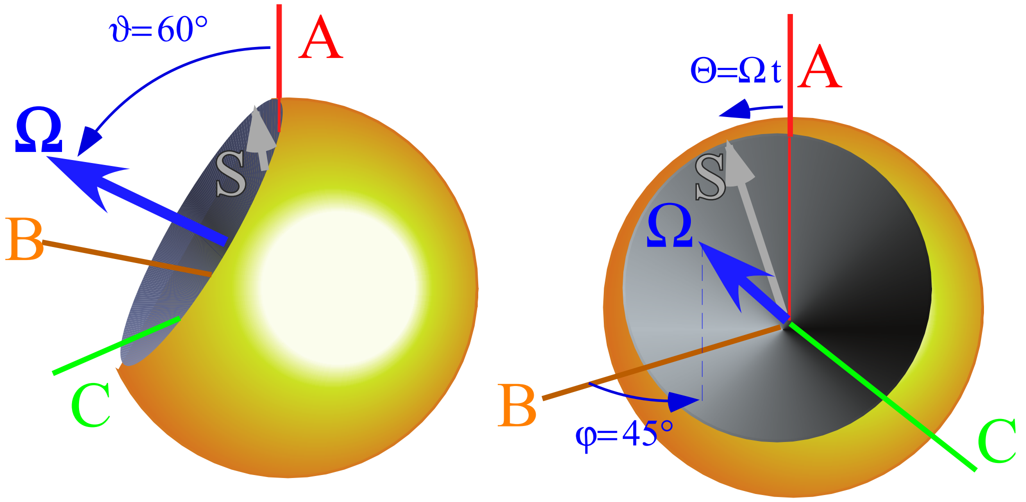
$$\boldsymbol{\rho}\mathbf{H} = \left(\frac{N}{2}\mathbf{1} + \vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right)\left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\Omega}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\Omega}\bullet\boldsymbol{\sigma} + \hbar\Omega_0\vec{\mathrm{S}}\bullet\boldsymbol{\sigma} + \frac{\hbar}{2}(\vec{\mathrm{S}}\bullet\boldsymbol{\sigma})(\vec{\Omega}\bullet\boldsymbol{\sigma})$$

Last terms don't cancel if the *spin* **S** and *crank* **Ω** *point in different directions*.

$$\mathbf{H}\boldsymbol{\rho} - \boldsymbol{\rho}\mathbf{H} = \frac{\hbar}{2}(\vec{\Omega}\bullet\boldsymbol{\sigma})(\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}) - \frac{\hbar}{2}(\vec{\mathrm{S}}\bullet\boldsymbol{\sigma})(\vec{\Omega}\bullet\boldsymbol{\sigma})$$

$$i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho} = i\hbar\dot{\boldsymbol{\rho}} = \frac{i\hbar}{2}(\vec{\Omega}\times\vec{\mathrm{S}})\bullet\boldsymbol{\sigma} - \frac{i\hbar}{2}(\vec{\mathrm{S}}\times\vec{\Omega})\bullet\boldsymbol{\sigma}$$

$$i\hbar\frac{\partial}{\partial t}\left(\frac{N}{2}\mathbf{1} + \vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right) = i\hbar\dot{\vec{\mathrm{S}}}\bullet\boldsymbol{\sigma} = i\hbar(\vec{\Omega}\times\mathbf{S})\bullet\boldsymbol{\sigma}$$



Factoring out •σ gives a classical/quantum *gyro-precession equation*. $\frac{\partial\vec{\mathrm{S}}}{\partial t} = \dot{\vec{\mathrm{S}}} = \vec{\Omega}\times\vec{\mathrm{S}}$

*Review: Fundamental Euler* $\mathbf{R}(\alpha\beta\gamma)$ *and Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

- *Euler* $\mathbf{R}(\alpha\beta\gamma)$ *derived from Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *and vice versa*
- *Euler* $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*
- *R(3)-U(2) slide rule for converting* $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$ *and Sundial*

*U(2) density operator approach to symmetry dynamics*

- *Bloch equation for density operator*

*The ABC's of U(2) dynamics-Archetypes*

- *Asymmetric-Diagonal A-Type motion*
- *Bilateral-Balanced B-Type motion*
- *Circular-Coriolis… C-Type motion*

*The ABC's of U(2) dynamics-Mixed modes*

- *AB-Type motion and Wigner's Avoided-Symmetry-Crossings*
- *ABC-Type elliptical polarized motion*

*Ellipsometry using U(2) symmetry coordinates*

- *Conventional amp-phase ellipse coordinates*
- *Euler Angle* $(\alpha\beta\gamma)$ *ellipse coordinates*

# The ABC's of U(2) dynamics

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

## Asymmetric Diagonal A-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^A|1\rangle & \langle 1|\mathbf{H}^A|2\rangle \\ \langle 2|\mathbf{H}^A|1\rangle & \langle 2|\mathbf{H}^A|2\rangle \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_A}{2}\sigma_A$$

# *The ABC's of U(2) dynamics*

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

## *Asymmetric Diagonal A-Type motion*

$$\begin{pmatrix} \langle 1|\mathbf{H}^A|1\rangle & \langle 1|\mathbf{H}^A|2\rangle \\ \langle 2|\mathbf{H}^A|1\rangle & \langle 2|\mathbf{H}^A|2\rangle \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_A}{2}\sigma_A$$

$$Crank:\ \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 0 \\ 0 \end{pmatrix} \qquad Eigen-Spin:\vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} \pm S \\ 0 \\ 0 \end{pmatrix}$$

# The ABC's of U(2) dynamics

$$\boxed{\begin{aligned}\rho &= \frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\\ \mathbf{H} &= \Omega_0\mathbf{1}+\frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}\end{aligned}}$$

$$\begin{pmatrix}\langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle\\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle\end{pmatrix} = \begin{pmatrix} A & B-iC\\ B+iC & D\end{pmatrix} = \frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix} + B\begin{pmatrix}0&1\\1&0\end{pmatrix} + C\begin{pmatrix}0&-i\\i&0\end{pmatrix}+\frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

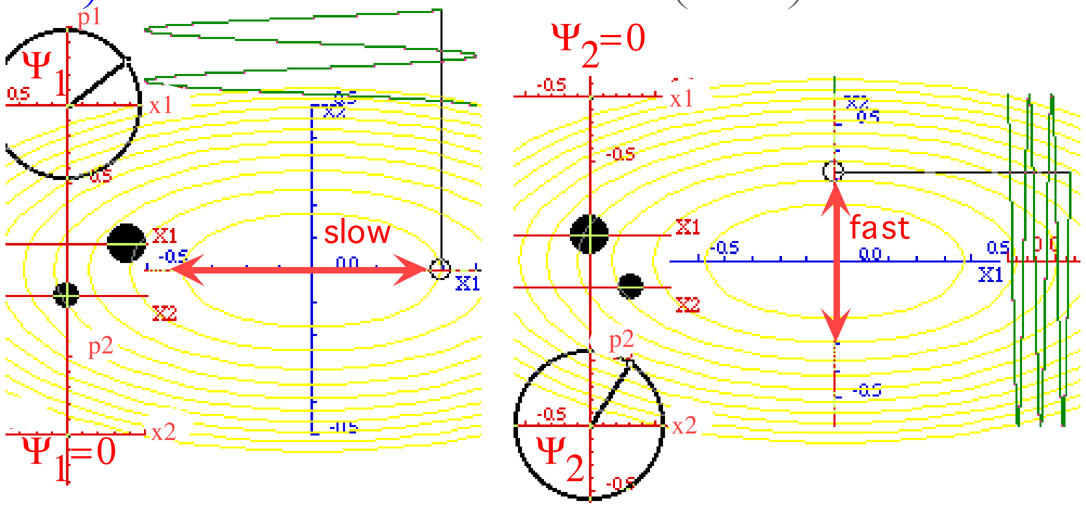
$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

$$\vec{\Omega} = \begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix} = \begin{pmatrix}A-D\\ 2B\\ 2C\end{pmatrix}$$

## Asymmetric Diagonal A-Type motion

$$\begin{pmatrix}\langle 1|\mathbf{H}^A|1\rangle & \langle 1|\mathbf{H}^A|2\rangle\\ \langle 2|\mathbf{H}^A|1\rangle & \langle 2|\mathbf{H}^A|2\rangle\end{pmatrix} = \begin{pmatrix} A & 0\\ 0 & D\end{pmatrix} = \frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix} + \frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_A}{2}\sigma_A$$

$$Crank:\ \vec{\Omega} = \begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix} = \begin{pmatrix}A-D\\ 0\\ 0\end{pmatrix} \qquad Eigen-Spin:\vec{\mathbf{S}} = \begin{pmatrix}S_A\\ S_B\\ S_C\end{pmatrix} = \begin{pmatrix}\pm S\\ 0\\ 0\end{pmatrix}$$



# The ABC's of U(2) dynamics

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix}\langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle\\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle\end{pmatrix} = \begin{pmatrix} A & B-iC\\ B+iC & D\end{pmatrix} = \frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix} + B\begin{pmatrix}0&1\\1&0\end{pmatrix} + C\begin{pmatrix}0&-i\\i&0\end{pmatrix} + \frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$= \frac{A+D}{2}\,\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\,\sigma_A$$

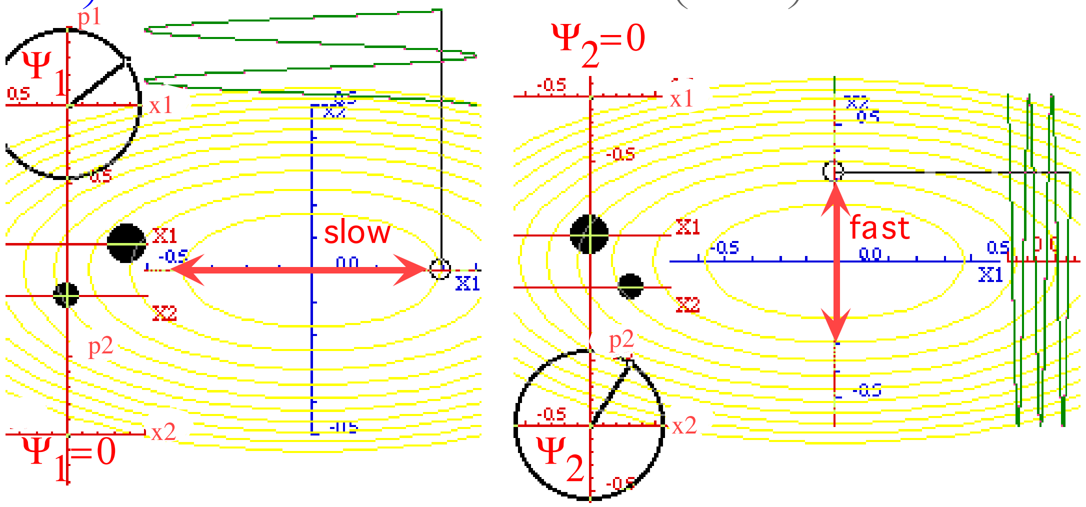
$$= \frac{A+D}{2}\,\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\,\sigma_B + \frac{\Omega_C}{2}\,\sigma_C + \frac{\Omega_A}{2}\,\sigma_A$$

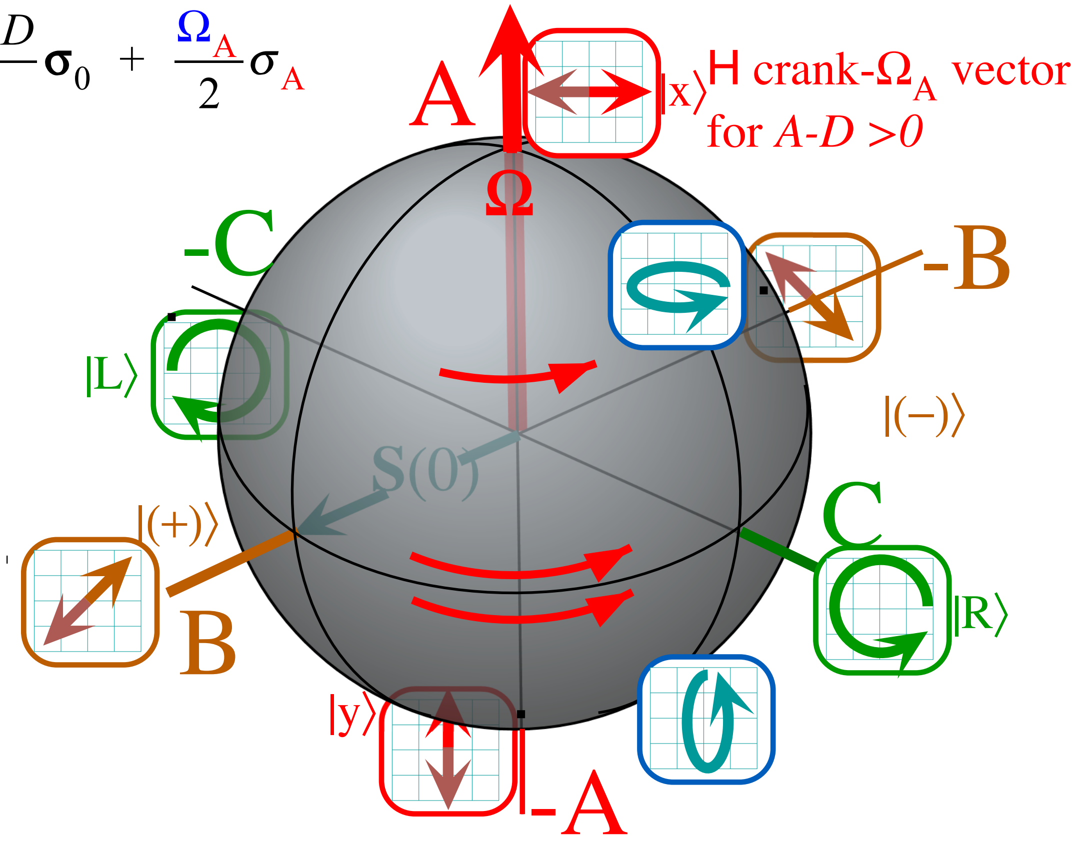
$$\vec{\Omega} = \begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix} = \begin{pmatrix}A-D\\ 2B\\ 2C\end{pmatrix}$$

## Asymmetric Diagonal *A*-Type motion

$$\begin{pmatrix}\langle 1|\mathbf{H}^A|1\rangle & \langle 1|\mathbf{H}^A|2\rangle\\ \langle 2|\mathbf{H}^A|1\rangle & \langle 2|\mathbf{H}^A|2\rangle\end{pmatrix} = \begin{pmatrix} A & 0\\ 0 & D\end{pmatrix} = \frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix} + \frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_A}{2}\sigma_A$$

$$Crank:\ \vec{\Omega} = \begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix} = \begin{pmatrix}A-D\\ 0\\ 0\end{pmatrix} \qquad Eigen-Spin:\vec{\mathbf{S}} = \begin{pmatrix}S_A\\ S_B\\ S_C\end{pmatrix} = \begin{pmatrix}\pm S\\ 0\\ 0\end{pmatrix}$$





H crank-$\Omega_A$ vector for *A-D* >0

# The ABC's of U(2) dynamics

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix}\langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle\\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle\end{pmatrix} = \begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix} = \frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix} + B\begin{pmatrix}0&1\\1&0\end{pmatrix} + C\begin{pmatrix}0&-i\\i&0\end{pmatrix} + \frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$= \frac{A+D}{2}\,\mathbf{1} \quad + \quad B\ \sigma_B \quad + \quad C\ \sigma_C \quad + \quad \frac{A-D}{2}\,\sigma_A$$

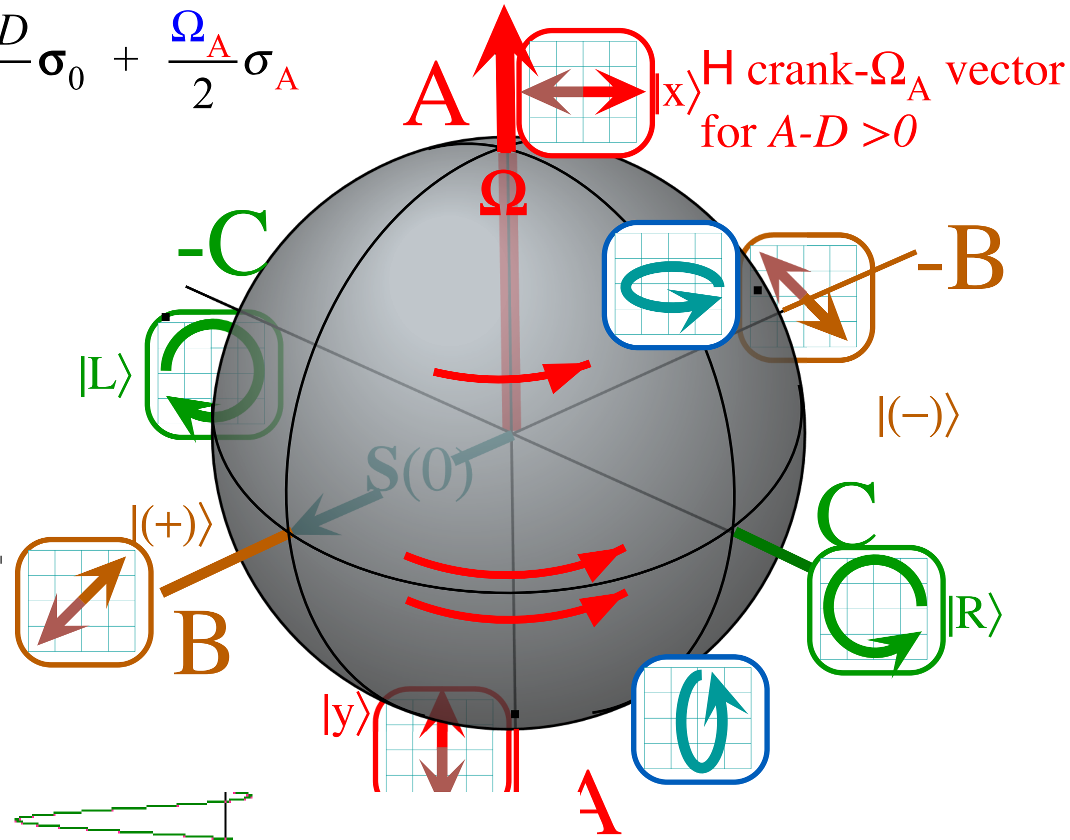
$$= \frac{A+D}{2}\,\boldsymbol{\sigma}_0 \quad + \quad \frac{\Omega_B}{2}\,\sigma_B \quad + \quad \frac{\Omega_C}{2}\,\sigma_C \quad + \quad \frac{\Omega_A}{2}\,\sigma_A$$

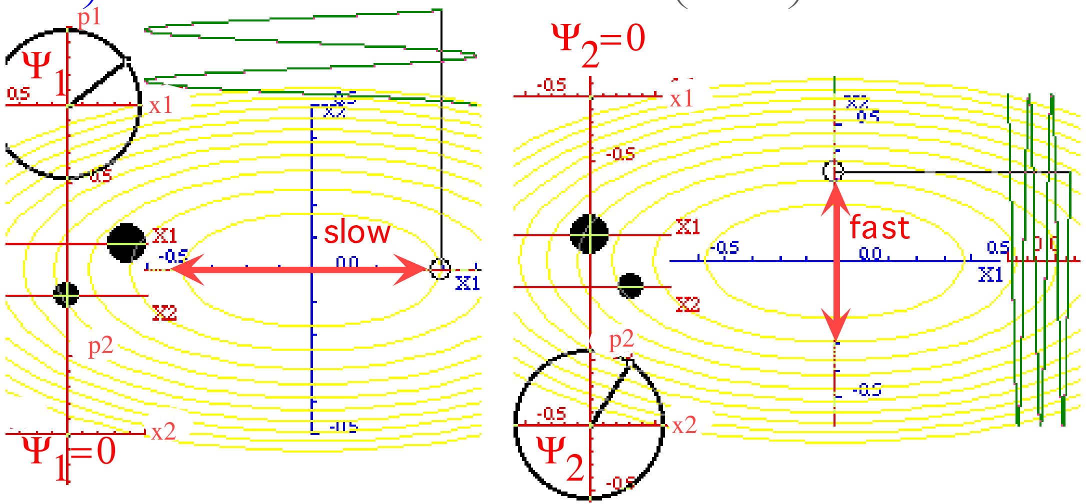
$$\vec{\Omega} = \begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix} = \begin{pmatrix}A-D\\ 2B\\ 2C\end{pmatrix}$$

## Asymmetric Diagonal A-Type motion

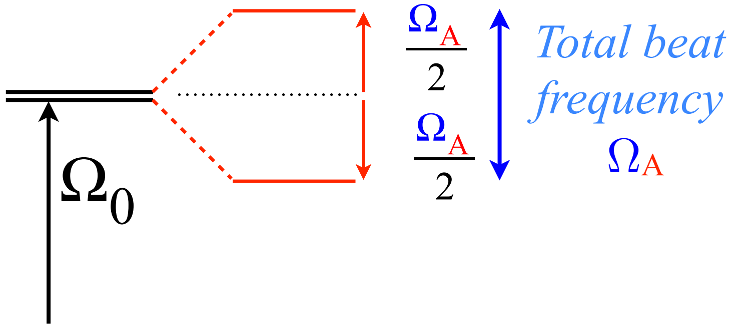
$$\begin{pmatrix}\langle 1|\mathbf{H}^A|1\rangle & \langle 1|\mathbf{H}^A|2\rangle\\ \langle 2|\mathbf{H}^A|1\rangle & \langle 2|\mathbf{H}^A|2\rangle\end{pmatrix} = \begin{pmatrix}A & 0\\ 0 & D\end{pmatrix} = \frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix} + \frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_A}{2}\sigma_A$$

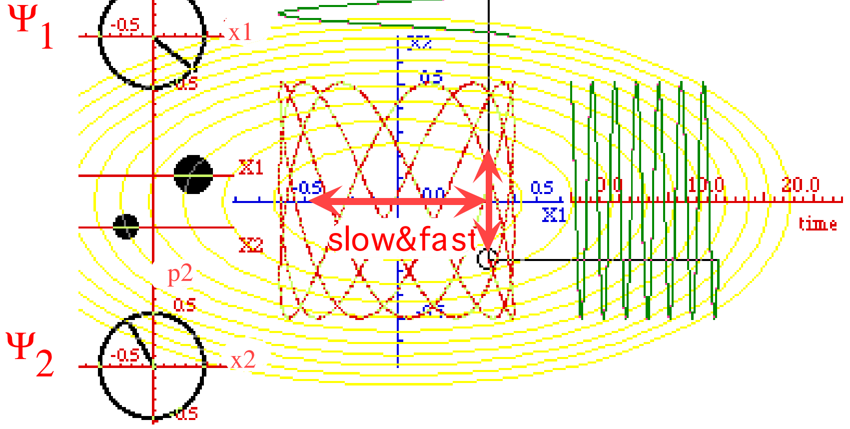
$$Crank:\ \vec{\Omega} = \begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix} = \begin{pmatrix}A-D\\ 0\\ 0\end{pmatrix} \qquad Eigen-Spin:\vec{\mathbf{S}} = \begin{pmatrix}S_A\\ S_B\\ S_C\end{pmatrix} = \begin{pmatrix}\pm S\\ 0\\ 0\end{pmatrix}$$





## Beat dynamics:





*Review: Fundamental Euler* $\mathbf{R}(\alpha\beta\gamma)$ *and Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*
- *Euler* $\mathbf{R}(\alpha\beta\gamma)$ *derived from Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *and vice versa*
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- *Asymmetric-Diagonal A-Type motion*
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$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

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$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_{\mathrm{B}} + C\,\sigma_{\mathrm{C}} + \frac{A-D}{2}\sigma_{\mathrm{A}}$$

$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_{\mathrm{B}} + \frac{\Omega_{\mathrm{C}}}{2}\sigma_{\mathrm{C}} + \frac{\Omega_{\mathrm{A}}}{2}\sigma_{\mathrm{A}}$$

$$\vec{\Omega} = \begin{pmatrix} \Omega_{\mathrm{A}} \\ \Omega_B \\ \Omega_{\mathrm{C}} \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

## Bilateral-Balanced B-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^B|1\rangle & \langle 1|\mathbf{H}^B|2\rangle \\ \langle 2|\mathbf{H}^B|1\rangle & \langle 2|\mathbf{H}^B|2\rangle \end{pmatrix} = \begin{pmatrix} \Omega_0 & B \\ B & \Omega_0 \end{pmatrix} = \Omega_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_{\mathrm{B}}$$

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$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

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$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

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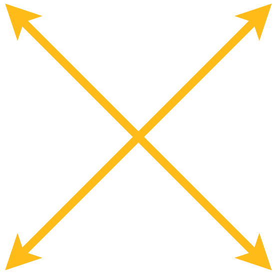
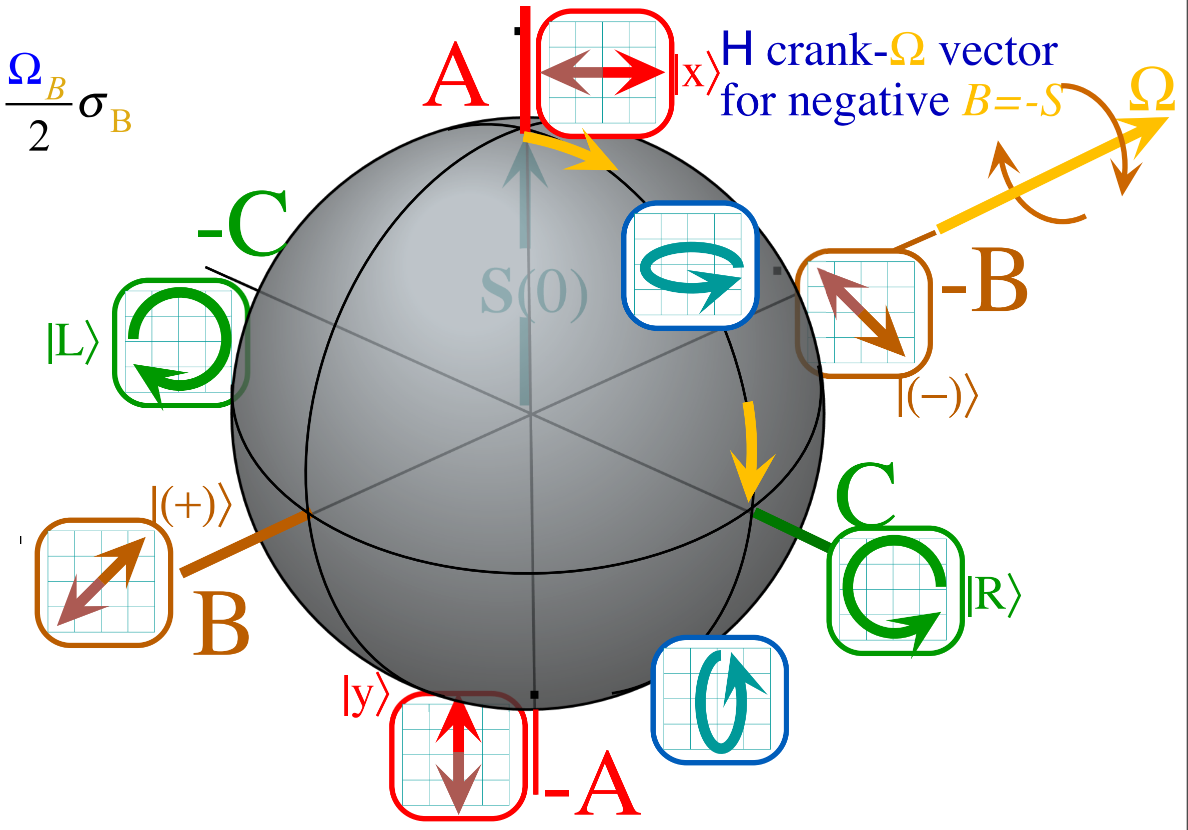
$$Crank:\ \vec{\Omega} = \begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix} = \begin{pmatrix}0\\ 2B\\ 0\end{pmatrix} \qquad Eigen-Spin:\vec{\mathbf{S}} = \begin{pmatrix}S_A\\ S_B\\ S_C\end{pmatrix} = \begin{pmatrix}0\\ \pm S\\ 0\end{pmatrix}$$





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$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

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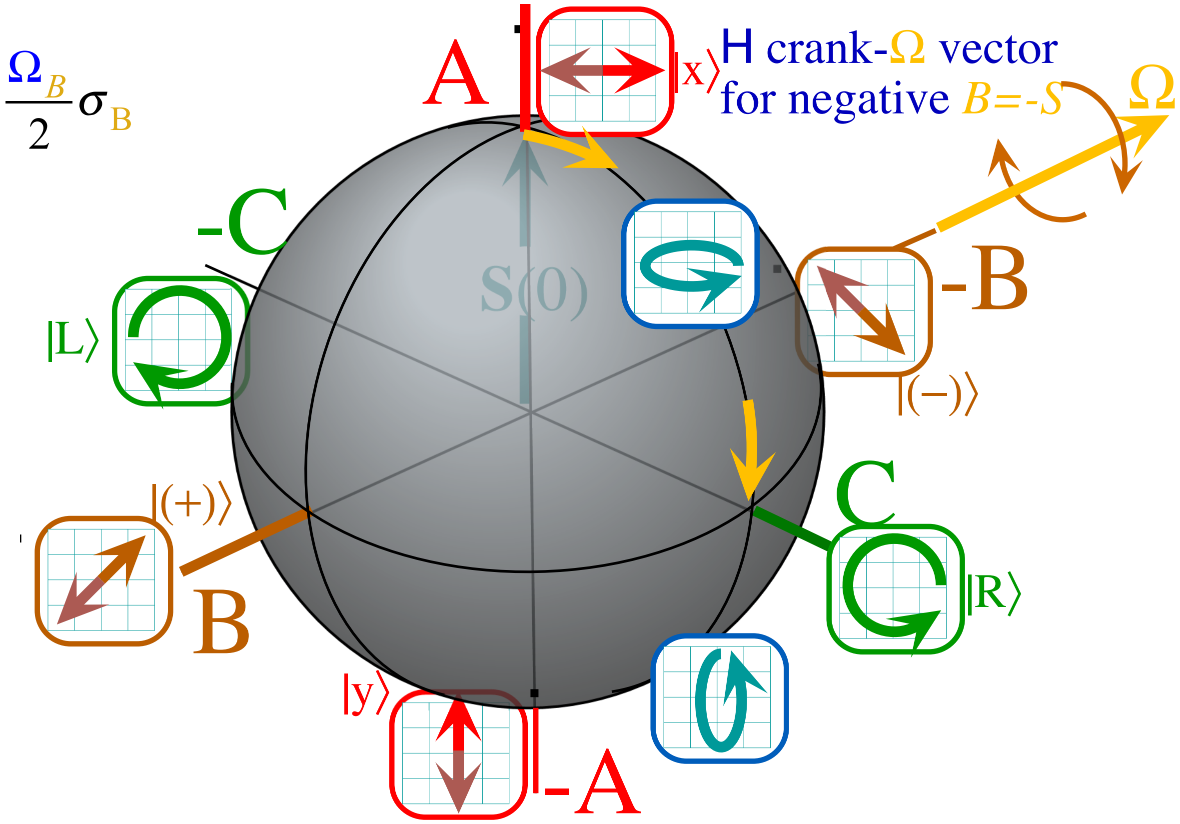
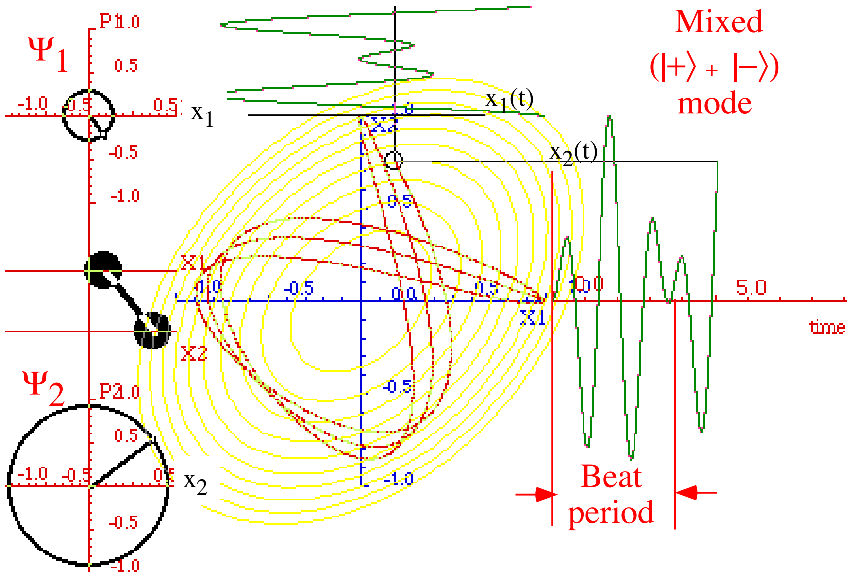


## Beat dynamics:



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$$= \frac{A+D}{2}\,\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\,\sigma_B + \frac{\Omega_C}{2}\,\sigma_C + \frac{\Omega_A}{2}\,\sigma_A$$

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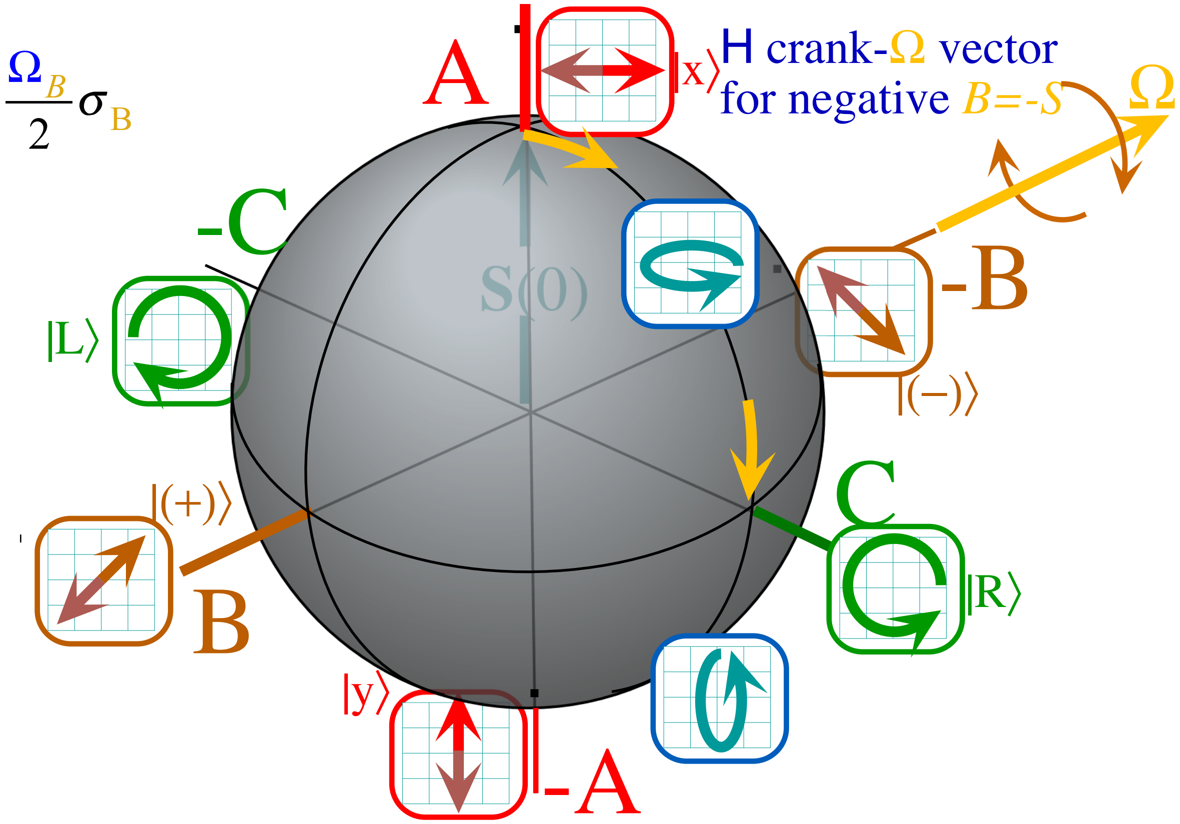
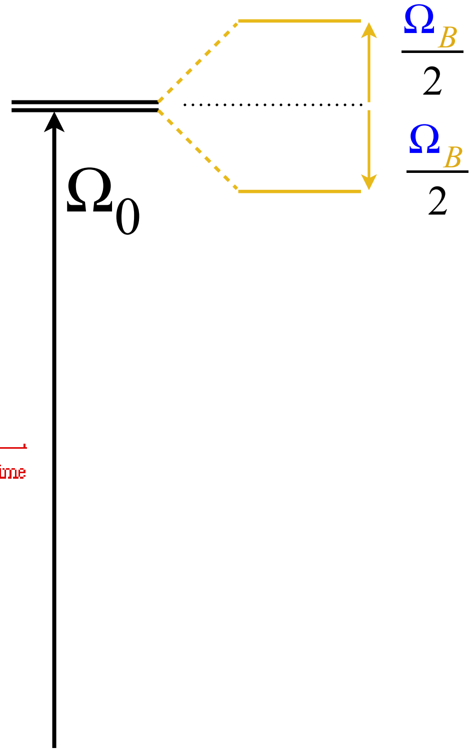
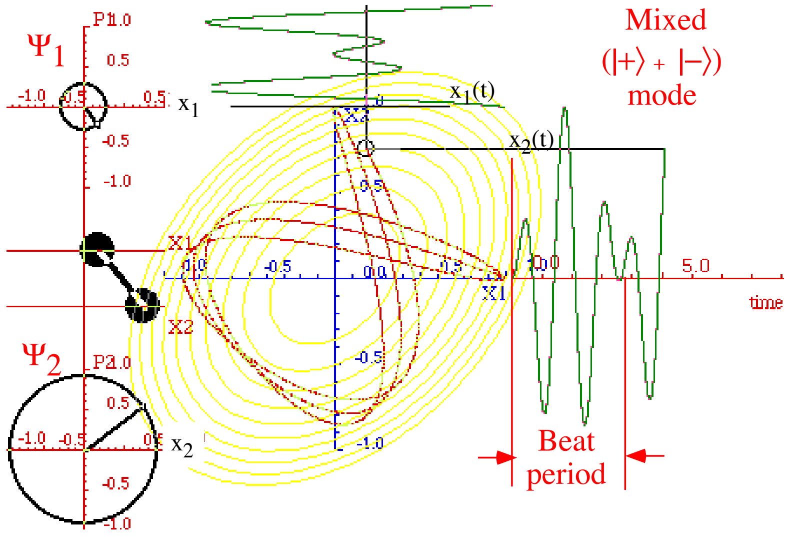
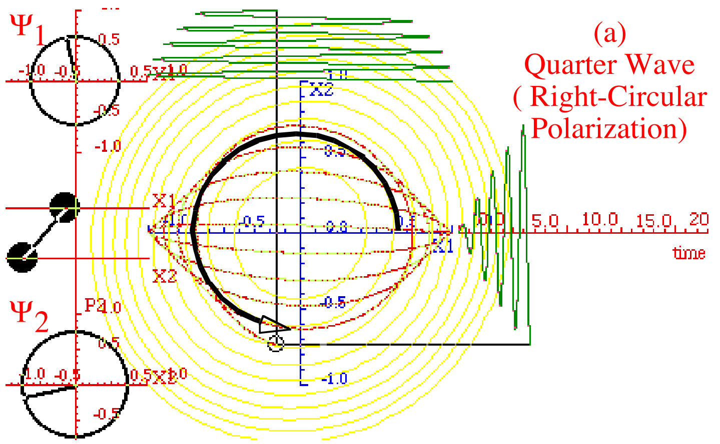


## Beat dynamics:

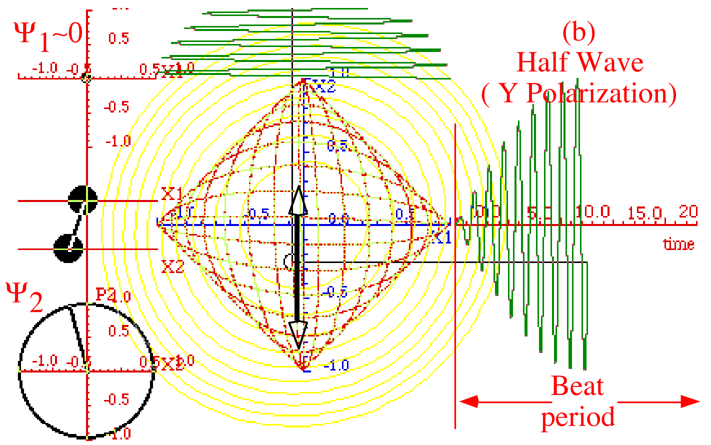


Total beat frequency $\Omega_B$





(a)
Quarter Wave
( Right-Circular
Polarization)



(b)
Half Wave
( Y Polarization)

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## Circular-Coriolis... C-Type motion

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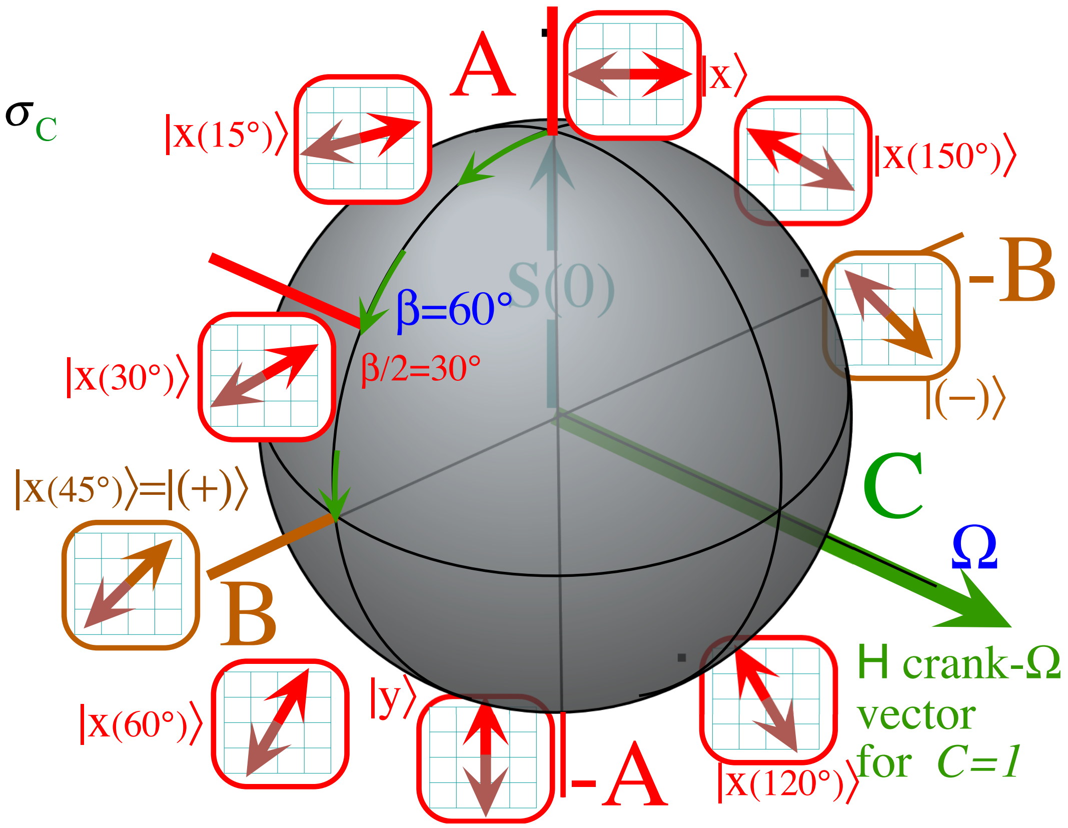

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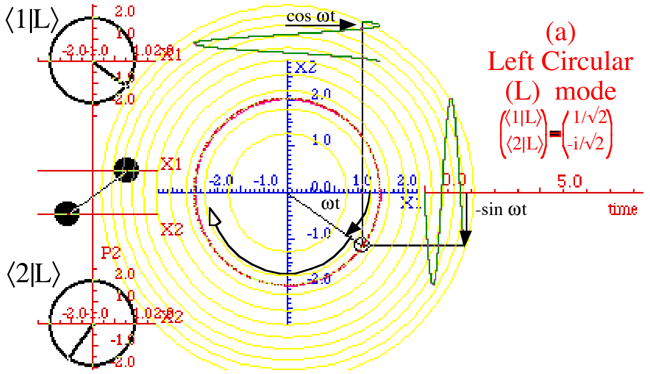

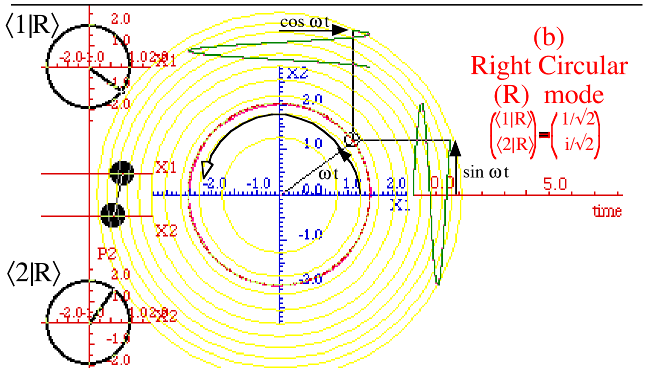

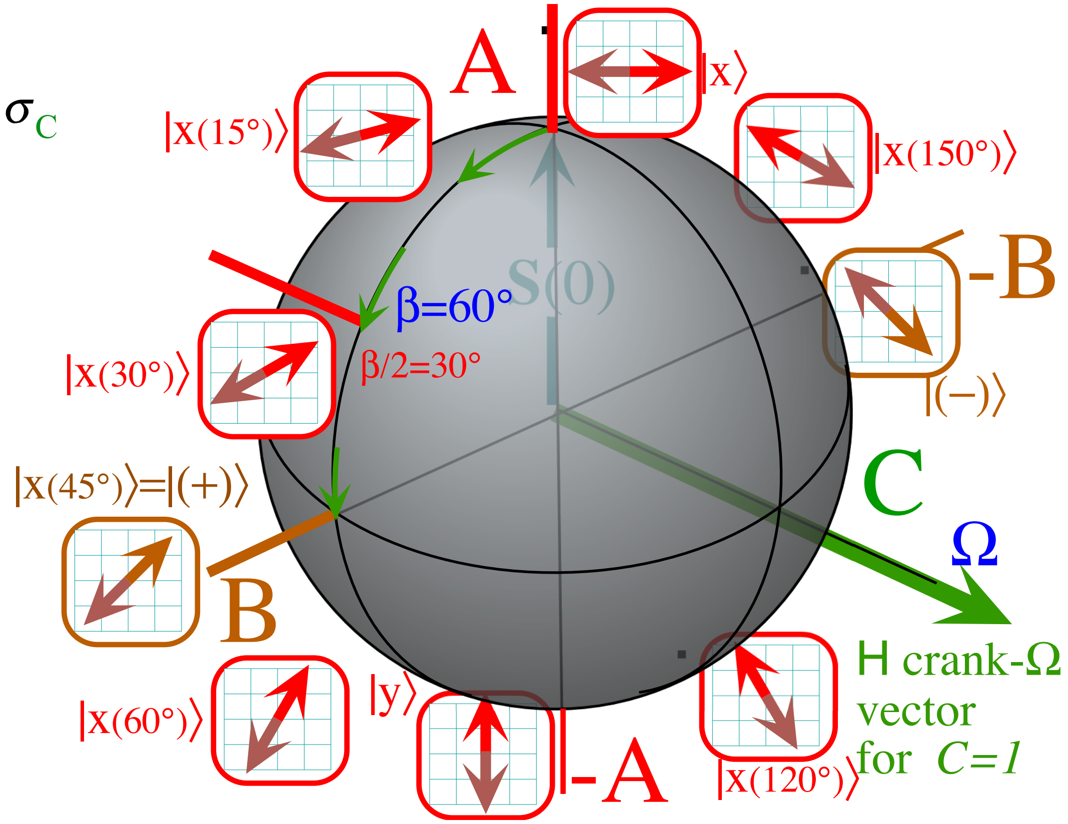

H crank-Ω vector for $C=1$

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$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

$$= \frac{A+D}{2}\sigma_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C+\frac{\Omega_A}{2}\sigma_A$$

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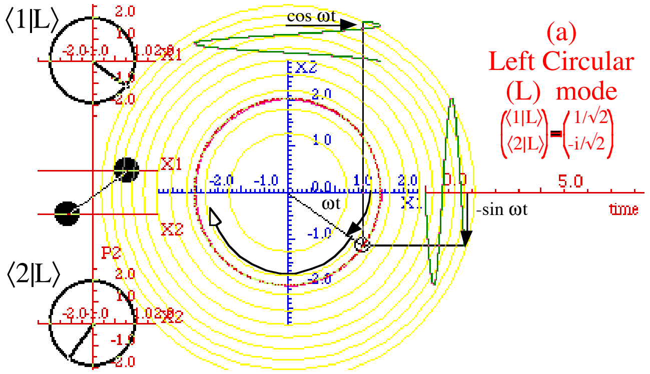
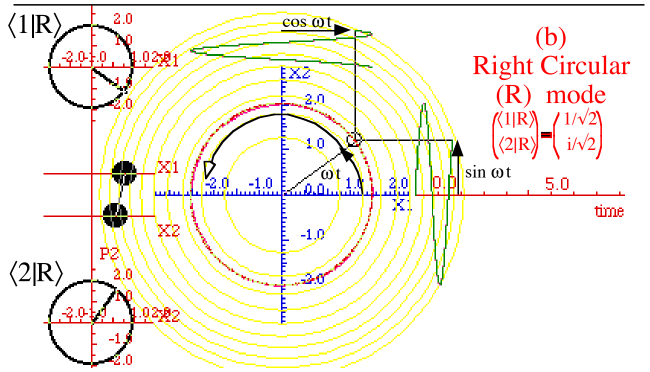


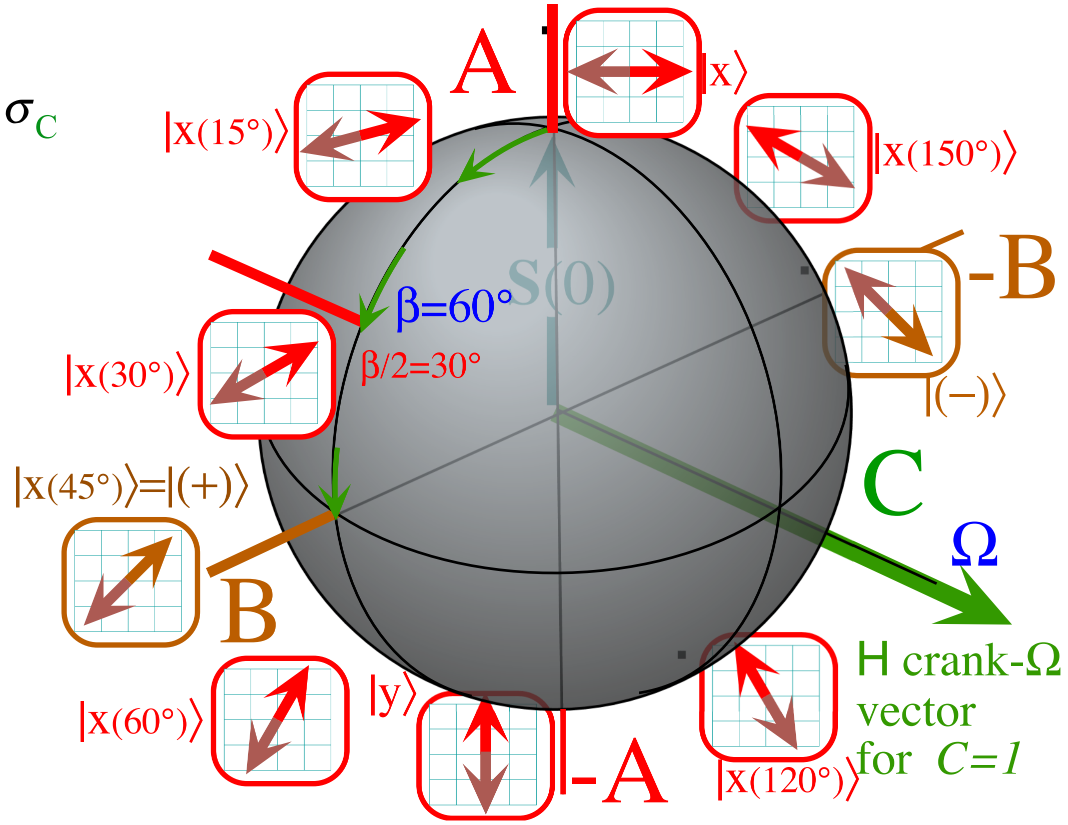
(a) Left Circular (L) mode

$$\begin{pmatrix}\langle 1|L\rangle\\ \langle 2|L\rangle\end{pmatrix}=\begin{pmatrix}1/\sqrt{2}\\ -i/\sqrt{2}\end{pmatrix}$$



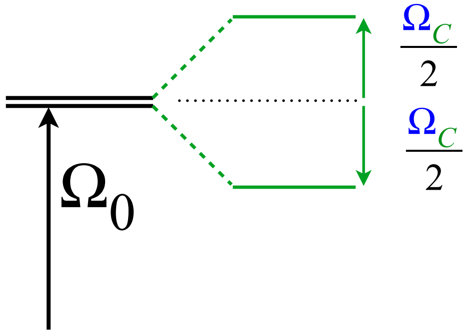
(b) Right Circular (R) mode

$$\begin{pmatrix}\langle 1|R\rangle\\ \langle 2|R\rangle\end{pmatrix}=\begin{pmatrix}1/\sqrt{2}\\ i/\sqrt{2}\end{pmatrix}$$

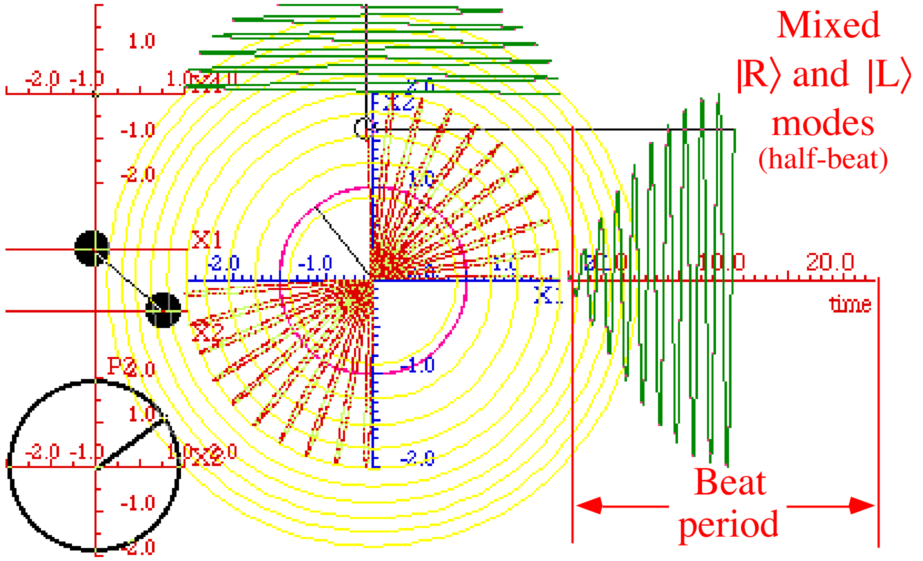


H crank-Ω vector for C=1

## Beat dynamics:



Total beat frequency $\Omega_C$



Mixed |R⟩ and |L⟩ modes (half-beat)

Beat period

*Review: Fundamental Euler* **R***(αβγ) and Darboux* **R**[φϑΘ] *representations of U(2) and R(3)*

*Euler* **R***(αβγ) derived from Darboux* **R**[φϑΘ] *and vice versa*

*Euler* **R***(αβγ) rotation* Θ=0−4π*-sequence* [φϑ] *fixed*

*R(3)-U(2) slide rule for converting* **R***(αβγ)* ↔ **R**[φϑΘ] *and Sundial*

*U(2) density operator approach to symmetry dynamics*

*Bloch equation for density operator*

*The ABC's of U(2) dynamics-Archetypes*

*Asymmetric-Diagonal A-Type motion*

*Bilateral-Balanced B-Type motion*

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*AB-Type motion and Wigner's Avoided-Symmetry-Crossings*

*ABC-Type elliptical polarized motion*

*Ellipsometry using U(2) symmetry coordinates*

*Conventional amp-phase ellipse coordinates*

*Euler Angle (αβγ) ellipse coordinates*

# The ABC's of U(2) dynamics-Mixed modes

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

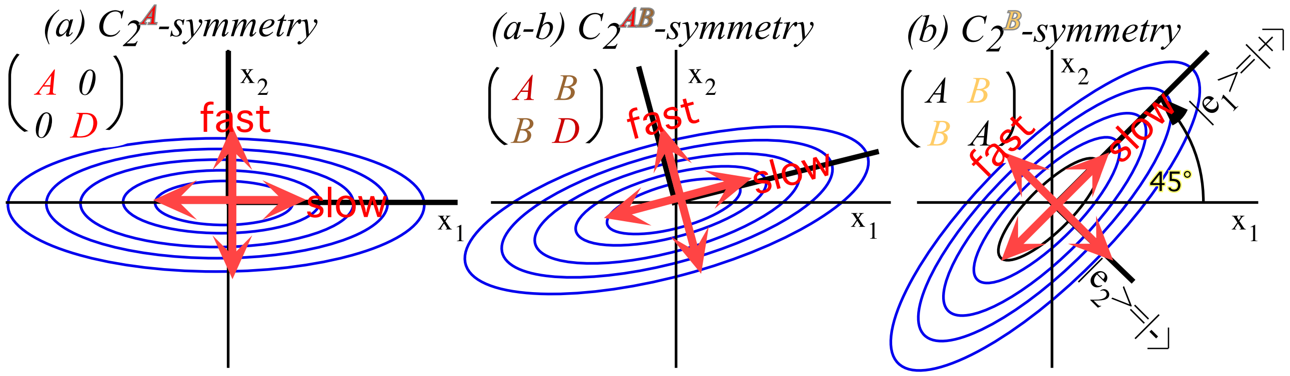
$$= \frac{A+D}{2}\sigma_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

## Tilted-plane polarization AB-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^{AB}|1\rangle & \langle 1|\mathbf{H}^{AB}|2\rangle \\ \langle 2|\mathbf{H}^{AB}|1\rangle & \langle 2|\mathbf{H}^{AB}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B \\ B & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_A}{2}\sigma_A + \frac{\Omega_B}{2}\sigma_B$$

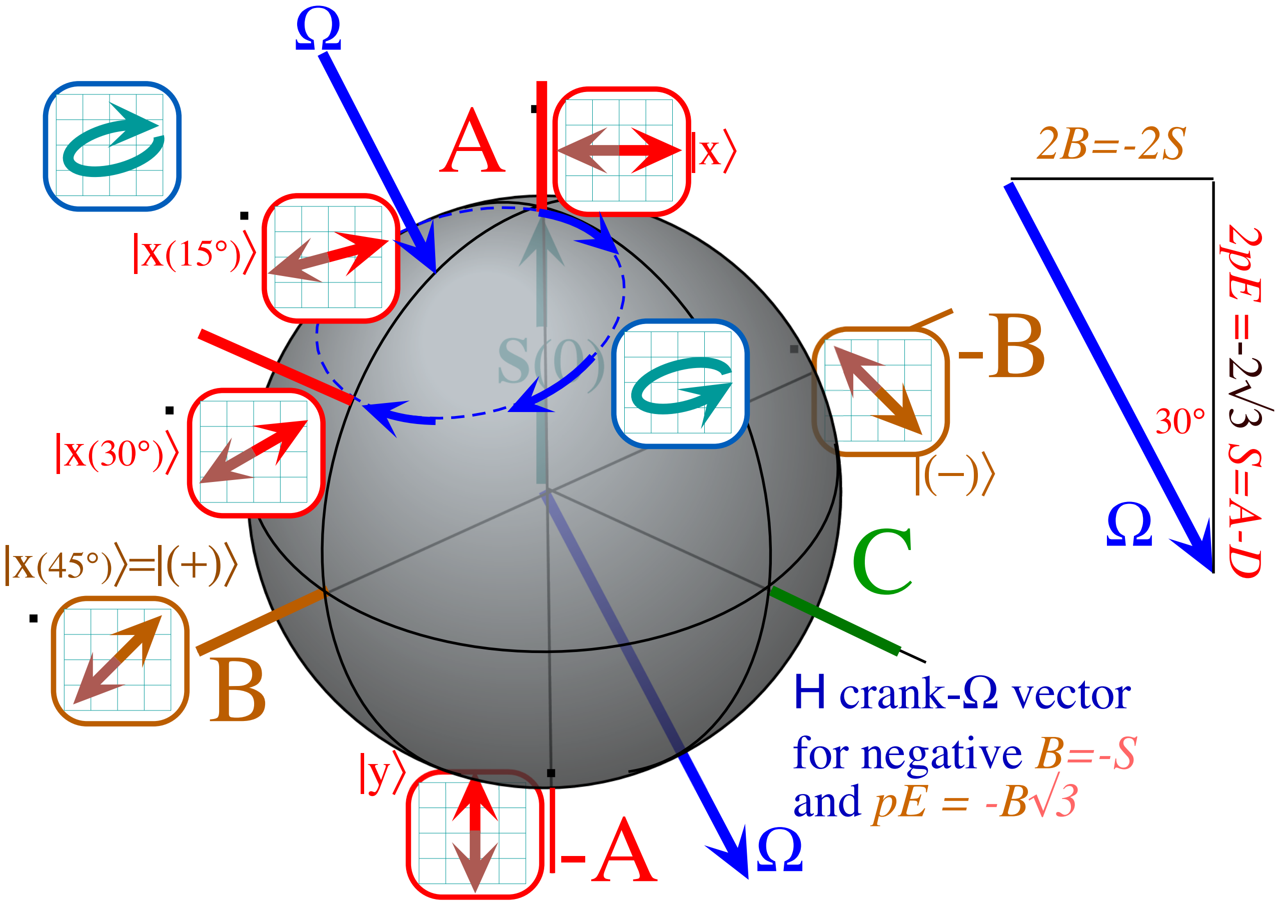
$$Crank:\ \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 0 \end{pmatrix} \qquad Eigen-Spin: \vec{\mathbf{S}} = \pm S\cdot\hat{\Omega}$$


(a) $C_2^A$-symmetry

(a-b) $C_2^{AB}$-symmetry

(b) $C_2^B$-symmetry

## Beat dynamics:



H crank-Ω vector for negative B=-S and pE = -B√3

# *A to B to A Symmetry breaking described by hyperbolic eigenvalues of* $A\boldsymbol{\sigma}_A+B\boldsymbol{\sigma}_B=\mathbf{H}=\begin{pmatrix}+A & B\\ B & -A\end{pmatrix}$

$$\mathbf{H}=\begin{pmatrix}+A & B\\ B & -A\end{pmatrix}$$ Secular equation: $\varepsilon^2-0\cdot\varepsilon-(A^2+B^2)$ gives *hyperbolic* energy levels: $\varepsilon=\pm\sqrt{A^2+B^2}$

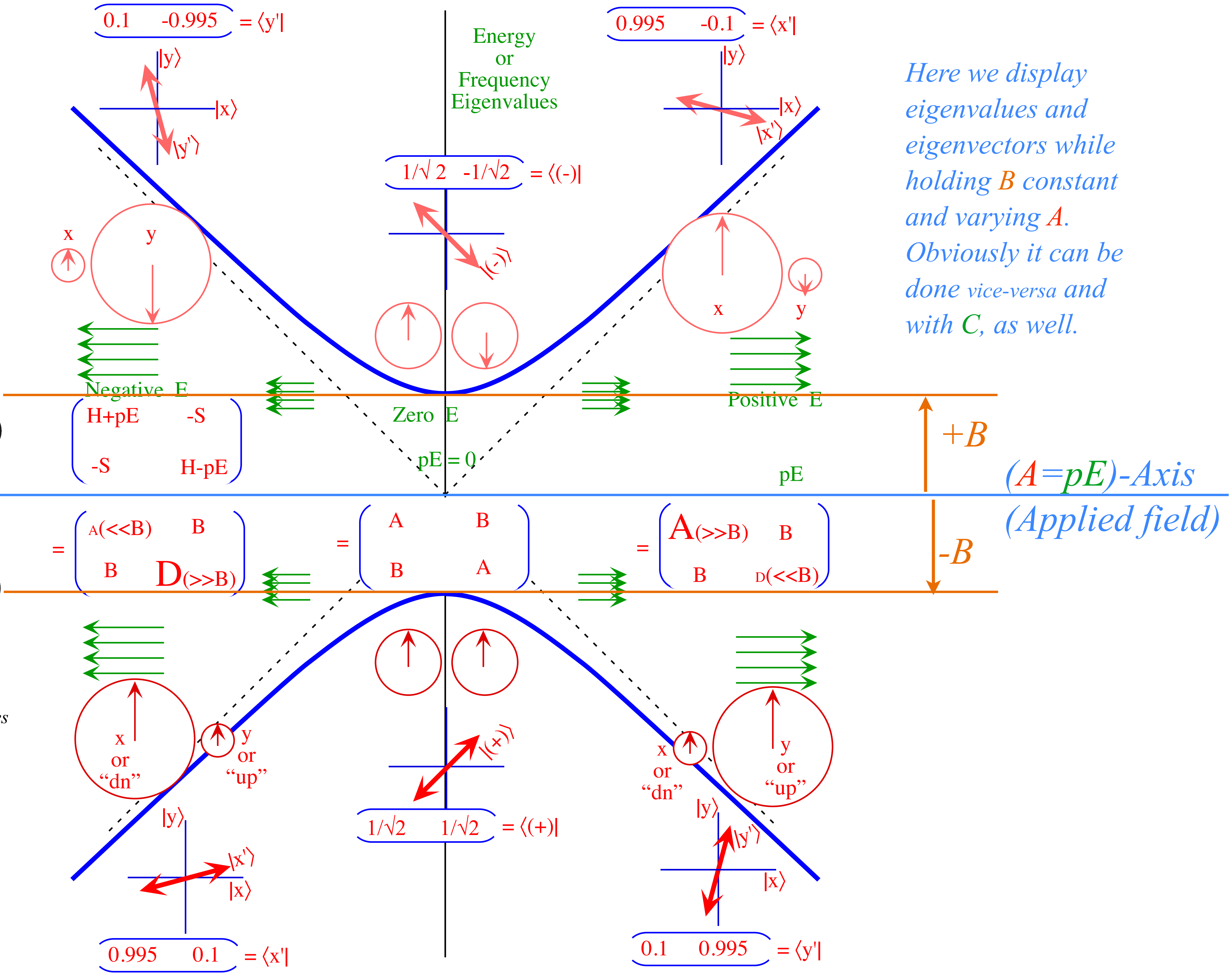


*Fig. 10.3.1 (b) Wigner avoided level crossing. (Fixed tunneling B=-S and variable A-D=pE field.)*

*Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa and with C, as well.*

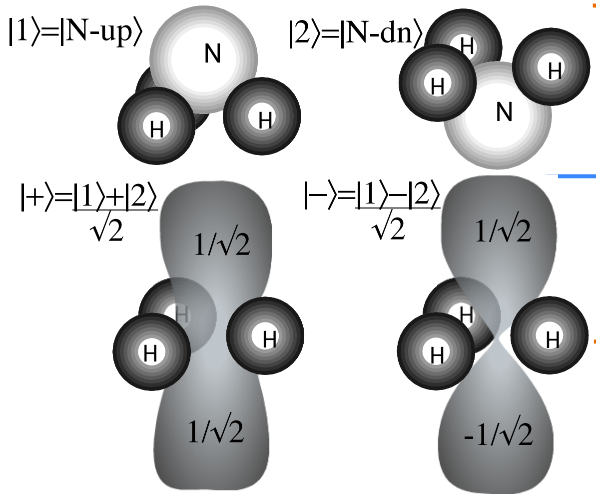


*Fig. 10.3.2 Ammonia ($NH_3$) inversion states*
*(a) Base states (b) $C_2$-Eigenstates*

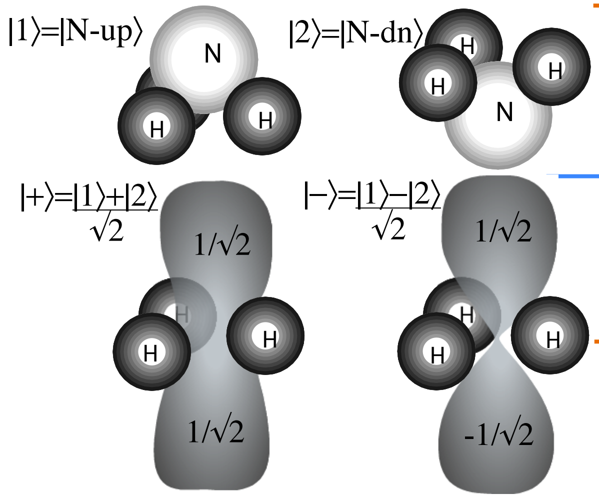
# A to B to A Symmetry breaking described by hyperbolic eigenvalues of $A\boldsymbol{\sigma}_A + B\boldsymbol{\sigma}_B = \mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$

$\mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$ Secular equation: $\varepsilon^2 - 0\cdot\varepsilon - (A^2 + B^2)$ gives *hyperbolic* energy levels: $\varepsilon = \pm\sqrt{A^2 + B^2}$

$\mathbf{H}(B-basis)$ $\mathbf{H}(A-basis)$

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}} \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}}$$

*Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa and with C, as well.*



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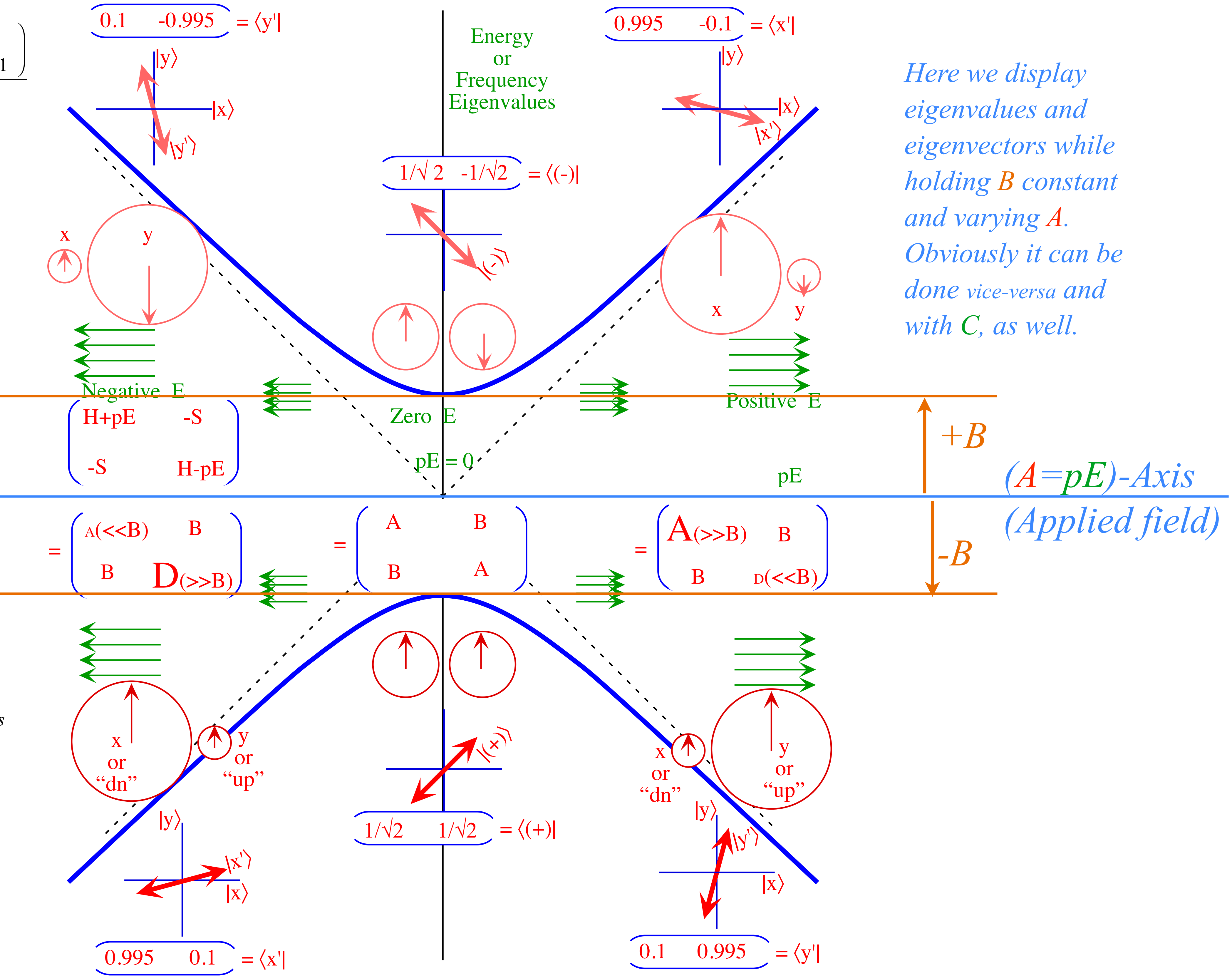

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$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}} \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} +A & B \\ B & -A \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} +A+B & B-A \\ +A-B & B+A \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

*Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa and with C, as well.*

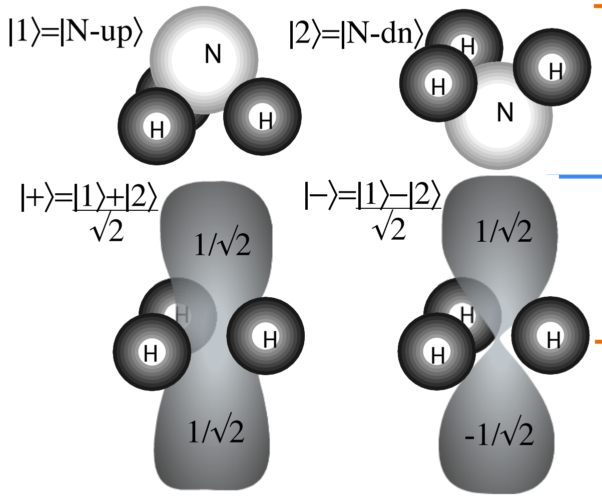
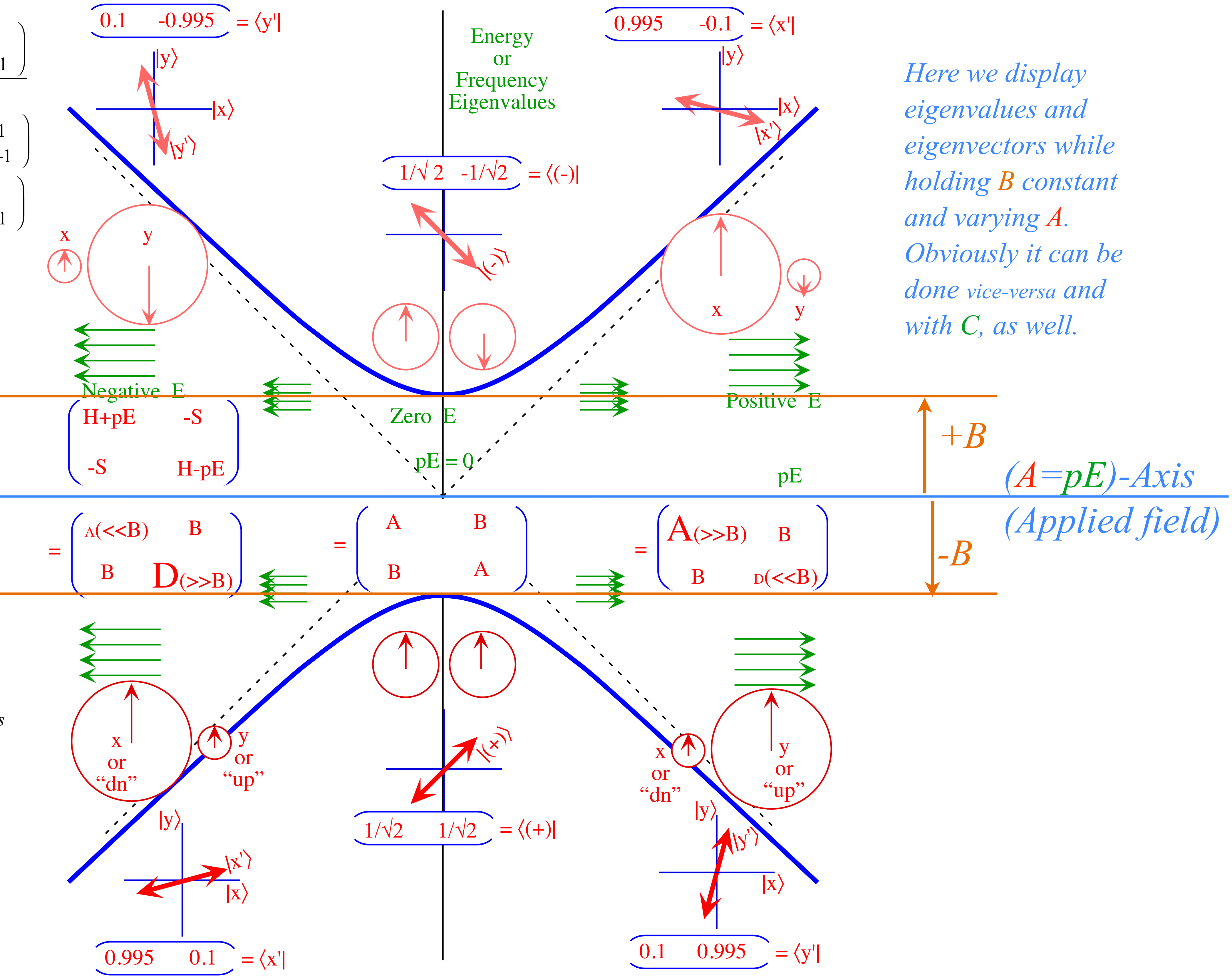


*Fig. 10.3.2 Ammonia (NH$_3$) inversion states*
*(a) Base states (b) C$_2$-Eigenstates*



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$\mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$ Secular equation: $\varepsilon^2 - 0 \cdot \varepsilon - (A^2 + B^2)$ gives *hyperbolic* energy levels: $\varepsilon = \pm\sqrt{A^2 + B^2}$

$\mathbf{H}(B-basis)$ $\mathbf{H}(A-basis)$

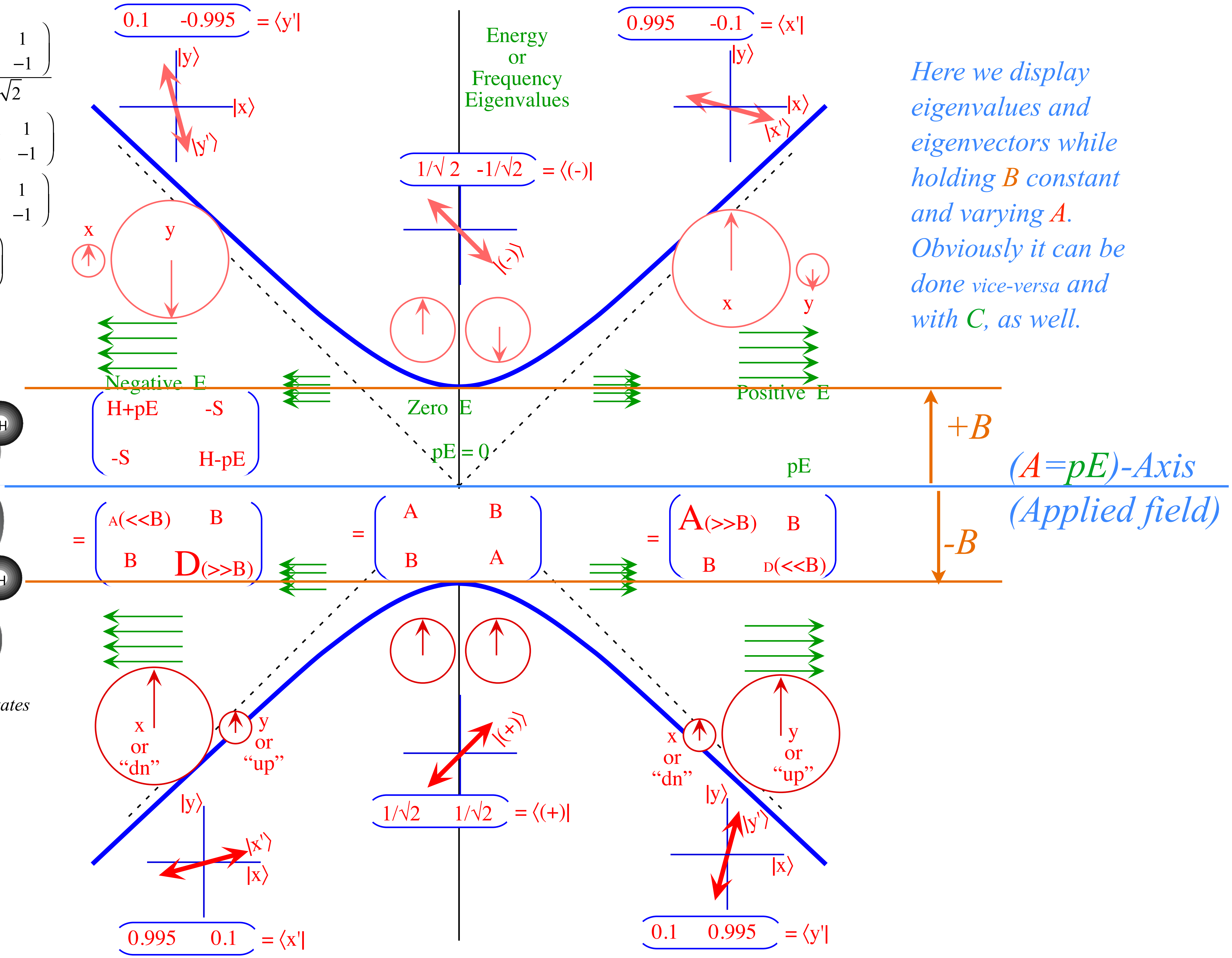
$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}} \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} +A+B & B-A \\ +A-B & B+A \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 2B & 2A \\ 2A & -2B \end{pmatrix}$$

*Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa and with C, as well.*



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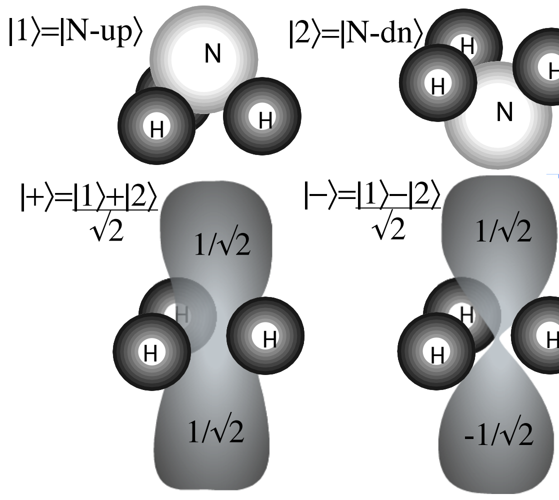

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*(a) Base states (b) $C_2$-Eigenstates*

# A to B to A Symmetry breaking described by hyperbolic eigenvalues of $A\boldsymbol{\sigma}_A + B\boldsymbol{\sigma}_B = \mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$

$$\mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \quad \text{Secular equation: } \varepsilon^2 - 0\cdot\varepsilon - (A^2 + B^2) \quad \text{gives } \textit{hyperbolic} \text{ energy levels: } \varepsilon = \pm\sqrt{A^2 + B^2}$$

$\mathbf{H}(B-basis)$ $\qquad$ $\mathbf{H}(A-basis)$

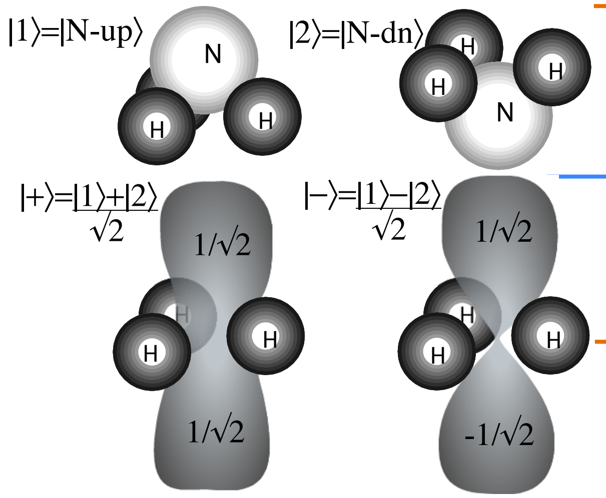
$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}} \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} +A+B & B-A \\ +A-B & B+A \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2B & 2A \\ 2A & -2B \end{pmatrix}$$

$$= \begin{pmatrix} +B & A \\ A & -B \end{pmatrix}$$



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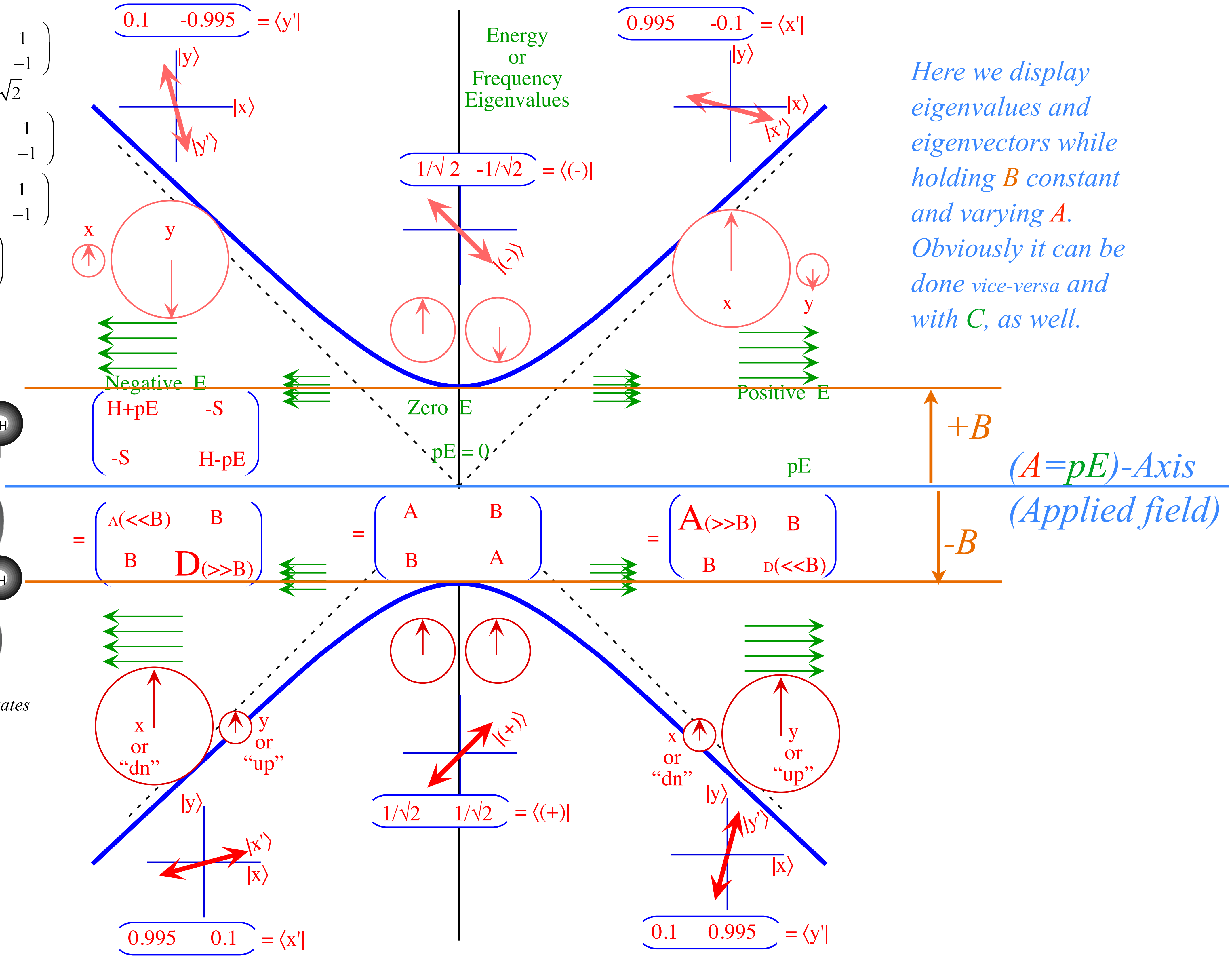


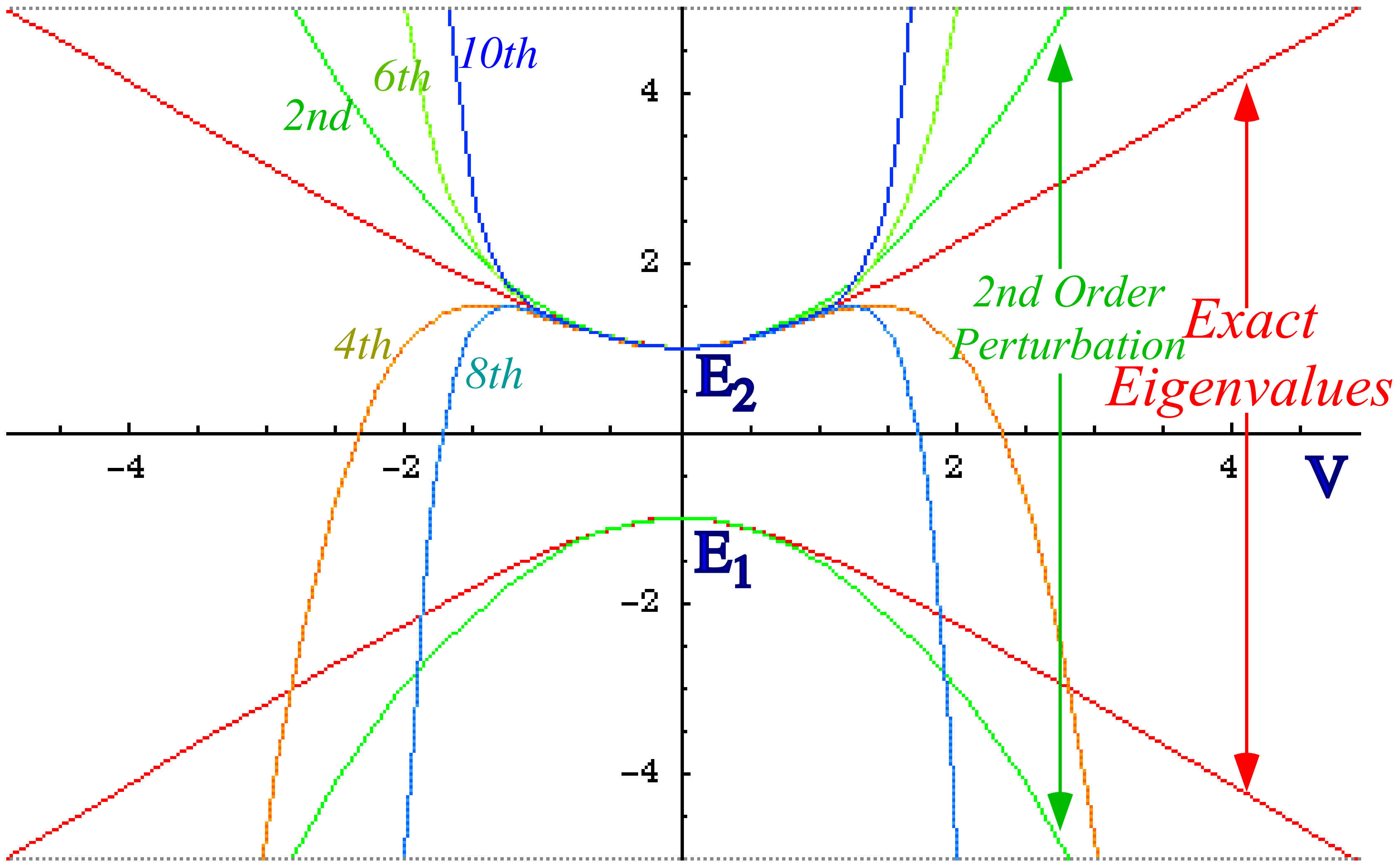
*Fig. 10.3.1 (b) Wigner avoided level crossing. (Fixed tunneling B=-S and variable A-D=pE field.)*

$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} E_1 & V \\ V & E_2 \end{pmatrix}$$

## 2nd order perturbation terms

$$\lambda_1 = E_1 + \frac{V^2}{E_1 - E_2} ,$$

$$\lambda_2 = E_2 + \frac{V^2}{E_2 - E_1} .$$



$$\lambda^2 - (Trace\mathbf{H})\lambda + \det|\mathbf{H}| = 0 = \lambda^2 - (E_1 + E_2)\lambda + (E_1E_2 - V^2)$$

$$\lambda_{1,2} = \frac{E_1 + E_2 \pm \sqrt{(E_1 + E_2)^2 - 4E_1E_2 + 4V^2}}{2} = \frac{E_1 + E_2 \pm \sqrt{(E_1 - E_2)^2 + 4V^2}}{2} ,$$

*Fig. 3.2.2 Comparison of exact vs. 2nd-order thru 10th-order perturbation approximations*

$$E_2 = \frac{\Delta}{2} + \frac{V^2}{\Delta} - \frac{V^4}{\Delta^3} + \frac{V^6}{\Delta^5} - \frac{V^8}{\Delta^7} + \frac{V^{10}}{\Delta^9} \cdots \text{, where: } \Delta = |E_1 - E_2|$$

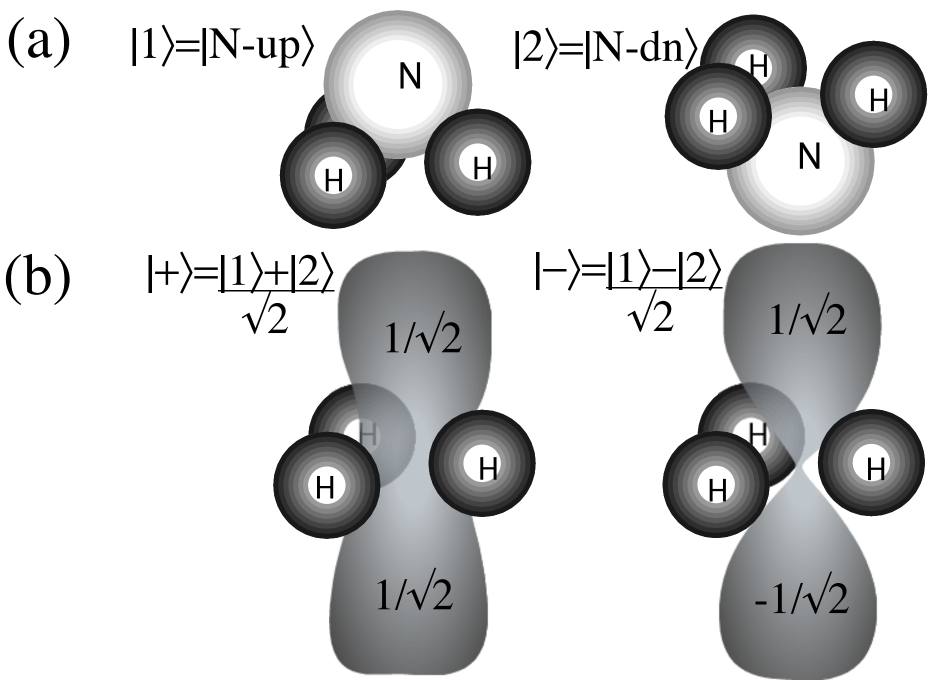
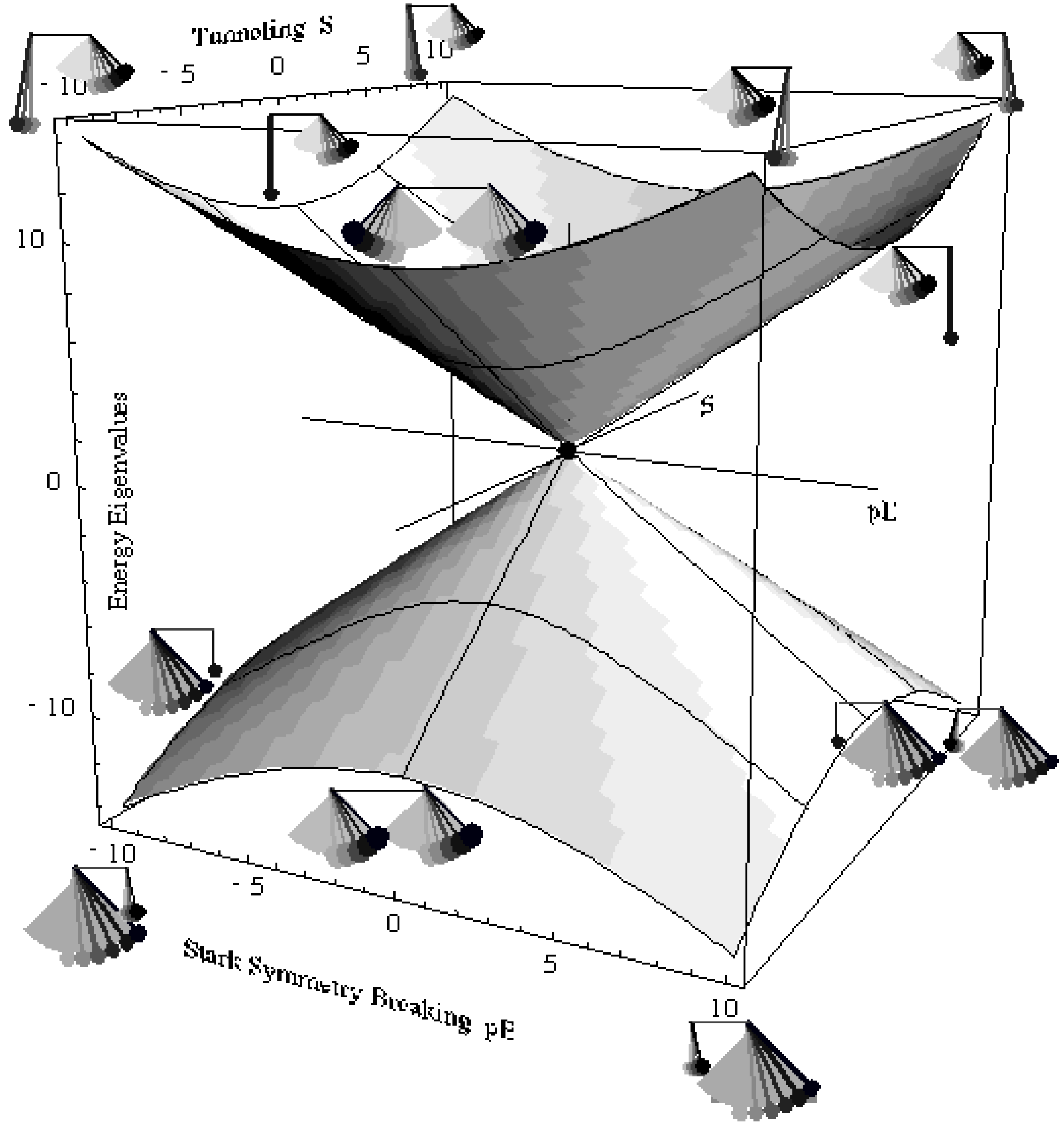


*Fig. 10.3.2 Ammonia ($NH_3$) inversion states*
*(a) Base states (b) $C_2$-Eigenstates*



*10.3.1 (a) Two state eigenvalue "diablo" surfaces and conical intersection and pendulum eigenstates.*

*Review: Fundamental Euler* **R**$(\alpha\beta\gamma)$ *and Darboux* **R**$[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

*Euler* **R**$(\alpha\beta\gamma)$ *derived from Darboux* **R**$[\varphi\vartheta\Theta]$ *and vice versa*
*Euler* **R**$(\alpha\beta\gamma)$ *rotation* $\Theta$=0−4π*-sequence* $[\varphi\vartheta]$ *fixed*
*R(3)-U(2) slide rule for converting* **R**$(\alpha\beta\gamma)$ ↔ **R**$[\varphi\vartheta\Theta]$ *and Sundial*

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→ *ABC-Type elliptical polarized motion*

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*Conventional amp-phase ellipse coordinates*
*Euler Angle* $(\alpha\beta\gamma)$ *ellipse coordinates*

# ABC-Type elliptical polarized motion

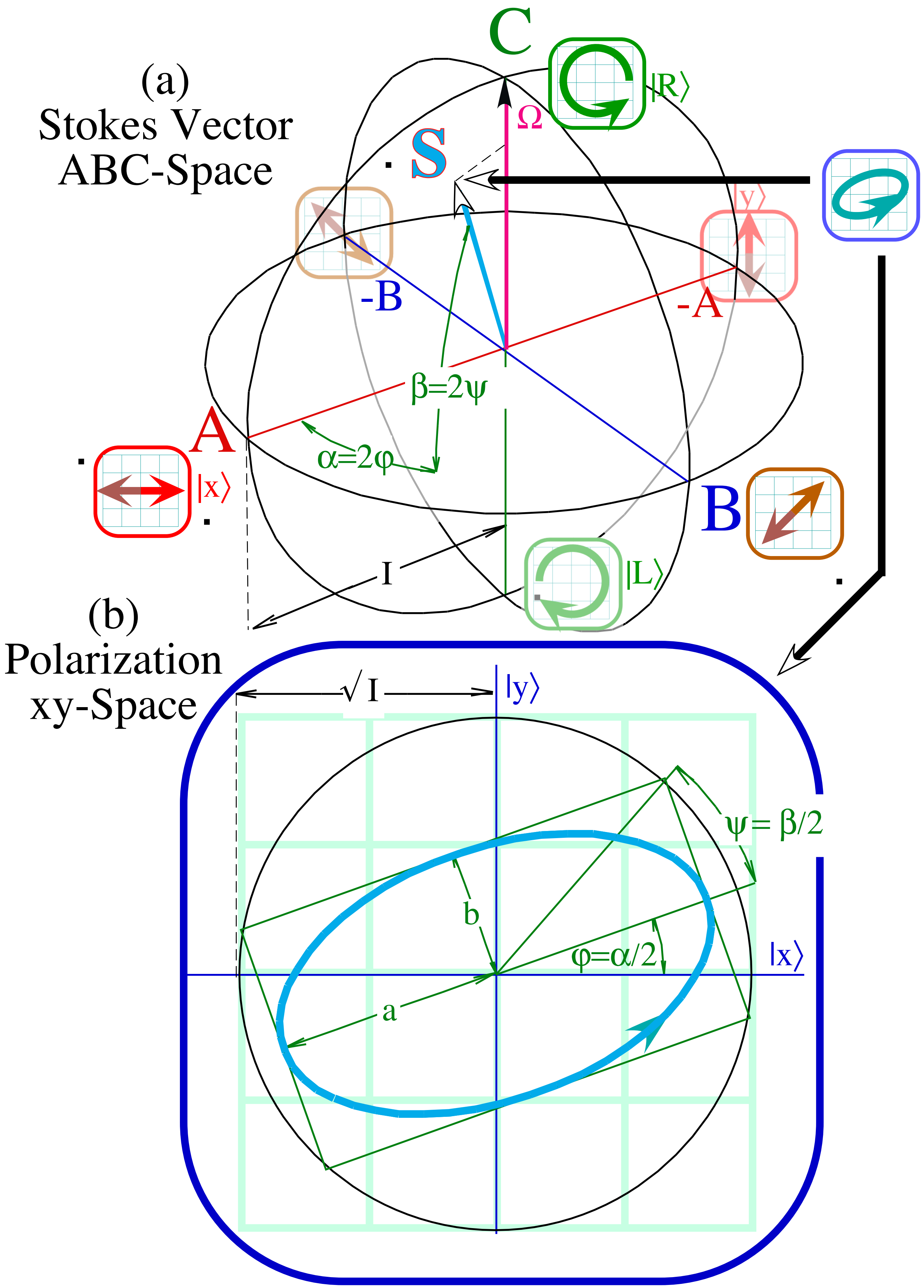


*Fig. 10.B.3*

*Euler-like coordinates for*
*(a) R(3) spin vector*
*(b) U(2) polarization ellipse*

# *ABC-Type elliptical polarized motion*

*(from Principles of Symmetry, Dynamics, and Spectroscopy)*

642 THEORY AND APPLICATION OF SYMMETRY REPRESENTATION PRODUCTS

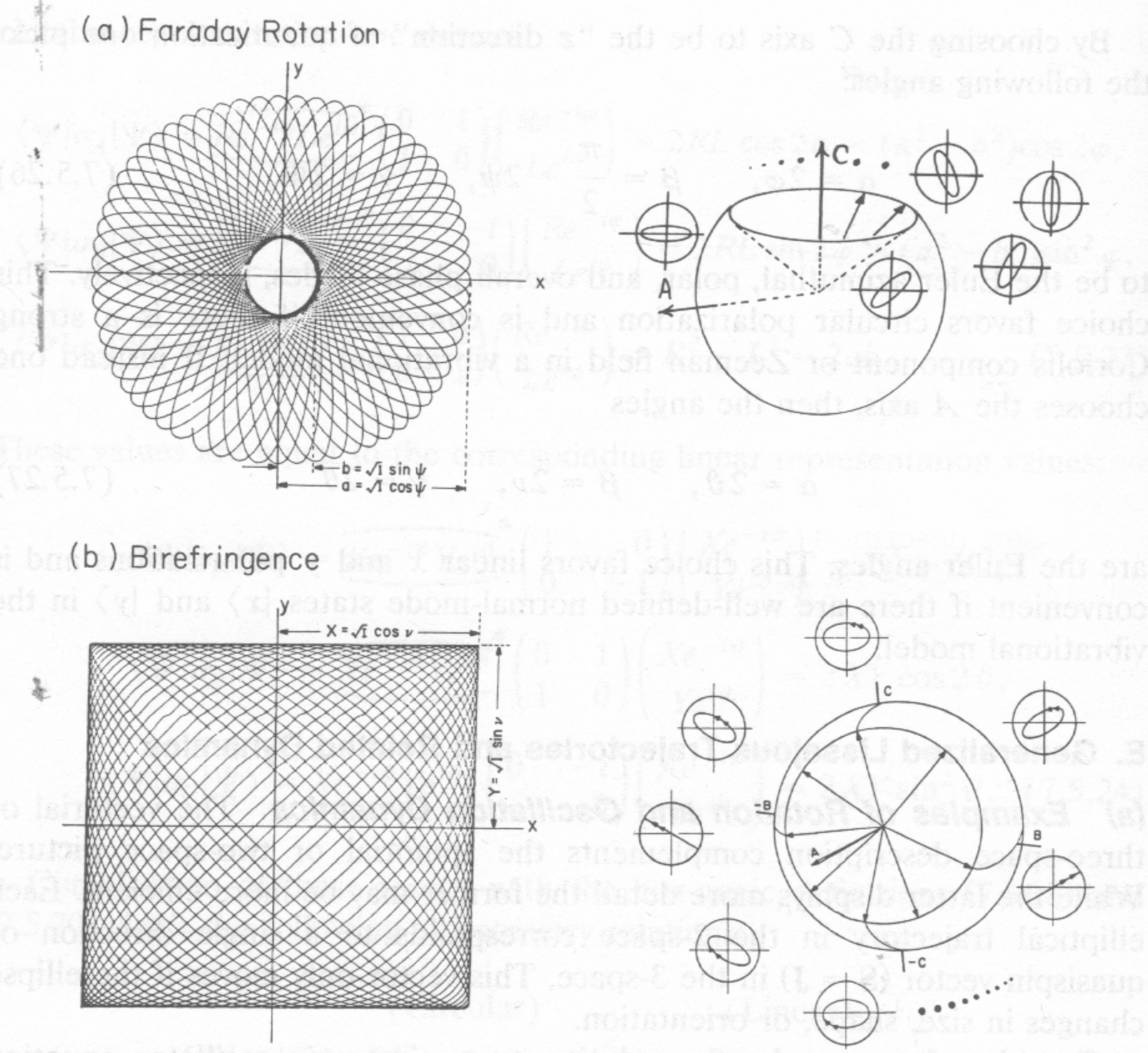
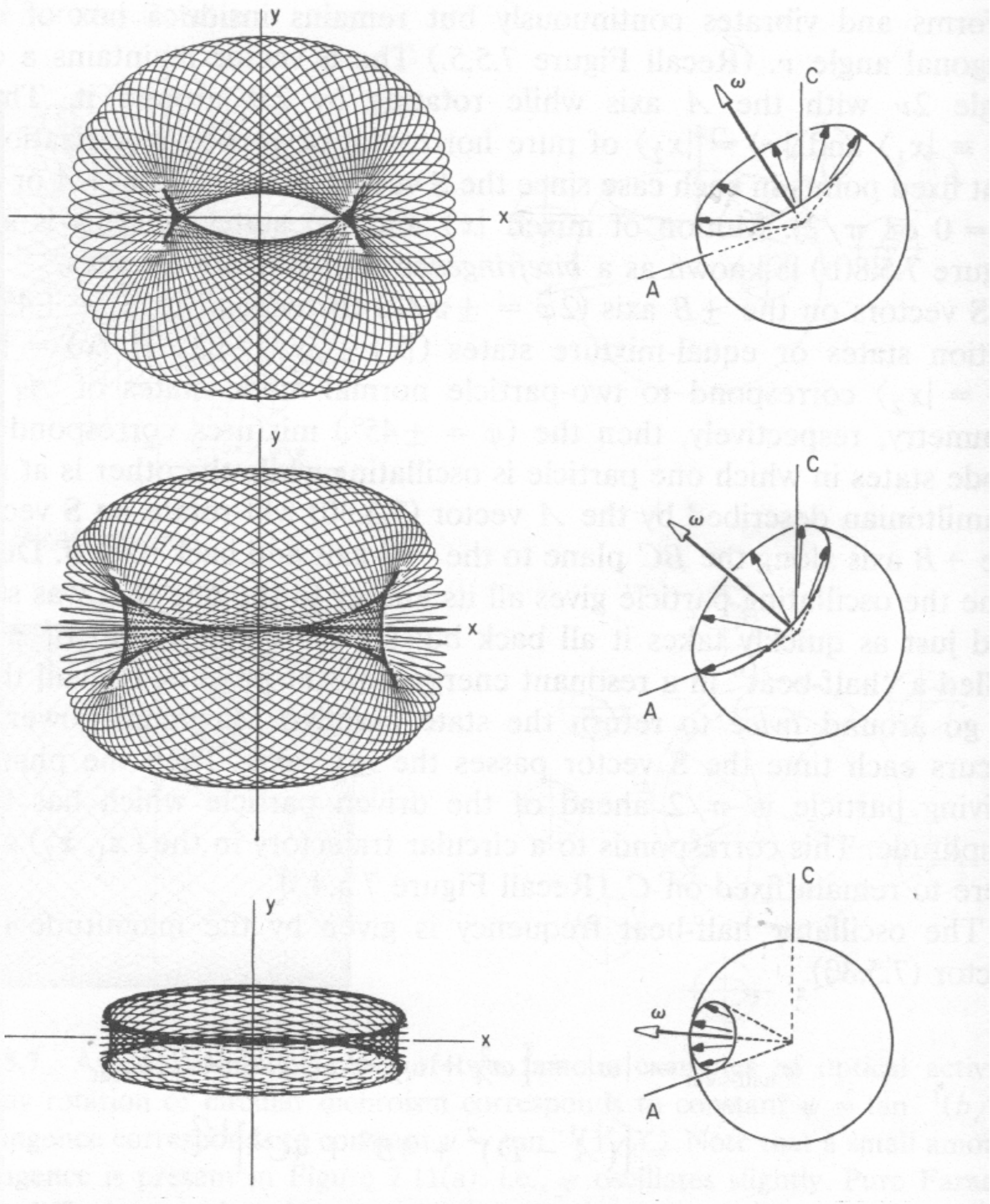
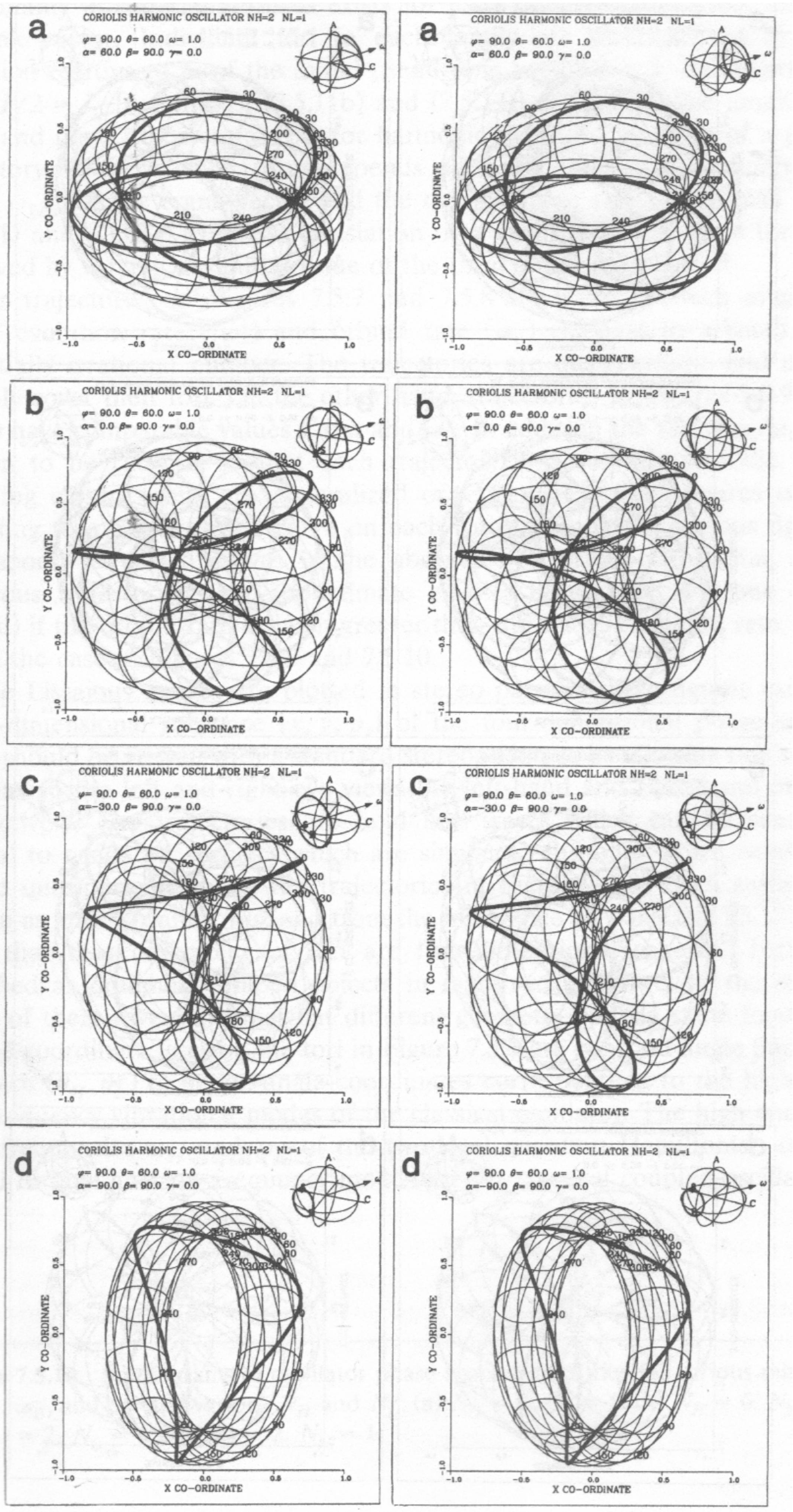


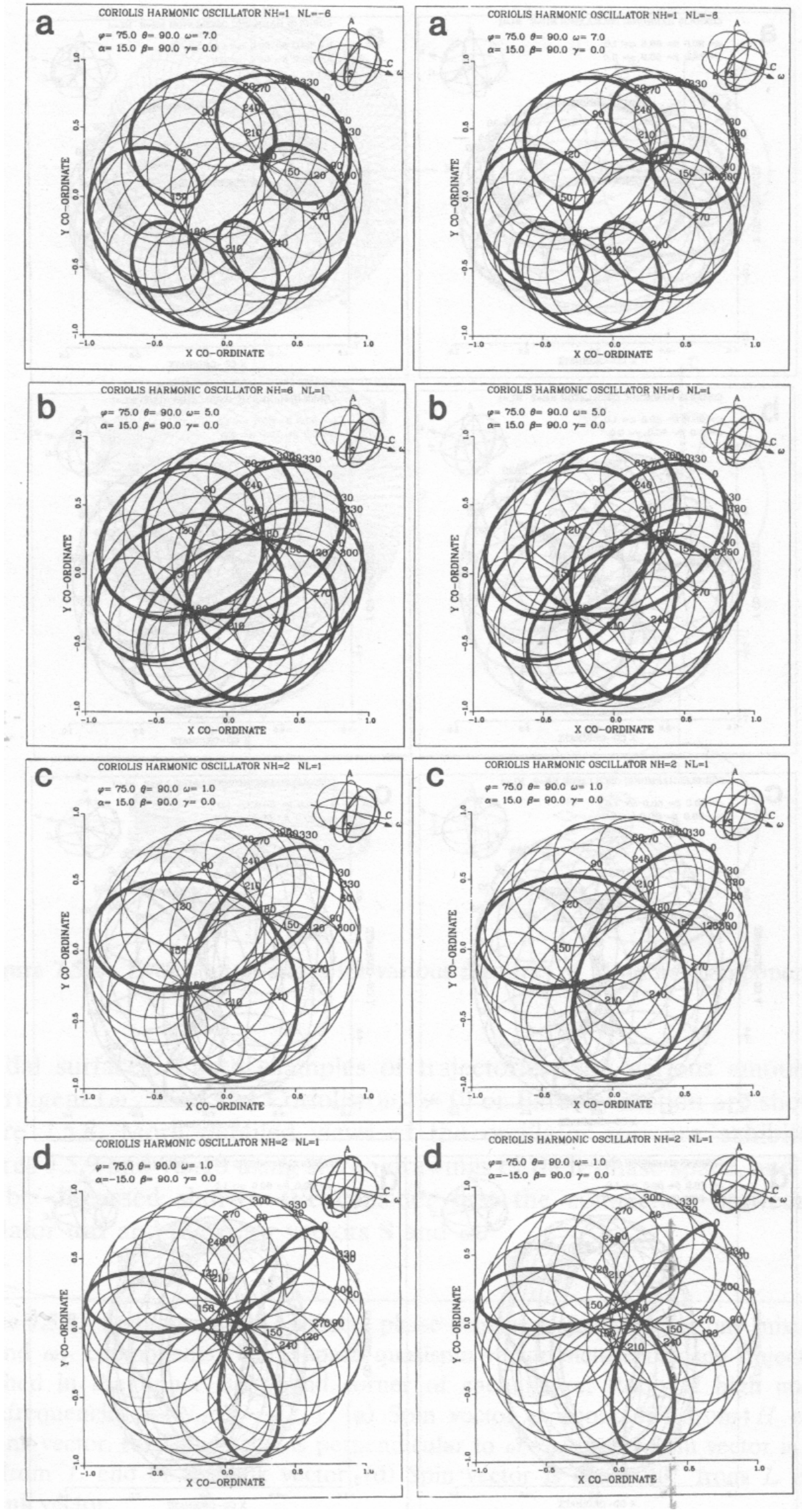
**Figure 7.5.7** Analog computer plots of two famous examples of optical activity. (a) Faraday rotation or circular dichroism corresponds to constant $\psi = \tan^{-1}(b/a)$. (b) Birefringence corresponds to constant $\nu = \tan^{-1}(Y/X)$. Note that a small amount of birefringence is present in Figure 7.11(a); i.e., $\psi$ oscillates slightly. Pure Faraday rotation is difficult to achieve on an analog computer.



**7.5.8** Evolution of states for various mixtures of $A$ and $C$ components.



*ABC-Type elliptical polarized dynamics*



*Review: Fundamental Euler* **R***(αβγ) and Darboux* **R**[φϑΘ] *representations of U(2) and R(3)*

- *Euler* **R***(αβγ) derived from Darboux* **R**[φϑΘ] *and vice versa*
- *Euler* **R***(αβγ) rotation* Θ=0−4*π-sequence* [φϑ] *fixed*
- *R(3)-U(2) slide rule for converting* **R***(αβγ)* ↔ **R**[φϑΘ] *and Sundial*

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*Ellipsometry using U(2) symmetry coordinates*

- ➡ *Conventional amp-phase ellipse coordinates*
- *Euler Angle (αβγ) ellipse coordinates*

# Ellipsometry using U(2) symmetry coordinates

## Conventional amp-phase ellipse coordinates and related to Euler Angles $(\alpha\beta\gamma)$

*2D* elliptic frequency $\omega$ orbit has amplitudes

$A_1$ and $A_2$, and phase shifts $\rho_1$ and $\rho_2=-\rho_1$.

$$x_1= A_1\cos(\omega t+\rho_1)$$

$$-p_1=A_1 \sin(\omega t+\rho_1)$$

$$x_2= A_2\cos(\omega t-\rho_1)$$

$$-p_2=A_2 \sin(\omega t-\rho_1)$$

*Amp-phase parameters* $(A_1,A_2, \omega t,\rho_1 )$

$$\begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ \\ x_2+ip_2 \end{pmatrix}$$

$p_1 = -A_1 \sin(\omega t + \rho_1)$

$p_2 = -A_2 \sin(\omega t - \rho_1)$

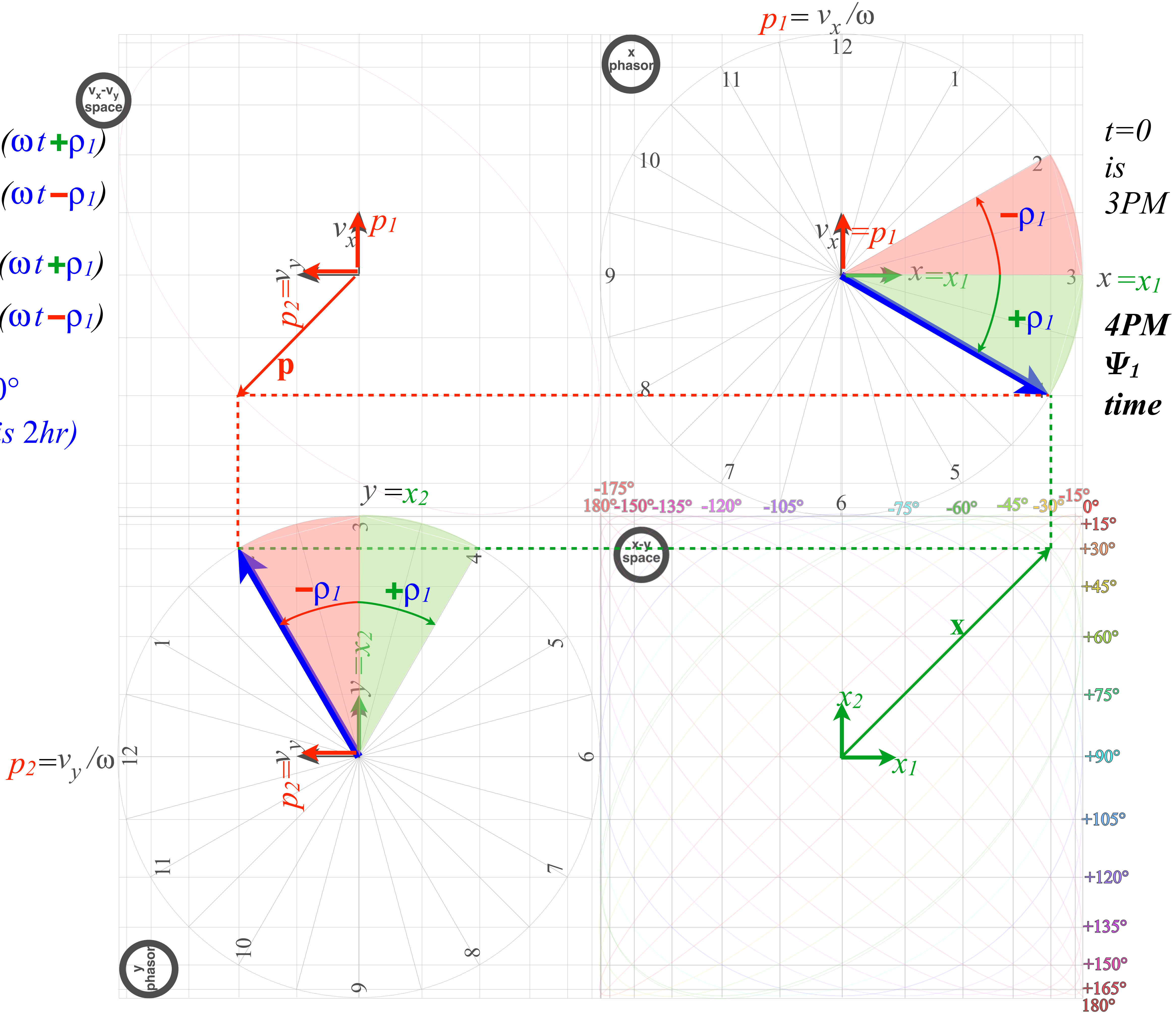
$x_1 = A_1 \cos(\omega t + \rho_1)$

$x_2 = A_2 \cos(\omega t - \rho_1)$

$2\rho_1 = 60°$

*(phase lag is 2hr)*

***2PM***
***$\Psi_2$***
***time***



$p_1=-A_1\sin(\omega t+\rho_1)$

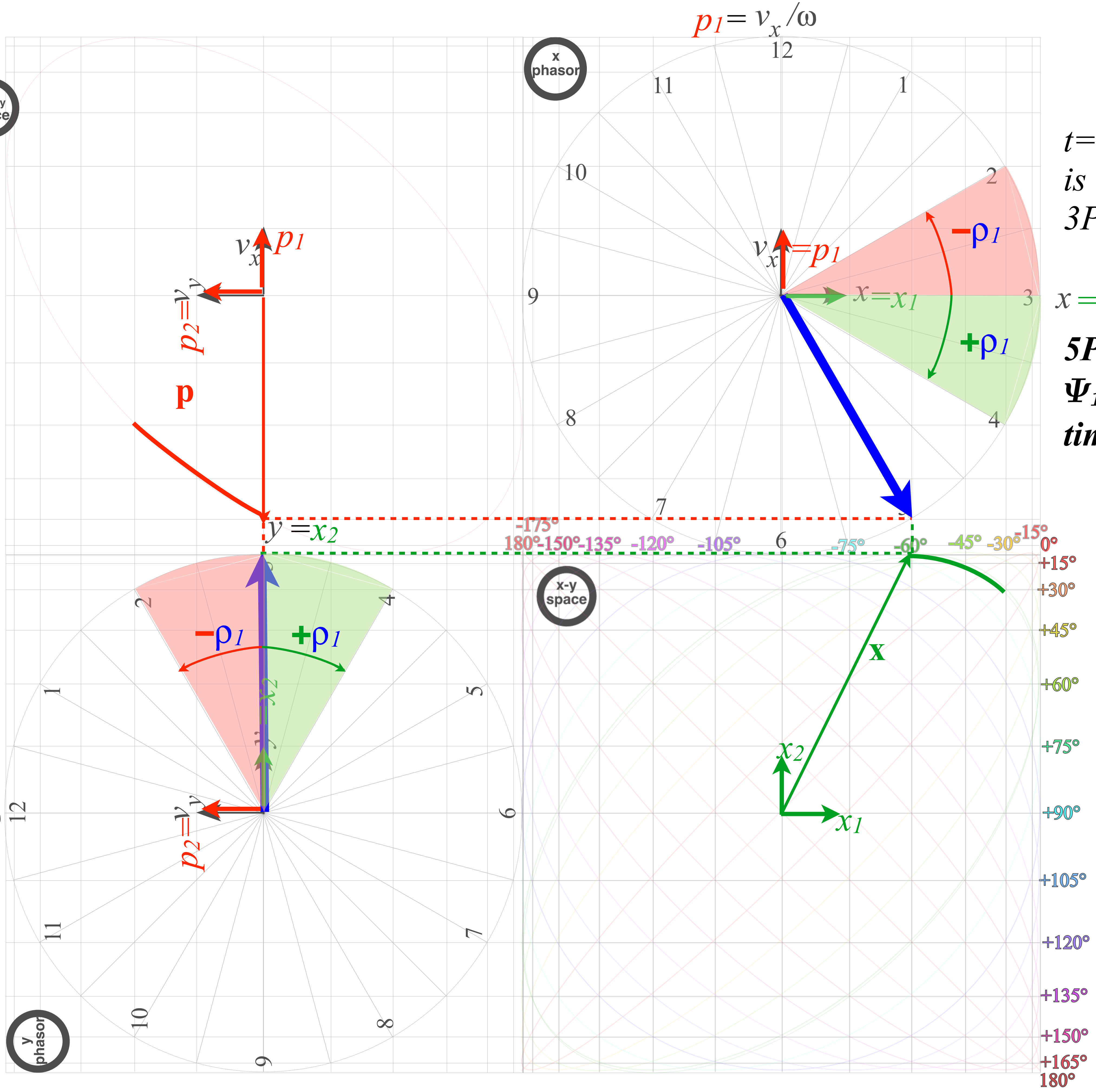
$p_2=-A_2\sin(\omega t-\rho_1)$

$x_1=A_1\cos(\omega t+\rho_1)$

$x_2=A_2\cos(\omega t-\rho_1)$

$2\rho_1=60°$

*(phase lag is 2hr)*



*t=0*
*is*
*3PM*

*x =x₁*

***5PM***
***Ψ₁***
***time***

***3PM***
***Ψ₂***
***time***

$p_1$=-$A_1 \sin(\omega t + \rho_1)$

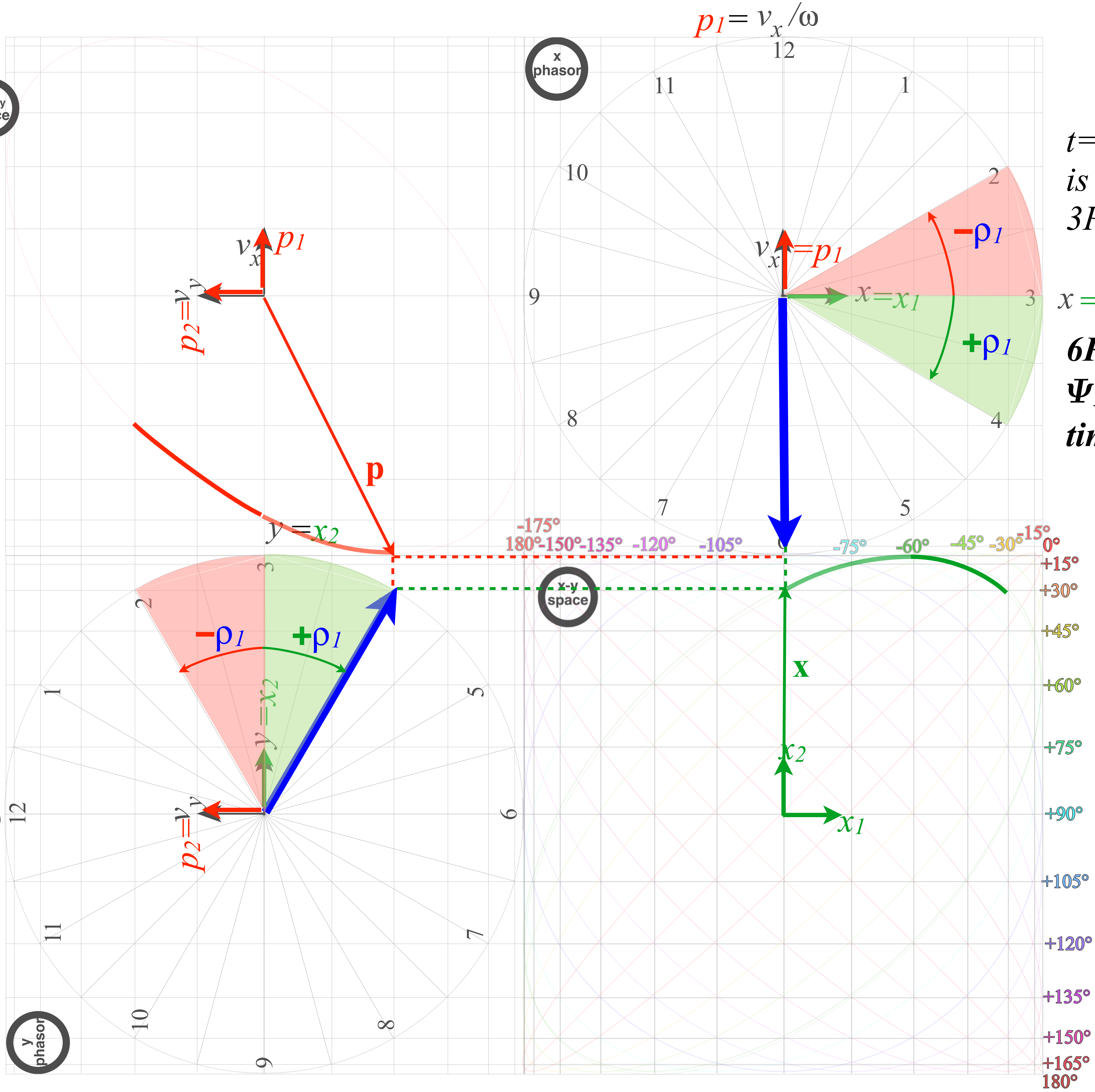
$p_2$=-$A_2 \sin(\omega t - \rho_1)$

$x_1 = A_1 \cos(\omega t + \rho_1)$

$x_2 = A_2 \cos(\omega t - \rho_1)$

$2\rho_1 = 60°$

*(phase lag is 2hr)*



***t=0* *is* *3PM***

***6PM***

***$\Psi_1$ time***

***4PM***

***$\Psi_2$ time***

$p_1=-A_1\sin(\omega t+\rho_1)$

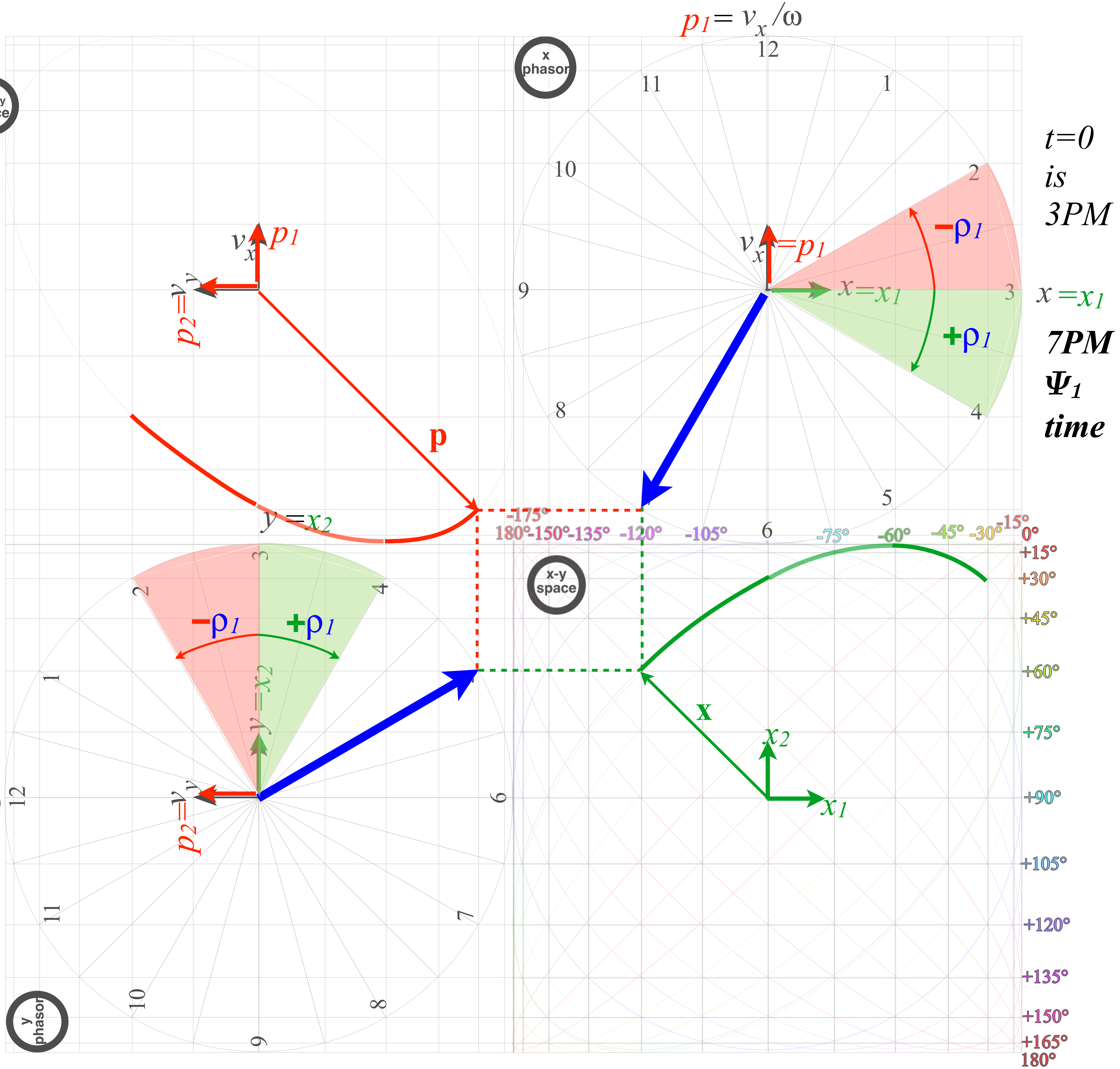
$p_2=-A_2\sin(\omega t-\rho_1)$

$x_1= A_1\cos(\omega t+\rho_1)$

$x_2= A_2\cos(\omega t-\rho_1)$

$2\rho_1=60°$

*(phase lag is 2hr)*



$p_1 = -A_1 \sin(\omega t + \rho_1)$

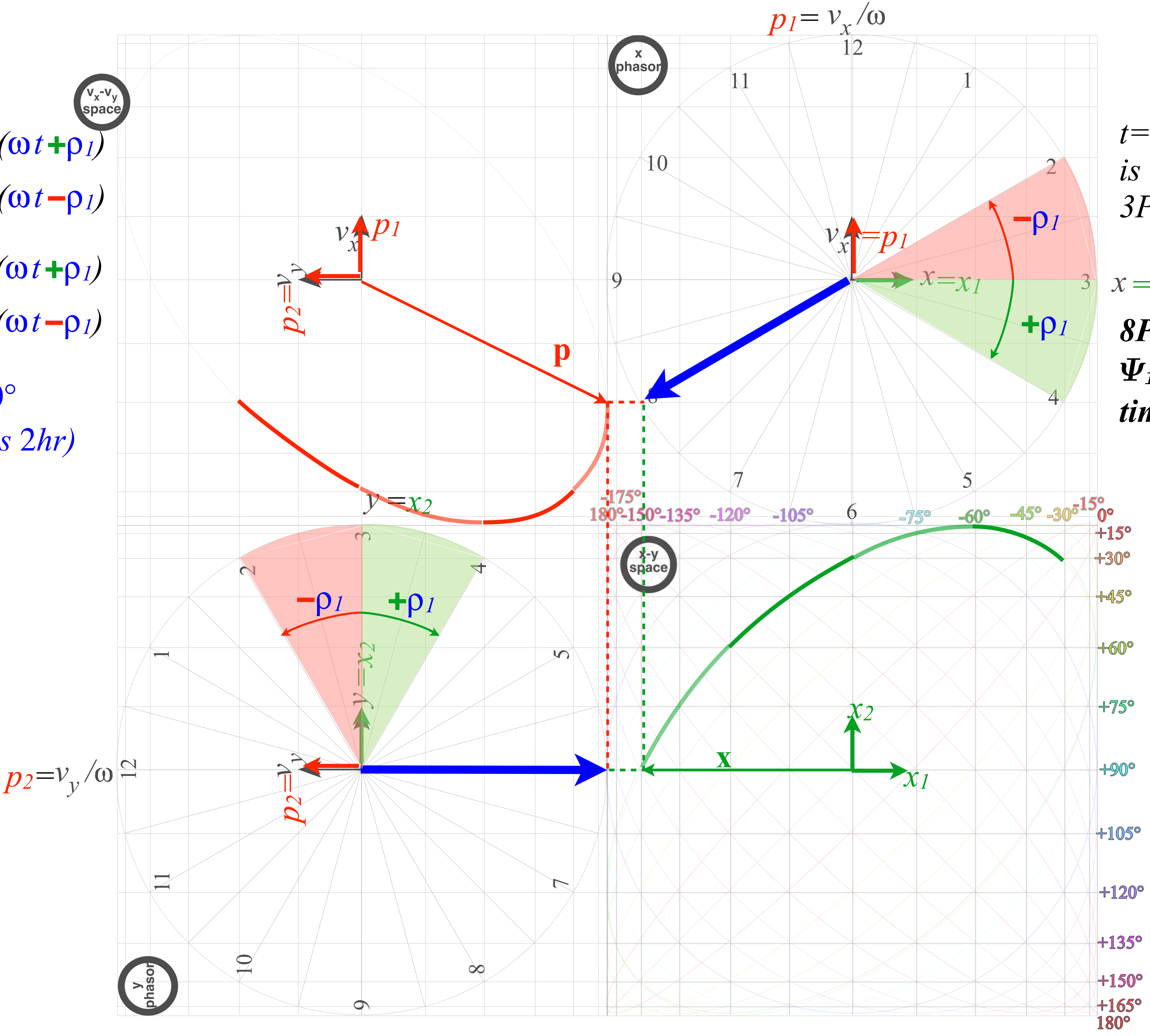
$p_2 = -A_2 \sin(\omega t - \rho_1)$

$x_1 = A_1 \cos(\omega t + \rho_1)$

$x_2 = A_2 \cos(\omega t - \rho_1)$

$2\rho_1 = 60°$

*(phase lag is 2hr)*



*t=0*
*is*
*3PM*

***8PM***
***Ψ₁***
***time***

***6PM***
***Ψ₂***
***time***

$p_1$=-$A_1\sin(\omega t+\rho_1)$

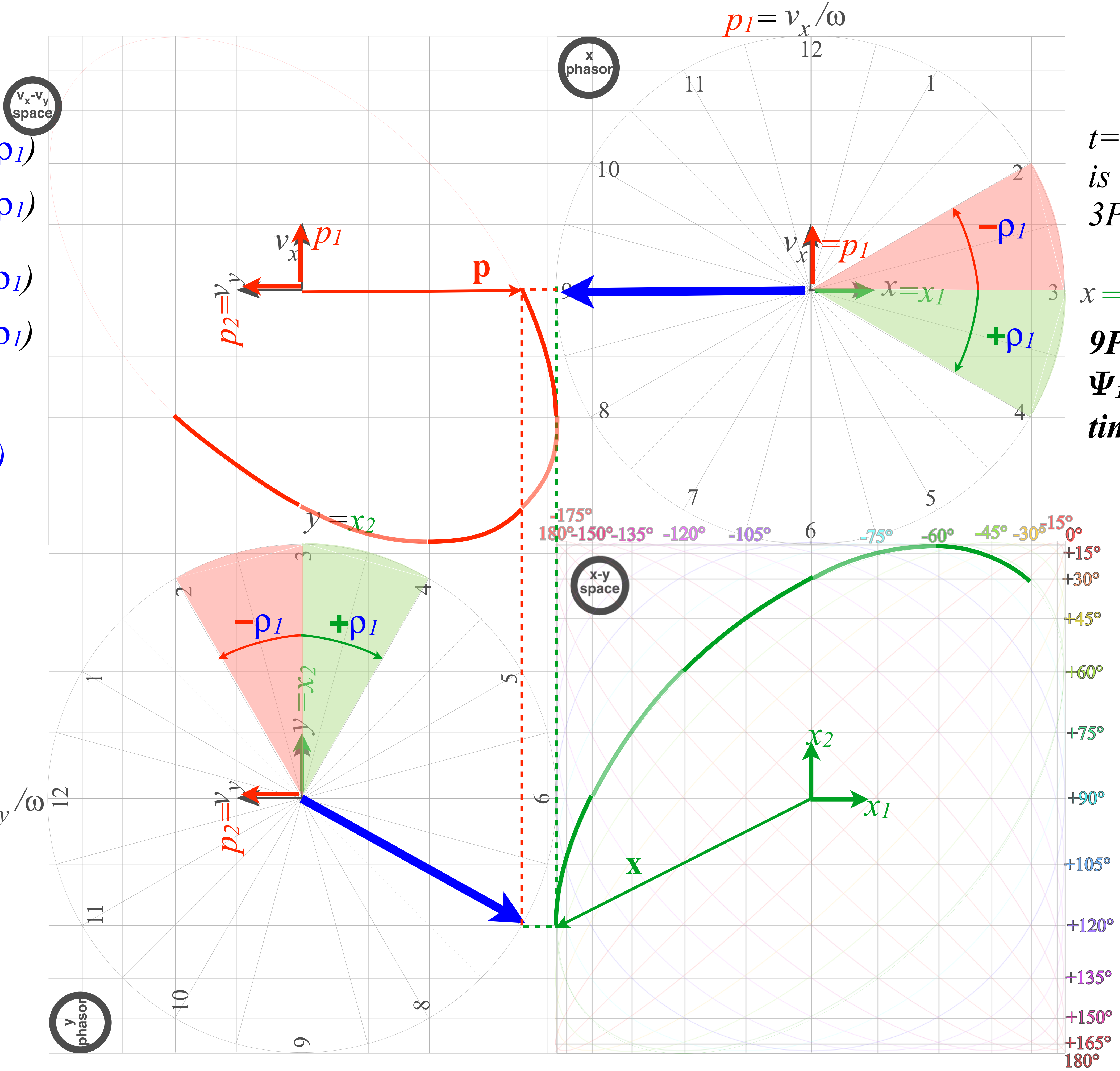
$p_2$=-$A_2\sin(\omega t-\rho_1)$

$x_1 = A_1\cos(\omega t+\rho_1)$

$x_2 = A_2\cos(\omega t-\rho_1)$

$2\rho_1=60°$

*(phase lag is 2hr)*



*t=0*
*is*
*3PM*

***9PM***
***$\Psi_1$***
***time***

***7PM***
***$\Psi_2$***
***time***

$p_1 = -A_1 \sin(\omega t + \rho_1)$

$p_2 = -A_2 \sin(\omega t - \rho_1)$

$x_1 = A_1 \cos(\omega t + \rho_1)$

$x_2 = A_2 \cos(\omega t - \rho_1)$

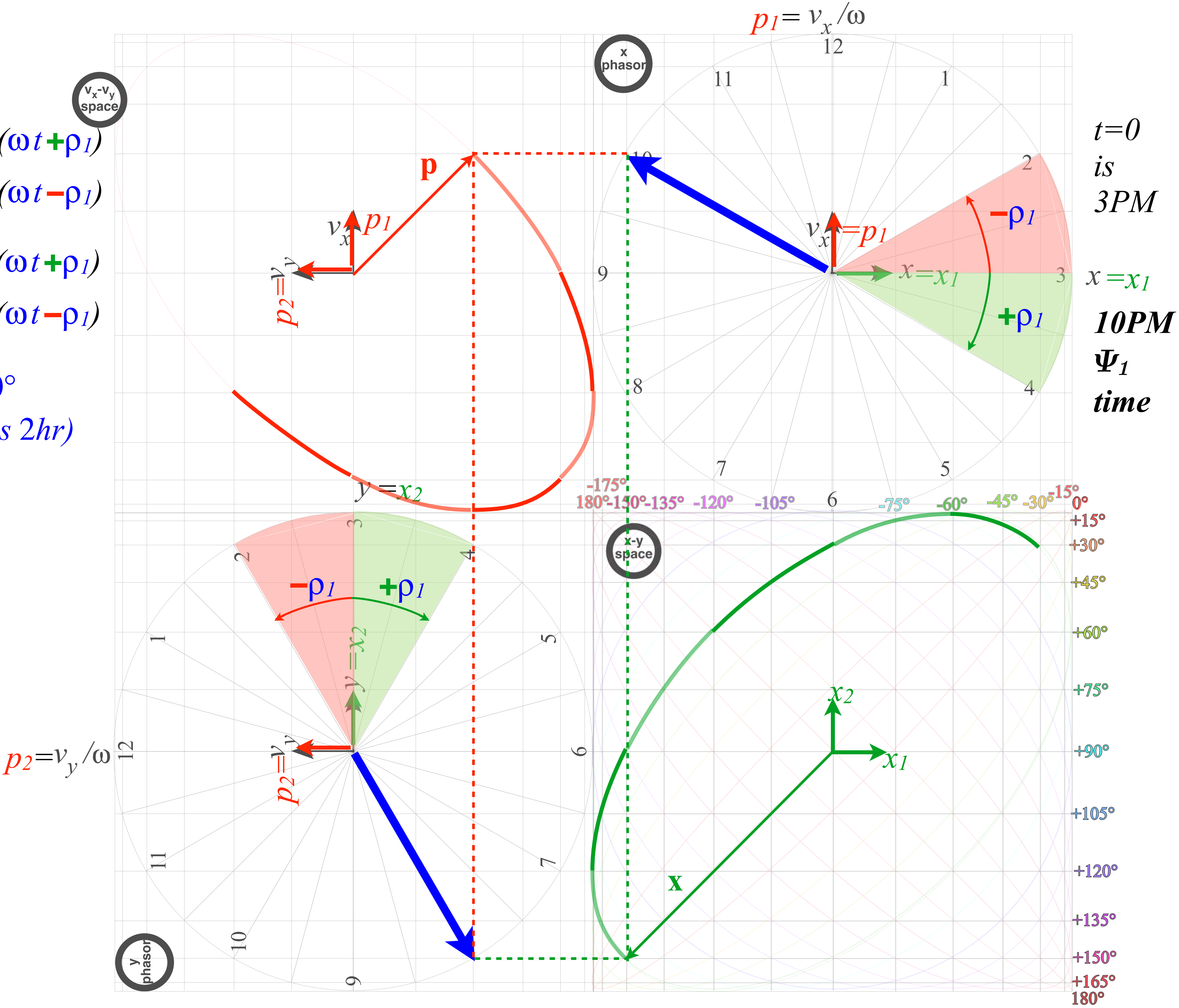
$2\rho_1 = 60°$

*(phase lag is 2hr)*

***8PM***
***$\Psi_2$***
***time***

*t=0*
*is*
*3PM*

***10PM***
***$\Psi_1$***
***time***



$p_1 = -A_1 \sin(\omega t + \rho_1)$

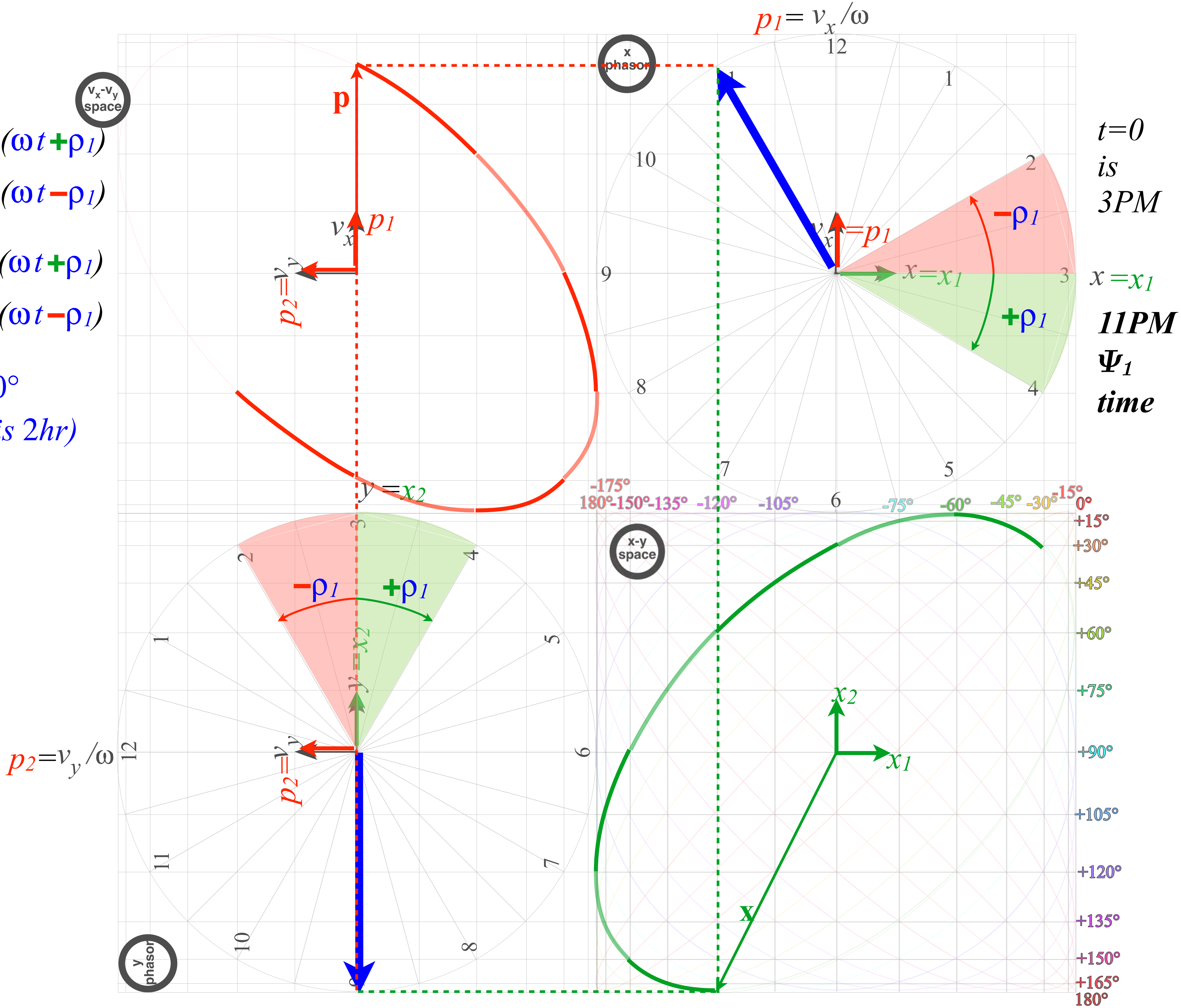
$p_2 = -A_2 \sin(\omega t - \rho_1)$

$x_1 = A_1 \cos(\omega t + \rho_1)$

$x_2 = A_2 \cos(\omega t - \rho_1)$

$2\rho_1 = 60°$

*(phase lag is 2hr)*



***t=0* is *3PM***

***11PM***
***$\Psi_1$***
***time***

***9PM***
***$\Psi_2$***
***time***

$p_1$=$-A_1\sin(\omega t+\rho_1)$

$p_2$=$-A_2\sin(\omega t-\rho_1)$

$x_1$= $A_1\cos(\omega t+\rho_1)$

$x_2$= $A_2\cos(\omega t-\rho_1)$

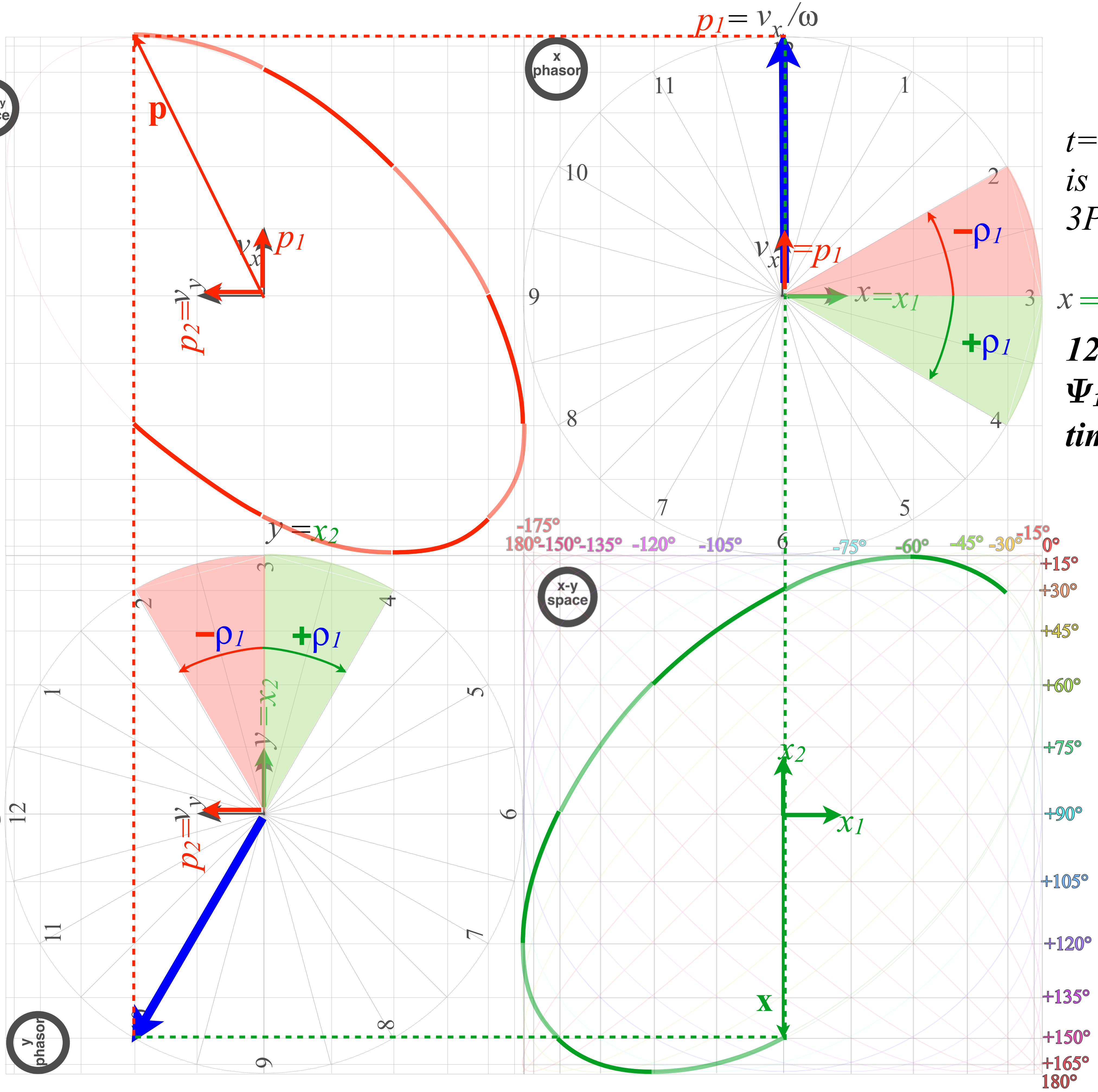
$2\rho_1$=60°

*(phase lag is 2hr)*

*t=0*
*is*
*3PM*

***12PM***
***$\Psi_1$***
***time***

***10PM***
***$\Psi_2$***
***time***



$p_1 = -A_1 \sin(\omega t + \rho_1)$

$p_2 = -A_2 \sin(\omega t - \rho_1)$

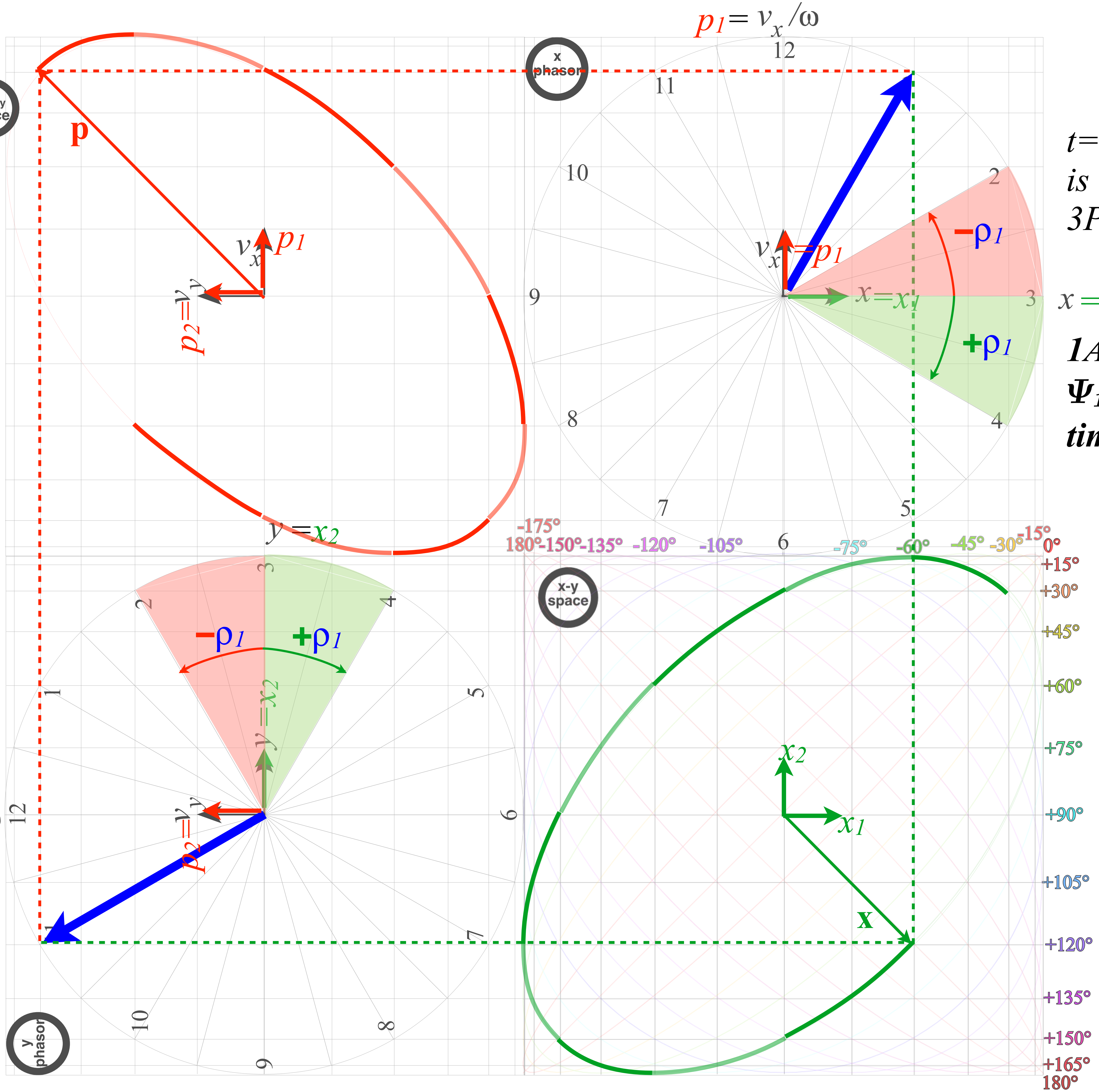
$x_1 = A_1 \cos(\omega t + \rho_1)$

$x_2 = A_2 \cos(\omega t - \rho_1)$

$2\rho_1 = 60°$

*(phase lag is 2hr)*

**11PM**
**$\Psi_2$**
**time**



*t=0*
*is*
*3PM*

**1AM**
**$\Psi_1$**
**time**

$p_1 = -A_1 \sin(\omega t + \rho_1)$

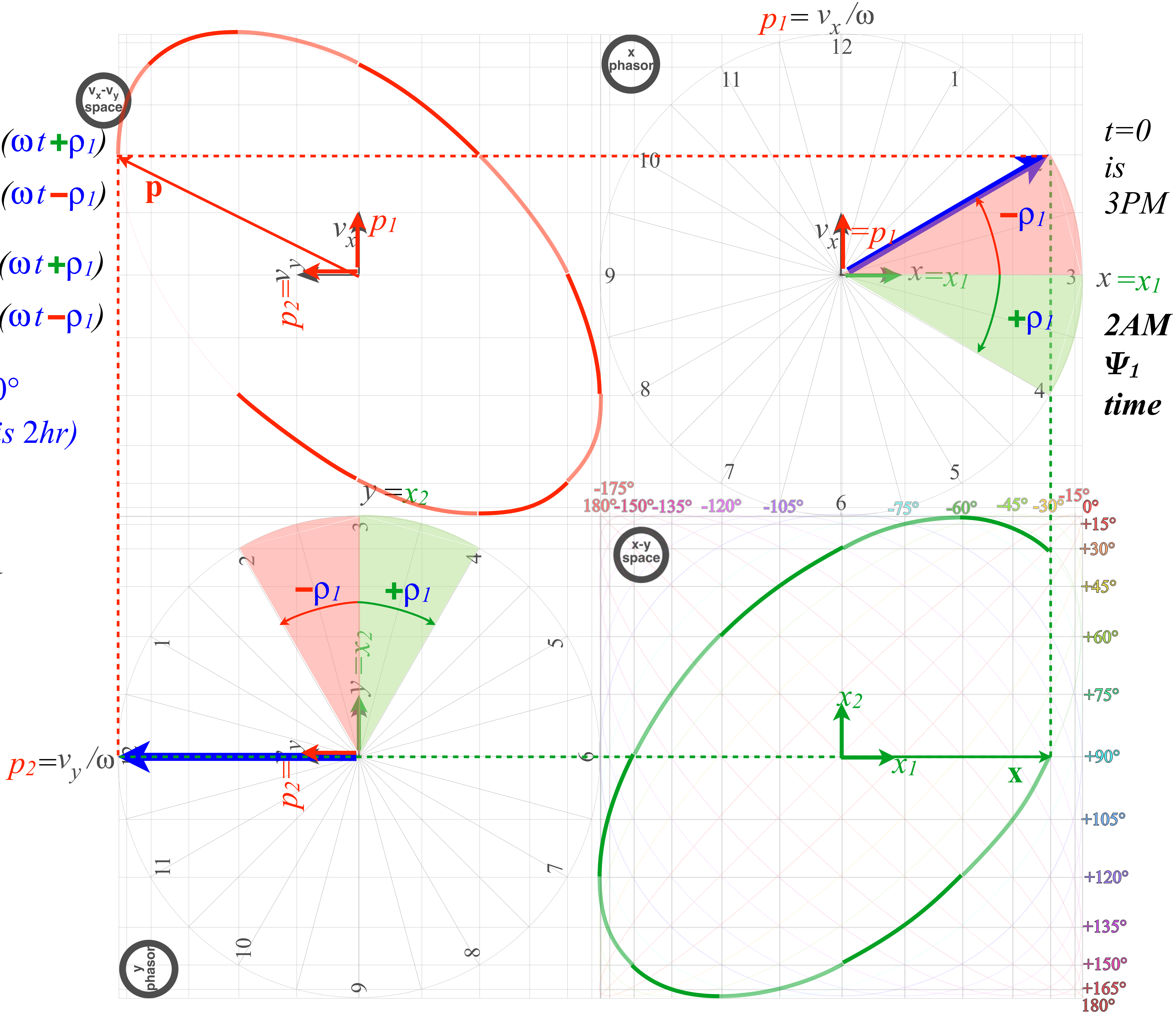
$p_2 = -A_2 \sin(\omega t - \rho_1)$

$x_1 = A_1 \cos(\omega t + \rho_1)$

$x_2 = A_2 \cos(\omega t - \rho_1)$

$2\rho_1 = 60°$

(phase lag is 2hr)



$p_1 = -A_1 \sin(\omega t + \rho_1)$

$p_2 = -A_2 \sin(\omega t - \rho_1)$

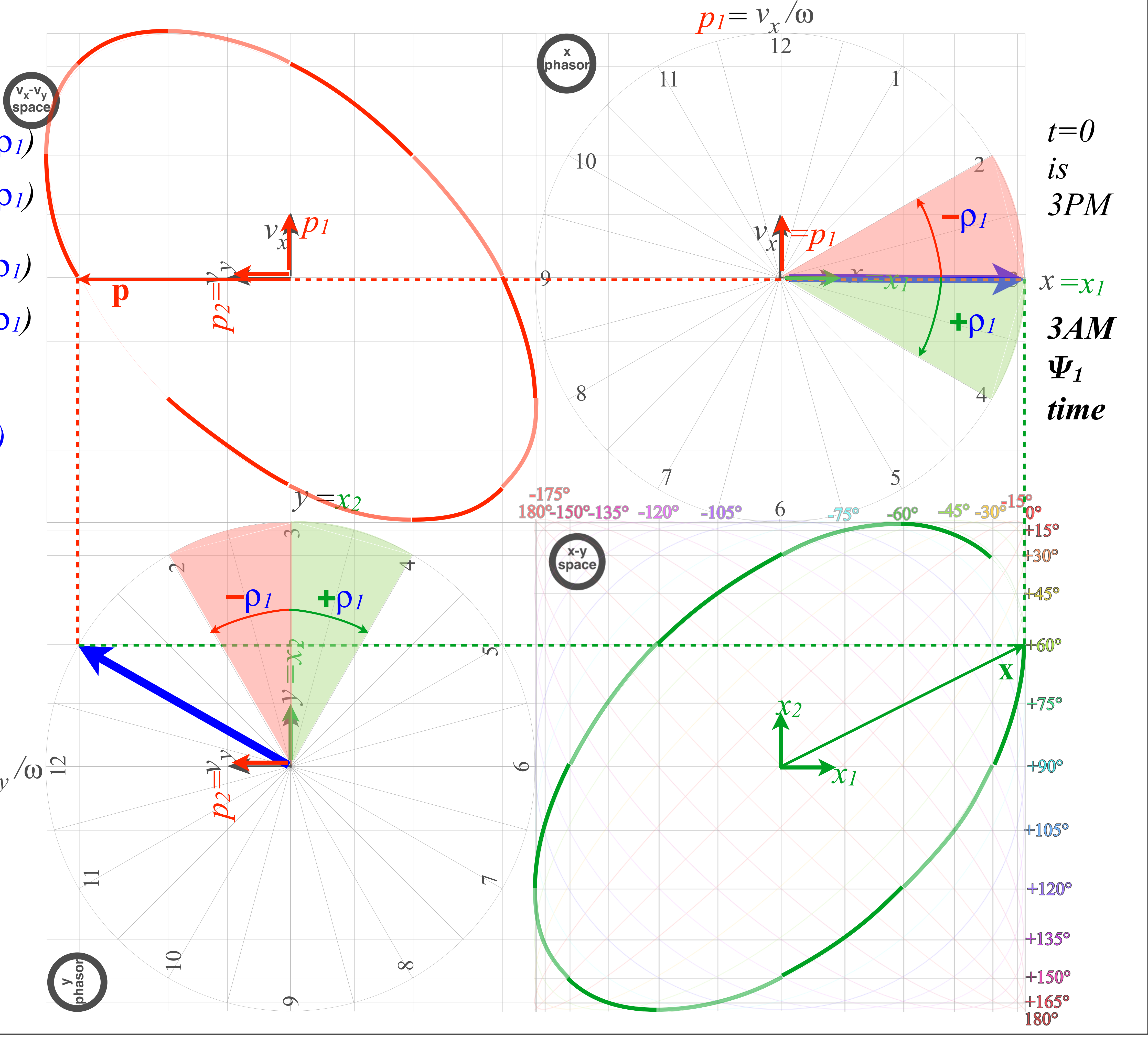
$x_1 = A_1 \cos(\omega t + \rho_1)$

$x_2 = A_2 \cos(\omega t - \rho_1)$

$2\rho_1 = 60°$

(phase lag is 2hr)

***1AM***
***$\Psi_2$***
***time***



$p_1 = -A_1 \sin(\omega t + \rho_1)$

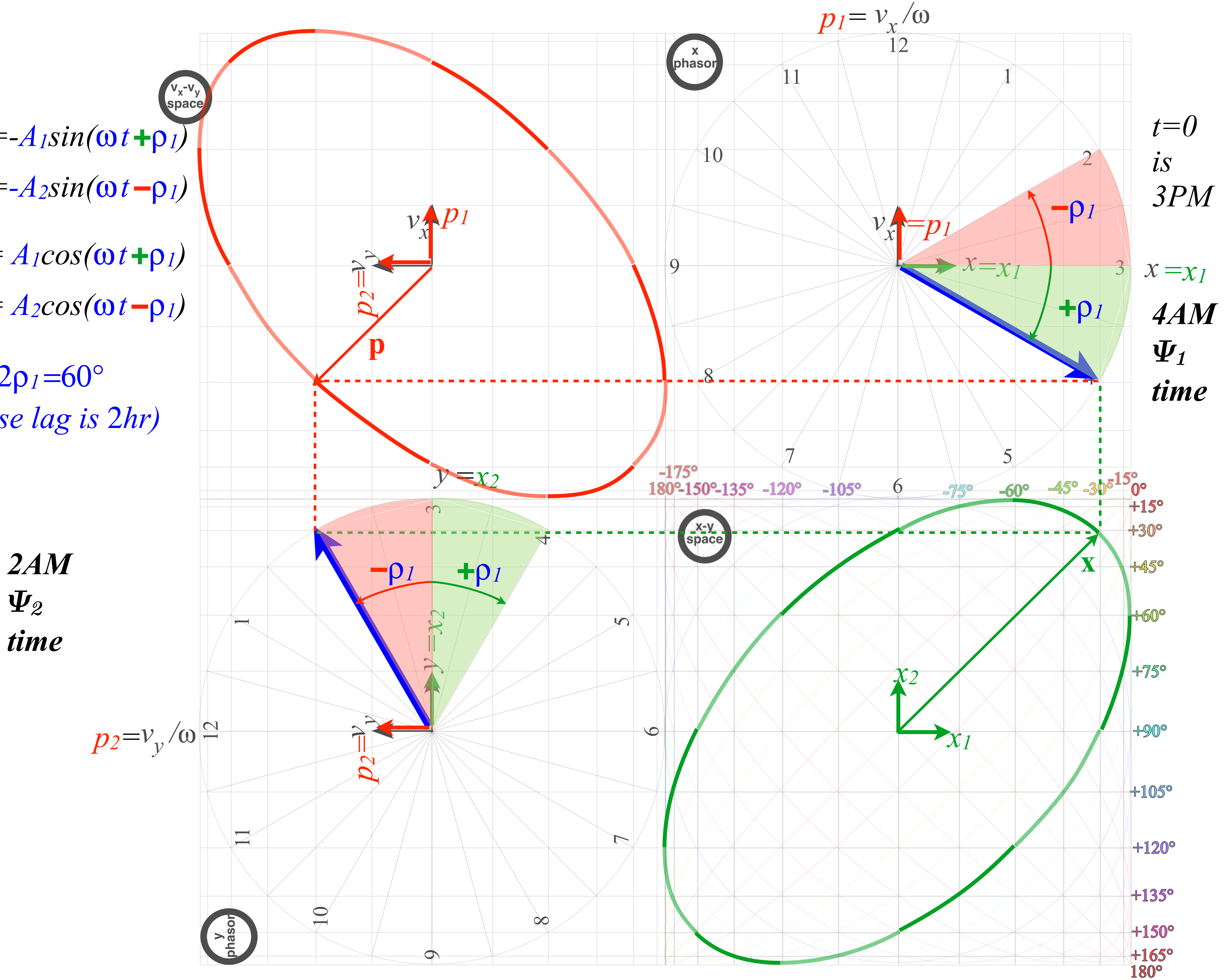
$p_2 = -A_2 \sin(\omega t - \rho_1)$

$x_1 = A_1 \cos(\omega t + \rho_1)$

$x_2 = A_2 \cos(\omega t - \rho_1)$

$2\rho_1 = 60°$

(phase lag is 2hr)



$p_1 = -A_1 \sin(\omega t + \rho_1)$

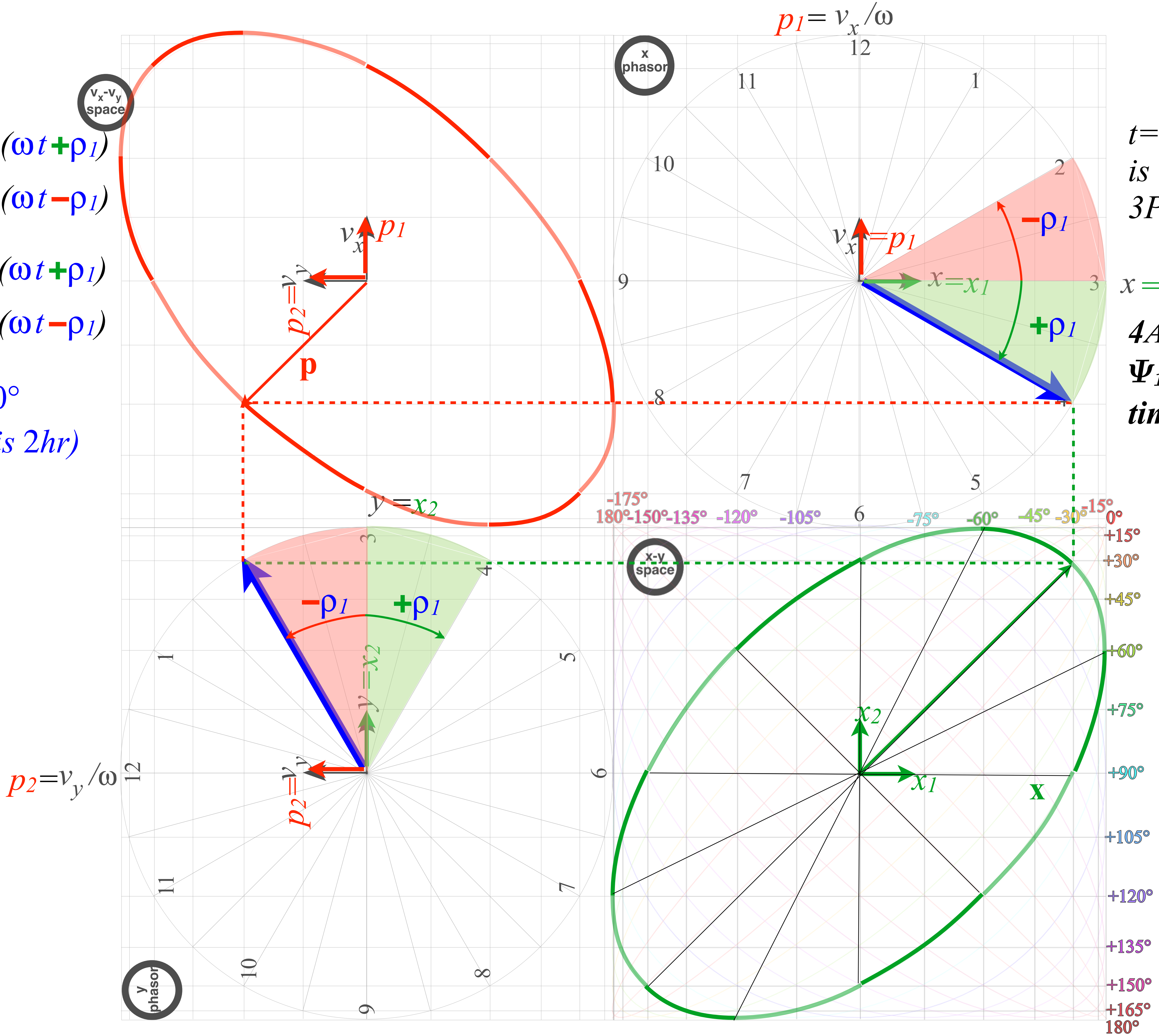
$p_2 = -A_2 \sin(\omega t - \rho_1)$

$x_1 = A_1 \cos(\omega t + \rho_1)$

$x_2 = A_2 \cos(\omega t - \rho_1)$

$2\rho_1 = 60°$

*(phase lag is 2hr)*



*t=0*
*is*
*3PM*

***4AM***
***Ψ₁***
***time***

***2AM***
***Ψ₂***
***time***

$p_1=-A_1 \sin(\omega t+\rho_1)$

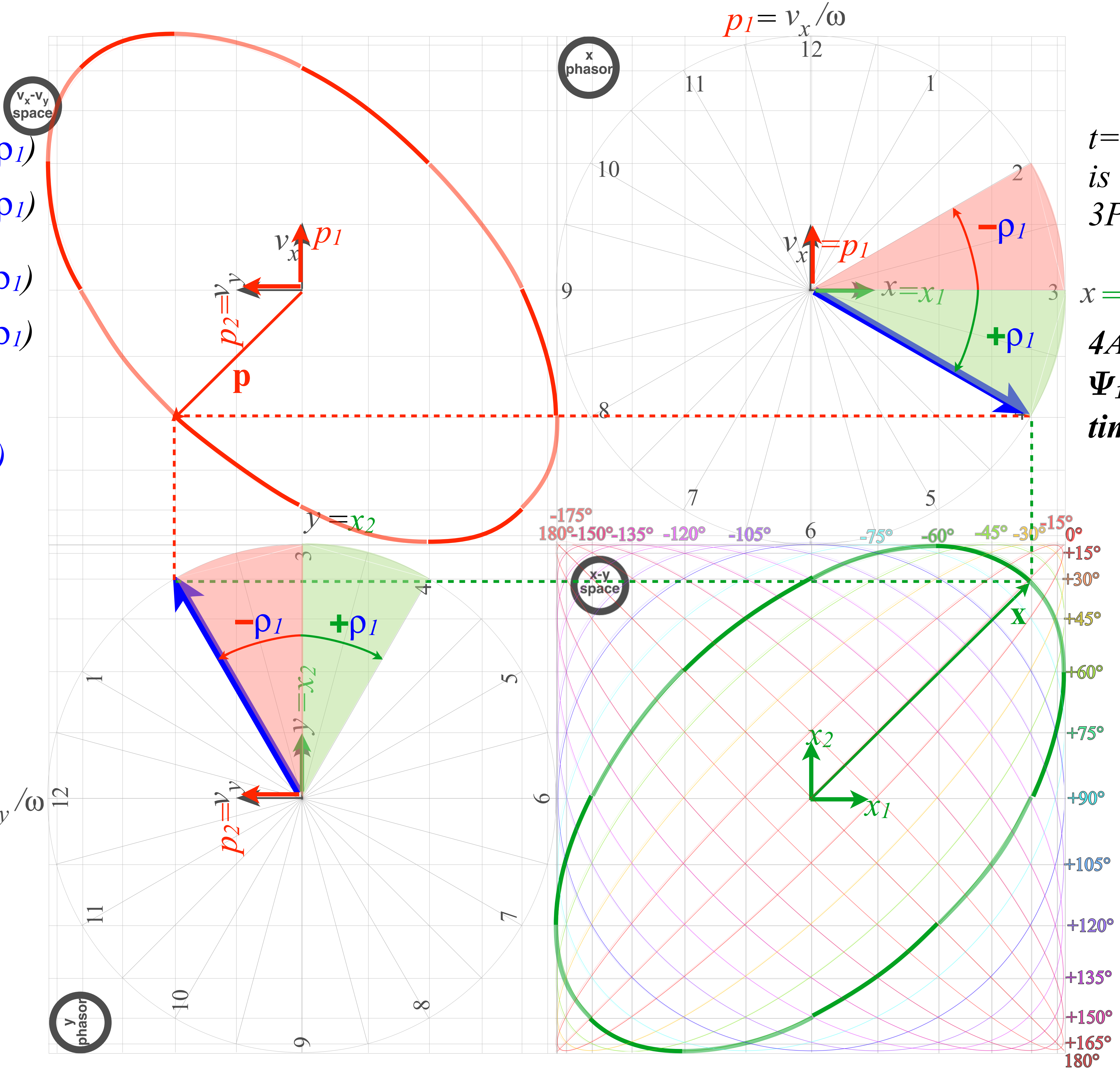
$p_2=-A_2 \sin(\omega t-\rho_1)$

$x_1= A_1 \cos(\omega t+\rho_1)$

$x_2= A_2 \cos(\omega t-\rho_1)$

$2\rho_1=60°$

*(phase lag is 2hr)*



*t=0*
*is*
*3PM*

***4AM***
***$\Psi_1$***
***time***

***2AM***
***$\Psi_2$***
***time***

*Review: Fundamental Euler* **R**$(\alpha\beta\gamma)$ *and Darboux* **R**$[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

- *Euler* **R**$(\alpha\beta\gamma)$ *derived from Darboux* **R**$[\varphi\vartheta\Theta]$ *and vice versa*
- *Euler* **R**$(\alpha\beta\gamma)$ *rotation* $\Theta$=0−4π*-sequence* $[\varphi\vartheta]$ *fixed*
- *R(3)-U(2) slide rule for converting* **R**$(\alpha\beta\gamma)$ ↔ **R**$[\varphi\vartheta\Theta]$ *and Sundial*

*U(2) density operator approach to symmetry dynamics*

- *Bloch equation for density operator*

*The ABC's of U(2) dynamics-Archetypes*

- *Asymmetric-Diagonal A-Type motion*
- *Bilateral-Balanced B-Type motion*
- *Circular-Coriolis… C-Type motion*

*The ABC's of U(2) dynamics-Mixed modes*

- *AB-Type motion and Wigner's Avoided-Symmetry-Crossings*
- *ABC-Type elliptical polarized motion*

*Ellipsometry using U(2) symmetry coordinates*

- *Conventional amp-phase ellipse coordinates*
- → *Euler Angle* $(\alpha\beta\gamma)$ *ellipse coordinates*

# *Ellipsometry using U(2) symmetry coordinates*

## *Conventional amp-phase ellipse coordinates related to Euler Angles* $(\alpha\beta\gamma)$

*2D* elliptic frequency $\omega$ orbit has amplitudes $A_1$ and $A_2$, and phase shifts $\rho_1$ and $\rho_2 = -\rho_1$.

$$\begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

$$x_1 = A_1 \cos(\omega t + \rho_1)$$

$$-p_1 = A_1 \sin(\omega t + \rho_1)$$

$$x_2 = A_2 \cos(\omega t - \rho_1)$$

$$-p_2 = A_2 \sin(\omega t - \rho_1)$$

Real $x_k$ and imaginary $p_k$ parts of phasor amplitudes $a_k = x_k + ip_k$ depend on Euler angles $(\alpha\beta\gamma)$ and $A$.

# Ellipsometry using U(2) symmetry coordinates

## Conventional amp-phase ellipse coordinates related to Euler Angles (αβγ)

2D elliptic frequency ω orbit has amplitudes $A_1$ and $A_2$, and phase shifts $\rho_1$ and $\rho_2=-\rho_1$.

$$\begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = A_1 \cos(\omega t+\rho_1)$$
$$-p_1 = A_1 \sin(\omega t+\rho_1)$$
$$x_2 = A_2 \cos(\omega t-\rho_1)$$
$$-p_2 = A_2 \sin(\omega t-\rho_1)$$

Real $x_k$ and imaginary $p_k$ parts of phasor amplitudes $a_k=x_k+ip_k$ depend on Euler angles (αβγ) and $A$.

$$x_1 = A\cos\beta/2\cos[(\gamma+\alpha)/2]$$
$$-p_1 = A\cos\beta/2\sin[(\gamma+\alpha)/2]$$
$$x_2 = A\sin\beta/2\cos[(\gamma-\alpha)/2]$$
$$-p_2 = A\sin\beta/2\sin[(\gamma-\alpha)/2]$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

# Ellipsometry using U(2) symmetry coordinates

## Conventional amp-phase ellipse coordinates related to Euler Angles $(\alpha\beta\gamma)$

*2D* elliptic frequency $\omega$ orbit has amplitudes $A_1$ and $A_2$, and phase shifts $\rho_1$ and $\rho_2=-\rho_1$.

$$\begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = A_1 \cos(\omega t+\rho_1)$$
$$-p_1 = A_1 \sin(\omega t+\rho_1)$$
$$x_2 = A_2 \cos(\omega t-\rho_1)$$
$$-p_2 = A_2 \sin(\omega t-\rho_1)$$

Let: $A_1 = A\cos\beta/2$

Real $x_k$ and imaginary $p_k$ parts of phasor amplitudes $a_k=x_k+ip_k$ depend on Euler angles $(\alpha\beta\gamma)$ and $A$.

$$x_1 = A\cos\beta/2\cos[(\gamma+\alpha)/2]$$
$$-p_1 = A\cos\beta/2\sin[(\gamma+\alpha)/2]$$
$$x_2 = A\sin\beta/2\cos[(\gamma-\alpha)/2]$$
$$-p_2 = A\sin\beta/2\sin[(\gamma-\alpha)/2]$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

# Ellipsometry using U(2) symmetry coordinates

## Conventional amp-phase ellipse coordinates related to Euler Angles $(\alpha\beta\gamma)$

*2D* elliptic frequency ω orbit has amplitudes $A_1$ and $A_2$, and phase shifts $\rho_1$ and $\rho_2=-\rho_1$.

Real $x_k$ and imaginary $p_k$ parts of phasor amplitudes $a_k=x_k+ip_k$ depend on Euler angles $(\alpha\beta\gamma)$ and $A$.

$$\begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = A_1 \cos(\omega t+\rho_1)$$
$$-p_1 = A_1 \sin(\omega t+\rho_1)$$
$$x_2 = A_2 \cos(\omega t-\rho_1)$$
$$-p_2 = A_2 \sin(\omega t-\rho_1)$$

*Let:* $A_1 = A\cos\beta/2$

$A_2 = A\sin\beta/2$

$$x_1 = A\cos\beta/2\cos[(\gamma+\alpha)/2]$$
$$-p_1 = A\cos\beta/2\sin[(\gamma+\alpha)/2]$$
$$x_2 = A\sin\beta/2\cos[(\gamma-\alpha)/2]$$
$$-p_2 = A\sin\beta/2\sin[(\gamma-\alpha)/2]$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

# Ellipsometry using U(2) symmetry coordinates

## Conventional amp-phase ellipse coordinates related to Euler Angles ($\alpha\beta\gamma$)

2D elliptic frequency $\omega$ orbit has amplitudes $A_1$ and $A_2$, and phase shifts $\rho_1$ and $\rho_2=-\rho_1$.

Real $x_k$ and imaginary $p_k$ parts of phasor amplitudes $a_k=x_k+ip_k$ depend on Euler angles ($\alpha\beta\gamma$) and $A$.

$$\begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1= A_1 \cos(\omega t+\rho_1)$$
$$-p_1=A_1 \sin(\omega t+\rho_1)$$
$$x_2= A_2 \cos(\omega t-\rho_1)$$
$$-p_2=A_2 \sin(\omega t-\rho_1)$$

Let: $A_1 = A\cos\beta/2$

$A_2 = A\sin\beta/2$

$$x_1= A\cos\beta/2\cos[(\gamma+\alpha)/2]$$
$$-p_1= A\cos\beta/2\sin[(\gamma+\alpha)/2]$$
$$x_2= A\sin\beta/2\cos[(\gamma-\alpha)/2]$$
$$-p_2= A\sin\beta/2\sin[(\gamma-\alpha)/2]$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

Let: $\omega t+\rho_1=(\gamma+\alpha)/2$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

# Ellipsometry using U(2) symmetry coordinates

## Conventional amp-phase ellipse coordinates related to Euler Angles (αβγ)

*2D* elliptic frequency ω orbit has amplitudes $A_1$ and $A_2$, and phase shifts $\rho_1$ and $\rho_2=-\rho_1$.

$$\begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = A_1 \cos(\omega t+\rho_1)$$

$$-p_1 = A_1 \sin(\omega t+\rho_1)$$

$$x_2 = A_2 \cos(\omega t-\rho_1)$$

$$-p_2 = A_2 \sin(\omega t-\rho_1)$$

*Let:* $A_1 = A\cos\beta/2$

$A_2 = A\sin\beta/2$

Real $x_k$ and imaginary $p_k$ parts of phasor amplitudes $a_k=x_k+ip_k$ depend on Euler angles (αβγ) and $A$.

$$x_1 = A\cos\beta/2\cos[(\gamma+\alpha)/2]$$

$$-p_1 = A\cos\beta/2\sin[(\gamma+\alpha)/2]$$

$$x_2 = A\sin\beta/2\cos[(\gamma-\alpha)/2]$$

$$-p_2 = A\sin\beta/2\sin[(\gamma-\alpha)/2]$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

*Let:* $\omega t+\rho_1 = (\gamma+\alpha)/2$

$\omega t-\rho_1 = (\gamma-\alpha)/2$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

# Ellipsometry using U(2) symmetry coordinates

## Conventional amp-phase ellipse coordinates related to Euler Angles $(\alpha\beta\gamma)$

*2D* elliptic frequency $\omega$ orbit has amplitudes $A_1$ and $A_2$, and phase shifts $\rho_1$ and $\rho_2=-\rho_1$.

Real $x_k$ and imaginary $p_k$ parts of phasor amplitudes $a_k=x_k+ip_k$ depend on Euler angles $(\alpha\beta\gamma)$ and $A$.

$$\begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1= A_1 \cos(\omega t+\rho_1)$$
$$-p_1= A_1 \sin(\omega t+\rho_1)$$
$$x_2= A_2 \cos(\omega t-\rho_1)$$
$$-p_2= A_2 \sin(\omega t-\rho_1)$$

$$x_1= A\cos\beta/2\cos[(\gamma+\alpha)/2]$$
$$-p_1= A\cos\beta/2\sin[(\gamma+\alpha)/2]$$
$$x_2= A\sin\beta/2\cos[(\gamma-\alpha)/2]$$
$$-p_2= A\sin\beta/2\sin[(\gamma-\alpha)/2]$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

*Let:* $A_1 = A\cos\beta/2$

$A_2 = A\sin\beta/2$

*Let:* $\omega t+\rho_1 = (\gamma+\alpha)/2$

$\omega t-\rho_1 = (\gamma-\alpha)/2$

$\tan\beta/2=A_2/A_1$ $\quad A^2=A_1^2+A_2^2$

$\alpha=2\rho_1$ $\quad \gamma=2\omega\cdot t$

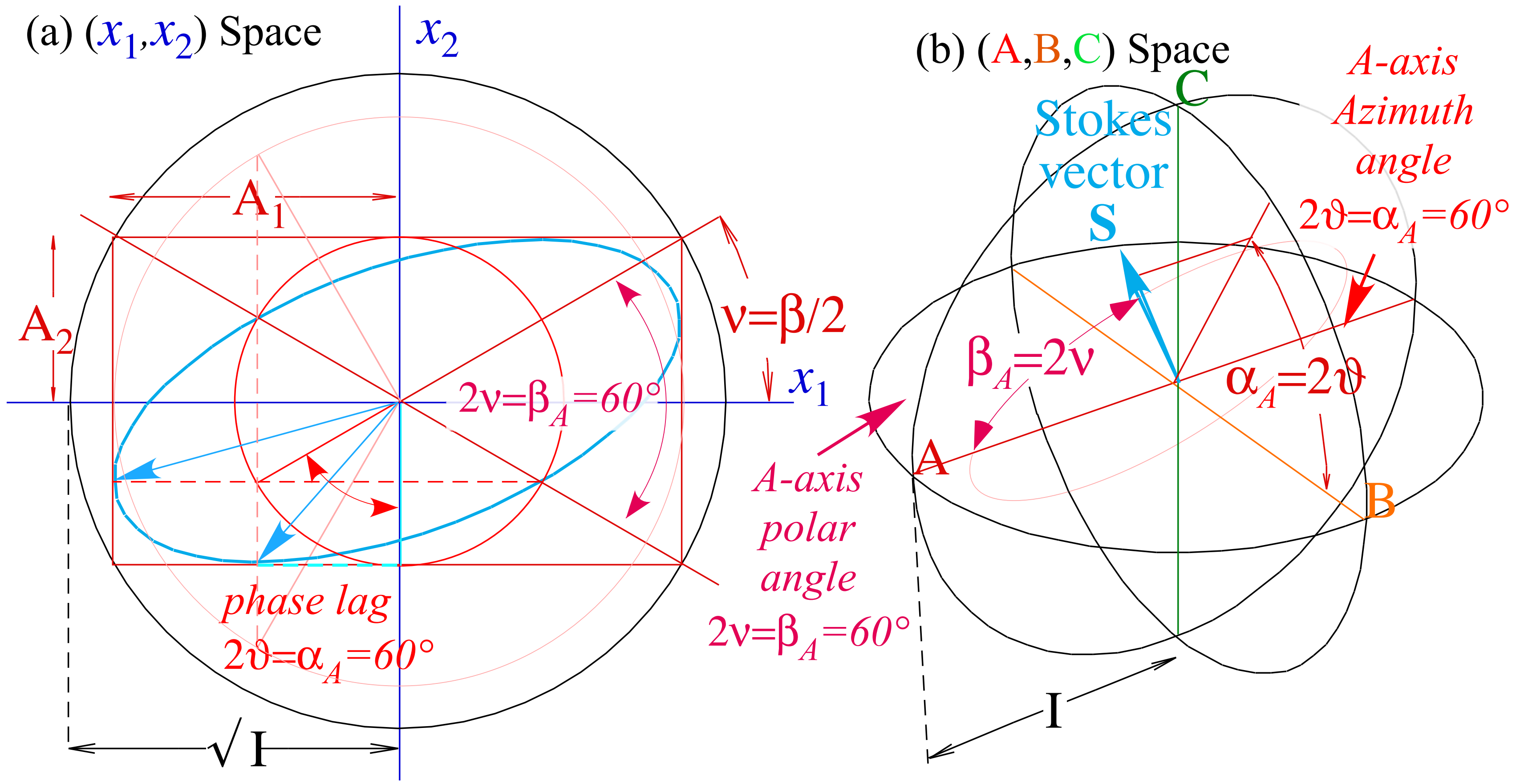
Euler parameters $(\alpha,\beta,\gamma, A)$ in terms of *amp-phase parameters* $(A_1,A_2, \omega t,\rho_1)$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

# *The A-view in* $\{x_1,x_2\}$*-basis*

Angles $\alpha_A=\rho_1-\rho_2=2\rho_1$, $\beta_A=2\tan^{-1}A_2/A_1$, $\gamma_A=2\omega\cdot t$
define ellipses with intensity $I=A^2=A_1^2+A_2^2$.

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = A \begin{pmatrix} e^{-i\alpha_A/2}\cos\frac{\beta_A}{2} \\ e^{+i\alpha_A/2}\sin\frac{\beta_A}{2} \end{pmatrix} e^{-i\omega t} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$



(a) $(x_1,x_2)$ Space

(b) (A,B,C) Space

*A* or *Z*-axis Euler angles

$\alpha=\alpha_A=\rho_1-\rho_2=2\rho_1=60°$

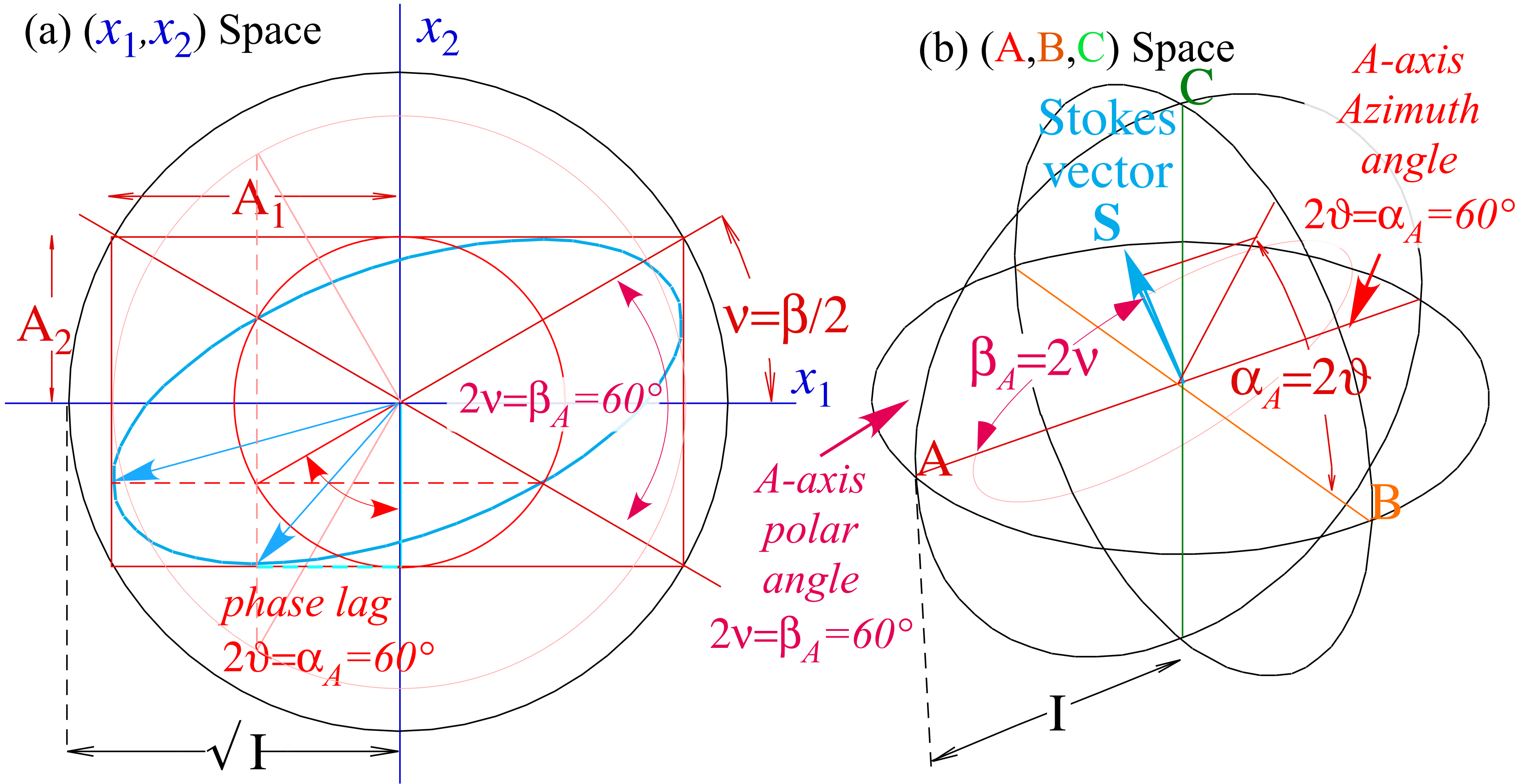
$\beta=\beta_A=2\tan^{-1}A_2/A_1=60°$

$\gamma_A=2\omega\cdot t$

# The A-view in $\{x_1,x_2\}$-basis

Angles $\alpha_A=\rho_1-\rho_2=2\rho_1$, $\beta_A=2\tan^{-1}A_2/A_1$, $\gamma_A=2\omega\cdot t$ define ellipses with intensity $I=A^2=A_1^2+A_2^2$.

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = A \begin{pmatrix} e^{-i\alpha_A/2}\cos\frac{\beta_A}{2} \\ e^{+i\alpha_A/2}\sin\frac{\beta_A}{2} \end{pmatrix} e^{-i\omega t} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$



*A* or *Z*-axis Euler angles

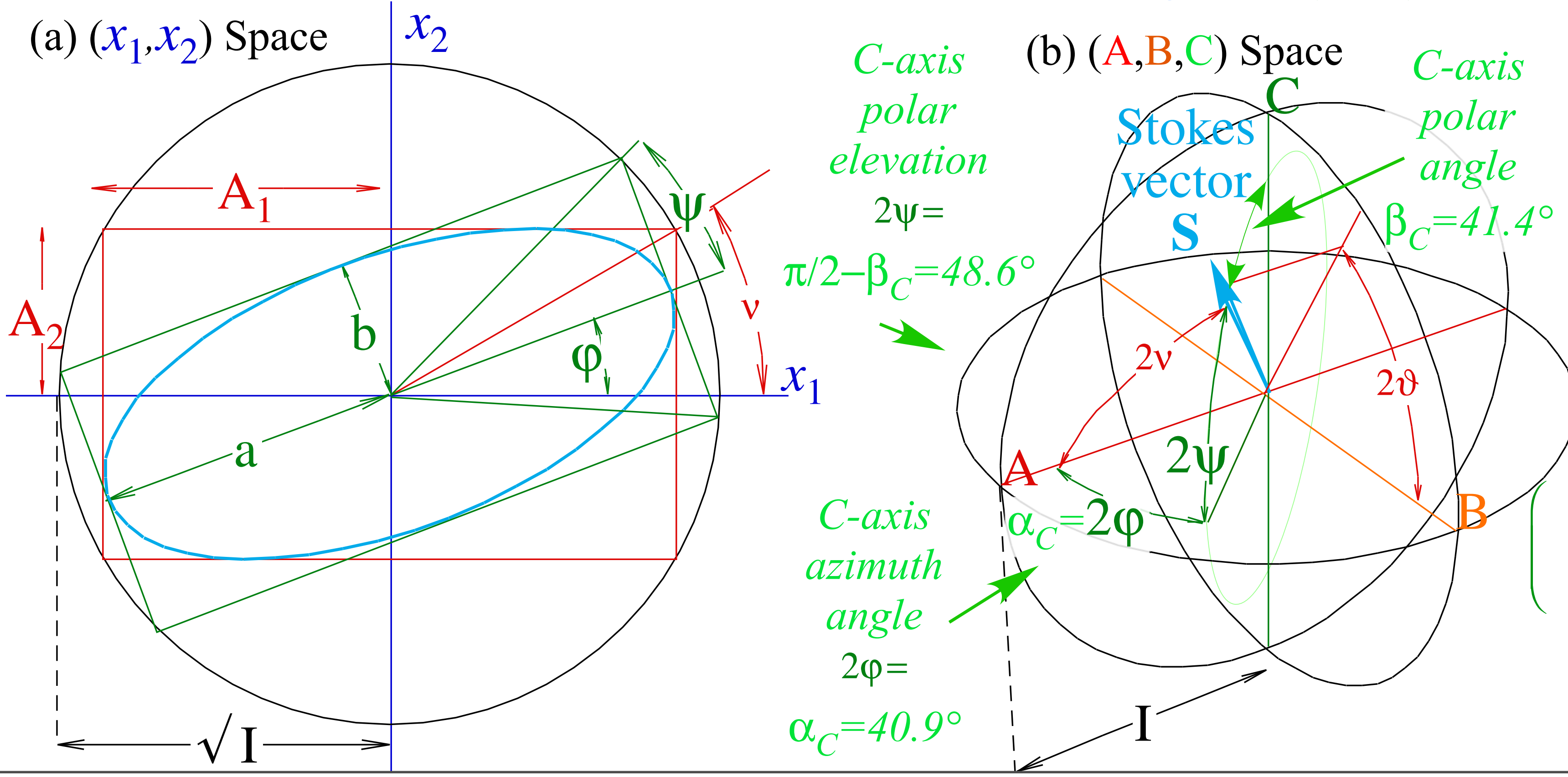
$\alpha=\alpha_A=\rho_1-\rho_2=2\rho_1=60°$

$\beta=\beta_A=2\tan^{-1}A_2/A_1=60°$

$\gamma_A=2\omega\cdot t$

# The C-view in $\{x_R,x_L\}$-basis

The same orbit viewed in right-left $\{x_R,x_L\}$-basis of circular polarization with angles $(\alpha_C,\beta_C,\gamma_C)$.



$$\begin{pmatrix} a_R \\ a_L \end{pmatrix} = A \begin{pmatrix} e^{-i\alpha_C/2}\cos\frac{\beta_C}{2} \\ e^{+i\alpha_C/2}\sin\frac{\beta_C}{2} \end{pmatrix} e^{-i\frac{\gamma_C}{2}} = \begin{pmatrix} x_R+ip_R \\ x_R+ip_R \end{pmatrix}$$

# Converting an *A*-based set of Stokes parameters into a *C*-based set or a *B*-based set involves cyclic permutation of *A*, *B*, and *C* polar formulas

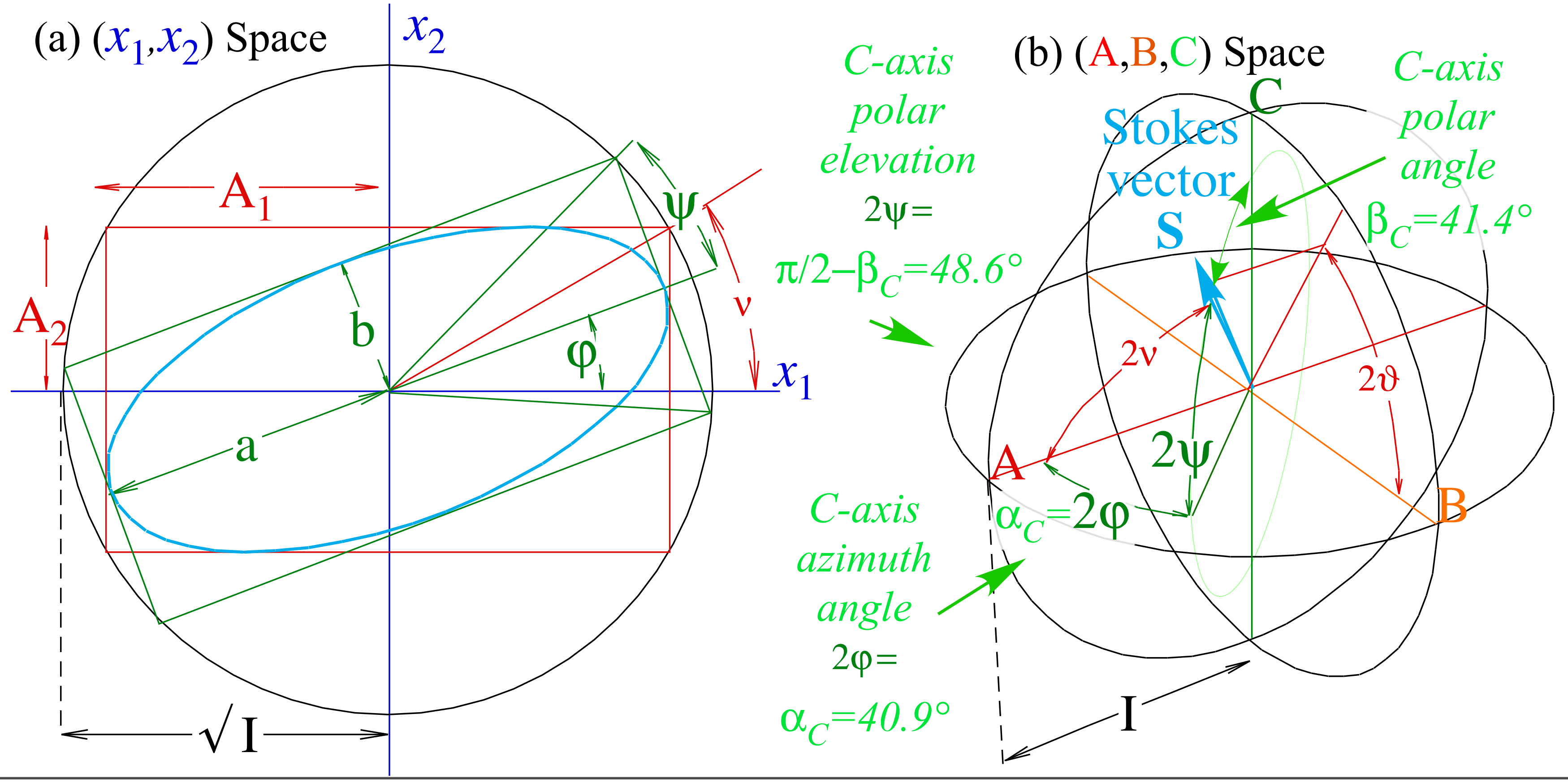
*Asymmetry* $S_A = \frac{I}{2}\cos\beta_A = \frac{I}{2}\sin\alpha_B \sin\beta_B = \frac{I}{2}\cos\alpha_C \sin\beta_C$

*Balance* $S_B = \frac{I}{2}\cos\alpha_A \sin\beta_A = \frac{I}{2}\cos\beta_B = \frac{I}{2}\sin\alpha_C \sin\beta_C$

*Chirality* $S_C = \frac{I}{2}\sin\alpha_A \sin\beta_A = \frac{I}{2}\cos\alpha_B \sin\beta_B = \frac{I}{2}\cos\beta_C$

*The C-view in* $\{x_R,x_L\}$*-basis*

The same orbit viewed in right and left circular polarization $\{x_R,x_L\}$-bases using angles $(\alpha_C, \beta_C, \gamma_C)$.



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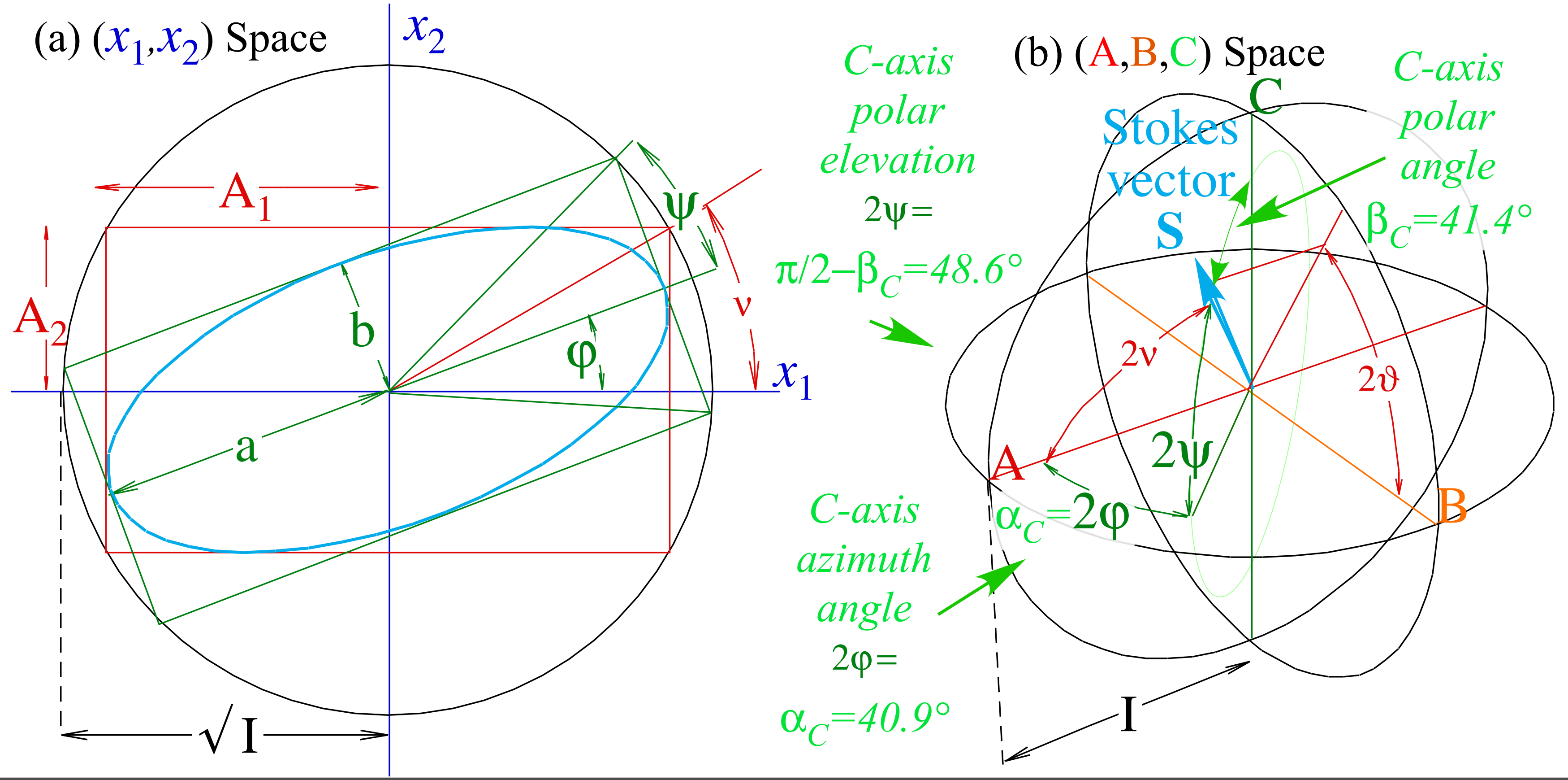
### *The* C*-view in* $\{x_R,x_L\}$*-basis*

The same orbit viewed in right and left circular polarization $\{x_R,x_L\}$-bases using angles $(\alpha_C,\beta_C,\gamma_C)$.

Angles $(\alpha_C,\beta_C)$: *C*-axial polar angle $\beta_C$ from above.

$$\sin\alpha_A\sin\beta_A = \cos\beta_C \qquad \text{or:}\quad \beta_C = \cos^{-1}(\sin\alpha_A\sin\beta_A) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\right) = 41.4°$$



## Converting an *A*-based set of Stokes parameters into a *C*-based set or a *B*-based set involves cyclic permutation of *A*, *B*, and *C* polar formulas

*Asymmetry* $S_A = \frac{I}{2}\cos\beta_A \qquad = \frac{I}{2}\sin\alpha_B \sin\beta_B = \frac{I}{2}\cos\alpha_C \sin\beta_C$

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## *The* C*-view in* $\{x_R, x_L\}$*-basis*

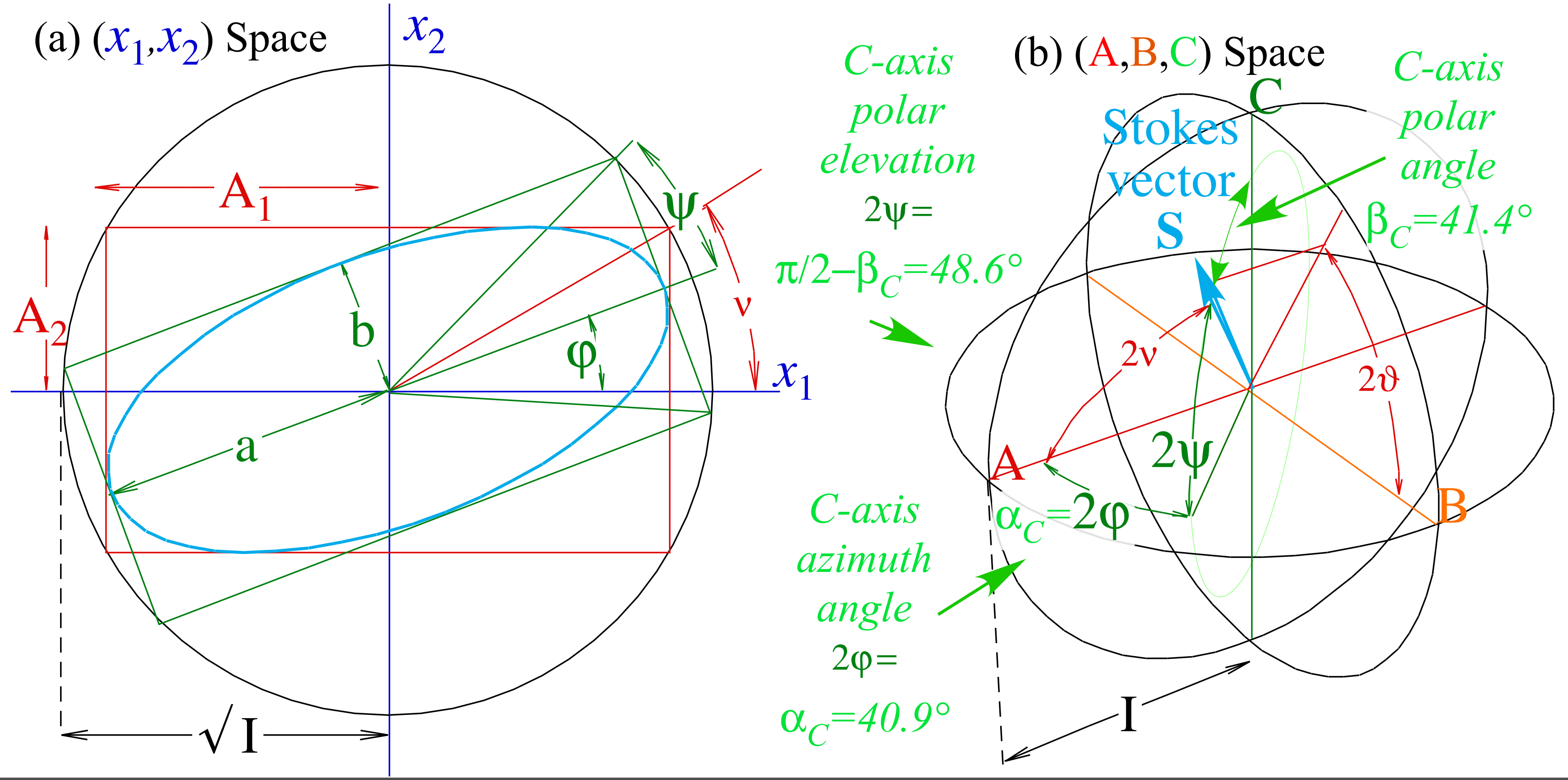
The same orbit viewed in right and left circular polarization $\{x_R, x_L\}$-bases using angles $(\alpha_C, \beta_C, \gamma_C)$.

Angles $(\alpha_C, \beta_C)$: *C*-axial polar angle $\beta_C$ from above.

$$\sin\alpha_A \sin\beta_A = \cos\beta_C \qquad \text{or:} \quad \beta_C = \cos^{-1}(\sin\alpha_A \sin\beta_A) = \cos^{-1}(\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}) = 41.4°$$

*C*-axis azimuth angle $\alpha_C$ relates to *A*-axis angles $\alpha_A$ and $\beta_A$ . See $\alpha_C = 2\varphi$ below.

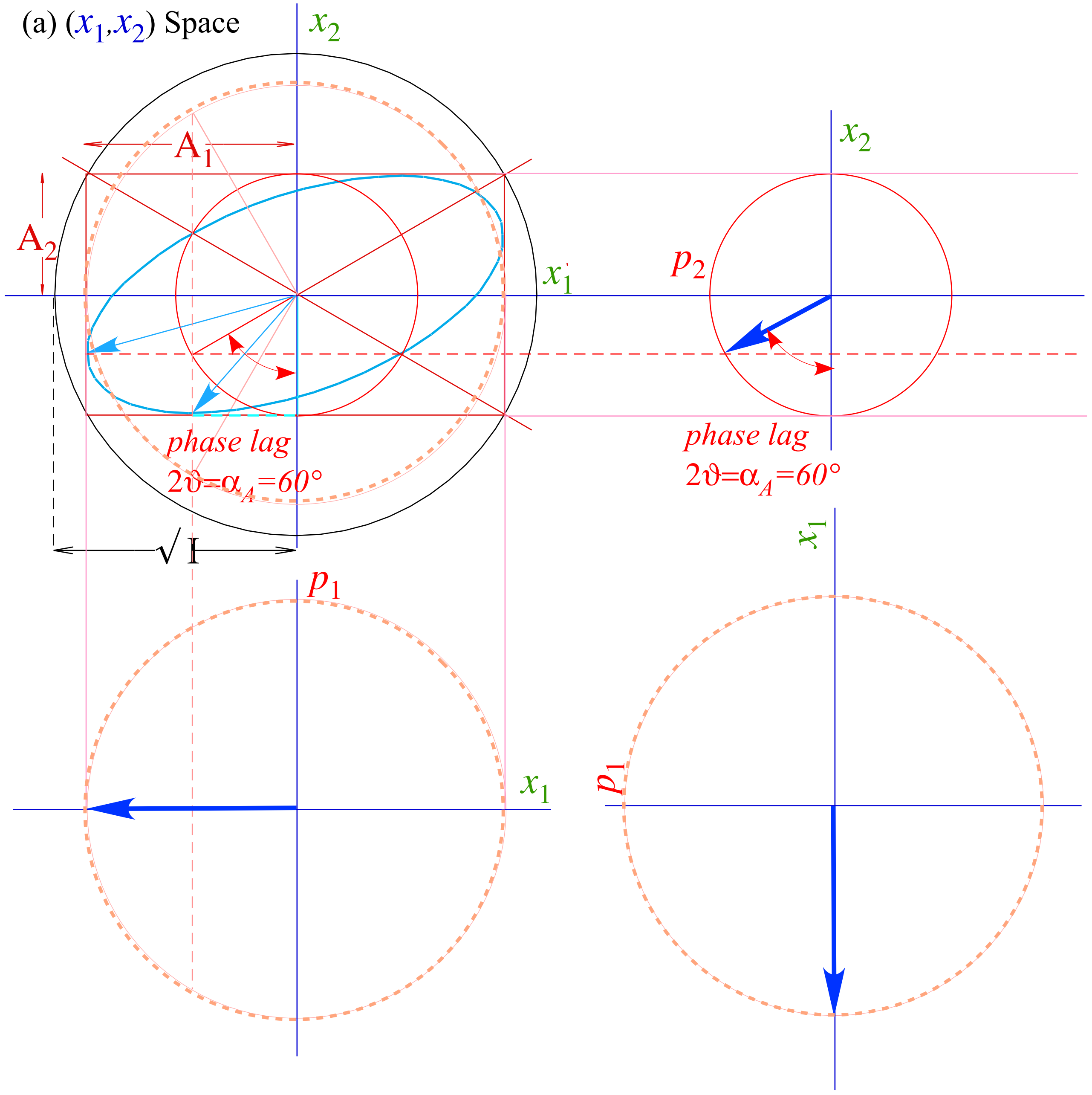
$$\frac{\cos\alpha_A \sin\beta_A}{\cos\beta_A} = \tan\alpha_C \qquad \text{or:} \quad \alpha_C = ATN2(\cos\alpha_A \sin\beta_A / \cos\beta_A) = ATN2(\frac{1}{2}\cdot\frac{\sqrt{3}}{2}/\frac{1}{2}) = 40.9°$$

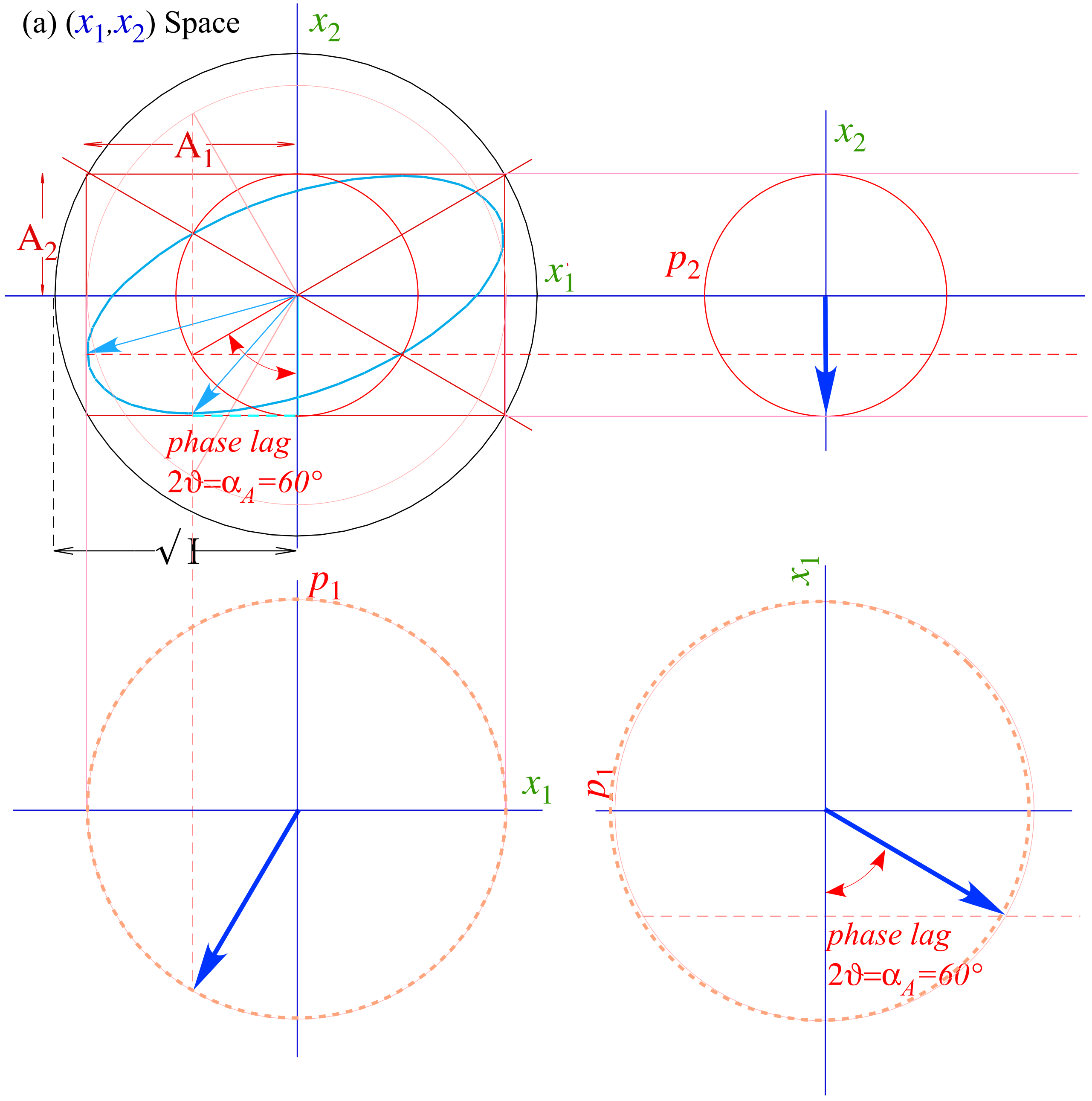


*C-axis polar elevation*
2ψ=
*π/2–$\beta_C$=48.6°*

*C-axis polar angle*
*$\beta_C$=41.4°*

*C-axis azimuth angle*
2φ=
*$\alpha_C$=40.9°*


(a) (x1,x2) Space
x2
A1
A2
x1
p2
x2
phase lag
2ϑ=αA=60°
phase lag
2ϑ=αA=60°
√ I
p1
x1
x1
p1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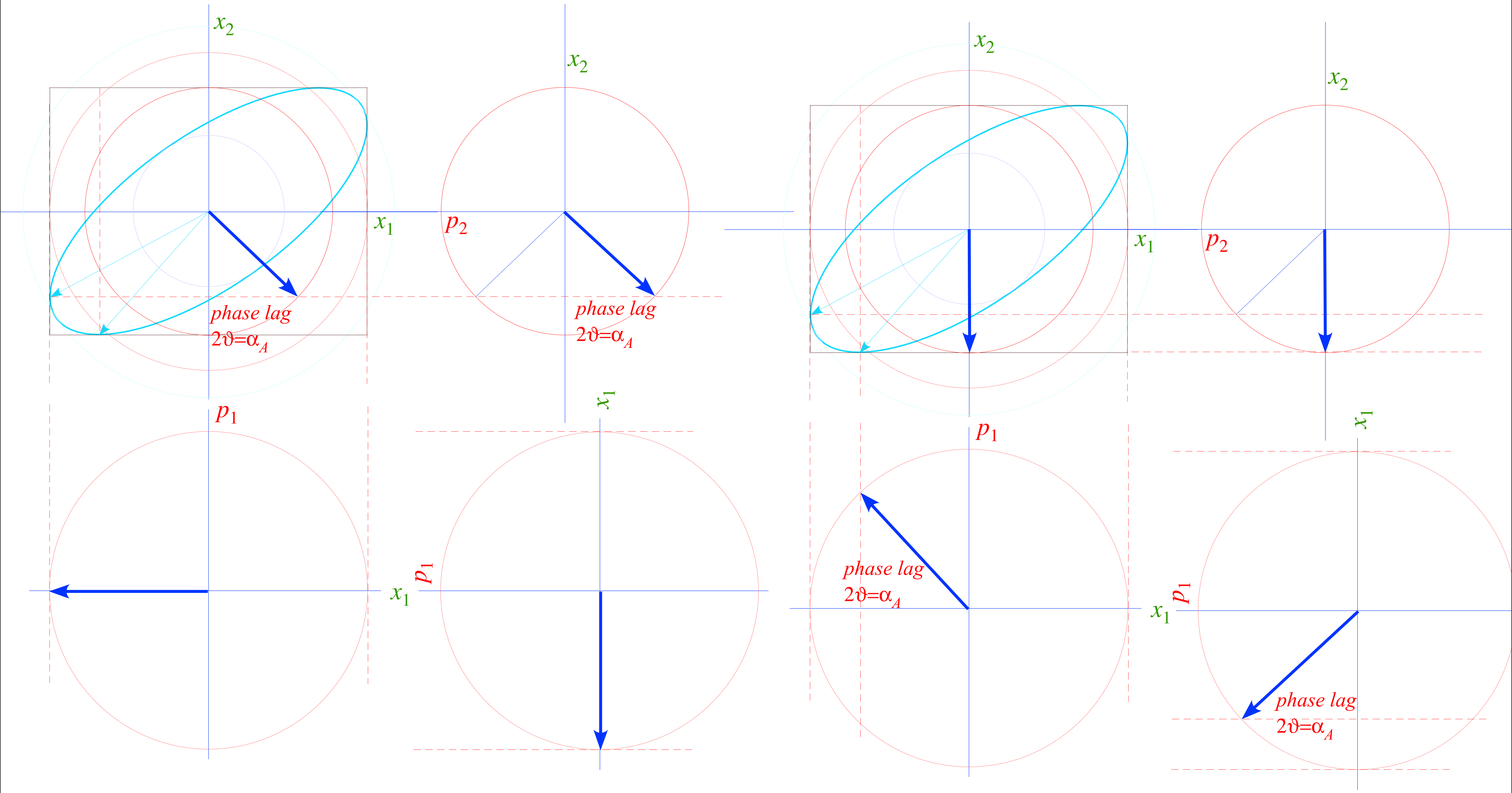
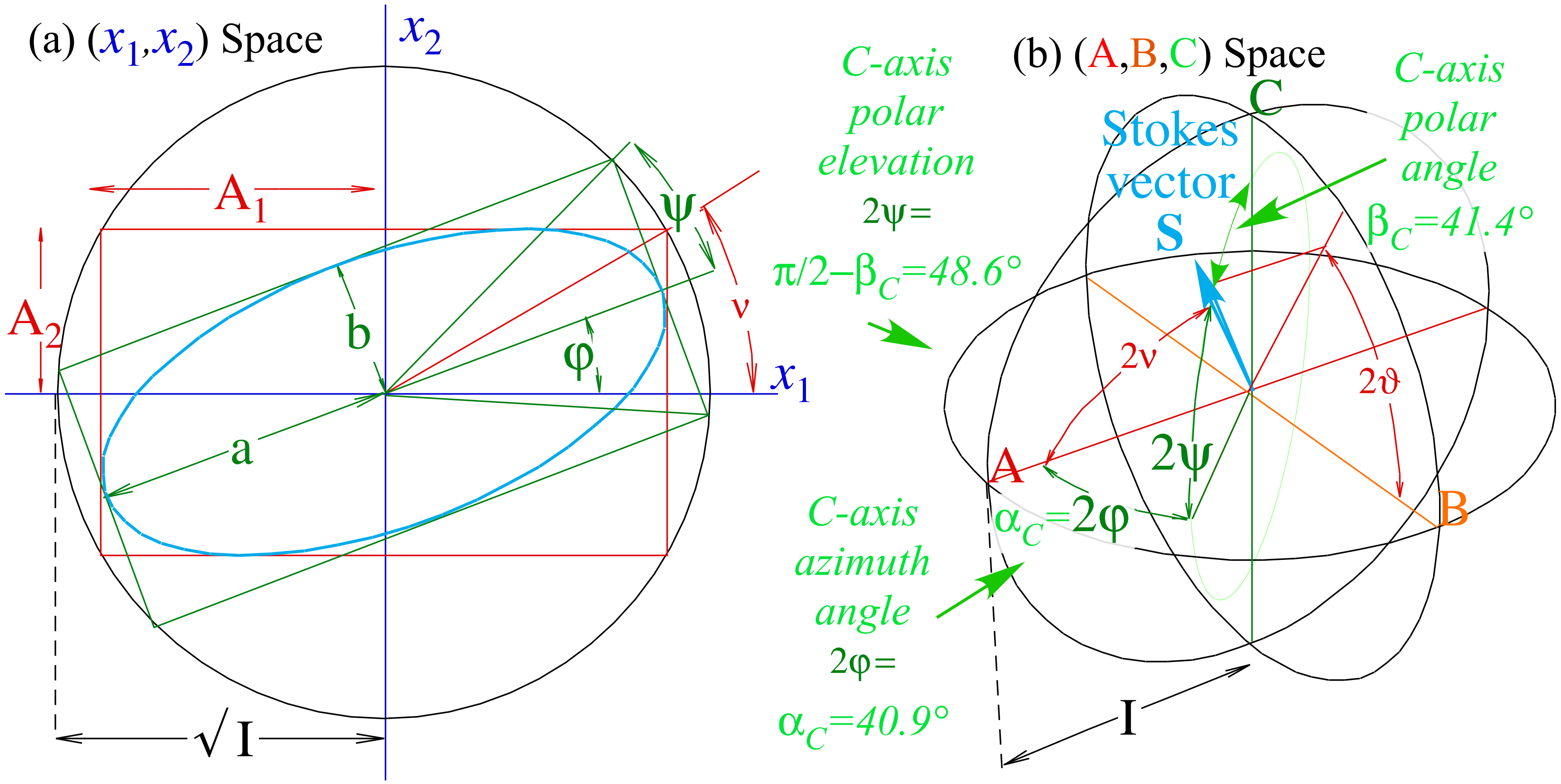

$x_2$
$x_1$
$p_2$
$x_2$
phase lag
$2\vartheta=\alpha_A$
phase lag
$2\vartheta=\alpha_A$
$p_1$
$x_1$
$x_1$
$p_1$
$x_2$
$x_1$
$p_2$
$x_2$
$p_1$
phase lag
$2\vartheta=\alpha_A$
$x_1$
$x_1$
$p_1$
phase lag
$2\vartheta=\alpha_A$

## *The C-view in {xR,xL}-basis*

The same orbit viewed in right and left circular polarization $\{x_R,x_L\}$-bases using angles $(\alpha_C, \beta_C, \gamma_C)$.

$$\begin{pmatrix} a_R \\ a_L \end{pmatrix} = A \begin{pmatrix} e^{-i\alpha_C/2}\cos\frac{\beta_C}{2} \\ e^{+i\alpha_C/2}\sin\frac{\beta_C}{2} \end{pmatrix} e^{-i\frac{\gamma_C}{2}} = \begin{pmatrix} x_R + ip_R \\ x_R + ip_R \end{pmatrix}$$



A 90° *B* –rotation $\mathbf{R}(\pi/4)\left|x_1\right\rangle = \left|x_R\right\rangle$ of axis *A* into *C* gets $(\alpha_C, \beta_C, \gamma_C)$ from $(\alpha_A, \beta_A, \gamma_A)$ all at once.

$$\begin{pmatrix} \cos\frac{\pi}{4} & i\sin\frac{\pi}{4} \\ i\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} Ae^{-i\alpha_A/2}\cos\frac{\beta_A}{2} \\ Ae^{+i\alpha_A/2}\sin\frac{\beta_A}{2} \end{pmatrix} e^{-i\frac{\gamma_A}{2}} = \begin{pmatrix} Ae^{-i\alpha_C/2}\cos\frac{\beta_C}{2} \\ Ae^{+i\alpha_C/2}\sin\frac{\beta_C}{2} \end{pmatrix} e^{-i\frac{\gamma_C}{2}} = \begin{pmatrix} x_R + ip_R \\ x_R + ip_R \end{pmatrix}$$

*How spinors give rotation products*

Now we find the product $\mathbf{R}_a$ $\mathbf{R}_b$ of rotation $\mathbf{R}_a$ by crank-axis $\vec{\Theta}_a = \hat{\Theta}_a \Theta_a$ following $\mathbf{R}_b$ by axis $\vec{\Theta}_b = \hat{\Theta}_b \Theta_b$ .

$$\mathbf{R}_a(\mathbf{\Theta}_a)\cdot\mathbf{R}_b(\mathbf{\Theta}_b) = e^{-i(\sigma\bullet\vec{\Theta}_a)/2} e^{-i(\sigma\bullet\vec{\Theta}_b)/2} = (\mathbf{1}\cos\frac{\Theta_a}{2} \quad - \quad i\ (\sigma\bullet\hat{\Theta}_a)\sin\frac{\Theta_a}{2})\ (\mathbf{1}\cos\frac{\Theta_b}{2} \quad - \quad i\ (\sigma\bullet\hat{\Theta}_b)\sin\frac{\Theta_b}{2}) = \mathbf{R}_{ab}(\mathbf{\Theta}_{ab})$$

$$=(\mathbf{1}\cos\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} - i\ (\sigma\bullet\hat{\Theta}_a)\sin\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} - i\ (\sigma\bullet\hat{\Theta}_b)\cos\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2} - (\sigma\bullet\hat{\Theta}_a)(\sigma\bullet\hat{\Theta}_b)\sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2})$$

$$=\mathbf{1}\left(\cos\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} - (\hat{\Theta}_a\bullet\hat{\Theta}_b)\sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\right) - i\ \sigma\bullet\left[\hat{\Theta}_a\sin\frac{\Theta_a}{2}\cos\frac{\Theta_b}{2} + \hat{\Theta}_b\cos\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2} + (\hat{\Theta}_a\times\hat{\Theta}_b)\sin\frac{\Theta_a}{2}\sin\frac{\Theta_b}{2}\right]$$

$$=\mathbf{1}\left(\qquad\qquad\cos\frac{\Theta_{ab}}{2}\qquad\qquad\right) - i\ \sigma\bullet\left[\qquad\qquad\qquad\hat{\Theta}_{ab}\sin\frac{\Theta_{ab}}{2}\qquad\qquad\qquad\right]$$