Group Theory in Quantum Mechanics
Lecture 3 *(1.22.13)*
Analyzers, operators, and group axioms

(Quantum Theory for Computer Age - Ch. 1-2 of Unit 1)
(Principles of Symmetry, Dynamics, and Spectroscopy - Sec. 1-3 of Ch. 1)

Review: Axioms 1-4 and "Do-Nothing" vs " Do-Something" analyzers

Abstraction of Axiom-4 to define projection and unitary operators
- *Projection operators and resolution of identity*

Unitary operators and matrices that do something (or "nothing")
- *Diagonal unitary operators*
- *Non-diagonal unitary operators and †-conjugation relations*
- *Non-diagonal projection operators and Kronecker ⊗−products*
- *Axiom-4 similarity transformation*

Matrix representation of beam analyzers
- *Non-unitary "killer" devices: Sorter-counter, filter*
- *Unitary "non-killer" devices: 1/2-wave plate, 1/4-wave plate*

How analyzers "peek" and how that changes outcomes
- *Peeking polarizers and coherence loss*
  - *Classical Bayesian probability vs. Quantum probability*

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Feynman amplitude axioms 1-4

Feynman-Dirac Interpretation of

$$\langle j | k' \rangle$$

=Amplitude of state-j after state-k′ is forced to choose from available m-type states

(1) The probability axiom

The first axiom deals with physical interpretation of amplitudes $\langle j | k' \rangle$.

Axiom 1: The absolute square $|\langle j | k' \rangle|^2 = \langle j | k' \rangle^* \langle j | k' \rangle$ *gives probability for occurrence in state-j of a system that started in state-k'=1',2',..,or n' from one sorter and then was forced to choose between states j=1,2,...,n by another sorter.*

(2) The conjugation or inversion axiom *(time reversal symmetry)*

The second axiom concerns going backwards through a sorter or the reversal of amplitudes.

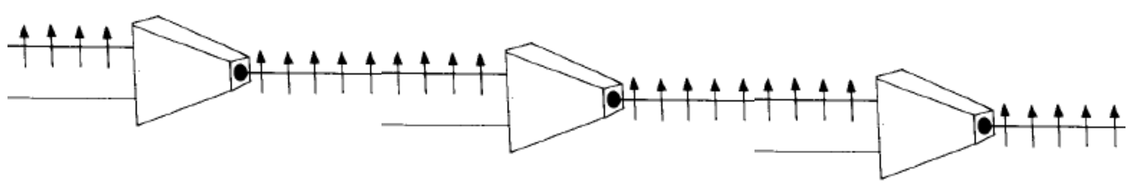
Axiom 2: The complex conjugate $\langle j | k' \rangle^*$ *of an amplitude* $\langle j | k' \rangle$ *equals its reverse:* $\langle j | k' \rangle^* = \langle k' | j \rangle$

(3) The orthonormality or identity axiom

The third axiom concerns the amplitude for "re measurement" by the same analyzer.

Axiom 3: If identical analyzers are used twice or more the amplitude for a passed state-k is one, and for all others it is zero:

$$\langle j | k \rangle = \delta_{jk} = \begin{cases} 1 \text{ if: } j=k \\ 0 \text{ if: } j \neq k \end{cases} = \langle j' | k' \rangle$$

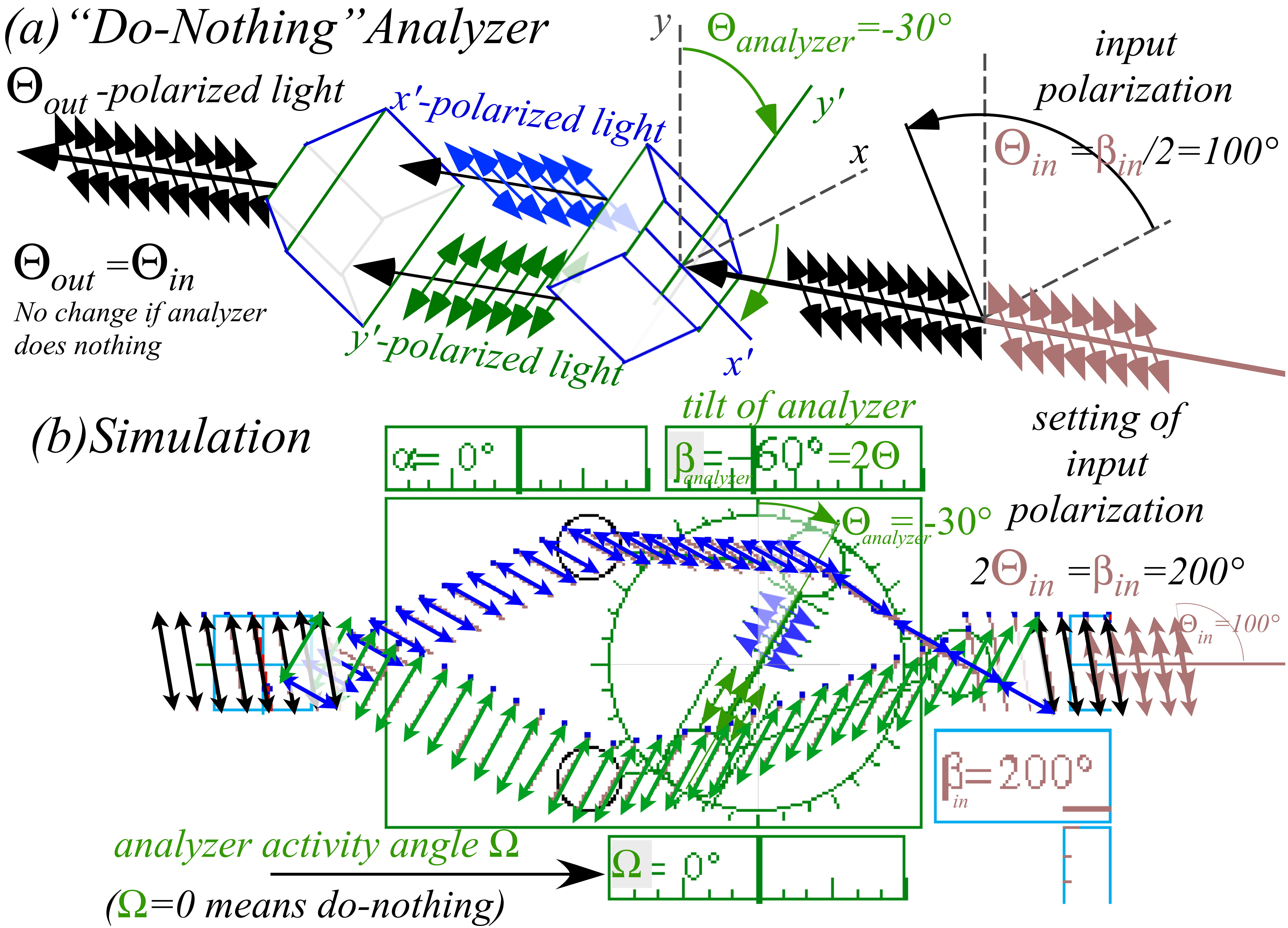


(4) The completeness or closure axiom

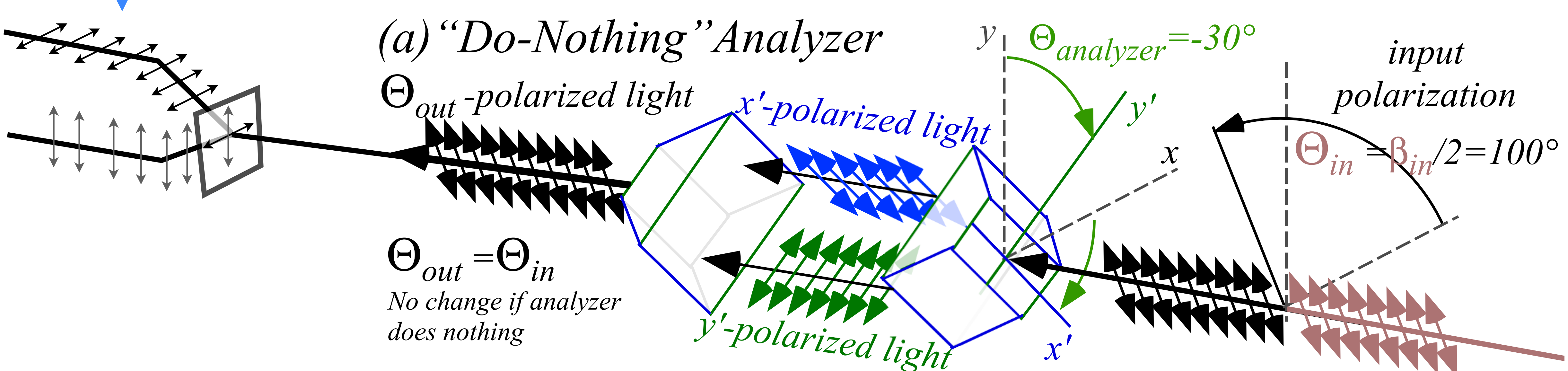
The fourth axiom concerns the "*Do-nothing*" property of an ideal analyzer, that is, a sorter followed by an "unsorter" or "put-back-togetherer" as sketched above.

Axiom 4. Ideal sorting followed by ideal recombination of amplitudes has no effect:

$$\langle j'' | m' \rangle = \sum_{k=1}^{n} \langle j'' | k \rangle \langle k | m' \rangle$$


(a) "Do-Nothing" Analyzer
Θout -polarized light
x'-polarized light
y
Θanalyzer=-30°
input polarization
y'
x
Θin =βin/2=100°
Θout =Θin
No change if analyzer does nothing
y'-polarized light
x'
(b)Simulation
tilt of analyzer
setting of input polarization
α= 0°
βanalyzer=-60°=2Θ
Θanalyzer=-30°
2Θin =βin=200°
Θin=100°
βin=200°
analyzer activity angle Ω
Ω= 0°
(Ω=0 means do-nothing)

Imagine final *xy*-sorter analyzes output beam into *x* and *y*-components.



Amplitude in *x* or *y*-channel is sum over *x'* and *y'*-amplitudes

$\langle x'|\Theta_{in}\rangle = \cos(\Theta_{in} - \Theta)$

$\langle y'|\Theta_{in}\rangle = \sin(\Theta_{in} - \Theta)$

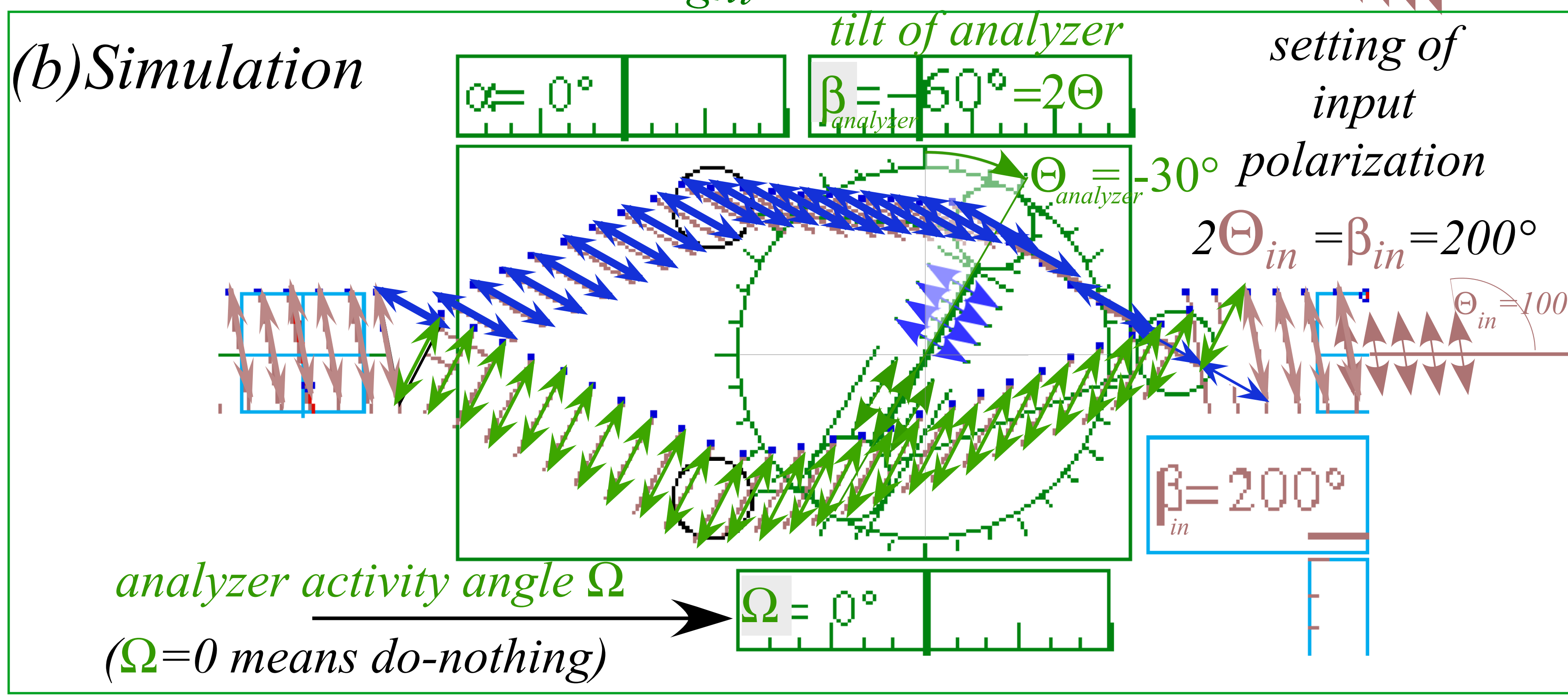
with relative angle $\Theta_{in} - \Theta$

of Θ_{in} to Θ-analyzer axes-(x',y')

in products with final *xy*-sorter:

lab x-axis: $\langle x|x'\rangle = \cos\Theta = \langle y|y'\rangle$

y-axis: $\langle y|x'\rangle = \sin\Theta = -\langle x|y'\rangle$.



x-Output is: $\langle x|\Theta_{out}\rangle = \langle x|x'\rangle\langle x'|\Theta_{in}\rangle + \langle x|y'\rangle\langle y'|\Theta_{in}\rangle = \cos\Theta\cos(\Theta_{in}-\Theta) - \sin\Theta\sin(\Theta_{in}-\Theta) = \cos\Theta_{in}$

y-Output is: $\langle y|\Theta_{out}\rangle = \langle y|x'\rangle\langle x'|\Theta_{in}\rangle + \langle y|y'\rangle\langle y'|\Theta_{in}\rangle = \sin\Theta\cos(\Theta_{in}-\Theta) - \cos\Theta\sin(\Theta_{in}-\Theta) = \sin\Theta_{in}$.

(Recall $\cos(a+b) = \cos a \cos b - \sin a \sin b$ and $\sin(a+b) = \sin a \cos b + \cos a \sin b$)

Conclusion:

$\langle x|\Theta_{out}\rangle = \cos\Theta_{out} = \cos\Theta_{in}$ or: $\Theta_{out} = \Theta_{in}$ so "Do-Nothing" Analyzer in fact does nothing.

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Center abstraction gives ket-bra ***identity operator****:*

$$\mathbf{1} = \sum_{k=1}^{n} |k\rangle\langle k| = \sum_{k=1}^{n} |k'\rangle\langle k'| = \sum_{k=1}^{n} |k''\rangle\langle k''| = \ldots$$

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Resolution of Identity into Projectors $\{|1\rangle\langle 1|, |2\rangle\langle 2|..\}$ or $\{|1'\rangle\langle 1'|, |2'\rangle\langle 2'|..\}$ or $\{|1''\rangle\langle 1''|, |2''\rangle\langle 2''|..\}$

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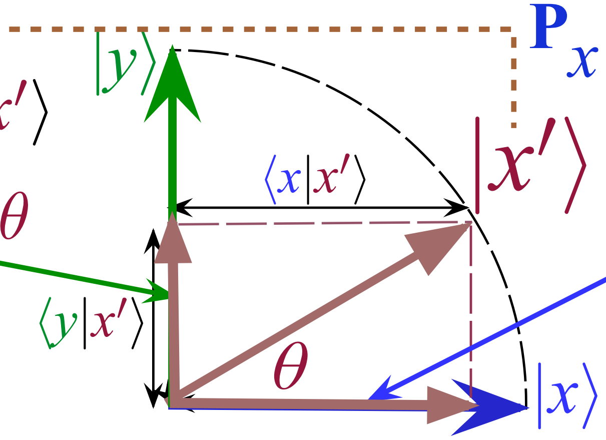

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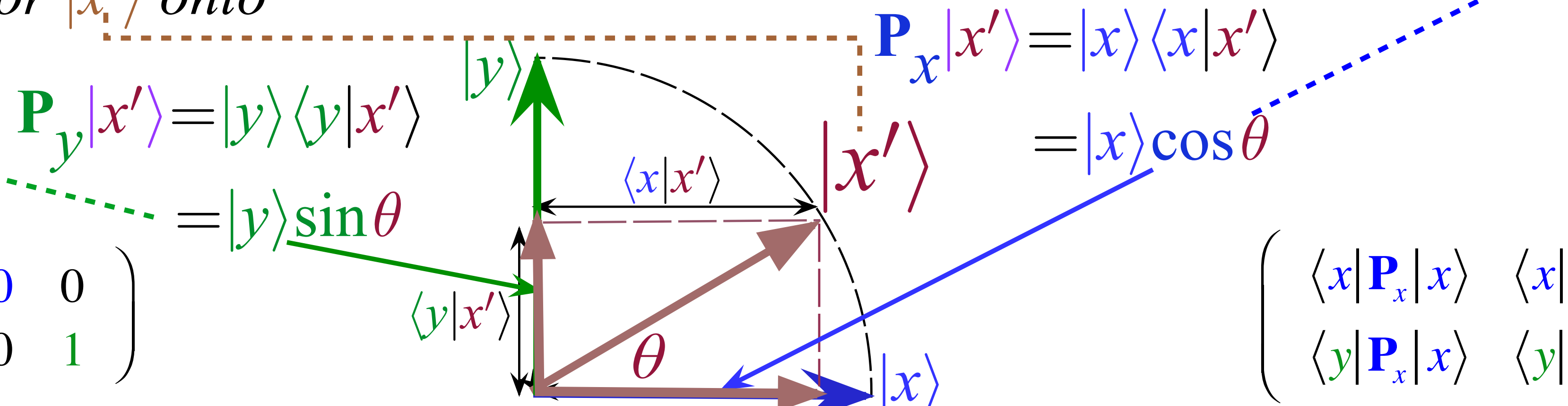

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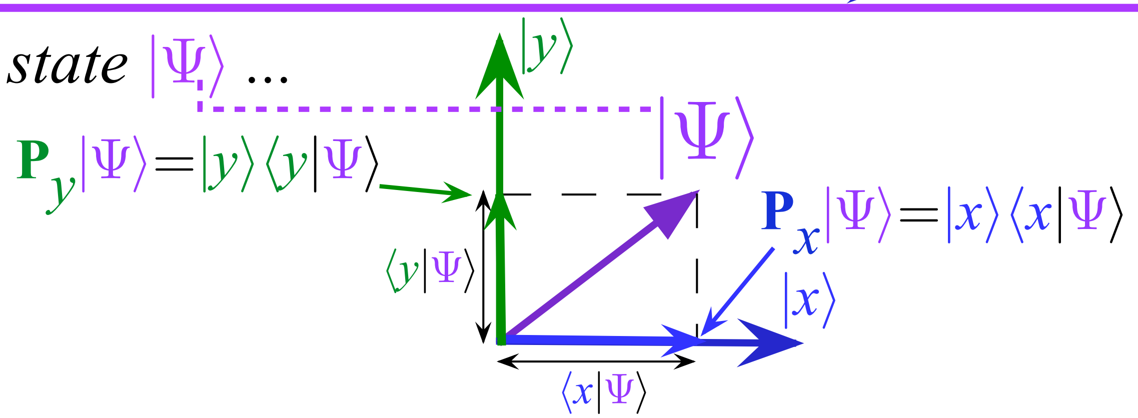
Projections of general state $|\Psi\rangle$ ...



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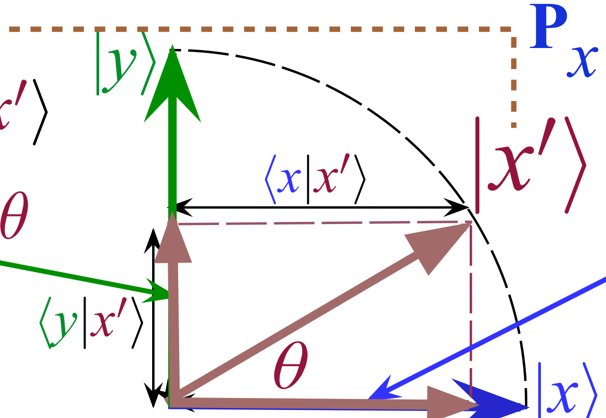

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...must add up to $|\Psi\rangle$

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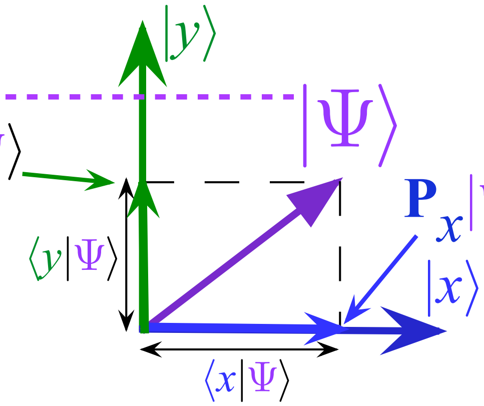

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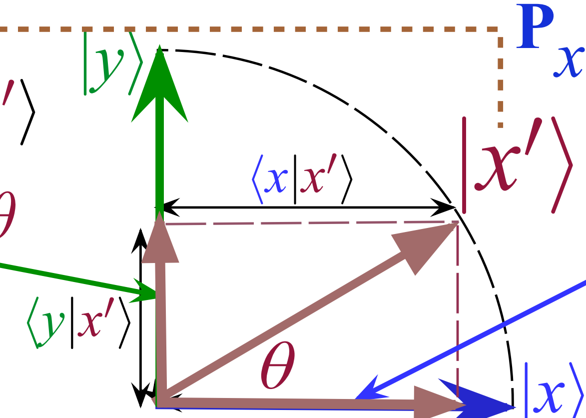

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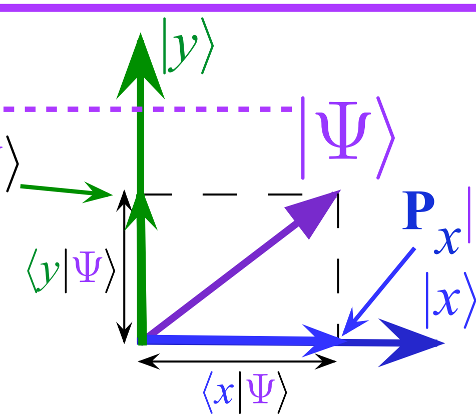

...and so $\mathbf{P}_m$ *projectors must add up to identity operator...*

$$\mathbf{1} = \mathbf{P}_x + \mathbf{P}_y$$

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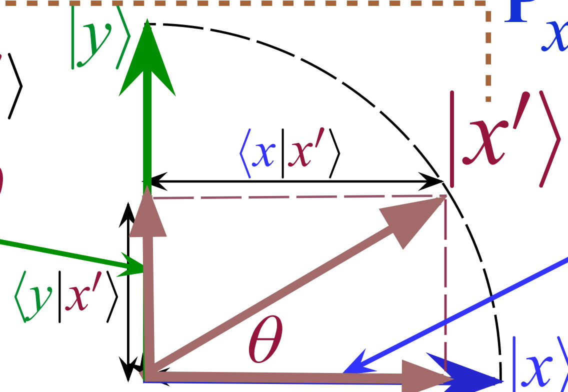

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$$\begin{pmatrix} \langle x|\mathbf{P}_x|x\rangle & \langle x|\mathbf{P}_x|y\rangle \\ \langle y|\mathbf{P}_x|x\rangle & \langle y|\mathbf{P}_x|y\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Projections of general state $|\Psi\rangle$ ...

...must add up to $|\Psi\rangle$

$\mathbf{P}_x|\Psi\rangle + \mathbf{P}_y|\Psi\rangle = |\Psi\rangle$

$(\mathbf{P}_x + \mathbf{P}_y)|\Psi\rangle = |\Psi\rangle$

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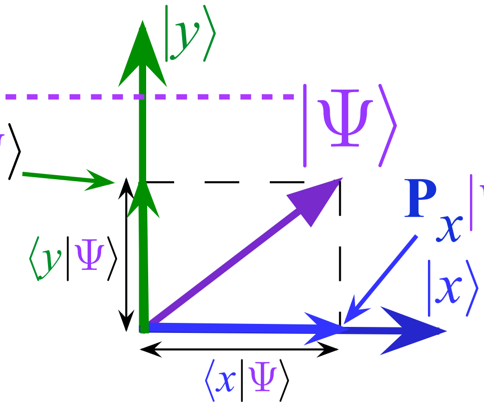


...and so $\mathbf{P}_m$ *projectors must add up to identity operator...*

and identity matrix...

$$\mathbf{1} = \mathbf{P}_x + \mathbf{P}_y$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Abstraction of Axiom 4 to define projection and unitary operators

Axiom 4: $\langle j''|m'\rangle = \sum_{k=1}^{n} \langle j''|k\rangle\langle k|m'\rangle$ *may be "abstracted" three different ways*

Left abstraction gives **bra-transform:**

$$\langle j''| = \sum_{k=1}^{n} \langle j''|k\rangle\langle k|$$

Right abstraction gives **ket-transform:**

$$|m'\rangle = \sum_{k=1}^{n} |k\rangle\langle k|m'\rangle$$

Center abstraction gives ket-bra **identity operator:**

$$\mathbf{1} = \sum_{k=1}^{n} |k\rangle\langle k| = \sum_{k=1}^{n} |k'\rangle\langle k'| = \sum_{k=1}^{n} |k''\rangle\langle k''| = \ldots$$

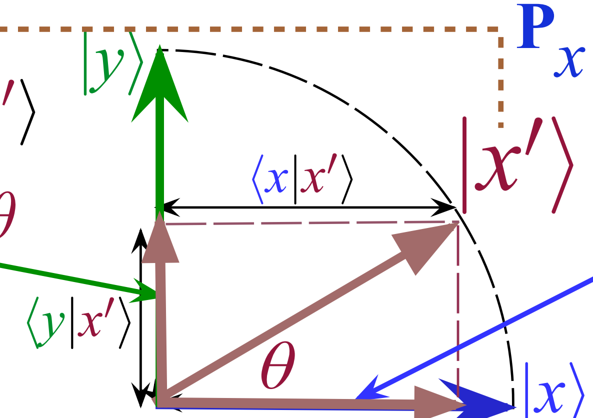
Resolution of Identity into Projectors $\{|1\rangle\langle 1|, |2\rangle\langle 2|..\}$ or $\{|1'\rangle\langle 1'|, |2'\rangle\langle 2'|..\}$ or $\{|1''\rangle\langle 1''|, |2''\rangle\langle 2''|..\}$

$\mathbf{P}_1 = |1\rangle\langle 1|$, $\mathbf{P}_2 = |2\rangle\langle 2|$,.. or $\mathbf{P}_{1'} = |1'\rangle\langle 1'|$, $\mathbf{P}_{2'} = |2'\rangle\langle 2'|$ *etc.*

Projections of unit vector $|x'\rangle$ *onto unit kets* $|x\rangle$ *and* $|y\rangle$

$\mathbf{P}_x|x'\rangle = |x\rangle\langle x|x'\rangle = |x\rangle\cos\theta$

$\mathbf{P}_y|x'\rangle = |y\rangle\langle y|x'\rangle = |y\rangle\sin\theta$



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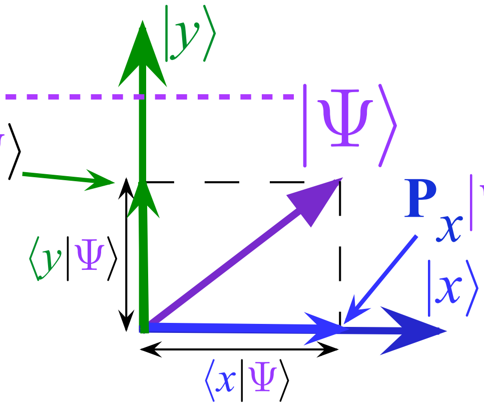

...and so $\mathbf{P}_m$ *projectors must add up to identity operator...*

$$\mathbf{1} = \mathbf{P}_x + \mathbf{P}_y$$

and identity matrix... $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

..as required by Axiom 4:

Review: Axioms 1-4 and "Do-Nothing" vs " Do-Something" analyzers

Abstraction of Axiom-4 to define projection and unitary operators
  Projection operators and resolution of identity

→ *Unitary operators and matrices that do something (or "nothing")*
  Diagonal unitary operators
  Non-diagonal unitary operators and †-conjugation relations
  Non-diagonal projection operators and Kronecker ⊗−products
  Axiom-4 similarity transformation

Matrix representation of beam analyzers
  Non-unitary "killer" devices: Sorter-counter, filter
  Unitary "non-killer" devices: 1/2-wave plate, 1/4-wave plate

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  Peeking polarizers and coherence loss
    Classical Bayesian probability vs. Quantum probability

Unitary operators and matrices that do something (or "nothing")

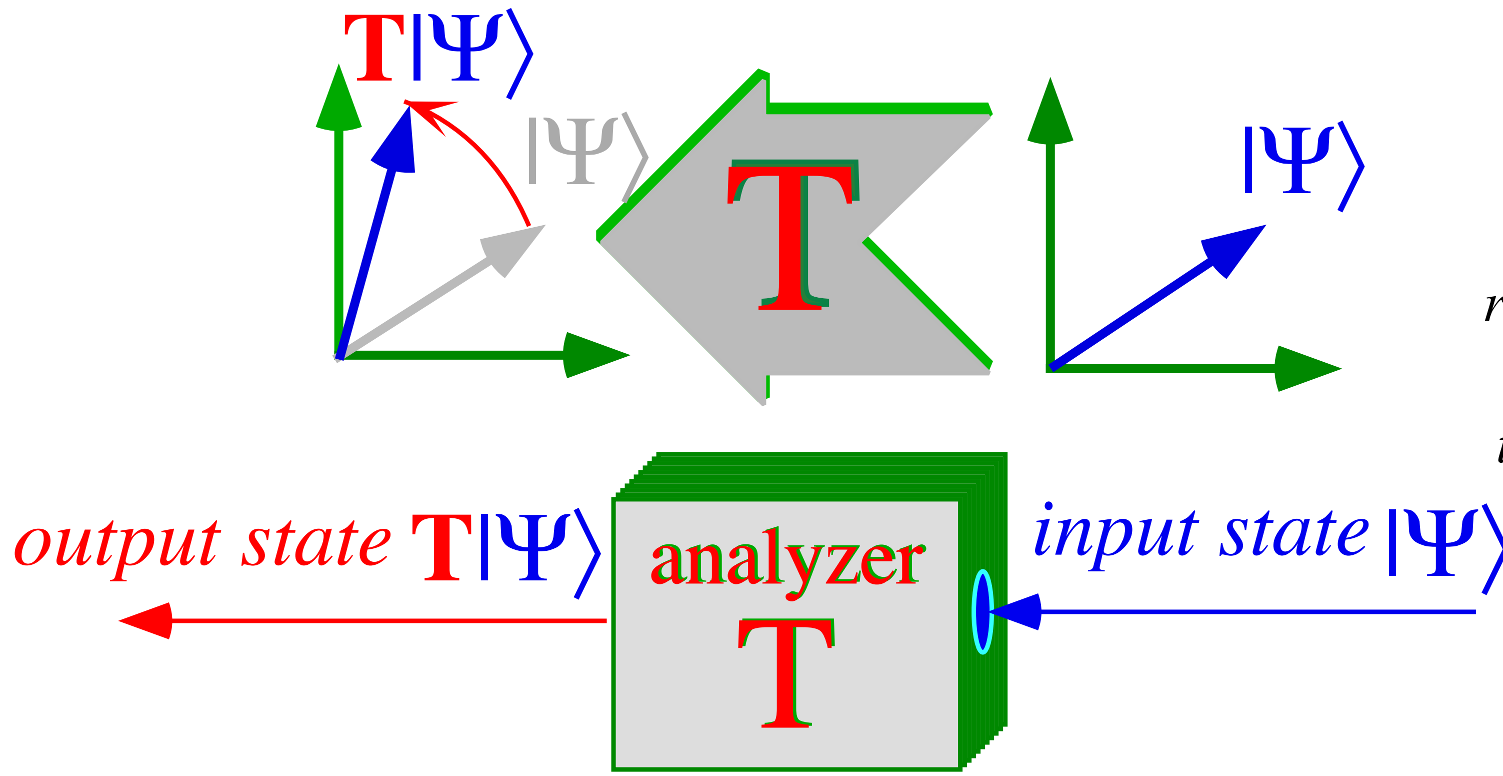

Fig. 3.1.1 Effect of analyzer represented by ket vector transformation of $|\Psi\rangle$ *to new ket vector* $\mathbf{T}|\Psi\rangle$.

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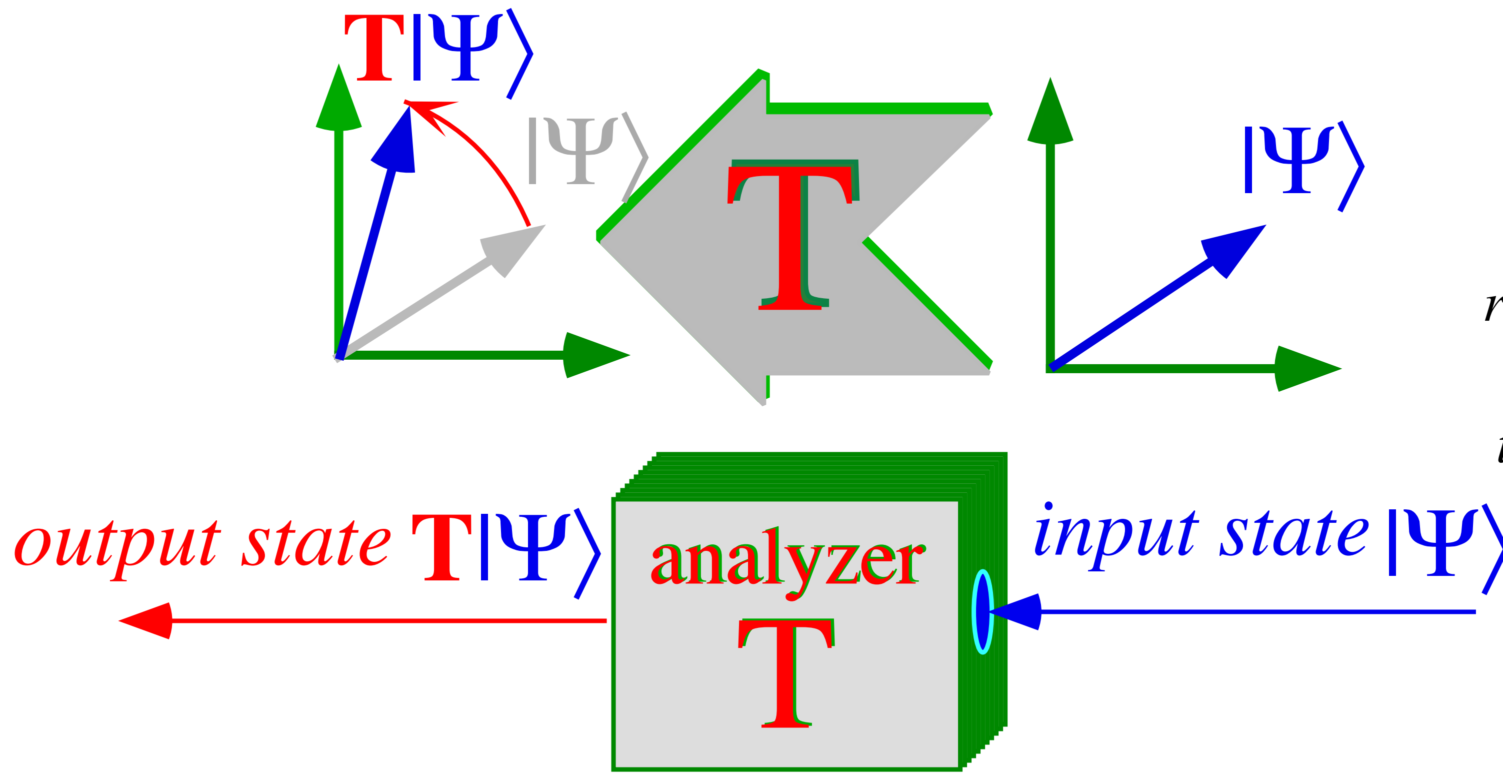

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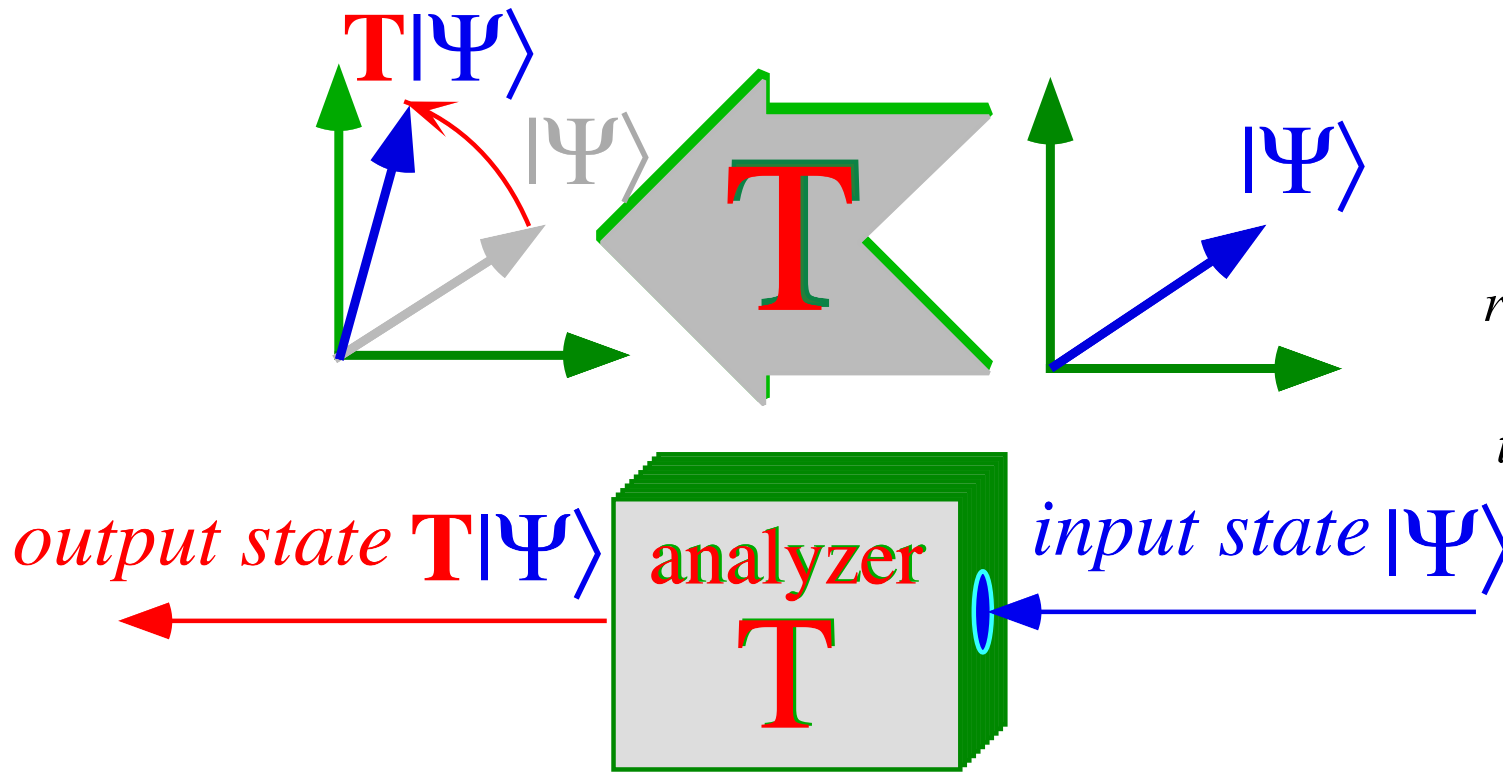

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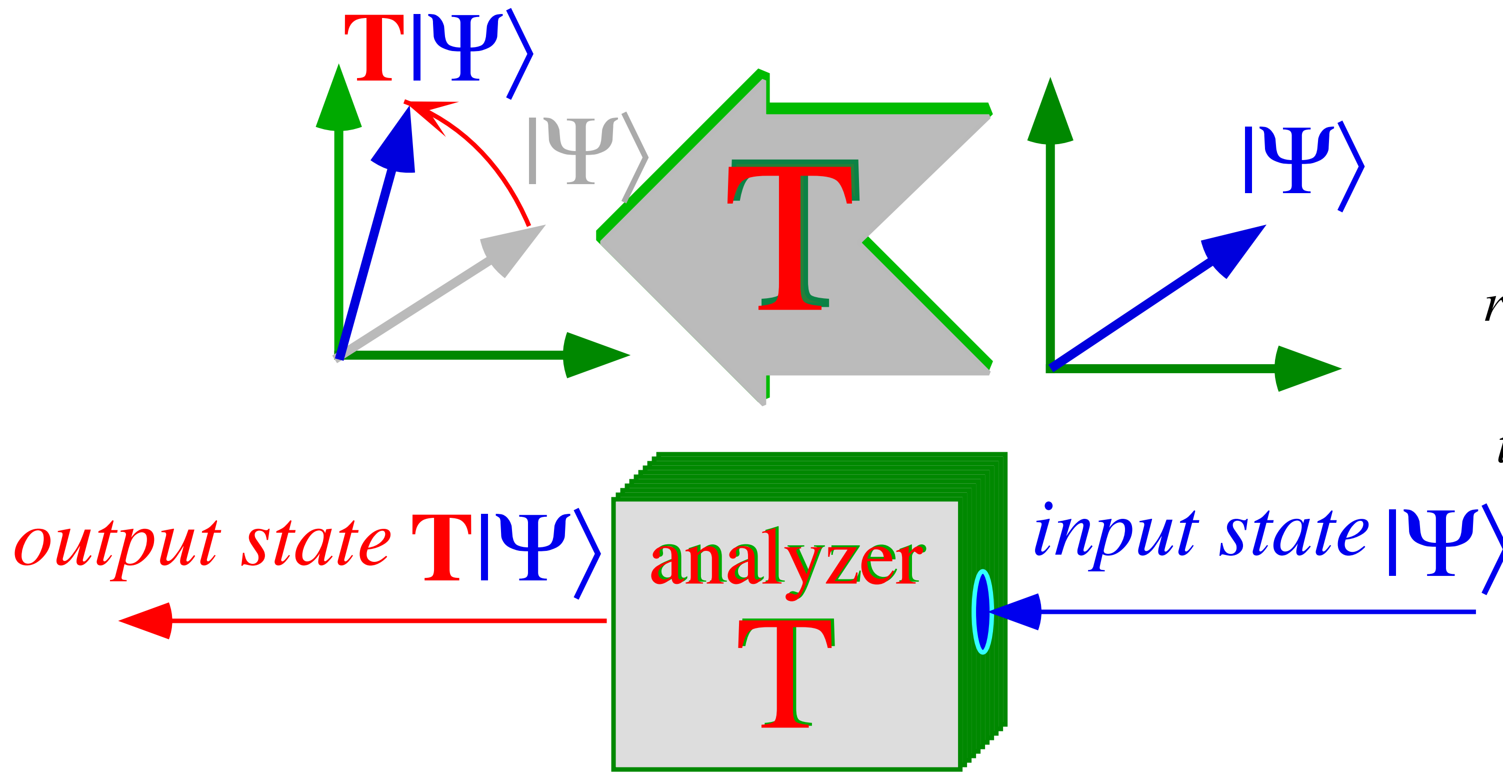

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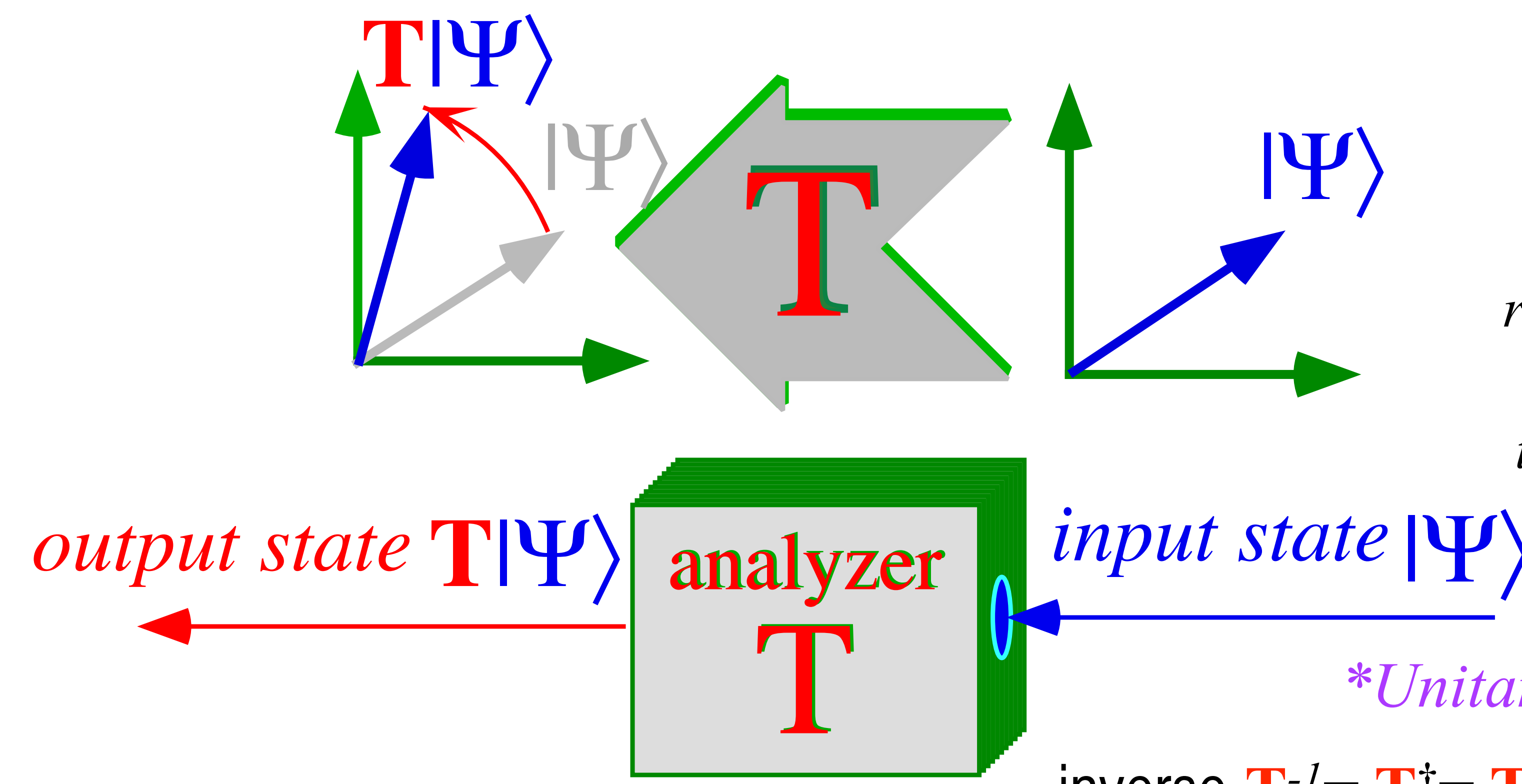

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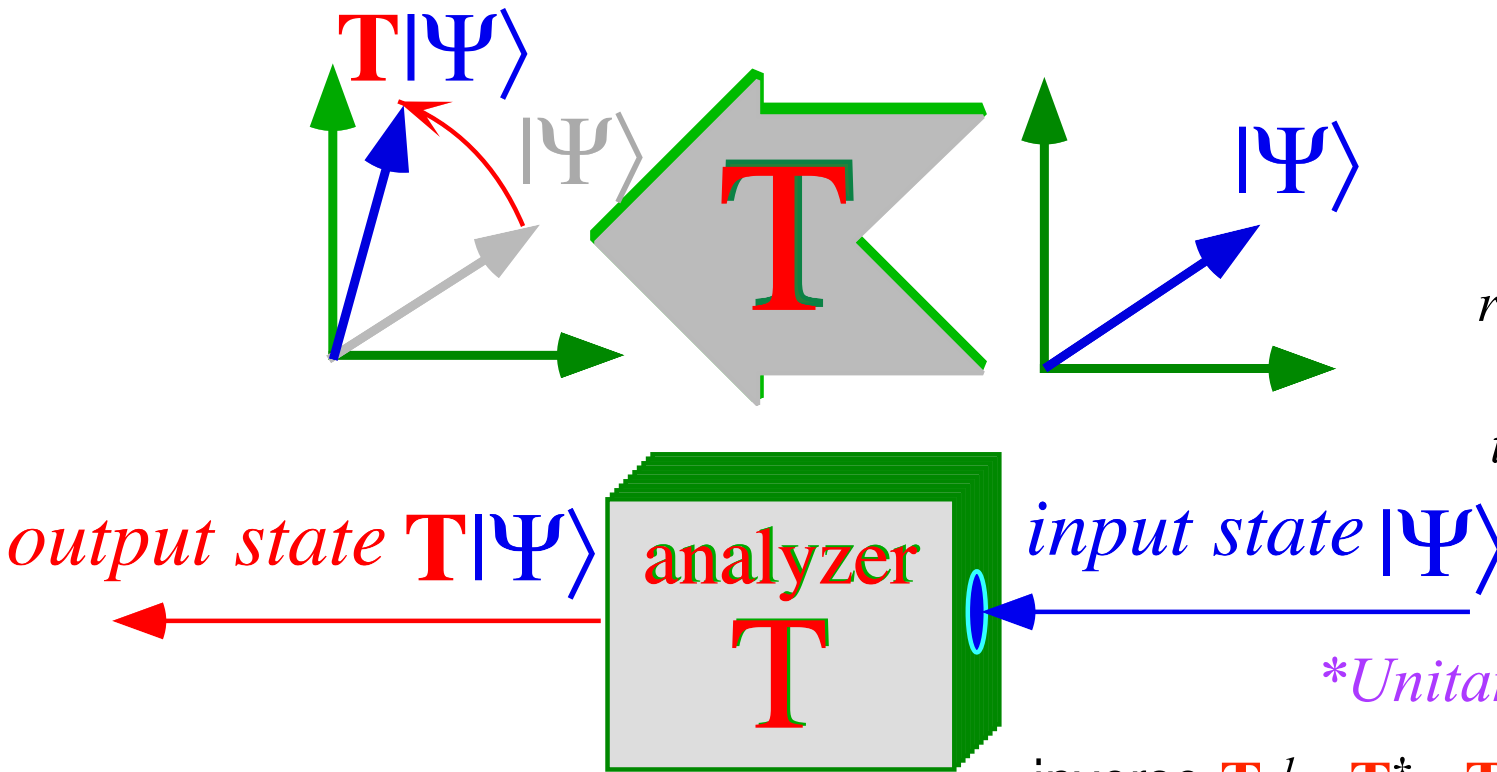

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Most "do-something" operators $\mathbf{T}'$ *are not diagonal, that is, not just* $|x\rangle\langle x|$ *and* $|y\rangle\langle y|$ *combinations.*

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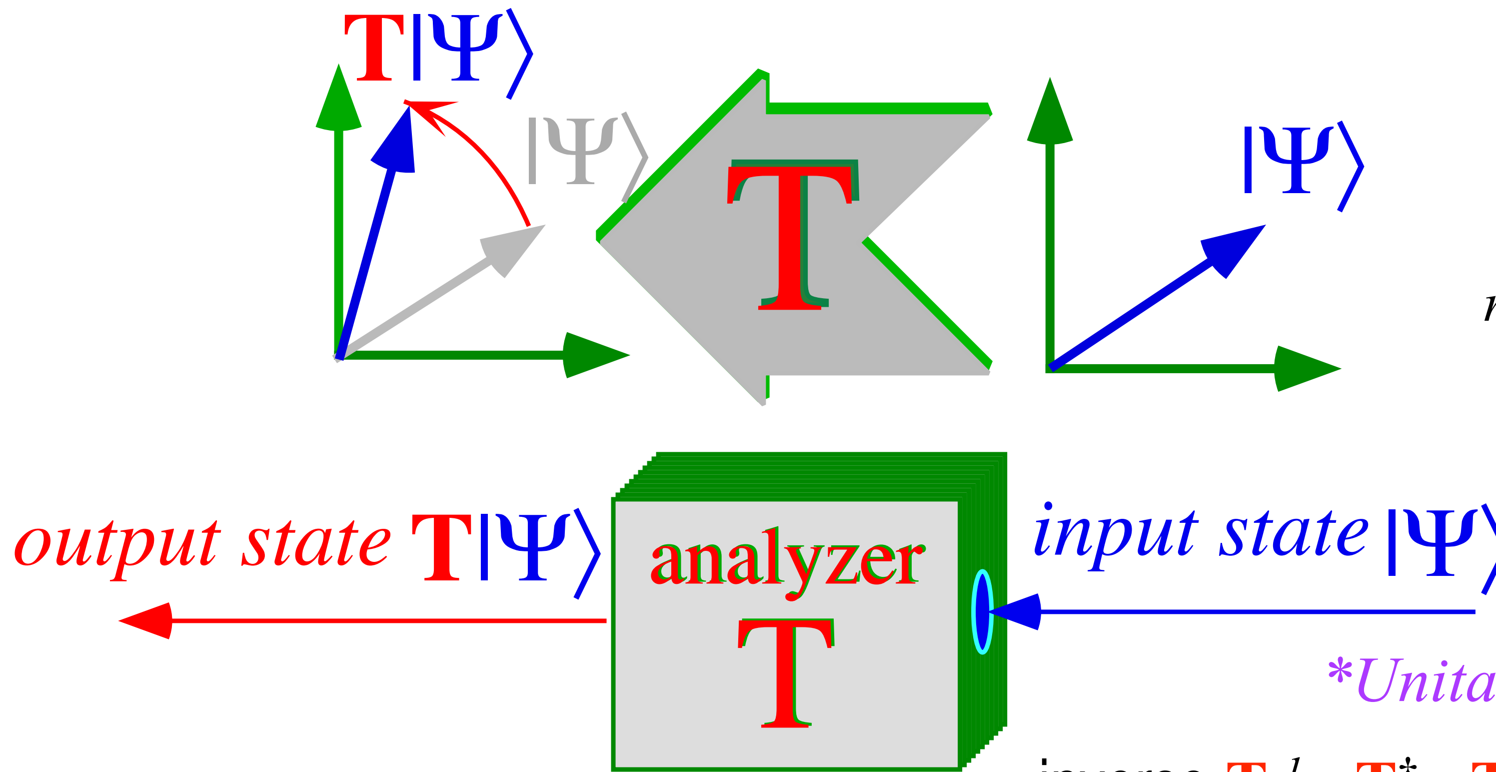
Unitary operators and matrices that do something (or "nothing")



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(Matrix representation of $\mathbf{T}'$ *is a little more complicated. See following pages.)*

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- *Projection operators and resolution of identity*

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Unitary operators **U** *satisfy "easy inversion" relations:* $\mathbf{U}^{-1} = \mathbf{U}^{\dagger} = \mathbf{U}^{T*}$

They are "designed" to conserve *probability* and *overlap*

so each transformed ket $|\Psi'\rangle = \mathbf{U}|\Psi\rangle$ has the same *probability* $\langle\Psi|\Psi\rangle = \langle\Psi'|\Psi'\rangle = \langle\Psi|\mathbf{U}^{\dagger}\mathbf{U}|\Psi\rangle$

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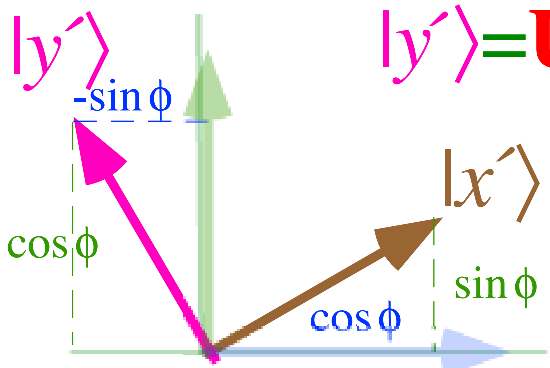

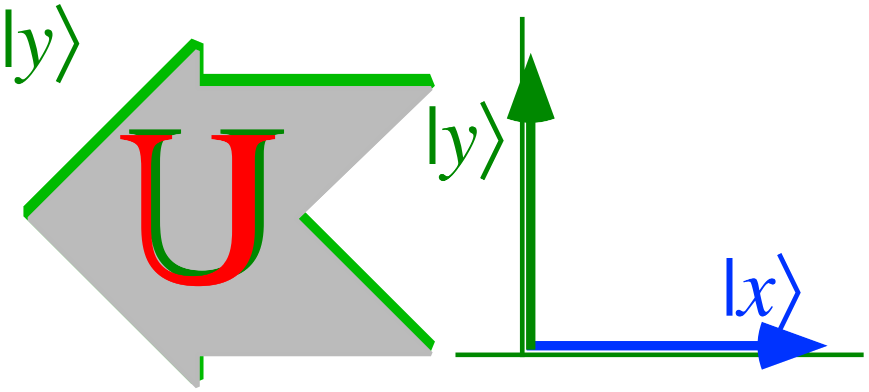

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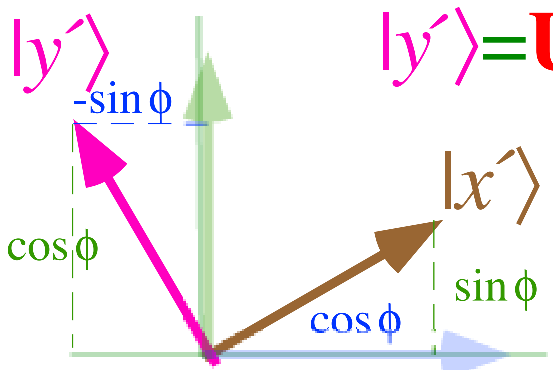

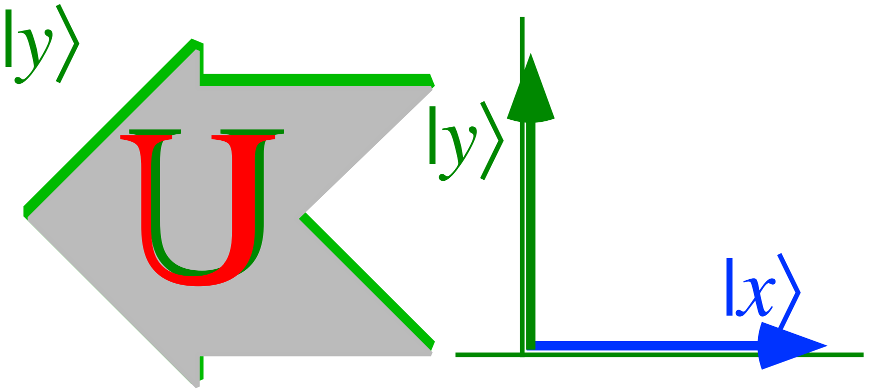

Ket definition: $|x'\rangle = \mathbf{U}|x\rangle$ implies: $\mathbf{U}^\dagger|x'\rangle = |x\rangle$ implies: $\langle x| = \langle x'|\mathbf{U}$ implies: $\langle x|\mathbf{U}^\dagger = \langle x'|$

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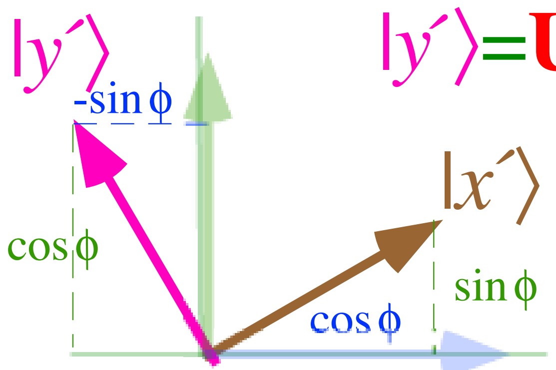

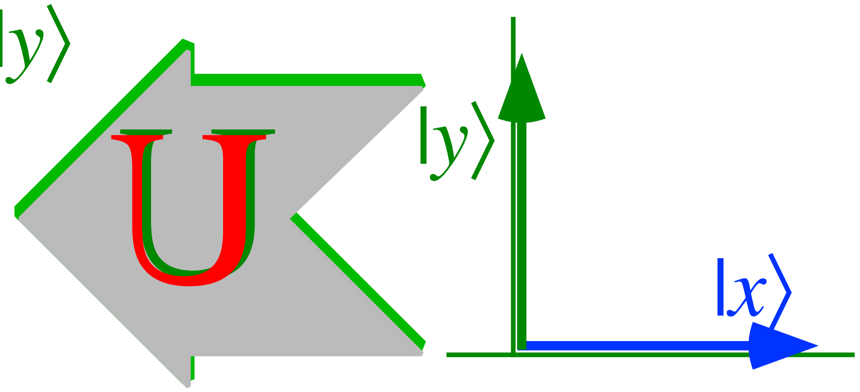

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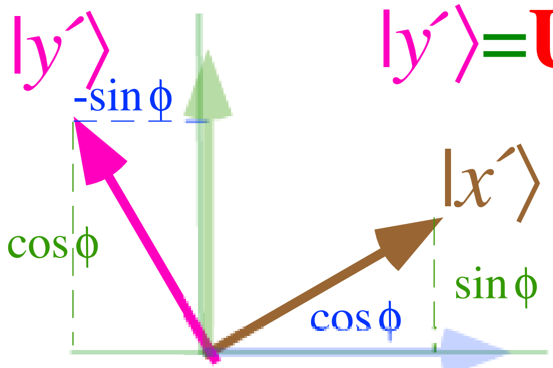

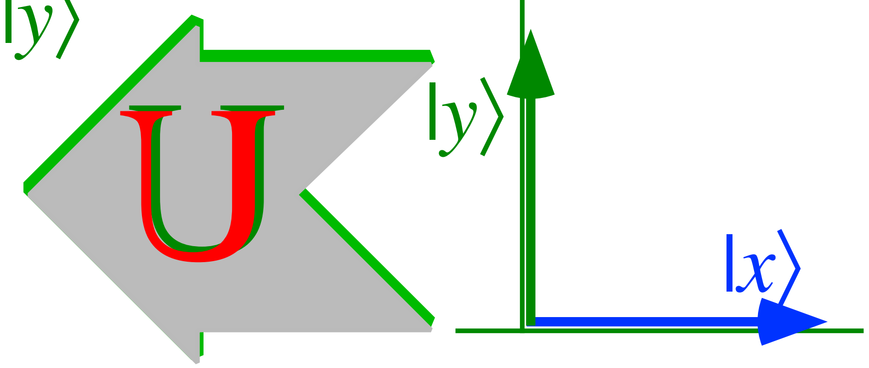

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...implies matrix representation of operator **U**

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Example **U** *transfomation:* *(Rotation by φ=30°)*

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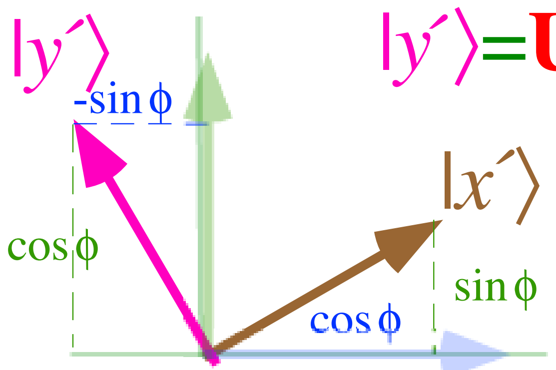

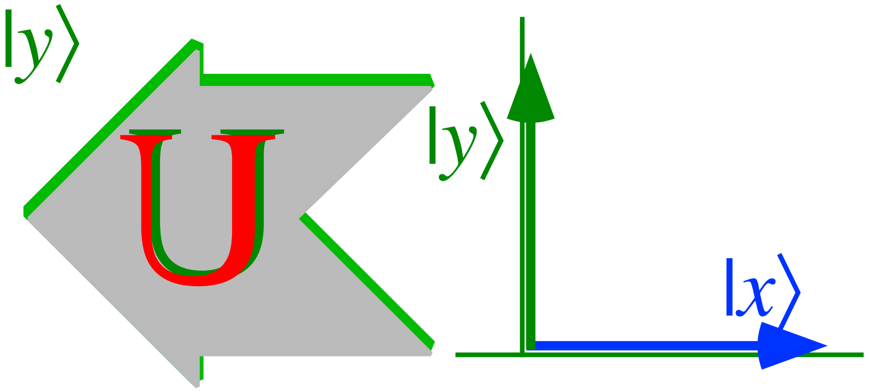

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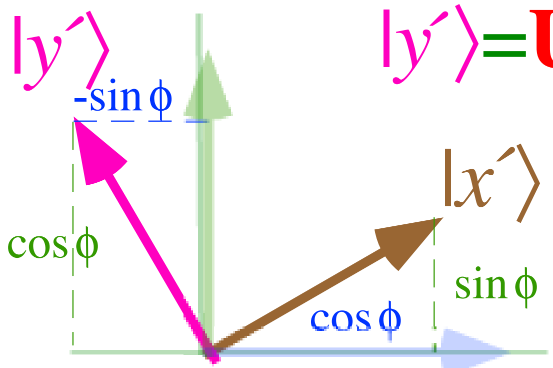

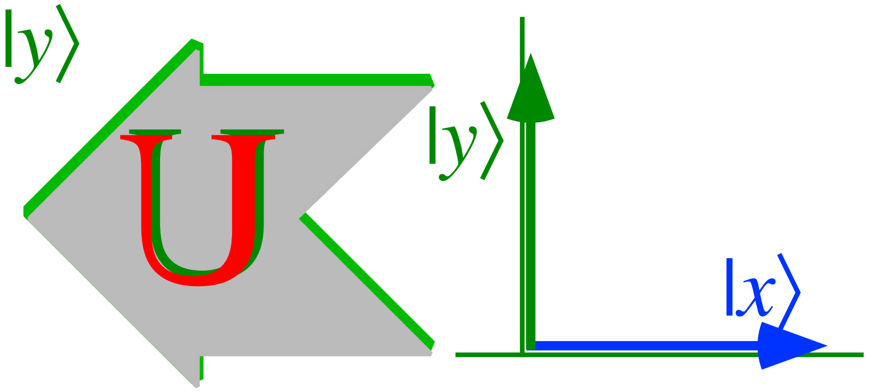

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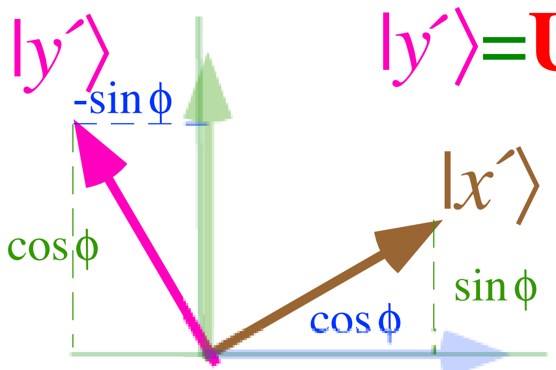

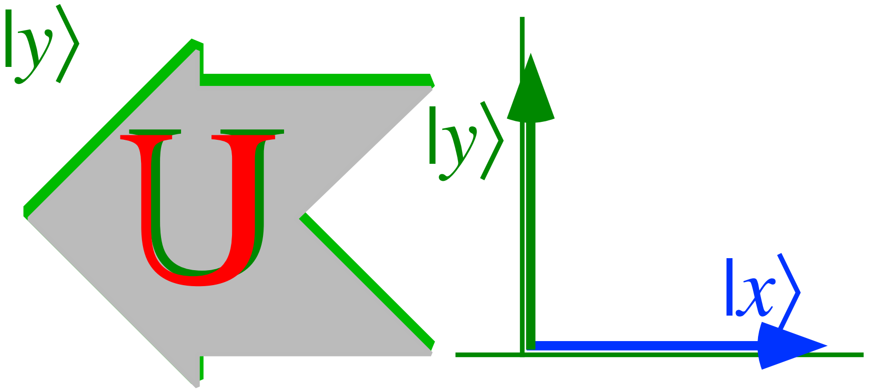

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Axiom-3 consistent with

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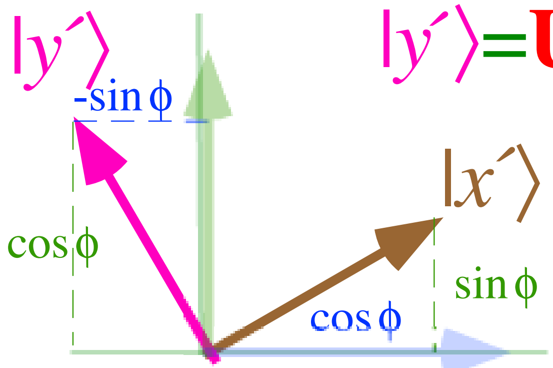

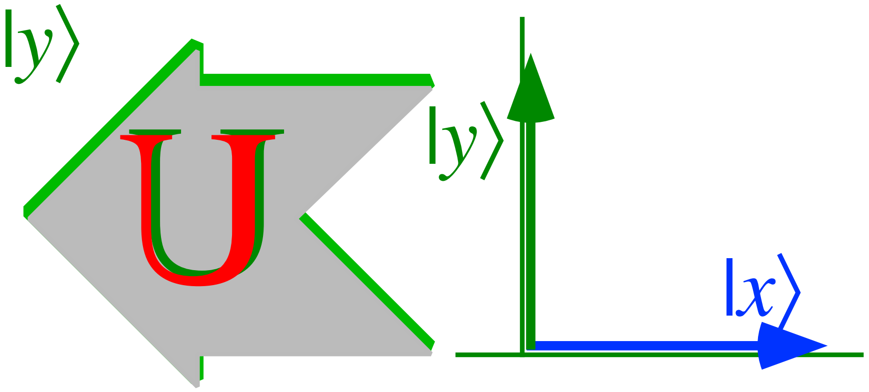

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Review: Axioms 1-4 and "Do-Nothing" vs " Do-Something" analyzers

Abstraction of Axiom-4 to define projection and unitary operators
- *Projection operators and resolution of identity*

Unitary operators and matrices that do something (or "nothing")
- *Diagonal unitary operators*
- *Non-diagonal unitary operators and †-conjugation relations*
- → *Non-diagonal projection operators and Kronecker ⊗−products*
- *Axiom-4 similarity transformation*

Matrix representation of beam analyzers
- *Non-unitary "killer" devices: Sorter-counter, filter*
- *Unitary "non-killer" devices: 1/2-wave plate, 1/4-wave plate*

How analyzers "peek" and how that changes outcomes
- *Peeking polarizers and coherence loss*
  - *Classical Bayesian probability vs. Quantum probability*

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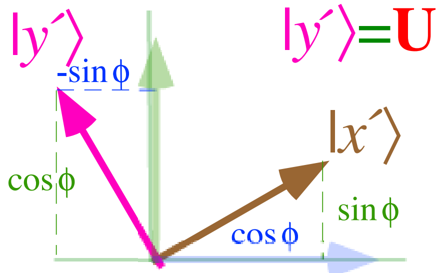

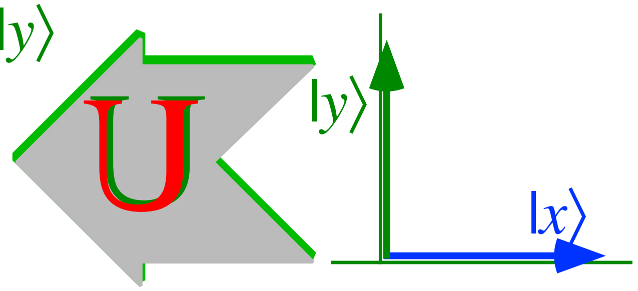

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Projector $\mathbf{P}_x = |x\rangle\langle x|$ in ϕ-tilted polarization bases $\{|x'\rangle, |y'\rangle\}$ is not diagonal.

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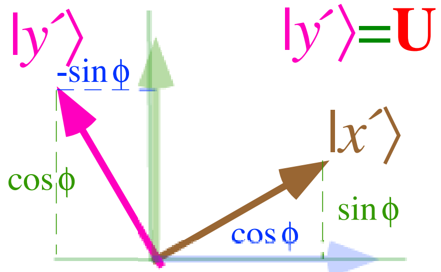

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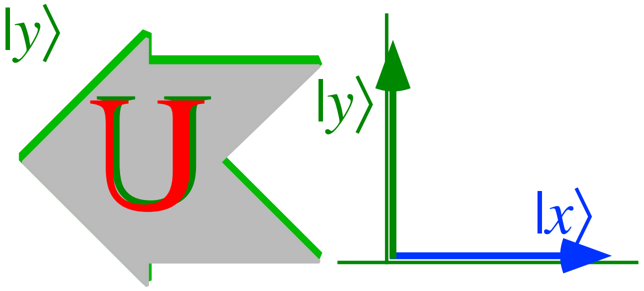

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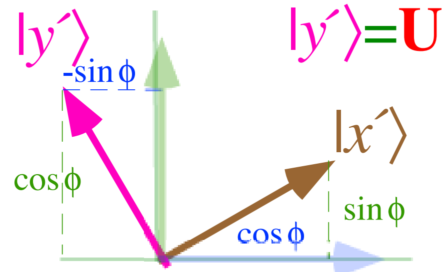

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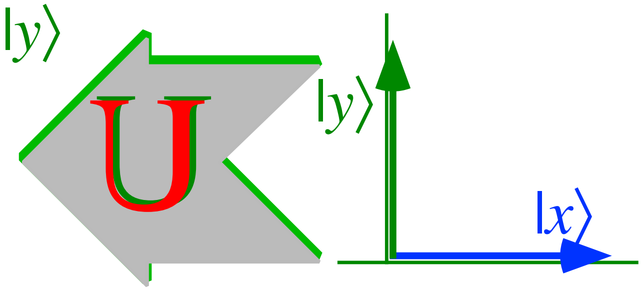

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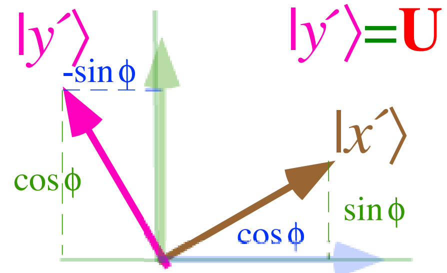

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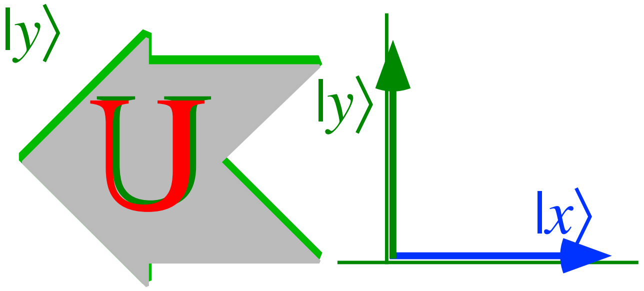

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Projector $\mathbf{P}_x = |x\rangle\langle x|$ is what is called an *outer* or *Kronecker tensor (⊗) product* of ket $|x\rangle$ and bra $\langle x|$.

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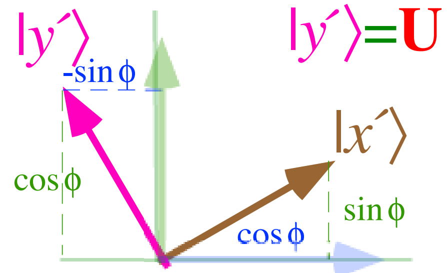
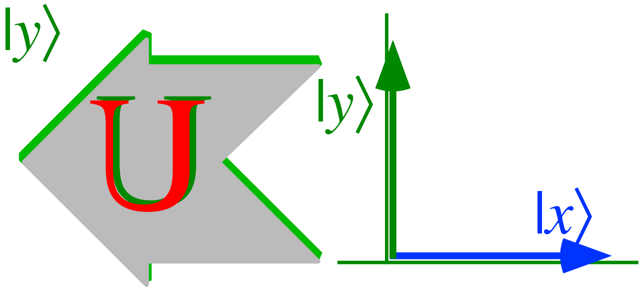
The $x'y'$-representation of $\mathbf{P}_X$:

$$\mathbf{P}_x = |x\rangle\langle x| \rightarrow \begin{pmatrix} \cos\phi \\ -\sin\phi \end{pmatrix} \otimes \begin{pmatrix} \cos\phi & -\sin\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\phi & -\sin\phi\cos\phi \\ -\sin\phi\cos\phi & \sin^2\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{(\text{for } \phi=0)}$$

$$|x'\rangle = \mathbf{U}|x\rangle = \cos\phi\,|x\rangle + \sin\phi\,|y\rangle$$
$$|y'\rangle = \mathbf{U}|y\rangle = -\sin\phi\,|x\rangle + \cos\phi\,|y\rangle$$





$$\begin{pmatrix} \langle x|\mathbf{U}|x\rangle & \langle x|\mathbf{U}|y\rangle \\ \langle y|\mathbf{U}|x\rangle & \langle y|\mathbf{U}|y\rangle \end{pmatrix} = \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix}$$

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

$$= \begin{pmatrix} \langle x'|\mathbf{U}|x'\rangle & \langle x'|\mathbf{U}|y'\rangle \\ \langle y'|\mathbf{U}|x'\rangle & \langle y'|\mathbf{U}|y'\rangle \end{pmatrix}$$

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The $x'y'$-representation of $\mathbf{P}_y$:

$$\mathbf{P}_y = |y\rangle\langle y| \rightarrow \begin{pmatrix} \sin\phi \\ \cos\phi \end{pmatrix} \otimes \begin{pmatrix} \sin\phi & \cos\phi \end{pmatrix}$$

$$= \begin{pmatrix} \sin^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \cos^2\phi \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_{(\text{for } \phi=0)}$$

Review: Axioms 1-4 and "Do-Nothing" vs " Do-Something" analyzers

Abstraction of Axiom-4 to define projection and unitary operators
- *Projection operators and resolution of identity*

Unitary operators and matrices that do something (or "nothing")
- *Diagonal unitary operators*
- *Non-diagonal unitary operators and †-conjugation relations*
- *Non-diagonal projection operators and Kronecker ⊗−products*
- ➡ *Axiom-4 similarity transformation*

Matrix representation of beam analyzers
- *Non-unitary "killer" devices: Sorter-counter, filter*
- *Unitary "non-killer" devices: 1/2-wave plate, 1/4-wave plate*

How analyzers "peek" and how that changes outcomes
- *Peeking polarizers and coherence loss*
  - *Classical Bayesian probability vs. Quantum probability*

Axiom-4 similarity transformations (Using: $\mathbf{1}=\sum|k\rangle\langle k|$ *)*

Axiom-4 is basically a matrix product as seen by comparing the following.

$$\langle j''|m'\rangle = \langle j''|\mathbf{1}|m'\rangle = \sum_{k=1}^{n} \langle j''|k\rangle\langle k|m'\rangle$$

$$\begin{pmatrix} \langle 1''|1'\rangle & \langle 1''|2'\rangle & \cdots & \langle 1''|n'\rangle \\ \langle 2''|1'\rangle & \langle 2''|2'\rangle & \cdots & \langle 2''|n'\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle n''|1'\rangle & \langle n''|2'\rangle & \cdots & \langle n''|n'\rangle \end{pmatrix} = \begin{pmatrix} \langle 1''|1\rangle & \langle 1''|2\rangle & \cdots & \langle 1''|n\rangle \\ \langle 2''|1\rangle & \langle 2''|2\rangle & \cdots & \langle 2''|n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle n''|1\rangle & \langle n''|2\rangle & \cdots & \langle n''|n\rangle \end{pmatrix} \bullet \begin{pmatrix} \langle 1|1'\rangle & \langle 1|2'\rangle & \cdots & \langle 1|n'\rangle \\ \langle 2|1'\rangle & \langle 2|2'\rangle & \cdots & \langle 2|n'\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle n|1'\rangle & \langle n|2'\rangle & \cdots & \langle n|n'\rangle \end{pmatrix}$$

$$T_{j''m'}\begin{pmatrix} prime \\ to \\ double-prime \end{pmatrix} = \sum_{k=1}^{n} T_{j''k}\begin{pmatrix} unprimed \\ to \\ double-prime \end{pmatrix} T_{k\,m'}\begin{pmatrix} prime \\ to \\ unprimed \end{pmatrix}$$

$$\mathbf{T}(b''\leftarrow b') = \mathbf{T}(b''\leftarrow b)\bullet\mathbf{T}(b\leftarrow b')$$

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$$\mathbf{T}(b''\leftarrow b') = \mathbf{T}(b''\leftarrow b)\bullet\mathbf{T}(b\leftarrow b')$$

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(1) The closure axiom

Products ab = c are defined between any two group elements a and b, and the result c is contained in the group.

(2) The associativity axiom

Products (ab)c and a(bc) are equal for all elements a, b, and c in the group .

(3) The identity axiom

There is a unique element 1 (the identity) such that 1·a = a = a·1 for all elements a in the group ..

4) The inverse axiom

For all elements a in the group there is an inverse element a^{-1} *such that* $a^{-1}a = 1 = a\cdot a^{-1}$.

Transformation Group axioms

Axiom-4 is applied twice to transform operator matrix representation.

Example: *Find:*

$$\begin{pmatrix} \langle x'|\mathbf{P}_x|x'\rangle & \langle x'|\mathbf{P}_x|y'\rangle \\ \langle y'|\mathbf{P}_x|x'\rangle & \langle y'|\mathbf{P}_x|y'\rangle \end{pmatrix}$$

given:

$$\begin{pmatrix} \langle x|\mathbf{P}_x|x\rangle & \langle x|\mathbf{P}_x|y\rangle \\ \langle y|\mathbf{P}_x|x\rangle & \langle y|\mathbf{P}_x|y\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and T-matrix:

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

The old "**P=1·P·1**-trick" where: $\mathbf{1} = \sum |k\rangle\langle k| = \quad |x\rangle\langle x| \; + \; |y\rangle\langle y|;$

Axiom-4 is applied twice to transform operator matrix representation.

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$$\begin{pmatrix} \langle x'|\mathbf{P}_x|x'\rangle & \langle x'|\mathbf{P}_x|y'\rangle \\ \langle y'|\mathbf{P}_x|x'\rangle & \langle y'|\mathbf{P}_x|y'\rangle \end{pmatrix}$$
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$$\langle x'|\mathbf{P}_x|y'\rangle = \langle x'|\mathbf{1}\cdot\mathbf{P}_x\cdot\mathbf{1}|y'\rangle = \langle x'|\left(|x\rangle\langle x| + |y\rangle\langle y|\right)\cdot\mathbf{P}_x\cdot\left(|x\rangle\langle x| + |y\rangle\langle y|\right)|y'\rangle$$

Axiom-4 is applied twice to transform operator matrix representation.

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$$= \langle x'|x\rangle\langle x|\mathbf{P}_x|x\rangle\langle x|y'\rangle + \langle x'|y\rangle\langle y|\mathbf{P}_x|x\rangle\langle x|y'\rangle + \langle x'|x\rangle\langle x|\mathbf{P}_x|y\rangle\langle y|y'\rangle + \langle x'|y\rangle\langle y|\mathbf{P}_x|y\rangle\langle y|y'\rangle$$

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More elegant matrix product:

$$\begin{pmatrix} \langle x'|\mathbf{P}_x|x'\rangle & \langle x'|\mathbf{P}_x|y'\rangle \\ \langle y'|\mathbf{P}_x|x'\rangle & \langle y'|\mathbf{P}_x|y'\rangle \end{pmatrix} = \begin{pmatrix} \langle x'|x\rangle & \langle x'|y\rangle \\ \langle y'|x\rangle & \langle y'|y\rangle \end{pmatrix} \begin{pmatrix} \langle x|\mathbf{P}_x|x\rangle & \langle x|\mathbf{P}_x|y\rangle \\ \langle y|\mathbf{P}_x|x\rangle & \langle y|\mathbf{P}_x|y\rangle \end{pmatrix} \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix}$$

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$$= \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \langle x|\mathbf{P}_x|x\rangle & \langle x|\mathbf{P}_x|y\rangle \\ \langle y|\mathbf{P}_x|x\rangle & \langle y|\mathbf{P}_x|y\rangle \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

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Axiom-4 is applied twice to transform operator matrix representation.

Example: *Find:* $\begin{pmatrix} \langle x'|\mathbf{P}_x|x'\rangle & \langle x'|\mathbf{P}_x|y'\rangle \\ \langle y'|\mathbf{P}_x|x'\rangle & \langle y'|\mathbf{P}_x|y'\rangle \end{pmatrix}$ *given:* $\begin{pmatrix} \langle x|\mathbf{P}_x|x\rangle & \langle x|\mathbf{P}_x|y\rangle \\ \langle y|\mathbf{P}_x|x\rangle & \langle y|\mathbf{P}_x|y\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *and T-matrix:* $\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$

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$$= \langle x'|x\rangle\langle x|\mathbf{P}_x|x\rangle\langle x|y'\rangle + \langle x'|y\rangle\langle y|\mathbf{P}_x|x\rangle\langle x|y'\rangle + \langle x'|x\rangle\langle x|\mathbf{P}_x|y\rangle\langle y|y'\rangle + \langle x'|y\rangle\langle y|\mathbf{P}_x|y\rangle\langle y|y'\rangle$$

More elegant matrix product:

$$\begin{pmatrix} \langle x'|\mathbf{P}_x|x'\rangle & \langle x'|\mathbf{P}_x|y'\rangle \\ \langle y'|\mathbf{P}_x|x'\rangle & \langle y'|\mathbf{P}_x|y'\rangle \end{pmatrix} = \begin{pmatrix} \langle x'|x\rangle & \langle x'|y\rangle \\ \langle y'|x\rangle & \langle y'|y\rangle \end{pmatrix} \begin{pmatrix} \langle x|\mathbf{P}_x|x\rangle & \langle x|\mathbf{P}_x|y\rangle \\ \langle y|\mathbf{P}_x|x\rangle & \langle y|\mathbf{P}_x|y\rangle \end{pmatrix} \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \langle x|\mathbf{P}_x|x\rangle & \langle x|\mathbf{P}_x|y\rangle \\ \langle y|\mathbf{P}_x|x\rangle & \langle y|\mathbf{P}_x|y\rangle \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi & 0 \\ -\sin\phi & 0 \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} \cos^2\phi & -\cos\phi\sin\phi \\ -\sin\phi\cos\phi & \sin^2\phi \end{pmatrix}$$

Axiom-4 is applied twice to transform operator matrix representation.

Example: *Find:* $\begin{pmatrix} \langle x'|\mathbf{P}_x|x'\rangle & \langle x'|\mathbf{P}_x|y'\rangle \\ \langle y'|\mathbf{P}_x|x'\rangle & \langle y'|\mathbf{P}_x|y'\rangle \end{pmatrix}$ *given:* $\begin{pmatrix} \langle x|\mathbf{P}_x|x\rangle & \langle x|\mathbf{P}_x|y\rangle \\ \langle y|\mathbf{P}_x|x\rangle & \langle y|\mathbf{P}_x|y\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *and T-matrix:* $\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$

The old "**P=1·P·1**-trick" where: $\mathbf{1} = \sum |k\rangle\langle k| = |x\rangle\langle x| + |y\rangle\langle y|$;

$$\langle x'|\mathbf{P}_x|y'\rangle = \langle x'|\mathbf{1}\cdot\mathbf{P}_x\cdot\mathbf{1}|y'\rangle = \langle x'|\big(|x\rangle\langle x| + |y\rangle\langle y|\big)\cdot\mathbf{P}_x\cdot\big(|x\rangle\langle x| + |y\rangle\langle y|\big)|y'\rangle = \big(\langle x'|x\rangle\langle x| + \langle x'|y\rangle\langle y|\big)\cdot\mathbf{P}_x\cdot\big(|x\rangle\langle x|y'\rangle + |y\rangle\langle y|y'\rangle\big)$$

$$= \langle x'|x\rangle\langle x|\mathbf{P}_x|x\rangle\langle x|y'\rangle + \langle x'|y\rangle\langle y|\mathbf{P}_x|x\rangle\langle x|y'\rangle + \langle x'|x\rangle\langle x|\mathbf{P}_x|y\rangle\langle y|y'\rangle + \langle x'|y\rangle\langle y|\mathbf{P}_x|y\rangle\langle y|y'\rangle$$

More elegant matrix product:

$$\begin{pmatrix} \langle x'|\mathbf{P}_x|x'\rangle & \langle x'|\mathbf{P}_x|y'\rangle \\ \langle y'|\mathbf{P}_x|x'\rangle & \langle y'|\mathbf{P}_x|y'\rangle \end{pmatrix} = \begin{pmatrix} \langle x'|x\rangle & \langle x'|y\rangle \\ \langle y'|x\rangle & \langle y'|y\rangle \end{pmatrix} \begin{pmatrix} \langle x|\mathbf{P}_x|x\rangle & \langle x|\mathbf{P}_x|y\rangle \\ \langle y|\mathbf{P}_x|x\rangle & \langle y|\mathbf{P}_x|y\rangle \end{pmatrix} \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \langle x|\mathbf{P}_x|x\rangle & \langle x|\mathbf{P}_x|y\rangle \\ \langle y|\mathbf{P}_x|x\rangle & \langle y|\mathbf{P}_x|y\rangle \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi & 0 \\ -\sin\phi & 0 \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} \cos^2\phi & -\cos\phi\sin\phi \\ -\sin\phi\cos\phi & \sin^2\phi \end{pmatrix}$$

This checks with the previous result 4-pages back:

$$\mathbf{P}_x = |x\rangle\langle x| \rightarrow \begin{pmatrix} \cos\phi \\ -\sin\phi \end{pmatrix} \otimes \begin{pmatrix} \cos\phi & -\sin\phi \end{pmatrix} = \begin{pmatrix} \cos^2\phi & -\sin\phi\cos\phi \\ -\sin\phi\cos\phi & \sin^2\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{(\text{for } \phi=0)}$$

Review: Axioms 1-4 and "Do-Nothing" vs " Do-Something" analyzers

Abstraction of Axiom-4 to define projection and unitary operators
- *Projection operators and resolution of identity*

Unitary operators and matrices that do something (or "nothing")
- *Diagonal unitary operators*
- *Non-diagonal unitary operators and $\dagger$-conjugation relations*
- *Non-diagonal projection operators and Kronecker $\otimes$-products*
- *Axiom-4 similarity transformation*

Matrix representation of beam analyzers
- ➡ *Non-unitary "killer" devices: Sorter-counter, filter*
- *Unitary "non-killer" devices: 1/2-wave plate, 1/4-wave plate*

How analyzers "peek" and how that changes outcomes
- *Peeking polarizers and coherence loss*
  - *Classical Bayesian probability vs. Quantum probability*

(1) Optical analyzer in sorter-counter configuration

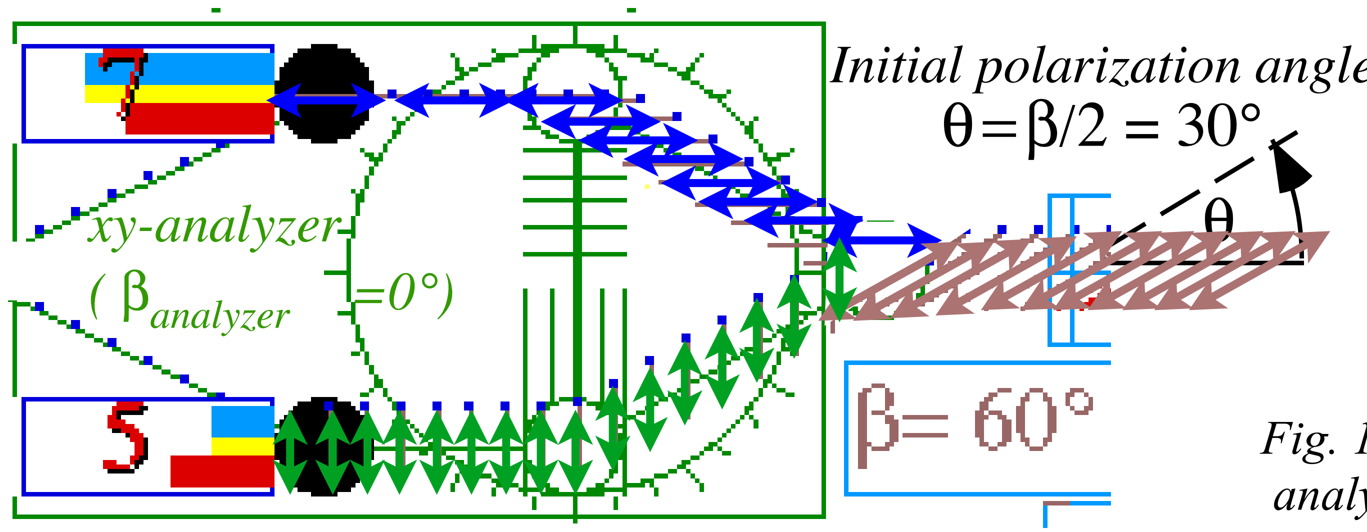
Analyzer reduced to a simple sorter-counter by blocking output of *x*-high-road and *y*-low-road with counters

$x\text{-counts} \sim \left| \langle x | x' \rangle \right|^2$

$= \cos^2\theta = 0.75$

$y\text{-counts} \sim \left| \langle y | x' \rangle \right|^2$

$= \sin^2\theta = 0.25$



Analyzer matrix:

$$\begin{pmatrix} \langle x|\mathbf{T}|x\rangle & \langle x|\mathbf{T}|y\rangle \\ \langle y|\mathbf{T}|x\rangle & \langle y|\mathbf{T}|y\rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Fig. 1.3.3 Simulated polarization analyzer set up as a sorter-counter

(1) Optical analyzer in sorter-counter configuration

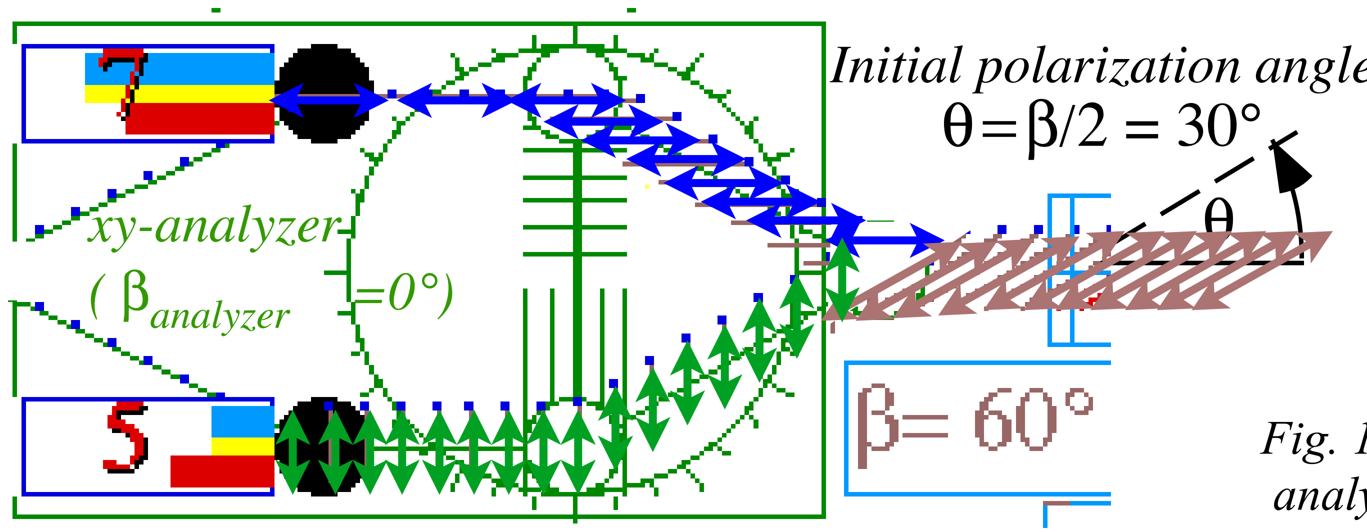
Analyzer reduced to a simple sorter-counter by blocking output of *x*-high-road and *y*-low-road with counters

$x\text{-counts} \sim |\langle x|x'\rangle|^2$

$= \cos^2\theta = 0.75$

$y\text{-counts} \sim |\langle y|x'\rangle|^2$

$= \sin^2\theta = 0.25$



Analyzer matrix:

$$\begin{pmatrix} \langle x|\mathbf{T}|x\rangle & \langle x|\mathbf{T}|y\rangle \\ \langle y|\mathbf{T}|x\rangle & \langle y|\mathbf{T}|y\rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Fig. 1.3.3 Simulated polarization analyzer set up as a sorter-counter

(2) Optical analyzer in a filter configuration (Polaroid© sunglasses)

Analyzer blocks one path which may have photon counter without affecting function.

Analyzer matrix:

$$\begin{pmatrix} \langle x|\mathbf{P}_y|x\rangle & \langle x|\mathbf{P}_y|y\rangle \\ \langle y|\mathbf{P}_y|x\rangle & \langle y|\mathbf{P}_y|y\rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$x\text{-counts} \sim |\langle y|x'\rangle|^2 = 0.75$

(Blocked and filtered out)

$y\text{-output} \sim |\langle y|x'\rangle|^2$

$= \sin^2\theta = 0.25$

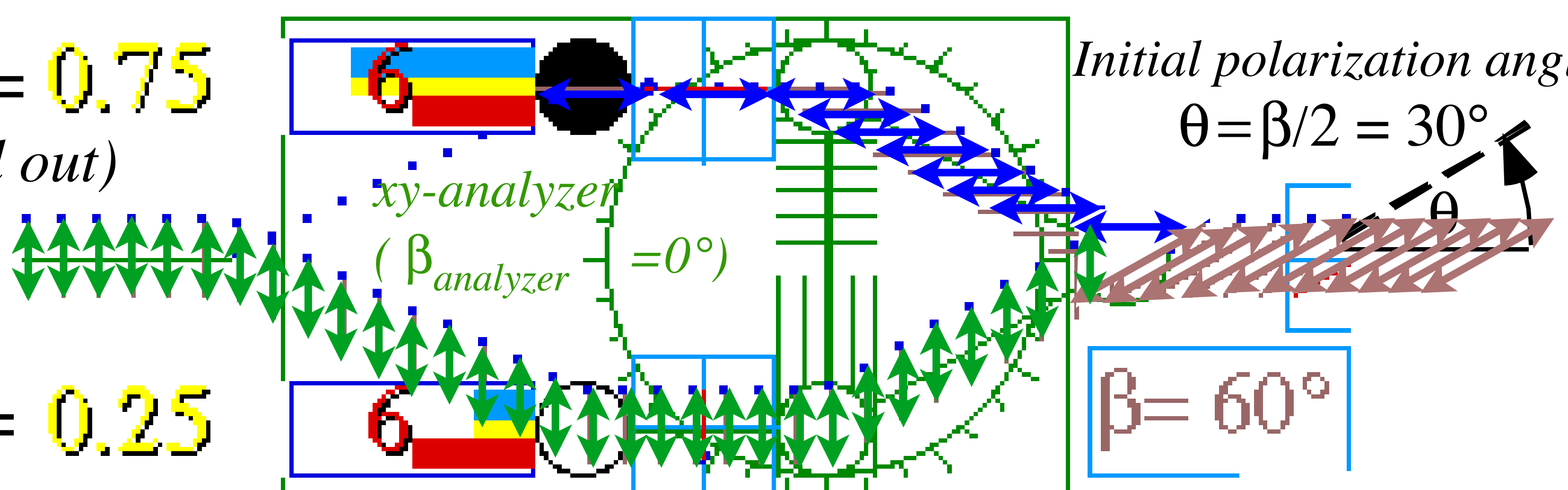


Fig. 1.3.4 Simulated polarization analyzer set up to filter out the x-polarized photons

Review: Axioms 1-4 and "Do-Nothing" vs "Do-Something" analyzers

Abstraction of Axiom-4 to define projection and unitary operators
- *Projection operators and resolution of identity*

Unitary operators and matrices that do something (or "nothing")
- *Diagonal unitary operators*
- *Non-diagonal unitary operators and †-conjugation relations*
- *Non-diagonal projection operators and Kronecker ⊗−products*
- *Axiom-4 similarity transformation*

Matrix representation of beam analyzers
- *Non-unitary "killer" devices: Sorter-counter, filter*
- → *Unitary "non-killer" devices: 1/2-wave plate, 1/4-wave plate*

How analyzers "peek" and how that changes outcomes
- *Peeking polarizers and coherence loss*
  - *Classical Bayesian probability vs. Quantum probability*

(3) Optical analyzers in the "control" configuration: Half or Quarter wave plates

(a)

Half-wave plate

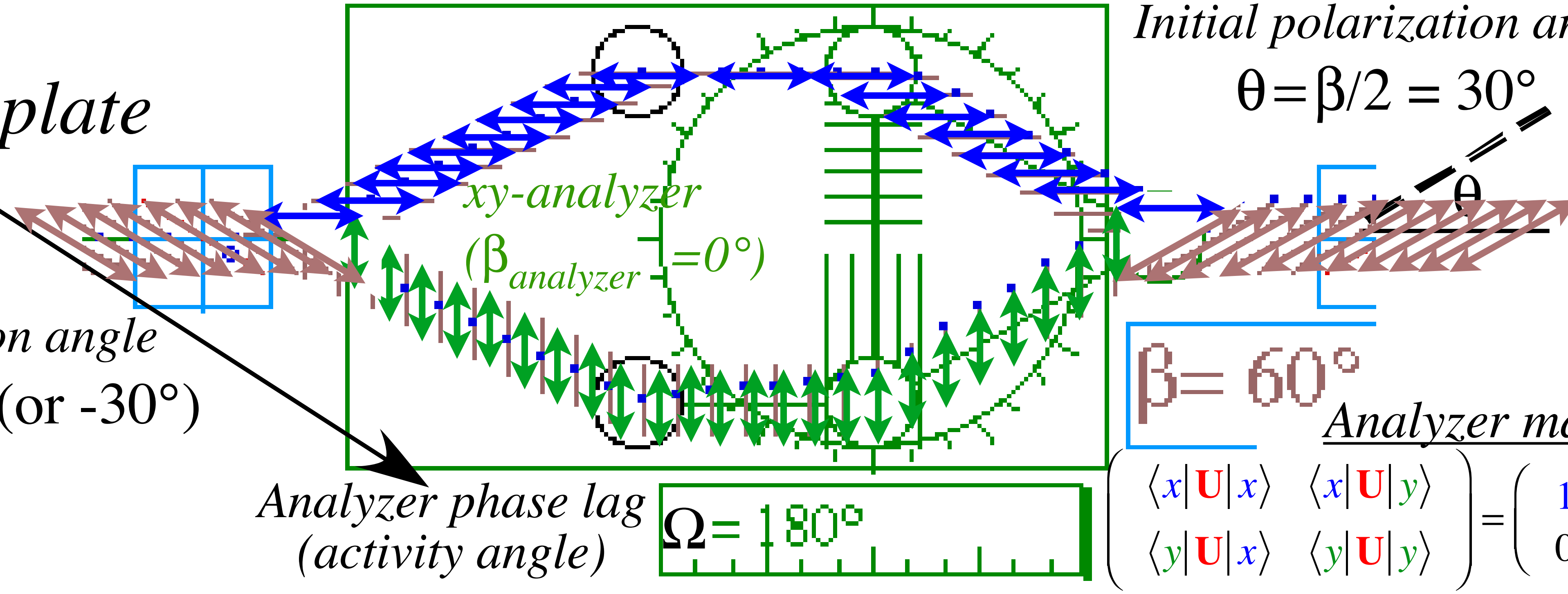
$(\Omega = \pi)$

Initial polarization angle

$\theta = \beta/2 = 30°$

Final polarization angle

$\theta = \beta/2 = 150°$(or -30°)

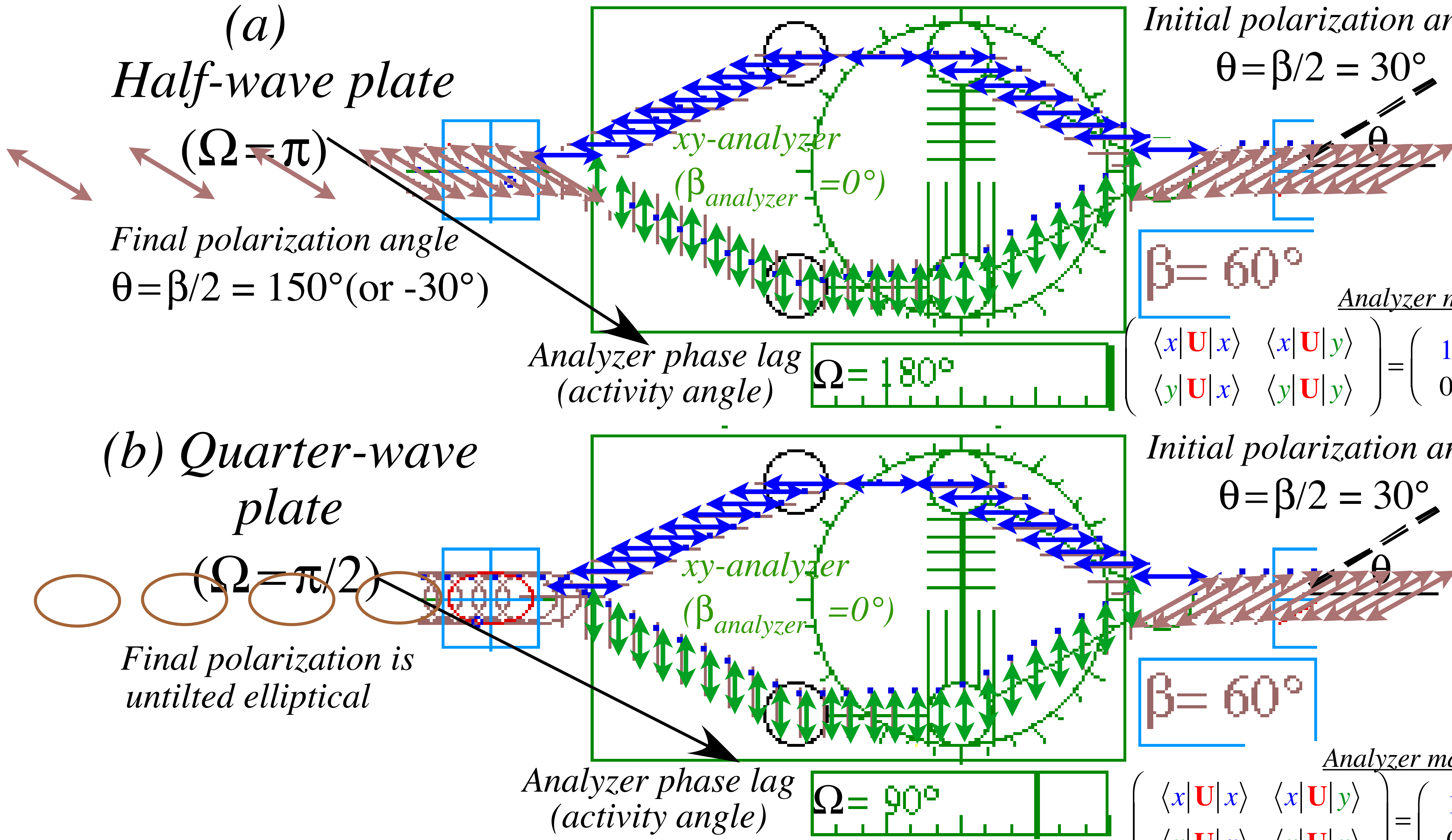


Analyzer phase lag
(activity angle)

Analyzer matrix:

$$\begin{pmatrix} \langle x|\mathbf{U}|x\rangle & \langle x|\mathbf{U}|y\rangle \\ \langle y|\mathbf{U}|x\rangle & \langle y|\mathbf{U}|y\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(3) Optical analyzers in the "control" configuration: Half or Quarter wave plates



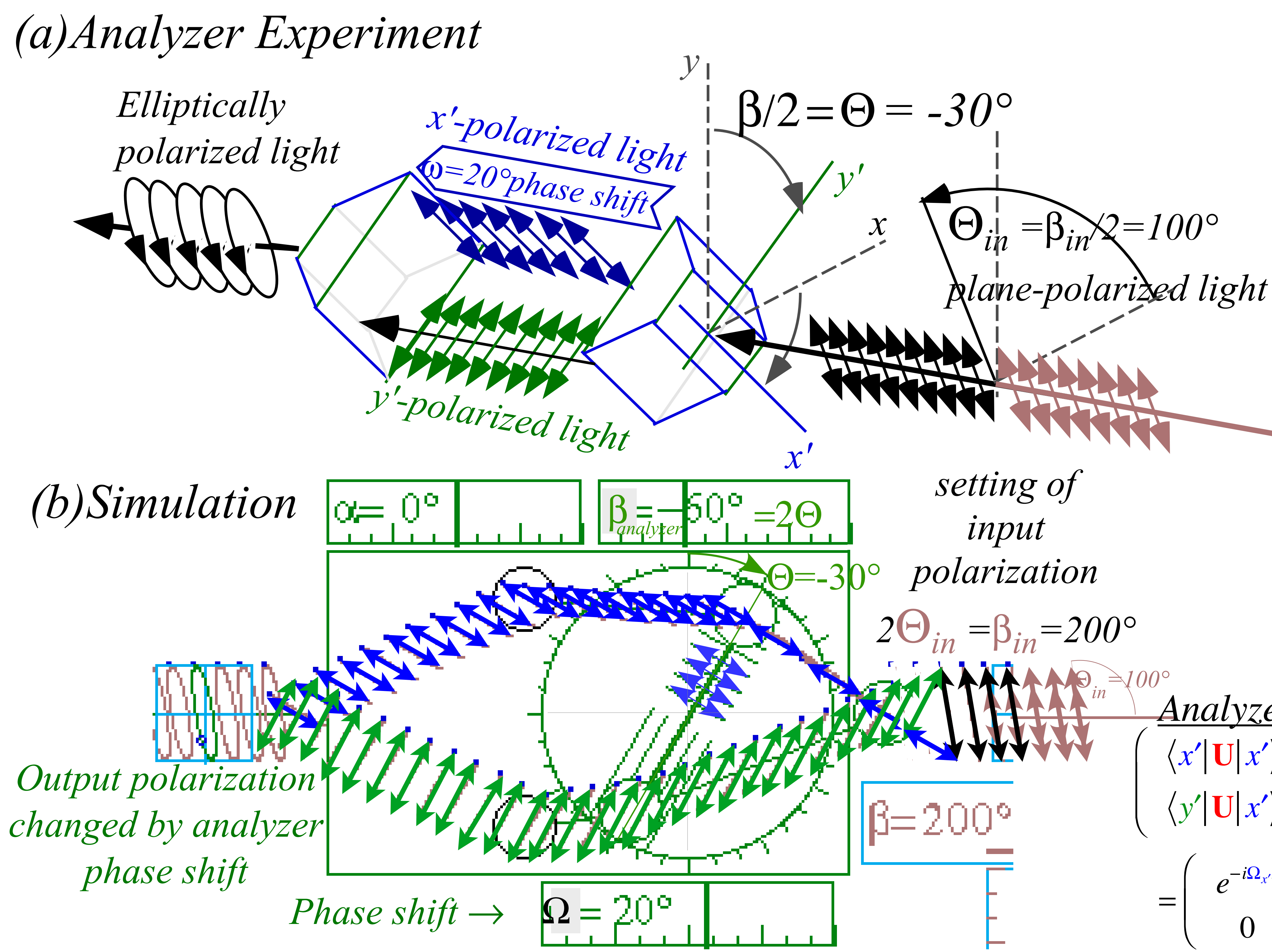
$$\begin{pmatrix} \langle x|\mathbf{U}|x\rangle & \langle x|\mathbf{U}|y\rangle \\ \langle y|\mathbf{U}|x\rangle & \langle y|\mathbf{U}|y\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} \langle x|\mathbf{U}|x\rangle & \langle x|\mathbf{U}|y\rangle \\ \langle y|\mathbf{U}|x\rangle & \langle y|\mathbf{U}|y\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$$

Analyzer matrix:

$$\begin{pmatrix} \langle x|\mathbf{U}|x\rangle & \langle x|\mathbf{U}|y\rangle \\ \langle y|\mathbf{U}|x\rangle & \langle y|\mathbf{U}|y\rangle \end{pmatrix} = \begin{pmatrix} e^{-i\Omega_x t} & 0 \\ 0 & e^{-i\Omega_x t} \end{pmatrix}$$

Fig. 1.3.5 Polarization control set to shift phase by (a) Half-wave (Ω=π) , (b) Quarter wave (Ω=π/2)



$$\begin{pmatrix} \langle x'|\mathbf{U}|x'\rangle & \langle x'|\mathbf{U}|y'\rangle \\ \langle y'|\mathbf{U}|x'\rangle & \langle y'|\mathbf{U}|y'\rangle \end{pmatrix} = \begin{pmatrix} e^{-i\Omega_{x'}t} & 0 \\ 0 & e^{-i\Omega_{x'}t} \end{pmatrix}$$

Similar to "do-nothing" analyzer but has extra phase factor $e^{-i\Omega_{x'}} = 0.94-i\,0.34$ on the x'-path (top).

x-output: $\langle x|\Psi_{out}\rangle = \langle x|x'\rangle e^{-i\Omega_{x'}}\langle x'|\Psi_{in}\rangle + \langle x|y'\rangle\langle y'|\Psi_{in}\rangle = e^{-i\Omega_{x'}}\cos\Theta\cos(\Theta_{in}-\Theta) - \sin\Theta\sin(\Theta_{in}-\Theta)$

y-output: $\langle y|\Psi_{out}\rangle = \langle y|x'\rangle e^{-i\Omega_{x'}}\langle x'|\Psi_{in}\rangle + \langle x|y'\rangle\langle y'|\Psi_{in}\rangle = e^{-i\Omega_{x'}}\sin\Theta\cos(\Theta_{in}-\Theta) + \cos\Theta\sin(\Theta_{in}-\Theta)$

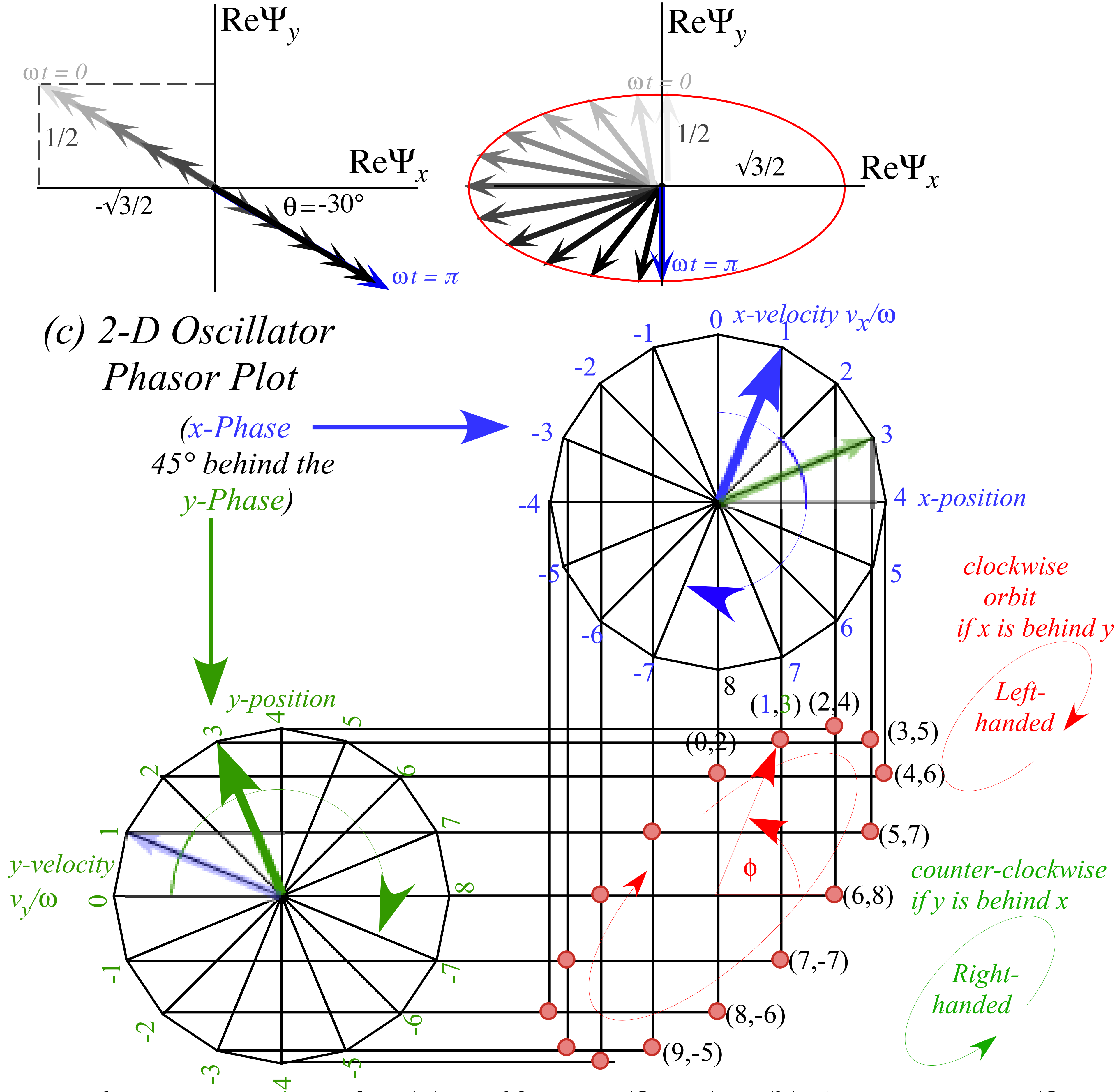


Fig. 1.3.6 Polarization states for (a) Half-wave (Ω=π) , (b) Quarter wave (Ω=π/2) (c) (Ω=−π/4)

Review: *Axioms 1-4 and "Do-Nothing" vs " Do-Something" analyzers*

Abstraction of Axiom-4 to define projection and unitary operators
    Projection operators and resolution of identity

Unitary operators and matrices that do something (or "nothing")
    Diagonal unitary operators
    Non-diagonal unitary operators and †-conjugation relations
    Non-diagonal projection operators and Kronecker ⊗−products
    Axiom-4 similarity transformation

Matrix representation of beam analyzers
    Non-unitary "killer" devices: Sorter-counter, filter
    Unitary "non-killer" devices: 1/2-wave plate, 1/4-wave plate

→ *How analyzers "peek" and how that changes outcomes*
    Peeking polarizers and coherence loss
        Classical Bayesian probability vs. Quantum probability

How analyzers may "peek" and how that changes outcomes

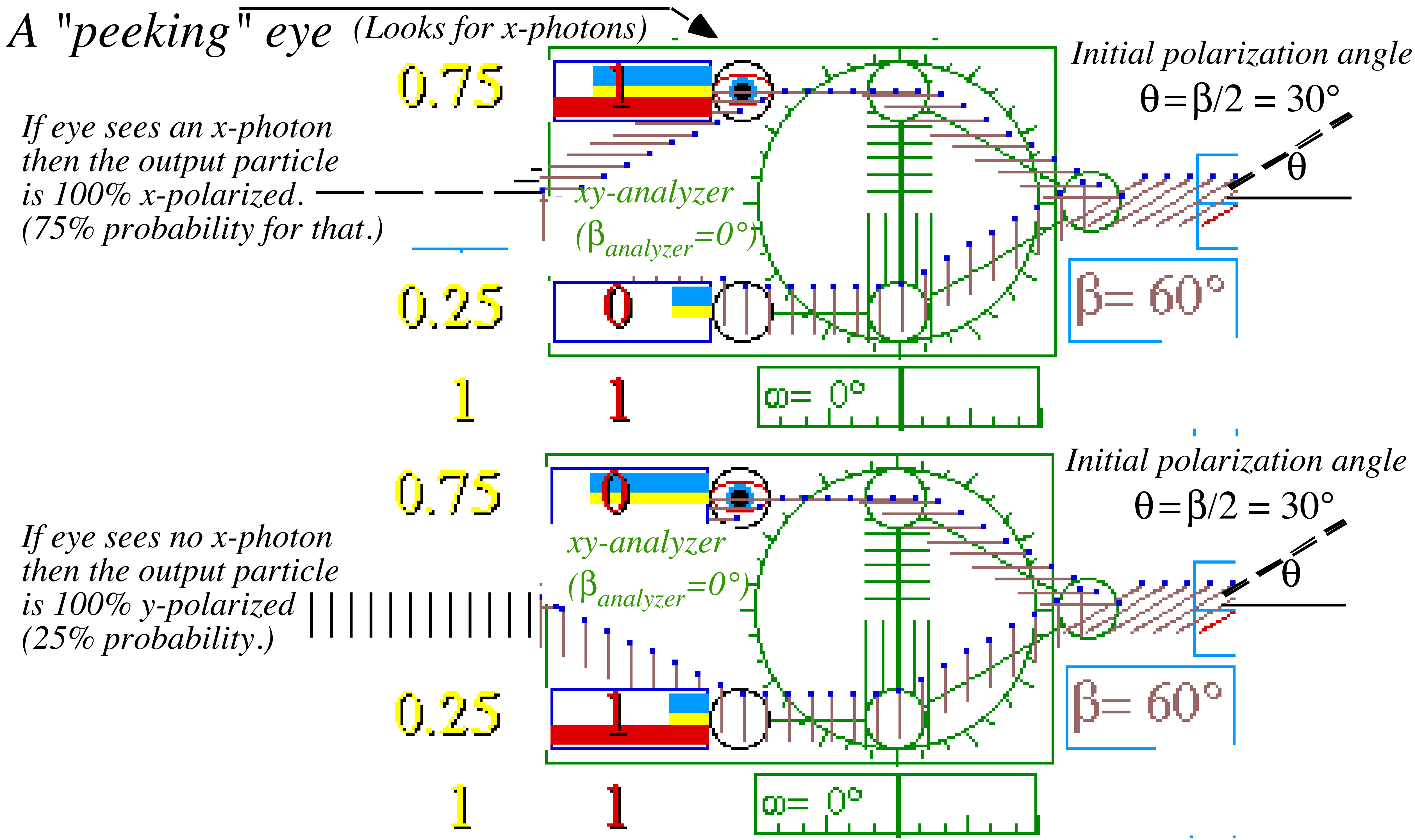


Fig. 1.3.7 Simulated polarization analyzer set up to "peek" if the photon is x-or y-polarized

How analyzers "peek" and how that changes outcomes

Simulations

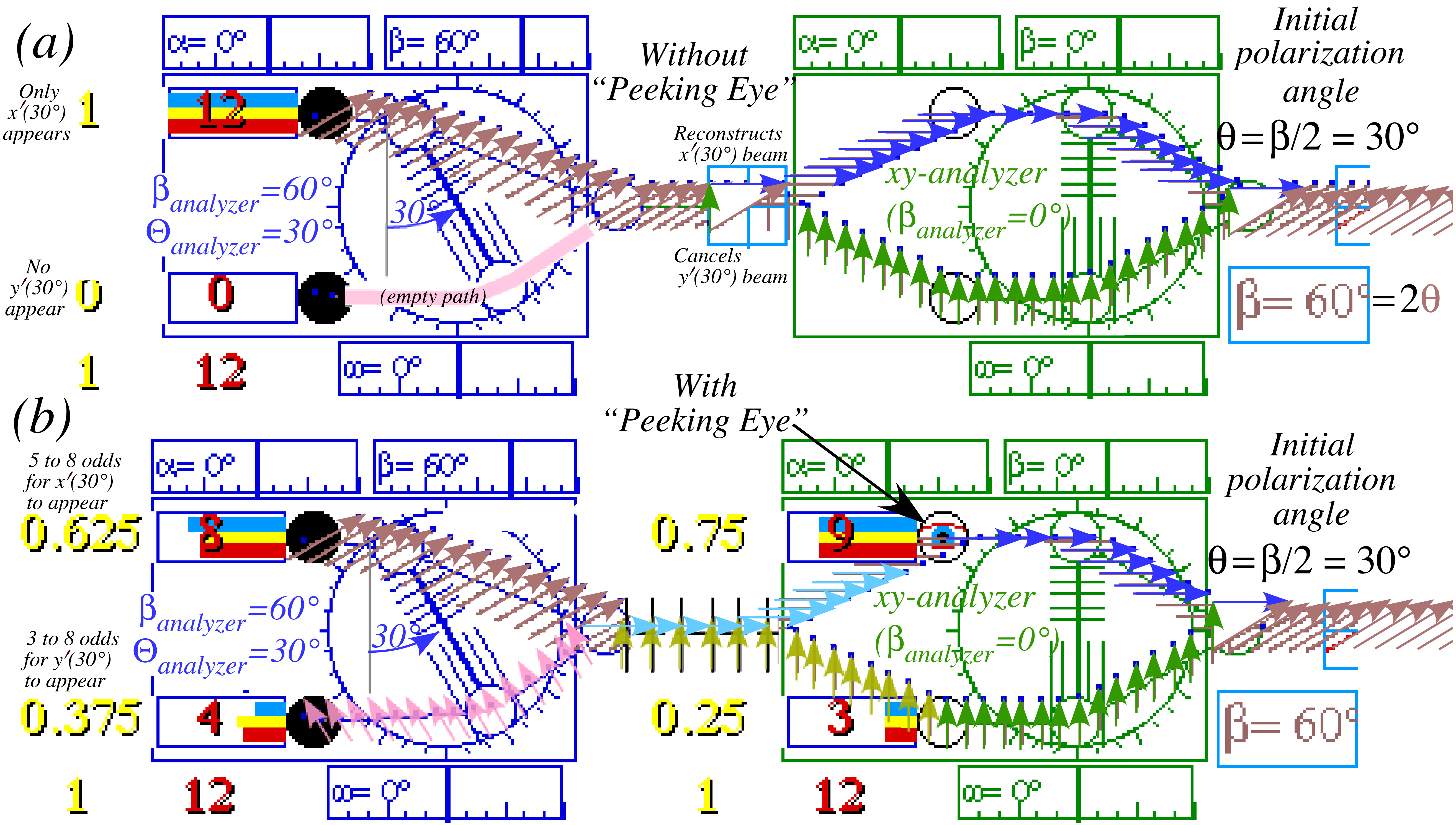
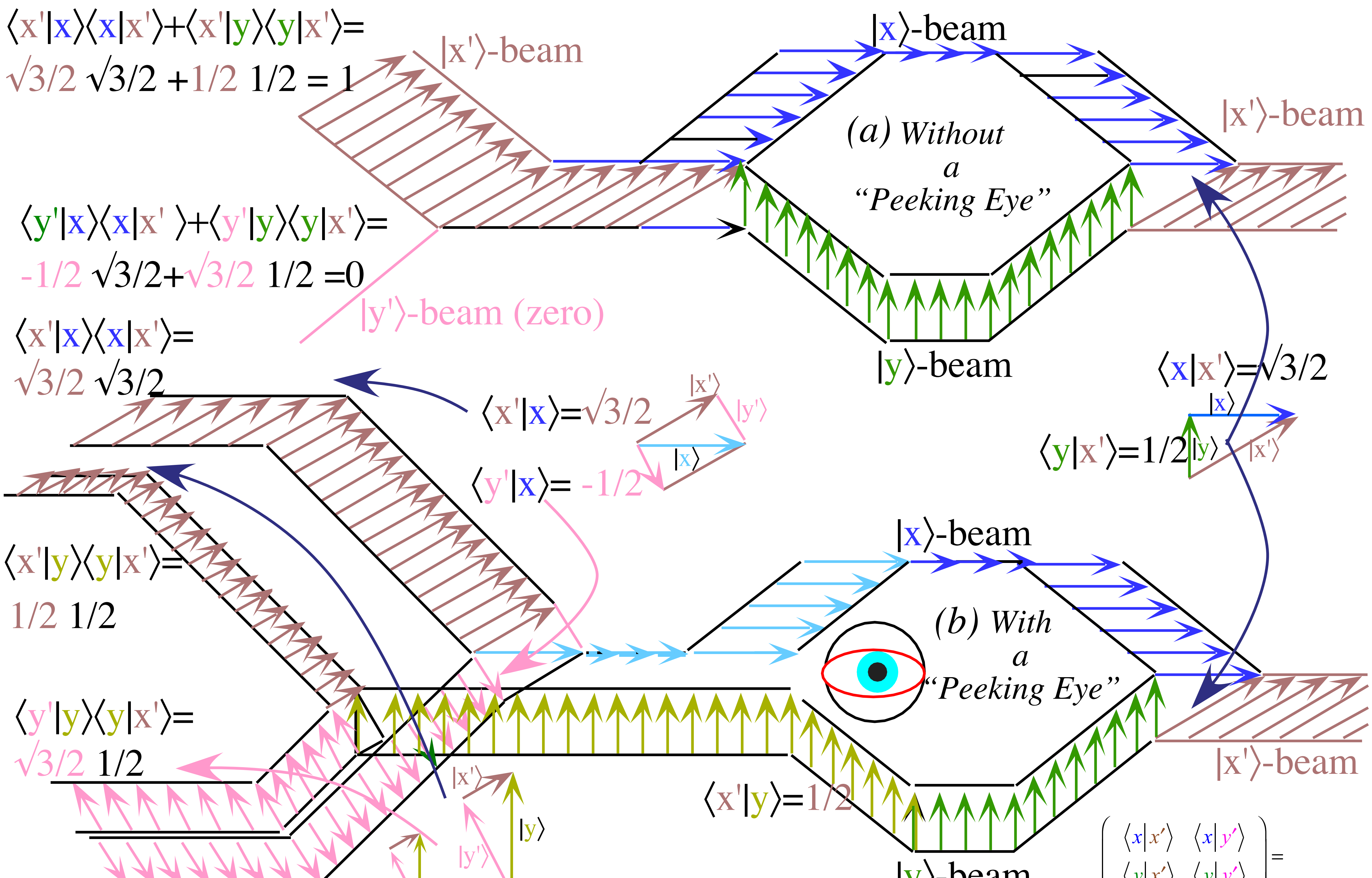


Fig. 1.3.8 Output with β/2=30° input to: (a) Coherent xy-"Do nothing" or
(b) Incoherent xy-"Peeking" devices

How analyzers "peek" and how that changes outcomes



$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

Fig. 1.3.9 Beams-amplitudes of (a) xy-"Do nothing" and (b) xy-"Peeking" analyzer each with input $|x'\rangle$

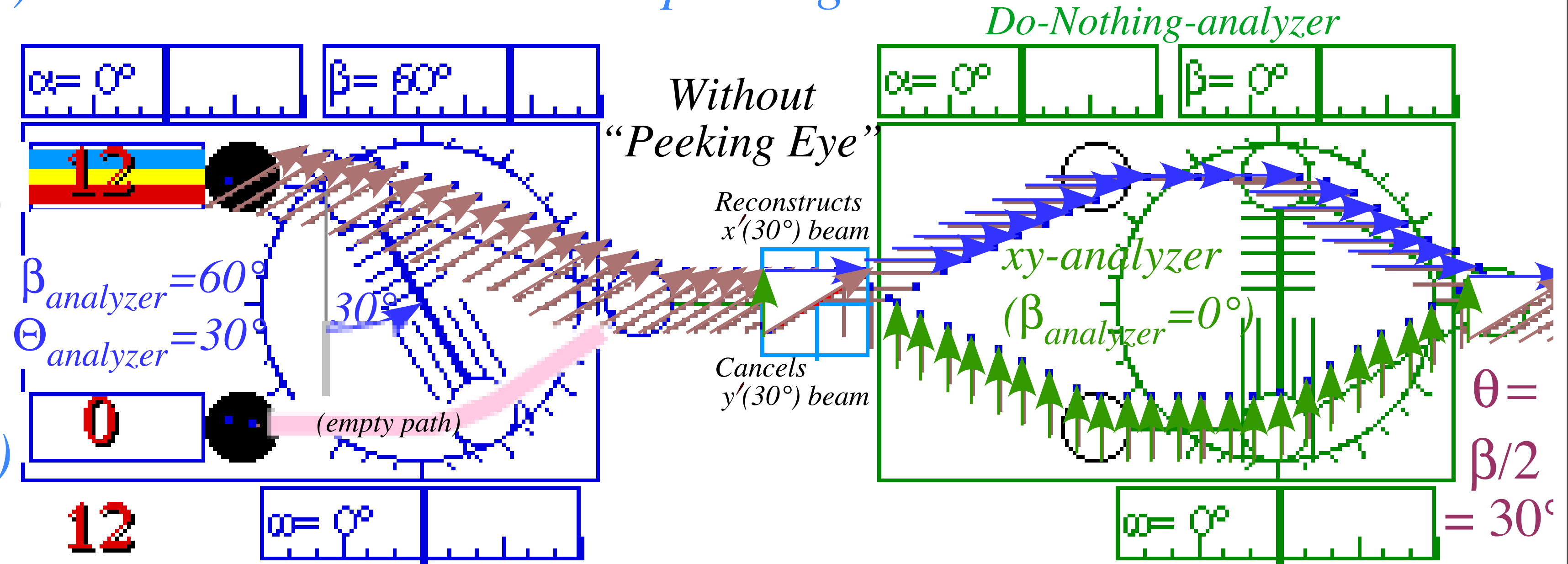
Amplitude $A(n')$ and Probability $P(n')$ at counter n' WITHOUT "peeking"

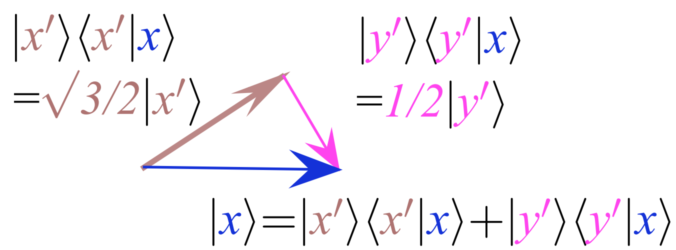
$$A(x') = \langle x'|x\rangle\langle x|x'\rangle + \langle x'|y\rangle\langle y|x'\rangle$$

$$= \quad \frac{3}{4} \quad + \quad \frac{1}{4} \quad = 1 = P(x')$$

$$A(y') = \langle y'|x\rangle\langle x|x'\rangle + \langle y'|y\rangle\langle y|x'\rangle$$

$$= \quad \frac{-\sqrt{3}}{4} \quad + \quad \frac{\sqrt{3}}{4} \quad = 0 = P(y')$$





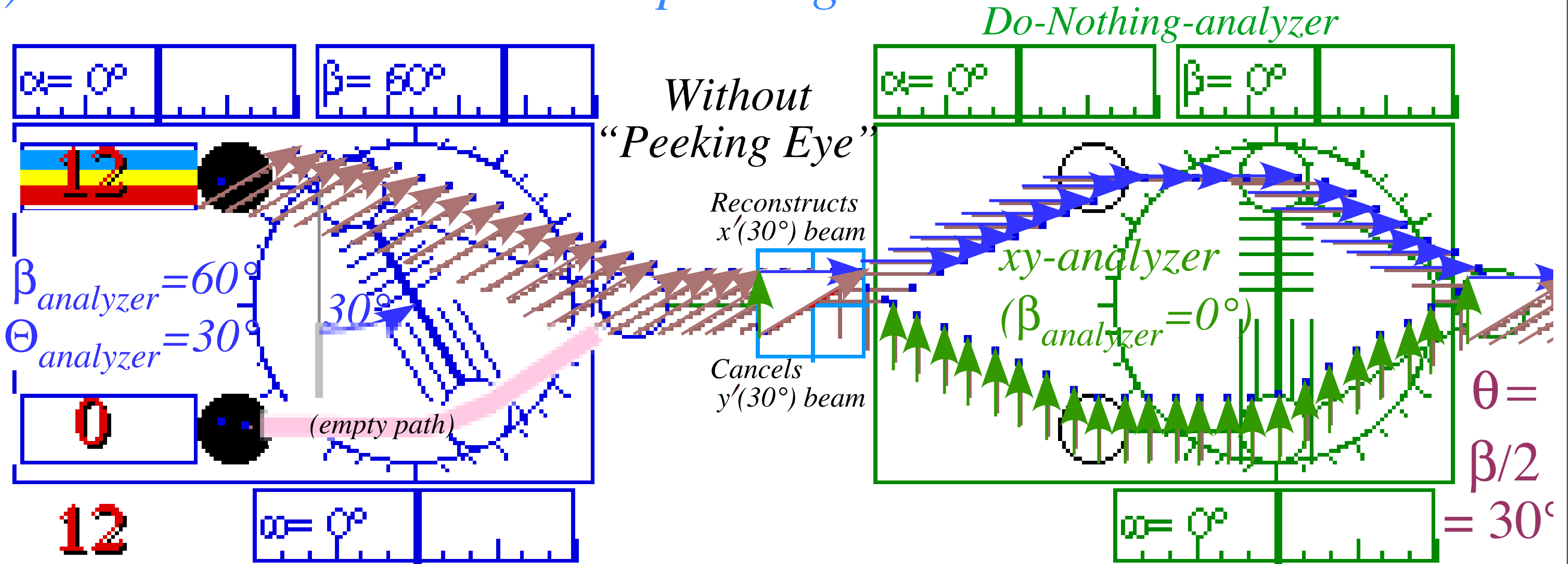
$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$$

Amplitude $A(n')$ and Probability $P(n')$ at counter n' WITHOUT "peeking"

$$A(x') = \langle x'|x\rangle(1)\langle x|x'\rangle + \langle x'|y\rangle\langle y|x'\rangle$$
$$= \tfrac{3}{4}(1) \quad + \quad \tfrac{1}{4} = 1 = P(x')$$

$$A(y') = \langle y'|x\rangle(1)\langle x|x'\rangle + \langle y'|y\rangle\langle y|x'\rangle$$
$$= \tfrac{-\sqrt{3}}{4}(1) \quad + \quad \tfrac{\sqrt{3}}{4} = 0 = P(y')$$

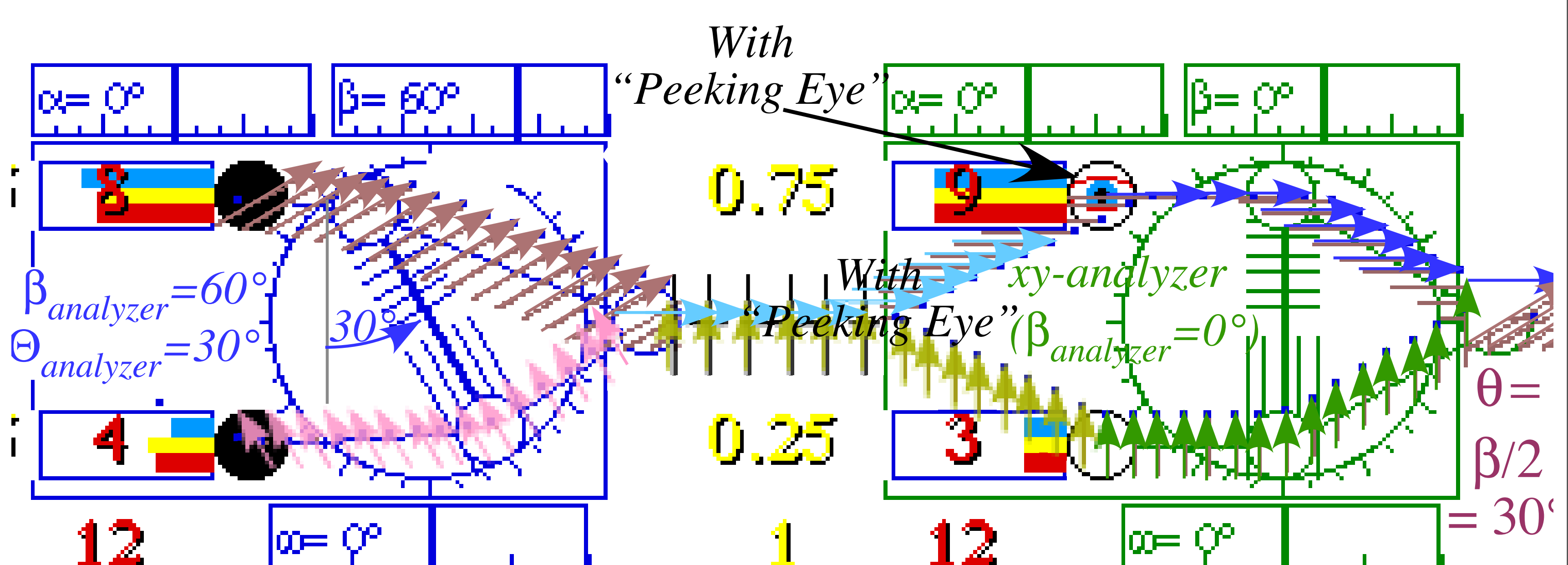


Amplitude $A(n')$ and Probability $P(n')$ at counter n' WITH "peeking"

Suppose "*x-eye*" puts phase $e^{i\phi}$ on each x-photon with random ϕ distributed over unit circle ($-\pi < \phi < \pi$).

So $e^{i\phi}$ averages to zero!

$$A(x') = \langle x'|x\rangle\left(e^{i\phi}\right)\langle x|x'\rangle + \langle x'|y\rangle\langle y|x'\rangle$$
$$= \tfrac{3}{4}\left(e^{i\phi}\right) \quad + \quad \tfrac{1}{4}$$

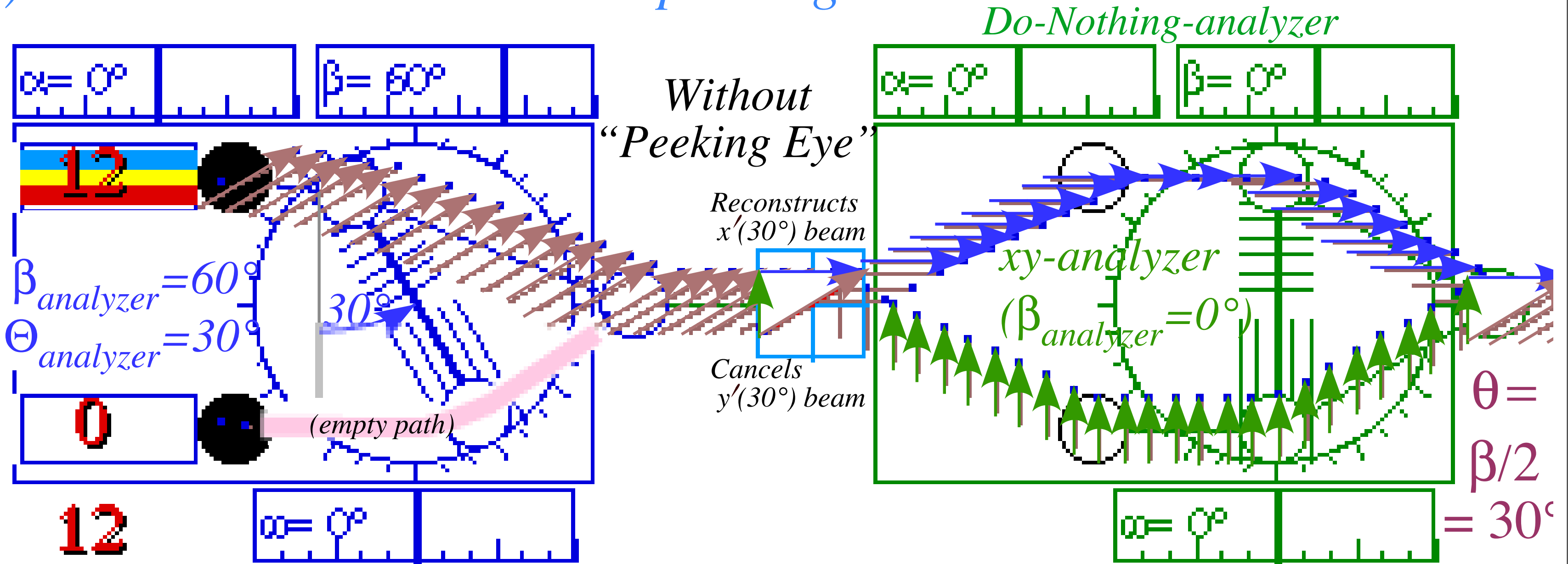


$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$$

Amplitude $A(n')$ and Probability $P(n')$ at counter n' WITHOUT "peeking"

$$A(x') = \langle x'|x\rangle(1)\langle x|x'\rangle + \langle x'|y\rangle\langle y|x'\rangle$$
$$= \quad \tfrac{3}{4}(1) \quad + \quad \tfrac{1}{4} = 1 = P(x')$$

$$A(y') = \langle y'|x\rangle(1)\langle x|x'\rangle + \langle y'|y\rangle\langle y|x'\rangle$$
$$= \quad \tfrac{-\sqrt{3}}{4}(1) \quad + \quad \tfrac{\sqrt{3}}{4} = 0 = P(y')$$

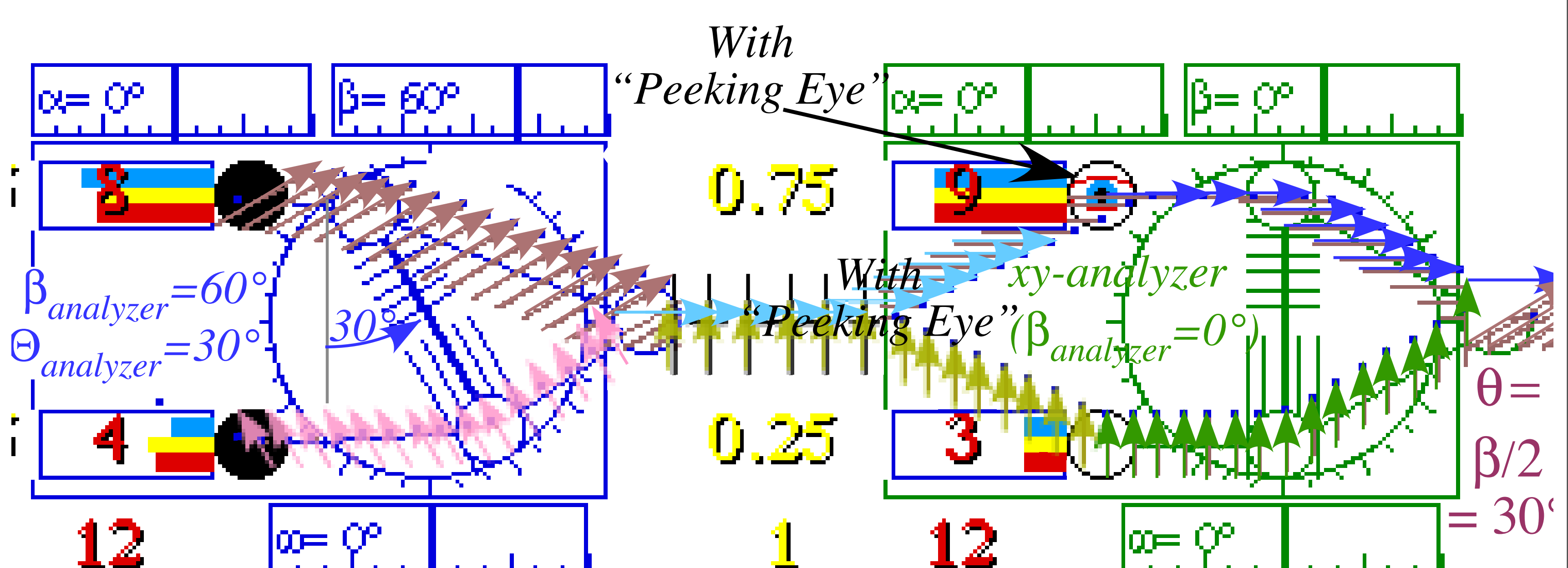


Amplitude $A(n')$ and Probability $P(n')$ at counter n' WITH "peeking"

Suppose "*x-eye*" puts phase $e^{i\phi}$ on each x-photon with random ϕ distributed over unit circle ($-\pi < \phi < \pi$).

So $e^{i\phi}$ averages to zero!

$$A(x') = \langle x'|x\rangle\left(e^{i\phi}\right)\langle x|x'\rangle + \langle x'|y\rangle\langle y|x'\rangle$$
$$= \quad \tfrac{3}{4}\left(e^{i\phi}\right) \quad + \quad \tfrac{1}{4}$$

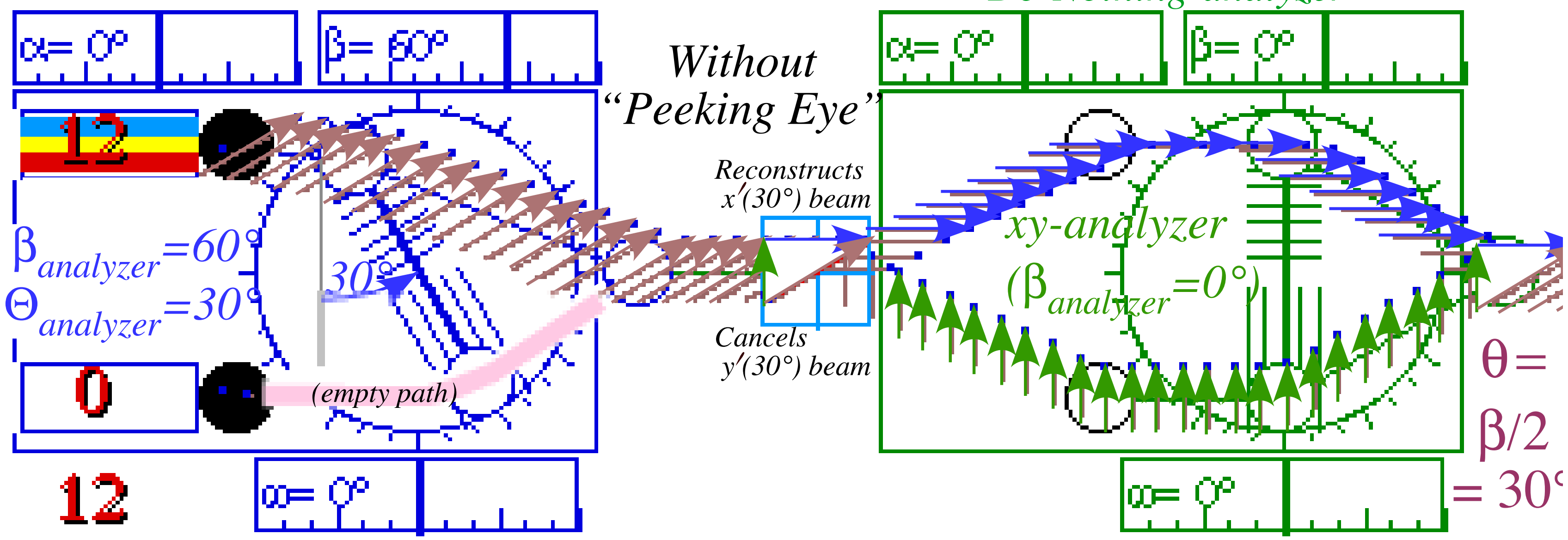


$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$$

Amplitude $A(n')$ and Probability $P(n')$ at counter n' WITHOUT "peeking"

$$A(x') = \langle x'|x\rangle(1)\langle x|x'\rangle + \langle x'|y\rangle\langle y|x'\rangle$$
$$= \tfrac{3}{4}(1) \quad + \quad \tfrac{1}{4} = 1 = P(x')$$

$$A(y') = \langle y'|x\rangle(1)\langle x|x'\rangle + \langle y'|y\rangle\langle y|x'\rangle$$
$$= \tfrac{-\sqrt{3}}{4}(1) \quad + \quad \tfrac{\sqrt{3}}{4} = 0 = P(y')$$



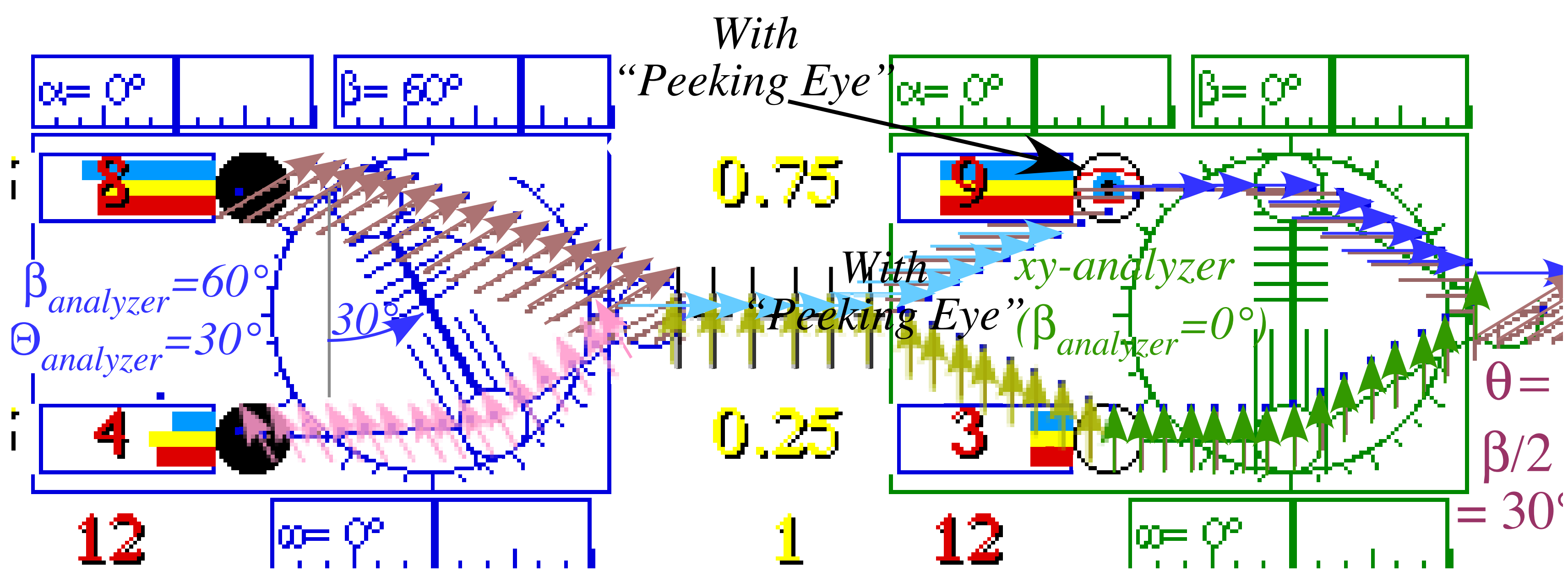
Amplitude $A(n')$ and Probability $P(n')$ at counter n' WITH "peeking"

Suppose "*x-eye*" puts phase $e^{i\phi}$ on each x-photon with random ϕ distributed over unit circle ($-\pi < \phi < \pi$).

So $e^{i\phi}$ averages to zero!

$$A(x') = \langle x'|x\rangle\left(e^{i\phi}\right)\langle x|x'\rangle + \langle x'|y\rangle\langle y|x'\rangle$$
$$= \tfrac{3}{4}\left(e^{i\phi}\right) \quad + \quad \tfrac{1}{4}$$

$$P(x') = \left(\tfrac{3}{4}\left(e^{i\phi}\right) + \tfrac{1}{4}\right)^* \left(\tfrac{3}{4}\left(e^{i\phi}\right) + \tfrac{1}{4}\right)$$
$$= \frac{5}{8} + \frac{3}{16}\left(e^{-i\phi} + e^{i\phi}\right) = \frac{5 + 3\cos\phi}{8}$$



$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$$

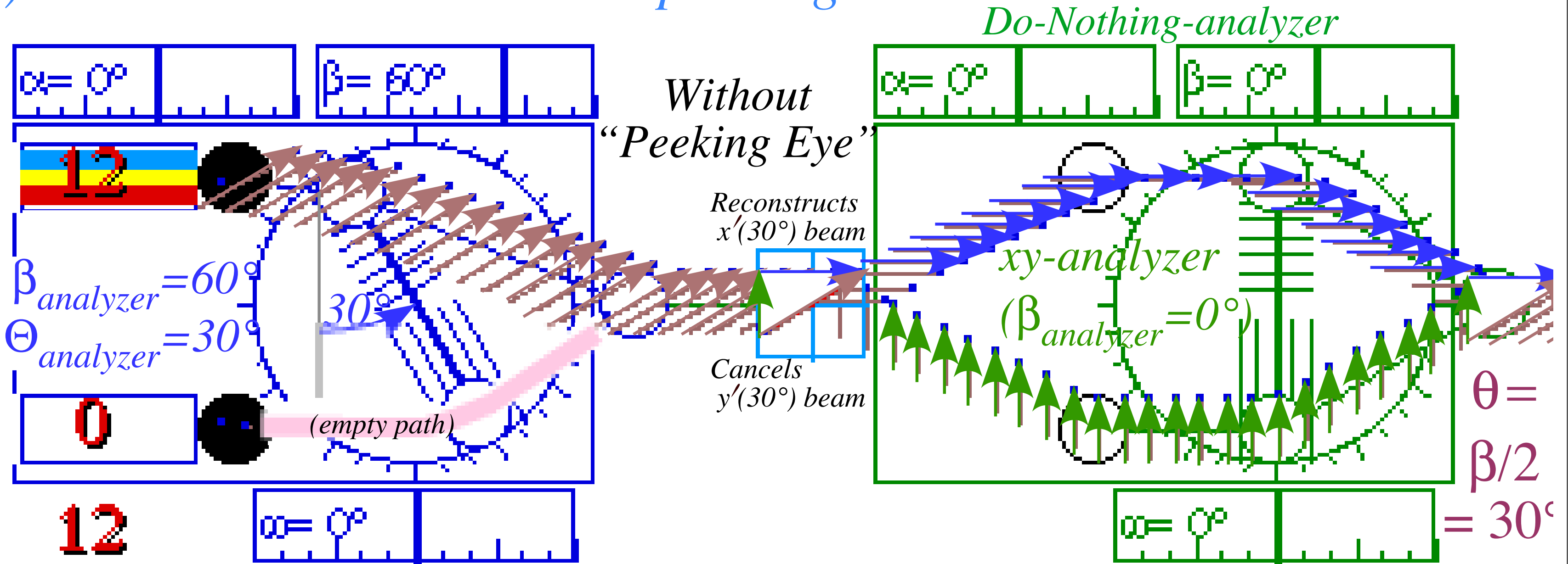
Amplitude A(n′) and Probability P(n′) at counter n′ WITHOUT "peeking"

$$A(x')=\langle x'|x\rangle(1)\langle x|x'\rangle+\langle x'|y\rangle\langle y|x'\rangle$$

$$=\quad \tfrac{3}{4}(1)\quad+\quad\tfrac{1}{4}=1=P(x')$$

$$A(y')=\langle y'|x\rangle(1)\langle x|x'\rangle+\langle y'|y\rangle\langle y|x'\rangle$$

$$=\quad \tfrac{-\sqrt{3}}{4}(1)\quad+\quad\tfrac{\sqrt{3}}{4}=0=P(y')$$



Amplitude A(n′) and Probability P(n′) at counter n′ WITH "peeking"

Suppose *"x-eye"* puts phase $e^{i\phi}$ on each x-photon with random ϕ distributed over unit circle ($-\pi<\phi<\pi$).

So $e^{i\phi}$ averages to zero!

$$A(x')=\langle x'|x\rangle\left(e^{i\phi}\right)\langle x|x'\rangle+\langle x'|y\rangle\langle y|x'\rangle$$

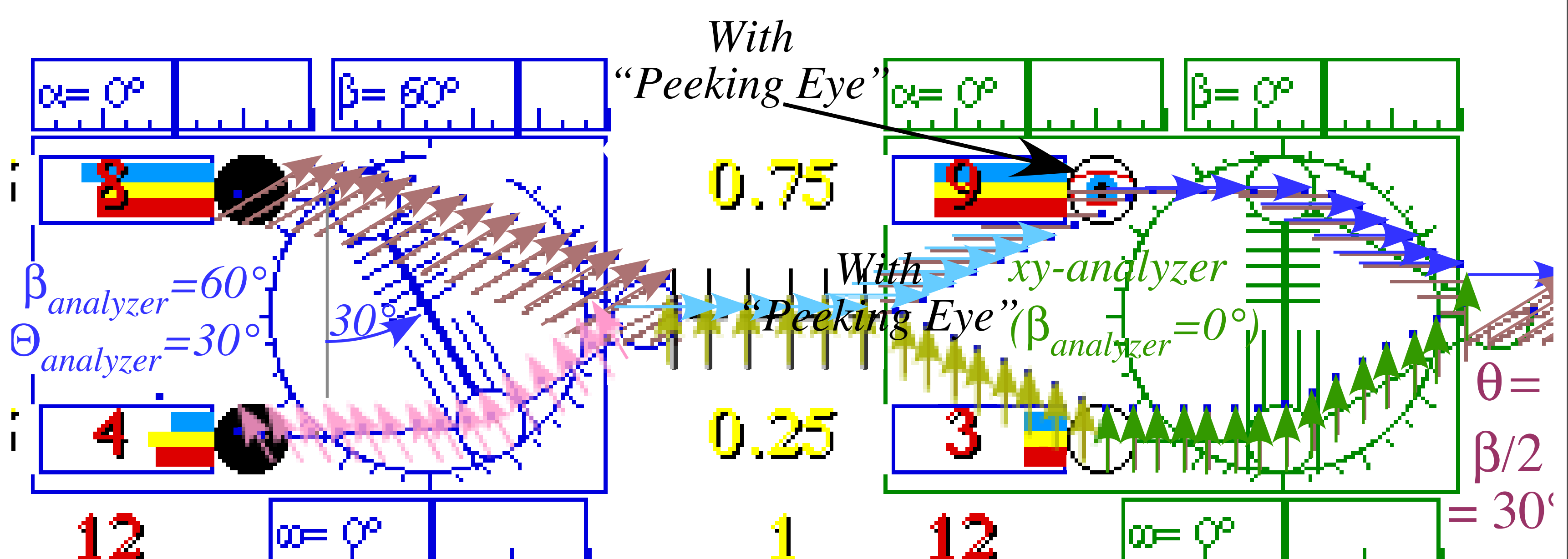
$$=\quad \tfrac{3}{4}\left(e^{i\phi}\right)\quad+\quad\tfrac{1}{4}$$

$$P(x')=\left(\tfrac{3}{4}\left(e^{i\phi}\right)+\tfrac{1}{4}\right)^*\left(\tfrac{3}{4}\left(e^{i\phi}\right)+\tfrac{1}{4}\right)$$

$$=\frac{5}{8}+\frac{3}{16}\left(e^{-i\phi}+e^{i\phi}\right)=\frac{5+3\cos\phi}{8}$$

$$A(y')=\langle y'|x\rangle\left(e^{i\phi}\right)\langle x|x'\rangle+\langle y'|y\rangle\langle y|x'\rangle$$

$$=\quad \tfrac{-\sqrt{3}}{4}\left(e^{i\phi}\right)\quad+\quad\tfrac{\sqrt{3}}{4}$$



$$\begin{pmatrix}\langle x|x'\rangle & \langle x|y'\rangle\\ \langle y|x'\rangle & \langle y|y'\rangle\end{pmatrix}=\begin{pmatrix}\sqrt{3}/2 & -1/2\\ 1/2 & \sqrt{3}/2\end{pmatrix}=\begin{pmatrix}\sqrt{3}/2 & -1/2\\ 1/2 & \sqrt{3}/2\end{pmatrix}$$

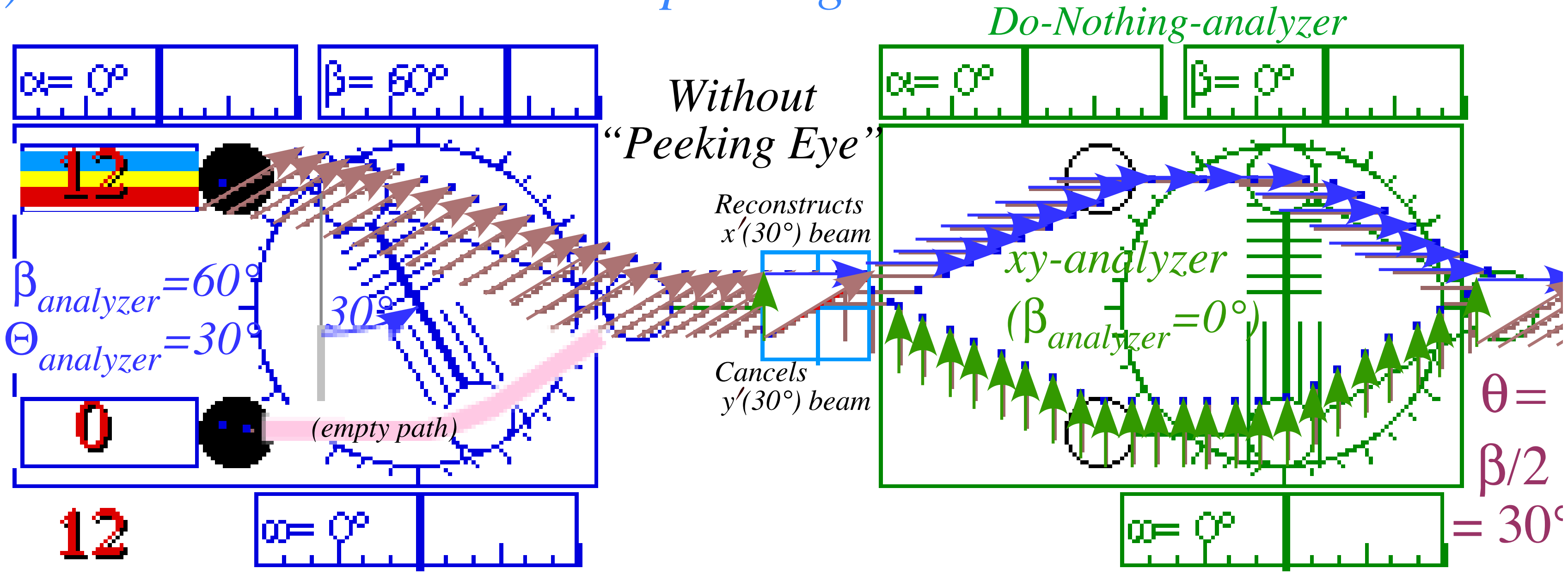
Amplitude $A(n')$ and Probability $P(n')$ at counter n' WITHOUT "peeking"

$$A(x') = \langle x'|x\rangle(1)\langle x|x'\rangle + \langle x'|y\rangle\langle y|x'\rangle$$

$$= \tfrac{3}{4}(1) \quad + \quad \tfrac{1}{4} = 1 = P(x')$$

$$A(y') = \langle y'|x\rangle(1)\langle x|x'\rangle + \langle y'|y\rangle\langle y|x'\rangle$$

$$= \tfrac{-\sqrt{3}}{4}(1) \quad + \quad \tfrac{\sqrt{3}}{4} = 0 = P(y')$$



Amplitude $A(n')$ and Probability $P(n')$ at counter n' WITH "peeking"

Suppose "*x-eye*" puts phase $e^{i\phi}$ on each x-photon with random ϕ distributed over unit circle ($-\pi < \phi < \pi$).

So $e^{i\phi}$ averages to zero!

$$A(x') = \langle x'|x\rangle\left(e^{i\phi}\right)\langle x|x'\rangle + \langle x'|y\rangle\langle y|x'\rangle$$

$$= \tfrac{3}{4}\left(e^{i\phi}\right) \quad + \quad \tfrac{1}{4}$$

$$P(x') = \left(\tfrac{3}{4}\left(e^{i\phi}\right) + \tfrac{1}{4}\right)^{*}\left(\tfrac{3}{4}\left(e^{i\phi}\right) + \tfrac{1}{4}\right)$$

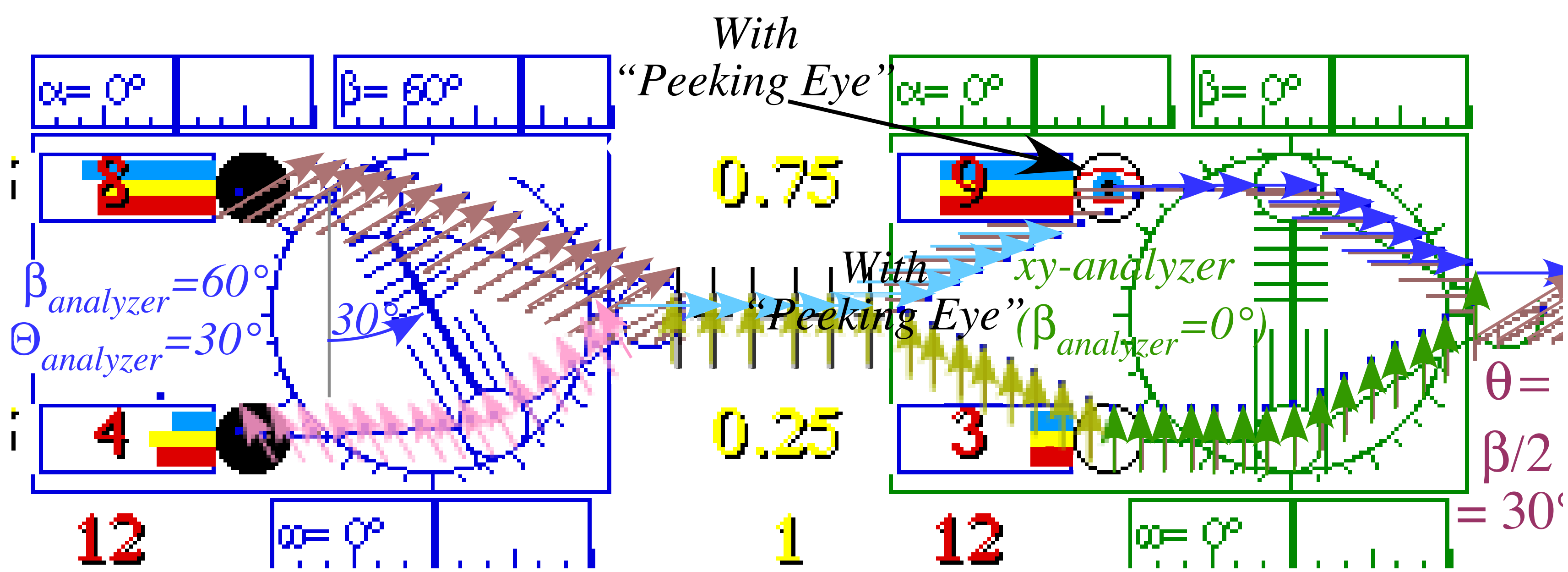
$$= \frac{5}{8} + \frac{3}{16}\left(e^{-i\phi} + e^{i\phi}\right) = \frac{5 + 3\cos\phi}{8}$$

$$A(y') = \langle y'|x\rangle\left(e^{i\phi}\right)\langle x|x'\rangle + \langle y'|y\rangle\langle y|x'\rangle$$

$$= \tfrac{-\sqrt{3}}{4}\left(e^{i\phi}\right) \quad + \quad \tfrac{\sqrt{3}}{4}$$

$$P(y') = \left(\tfrac{-\sqrt{3}}{4}\left(e^{i\phi}\right) + \tfrac{\sqrt{3}}{4}\right)^{*}\left(\tfrac{-\sqrt{3}}{4}\left(e^{i\phi}\right) + \tfrac{\sqrt{3}}{4}\right)$$

$$= \frac{3}{8} - \frac{3}{16}\left(e^{-i\phi} + e^{i\phi}\right) = \frac{3 - 3\cos\phi}{8}$$



$$\begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$$

Review: Axioms 1-4 and "Do-Nothing" vs " Do-Something" analyzers

Abstraction of Axiom-4 to define projection and unitary operators
- *Projection operators and resolution of identity*

Unitary operators and matrices that do something (or "nothing")
- *Diagonal unitary operators*
- *Non-diagonal unitary operators and †-conjugation relations*
- *Non-diagonal projection operators and Kronecker ⊗−products*
- *Axiom-4 similarity transformation*

Matrix representation of beam analyzers
- *Non-unitary "killer" devices: Sorter-counter, filter*
- *Unitary "non-killer" devices: 1/2-wave plate, 1/4-wave plate*

How analyzers "peek" and how that changes outcomes
- ➡ *Peeking polarizers and coherence loss*
  - *Classical Bayesian probability vs. Quantum probability*

Classical Bayesian probability vs. Quantum probability

$$\begin{pmatrix} \text{Probability that} \\ \text{photon in } x'\text{-input} \\ \textit{becomes} \\ \text{photon in } x'\text{-counter} \end{pmatrix}_{classical} =$$

$$\begin{pmatrix} \text{probability that} \\ \text{photon in } x\text{-beam} \\ \textit{becomes} \\ \text{photon in } x'\text{-counter} \end{pmatrix} * \begin{pmatrix} \text{probability that} \\ \text{photon in } x'\text{-input} \\ \textit{becomes} \\ \text{photon in } x\text{-beam} \end{pmatrix} + \begin{pmatrix} \text{probability that} \\ \text{photon in } y\text{-beam} \\ \textit{becomes} \\ \text{photon in } x'\text{-counter} \end{pmatrix} * \begin{pmatrix} \text{probability that} \\ \text{photon in } x'\text{-input} \\ \textit{becomes} \\ \text{photon in } y\text{-beam} \end{pmatrix}$$

$$\begin{pmatrix} \text{Probability that} \\ \text{photon in } x'\text{-input} \\ \textit{becomes} \\ \text{photon in } x'\text{-counter} \end{pmatrix}_{classical} = \left(\left|\langle x'|x\rangle\right|^2\right) * \left(\left|\langle x|x'\rangle\right|^2\right) + \left(\left|\langle x'|y\rangle\right|^2\right) * \left(\left|\langle y|x'\rangle\right|^2\right)$$

Classical Bayesian probability vs. Quantum probability

$$\begin{pmatrix} \text{Probability that} \\ \text{photon in } x'\text{-input} \\ \textit{becomes} \\ \text{photon in } x'\text{-counter} \end{pmatrix}_{classical} = \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$$

$$\begin{pmatrix} \text{probability that} \\ \text{photon in } x\text{-beam} \\ \textit{becomes} \\ \text{photon in } x'\text{-counter} \end{pmatrix} * \begin{pmatrix} \text{probability that} \\ \text{photon in } x'\text{-input} \\ \textit{becomes} \\ \text{photon in } x\text{-beam} \end{pmatrix} + \begin{pmatrix} \text{probability that} \\ \text{photon in } y\text{-beam} \\ \textit{becomes} \\ \text{photon in } x'\text{-counter} \end{pmatrix} * \begin{pmatrix} \text{probability that} \\ \text{photon in } x'\text{-input} \\ \textit{becomes} \\ \text{photon in } y\text{-beam} \end{pmatrix}$$

$$\begin{pmatrix} \text{Probability that} \\ \text{photon in } x'\text{-input} \\ \textit{becomes} \\ \text{photon in } x'\text{-counter} \end{pmatrix}_{classical} = \left(\left|\langle x'|x\rangle\right|^2\right) * \left(\left|\langle x|x'\rangle\right|^2\right) + \left(\left|\langle x'|y\rangle\right|^2\right) * \left(\left|\langle y|x'\rangle\right|^2\right) = \left(\left|\frac{\sqrt{3}}{2}\right|^2\right) * \left(\left|\frac{\sqrt{3}}{2}\right|^2\right) + \left(\left|\frac{-1}{2}\right|^2\right) * \left(\left|\frac{1}{2}\right|^2\right) = \frac{5}{8}$$

Classical Bayesian probability vs. Quantum probability

$$\begin{pmatrix}\text{Probability that}\\ \text{photon in } x'\text{-input}\\ \textit{becomes}\\ \text{photon in } x'\text{-counter}\end{pmatrix}_{classical} = \qquad \begin{pmatrix}\langle x|x'\rangle & \langle x|y'\rangle\\ \langle y|x'\rangle & \langle y|y'\rangle\end{pmatrix} = \begin{pmatrix}\sqrt{3}/2 & -1/2\\ 1/2 & \sqrt{3}/2\end{pmatrix} = \begin{pmatrix}\sqrt{3}/2 & -1/2\\ 1/2 & \sqrt{3}/2\end{pmatrix}$$

$$\begin{pmatrix}\text{probability that}\\ \text{photon in } x\text{-beam}\\ \textit{becomes}\\ \text{photon in } x'\text{-counter}\end{pmatrix} * \begin{pmatrix}\text{probability that}\\ \text{photon in } x'\text{-input}\\ \textit{becomes}\\ \text{photon in } x\text{-beam}\end{pmatrix} + \begin{pmatrix}\text{probability that}\\ \text{photon in } y\text{-beam}\\ \textit{becomes}\\ \text{photon in } x'\text{-counter}\end{pmatrix} * \begin{pmatrix}\text{probability that}\\ \text{photon in } x'\text{-input}\\ \textit{becomes}\\ \text{photon in } y\text{-beam}\end{pmatrix}$$

$$\begin{pmatrix}\text{Probability that}\\ \text{photon in } x'\text{-input}\\ \textit{becomes}\\ \text{photon in } x'\text{-counter}\end{pmatrix}_{classical} = \left(\left|\langle x'|x\rangle\right|^2\right) * \left(\left|\langle x|x'\rangle\right|^2\right) + \left(\left|\langle x'|y\rangle\right|^2\right) * \left(\left|\langle y|x'\rangle\right|^2\right) = \left(\left|\frac{\sqrt{3}}{2}\right|^2\right) * \left(\left|\frac{\sqrt{3}}{2}\right|^2\right) + \left(\left|\frac{-1}{2}\right|^2\right) * \left(\left|\frac{1}{2}\right|^2\right) = \frac{5}{8}$$

$$\begin{pmatrix}\text{Quantum probability}\\ \text{at } x'\text{-counter}\end{pmatrix} = \left|\langle x'|x\rangle\left(e^{i\phi}\right)\langle x|x'\rangle + \langle x'|y\rangle\langle y|x'\rangle\right|^2$$

Classical Bayesian probability vs. Quantum probability

$$\begin{pmatrix}\text{Probability that}\\ \text{photon in } x'\text{-input}\\ \textit{becomes}\\ \text{photon in } x'\text{-counter}\end{pmatrix}_{classical} = \begin{pmatrix}\langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle\end{pmatrix} = \begin{pmatrix}\sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2\end{pmatrix} = \begin{pmatrix}\sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2\end{pmatrix}$$

$$\begin{pmatrix}\text{probability that}\\ \text{photon in } x\text{-beam}\\ \textit{becomes}\\ \text{photon in } x'\text{-counter}\end{pmatrix} * \begin{pmatrix}\text{probability that}\\ \text{photon in } x'\text{-input}\\ \textit{becomes}\\ \text{photon in } x\text{-beam}\end{pmatrix} + \begin{pmatrix}\text{probability that}\\ \text{photon in } y\text{-beam}\\ \textit{becomes}\\ \text{photon in } x'\text{-counter}\end{pmatrix} * \begin{pmatrix}\text{probability that}\\ \text{photon in } x'\text{-input}\\ \textit{becomes}\\ \text{photon in } y\text{-beam}\end{pmatrix}$$

$$\begin{pmatrix}\text{Probability that}\\ \text{photon in } x'\text{-input}\\ \textit{becomes}\\ \text{photon in } x'\text{-counter}\end{pmatrix}_{classical} = \left(\left|\langle x'|x\rangle\right|^2\right) * \left(\left|\langle x|x'\rangle\right|^2\right) + \left(\left|\langle x'|y\rangle\right|^2\right) * \left(\left|\langle y|x'\rangle\right|^2\right) = \left(\left|\frac{\sqrt{3}}{2}\right|^2\right) * \left(\left|\frac{\sqrt{3}}{2}\right|^2\right) + \left(\left|\frac{-1}{2}\right|^2\right) * \left(\left|\frac{1}{2}\right|^2\right) = \frac{5}{8}$$

$$\begin{pmatrix}\text{Quantum probability}\\ \text{at } x'\text{-counter}\end{pmatrix} = \left|\langle x'|x\rangle\left(e^{i\phi}\right)\langle x|x'\rangle + \langle x'|y\rangle\langle y|x'\rangle\right|^2$$

$$= \left|\langle x'|x\rangle\langle x|x'\rangle\right|^2 + \left|\langle x'|y\rangle\langle y|x'\rangle\right|^2 + e^{-i\phi}\langle x'|x\rangle^*\langle x|x'\rangle^*\langle x'|y\rangle\langle y|x'\rangle + e^{i\phi}\langle x'|x\rangle\langle x|x'\rangle\langle x'|y\rangle^*\langle y|x'\rangle^* = 1$$

=(classical probability) + (Phase-sensitive or *quantum interference* terms)

Classical Bayesian probability vs. Quantum probability

$$\begin{pmatrix}\text{Probability that}\\ \text{photon in } x'\text{-input}\\ \textit{becomes}\\ \text{photon in } x'\text{-counter}\end{pmatrix}_{classical} = \begin{pmatrix}\langle x|x'\rangle & \langle x|y'\rangle\\ \langle y|x'\rangle & \langle y|y'\rangle\end{pmatrix} = \begin{pmatrix}\sqrt{3}/2 & -1/2\\ 1/2 & \sqrt{3}/2\end{pmatrix} = \begin{pmatrix}\sqrt{3}/2 & -1/2\\ 1/2 & \sqrt{3}/2\end{pmatrix}$$

$$\begin{pmatrix}\text{probability that}\\ \text{photon in } x\text{-beam}\\ \textit{becomes}\\ \text{photon in } x'\text{-counter}\end{pmatrix} * \begin{pmatrix}\text{probability that}\\ \text{photon in } x'\text{-input}\\ \textit{becomes}\\ \text{photon in } x\text{-beam}\end{pmatrix} + \begin{pmatrix}\text{probability that}\\ \text{photon in } y\text{-beam}\\ \textit{becomes}\\ \text{photon in } x'\text{-counter}\end{pmatrix} * \begin{pmatrix}\text{probability that}\\ \text{photon in } x'\text{-input}\\ \textit{becomes}\\ \text{photon in } y\text{-beam}\end{pmatrix}$$

$$\begin{pmatrix}\text{Probability that}\\ \text{photon in } x'\text{-input}\\ \textit{becomes}\\ \text{photon in } x'\text{-counter}\end{pmatrix}_{classical} = \left(\left|\langle x'|x\rangle\right|^2\right) * \left(\left|\langle x|x'\rangle\right|^2\right) + \left(\left|\langle x'|y\rangle\right|^2\right) * \left(\left|\langle y|x'\rangle\right|^2\right) = \left(\left|\frac{\sqrt{3}}{2}\right|^2\right) * \left(\left|\frac{\sqrt{3}}{2}\right|^2\right) + \left(\left|\frac{-1}{2}\right|^2\right) * \left(\left|\frac{1}{2}\right|^2\right) = \frac{5}{8}$$

$$\begin{pmatrix}\text{Quantum probability}\\ \text{at } x'\text{-counter}\end{pmatrix} = \left|\langle x'|x\rangle\left(e^{i\phi}\right)\langle x|x'\rangle + \langle x'|y\rangle\langle y|x'\rangle\right|^2$$

$$= \left|\langle x'|x\rangle\langle x|x'\rangle\right|^2 + \left|\langle x'|y\rangle\langle y|x'\rangle\right|^2 + e^{-i\phi}\langle x'|x\rangle^*\langle x|x'\rangle^*\langle x'|y\rangle\langle y|x'\rangle + e^{i\phi}\langle x'|x\rangle\langle x|x'\rangle\langle x'|y\rangle^*\langle y|x'\rangle^* = 1$$

=(classical probability) + (Phase-sensitive or *quantum interference* terms)

$$\begin{pmatrix}\text{Quantum probability}\\ \text{at } x'\text{-counter}\end{pmatrix} = \left|\langle x'|x\rangle\langle x|x'\rangle\right| + \left|\langle x'|y\rangle\langle y|x'\rangle\right|^2$$

Classical Bayesian probability vs. Quantum probability

$$\begin{pmatrix}\text{Probability that}\\ \text{photon in } x'\text{-input}\\ \textit{becomes}\\ \text{photon in } x'\text{-counter}\end{pmatrix}_{classical} = \qquad \begin{pmatrix}\langle x|x'\rangle & \langle x|y'\rangle\\ \langle y|x'\rangle & \langle y|y'\rangle\end{pmatrix} = \begin{pmatrix}\sqrt{3}/2 & -1/2\\ 1/2 & \sqrt{3}/2\end{pmatrix} = \begin{pmatrix}\sqrt{3}/2 & -1/2\\ 1/2 & \sqrt{3}/2\end{pmatrix}$$

$$\begin{pmatrix}\text{probability that}\\ \text{photon in } x\text{-beam}\\ \textit{becomes}\\ \text{photon in } x'\text{-counter}\end{pmatrix} * \begin{pmatrix}\text{probability that}\\ \text{photon in } x'\text{-input}\\ \textit{becomes}\\ \text{photon in } x\text{-beam}\end{pmatrix} + \begin{pmatrix}\text{probability that}\\ \text{photon in } y\text{-beam}\\ \textit{becomes}\\ \text{photon in } x'\text{-counter}\end{pmatrix} * \begin{pmatrix}\text{probability that}\\ \text{photon in } x'\text{-input}\\ \textit{becomes}\\ \text{photon in } y\text{-beam}\end{pmatrix}$$

$$\begin{pmatrix}\text{Probability that}\\ \text{photon in } x'\text{-input}\\ \textit{becomes}\\ \text{photon in } x'\text{-counter}\end{pmatrix}_{classical} = \left(\left|\langle x'|x\rangle\right|^2\right) * \left(\left|\langle x|x'\rangle\right|^2\right) + \left(\left|\langle x'|y\rangle\right|^2\right) * \left(\left|\langle y|x'\rangle\right|^2\right) = \left(\left|\frac{\sqrt{3}}{2}\right|^2\right) * \left(\left|\frac{\sqrt{3}}{2}\right|^2\right) + \left(\left|\frac{-1}{2}\right|^2\right) * \left(\left|\frac{1}{2}\right|^2\right) = \frac{5}{8}$$

$$\begin{pmatrix}\text{Quantum probability}\\ \text{at } x'\text{-counter}\end{pmatrix} = \left|\langle x'|x\rangle\left(e^{i\phi}\right)\langle x|x'\rangle + \langle x'|y\rangle\langle y|x'\rangle\right|^2$$

$$= \left|\langle x'|x\rangle\langle x|x'\rangle\right|^2 + \left|\langle x'|y\rangle\langle y|x'\rangle\right|^2 + e^{-i\phi}\langle x'|x\rangle^*\langle x|x'\rangle^*\langle x'|y\rangle\langle y|x'\rangle + e^{i\phi}\langle x'|x\rangle\langle x|x'\rangle\langle x'|y\rangle^*\langle y|x'\rangle^* = 1$$

=(classical probability) + (Phase-sensitive or *quantum interference* terms)

$$\begin{pmatrix}\text{Quantum probability}\\ \text{at } x'\text{-counter}\end{pmatrix} = \left|\langle x'|x\rangle\langle x|x'\rangle + \langle x'|y\rangle\langle y|x'\rangle\right|^2$$

Square of sum

$$\begin{pmatrix}\text{Classical probability}\\ \text{at } x'\text{-counter}\end{pmatrix} = \left|\langle x'|x\rangle\langle x|x'\rangle\right|^2 + \left|\langle x'|y\rangle\langle y|x'\rangle\right|^2$$

Sum of squares

Group axioms

(1) The closure axiom

Products ab = c are defined between any two group elements a and b, and the result c is contained in the group.

(2) The associativity axiom

Products (ab)c and a(bc) are equal for all elements a, b, and c in the group .

(3) The identity axiom

There is a unique element 1 (the identity) such that $1 \cdot a = a = a \cdot 1$ for all elements a in the group ..

4) The inverse axiom

For all elements a in the group there is an inverse element a^{-1} such that $a^{-1}a = 1 = a \cdot a^{-1}$.

(5) The commutative axiom (Abelian groups only)

All elements a in an Abelian group are mutually commuting: $a \cdot b = b \cdot a$.