Group Theory in Quantum Mechanics

Lecture 26 *(5.2.13)*

*Symmetry product analysis U(m)*Sn tensors*

(Int.J.Mol.Sci, 14, 714(2013) p.755-774 , QTCA Unit 7 Ch. 23-26)
(PSDS - Ch. 5, 7)

Review : *2-D* **a†a** *algebra of U(2) representations*

Spin-spin $(1/2)^2$ product states: Hydrogen hyperfine structure

Kronecker product states and operators

Spin-spin interaction reduces symmetry $U(2)^{proton} \times U(2)^{electron}$ to $U(2)^{e+p}$

Clebsch-Gordan Coefficients

Hydrogen hyperfine structure: Fermi-contact interaction

plus B-field gives avoided crossing

Higher-J product states

$(J=1)\otimes(J=1)=2\oplus 1\oplus 0$ case

General U(2) case

Multi-spin $(1/2)^N$ product states

Magic squares - Intro to Young Tableaus

Tensor operators for spin-1/2 states: Outer products give Hamilton-Pauli-spinors

Tensor operators for spin-1 states: U(3) generalization of Pauli spinors

Spin-spin $(1/2)^2$ product states: Hydrogen hyperfine structure

electron-proton spin-spin interaction gives a simple example of *hyperfine* spectra

Ket-kets for spin-up and spin-dn states and column matrix representations..

$$\left|\uparrow\right\rangle\left|\uparrow\right\rangle = \left|\begin{matrix}\frac{1}{2}\\ \frac{1}{2}\end{matrix}\right\rangle^{proton}\left|\begin{matrix}\frac{1}{2}\\ \frac{1}{2}\end{matrix}\right\rangle^{electron}, \left|\uparrow\right\rangle\left|\downarrow\right\rangle = \left|\begin{matrix}\frac{1}{2}\\ \frac{1}{2}\end{matrix}\right\rangle^{proton}\left|\begin{matrix}\frac{1}{2}\\ -\frac{1}{2}\end{matrix}\right\rangle^{electron}, \left|\downarrow\right\rangle\left|\uparrow\right\rangle = \left|\begin{matrix}\frac{1}{2}\\ -\frac{1}{2}\end{matrix}\right\rangle^{proton}\left|\begin{matrix}\frac{1}{2}\\ \frac{1}{2}\end{matrix}\right\rangle^{electron}, \left|\downarrow\right\rangle\left|\downarrow\right\rangle = \left|\begin{matrix}\frac{1}{2}\\ -\frac{1}{2}\end{matrix}\right\rangle^{proton}\left|\begin{matrix}\frac{1}{2}\\ -\frac{1}{2}\end{matrix}\right\rangle^{electron}$$

$$\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

Same spin-1/2 representation applies to either proton or electron kets.

$$D^{1/2}(\alpha\beta\gamma)=\begin{pmatrix}D^{1/2}_{+1/2,+1/2} & D^{1/2}_{+1/2,-1/2}\\ D^{1/2}_{-1/2,+1/2} & D^{1/2}_{-1/2,-1/2}\end{pmatrix}=\begin{pmatrix}e^{\frac{-i(\alpha+\gamma)}{2}}\cos\frac{\beta}{2} & -e^{\frac{-i(\alpha-\gamma)}{2}}\sin\frac{\beta}{2}\\ e^{\frac{i(\alpha-\gamma)}{2}}\sin\frac{\beta}{2} & e^{\frac{i(\alpha+\gamma)}{2}}\cos\frac{\beta}{2}\end{pmatrix}$$

Kronecker product $D^{\frac{1}{2}} \otimes D^{\frac{1}{2}}$

Applies to *outer product symmetry* $U(2)^{proton}\times U(2)^{electron}$ for NO interaction.

$$\begin{pmatrix}\cos\frac{\beta_p}{2} & -\sin\frac{\beta_p}{2}\\ \sin\frac{\beta_p}{2} & \cos\frac{\beta_p}{2}\end{pmatrix}\otimes\begin{pmatrix}\cos\frac{\beta_e}{2} & -\sin\frac{\beta_e}{2}\\ \sin\frac{\beta_e}{2} & \cos\frac{\beta_e}{2}\end{pmatrix}=\begin{pmatrix}\cos\frac{\beta_p}{2}\cos\frac{\beta_e}{2} & -\cos\frac{\beta_p}{2}\sin\frac{\beta_e}{2} & -\sin\frac{\beta_p}{2}\cos\frac{\beta_e}{2} & \sin\frac{\beta_p}{2}\sin\frac{\beta_e}{2}\\ \cos\frac{\beta_p}{2}\sin\frac{\beta_e}{2} & \cos\frac{\beta_p}{2}\cos\frac{\beta_e}{2} & -\sin\frac{\beta_p}{2}\sin\frac{\beta_e}{2} & -\sin\frac{\beta_p}{2}\cos\frac{\beta_e}{2}\\ \sin\frac{\beta_p}{2}\cos\frac{\beta_e}{2} & -\sin\frac{\beta_p}{2}\sin\frac{\beta_e}{2} & \cos\frac{\beta_p}{2}\cos\frac{\beta_e}{2} & -\cos\frac{\beta_p}{2}\sin\frac{\beta_e}{2}\\ \sin\frac{\beta_p}{2}\sin\frac{\beta_e}{2} & \sin\frac{\beta_p}{2}\cos\frac{\beta_e}{2} & \cos\frac{\beta_p}{2}\sin\frac{\beta_e}{2} & \cos\frac{\beta_p}{2}\cos\frac{\beta_e}{2}\end{pmatrix}$$

Spin-spin interaction reduces symmetry $U(2)^{proton}\times U(2)^{electron}$ to $U(2)^{e+p}$

$$\begin{pmatrix}1&0&0&0\\0&\frac{1}{\sqrt2}&\frac{1}{\sqrt2}&0\\0&0&0&1\\0&\frac{1}{\sqrt2}&\frac{-1}{\sqrt2}&0\end{pmatrix}\cdot\begin{pmatrix}\cos^2\frac{\beta}{2} & -\sin\frac{\beta}{2}\cos\frac{\beta}{2} & -\sin\frac{\beta}{2}\cos\frac{\beta}{2} & \sin^2\frac{\beta}{2}\\ \sin\frac{\beta}{2}\cos\frac{\beta}{2} & \cos^2\frac{\beta}{2} & -\sin^2\frac{\beta}{2} & -\sin\frac{\beta}{2}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\cos\frac{\beta}{2} & -\sin^2\frac{\beta}{2} & \cos^2\frac{\beta}{2} & -\sin\frac{\beta}{2}\cos\frac{\beta}{2}\\ \sin^2\frac{\beta}{2} & \sin\frac{\beta}{2}\cos\frac{\beta}{2} & \sin\frac{\beta}{2}\cos\frac{\beta}{2} & \cos^2\frac{\beta}{2}\end{pmatrix}\cdot\begin{pmatrix}1&0&0&0\\0&\frac{1}{\sqrt2}&0&\frac{1}{\sqrt2}\\0&\frac{1}{\sqrt2}&0&\frac{-1}{\sqrt2}\\0&0&1&0\end{pmatrix}=\begin{pmatrix}\sin^2\frac{\beta}{2} & \frac{-\sin\beta}{\sqrt2} & \sin^2\frac{\beta}{2} & 0\\ \frac{\sin\beta}{\sqrt2} & \cos\beta & \frac{-\sin\beta}{\sqrt2} & 0\\ \sin^2\frac{\beta}{2} & \frac{\sin\beta}{\sqrt2} & \cos^2\frac{\beta}{2} & 0\\ 0&0&0&1\end{pmatrix}=\left(\begin{array}{c|c} D^{J=1}_{(0\beta0)} & \\ \hline & D^{J=0}\end{array}\right)$$

Spin-spin interaction reduces symmetry $U(2)^{proton}\times U(2)^{electron}$ to $U(2)^{e+p}$

$$
\begin{pmatrix}
1 & 0 & 0 & 0\\
0 & \frac{1}{\sqrt2} & \frac{1}{\sqrt2} & 0\\
0 & 0 & 0 & 1\\
0 & \frac{1}{\sqrt2} & \frac{-1}{\sqrt2} & 0
\end{pmatrix}
\cdot
\begin{pmatrix}
\cos^2\frac{\beta}{2} & -\sin\frac{\beta}{2}\cos\frac{\beta}{2} & -\sin\frac{\beta}{2}\cos\frac{\beta}{2} & \sin^2\frac{\beta}{2}\\
\sin\frac{\beta}{2}\cos\frac{\beta}{2} & \cos^2\frac{\beta}{2} & -\sin^2\frac{\beta}{2} & -\sin\frac{\beta}{2}\cos\frac{\beta}{2}\\
\sin\frac{\beta}{2}\cos\frac{\beta}{2} & -\sin^2\frac{\beta}{2} & \cos^2\frac{\beta}{2} & -\sin\frac{\beta}{2}\cos\frac{\beta}{2}\\
\sin^2\frac{\beta}{2} & \sin\frac{\beta}{2}\cos\frac{\beta}{2} & \sin\frac{\beta}{2}\cos\frac{\beta}{2} & \cos^2\frac{\beta}{2}
\end{pmatrix}
\cdot
\begin{pmatrix}
1 & 0 & 0 & 0\\
0 & \frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2}\\
0 & \frac{1}{\sqrt2} & 0 & \frac{-1}{\sqrt2}\\
0 & 0 & 1 & 0
\end{pmatrix}
=
\begin{pmatrix}
\sin^2\frac{\beta}{2} & \frac{-\sin\beta}{\sqrt2} & \sin^2\frac{\beta}{2} & 0\\
\frac{\sin\beta}{\sqrt2} & \cos\beta & \frac{-\sin\beta}{\sqrt2} & 0\\
\sin^2\frac{\beta}{2} & \frac{\sin\beta}{\sqrt2} & \cos^2\frac{\beta}{2} & 0\\
0 & 0 & 0 & 1
\end{pmatrix}
=
\begin{pmatrix}
D^{J=1}_{(0\beta 0)} & \\
 & D^{J=0}
\end{pmatrix}
$$

Clebsch-Gordan coefficients(CGC)

$$C^{\frac12\ \frac12\ J}_{m_p\ m_e\ M} \equiv \left\langle \begin{matrix} \frac12 & \frac12 \\ m_p & m_e \end{matrix} \middle| \begin{matrix} J \\ M \end{matrix} \right\rangle$$

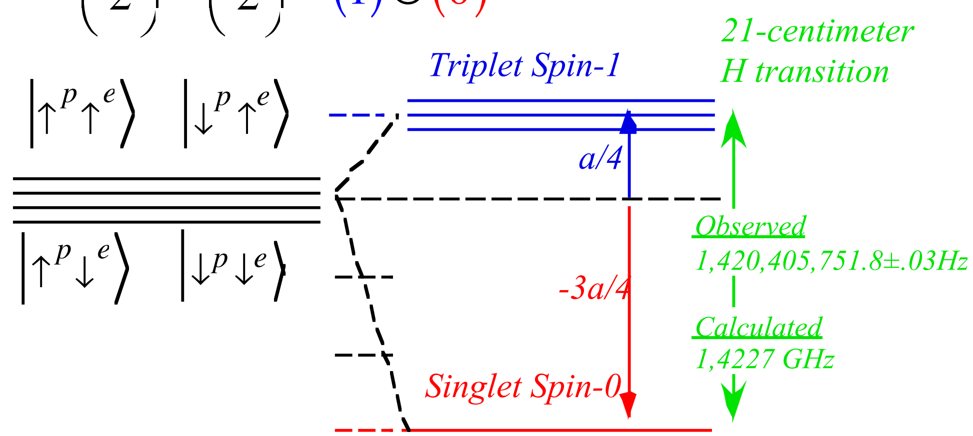
reduce $D^{1/2}\otimes D^{1/2}$ to $D^{1}\oplus D^{0}$

$\frac12\otimes\frac12$	$J=1$, $M=1$	1, 0	1, -1	0, 0
$\frac12,\frac12$	1	0	0	0
$\frac12,\frac{-1}{2}$	0	$\frac{1}{\sqrt2}$	0	$\frac{1}{\sqrt2}$
$\frac{-1}{2},\frac12$	0	$\frac{1}{\sqrt2}$	0	$\frac{-1}{\sqrt2}$
$\frac{-1}{2},\frac{-1}{2}$	0	0	1	0

$$=\left\langle C^{\frac12\ \frac12}_{m_p\ m_e} \middle| \begin{matrix} J \\ M \end{matrix} \right\rangle$$

$$\sum_{m_1 m_1'}\sum_{m_2 m_2'} C^{\frac12\ \frac12\ J}_{m_1\ m_1'\ M}\, D^{\frac12}_{m_1 m_2}\, D^{\frac12}_{m_1' m_2'}\, C^{\frac12\ \frac12\ J'}_{m_2\ m_2'\ M'} = \delta^{JJ'} D^{J}_{M\,M'}$$

$$\left(\frac12\right)\otimes\left(\frac12\right) = (1)\oplus(0)$$



$$\left|\begin{matrix} J \\ M \end{matrix}\ {}^{(1/2\otimes 1/2)}\right\rangle = \sum_{m_1,m_2} C^{1/2\ 1/2\ J}_{m_1\ m_2\ M} \left|\begin{matrix}1/2\\ m_1\end{matrix}\right\rangle \left|\begin{matrix}1/2\\ m_2\end{matrix}\right\rangle$$

$$\left|\begin{matrix}1\\ +1\end{matrix}\right\rangle = |\uparrow^p\uparrow^e\rangle$$

$$\left|\begin{matrix}1\\ 0\end{matrix}\right\rangle = \left(|\uparrow^p\downarrow^e\rangle + |\downarrow^p\uparrow^e\rangle\right)/\sqrt2$$

$$\left|\begin{matrix}1\\ -1\end{matrix}\right\rangle = |\downarrow^p\downarrow^e\rangle$$

$$\left|\begin{matrix}0\\ 0\end{matrix}\right\rangle = \left(|\uparrow^p\downarrow^e\rangle - |\downarrow^p\uparrow^e\rangle\right)/\sqrt2$$

Hydrogen hyperfine structure: Fermi-contact interaction

Racah's trick for energy eigenvalues

$$a_{ep}\mathbf{J}^{proton}\bullet\mathbf{J}^{electron}=\frac{a_{ep}}{2}\left[\left(\mathbf{J}^{proton}+\mathbf{J}^{electron}\right)^2-\left(\mathbf{J}^{proton}\right)^2-\left(\mathbf{J}^{electron}\right)^2\right]$$

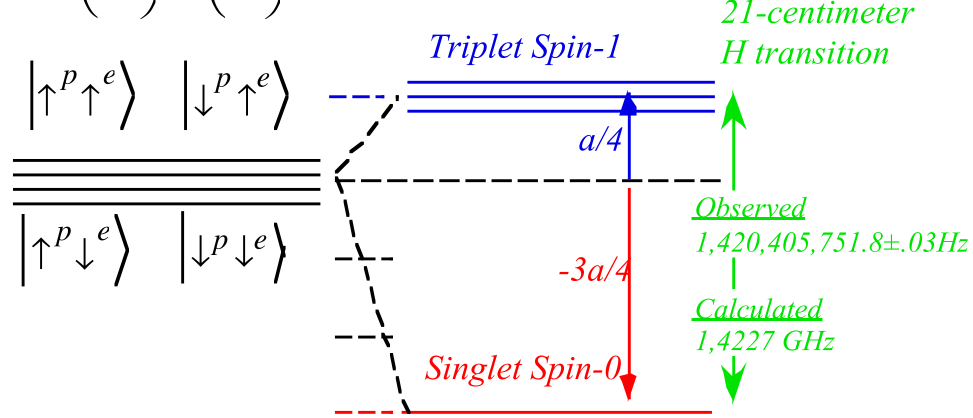
$$=\frac{a_{ep}}{2}\left[\left(\mathbf{J}^{total}\right)^2-\left(\mathbf{J}^{proton}\right)^2-\left(\mathbf{J}^{electron}\right)^2\right].$$

$$\left\langle {J \atop M}^{(1/2\otimes1/2)}\right| H_{contact}\left| {J \atop M}^{(1/2\otimes1/2)}\right\rangle=\frac{a_{ep}}{2}\left[J(J+1)-\tfrac{1}{2}\left(\tfrac{1}{2}+1\right)-\tfrac{1}{2}\left(\tfrac{1}{2}+1\right)\right]$$

$$=\begin{cases} a_{ep}/4 \text{ for the } (J=1) & \text{triplet state,}\\ -3a_{ep}/4 \text{ for the } (J=0) & \text{singlet state.}\end{cases}$$

$$\left|{J \atop M}^{(1/2\otimes1/2)}\right\rangle=\sum_{m_1,m_2}C^{1/2\ 1/2\ J}_{m_1\ m_2\ M}\left|{1/2 \atop m_1}\right\rangle\left|{1/2 \atop m_2}\right\rangle$$

$$\left(\frac{1}{2}\right)\otimes\left(\frac{1}{2}\right)=(1)\oplus(0)$$



$$\left|{1 \atop +1}\right\rangle=\left|\uparrow^p\uparrow^e\right\rangle$$

$$\left|{1 \atop 0}\right\rangle=\left(\left|\uparrow^p\downarrow^e\right\rangle+\left|\downarrow^p\uparrow^e\right\rangle\right)/\sqrt{2}$$

$$\left|{1 \atop -1}\right\rangle=\left|\downarrow^p\downarrow^e\right\rangle$$

$$\left|{0 \atop 0}\right\rangle=\left(\left|\uparrow^p\downarrow^e\right\rangle-\left|\downarrow^p\uparrow^e\right\rangle\right)/\sqrt{2}$$

Hydrogen hyperfine structure: Fermi-contact interaction + B-field

$$H_{1s-B-field} = -a_p B_z J_z^{proton} + a_e B_z J_z^{electron} + a_{ep} \mathbf{J}^{proton} \bullet \mathbf{J}^{electron}$$

	$g-factor$	$Bohr-magneton$	$gyromagnetic\ factor$
$electron$	$g_e = 2.0023$	$\mu_e = \frac{e\hbar}{2m_e} = 9.27401 \cdot 10^{-24}\frac{J}{T}$	$a_e = g_e\mu_e = 1.8570 \cdot 10^{-23}\frac{J}{T}$
$proton$	$g_p = 5.585$	$\mu_p = \frac{e\hbar}{2m_p} = 5.05078 \cdot 10^{-27}\frac{J}{T}$	$a_p = g_p\mu_p = 2.8209 \cdot 10^{-26}\frac{J}{T}$

$Magnetic\ constant:\ \mu_0 / 4\pi = 10^{-7} N/A^2$

$Fermi-contact\ factor$
$a_{ep} = \mu_0 \frac{2}{3}\frac{1}{\pi a_0^3} a_e a_p = 9.427 \cdot 10^{-25} J$
$\mu_0 \frac{2}{3}\frac{1}{\pi a_0^3}\frac{a_e a_p}{h} = 1.4227 \cdot 10^9 Hz$
$\mu_0 \frac{2}{3}\frac{1}{\pi a_0^3}\frac{a_e a_p}{hc} = 4.746 m^{-1}$
$= \frac{1}{21.1} cm^{-1}$

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$Magnetic\ constant:\ \mu_0 / 4\pi = 10^{-7} N / A^2$

$Fermi-contact\ factor$
$a_{ep} = \mu_0 \frac{2}{3} \frac{1}{\pi a_0^3} a_e a_p = 9.427 \cdot 10^{-25} J$
$\mu_0 \frac{2}{3} \frac{1}{\pi a_0^3} \frac{a_e a_p}{h} = 1.4227 \cdot 10^9 Hz$
$\mu_0 \frac{2}{3} \frac{1}{\pi a_0^3} \frac{a_e a_p}{hc} = 4.746 m^{-1} = \frac{1}{21.1} cm^{-1}$

$$\left\langle -a_p B_z J_z^{proton} + a_e B_z J_z^{electron} \right\rangle =$$

	$\left\lvert \uparrow^p \uparrow^e \right\rangle$	$\left\lvert \uparrow^p \downarrow^e \right\rangle$	$\left\lvert \downarrow^p \uparrow^e \right\rangle$	$\left\lvert \downarrow^p \downarrow^e \right\rangle$
$\left\langle \uparrow^p \uparrow^e \right\rvert$	$\frac{1}{2}(a_e - a_p) B_z$	.	.	.
$\left\langle \uparrow^p \downarrow^e \right\rvert$	.	$\frac{-1}{2}(a_e + a_p) B_z$	0	.
$\left\langle \downarrow^p \uparrow^e \right\rvert$	.	0	$\frac{1}{2}(a_e + a_p) B_z$	.
$\left\langle \downarrow^p \downarrow^e \right\rvert$	.	.	.	$\frac{-1}{2}(a_e - a_p) B_z$

$$\left\langle a_{ep} \mathbf{J}^{proton} \bullet \mathbf{J}^{electron} \right\rangle =$$

	$\left\lvert {1 \atop 1} \right\rangle$	$\left\lvert {1 \atop 0} \right\rangle$	$\left\lvert {0 \atop 0} \right\rangle$	$\left\lvert {1 \atop -1} \right\rangle$
$\left\langle {1 \atop 1} \right\rvert$	$\frac{a_{ep}}{4}$	.	.	.
$\left\langle {1 \atop 0} \right\rvert$	.	$\frac{a_{ep}}{4}$	0	.
$\left\langle {0 \atop 0} \right\rvert$	.	0	$\frac{-3a_{ep}}{4}$	.
$\left\langle {1 \atop -1} \right\rvert$	.	.	.	$\frac{a_{ep}}{4}$

$$H_{1s-B-field} = -a_p B_z J_z^{proton} + a_e B_z J_z^{electron} + a_{ep}\mathbf{J}^{proton} \bullet \mathbf{J}^{electron}$$

	$g-factor$	$Bohr-magneton$	$gyromagnetic\ factor$
electron	$g_e = 2.0023$	$\mu_e = \frac{e\hbar}{2m_e} = 9.27401\cdot 10^{-24}\frac{J}{T}$	$a_e = g_e\mu_e = 1.8570\cdot 10^{-23}\frac{J}{T}$
proton	$g_p = 5.585$	$\mu_p = \frac{e\hbar}{2m_p} = 5.05078\cdot 10^{-27}\frac{J}{T}$	$a_p = g_p\mu_p = 2.8209\cdot 10^{-26}\frac{J}{T}$

$Magnetic\ constant:\ \mu_0 / 4\pi = 10^{-7}\,N / A^2$

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$a_{ep} = \mu_0\frac{2}{3}\frac{1}{\pi a_0^3}a_e a_p = 9.427\cdot 10^{-25}\,J$
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$\mu_0\frac{2}{3}\frac{1}{\pi a_0^3}\frac{a_e a_p}{hc} = 4.746\,m^{-1}$
$= \frac{1}{21.1}cm^{-1}$

$\frac{1}{2}\otimes\frac{1}{2}$	$J=1$, $M=1$	1, 0	1, -1	0, 0
$\frac{1}{2},\frac{1}{2}$	1	0	0	0
$\frac{1}{2},\frac{-1}{2}$	0	$\frac{1}{\sqrt{2}}$		$\frac{1}{\sqrt{2}}$
$\frac{-1}{2},\frac{1}{2}$	0	$\frac{1}{\sqrt{2}}$		$\frac{-1}{\sqrt{2}}$
$\frac{-1}{2},\frac{-1}{2}$	0	0	1	0

$= \left\langle C^{\frac{1}{2}\ \frac{1}{2}}_{m_p\ m_e}\middle| {}^{J}_{M}\right\rangle$

$$\left\langle -a_p B_z J_z^{proton} + a_e B_z J_z^{electron}\right\rangle =$$

	$\lvert\uparrow^p\uparrow^e\rangle$	$\lvert\uparrow^p\downarrow^e\rangle$	$\lvert\downarrow^p\uparrow^e\rangle$	$\lvert\downarrow^p\downarrow^e\rangle$
$\langle\uparrow^p\uparrow^e\rvert$	$\frac{1}{2}(a_e - a_p)B_z$	.	.	.
$\langle\uparrow^p\downarrow^e\rvert$	.	$\frac{-1}{2}(a_e + a_p)B_z$	0	.
$\langle\downarrow^p\uparrow^e\rvert$	.	0	$\frac{1}{2}(a_e + a_p)B_z$	.
$\langle\downarrow^p\downarrow^e\rvert$	.	.	.	$\frac{-1}{2}(a_e - a_p)B_z$

$$\left\langle a_{ep}\mathbf{J}^{proton}\bullet\mathbf{J}^{electron}\right\rangle =$$

	$\lvert{}^1_1\rangle$	$\lvert{}^1_0\rangle$	$\lvert{}^0_0\rangle$	$\lvert{}^{1}_{-1}\rangle$
$\langle{}^1_1\rvert$	$\frac{a_{ep}}{4}$	.	.	.
$\langle{}^1_0\rvert$	.	$\frac{a_{ep}}{4}$	0	.
$\langle{}^0_0\rvert$	.	0	$\frac{-3a_{ep}}{4}$	.
$\langle{}^{1}_{-1}\rvert$	.	.	.	$\frac{a_{ep}}{4}$

$$H_{1s-B-field} = -a_p B_z J_z^{proton} + a_e B_z J_z^{electron} + a_{ep} \mathbf{J}^{proton} \bullet \mathbf{J}^{electron}$$

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$\frac{1}{2}\otimes\frac{1}{2}$	$J=1$, $M=1$	1, 0	1, -1	0, 0
$\frac{1}{2},\frac{1}{2}$	1	0	0	0
$\frac{1}{2},\frac{-1}{2}$	0	$\frac{1}{\sqrt{2}}$		$\frac{1}{\sqrt{2}}$
$\frac{-1}{2},\frac{1}{2}$	0	$\frac{1}{\sqrt{2}}$		$\frac{-1}{\sqrt{2}}$
$\frac{-1}{2},\frac{-1}{2}$	0	0	1	0

$= \left\langle C^{\frac{1}{2}\ \frac{1}{2}}_{m_p\ m_e} \middle| {J \atop M} \right\rangle$

$\left\langle -a_p B_z J_z^{proton} + a_e B_z J_z^{electron} \right\rangle =$

	$\lvert\uparrow^p\uparrow^e\rangle$	$\lvert\uparrow^p\downarrow^e\rangle$	$\lvert\downarrow^p\uparrow^e\rangle$	$\lvert\downarrow^p\downarrow^e\rangle$
$\langle\uparrow^p\uparrow^e\rvert$	$\frac{1}{2}(a_e - a_p)B_z$	.	.	.
$\langle\uparrow^p\downarrow^e\rvert$	.	$\frac{-1}{2}(a_e + a_p)B_z$	0	.
$\langle\downarrow^p\uparrow^e\rvert$	.	0	$\frac{1}{2}(a_e + a_p)B_z$	.
$\langle\downarrow^p\downarrow^e\rvert$	.	.	.	$\frac{-1}{2}(a_e - a_p)B_z$

$\left\langle a_{ep}\mathbf{J}^{proton} \bullet \mathbf{J}^{electron} \right\rangle =$

	$\lvert\uparrow^p\uparrow^e\rangle$	$\lvert\uparrow^p\downarrow^e\rangle$	$\lvert\downarrow^p\uparrow^e\rangle$	$\lvert\downarrow^p\downarrow^e\rangle$
$\langle\uparrow^p\uparrow^e\rvert$	$\frac{a_{ep}}{4}$	.	.	.
$\langle\uparrow^p\downarrow^e\rvert$	.	$\frac{-a_{ep}}{4}$	$\frac{a_{ep}}{2}$	.
$\langle\downarrow^p\uparrow^e\rvert$	.	$\frac{a_{ep}}{2}$	$\frac{-a_{ep}}{4}$	.
$\langle\downarrow^p\downarrow^e\rvert$	.	.	.	$\frac{a_{ep}}{4}$

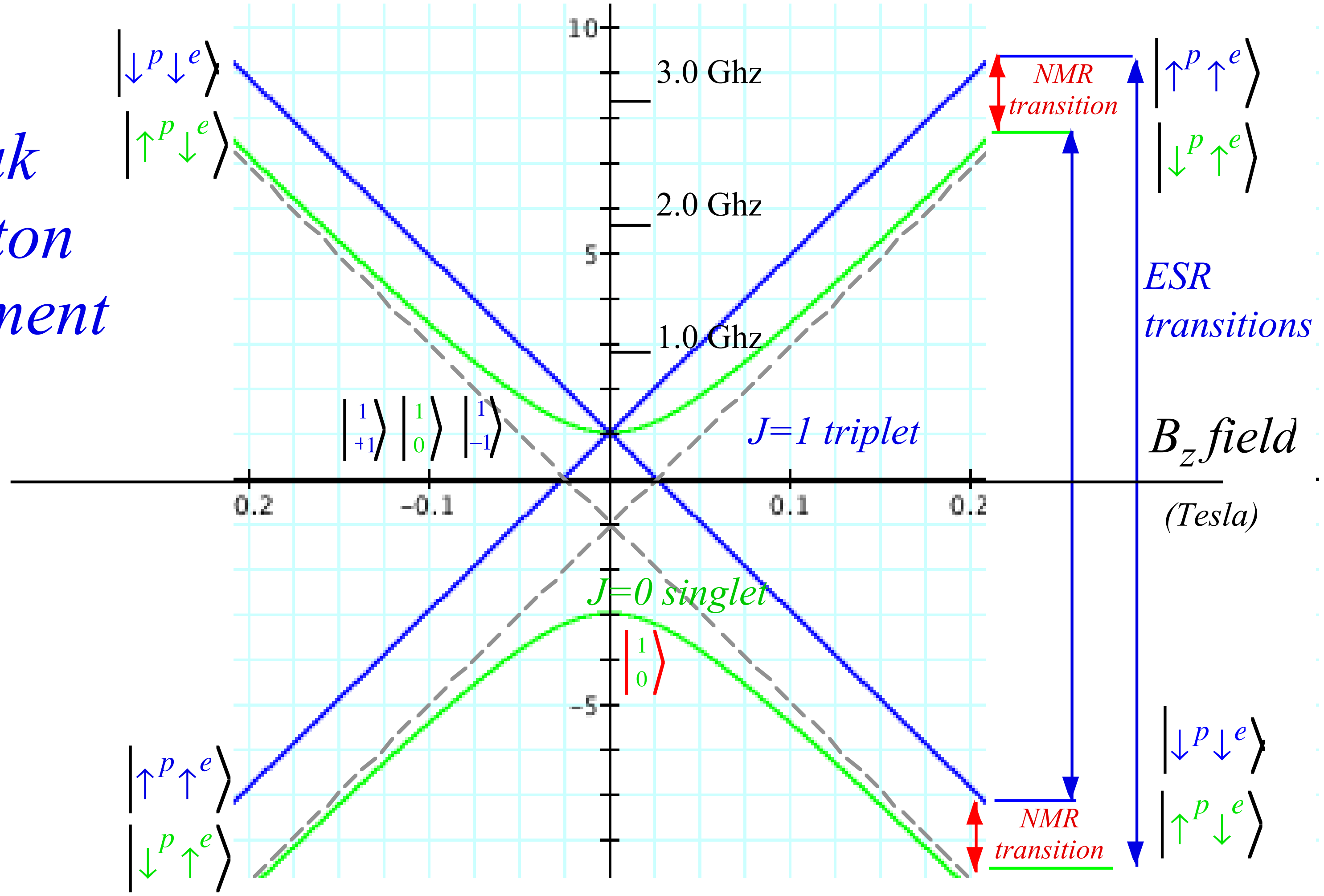
$\left\langle -a_p B_z J_z^{proton} + a_e B_z J_z^{electron} \right\rangle =$

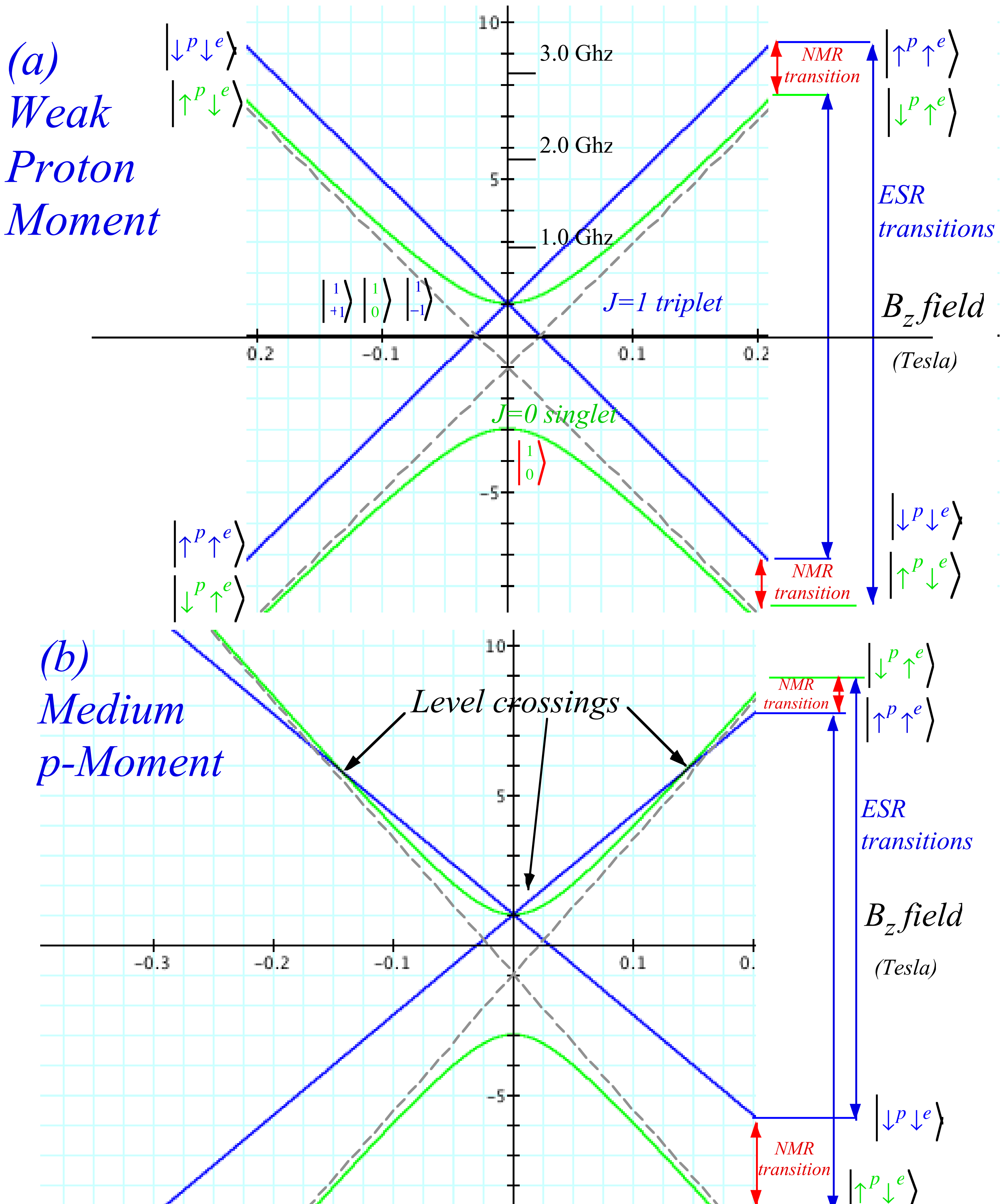
	$\lvert{1 \atop 1}\rangle$	$\lvert{1 \atop 0}\rangle$	$\lvert{0 \atop 0}\rangle$	$\lvert{1 \atop -1}\rangle$
$\langle{1 \atop 1}\rvert$	$\frac{1}{2}(a_e - a_p)B_z$	.	.	.
$\langle{1 \atop 0}\rvert$	.	0	$\frac{-1}{2}(a_e + a_p)B_z$	.
$\langle{0 \atop 0}\rvert$	.	$\frac{-1}{2}(a_e + a_p)B_z$	0	.
$\langle{1 \atop -1}\rvert$	.	.	.	$\frac{-1}{2}(a_e - a_p)B_z$

$\left\langle a_{ep}\mathbf{J}^{proton} \bullet \mathbf{J}^{electron} \right\rangle =$

	$\lvert{1 \atop 1}\rangle$	$\lvert{1 \atop 0}\rangle$	$\lvert{0 \atop 0}\rangle$	$\lvert{1 \atop -1}\rangle$
$\langle{1 \atop 1}\rvert$	$\frac{a_{ep}}{4}$	.	.	.
$\langle{1 \atop 0}\rvert$	.	$\frac{a_{ep}}{4}$	0	.
$\langle{0 \atop 0}\rvert$	.	0	$\frac{-3a_{ep}}{4}$	.
$\langle{1 \atop -1}\rvert$	.	.	.	$\frac{a_{ep}}{4}$

(a) Weak Proton Moment




(a)
Weak
Proton
Moment
3.0 Ghz
2.0 Ghz
1.0 Ghz
J=1 triplet
J=0 singlet
NMR transition
ESR transitions
NMR transition
B_z field
(Tesla)
(b)
Medium
p-Moment
Level crossings
NMR transition
ESR transitions
B_z field
(Tesla)
NMR transition

Higher-J product states

(J=1)⊗(J=1)=2⊕1⊕0 case

$\left|C^{1\;1\;L}_{m_1\,m_2\,M}\right\rangle =$

$1 \otimes 1$		2	2	2	2	2	1	1	1	0
		2	1	0	−1	−2	1	0	−1	0
1	1	1	·	·	·	·	·	·	·	·
1	0	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·
1	−1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
0	1	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	·	·
0	0	·	·	$\sqrt{\frac{2}{3}}$	·	·	·	·	·	$-\frac{1}{\sqrt{3}}$
0	−1	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·
−1	1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
−1	0	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·
−1	−1	·	·	·	·	1	·	·	·	·

..

Higher-J product states

$(J=1)\otimes(J=1)=2\oplus 1\oplus 0$ case

$\left|C^{\,1\;1\;L}_{m_1\,m_2\,M}\right\rangle =$

$1 \otimes 1$		2 2	2 1	2 0	2 −1	2 −2	1 1	1 0	1 −1	0 0
1	1	1	·	·	·	·	·	·	·	·
1	0	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·
1	−1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
0	1	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	·	·
0	0	·	·	$\sqrt{\frac{2}{3}}$	·	·	·	·	·	$-\frac{1}{\sqrt{3}}$
0	−1	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·
−1	1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
−1	0	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·
−1	−1	·	·	·	·	1	·	·	·	·

..

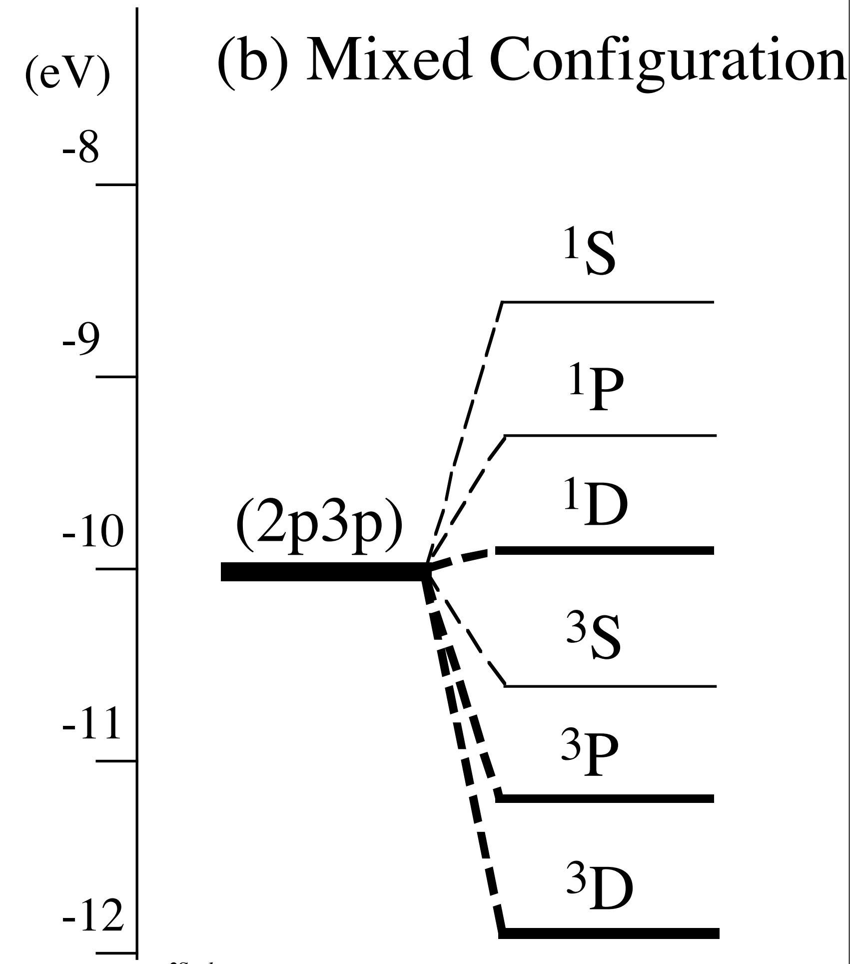


Figure 24.1.3 *Atomic* $^{2S+1}$L *multiplet levels for two* (ℓ = 1) p *electrons.*

$(J=1)\otimes(J=1)=2\oplus 1\oplus 0$ case

$\left|C^{\,1\ \ 1\ \ L}_{\,m_1\,m_2\,M}\right\rangle =$

$1\otimes 1$		$L=2$	2	2	2	2	1	1	1	0
m_1	m_2	$M=2$	1	0	−1	−2	1	0	−1	0
1	1	1	·	·	·	·	·	·	·	·
1	0	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·
1	−1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
0	1	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	·	·
0	0	·	·	$\sqrt{\frac{2}{3}}$	·	·	·	·	·	$-\frac{1}{\sqrt{3}}$
0	−1	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·
−1	1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
−1	0	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·
−1	−1	·	·	·	·	1	·	·	·	·

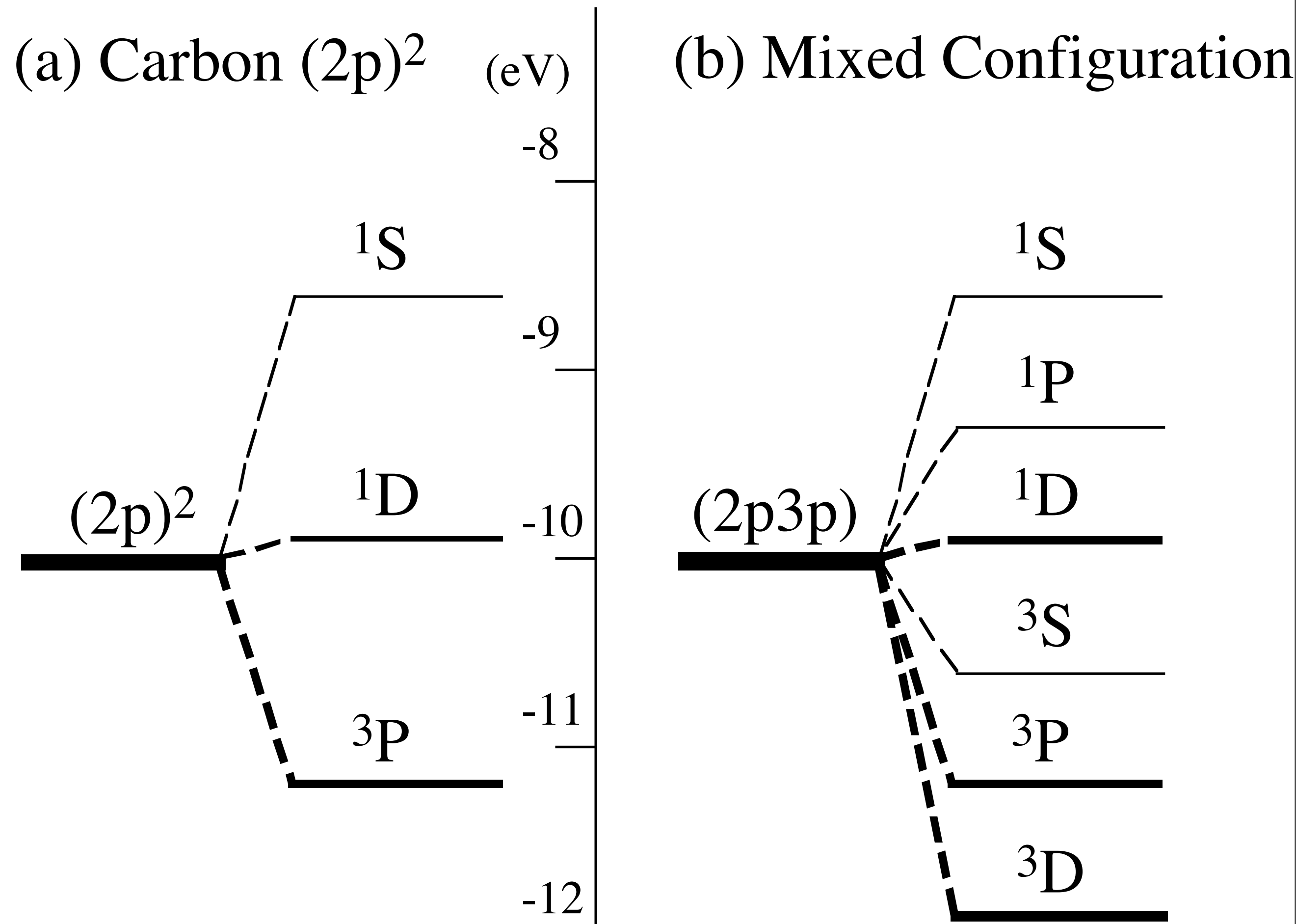


Figure 24.1.3 *Atomic* ^{2S+1}L *multiplet levels for two* ($l = 1$) p *electrons.*

Higher-J product states

$(J{=}1)\otimes(J{=}1)=2\oplus1\oplus0$ case

$$\left|C^{\,1\ \,1\ \,L}_{m_1\,m_2\,M}\right\rangle =$$

$1\ \otimes\ 1$		2	2	2	2	2	1	1	1	0
		2	1	0	−1	−2	1	0	−1	0
1	1	1	·	·	·	·	·	·	·	·
1	0	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·
1	−1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
0	1	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	·	·
0	0	·	·	$\sqrt{\frac{2}{3}}$	·	·	·	·	·	$-\frac{1}{\sqrt{3}}$
0	−1	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·
−1	1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
−1	0	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·
−1	−1	·	·	·	·	1	·	·	·	·

..

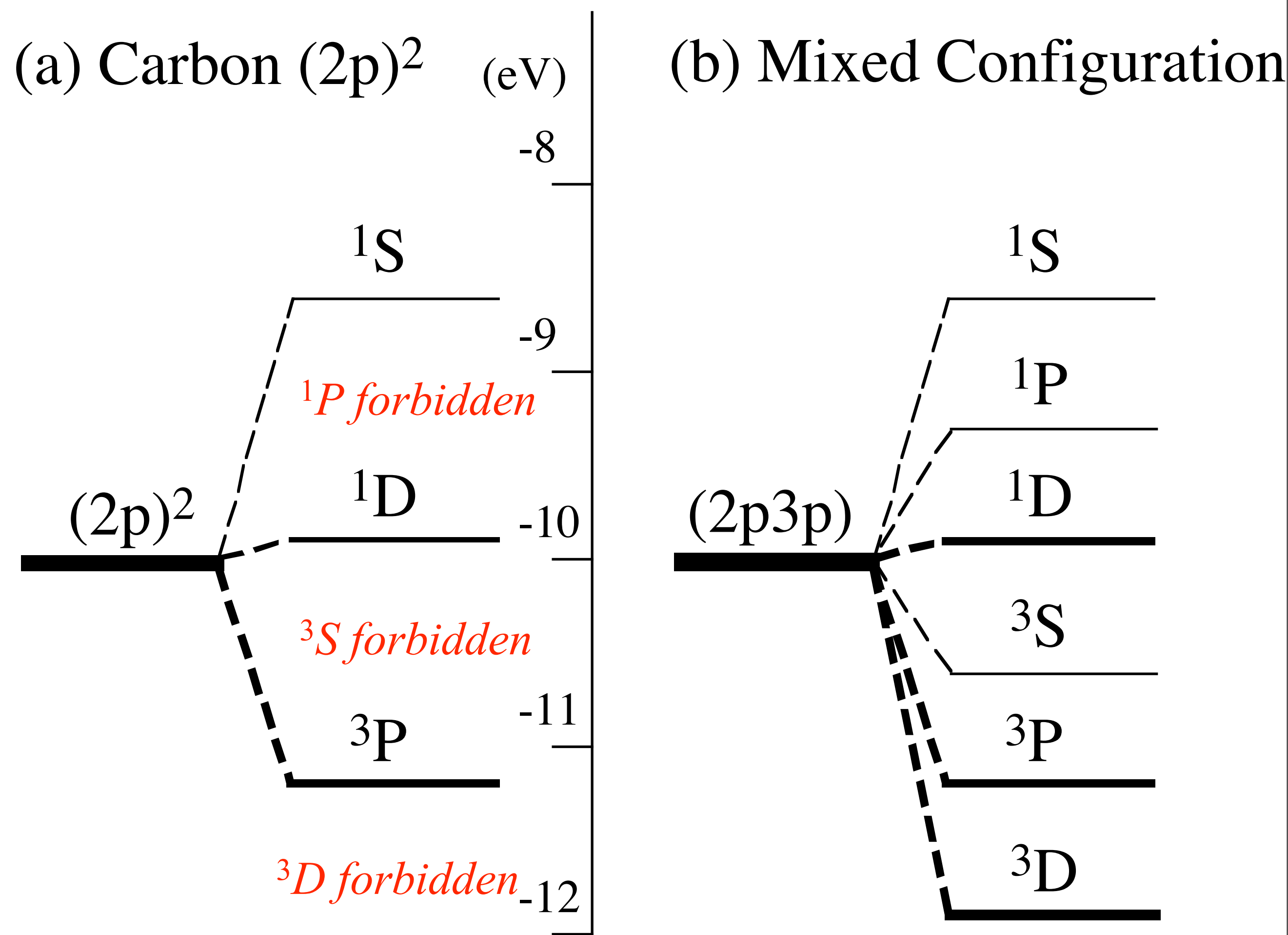


Figure 24.1.3 *Atomic* $^{2S+1}$L *multiplet levels for two (ℓ = 1)* p *electrons.*

Pauli-Fermi *selection rules*
requires total anti-symmetry

Higher-J product states

$(J=1)\otimes(J=1)=2\oplus 1\oplus 0$ case

$$\left|C^{1\ \ 1\ \ L}_{m_1 m_2 M}\right\rangle =$$

$1 \otimes 1$		2, 2	2, 1	2, 0	2, −1	2, −2	1, 1	1, 0	1, −1	0, 0
1	1	1	·	·	·	·	·	·	·	·
1	0	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·
1	−1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
0	1	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	·	·
0	0	·	·	$\sqrt{\frac{2}{3}}$	·	·	·	·	·	$-\frac{1}{\sqrt{3}}$
0	−1	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·
−1	1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
−1	0	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·
−1	−1	·	·	·	·	1	·	·	·	·

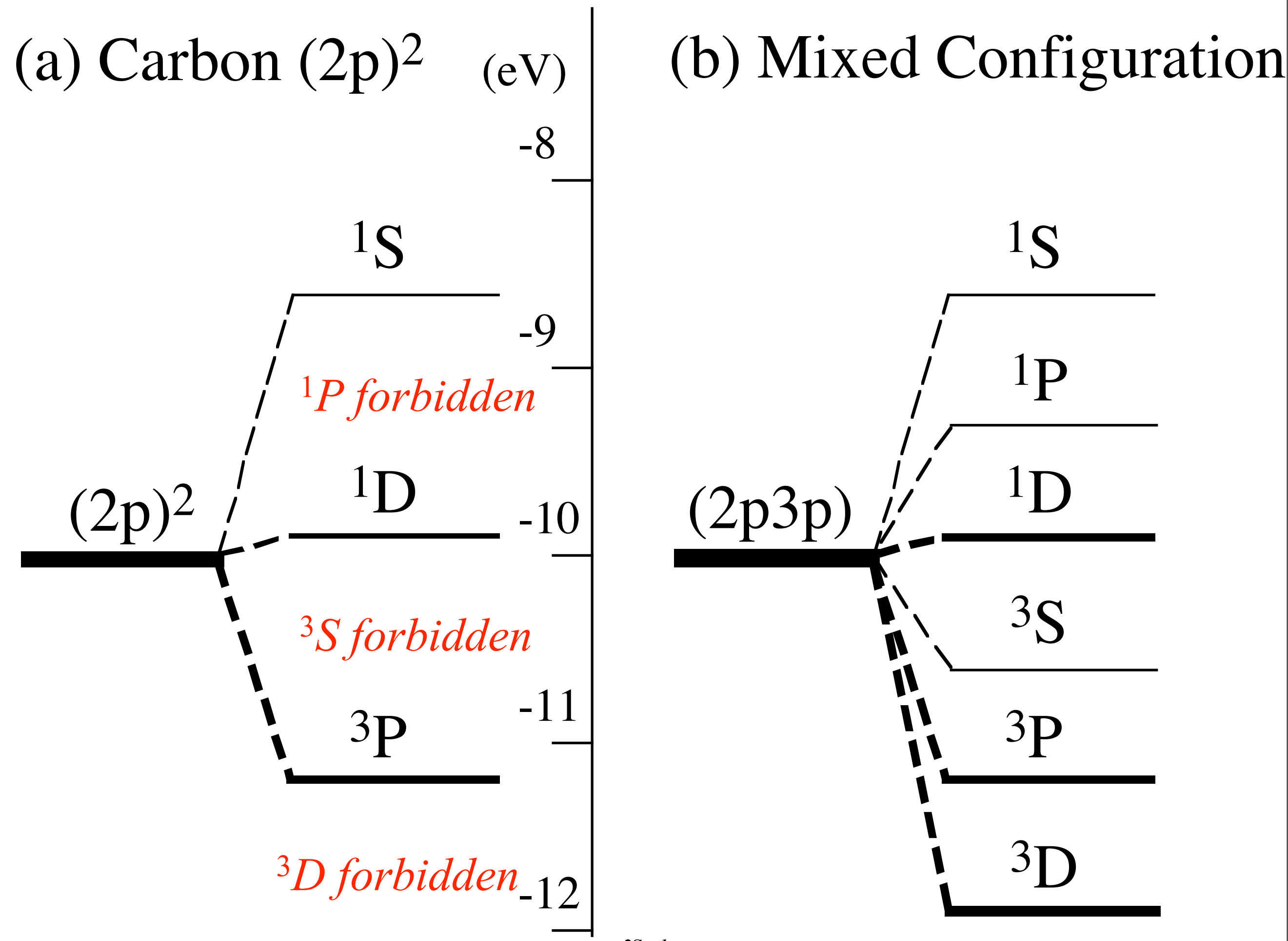
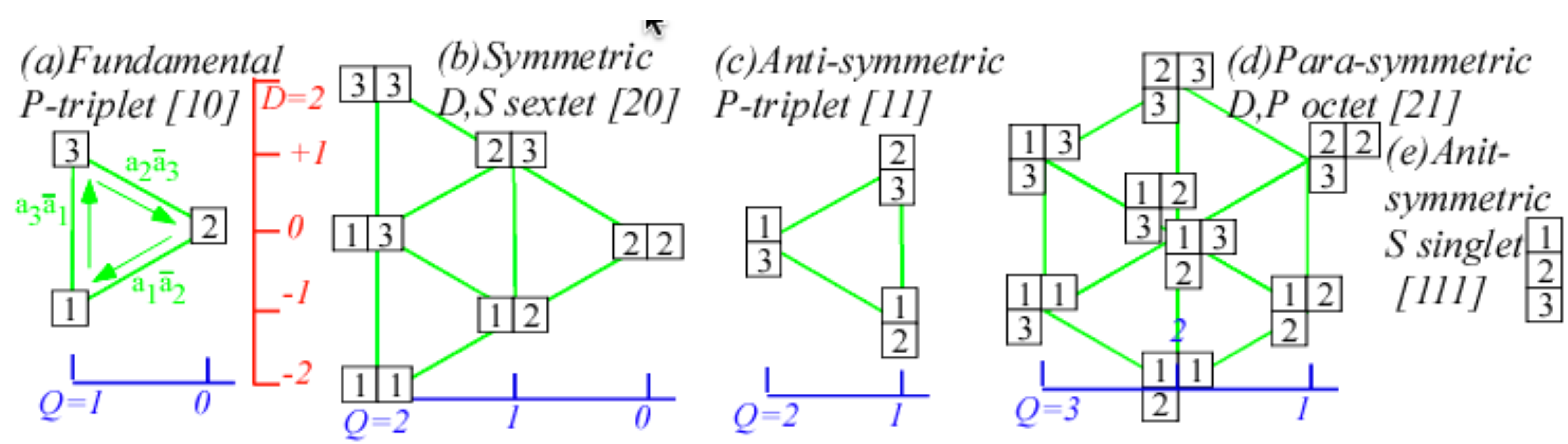


Figure 24.1.3 *Atomic* $^{2S+1}$L *multiplet levels for two (*l *= 1)* p *electrons.*

Pauli-Fermi *selection rules*
requires total anti-symmetry



Higher-J product states

(J=1)⊗(J=1)=2⊕1⊕0 case

$\left|C^{1\ \ 1\ \ L}_{m_1\ m_2\ M}\right\rangle =$

$1\otimes 1$		2	2	2	2	2	1	1	1	0
		2	1	0	−1	−2	1	0	−1	0
1	1	1	·	·	·	·	·	·	·	·
1	0	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·
1	−1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
0	1	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	·	·
0	0	·	·	$\sqrt{\frac{2}{3}}$	·	·	·	·	·	$-\frac{1}{\sqrt{3}}$
0	−1	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·
−1	1	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
−1	0	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·
−1	−1	·	·	·	·	1	·	·	·	·

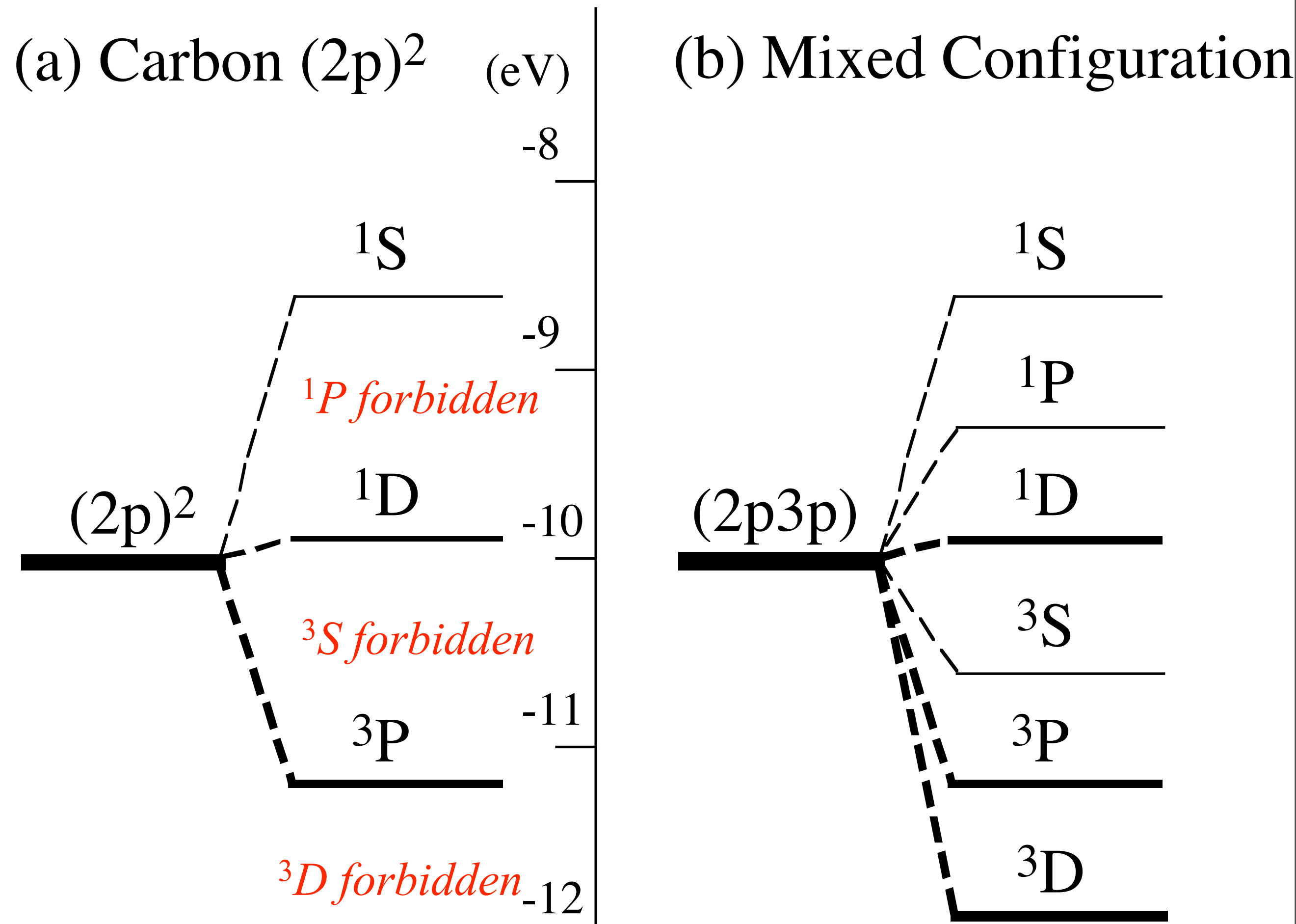


Figure 24.1.3 *Atomic* $^{2S+1}$L *multiplet levels for two (l = 1)* p *electrons.*

Pauli-Fermi *selection rules*
requires total anti-symmetry

General U(2) case

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1-j_2-m_3} C^{j_1\ \ j_2\ \ j_3}_{m_1\ m_2\ m_3} \Big/ (2j_3+1)^{\frac{1}{2}}$$

Wigner 3j vs. Clebsch-Gordon (CGC)

Higher-J product states

$(J=1)\otimes(J=1)=2\oplus 1\oplus 0$ case

$\left|C^{1\ \ 1\ \ L}_{m_1 m_2 M}\right\rangle =$

$1\ \otimes\ 1$	$J=2,\ M=2$	$2,\ 1$	$2,\ 0$	$2,\ -1$	$2,\ -2$	$1,\ 1$	$1,\ 0$	$1,\ -1$	$0,\ 0$
$1\ \ 1$	1	·	·	·	·	·	·	·	·
$1\ \ 0$	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·
$1\ \ -1$	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
$0\ \ 1$	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	·	·
$0\ \ 0$	·	·	$\sqrt{\frac{2}{3}}$	·	·	·	·	·	$-\frac{1}{\sqrt{3}}$
$0\ \ -1$	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$\frac{1}{\sqrt{2}}$	·
$-1\ \ 1$	·	·	$\frac{1}{\sqrt{6}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·	$\frac{1}{\sqrt{3}}$
$-1\ \ 0$	·	·	·	$\frac{1}{\sqrt{2}}$	·	·	·	$-\frac{1}{\sqrt{2}}$	·
$-1\ \ -1$	·	·	·	·	1	·	·	·	·

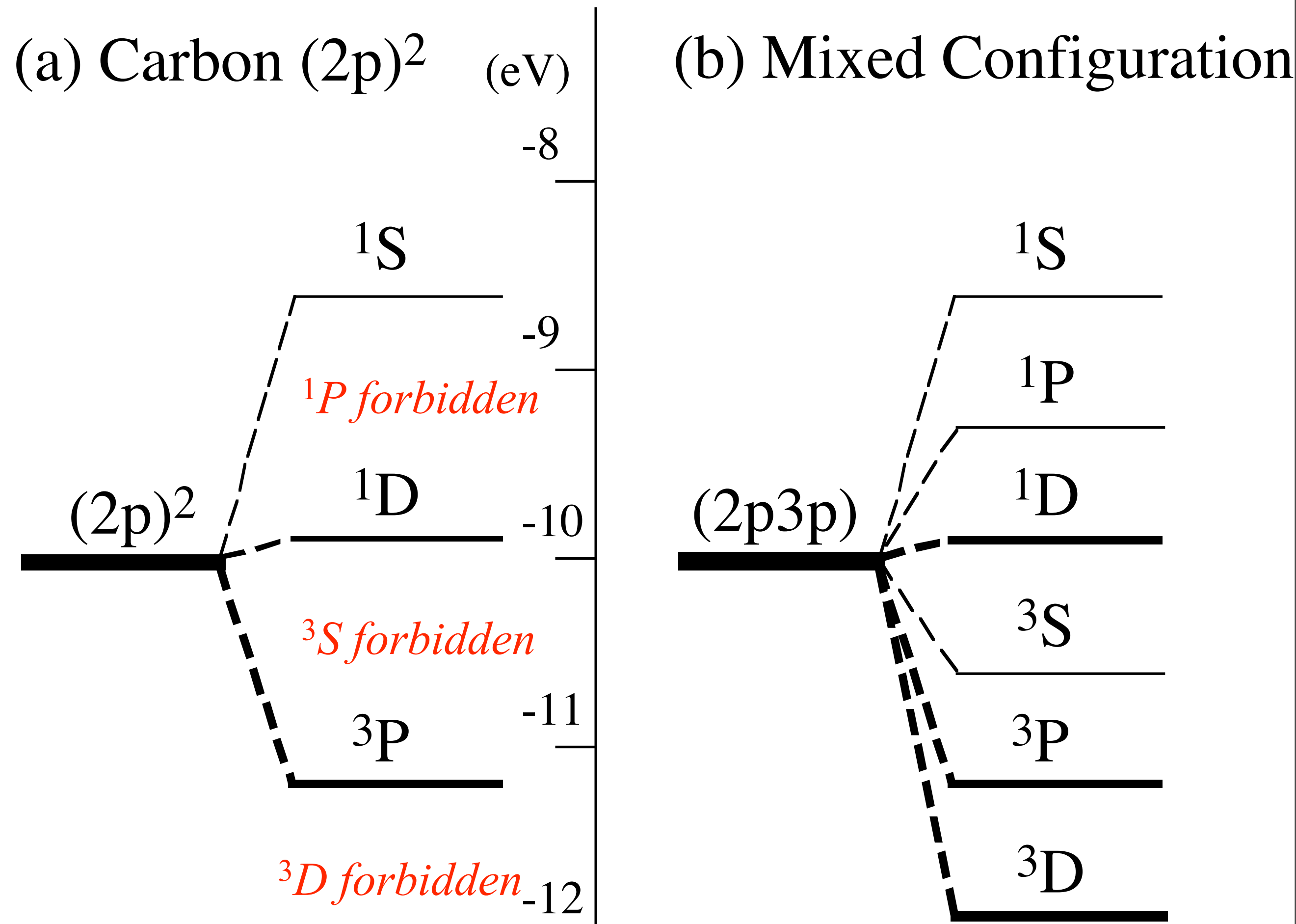


Figure 24.1.3 *Atomic* $^{2S+1}$L *multiplet levels for two (l = 1)* p *electrons.*

Pauli-Fermi *selection rules requires total anti-symmetry*

General U(2) case

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1-j_2-m_3}\, C^{j_1\ \ j_2\ \ j_3}_{m_1\ m_2\ m_3} \Big/ (2j_3+1)^{\frac{1}{2}}$$

Wigner 3j vs. Clebsch-Gordon (CGC)

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1-j_2-n_3} \sqrt{\frac{(j_1+j_2-j_3)!(j_1-j_2+j_3)(-j_1+j_2+j_3)}{(j_1+j_2+j_3+1)!}}$$

$$\sum_k \frac{(-1)^k}{k!} \frac{\sqrt{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(j_3+m_3)!(j_3-m_3)!}}{(j_1-m_1-k)!(j_2-m_2-k)!(j_1+j_2-j_3-k)!(j_3-j_2-m_1+k)!(j_3-j_1-m_2+k)!}$$

Higher-J product states

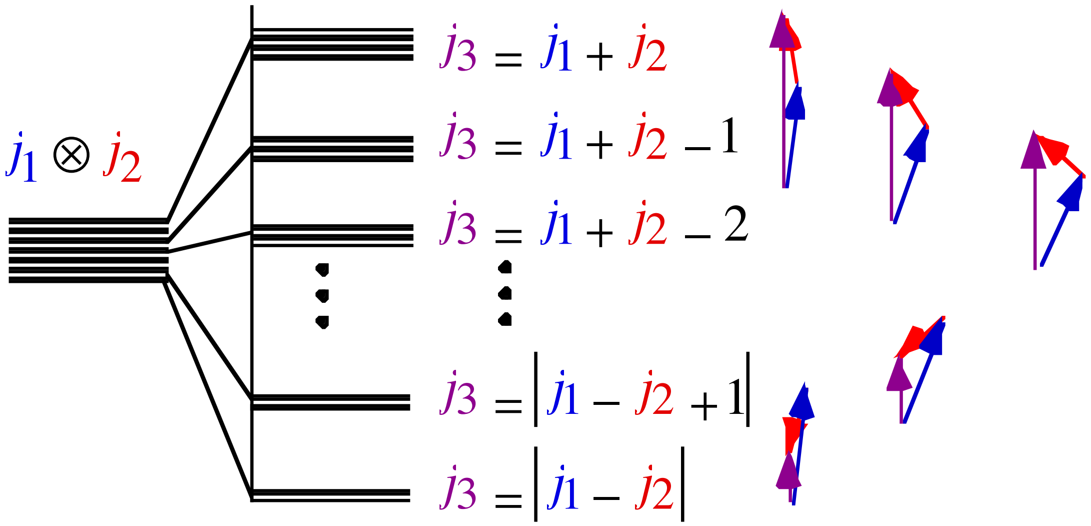


Figure 24.1.6 Level-splitting and vector-addition picture of angular-momentum coupling.

Higher-J product states

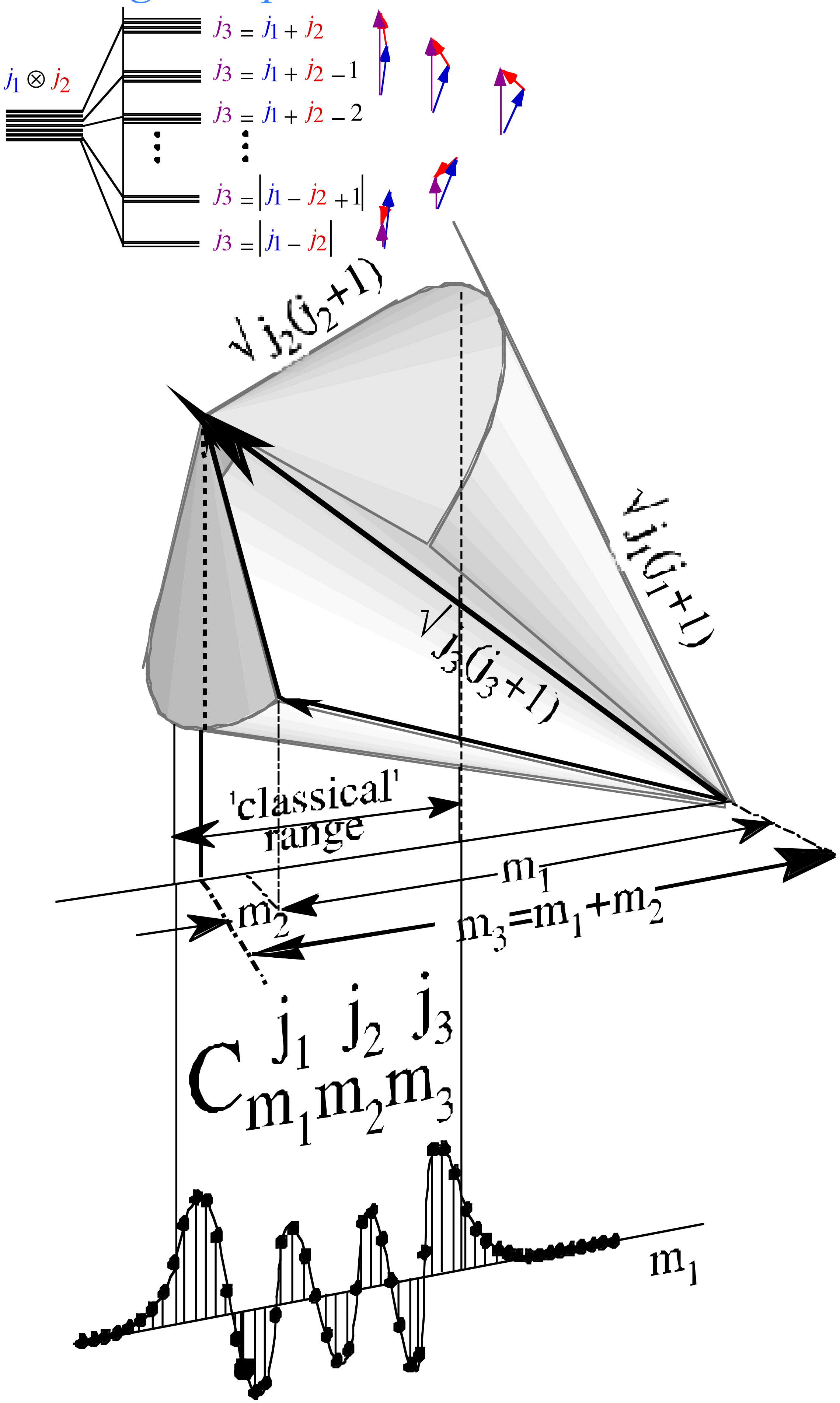


Figure 24.1.7 Angular-momentum cone picture of Clebsch-Gordan coupling amplitudes.

Higher-J product states

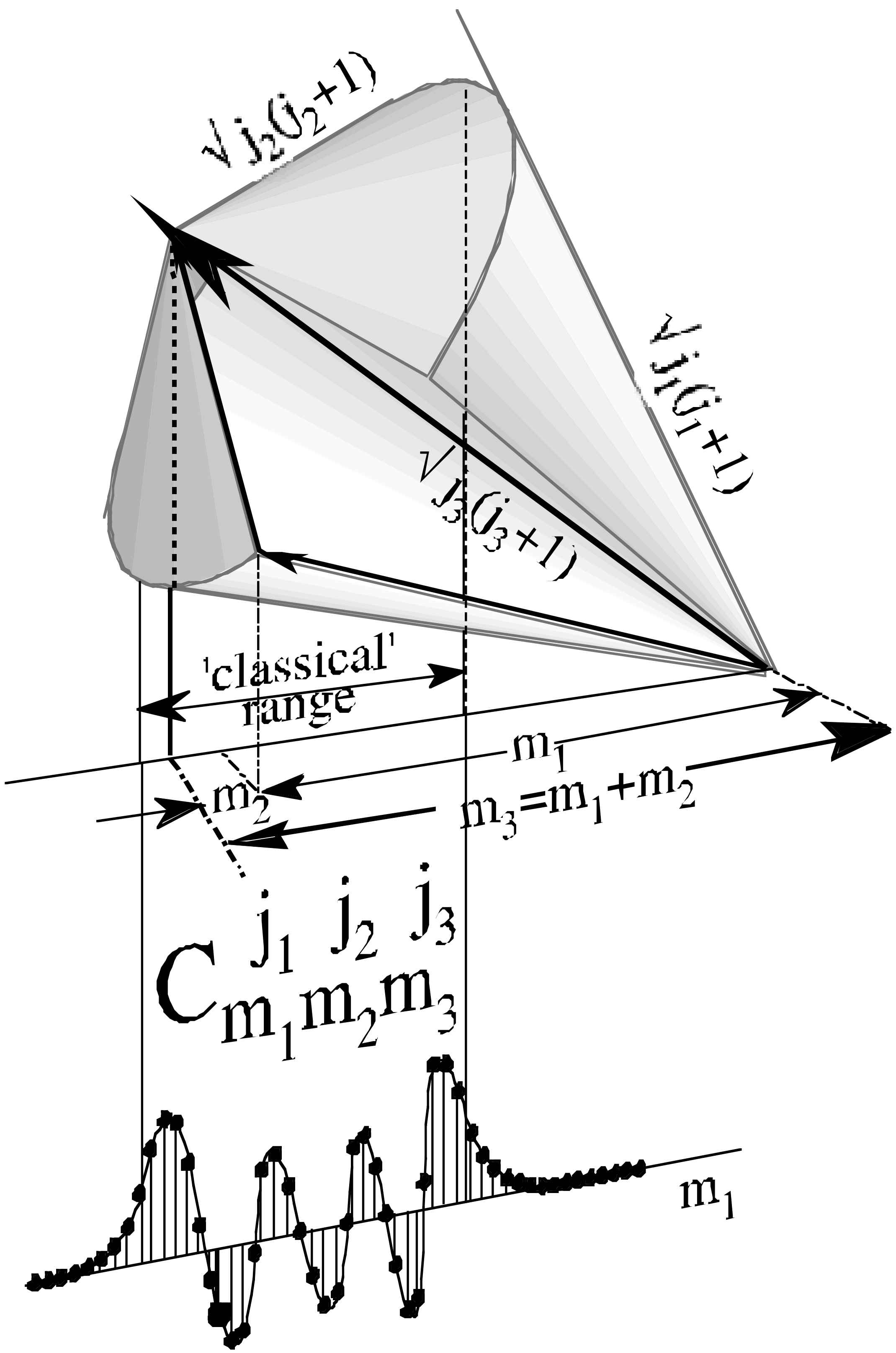


Figure 24.1.7 Angular-momentum cone picture of Clebsch-Gordan coupling amplitudes.

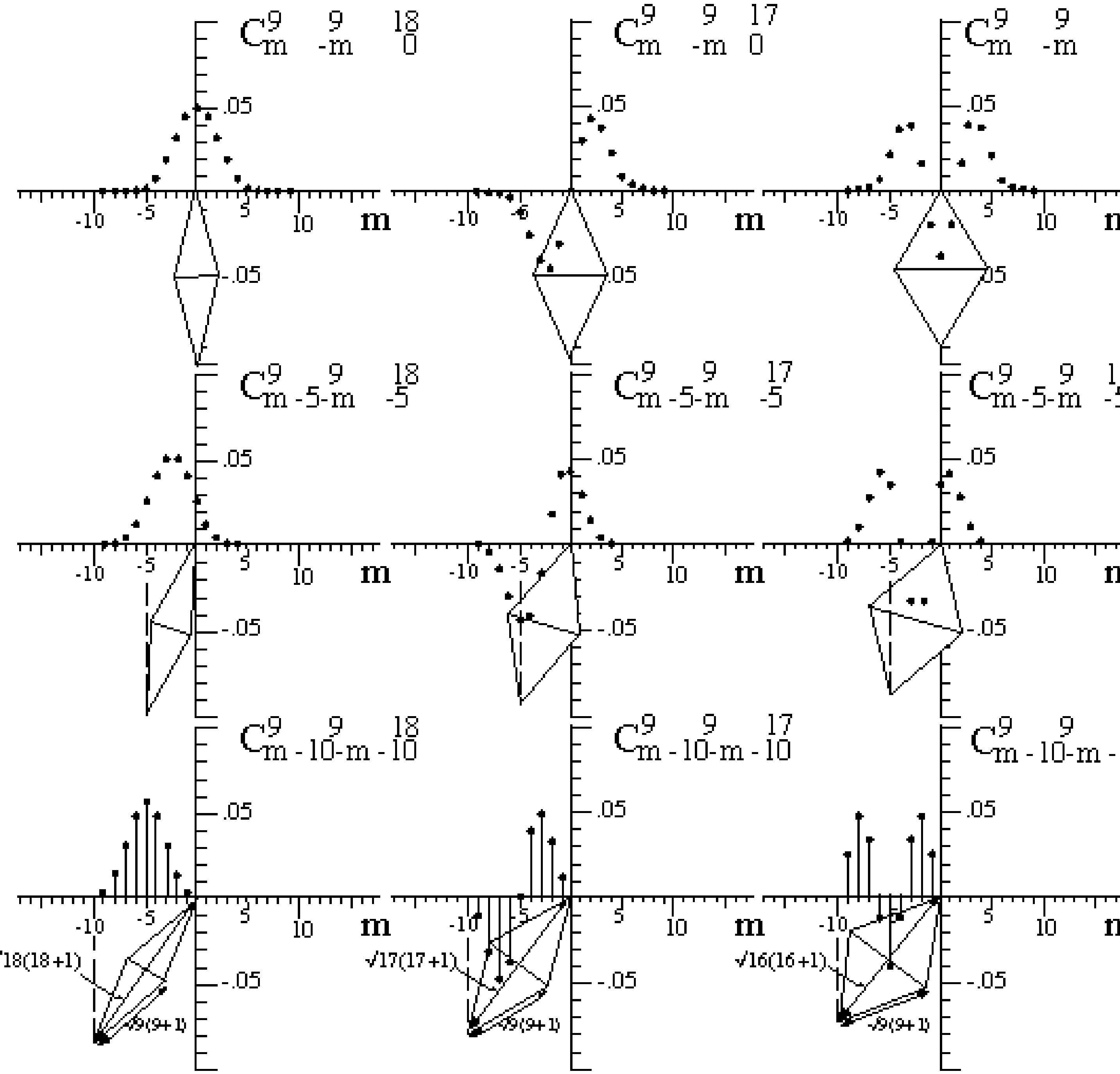
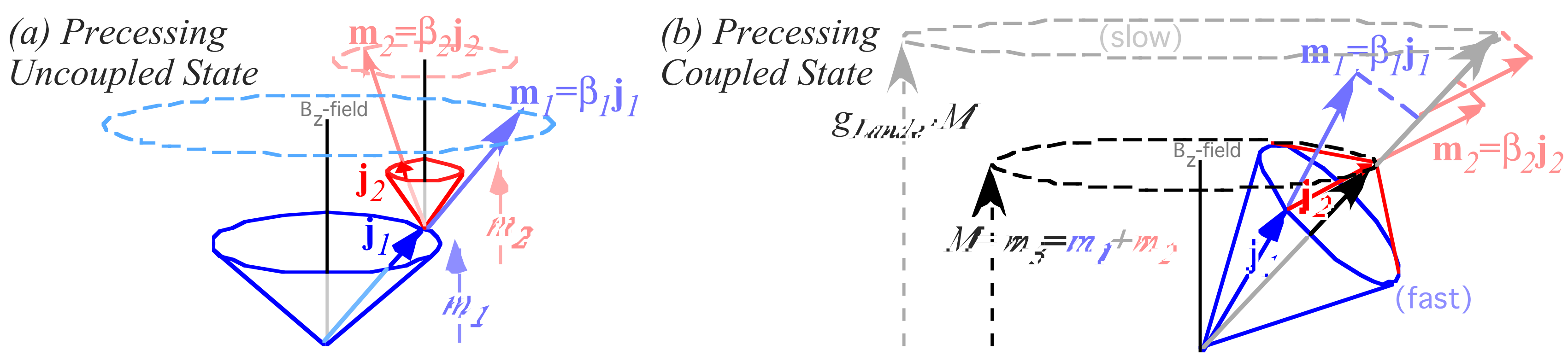


Figure 24.1.8 Clebsch-Gordan coefficients plotted next to their angular-momentum cones.

Higher-J product states



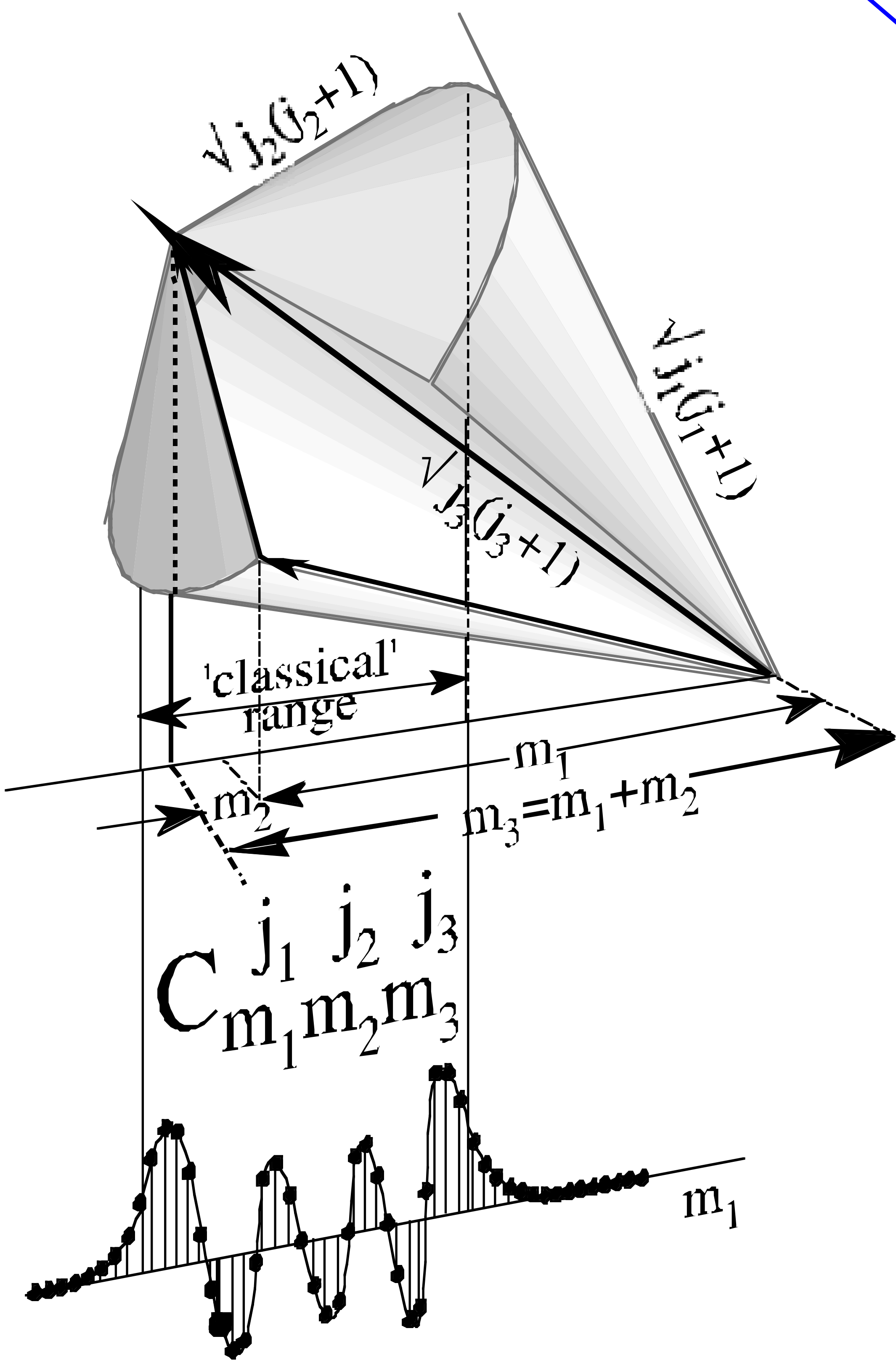


Figure 24.1.7 *Angular-momentum cone picture of Clebsch-Gordan coupling amplitudes.*

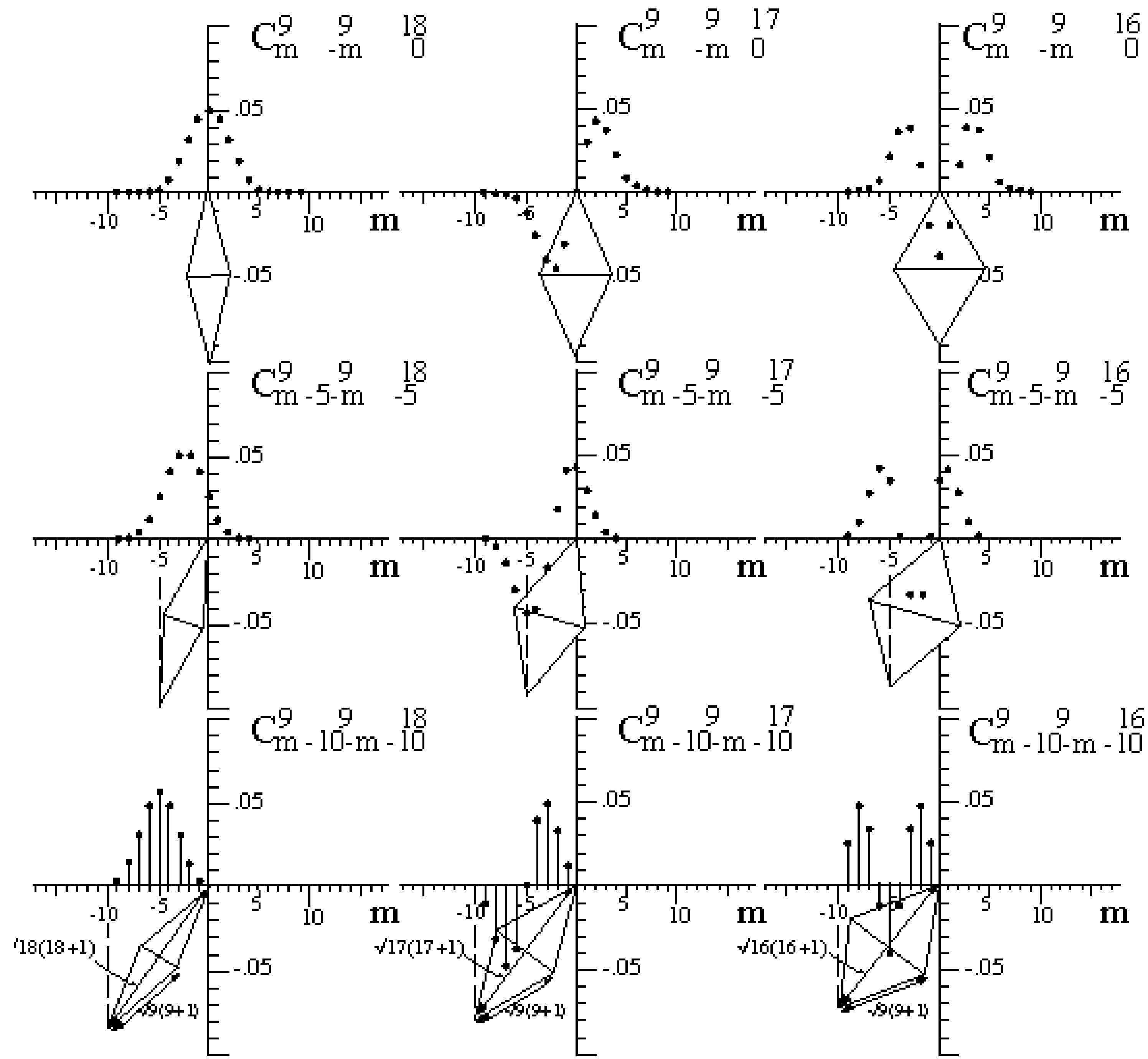


Figure 24.1.8 *Clebsch-Gordan coefficients plotted next to their angular-momentum cones.*

Multi-spin (1/2)^N product states

$$\left(\frac{1}{2}\otimes\frac{1}{2}\right)\otimes\frac{1}{2}$$

Multi-spin $(1/2)^N$ product states

$$\left(\frac{1}{2}\otimes\frac{1}{2}\right)\otimes\frac{1}{2}=(0\oplus 1)\otimes\frac{1}{2}$$

Multi-spin $(1/2)^N$ product states

$$\left(\frac{1}{2}\otimes\frac{1}{2}\right)\otimes\frac{1}{2}=\left(0\oplus 1\right)\otimes\frac{1}{2}=\left(0\otimes\frac{1}{2}\right)\oplus\quad\left(1\otimes\frac{1}{2}\right)$$

Multi-spin (1/2)^N product states

$$\left(\frac{1}{2}\otimes\frac{1}{2}\right)\otimes\frac{1}{2}=(0\oplus 1)\otimes\frac{1}{2}=\left(0\otimes\frac{1}{2}\right)\oplus\left(1\otimes\frac{1}{2}\right)$$

$$=\left(\frac{1}{2}\right)\oplus\left(\left(\frac{1}{2}\right)\oplus\left(\frac{3}{2}\right)\right)$$

Multi-spin (1/2)^N product states

$$\left(\frac{1}{2}\otimes\frac{1}{2}\right)\otimes\frac{1}{2}=(0\oplus 1)\otimes\frac{1}{2}=\left(0\otimes\frac{1}{2}\right)\oplus\quad\left(1\otimes\frac{1}{2}\right)$$

$$=\quad\left(\frac{1}{2}\right)\quad\oplus\left(\left(\frac{1}{2}\right)\oplus\left(\frac{3}{2}\right)\right)=\frac{1}{2}\oplus\ \frac{1}{2}\oplus\frac{3}{2}$$

Multi-spin $(1/2)^N$ product states

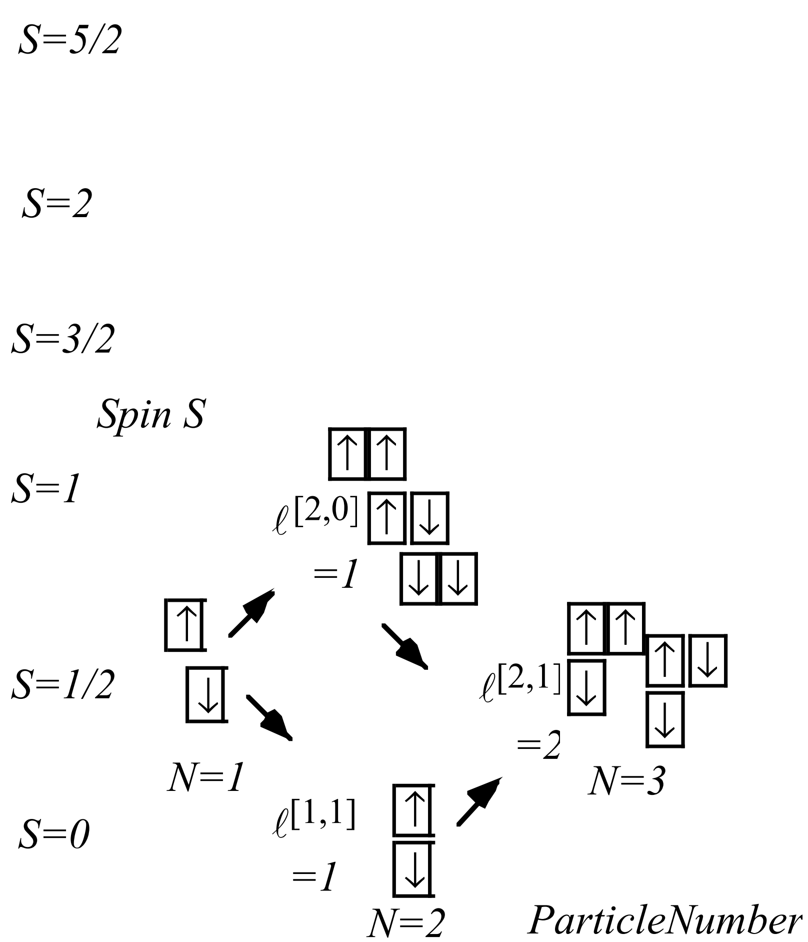
$$\left(\frac{1}{2}\otimes\frac{1}{2}\right)\otimes\frac{1}{2}=(0\oplus 1)\otimes\frac{1}{2}=\left(0\otimes\frac{1}{2}\right)\oplus\quad\left(1\otimes\frac{1}{2}\right)$$

$$=\quad\left(\frac{1}{2}\right)\quad\oplus\left(\left(\frac{1}{2}\right)\oplus\left(\frac{3}{2}\right)\right)=\frac{1}{2}\oplus\;\frac{1}{2}\oplus\frac{3}{2}=2\left(\frac{1}{2}\right)\oplus 1\left(\frac{3}{2}\right)$$

Multi-spin $(1/2)^N$ product states

$$\left(\frac{1}{2}\otimes\frac{1}{2}\right)\otimes\frac{1}{2}=(0\oplus 1)\otimes\frac{1}{2}=\left(0\otimes\frac{1}{2}\right)\oplus\left(1\otimes\frac{1}{2}\right)$$

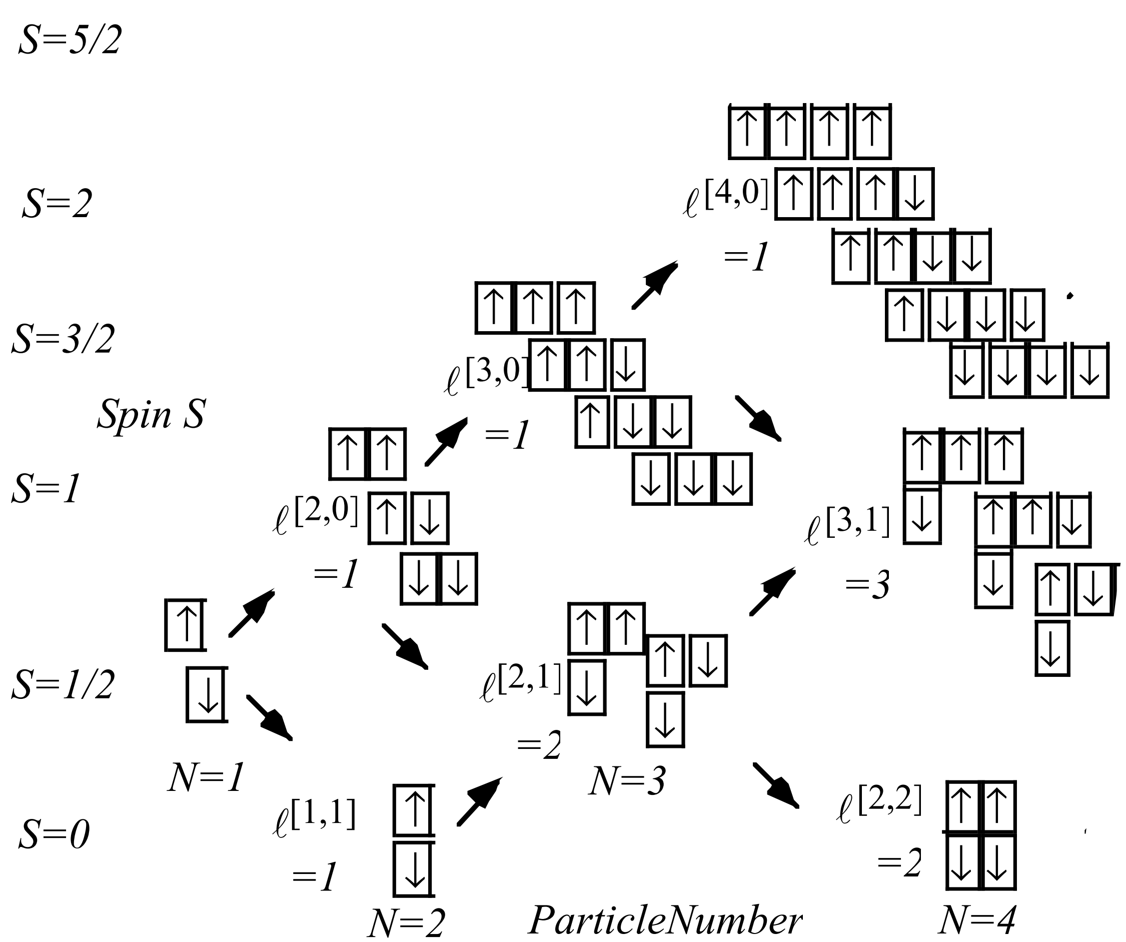
$$=\left(\frac{1}{2}\right)\oplus\left(\left(\frac{1}{2}\right)\oplus\left(\frac{3}{2}\right)\right)=\frac{1}{2}\oplus\frac{1}{2}\oplus\frac{3}{2}=2\left(\frac{1}{2}\right)\oplus 1\left(\frac{3}{2}\right)$$



Multi-spin $(1/2)^N$ product states

$$\left(\frac{1}{2}\otimes\frac{1}{2}\right)\otimes\frac{1}{2}=(0\oplus 1)\otimes\frac{1}{2}=\left(0\otimes\frac{1}{2}\right)\oplus\left(1\otimes\frac{1}{2}\right)$$

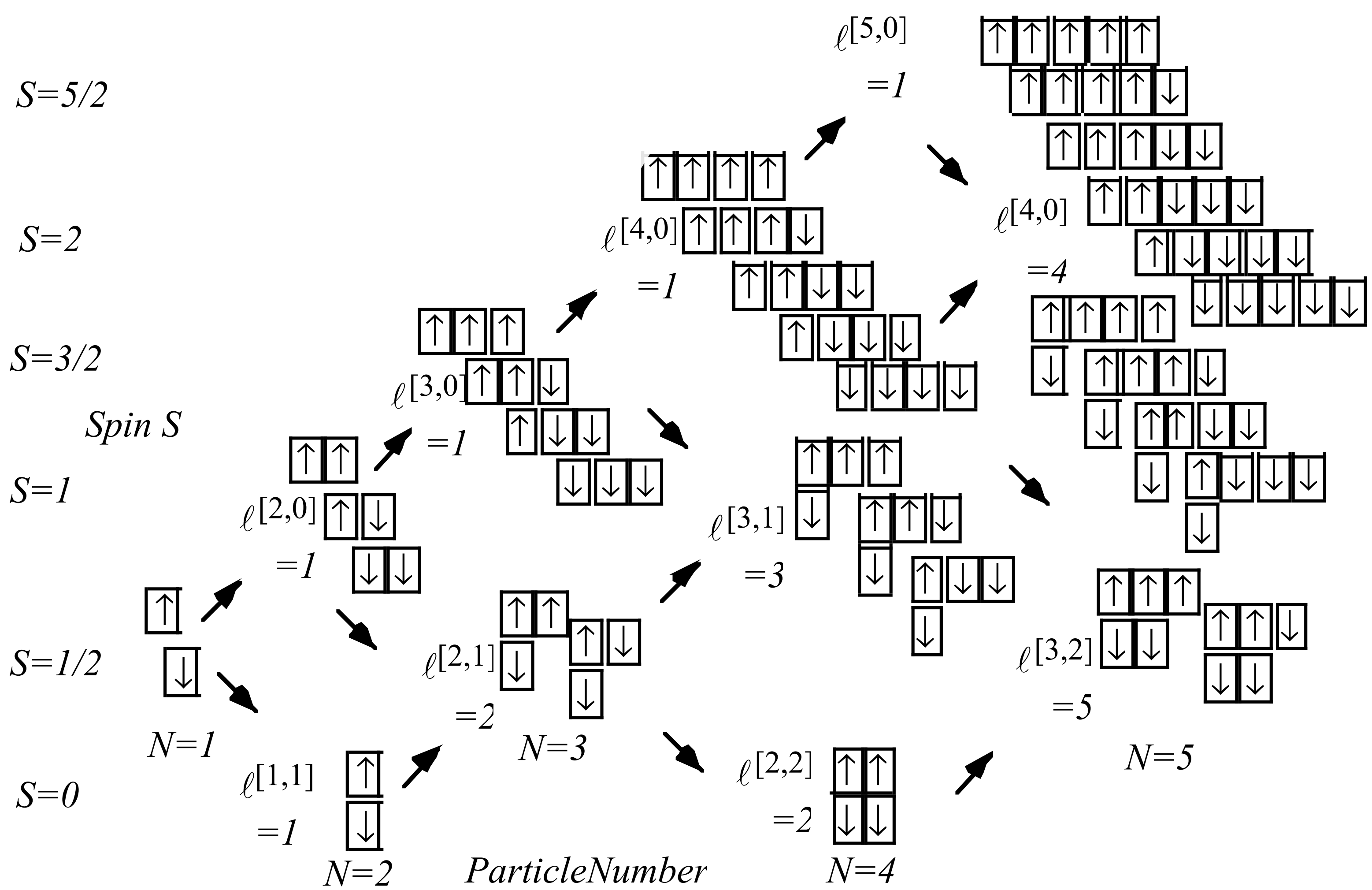
$$=\left(\frac{1}{2}\right)\oplus\left(\left(\frac{1}{2}\right)\oplus\left(\frac{3}{2}\right)\right)=\frac{1}{2}\oplus\frac{1}{2}\oplus\frac{3}{2}=2\left(\frac{1}{2}\right)\oplus 1\left(\frac{3}{2}\right)$$



Multi-spin $(1/2)^N$ product states

$$\left(\frac{1}{2}\otimes\frac{1}{2}\right)\otimes\frac{1}{2}=(0\oplus 1)\otimes\frac{1}{2}=\left(0\otimes\frac{1}{2}\right)\oplus\left(1\otimes\frac{1}{2}\right)$$

$$=\left(\frac{1}{2}\right)\oplus\left(\left(\frac{1}{2}\right)\oplus\left(\frac{3}{2}\right)\right)=\frac{1}{2}\oplus\frac{1}{2}\oplus\frac{3}{2}=2\left(\frac{1}{2}\right)\oplus 1\left(\frac{3}{2}\right)$$



Multi-spin $(1/2)^N$ product states

$$\left|\begin{array}{cccccc}\boxed{\uparrow} & \boxed{\uparrow} & \boxed{\uparrow} & \boxed{\uparrow} & \boxed{\uparrow} & \boxed{\downarrow} \\ \boxed{\downarrow} & \boxed{\downarrow} & & & & \end{array} \begin{array}{c} S=2 \\ M_S=1 \end{array}\right\rangle = C^{5/2\ 1/2\ 2}_{1/2\ 1/2\ 1}\left|\begin{array}{cccccc}\boxed{\uparrow} & \boxed{\uparrow} & \boxed{\uparrow} & \boxed{\uparrow} & \boxed{\downarrow} & \boxed{\downarrow} \\ \boxed{\downarrow} & \boxed{\cdot\cdot} & & & & \end{array} \begin{array}{c} 5/2 \\ 1/2 \end{array}\right\rangle \left|\boxed{\uparrow}^{1/2}_{1/2}\right\rangle + C^{5/2\ 1/2\ 2}_{3/2\ -1/2\ 1}\left|\begin{array}{cccccc}\boxed{\uparrow} & \boxed{\uparrow} & \boxed{\uparrow} & \boxed{\uparrow} & \boxed{\uparrow} & \boxed{\downarrow} \\ \boxed{\downarrow} & \boxed{\cdot\cdot} & & & & \end{array} \begin{array}{c} 5/2 \\ 1/2 \end{array}\right\rangle \left|\boxed{\downarrow}^{1/2}_{-1/2}\right\rangle$$

$$+C^{3/2\ 1/2\ 2}_{1/2\ 1/2\ 1}\left|\begin{array}{cccccc}\boxed{\uparrow} & \boxed{\uparrow} & \boxed{\uparrow} & \boxed{\uparrow} & \boxed{\downarrow} & \boxed{\cdot\cdot} \\ \boxed{\downarrow} & \boxed{\downarrow} & & & & \end{array} \begin{array}{c} 3/2 \\ 1/2 \end{array}\right\rangle \left|\boxed{\uparrow}^{1/2}_{1/2}\right\rangle + C^{3/2\ 1/2\ 2}_{3/2\ -1/2\ 1}\left|\begin{array}{cccccc}\boxed{\uparrow} & \boxed{\uparrow} & \boxed{\uparrow} & \boxed{\uparrow} & \boxed{\uparrow} & \boxed{\cdot\cdot} \\ \boxed{\downarrow} & \boxed{\downarrow} & & & & \end{array} \begin{array}{c} 3/2 \\ 1/2 \end{array}\right\rangle \left|\boxed{\downarrow}^{1/2}_{-1/2}\right\rangle$$

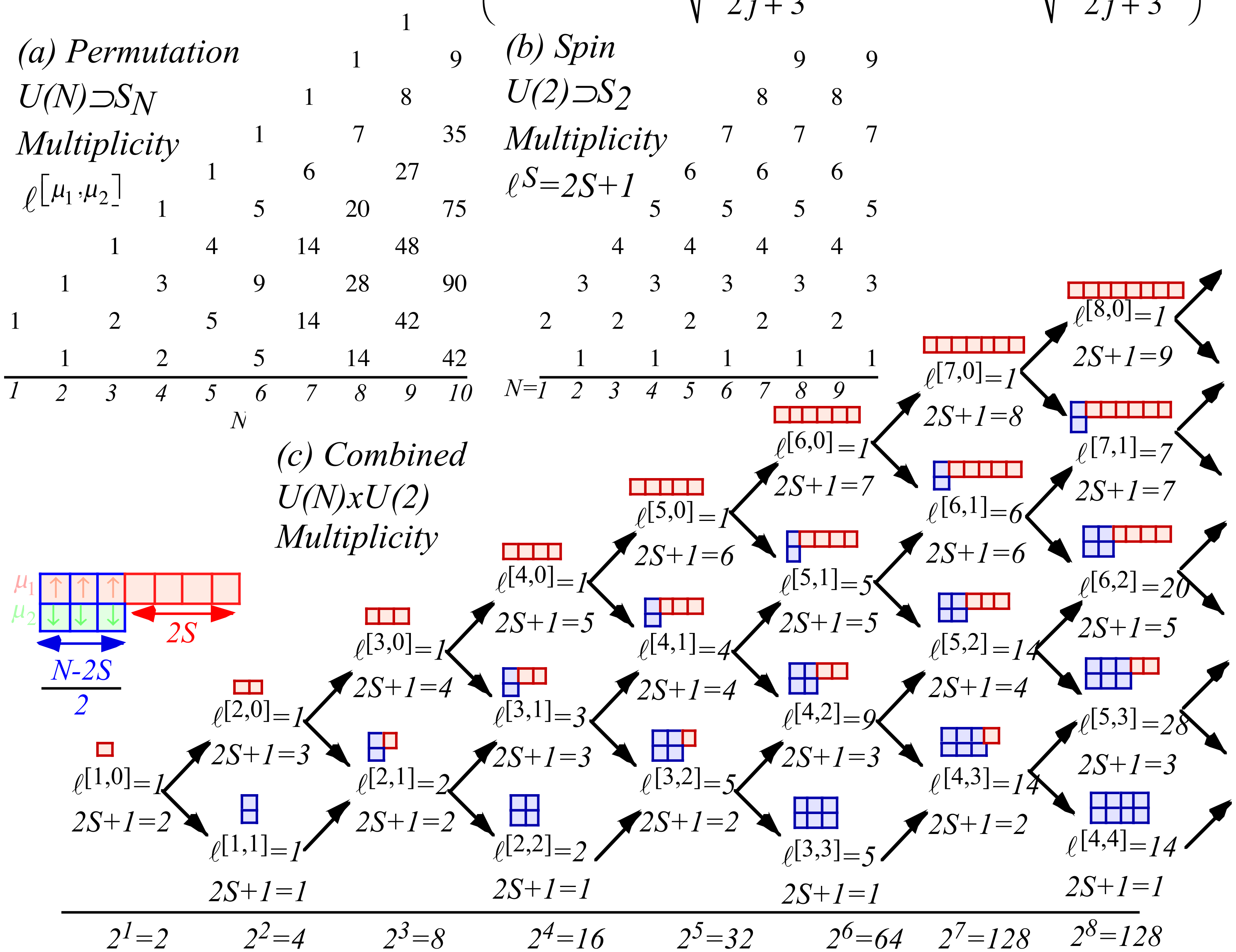
$$\begin{pmatrix} C^{j\ \ 1/2\ \ j+1/2}_{m\ 1/2\ m+1/2} = \sqrt{\dfrac{j+m+1}{2j+1}} & C^{j\ \ \ 1/2\ \ j+1/2}_{m+1\ -1/2\ m+1/2} = \sqrt{\dfrac{j-m}{2j+1}} \\ C^{j+1\ 1/2\ \ j+1/2}_{m\ \ 1/2\ m+1/2} = -\sqrt{\dfrac{j-m+1}{2j+3}} & C^{j+1\ \ 1/2\ \ j+1/2}_{m+1\ -1/2\ m+1/2} = \sqrt{\dfrac{j+m+2}{2j+3}} \end{pmatrix} \quad \text{example:} \quad \begin{pmatrix} C^{3/2\ 1/2\ 2}_{1/2\ 1/2\ 1} = \sqrt{\dfrac{2}{3}} & C^{3/2\ 1/2\ 2}_{1/2\ 1/2\ 1} = \sqrt{\dfrac{1}{3}} \\ C^{5/2\ 1/2\ 2}_{1/2\ 1/2\ 1} = -\sqrt{\dfrac{1}{3}} & C^{5/2\ 1/2\ 2}_{1/2\ 1/2\ 1} = \sqrt{\dfrac{2}{3}} \end{pmatrix}$$

Multi-spin $(1/2)^N$ product states

$$\left|\begin{array}{cccccc}\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\downarrow}\\ \boxed{\downarrow}&\boxed{\downarrow}&&&&\end{array}\;\begin{array}{c}S=2\\ M_S=1\end{array}\right\rangle = C^{5/2\;1/2\;2}_{1/2\;1/2\;1}\left|\begin{array}{cccccc}\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\downarrow}&\boxed{\downarrow}\\ \boxed{\downarrow}&\boxed{\cdot\cdot}&&&&\end{array}\begin{array}{c}5/2\\ 1/2\end{array}\right\rangle\left|\boxed{\uparrow}^{1/2}_{1/2}\right\rangle + C^{5/2\;1/2\;2}_{3/2\;-1/2\;1}\left|\begin{array}{cccccc}\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\downarrow}\\ \boxed{\downarrow}&\boxed{\cdot\cdot}&&&&\end{array}\begin{array}{c}5/2\\ 1/2\end{array}\right\rangle\left|\boxed{\downarrow}^{1/2}_{-1/2}\right\rangle$$

$$+C^{3/2\;1/2\;2}_{1/2\;1/2\;1}\left|\begin{array}{cccccc}\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\downarrow}&\boxed{\cdot\cdot}\\ \boxed{\downarrow}&\boxed{\downarrow}&&&&\end{array}\begin{array}{c}3/2\\ 1/2\end{array}\right\rangle\left|\boxed{\uparrow}^{1/2}_{1/2}\right\rangle + C^{3/2\;1/2\;2}_{3/2\;-1/2\;1}\left|\begin{array}{cccccc}\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\cdot\cdot}\\ \boxed{\downarrow}&\boxed{\downarrow}&&&&\end{array}\begin{array}{c}3/2\\ 1/2\end{array}\right\rangle\left|\boxed{\downarrow}^{1/2}_{-1/2}\right\rangle$$

$$\begin{pmatrix} C^{j\;\;1/2\;\;j+1/2}_{m\;1/2\;\;m+1/2}=\sqrt{\dfrac{j+m+1}{2j+1}} & C^{j\;\;\;\;\;1/2\;\;j+1/2}_{m+1\;-1/2\;\;m+1/2}=\sqrt{\dfrac{j-m}{2j+1}} \\ C^{j+1\;1/2\;\;j+1/2}_{m\;\;\;\;1/2\;\;m+1/2}=-\sqrt{\dfrac{j-m+1}{2j+3}} & C^{j+1\;\;\;1/2\;\;j+1/2}_{m+1\;-1/2\;\;m+1/2}=\sqrt{\dfrac{j+m+2}{2j+3}}\end{pmatrix}\quad \text{example:}\quad \begin{pmatrix} C^{3/2\;1/2\;2}_{1/2\;1/2\;1}=\sqrt{\dfrac{2}{3}} & C^{3/2\;1/2\;2}_{1/2\;1/2\;1}=\sqrt{\dfrac{1}{3}} \\ C^{5/2\;1/2\;2}_{1/2\;1/2\;1}=-\sqrt{\dfrac{1}{3}} & C^{5/2\;1/2\;2}_{1/2\;1/2\;1}=\sqrt{\dfrac{2}{3}}\end{pmatrix}$$



$$2^N=\sum_S^{N/2}\ell^{[S]}\ell^{[\mu_1,\mu_2]}$$

$$=\sum_S^{N/2}(2S+1)\ell^{\left[\frac{N+2S}{2},\frac{N-2S}{2}\right]}$$

Fig. 23.3.2 *Spin-1/2 and U(2) Tableau branching diagrams*

Magic squares - Intro to Young Tableaus

$$\left|\begin{matrix}\boxed{\uparrow}\,\boxed{\uparrow}\,\boxed{\uparrow}\,\boxed{\uparrow}\,\boxed{\uparrow}\,\boxed{\downarrow}\\ \boxed{\downarrow}\,\boxed{\downarrow}\end{matrix}\quad \begin{matrix}S=2\\ M_S=1\end{matrix}\right\rangle = C^{5/2\;1/2\;\;2}_{1/2\;\;1/2\;\;1}\left|\begin{matrix}\boxed{\uparrow}\,\boxed{\uparrow}\,\boxed{\uparrow}\,\boxed{\uparrow}\,\boxed{\downarrow}\,\boxed{\downarrow}\\ \boxed{\downarrow}\,\boxed{\cdot\cdot}\end{matrix}\;\begin{matrix}5/2\\1/2\end{matrix}\right\rangle\left|\boxed{\uparrow}^{1/2}_{1/2}\right\rangle + C^{5/2\;1/2\;\;2}_{3/2\;-1/2\;1}\left|\begin{matrix}\boxed{\uparrow}\,\boxed{\uparrow}\,\boxed{\uparrow}\,\boxed{\uparrow}\,\boxed{\uparrow}\,\boxed{\downarrow}\\ \boxed{\downarrow}\,\boxed{\cdot\cdot}\end{matrix}\;\begin{matrix}5/2\\1/2\end{matrix}\right\rangle\left|\boxed{\downarrow}^{1/2}_{-1/2}\right\rangle$$

$$+C^{3/2\;1/2\;\;2}_{1/2\;\;1/2\;\;1}\left|\begin{matrix}\boxed{\uparrow}\,\boxed{\uparrow}\,\boxed{\uparrow}\,\boxed{\uparrow}\,\boxed{\downarrow}\,\boxed{\cdot\cdot}\\ \boxed{\downarrow}\,\boxed{\downarrow}\end{matrix}\;\begin{matrix}3/2\\1/2\end{matrix}\right\rangle\left|\boxed{\uparrow}^{1/2}_{1/2}\right\rangle + C^{3/2\;1/2\;\;2}_{3/2\;-1/2\;1}\left|\begin{matrix}\boxed{\uparrow}\,\boxed{\uparrow}\,\boxed{\uparrow}\,\boxed{\uparrow}\,\boxed{\uparrow}\,\boxed{\cdot\cdot}\\ \boxed{\downarrow}\,\boxed{\downarrow}\end{matrix}\;\begin{matrix}3/2\\1/2\end{matrix}\right\rangle\left|\boxed{\downarrow}^{1/2}_{-1/2}\right\rangle$$

$$\begin{pmatrix} C^{j\;\;1/2\;\;j+1/2}_{m\;\;1/2\;\;m+1/2}=\sqrt{\dfrac{j+m+1}{2j+1}} & C^{j\;\;\;\;\;\;1/2\;\;\;j+1/2}_{m+1\;-1/2\;\;m+1/2}=\sqrt{\dfrac{j-m}{2j+1}}\\ C^{j+1\;1/2\;\;j+1/2}_{m\;\;\;\;1/2\;\;m+1/2}=-\sqrt{\dfrac{j-m+1}{2j+3}} & C^{j+1\;\;\;1/2\;\;\;j+1/2}_{m+1\;-1/2\;\;m+1/2}=\sqrt{\dfrac{j+m+2}{2j+3}}\end{pmatrix}$$

example:

$$\begin{pmatrix} C^{3/2\;1/2\;\;2}_{1/2\;\;1/2\;\;1}=\sqrt{\dfrac{2}{3}} & C^{3/2\;1/2\;\;2}_{1/2\;\;1/2\;\;1}=\sqrt{\dfrac{1}{3}}\\ C^{5/2\;1/2\;\;2}_{1/2\;\;1/2\;\;1}=-\sqrt{\dfrac{1}{3}} & C^{5/2\;1/2\;\;2}_{1/2\;\;1/2\;\;1}=\sqrt{\dfrac{2}{3}}\end{pmatrix}$$

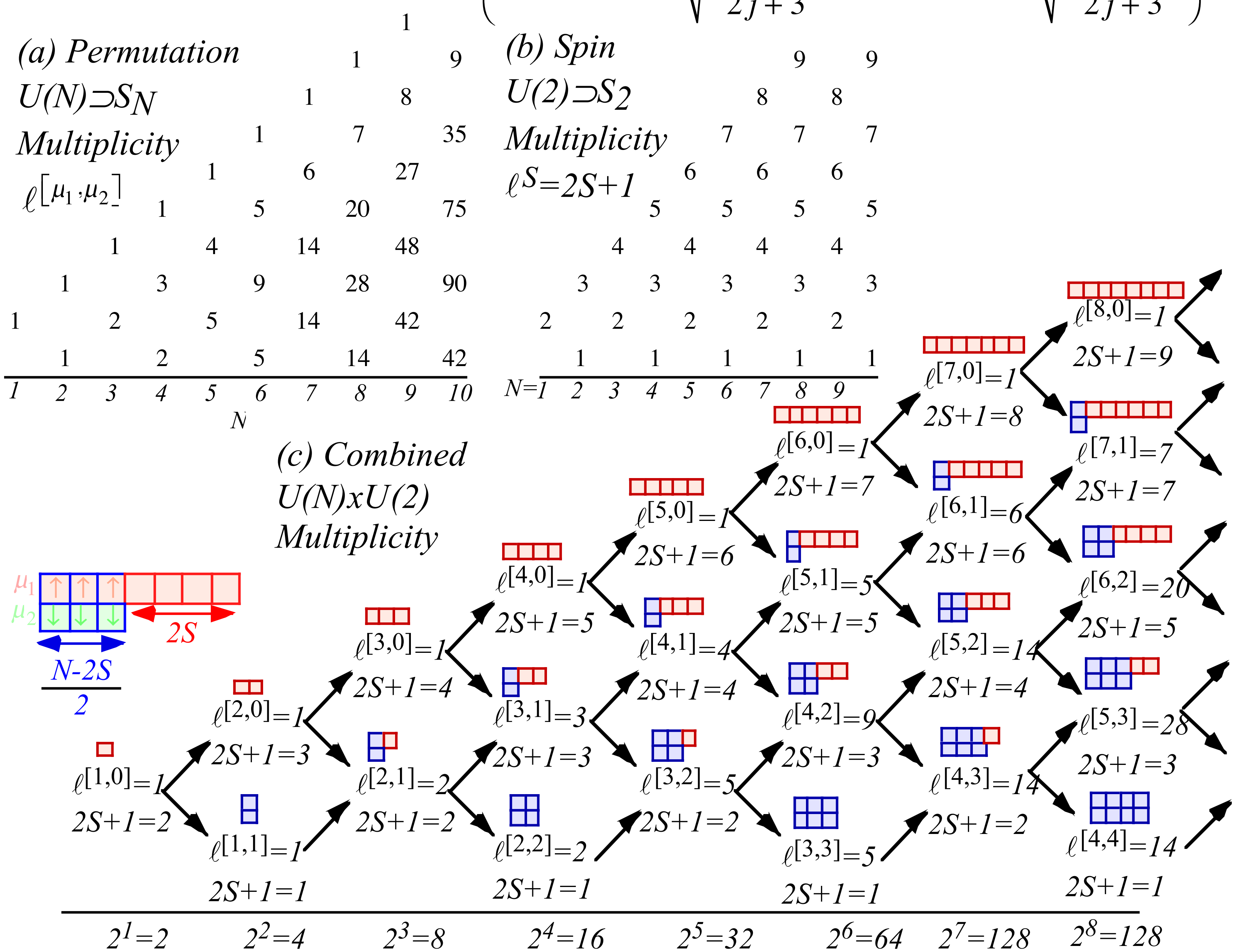


Fig. 23.3.2 *Spin-1/2 and U(2) Tableau branching diagrams*

2	3	4	5
1	2	3	4

———

5	4	3	2
4	3	2	1

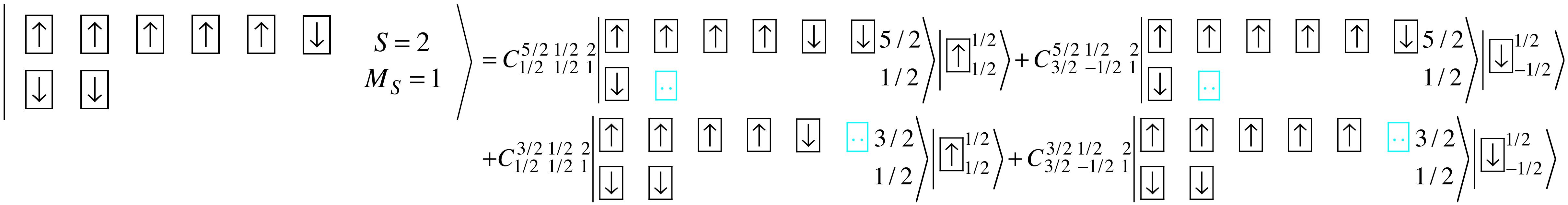
$$2^N=\sum_S^{N/2}\ell^{[S]}\ell^{[\mu_1,\mu_2]}$$

$$=\sum_S^{N/2}(2S+1)\ell^{\left[\frac{N+2S}{2},\frac{N-2S}{2}\right]}$$

$8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1$

———

5	4	3	2
4	3	2	1



$$\begin{pmatrix} C^{j\ \ 1/2\ \ j+1/2}_{m\ \ 1/2\ \ m+1/2} = \sqrt{\frac{j+m+1}{2j+1}} & C^{j\ \ \ 1/2\ \ j+1/2}_{m+1\ -1/2\ \ m+1/2} = \sqrt{\frac{j-m}{2j+1}} \\ C^{j+1\ 1/2\ \ j+1/2}_{m\ \ \ 1/2\ \ m+1/2} = -\sqrt{\frac{j-m+1}{2j+3}} & C^{j+1\ \ 1/2\ \ j+1/2}_{m+1\ -1/2\ \ m+1/2} = \sqrt{\frac{j+m+2}{2j+3}} \end{pmatrix}$$

example:

$$\begin{pmatrix} C^{3/2\ 1/2\ 2}_{1/2\ 1/2\ 1} = \sqrt{\frac{2}{3}} & C^{3/2\ 1/2\ 2}_{1/2\ 1/2\ 1} = \sqrt{\frac{1}{3}} \\ C^{5/2\ 1/2\ 2}_{1/2\ 1/2\ 1} = -\sqrt{\frac{1}{3}} & C^{5/2\ 1/2\ 2}_{1/2\ 1/2\ 1} = \sqrt{\frac{2}{3}} \end{pmatrix}$$

2	3	4	5
1	2	3	4
5	4	3	2
4	3	2	1

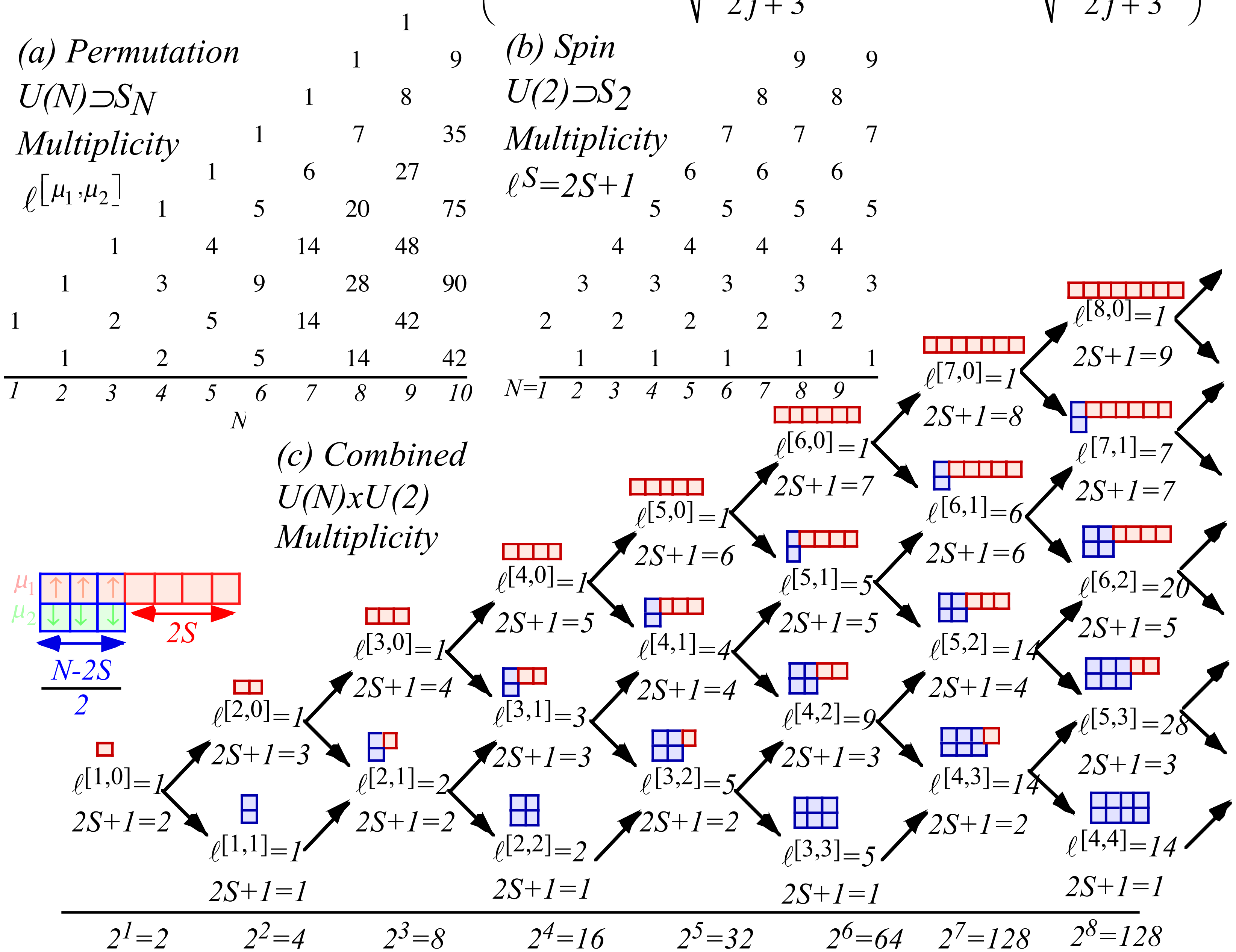


Fig. 23.3.2 Spin-1/2 and U(2) Tableau branching diagrams

$$2^N = \sum_S^{N/2} \ell^{[S]}\ell^{[\mu_1,\mu_2]}$$

$$= \sum_S^{N/2} (2S+1)\ell^{\left[\frac{N+2S}{2},\frac{N-2S}{2}\right]}$$

$8\cdot7\cdot\cancel{6\cdot5\cdot4\cdot3\cdot2\cdot1}$

~~5~~	~~4~~	~~3~~	~~2~~
4	~~3~~	~~2~~	1

$$\left|\begin{array}{cccccc}\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\downarrow}\\ \boxed{\downarrow}&\boxed{\downarrow}\end{array}\quad \begin{array}{c}S=2\\ M_S=1\end{array}\right\rangle = C^{5/2\ 1/2\ 2}_{1/2\ 1/2\ 1}\left|\begin{array}{cccccc}\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\downarrow}&\boxed{\downarrow}\\ \boxed{\downarrow}&\boxed{\cdot\cdot}\end{array}\ \begin{array}{c}5/2\\ 1/2\end{array}\right\rangle\left|\boxed{\uparrow}^{1/2}_{1/2}\right\rangle + C^{5/2\ 1/2\ 2}_{3/2\ -1/2\ 1}\left|\begin{array}{cccccc}\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\downarrow}\\ \boxed{\downarrow}&\boxed{\cdot\cdot}\end{array}\ \begin{array}{c}5/2\\ 1/2\end{array}\right\rangle\left|\boxed{\downarrow}^{1/2}_{-1/2}\right\rangle$$

$$+C^{3/2\ 1/2\ 2}_{1/2\ 1/2\ 1}\left|\begin{array}{cccccc}\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\downarrow}&\boxed{\cdot\cdot}\\ \boxed{\downarrow}&\boxed{\downarrow}\end{array}\ \begin{array}{c}3/2\\ 1/2\end{array}\right\rangle\left|\boxed{\uparrow}^{1/2}_{1/2}\right\rangle + C^{3/2\ 1/2\ 2}_{3/2\ -1/2\ 1}\left|\begin{array}{cccccc}\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\uparrow}&\boxed{\cdot\cdot}\\ \boxed{\downarrow}&\boxed{\downarrow}\end{array}\ \begin{array}{c}3/2\\ 1/2\end{array}\right\rangle\left|\boxed{\downarrow}^{1/2}_{-1/2}\right\rangle$$

$$\begin{pmatrix} C^{j\ \ 1/2\ \ j+1/2}_{m\ 1/2\ m+1/2}=\sqrt{\dfrac{j+m+1}{2j+1}} & C^{j\ \ \ \ 1/2\ \ j+1/2}_{m+1\ -1/2\ m+1/2}=\sqrt{\dfrac{j-m}{2j+1}} \\ C^{j+1\ 1/2\ j+1/2}_{m\ \ \ 1/2\ m+1/2}=-\sqrt{\dfrac{j-m+1}{2j+3}} & C^{j+1\ \ 1/2\ \ j+1/2}_{m+1\ -1/2\ m+1/2}=\sqrt{\dfrac{j+m+2}{2j+3}}\end{pmatrix} \quad \text{example:} \begin{pmatrix} C^{3/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=\sqrt{\dfrac{2}{3}} & C^{3/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=\sqrt{\dfrac{1}{3}} \\ C^{5/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=-\sqrt{\dfrac{1}{3}} & C^{5/2\ 1/2\ 2}_{1/2\ 1/2\ 1}=\sqrt{\dfrac{2}{3}}\end{pmatrix}$$

(a) Permutation
$U(N)\supset S_N$
Multiplicity
$\ell^{[\mu_1,\mu_2]}$

								1	
							1		9
						1		8	
					1		7		35
				1		6		27	
			1		5		20		75
		1		4		14		48	
	1		3		9		28		90
1		2		5		14		42	
	1		2		5		14		42
1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*

N

(b) Spin
$U(2)\supset S_2$
Multiplicity
$\ell^S=2S+1$

							9		9
						8		8	
					7		7		7
				6		6		6	
			5		5		5		5
		4		4		4		4	
	3		3		3		3		3
2		2		2		2		2	
	1		1		1		1		1
N=1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	

(c) Combined
U(N)xU(2)
Multiplicity

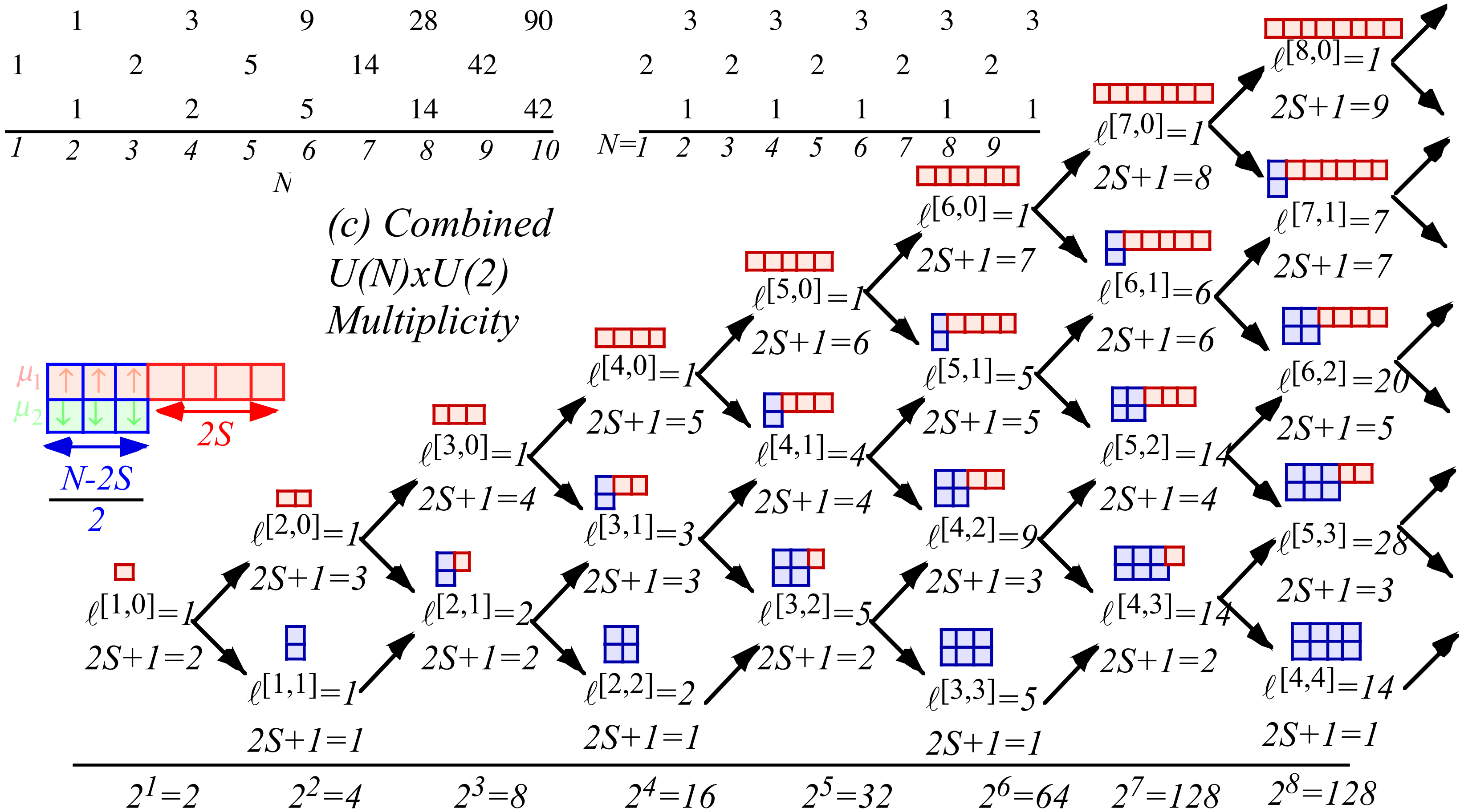


***Fig. 23.3.2** Spin-1/2 and U(2) Tableau branching diagrams*

2	3	4	5
1	2	3	4

5	4	3	2
4	3	2	1

$=1$

$$2^N=\sum_S^{N/2}\ell^{[S]}\ell^{[\mu_1,\mu_2]}$$

$$=\sum_S^{N/2}(2S+1)\ell^{\left[\frac{N+2S}{2},\frac{N-2S}{2}\right]}$$

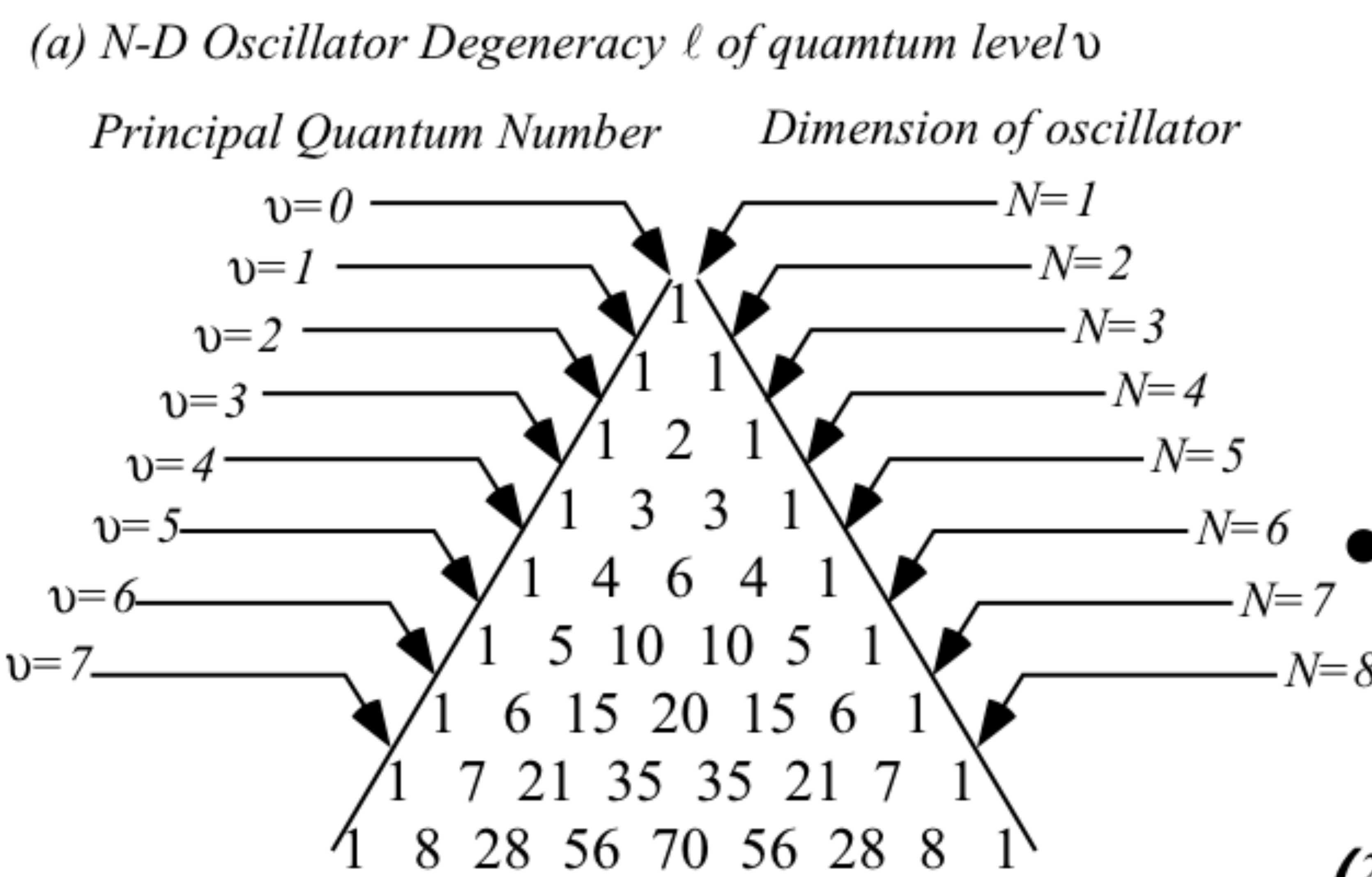
2

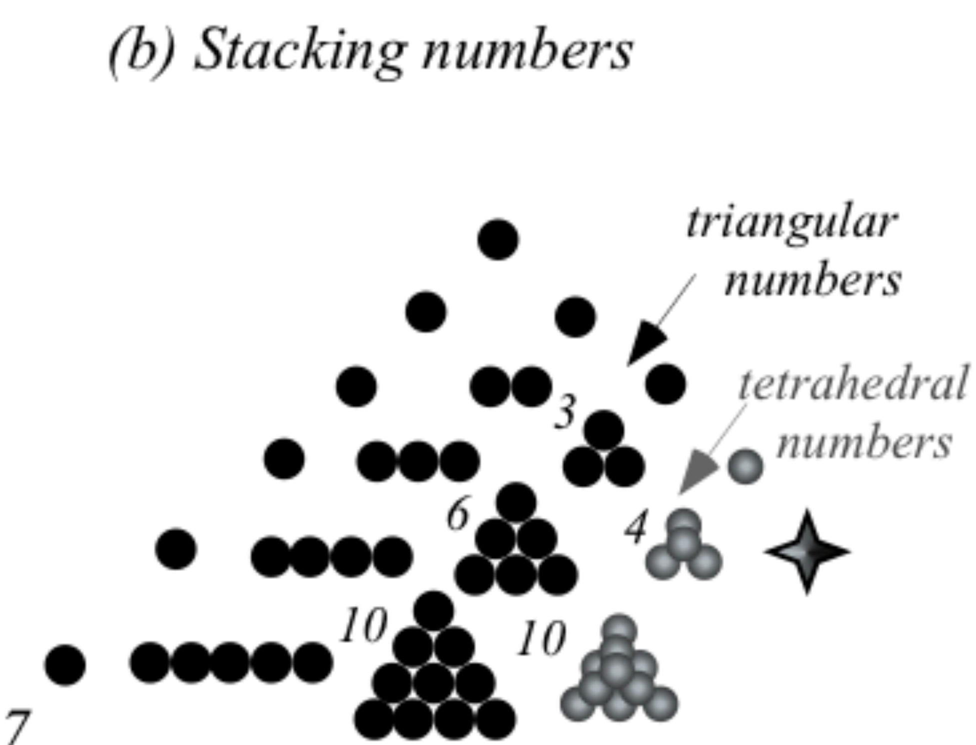
$8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1$

5	4	3	2
4	3	2	1

$=14$

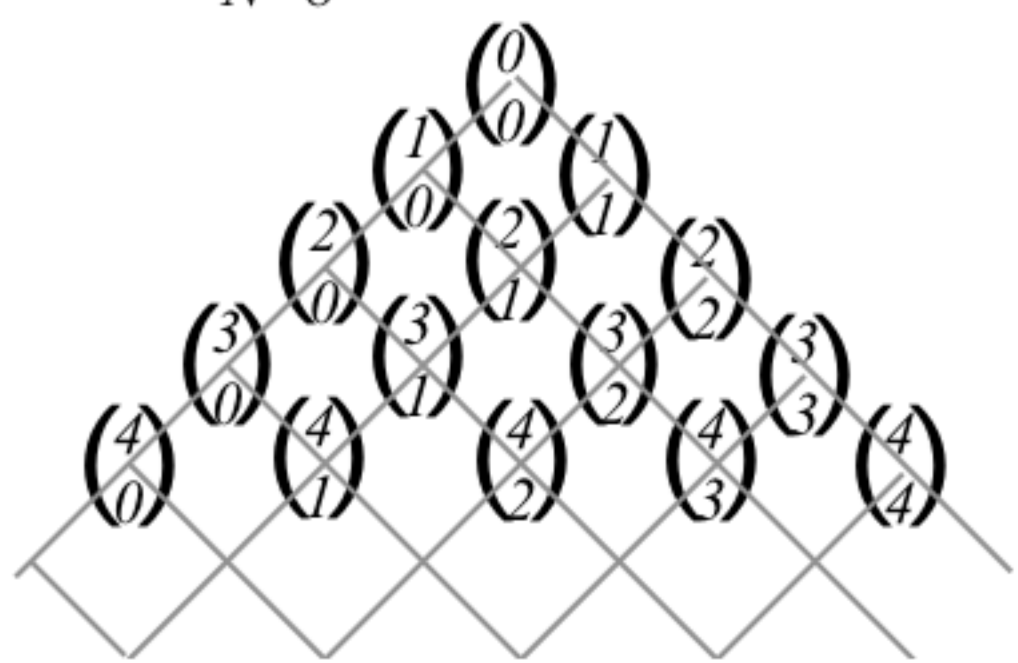
Introducing U(N)





(c) Binomial coefficients

$$\frac{(N\text{-}1+\upsilon)!}{(N\text{-}1)!\,\upsilon!} = \binom{N\text{-}1+\upsilon}{\upsilon} = \binom{N\text{-}1+\upsilon}{N\text{-}1}$$



Introducing U(3)

(b) N-particle 3-level states ...or spin-1 states

$\boxed{1} = |1\ 0\ 0\rangle = a_1^\dagger\ |0\ 0\ 0\rangle$

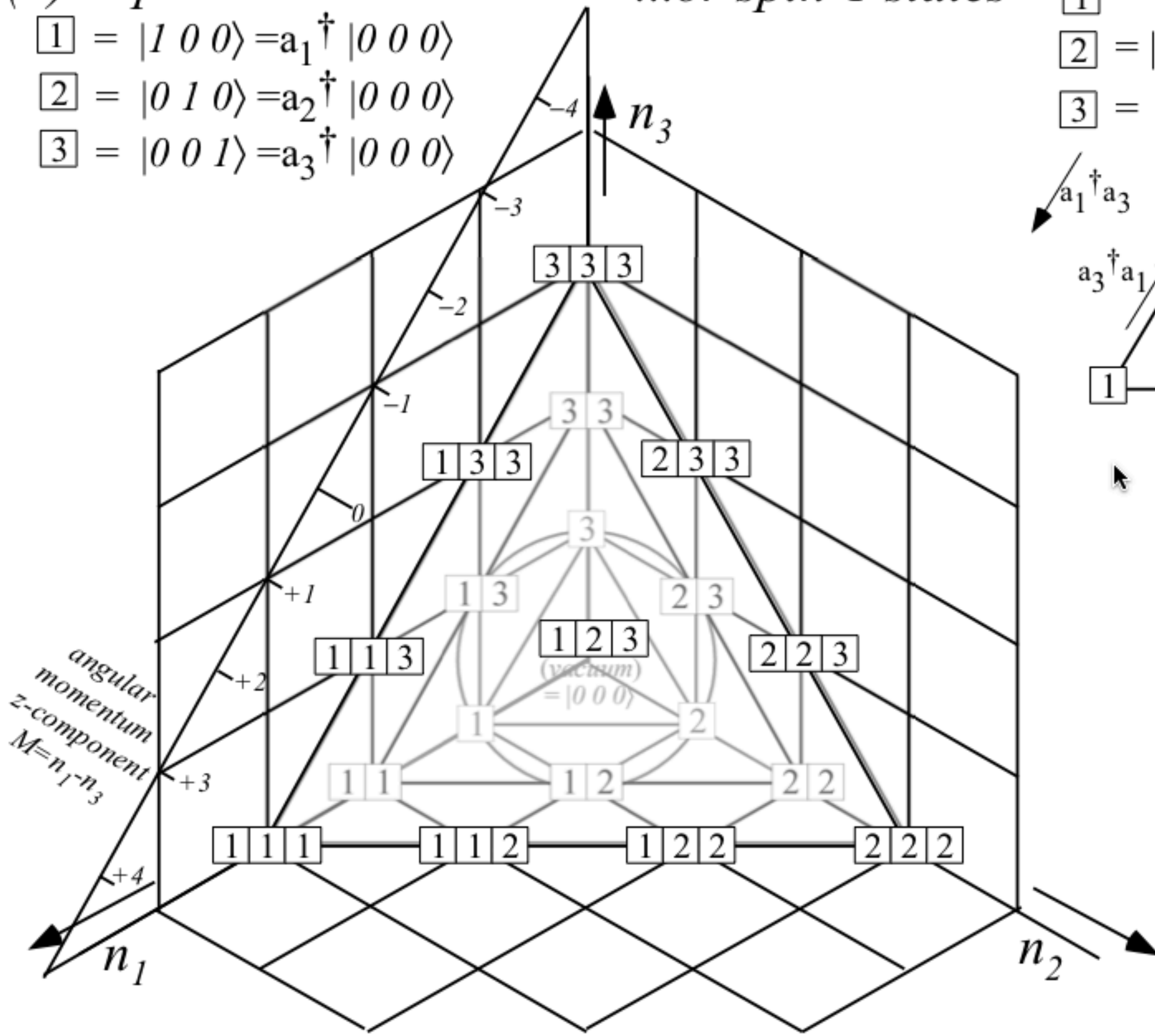
$\boxed{2} = |0\ 1\ 0\rangle = a_2^\dagger\ |0\ 0\ 0\rangle$

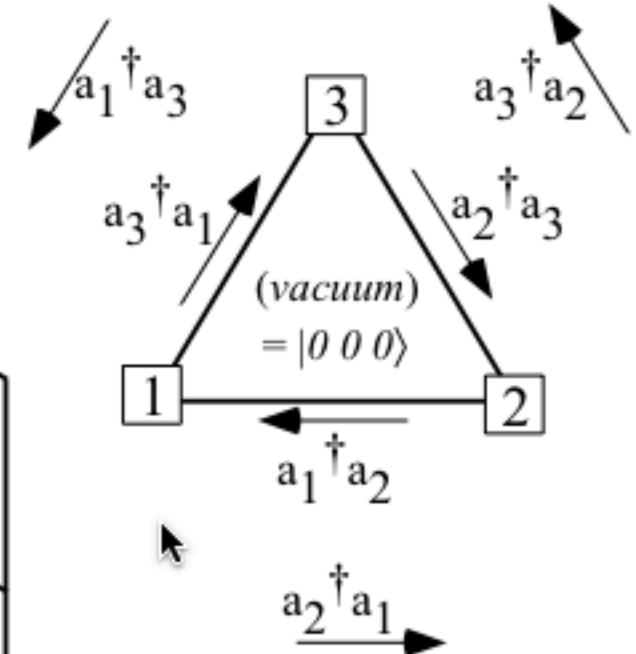
$\boxed{3} = |0\ 0\ 1\rangle = a_3^\dagger\ |0\ 0\ 0\rangle$

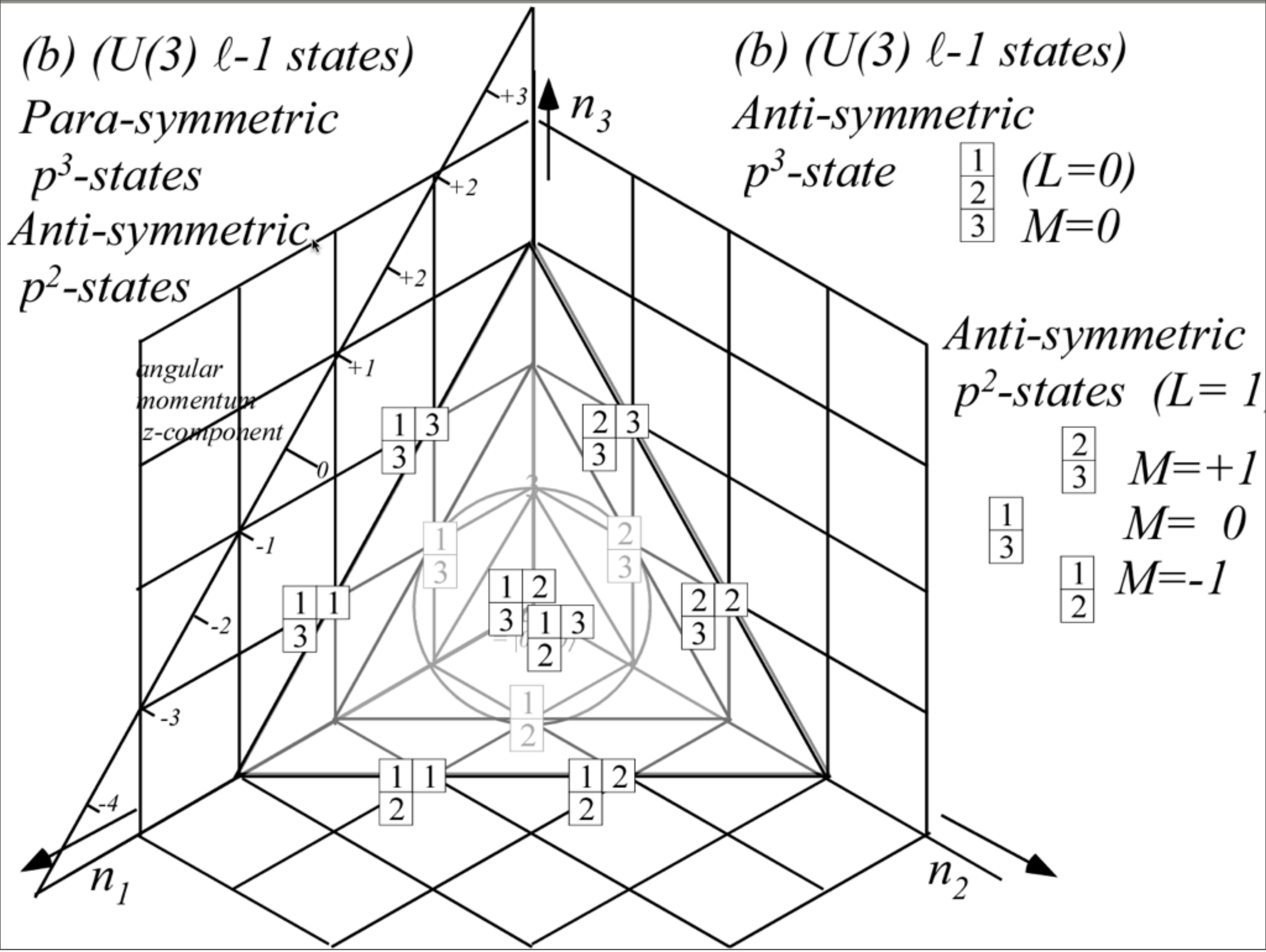
$\boxed{1} = |\uparrow\rangle = |{}^{j=1}_{m=+1}\rangle$

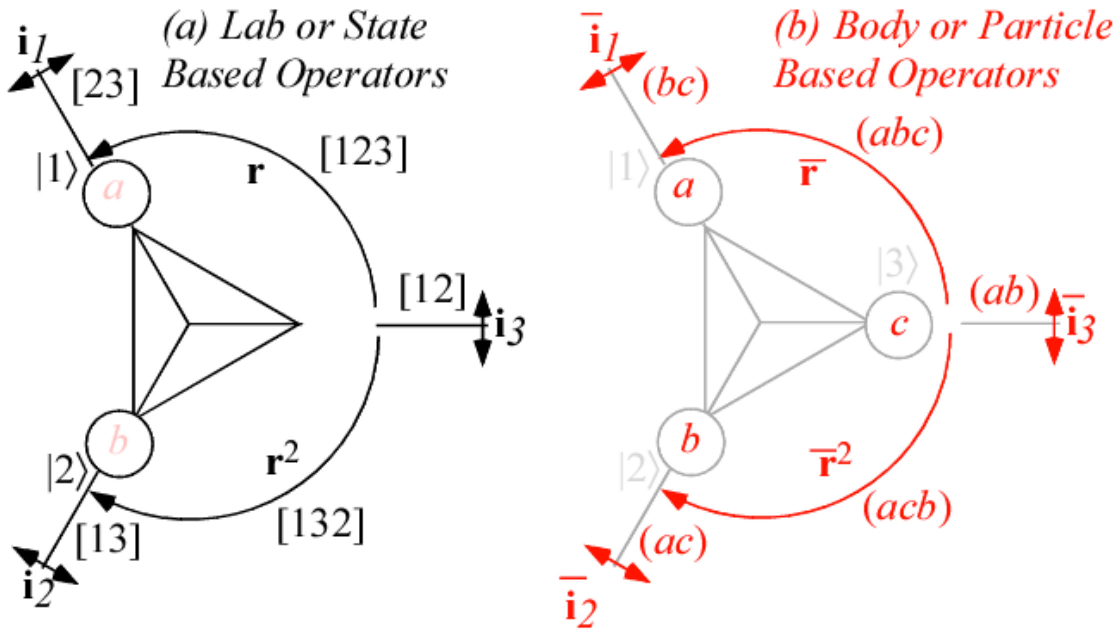
$\boxed{2} = |\leftrightarrow\rangle = |{}^{j=1}_{m=0}\rangle$

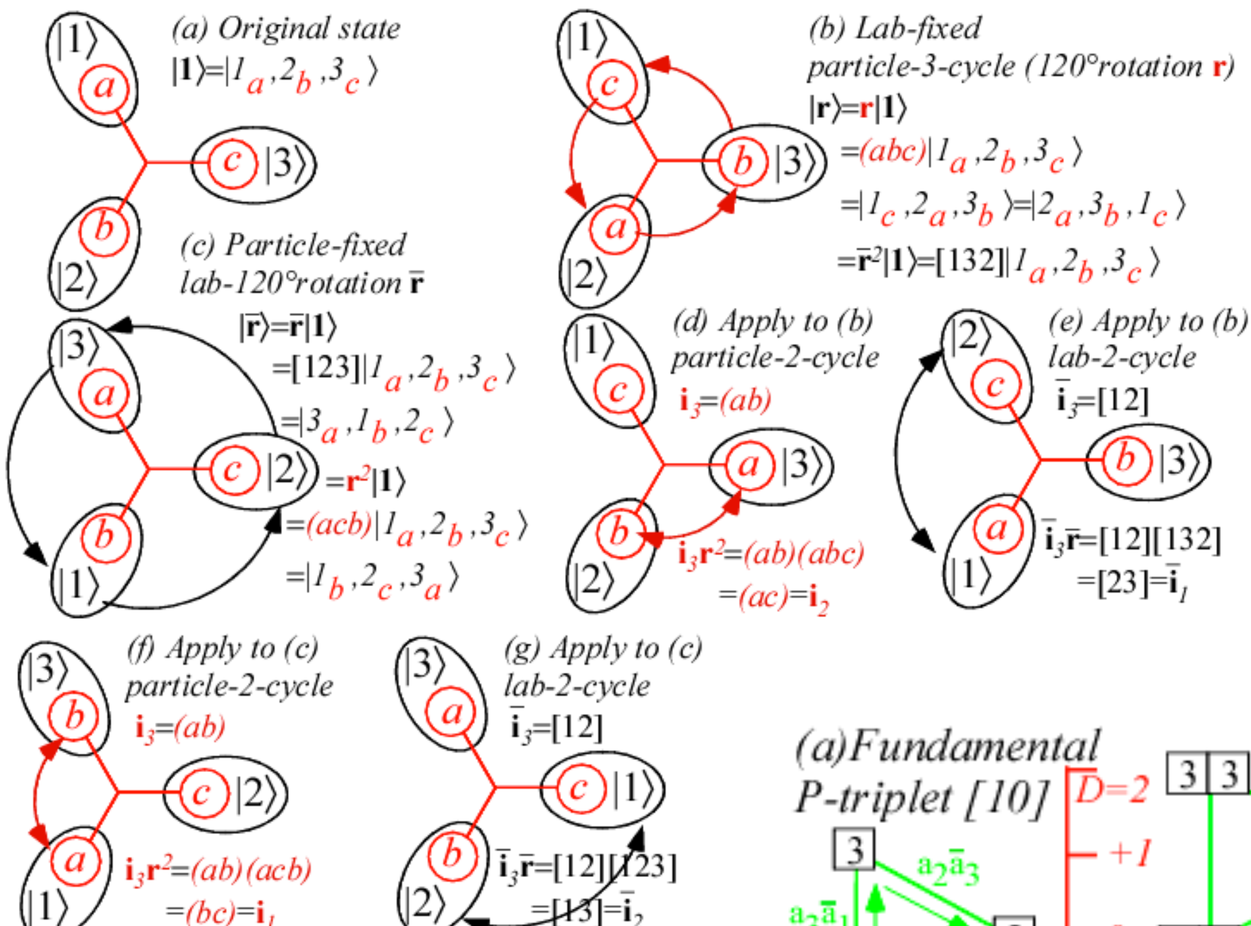
$\boxed{3} = |\downarrow\rangle = |{}^{j=1}_{m=-1}\rangle$

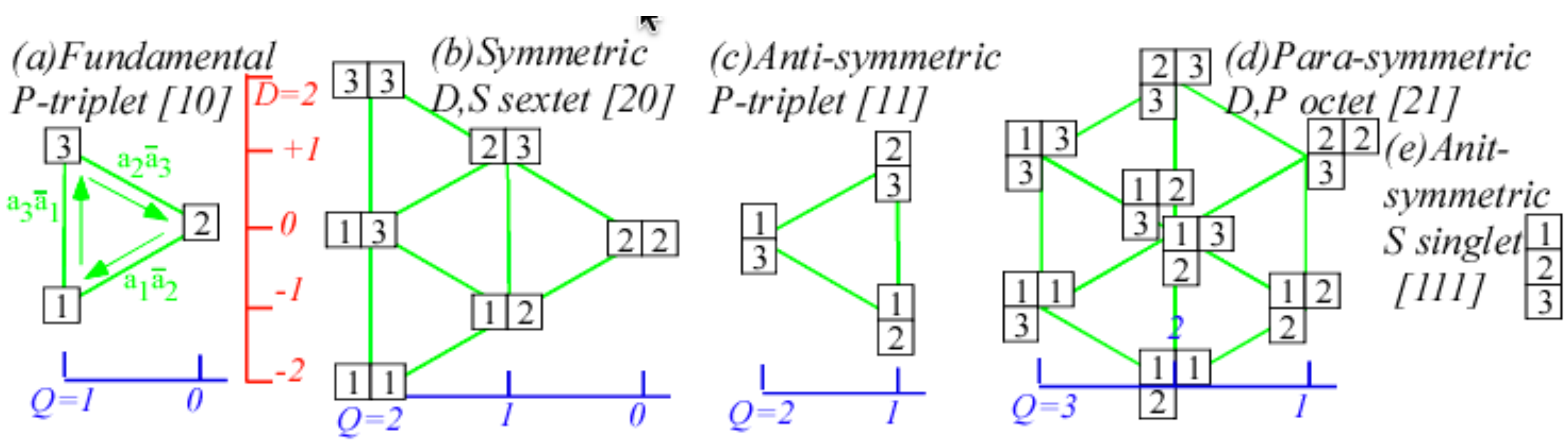





(b) (U(3) ℓ-1 states)
Para-symmetric
p3-states
Anti-symmetric
p2-states
angular
momentum
z-component
+3
+2
+2
+1
0
-1
-2
-3
-4
n3
n1
n2
(b) (U(3) ℓ-1 states)
Anti-symmetric
p3-state
(L=0)
M=0
Anti-symmetric
p2-states (L= 1)
M=+1
M= 0
M=-1


(a) Lab or State Based Operators
$\mathbf{i}_1$ [23]
$|1\rangle$ a
r [123]
[12] $\mathbf{i}_3$
$|2\rangle$ b
$\mathbf{r}^2$
[13] [132]
$\mathbf{i}_2$
(b) Body or Particle Based Operators
$\bar{\mathbf{i}}_1$ (bc)
(abc)
$|1\rangle$ a $\bar{\mathbf{r}}$
$|3\rangle$ c (ab) $\bar{\mathbf{i}}_3$
$|2\rangle$ b $\bar{\mathbf{r}}^2$
(acb)
(ac)
$\bar{\mathbf{i}}_2$


(a) Original state
$|\mathbf{1}\rangle = |1_a, 2_b, 3_c\rangle$
(b) Lab-fixed particle-3-cycle (120°rotation r)
$|\mathbf{r}\rangle = \mathbf{r}|\mathbf{1}\rangle$
$= (abc)|1_a, 2_b, 3_c\rangle$
$= |1_c, 2_a, 3_b\rangle = |2_a, 3_b, 1_c\rangle$
$= \bar{\mathbf{r}}^2|\mathbf{1}\rangle = [132]|1_a, 2_b, 3_c\rangle$
(c) Particle-fixed lab-120°rotation $\bar{\mathbf{r}}$
$|\bar{\mathbf{r}}\rangle = \bar{\mathbf{r}}|\mathbf{1}\rangle$
$= [123]|1_a, 2_b, 3_c\rangle$
$= |3_a, 1_b, 2_c\rangle$
$= \mathbf{r}^2|\mathbf{1}\rangle$
$= (acb)|1_a, 2_b, 3_c\rangle$
$= |1_b, 2_c, 3_a\rangle$
(d) Apply to (b) particle-2-cycle
$\mathbf{i}_3 = (ab)$
$\mathbf{i}_3\mathbf{r}^2 = (ab)(abc)$
$= (ac) = \mathbf{i}_2$
(e) Apply to (b) lab-2-cycle
$\bar{\mathbf{i}}_3 = [12]$
$\bar{\mathbf{i}}_3\bar{\mathbf{r}} = [12][132]$
$= [23] = \bar{\mathbf{i}}_1$
(f) Apply to (c) particle-2-cycle
$\mathbf{i}_3 = (ab)$
$\mathbf{i}_3\mathbf{r}^2 = (ab)(acb)$
$= (bc) = \mathbf{i}_1$
(g) Apply to (c) lab-2-cycle
$\bar{\mathbf{i}}_3 = [12]$
$\bar{\mathbf{i}}_3\bar{\mathbf{r}} = [12][123]$
$= [13] = \bar{\mathbf{i}}_2$


(a)Fundamental P-triplet [10]
$a_2\bar{a}_3$
$a_3\bar{a}_1$
$a_1\bar{a}_2$
D=2
+1
0
-1
-2
Q=1 0
(b)Symmetric D,S sextet [20]
Q=2 1 0
(c)Anti-symmetric P-triplet [11]
Q=2 1
(d)Para-symmetric D,P octet [21]
Q=3 1
(e)Anit-symmetric S singlet [111]

Tensor operators for spin-1/2 states: Outer products give Hamilton-Pauli-spinors

CG-Products of spin-1/2 **ket-bras** $\left\{ \left| \begin{smallmatrix} 1/2 \\ m_1 \end{smallmatrix} \right\rangle, \left\langle \begin{smallmatrix} 1/2 \\ m_2 \end{smallmatrix} \right| \right\}$ give scalar/vector operators analogous to: **ket-kets**

$$T^k_q = \sum_{m_1} C^{1/2\ 1/2\ k}_{m_1\ m_2\ q} \left| \begin{smallmatrix} 1/2 \\ m_1 \end{smallmatrix} \right\rangle \left\langle \begin{smallmatrix} 1/2 \\ -m_2 \end{smallmatrix} \right| (-1)^{\frac{1}{2}-m_2}$$

analogous to:

$$\left| \begin{smallmatrix} J \\ M \end{smallmatrix} (1/2\otimes 1/2) \right\rangle = \sum_{m_1, m_2} C^{1/2\ 1/2\ J}_{m_1\ m_2\ M} \left| \begin{smallmatrix} 1/2 \\ m_1 \end{smallmatrix} \right\rangle \left| \begin{smallmatrix} 1/2 \\ m_2 \end{smallmatrix} \right\rangle$$

$$T^1_{-1} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = -\left| \begin{smallmatrix} 1/2 \\ -1/2 \end{smallmatrix} \right\rangle \left\langle \begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right|, \quad T^1_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\frac{1}{\sqrt{2}} \left[\left| \begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right\rangle \left\langle \begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right| - \left| \begin{smallmatrix} 1/2 \\ -1/2 \end{smallmatrix} \right\rangle \left\langle \begin{smallmatrix} 1/2 \\ -1/2 \end{smallmatrix} \right| \right], \quad T^1_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \left| \begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right\rangle \left\langle \begin{smallmatrix} 1/2 \\ -1/2 \end{smallmatrix} \right|,$$

analogous to:

$$\left| \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} (1/2\otimes 1/2) \right\rangle = \left| \begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right\rangle \left| \begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right\rangle$$

$$\left| \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} (1/2\otimes 1/2) \right\rangle = \frac{1}{\sqrt{2}} \left| \begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right\rangle \left| \begin{smallmatrix} 1/2 \\ -1/2 \end{smallmatrix} \right\rangle + \frac{1}{\sqrt{2}} \left| \begin{smallmatrix} 1/2 \\ -1/2 \end{smallmatrix} \right\rangle \left| \begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right\rangle$$

$$\left| \begin{smallmatrix} 1 \\ -1 \end{smallmatrix} (1/2\otimes 1/2) \right\rangle = \left| \begin{smallmatrix} 1/2 \\ -1/2 \end{smallmatrix} \right\rangle \left| \begin{smallmatrix} 1/2 \\ -1/2 \end{smallmatrix} \right\rangle$$

$$T^0_0 = -\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\frac{1}{\sqrt{2}} \left[\left| \begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right\rangle \left\langle \begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right| + \left| \begin{smallmatrix} 1/2 \\ -1/2 \end{smallmatrix} \right\rangle \left\langle \begin{smallmatrix} 1/2 \\ -1/2 \end{smallmatrix} \right| \right].$$

analogous to:

$$\left| \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} (1/2\otimes 1/2) \right\rangle = \frac{1}{\sqrt{2}} \left| \begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right\rangle \left| \begin{smallmatrix} 1/2 \\ -1/2 \end{smallmatrix} \right\rangle + \frac{-1}{\sqrt{2}} \left| \begin{smallmatrix} 1/2 \\ -1/2 \end{smallmatrix} \right\rangle \left| \begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right\rangle$$

1st three operators are a *vector* set with following Cartesian combinations:

$$T_x \equiv -\frac{T^1_{-1} - T^1_1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \equiv \frac{1}{\sqrt{2}} \sigma_x \equiv \sqrt{2} J_x$$

$$T_y \equiv -i \frac{T^1_{-1} + T^1_1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \equiv \frac{1}{\sqrt{2}} \sigma_y \equiv \sqrt{2} J_y$$

$$T_z \equiv -T^1_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv \frac{1}{\sqrt{2}} \sigma_z \equiv \sqrt{2} J_z$$

(Some old friends!)

$$\sigma_X \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_Y \to \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_Z \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

Spherical vs. Cartesian operators

$$T^1_{-1} = J_-/2 = \left(J_x - iJ_y\right)/\sqrt{2}, \qquad T^1_0 = J_z/\sqrt{2}, \qquad T^1_{-1} = J_+/2 = \left(J_x + iJ_y\right)/2.$$

Tensor operators for spin-1/2 states: Outer products give Hamilton-Pauli-spinors

CG-Products of spin-1/2 **ket-bras** $\left\{\left|\begin{smallmatrix}1/2\\m_1\end{smallmatrix}\right\rangle, \left\langle\begin{smallmatrix}1/2\\m_2\end{smallmatrix}\right|\right\}$ give scalar/vector operators analogous to: **ket-kets**

$$T^k_q = \sum_{m_1} C^{1/2\ 1/2\ k}_{m_1\ m_2\ q} \left|\begin{matrix}1/2\\m_1\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\-m_2\end{matrix}\right| (-1)^{\frac{1}{2}-m_2}$$

analogous to:

$$\left|\begin{matrix}J\\M\end{matrix}\ (1/2\otimes 1/2)\right\rangle = \sum_{m_1,m_2} C^{1/2\ 1/2\ J}_{m_1\ m_2\ M}\left|\begin{matrix}1/2\\m_1\end{matrix}\right\rangle\left|\begin{matrix}1/2\\m_2\end{matrix}\right\rangle$$

$$T^1_{-1} = \begin{pmatrix}0 & 0\\ -1 & 0\end{pmatrix} = -\left|\begin{matrix}1/2\\-1/2\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\1/2\end{matrix}\right|,\quad T^1_0 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} = -\frac{1}{\sqrt{2}}\left[\left|\begin{matrix}1/2\\1/2\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\1/2\end{matrix}\right| - \left|\begin{matrix}1/2\\-1/2\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\-1/2\end{matrix}\right|\right],\quad T^1_1 = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = \left|\begin{matrix}1/2\\1/2\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\-1/2\end{matrix}\right|,$$

analogous to:

$$\left|\begin{matrix}1\\1\end{matrix}\ (1/2\otimes 1/2)\right\rangle = \left|\begin{matrix}1/2\\1/2\end{matrix}\right\rangle\left|\begin{matrix}1/2\\1/2\end{matrix}\right\rangle$$

$$\left|\begin{matrix}1\\0\end{matrix}\ (1/2\otimes 1/2)\right\rangle = \frac{1}{\sqrt{2}}\left|\begin{matrix}1/2\\1/2\end{matrix}\right\rangle\left|\begin{matrix}1/2\\-1/2\end{matrix}\right\rangle + \frac{1}{\sqrt{2}}\left|\begin{matrix}1/2\\-1/2\end{matrix}\right\rangle\left|\begin{matrix}1/2\\1/2\end{matrix}\right\rangle$$

$$\left|\begin{matrix}1\\-1\end{matrix}\ (1/2\otimes 1/2)\right\rangle = \left|\begin{matrix}1/2\\-1/2\end{matrix}\right\rangle\left|\begin{matrix}1/2\\-1/2\end{matrix}\right\rangle$$

$$T^0_0 = -\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = -\frac{1}{\sqrt{2}}\left[\left|\begin{matrix}1/2\\1/2\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\1/2\end{matrix}\right| + \left|\begin{matrix}1/2\\-1/2\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\-1/2\end{matrix}\right|\right].$$

analogous to:

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1st three operators are a *vector* set that *transform like a vector set*

$$\begin{array}{cccc}
R(0\beta 0) & T^1_0 & R^\dagger(0\beta 0) & = \quad T'_0\\
\downarrow & \downarrow & \downarrow & \downarrow\\
\begin{pmatrix}\cos\frac{\beta}{2} & -\sin\frac{\beta}{2}\\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix} & \begin{pmatrix}-1/\sqrt{2} & 0\\ 0 & 1/\sqrt{2}\end{pmatrix} & \begin{pmatrix}\cos\frac{\beta}{2} & \sin\frac{\beta}{2}\\ -\sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix} = & -\frac{1}{\sqrt{2}}\begin{pmatrix}\cos\beta & \sin\beta\\ \sin\beta & -\cos\beta\end{pmatrix}\\
= D^1_{10}(0\beta 0)T^1_1 & +D^1_{00}(0\beta 0)T^1_0 & +D^1_{-10}(0\beta 0)T^1_{-1} & \\
\downarrow & \downarrow & \downarrow & \\
= \frac{-\sin\beta}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} & +\cos\beta\begin{pmatrix}-1/\sqrt{2} & 0\\ 0 & 1/\sqrt{2}\end{pmatrix} & +\frac{\sin\beta}{\sqrt{2}}\begin{pmatrix}0 & 0\\ -1 & 0\end{pmatrix} &
\end{array}$$

Tensor operators for spin-1/2 states: Outer products give Hamilton-Pauli-spinors

CG-Products of spin-1/2 **ket-bras** $\left\{ \left|\begin{matrix}1/2\\m_1\end{matrix}\right\rangle , \left\langle\begin{matrix}1/2\\m_2\end{matrix}\right| \right\}$ give scalar/vector operators analogous to: **ket-kets**

$$T^k_q = \sum_{m_1} C^{1/2\;1/2\;k}_{m_1\;m_2\;q} \left|\begin{matrix}1/2\\m_1\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\-m_2\end{matrix}\right| (-1)^{\frac{1}{2}-m_2}$$

analogous to:

$$\left|\begin{matrix}J\\M\end{matrix}(1/2\otimes 1/2)\right\rangle = \sum_{m_1,m_2} C^{1/2\;1/2\;J}_{m_1\;m_2\;M}\left|\begin{matrix}1/2\\m_1\end{matrix}\right\rangle\left|\begin{matrix}1/2\\m_2\end{matrix}\right\rangle$$

$$T^1_{-1} = \begin{pmatrix}0&0\\-1&0\end{pmatrix} = -\left|\begin{matrix}1/2\\-1/2\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\1/2\end{matrix}\right|,$$

$$T^1_0 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1&0\\0&1\end{pmatrix} = -\frac{1}{\sqrt{2}}\left[\left|\begin{matrix}1/2\\1/2\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\1/2\end{matrix}\right| - \left|\begin{matrix}1/2\\-1/2\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\-1/2\end{matrix}\right|\right],$$

$$T^1_1 = \begin{pmatrix}0&1\\0&0\end{pmatrix} = \left|\begin{matrix}1/2\\1/2\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\-1/2\end{matrix}\right|,$$

analogous to:

$$\left|\begin{matrix}1\\1\end{matrix}(1/2\otimes 1/2)\right\rangle = \left|\begin{matrix}1/2\\1/2\end{matrix}\right\rangle\left|\begin{matrix}1/2\\1/2\end{matrix}\right\rangle$$

$$\left|\begin{matrix}1\\0\end{matrix}(1/2\otimes 1/2)\right\rangle = \frac{1}{\sqrt{2}}\left|\begin{matrix}1/2\\1/2\end{matrix}\right\rangle\left|\begin{matrix}1/2\\-1/2\end{matrix}\right\rangle + \frac{1}{\sqrt{2}}\left|\begin{matrix}1/2\\-1/2\end{matrix}\right\rangle\left|\begin{matrix}1/2\\1/2\end{matrix}\right\rangle$$

$$\left|\begin{matrix}1\\-1\end{matrix}(1/2\otimes 1/2)\right\rangle = \left|\begin{matrix}1/2\\-1/2\end{matrix}\right\rangle\left|\begin{matrix}1/2\\-1/2\end{matrix}\right\rangle$$

$$T^0_0 = -\frac{1}{\sqrt{2}}\begin{pmatrix}1&0\\0&1\end{pmatrix} = -\frac{1}{\sqrt{2}}\left[\left|\begin{matrix}1/2\\1/2\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\1/2\end{matrix}\right| + \left|\begin{matrix}1/2\\-1/2\end{matrix}\right\rangle\left\langle\begin{matrix}1/2\\-1/2\end{matrix}\right|\right].$$

analogous to:

$$\left|\begin{matrix}0\\0\end{matrix}(1/2\otimes 1/2)\right\rangle = \frac{1}{\sqrt{2}}\left|\begin{matrix}1/2\\1/2\end{matrix}\right\rangle\left|\begin{matrix}1/2\\-1/2\end{matrix}\right\rangle + \frac{-1}{\sqrt{2}}\left|\begin{matrix}1/2\\-1/2\end{matrix}\right\rangle\left|\begin{matrix}1/2\\1/2\end{matrix}\right\rangle$$

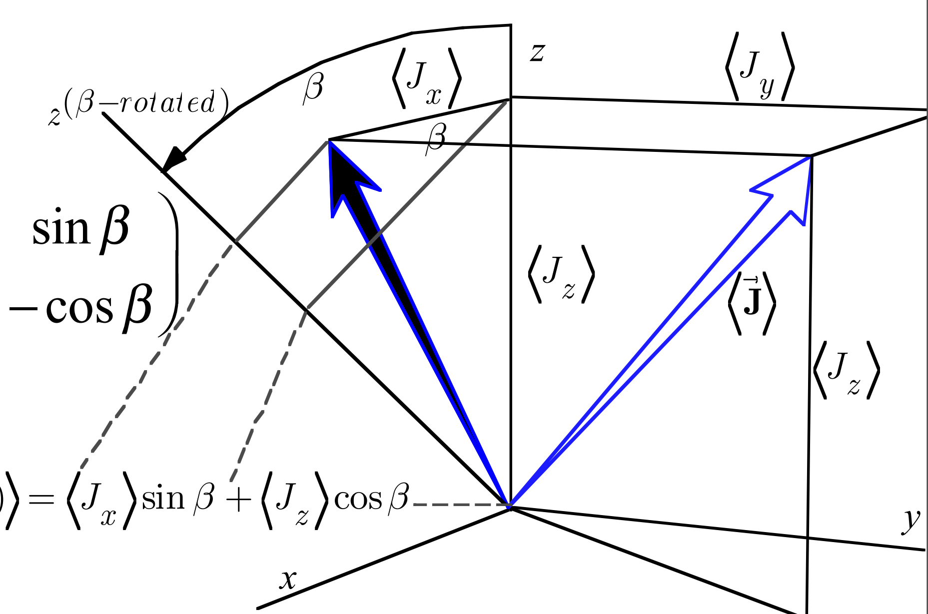
1st three operators are a *vector* set that *transform like a vector set* so do expectation values

$$R(0\beta 0)\; T^1_0\; R^\dagger(0\beta 0) = T'_0$$

$$\begin{pmatrix}\cos\frac{\beta}{2} & -\sin\frac{\beta}{2}\\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix}-1/\sqrt{2} & 0\\ 0 & 1/\sqrt{2}\end{pmatrix}\begin{pmatrix}\cos\frac{\beta}{2} & \sin\frac{\beta}{2}\\ -\sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix} = -\frac{1}{\sqrt{2}}\begin{pmatrix}\cos\beta & \sin\beta\\ \sin\beta & -\cos\beta\end{pmatrix}$$

$$= D^1_{10}(0\beta 0)T^1_1 + D^1_{00}(0\beta 0)T^1_0 + D^1_{-10}(0\beta 0)T^1_{-1}$$

$$= \frac{-\sin\beta}{\sqrt{2}}\begin{pmatrix}0&1\\0&0\end{pmatrix} + \cos\beta\begin{pmatrix}-1/\sqrt{2}&0\\0&1/\sqrt{2}\end{pmatrix} + \frac{\sin\beta}{\sqrt{2}}\begin{pmatrix}0&0\\-1&0\end{pmatrix}$$



$$\langle J_z(\beta - rotated)\rangle = \langle J_x\rangle \sin\beta + \langle J_z\rangle\cos\beta$$

Tensor operators for spin-1 states: U(1) generalization of Pauli spinors

CGC definition:

$$\mathbf{v}^k_q = \sum_{m,m'} C^{j\ j\ \ k}_{m\ -m'\ q} (-1)^{j-m'} \left|\begin{matrix} j \\ m \end{matrix}\right\rangle\left\langle\begin{matrix} j \\ m' \end{matrix}\right| = (-1)^{2j}\, T^k_q .$$

Wigner 3jm definition:

$$\mathbf{v}^k_q = \sum_{m,m'} (-1)^{j-m} \sqrt{2k+1} \begin{pmatrix} k & j & j \\ q & m' & -m \end{pmatrix} \left|\begin{matrix} j \\ m \end{matrix}\right\rangle\left\langle\begin{matrix} j \\ m' \end{matrix}\right|$$

$$T^2_{-2} = \left|\begin{matrix} 1 \\ -1 \end{matrix}\right\rangle\left\langle\begin{matrix} 1 \\ 1 \end{matrix}\right|, \quad T^2_{-1} = \frac{\left|\begin{matrix} 1 \\ 0 \end{matrix}\right\rangle\left\langle\begin{matrix} 1 \\ 1 \end{matrix}\right| - \left|\begin{matrix} 1 \\ -1 \end{matrix}\right\rangle\left\langle\begin{matrix} 1 \\ 0 \end{matrix}\right|}{\sqrt{2}}, \quad T^2_{0} = \frac{\left|\begin{matrix} 1 \\ 1 \end{matrix}\right\rangle\left\langle\begin{matrix} 1 \\ 1 \end{matrix}\right| - 2\left|\begin{matrix} 1 \\ 0 \end{matrix}\right\rangle\left\langle\begin{matrix} 1 \\ 0 \end{matrix}\right| + \left|\begin{matrix} 1 \\ -1 \end{matrix}\right\rangle\left\langle\begin{matrix} 1 \\ -1 \end{matrix}\right|}{\sqrt{6}}, \quad T^2_{1} = \frac{-\left|\begin{matrix} 1 \\ 1 \end{matrix}\right\rangle\left\langle\begin{matrix} 1 \\ 0 \end{matrix}\right| + \left|\begin{matrix} 1 \\ 0 \end{matrix}\right\rangle\left\langle\begin{matrix} 1 \\ -1 \end{matrix}\right|}{\sqrt{2}}, \quad T^2_{2} = \left|\begin{matrix} 1 \\ 1 \end{matrix}\right\rangle\left\langle\begin{matrix} 1 \\ -1 \end{matrix}\right|$$

$$\rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 \\ 0 & -1/\sqrt{2} & 0 \end{pmatrix} \quad \rightarrow \begin{pmatrix} 1/\sqrt{6} & 0 & 0 \\ 0 & -2/\sqrt{6} & 0 \\ 0 & 0 & 1/\sqrt{6} \end{pmatrix} \quad \rightarrow \begin{pmatrix} 0 & -1/\sqrt{2} & 0 \\ 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \quad \rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T^1_{-1} = \frac{\left|\begin{matrix} 1 \\ 0 \end{matrix}\right\rangle\left\langle\begin{matrix} 1 \\ 1 \end{matrix}\right| + \left|\begin{matrix} 1 \\ -1 \end{matrix}\right\rangle\left\langle\begin{matrix} 1 \\ 0 \end{matrix}\right|}{\sqrt{2}}, \quad T^1_{0} = \frac{\left|\begin{matrix} 1 \\ 1 \end{matrix}\right\rangle\left\langle\begin{matrix} 1 \\ 1 \end{matrix}\right| - \left|\begin{matrix} 1 \\ -1 \end{matrix}\right\rangle\left\langle\begin{matrix} 1 \\ -1 \end{matrix}\right|}{\sqrt{2}}, \quad T^1_{1} = \frac{-\left|\begin{matrix} 1 \\ 1 \end{matrix}\right\rangle\left\langle\begin{matrix} 1 \\ 0 \end{matrix}\right| - \left|\begin{matrix} 1 \\ 0 \end{matrix}\right\rangle\left\langle\begin{matrix} 1 \\ -1 \end{matrix}\right|}{\sqrt{2}}$$

$$\rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \end{pmatrix} \quad \rightarrow \begin{pmatrix} 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1/\sqrt{2} \end{pmatrix} \quad \rightarrow \begin{pmatrix} 0 & -1/\sqrt{2} & 0 \\ 0 & 0 & -1/\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$T^0_{0} = \frac{\left|\begin{matrix} 1 \\ 1 \end{matrix}\right\rangle\left\langle\begin{matrix} 1 \\ 1 \end{matrix}\right| + \left|\begin{matrix} 1 \\ 0 \end{matrix}\right\rangle\left\langle\begin{matrix} 1 \\ 0 \end{matrix}\right| + \left|\begin{matrix} 1 \\ -1 \end{matrix}\right\rangle\left\langle\begin{matrix} 1 \\ -1 \end{matrix}\right|}{\sqrt{2}}$$

$$\rightarrow \begin{pmatrix} 1/\sqrt{3} & 0 & 0 \\ 0 & 1/\sqrt{3} & 0 \\ 0 & 0 & 1/\sqrt{3} \end{pmatrix}$$

Tensor operators for spin-J states: U(2J+1) generalization of Pauli spinors

$$\mathbf{v}^k_q = \sum_{m,m'} (-1)^{j-m}\sqrt{2k+1}\begin{pmatrix} k & j & j \\ q & m' & -m \end{pmatrix} \left| \begin{matrix} j \\ m \end{matrix} \right\rangle \left\langle \begin{matrix} j \\ m' \end{matrix} \right|$$

for $j = 1,2,3.$

$$\begin{array}{cc}
 & \begin{array}{ccccccc} q=0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
-1 & \\
v_q^6 = & \left[\begin{array}{ccccccc}
1 & -\sqrt{2} & 1 & -\sqrt{2} & \sqrt{5} & -1 & 1 \\
\sqrt{2} & -6 & \sqrt{30} & -\sqrt{8} & 3 & -\sqrt{12} & 1 \\
1 & -\sqrt{30} & 15 & -10 & \sqrt{15} & -3 & \sqrt{5} \\
\sqrt{2} & -\sqrt{8} & 10 & -20 & 10 & -\sqrt{8} & \sqrt{2} \\
\sqrt{5} & -3 & \sqrt{15} & -10 & 15 & -\sqrt{30} & 1 \\
1 & -\sqrt{12} & 3 & -\sqrt{8} & \sqrt{30} & -6 & \sqrt{2} \\
1 & -1 & \sqrt{5} & -\sqrt{2} & 1 & -\sqrt{2} & 1
\end{array}\right]
\end{array}
\quad
\begin{array}{c} 1 \\ \sqrt{2} \\ \sqrt{22} \\ \sqrt{22} \\ \sqrt{33} \\ \sqrt{264} \\ \sqrt{924} \end{array}$$

$$v_q^5 = \left[\begin{array}{ccccccc}
1 & -\sqrt{5} & 1 & -\sqrt{2} & 1 & -1 & \cdot \\
\sqrt{5} & -4 & \sqrt{27} & -\sqrt{2} & 1 & 0 & -1 \\
1 & -\sqrt{27} & 5 & -\sqrt{10} & 0 & 1 & -1 \\
\sqrt{2} & -\sqrt{2} & \sqrt{10} & 0 & -\sqrt{10} & \sqrt{2} & \sqrt{2} \\
1 & -1 & 0 & \sqrt{10} & -5 & \sqrt{27} & -1 \\
1 & 0 & -1 & \sqrt{2} & -\sqrt{27} & 4 & -\sqrt{5} \\
\cdot & 1 & -1 & \sqrt{2} & -1 & \sqrt{5} & -1
\end{array}\right]
\quad
\begin{array}{c} \sqrt{2} \\ \sqrt{2} \\ \sqrt{6} \\ \sqrt{6} \\ \sqrt{84} \\ \sqrt{84} \end{array}$$

$$v_q^4 = \left[\begin{array}{ccccccc}
3 & -\sqrt{30} & \sqrt{54} & -3 & \sqrt{3} & \cdot & \cdot \\
\sqrt{30} & -7 & \sqrt{32} & -\sqrt{3} & -\sqrt{2} & \sqrt{5} & \cdot \\
\sqrt{54} & -\sqrt{32} & 1 & \sqrt{15} & -\sqrt{40} & \sqrt{2} & \sqrt{3} \\
3 & -\sqrt{3} & -\sqrt{15} & 6 & -\sqrt{15} & -\sqrt{3} & 3 \\
\sqrt{3} & \sqrt{2} & -\sqrt{40} & \sqrt{15} & 1 & -\sqrt{32} & \sqrt{54} \\
\cdot & \sqrt{5} & -\sqrt{2} & -\sqrt{3} & \sqrt{32} & -7 & \sqrt{30} \\
\cdot & \cdot & \sqrt{3} & -3 & \sqrt{54} & -\sqrt{30} & 3
\end{array}\right]
\quad
\begin{array}{c} \sqrt{11} \\ \sqrt{22} \\ \sqrt{154} \\ \sqrt{154} \\ \sqrt{154} \end{array}$$

$$v_q^3 = \left[\begin{array}{ccccccc}
1 & -\sqrt{2} & \sqrt{2} & -1 & \cdot & \cdot & \cdot \\
\sqrt{2} & -1 & 0 & 1 & -\sqrt{2} & \cdot & \cdot \\
\sqrt{2} & 0 & -1 & 1 & 0 & -\sqrt{2} & \cdot \\
1 & 1 & -1 & 0 & 1 & -1 & -1 \\
\cdot & \sqrt{2} & 0 & -1 & 1 & 0 & -\sqrt{2} \\
\cdot & \cdot & \sqrt{2} & -1 & 0 & 1 & -\sqrt{2} \\
\cdot & \cdot & \cdot & 1 & -\sqrt{2} & \sqrt{2} & -1
\end{array}\right]
\quad
\begin{array}{c} \sqrt{6} \\ \sqrt{6} \\ \sqrt{6} \\ \sqrt{6} \end{array}$$

$$v_q^2 = \left[\begin{array}{ccccccc}
5 & -5 & \sqrt{5} & \cdot & \cdot & \cdot & \cdot \\
5 & 0 & -\sqrt{15} & \sqrt{10} & \cdot & \cdot & \cdot \\
\sqrt{5} & \sqrt{15} & -3 & -\sqrt{2} & \sqrt{12} & \cdot & \cdot \\
\cdot & \sqrt{10} & \sqrt{2} & -4 & \sqrt{2} & \sqrt{10} & \cdot \\
\cdot & \cdot & \sqrt{12} & -\sqrt{2} & -3 & \sqrt{15} & \sqrt{5} \\
\cdot & \cdot & \cdot & \sqrt{10} & -\sqrt{15} & 0 & 5 \\
\cdot & \cdot & \cdot & \cdot & \sqrt{5} & -5 & 5
\end{array}\right]
\quad
\begin{array}{c} \sqrt{42} \\ \sqrt{84} \\ \sqrt{84} \end{array}$$

$$v_q^1 = \left[\begin{array}{ccccccc}
3 & -\sqrt{3} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\sqrt{3} & 2 & -\sqrt{5} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \sqrt{5} & 1 & -\sqrt{6} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \sqrt{6} & 0 & -\sqrt{6} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \sqrt{6} & -1 & -\sqrt{5} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \sqrt{5} & -2 & -\sqrt{3} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{3} & -3
\end{array}\right]
\quad
\begin{array}{c} \sqrt{28} \\ \sqrt{28} \end{array}$$

(f) $l = 3$

$$\begin{array}{c}
\begin{array}{ccccc} q=0 & 1 & 2 & 3 & 4 \end{array} \\
\left[\begin{array}{ccccc}
1 & -1 & \sqrt{3} & -1 & 1 \\
1 & -4 & \sqrt{6} & -\sqrt{8} & 1 \\
\sqrt{3} & -\sqrt{6} & 6 & -\sqrt{6} & \sqrt{3} \\
1 & -\sqrt{8} & \sqrt{6} & -4 & 1 \\
1 & -1 & \sqrt{3} & -1 & 1
\end{array}\right]
\end{array}
\quad
\begin{array}{c} 1 \\ \sqrt{2} \\ \sqrt{14} \\ \sqrt{14} \\ \sqrt{70} \end{array}$$

$$\left[\begin{array}{ccccc}
1 & -\sqrt{3} & 1 & -1 & \cdot \\
\sqrt{3} & -2 & \sqrt{2} & 0 & -1 \\
1 & -\sqrt{2} & 0 & \sqrt{2} & -1 \\
1 & 0 & -\sqrt{2} & 2 & -\sqrt{3} \\
\cdot & 1 & -1 & \sqrt{3} & -1
\end{array}\right]
\quad
\begin{array}{c} \sqrt{2} \\ \sqrt{2} \\ \sqrt{10} \\ \sqrt{10} \end{array}$$

$$\left[\begin{array}{ccccc}
2 & -\sqrt{6} & \sqrt{2} & \cdot & \cdot \\
\sqrt{6} & -1 & -1 & \sqrt{3} & \cdot \\
\sqrt{2} & 1 & -2 & 1 & \sqrt{2} \\
\cdot & \sqrt{3} & -1 & -1 & \sqrt{6} \\
\cdot & \cdot & \sqrt{2} & -\sqrt{6} & 2
\end{array}\right]
\quad
\begin{array}{c} \sqrt{7} \\ \sqrt{14} \\ \sqrt{14} \end{array}$$

$$\left[\begin{array}{ccccc}
2 & -\sqrt{2} & \cdot & \cdot & \cdot \\
\sqrt{2} & 1 & -\sqrt{3} & \cdot & \cdot \\
\cdot & \sqrt{3} & 0 & -\sqrt{3} & \cdot \\
\cdot & \cdot & \sqrt{3} & -1 & -\sqrt{2} \\
\cdot & \cdot & \cdot & \sqrt{2} & -2
\end{array}\right]
\quad
\begin{array}{c} \sqrt{10} \\ \sqrt{10} \end{array}$$

(d) $l = 2$

$$\begin{array}{c}
\begin{array}{ccc} q=0 & 1 & 2 \end{array} \\
\left[\begin{array}{ccc}
1 & -1 & 1 \\
1 & -2 & 1 \\
1 & -1 & -1
\end{array}\right]
\end{array}$$

$$\left[\begin{array}{ccc}
1 & -1 & \cdot \\
1 & 0 & -1 \\
\cdot & 1 & -1
\end{array}\right]$$

(p) $l = 1$

Tensor operators for spin-J states: U(2J+1) generalization of Pauli spinors

$$\mathbf{v}^k_q = \sum_{m,m'} (-1)^{j-m}\sqrt{2k+1}\begin{pmatrix} k & j & j\\ q & m' & -m\end{pmatrix} \left| \begin{matrix} j\\ m\end{matrix}\right\rangle\left\langle \begin{matrix} j \\ m'\end{matrix}\right|$$

for $j = 1,2,3.$

$q = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$

-1

$$v_q^6 = \left|\begin{array}{ccccccc} 1 & -\sqrt{2} & 1 & -\sqrt{2} & \sqrt{5} & -1 & 1 \\ \sqrt{2} & -6 & \sqrt{30} & -\sqrt{8} & 3 & -\sqrt{12} & 1 \\ 1 & -\sqrt{30} & 15 & -10 & \sqrt{15} & -3 & \sqrt{5} \\ \sqrt{2} & -\sqrt{8} & 10 & -20 & 10 & -\sqrt{8} & \sqrt{2} \\ \sqrt{5} & -3 & \sqrt{15} & -10 & 15 & -\sqrt{30} & 1 \\ 1 & -\sqrt{12} & 3 & -\sqrt{8} & \sqrt{30} & -6 & \sqrt{2} \\ 1 & -1 & \sqrt{5} & -\sqrt{2} & 1 & -\sqrt{2} & 1 \end{array}\right| \begin{array}{c} 1 \\ \sqrt{2} \\ \sqrt{22} \\ \sqrt{22} \\ \sqrt{33} \\ \sqrt{264} \\ \sqrt{924} \end{array}$$

$$v_q^5 = \left|\begin{array}{ccccccc} 1 & -\sqrt{5} & 1 & -\sqrt{2} & 1 & -1 & \cdot \\ \sqrt{5} & -4 & \sqrt{27} & -\sqrt{2} & 1 & 0 & -1 \\ 1 & -\sqrt{27} & 5 & -\sqrt{10} & 0 & 1 & -1 \\ \sqrt{2} & -\sqrt{2} & \sqrt{10} & 0 & -\sqrt{10} & \sqrt{2} & \sqrt{2} \\ 1 & -1 & 0 & \sqrt{10} & -5 & \sqrt{27} & -1 \\ 1 & 0 & -1 & \sqrt{2} & -\sqrt{27} & 4 & -\sqrt{5} \\ \cdot & 1 & -1 & \sqrt{2} & -1 & \sqrt{5} & -1 \end{array}\right| \begin{array}{c} \\ \sqrt{2} \\ \sqrt{2} \\ \sqrt{6} \\ \sqrt{6} \\ \sqrt{84} \\ \sqrt{84} \end{array}$$

$$v_q^4 = \left|\begin{array}{ccccccc} 3 & -\sqrt{30} & \sqrt{54} & -3 & \sqrt{3} & \cdot & \cdot \\ \sqrt{30} & -7 & \sqrt{32} & -\sqrt{3} & -\sqrt{2} & \sqrt{5} & \cdot \\ \sqrt{54} & -\sqrt{32} & 1 & \sqrt{15} & -\sqrt{40} & \sqrt{2} & \sqrt{3} \\ 3 & -\sqrt{3} & -\sqrt{15} & 6 & -\sqrt{15} & -\sqrt{3} & 3 \\ \sqrt{3} & \sqrt{2} & -\sqrt{40} & \sqrt{15} & 1 & -\sqrt{32} & \sqrt{54} \\ \cdot & \sqrt{5} & -\sqrt{2} & -\sqrt{3} & \sqrt{32} & -7 & \sqrt{30} \\ \cdot & \cdot & \sqrt{3} & -3 & \sqrt{54} & -\sqrt{30} & 3 \end{array}\right| \begin{array}{c} \\ \\ \sqrt{11} \\ \sqrt{22} \\ \sqrt{154} \\ \sqrt{154} \\ \sqrt{154} \end{array}$$

$$v_q^3 = \left|\begin{array}{ccccccc} 1 & -\sqrt{2} & \sqrt{2} & -1 & \cdot & \cdot & \cdot \\ \sqrt{2} & -1 & 0 & 1 & -\sqrt{2} & \cdot & \cdot \\ \sqrt{2} & 0 & -1 & 1 & 0 & -\sqrt{2} & \cdot \\ 1 & 1 & -1 & 0 & 1 & -1 & -1 \\ \cdot & \sqrt{2} & 0 & -1 & 1 & 0 & -\sqrt{2} \\ \cdot & \cdot & \sqrt{2} & -1 & 0 & 1 & -\sqrt{2} \\ \cdot & \cdot & \cdot & 1 & -\sqrt{2} & \sqrt{2} & -1 \end{array}\right| \begin{array}{c} \\ \\ \\ \sqrt{6} \\ \sqrt{6} \\ \sqrt{6} \\ \sqrt{6} \end{array}$$

$$v_q^2 = \left|\begin{array}{ccccccc} 5 & -5 & \sqrt{5} & \cdot & \cdot & \cdot & \cdot \\ 5 & 0 & -\sqrt{15} & \sqrt{10} & \cdot & \cdot & \cdot \\ \sqrt{5} & \sqrt{15} & -3 & -\sqrt{2} & \sqrt{12} & \cdot & \cdot \\ \cdot & \sqrt{10} & \sqrt{2} & -4 & \sqrt{2} & \sqrt{10} & \cdot \\ \cdot & \cdot & \sqrt{12} & -\sqrt{2} & -3 & \sqrt{15} & \sqrt{5} \\ \cdot & \cdot & \cdot & \sqrt{10} & -\sqrt{15} & 0 & 5 \\ \cdot & \cdot & \cdot & \cdot & \sqrt{5} & -5 & 5 \end{array}\right| \begin{array}{c} \\ \\ \\ \\ \sqrt{42} \\ \sqrt{84} \\ \sqrt{84} \end{array}$$

$$v_q^1 = \left|\begin{array}{ccccccc} 3 & -\sqrt{3} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \sqrt{3} & 2 & -\sqrt{5} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \sqrt{5} & 1 & -\sqrt{6} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \sqrt{6} & 0 & -\sqrt{6} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \sqrt{6} & -1 & -\sqrt{5} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \sqrt{5} & -2 & -\sqrt{3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{3} & -3 \end{array}\right| \begin{array}{c} \\ \\ \\ \\ \\ \sqrt{28} \\ \sqrt{28} \end{array}$$

(f) $l = 3$

$q = 0 \quad 1 \quad 2 \quad 3 \quad 4$

$$\left|\begin{array}{ccccc} 1 & -1 & \sqrt{3} & -1 & 1 \\ 1 & -4 & \sqrt{6} & -\sqrt{8} & 1 \\ \sqrt{3} & -\sqrt{6} & 6 & -\sqrt{6} & \sqrt{3} \\ 1 & -\sqrt{8} & \sqrt{6} & -4 & 1 \\ 1 & -1 & \sqrt{3} & -1 & 1 \end{array}\right| \begin{array}{c} 1 \\ \sqrt{2} \\ \sqrt{14} \\ \sqrt{14} \\ \sqrt{70} \end{array}$$

$$\left|\begin{array}{ccccc} 1 & -\sqrt{3} & 1 & -1 & \cdot \\ \sqrt{3} & -2 & \sqrt{2} & 0 & -1 \\ 1 & -\sqrt{2} & 0 & \sqrt{2} & -1 \\ 1 & 0 & -\sqrt{2} & 2 & -\sqrt{3} \\ \cdot & 1 & -1 & \sqrt{3} & -1 \end{array}\right| \begin{array}{c} \\ \sqrt{2} \\ \sqrt{2} \\ \sqrt{10} \\ \sqrt{10} \end{array}$$

$$\left|\begin{array}{ccccc} 2 & -\sqrt{6} & \sqrt{2} & \cdot & \cdot \\ \sqrt{6} & -1 & -1 & \sqrt{3} & \cdot \\ \sqrt{2} & 1 & -2 & 1 & \sqrt{2} \\ \cdot & \sqrt{3} & -1 & -1 & \sqrt{6} \\ \cdot & \cdot & \sqrt{2} & -\sqrt{6} & 2 \end{array}\right| \begin{array}{c} \\ \\ \sqrt{7} \\ \sqrt{14} \\ \sqrt{14} \end{array}$$

$$\left|\begin{array}{ccccc} 2 & -\sqrt{2} & \cdot & \cdot & \cdot \\ \sqrt{2} & 1 & -\sqrt{3} & \cdot & \cdot \\ \cdot & \sqrt{3} & 0 & -\sqrt{3} & \cdot \\ \cdot & \cdot & \sqrt{3} & -1 & -\sqrt{2} \\ \cdot & \cdot & \cdot & \sqrt{2} & -2 \end{array}\right| \begin{array}{c} \\ \\ \\ \sqrt{10} \\ \sqrt{10} \end{array}$$

(d) $l = 2$

$q = 0 \quad 1 \quad 2$

$$\left|\begin{array}{ccc} 1 & -1 & 1 \\ 1 & -2 & 1 \\ 1 & -1 & -1 \end{array}\right|$$

$$\left|\begin{array}{ccc} 1 & -1 & \cdot \\ 1 & 0 & -1 \\ \cdot & 1 & -1 \end{array}\right|$$

(p) $l = 1$

(a) $j = \frac{1}{2}$

$q = 0 \quad 1$

$$v_q^1 = \left|\begin{array}{cc} 1 & -1 \\ 1 & -1 \end{array}\right| \begin{array}{c} 1 \\ \sqrt{2} \end{array}$$

(b) $j = \frac{3}{2}$

$q = 0 \quad 1 \quad 2 \quad 3$

$$v_q^1 = \left|\begin{array}{cccc} 3 & -\sqrt{3} & \cdot & \cdot \\ \sqrt{3} & 1 & -2 & \cdot \\ \cdot & 2 & -1 & -\sqrt{3} \\ \cdot & \cdot & \sqrt{3} & -3 \end{array}\right| \begin{array}{c} \\ \\ \sqrt{10} \\ \sqrt{20} \end{array}$$

$$v_q^2 = \left|\begin{array}{cccc} 1 & -1 & 1 & \cdot \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ \cdot & 1 & -1 & 1 \end{array}\right| \begin{array}{c} \\ \sqrt{2} \\ \sqrt{2} \\ \sqrt{4} \end{array}$$

$$v_q^3 = \left|\begin{array}{cccc} 1 & -1 & 1 & -1 \\ 1 & -3 & \sqrt{3} & -1 \\ 1 & -\sqrt{3} & 3 & -1 \\ 1 & -1 & 1 & -1 \end{array}\right| \begin{array}{c} 1 \\ \sqrt{2} \\ \sqrt{5} \\ \sqrt{20} \end{array}$$

(c) $j = \frac{5}{2}$

$q = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5$

$$v^1q = \left|\begin{array}{cccccc} 5 & -\sqrt{5} & \cdot & \cdot & \cdot & \cdot \\ \sqrt{5} & 3 & -\sqrt{8} & \cdot & \cdot & \cdot \\ \cdot & \sqrt{8} & 1 & -3 & \cdot & \cdot \\ \cdot & \cdot & 3 & -1 & -\sqrt{8} & \cdot \\ \cdot & \cdot & \cdot & \sqrt{8} & -3 & -\sqrt{5} \\ \cdot & \cdot & \cdot & \cdot & \sqrt{5} & -5 \end{array}\right| \begin{array}{c} \\ \\ \\ \\ \sqrt{35} \\ \sqrt{70} \end{array}$$

$$v_q^2 = \left|\begin{array}{cccccc} 5 & -\sqrt{5} & \sqrt{5} & \cdot & \cdot & \cdot \\ \sqrt{5} & -1 & -\sqrt{2} & 3 & \cdot & \cdot \\ \sqrt{5} & \sqrt{2} & -4 & 0 & 3 & \cdot \\ \cdot & 3 & 0 & -4 & \sqrt{2} & \sqrt{5} \\ \cdot & \cdot & 3 & -\sqrt{2} & -1 & \sqrt{5} \\ \cdot & \cdot & \cdot & \sqrt{5} & -\sqrt{5} & 5 \end{array}\right| \begin{array}{c} \\ \\ \\ \sqrt{28} \\ \sqrt{14} \\ \sqrt{84} \end{array}$$

$$v_q^3 = \left|\begin{array}{cccccc} 5 & -\sqrt{10} & \sqrt{5} & -\sqrt{5} & \cdot & \cdot \\ \sqrt{10} & -7 & 1 & 1 & -\sqrt{8} & \cdot \\ \sqrt{5} & -1 & -4 & \sqrt{8} & -1 & -\sqrt{5} \\ \sqrt{5} & 1 & -\sqrt{8} & 4 & 1 & -\sqrt{5} \\ \cdot & \sqrt{8} & -1 & -1 & 7 & -\sqrt{10} \\ \cdot & \cdot & \sqrt{5} & -\sqrt{5} & \sqrt{10} & -5 \end{array}\right| \begin{array}{c} \\ \\ \sqrt{18} \\ \sqrt{12} \\ \sqrt{30} \\ \sqrt{180} \end{array}$$

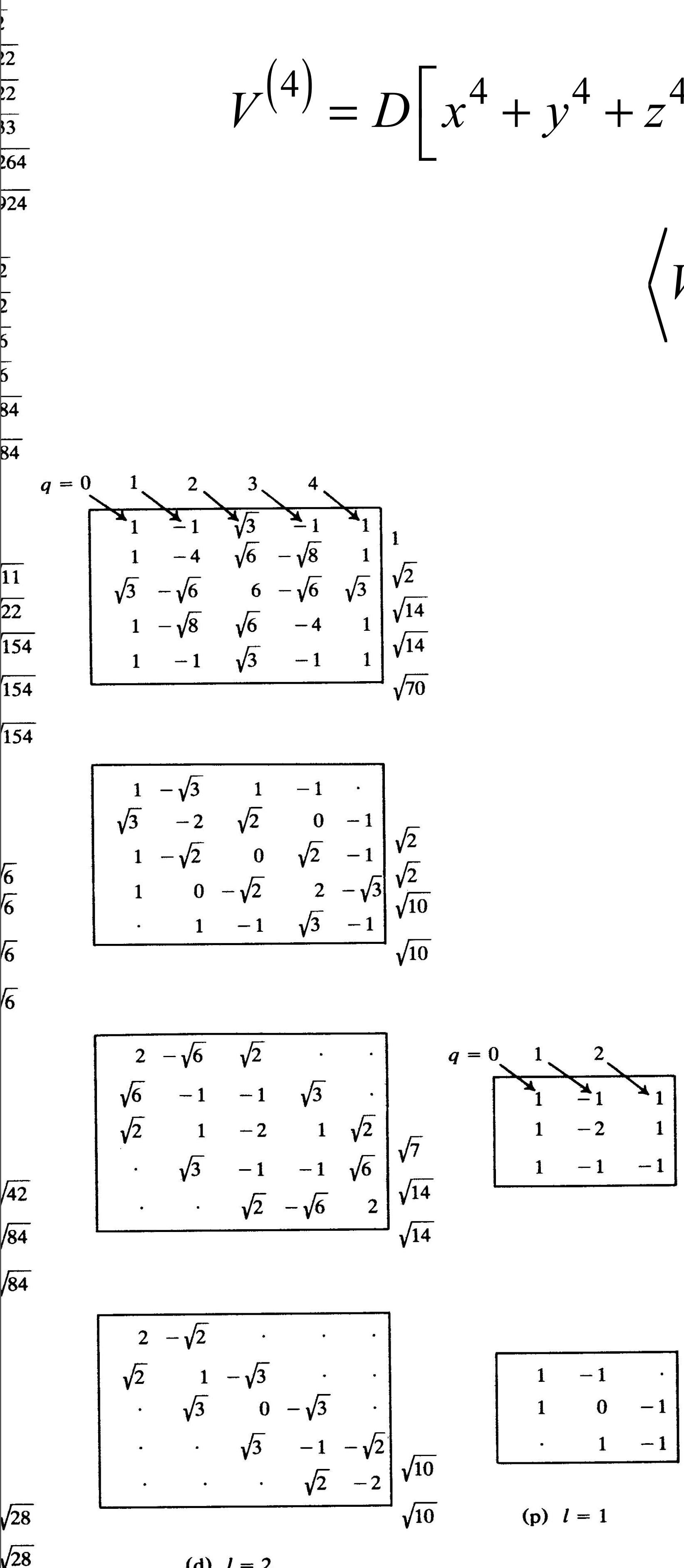
$$v_q^4 = \left|\begin{array}{cccccc} 1 & -\sqrt{2} & 3 & -1 & 1 & \cdot \\ \sqrt{2} & -3 & \sqrt{5} & -\sqrt{5} & 0 & 1 \\ 3 & -\sqrt{5} & 2 & 0 & -\sqrt{5} & 1 \\ 1 & -\sqrt{5} & 0 & 2 & -\sqrt{5} & 3 \\ 1 & 0 & -\sqrt{5} & \sqrt{5} & -3 & \sqrt{2} \\ \cdot & 1 & -1 & 3 & -\sqrt{2} & 1 \end{array}\right| \begin{array}{c} \\ \sqrt{2} \\ \sqrt{2} \\ \sqrt{28} \\ \sqrt{14} \\ \sqrt{28} \end{array}$$

$$v_q^5 = \left|\begin{array}{cccccc} 1 & -1 & 1 & -\sqrt{2} & 1 & -1 \\ 1 & -5 & \sqrt{10} & -\sqrt{5} & \sqrt{5} & -1 \\ 1 & -\sqrt{10} & 10 & -\sqrt{20} & \sqrt{5} & -\sqrt{2} \\ \sqrt{2} & -\sqrt{5} & \sqrt{20} & -10 & \sqrt{10} & -1 \\ 1 & -\sqrt{5} & \sqrt{5} & -\sqrt{10} & 5 & -1 \\ 1 & -1 & \sqrt{2} & -1 & 1 & -1 \end{array}\right| \begin{array}{c} 1 \\ \sqrt{2} \\ \sqrt{9} \\ \sqrt{12} \\ \sqrt{42} \\ \sqrt{252} \end{array}$$

$$V^{(4)} = D\left[x^4 + y^4 + z^4 - \tfrac{3}{4}r^4\right] = D\left[\frac{2}{\sqrt{70}}\left(X^4_4 + X^4_{-4}\right) + \tfrac{2}{5}X^4_0\right]$$

$$\left\langle V^{(4)}\right\rangle_{j=2} = D\left\langle \frac{2}{\sqrt{70}}\left(v^4_4 + v^4_{-4}\right) + \tfrac{2}{5}v^4_0\right\rangle_{j=2} \frac{\sqrt{5}}{3}\left\langle 2\right|\left|X^4\right|\left|2\right\rangle.$$

$$\left\langle V^{(4)}\right\rangle_{j=2} = \frac{D}{\sqrt{70}}\begin{pmatrix} \frac{2}{5} & \cdot & \cdot & \cdot & 2 \\ \cdot & -\frac{8}{5} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \frac{12}{5} & \cdot & \cdot \\ \cdot & \cdot & \cdot & -\frac{8}{5} & \cdot \\ 2 & \cdot & \cdot & \cdot & \frac{2}{5} \end{pmatrix}\frac{\sqrt{5}}{3}\left\langle 2\right\|X^4\left\|2\right\rangle.$$



q = 0	1	2	3	4	
1	−1	√3	−1	1	1
1	−4	√6	−√8	1	√2
√3	−√6	6	−√6	√3	√14
1	−√8	√6	−4	1	√14
1	−1	√3	−1	1	√70

1	−√3	1	−1	·	
√3	−2	√2	0	−1	√2
1	−√2	0	√2	−1	√2
1	0	−√2	2	−√3	√10
·	1	−1	√3	−1	√10

2	−√6	√2	·	·	
√6	−1	−1	√3	·	
√2	1	−2	1	√2	√7
·	√3	−1	−1	√6	√14
·	·	√2	−√6	2	√14

2	−√2	·	·	·	
√2	1	−√3	·	·	
·	√3	0	−√3	·	
·	·	√3	−1	−√2	√10
·	·	·	√2	−2	√10

(d) $l = 2$

q = 0	1	2
1	−1	1
1	−2	1
1	−1	−1

1	−1	·
1	0	−1
·	1	−1

(p) $l = 1$