Group Theory in Quantum Mechanics

Lecture 24 *(4.25.13)*

Harmonic oscillator symmetry U(1)⊂U(2)⊂U(3)...

(Int.J.Mol.Sci, 14, 714(2013) p.755-774 , QTCA Unit 7 Ch. 21-22)
(PSDS - Ch. 8)

Review : *1-D* $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(1) representations*

2-D $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(2) representations and R(3) angular momentum operators*

- *2D-Oscillator basics*
  - *Commutation relations*
  - *Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry*
    - *Anti-commutation relations*
  - *Two-dimensional (or 2-particle) base states: ket-kets and bra-bras*
    - *Outer product arrays*
    - *Entangled 2-particle states*
- *Two-particle (or 2-dimensional) matrix operators*
  - *U(2) Hamiltonian and irreducible representations*
  - *2D-Oscillator eigensolutions*

U(1) Oscillator coherent states ("Shoved" and "kicked" states) ← Left from 4.23.13

- *Translation operators vs. boost operators*
- *Applying boost-translation combinations*
- *Time evolution of coherent state*
- *Properties of coherent state and "squeezed" states*

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Left from 4.23.13

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Review : *Creation-Destruction* $\mathbf{a}^\dagger\mathbf{a}$ *algebra*

Define

$$\mathbf{a} = \frac{(\mathbf{X}+i\mathbf{P})}{\sqrt{\hbar\omega}} = \frac{\left(\sqrt{M\omega}\,\mathbf{x}+i\mathbf{p}/\sqrt{M\omega}\right)}{\sqrt{2\hbar}}$$

Destruction operator

and

$$\mathbf{a}^\dagger = \frac{(\mathbf{X}-i\mathbf{P})}{\sqrt{\hbar\omega}} = \frac{\left(\sqrt{M\omega}\,\mathbf{x}-i\mathbf{p}/\sqrt{M\omega}\right)}{\sqrt{2\hbar}}$$

Creation Operator

Commutation relations between $\mathbf{a} = (\mathbf{X}+i\mathbf{P})/2$ and $\mathbf{a}^\dagger = (\mathbf{X}-i\mathbf{P})/2$ with $\mathbf{X}\equiv\sqrt{M\omega}\mathbf{x}/\sqrt{2}$ and $\mathbf{P}\equiv\mathbf{p}/\sqrt{2M}$:

$$\left[\mathbf{a},\mathbf{a}^\dagger\right]\equiv\mathbf{a}\mathbf{a}^\dagger-\mathbf{a}^\dagger\mathbf{a}=\frac{1}{2\hbar}\left(\sqrt{M\omega}\,\mathbf{x}+i\mathbf{p}/\sqrt{M\omega}\right)\left(\sqrt{M\omega}\,\mathbf{x}-i\mathbf{p}/\sqrt{M\omega}\right)-\frac{1}{2\hbar}\left(\sqrt{M\omega}\,\mathbf{x}-i\mathbf{p}/\sqrt{M\omega}\right)\left(\sqrt{M\omega}\,\mathbf{x}+i\mathbf{p}/\sqrt{M\omega}\right)$$

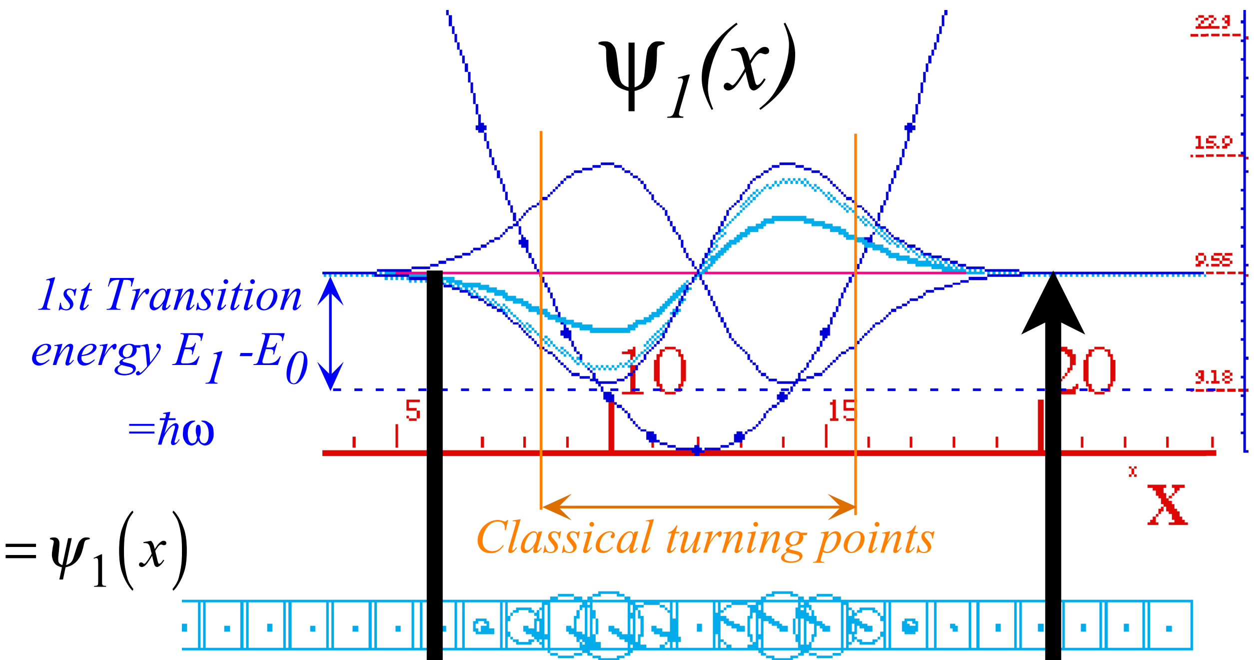
$$\left[\mathbf{a},\mathbf{a}^\dagger\right]=\frac{2i}{2\hbar}(\mathbf{px}-\mathbf{xp})=\frac{-i}{\hbar}\left[\mathbf{x},\mathbf{p}\right]=\mathbf{1}$$

$$\boxed{\left[\mathbf{a},\mathbf{a}^\dagger\right]=\mathbf{1}}\quad\text{or}\quad\boxed{\mathbf{a}\mathbf{a}^\dagger=\mathbf{a}^\dagger\mathbf{a}+\mathbf{1}}\qquad\left[\mathbf{x},\mathbf{p}\right]\equiv\mathbf{xp}-\mathbf{px}=\hbar i\,\mathbf{1}$$

Wavefunction creationism (1st Excited state)

1st excited state wavefunction $\psi_1(x) = \langle x|1\rangle$

$$\langle x|\mathbf{a}^\dagger|0\rangle = \langle x|1\rangle = \psi_1(x)$$



Expanding the creation operator

$$\langle x|\mathbf{a}^\dagger|0\rangle = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{M\omega}\,\langle x|\mathbf{x}|0\rangle - i\langle x|\mathbf{p}|0\rangle/\sqrt{M\omega}\right) = \langle x|1\rangle = \psi_1(x)$$

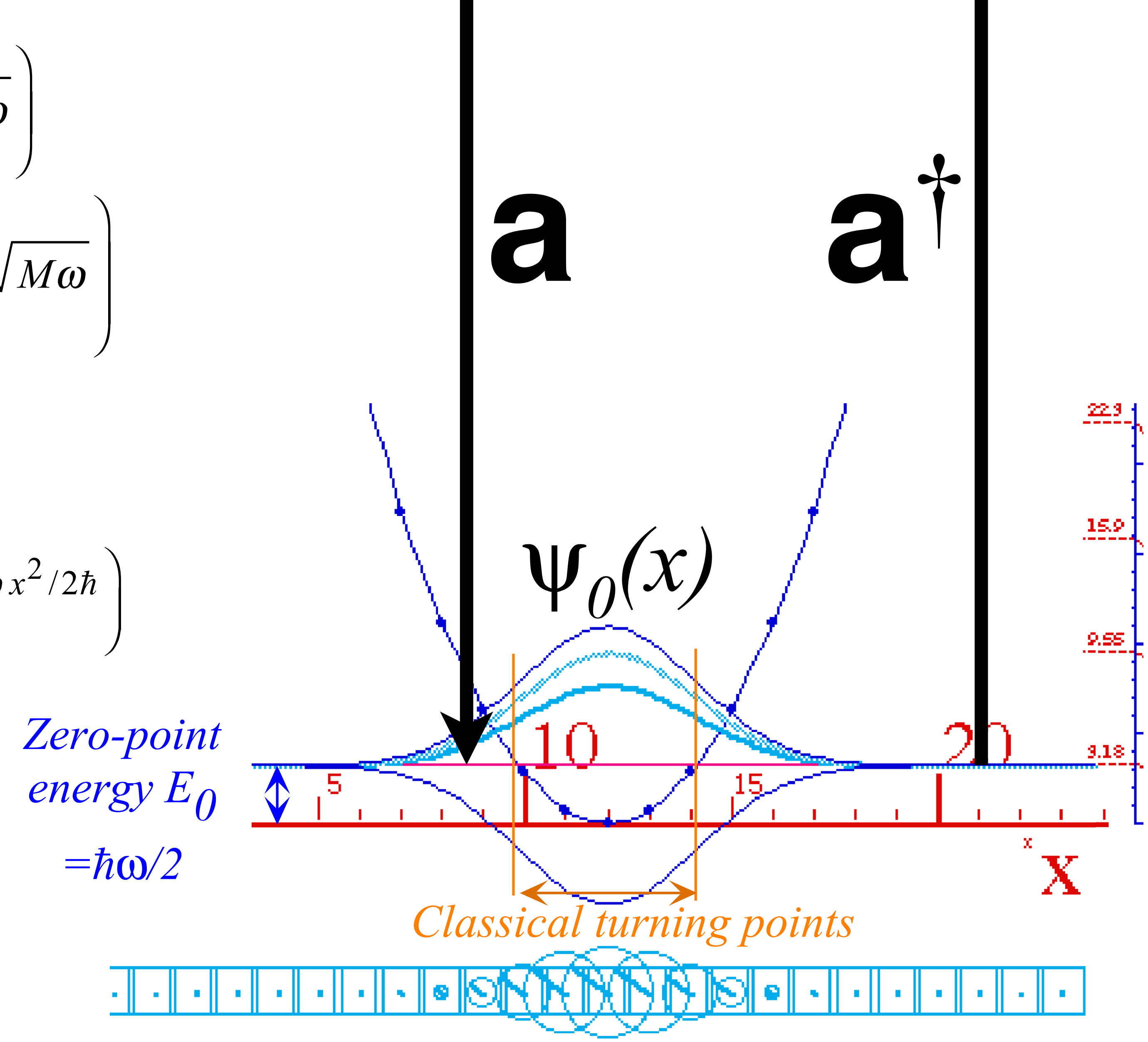
The operator coordinate representations generate the first excited state wavefunction.

$$\langle x|1\rangle = \psi_1(x) = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{M\omega}\;x\psi_0(x) - i\frac{\hbar}{i}\frac{\partial\psi_0(x)}{\partial x}/\sqrt{M\omega}\right)$$

$$= \frac{1}{\sqrt{2\hbar}}\left(\sqrt{M\omega}\;x\frac{e^{-M\omega x^2/2\hbar}}{const.} - i\frac{\hbar}{i}\frac{\partial}{\partial x}\frac{e^{-M\omega x^2/2\hbar}}{const.}/\sqrt{M\omega}\right)$$

$$= \frac{1}{\sqrt{2\hbar}}\frac{e^{-M\omega x^2/2\hbar}}{const.}\left(\sqrt{M\omega}\;x + i\frac{\hbar}{i}\frac{M\omega x}{\hbar}/\sqrt{M\omega}\right)$$

$$= \frac{\sqrt{M\omega}}{\sqrt{2\hbar}}\frac{e^{-M\omega x^2/2\hbar}}{const.}(2x) = \left(\frac{M\omega}{\pi\hbar}\right)^{3/4}\sqrt{2\pi}\left(x\,e^{-M\omega x^2/2\hbar}\right)$$



Review : *Matrix* $\langle \mathbf{a}^n \mathbf{a}^{\dagger n} \rangle$ *calculation*

Derive normalization for n^{th} state obtained by $(\mathbf{a}^\dagger)^n$ operator:

Use: $\mathbf{a}^n \mathbf{a}^{\dagger n} = n!\left(\mathbf{1} + n\mathbf{a}^\dagger \mathbf{a} + \frac{n(n-1)}{2!\cdot 2!}\mathbf{a}^{\dagger 2}\mathbf{a}^2 + \ldots\right)$

$$|n\rangle = \frac{\mathbf{a}^{\dagger n}|0\rangle}{const.}, \qquad \text{where:} \qquad 1 = \langle n|n\rangle = \frac{\langle 0|\mathbf{a}^n \mathbf{a}^{\dagger n}|0\rangle}{(const.)^2} = n!\frac{\langle 0|\mathbf{1} + n\mathbf{a}^\dagger \mathbf{a} + ..|0\rangle}{(const.)^2} = \frac{n!}{(const.)^2}$$

$$|n\rangle = \frac{\mathbf{a}^{\dagger n}|0\rangle}{\sqrt{n!}} \qquad \textit{Root-factorial normalization}$$

Use: $\mathbf{a}\mathbf{a}^{\dagger n} = n\mathbf{a}^{\dagger n-1} + \mathbf{a}^{\dagger n}\mathbf{a}$

Apply creation $\mathbf{a}^\dagger$:

$$\mathbf{a}^\dagger|n\rangle = \frac{\mathbf{a}^{\dagger n+1}|0\rangle}{\sqrt{n!}} = \sqrt{n+1}\,\frac{\mathbf{a}^{\dagger n+1}|0\rangle}{\sqrt{(n+1)!}}$$

Apply destruction $\mathbf{a}$:

$$\mathbf{a}|n\rangle = \frac{\mathbf{a}\mathbf{a}^{\dagger n}|0\rangle}{\sqrt{n!}} = \frac{(n\mathbf{a}^{\dagger n-1} + \mathbf{a}^{\dagger n}\mathbf{a})|0\rangle}{\sqrt{n!}} = \sqrt{n}\,\frac{\mathbf{a}^{\dagger n-1}|0\rangle}{\sqrt{(n-1)!}}$$

$$\mathbf{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle \qquad\qquad \mathbf{a}|n\rangle = \sqrt{n}\,|n-1\rangle$$

Feynman's mnemonic rule: Larger of two quanta goes in radical factor

$$\langle \mathbf{a}^\dagger \rangle = \begin{pmatrix} \cdot & & & & & \\ 1 & \cdot & & & & \\ & \sqrt{2} & \cdot & & & \\ & & \sqrt{3} & \cdot & & \\ & & & \sqrt{4} & \cdot & \\ & & & & \ddots & \cdot \end{pmatrix} \qquad \langle \mathbf{a} \rangle = \begin{pmatrix} \cdot & 1 & & & & \\ & \cdot & \sqrt{2} & & & \\ & & \cdot & \sqrt{3} & & \\ & & & \cdot & \sqrt{4} & \\ & & & & \cdot & \ddots \\ & & & & & \cdot \end{pmatrix}$$

Use: $\mathbf{a}\mathbf{a}^{\dagger n} = n\mathbf{a}^{\dagger n-1} + \mathbf{a}^{\dagger n}\mathbf{a}$

Number operator and Hamiltonian operator

Number operator $\mathbf{N}=\mathbf{a}^\dagger\mathbf{a}$ counts quanta.

$$\mathbf{a}^\dagger \mathbf{a}|n\rangle = \frac{\mathbf{a}^\dagger \mathbf{a}\mathbf{a}^{\dagger n}|0\rangle}{\sqrt{n!}} = n\frac{\mathbf{a}^\dagger \mathbf{a}^{\dagger n-1}|0\rangle}{\sqrt{n!}} = n\frac{\mathbf{a}^{\dagger n}|0\rangle}{\sqrt{n!}} = n|n\rangle$$

Hamiltonian operator

$$\mathbf{H}\,|n\rangle = \hbar\omega\, \mathbf{a}^\dagger \mathbf{a}\,|n\rangle + \hbar\omega/2\,\mathbf{1}\,|n\rangle = \hbar\omega(n+1/2)|n\rangle$$

$$\langle \mathbf{H} \rangle = \hbar\omega\langle \mathbf{a}^\dagger \mathbf{a} + \tfrac{1}{2}\mathbf{1}\rangle = \hbar\omega \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 2 & & \\ & & & 3 & \\ & & & & \ddots \end{pmatrix} + \hbar\omega \begin{pmatrix} 1/2 & & & & \\ & 1/2 & & & \\ & & 1/2 & & \\ & & & 1/2 & \\ & & & & \ddots \end{pmatrix}$$

Hamiltonian operator is $\hbar\omega$ $\mathbf{N}$ plus zero-point energy $\mathbf{1}\hbar\omega/2$.

Review : *Expectation values of position, momentum, and uncertainty for eigenstate* $|n\rangle$

Operator for position $\mathbf{x}$: $\sqrt{\frac{M\omega}{2\hbar}}\mathbf{x} = \frac{\mathbf{a}+\mathbf{a}^\dagger}{2}$

expectation for position $\langle\mathbf{x}\rangle$:

$$\bar{\mathbf{x}}|_n = \langle n|\mathbf{x}|n\rangle = \sqrt{\frac{\hbar}{2M\omega}}\langle n|\left(\mathbf{a}+\mathbf{a}^\dagger\right)|n\rangle = 0$$

expectation for (position)^2 $\langle\mathbf{x}^2\rangle$:

$$\overline{\mathbf{x}^2}|_n = \langle n|\mathbf{x}^2|n\rangle = \frac{\hbar}{2M\omega}\langle n|\left(\mathbf{a}+\mathbf{a}^\dagger\right)^2|n\rangle$$

$$= \frac{\hbar}{2M\omega}\langle n|\left(\mathbf{a}^2+\mathbf{a}^\dagger\mathbf{a}+\mathbf{a}\mathbf{a}^\dagger+\mathbf{a}^{\dagger 2}\right)|n\rangle$$

$$= \frac{\hbar}{2M\omega}\quad(2n+1)$$

Operator for momentum $\mathbf{p}$: $\sqrt{\frac{1}{2\hbar M\omega}}\mathbf{p} = \frac{\mathbf{a}-\mathbf{a}^\dagger}{2i}$

expectation for momentum $\langle\mathbf{p}\rangle$:

$$\bar{\mathbf{p}}|_n = \langle n|\mathbf{p}|n\rangle = i\sqrt{\frac{\hbar M\omega}{2}}\langle n|\left(\mathbf{a}^\dagger-\mathbf{a}\right)|n\rangle = 0$$

expectation for (momentum)^2 $\langle\mathbf{p}^2\rangle$:

$$\overline{\mathbf{p}^2}|_n = \langle n|\mathbf{p}^2|n\rangle = i^2\frac{\hbar M\omega}{2}\langle n|\left(\mathbf{a}^\dagger-\mathbf{a}\right)^2|n\rangle$$

$$= -\frac{\hbar M\omega}{2}\langle n|\left(\mathbf{a}^{\dagger 2}-\mathbf{a}^\dagger\mathbf{a}-\mathbf{a}\mathbf{a}^\dagger+\mathbf{a}^2\right)|n\rangle$$

$$= \frac{\hbar M\omega}{2}\quad(2n+1)$$

Use:
$\mathbf{a}\mathbf{a}^\dagger = \mathbf{1}+\mathbf{a}^\dagger\mathbf{a}$

Uncertainty or *standard deviation* Δq of a statistical quantity q is its root mean-square difference.

$$(\Delta q)^2 = \overline{(q-\bar{q})^2} \quad\text{or:}\quad \Delta q = \sqrt{\overline{(q-\bar{q})^2}}$$

$$\Delta x|_n = \sqrt{\overline{\mathbf{x}^2}} = \sqrt{\frac{\hbar(2n+1)}{2M\omega}} \qquad \Delta p|_n = \sqrt{\overline{\mathbf{p}^2}} = \sqrt{\frac{\hbar M\omega(2n+1)}{2}}$$

Heisenberg uncertainty product for the n-quantum eigenstate $|n\rangle$

$$(\Delta x\cdot\Delta p)|_n = \sqrt{\overline{\mathbf{x}^2}}\sqrt{\overline{\mathbf{p}^2}} = \sqrt{\frac{\hbar(2n+1)}{2M\omega}}\sqrt{\frac{\hbar M\omega(2n+1)}{2}}$$

$$\boxed{(\Delta x\cdot\Delta p)|_n = \hbar\left(n+\frac{1}{2}\right)}$$

Heisenberg minimum uncertainty product occurs for the *0*-quantum (ground) eigenstate.

$$\boxed{(\Delta x\cdot\Delta p)|_0 = \frac{\hbar}{2}}$$

 Harmonic oscillator beat dynamics of mixed states

$$|\Psi\rangle = |0\rangle\langle 0|\Psi\rangle + |1\rangle\langle 1|\Psi\rangle = |0\rangle\Psi_0 + |1\rangle\Psi_1$$

$$\Psi(x) = \langle x|\Psi\rangle = \langle x|0\rangle\langle 0|\Psi\rangle + \langle x|1\rangle\langle 1|\Psi\rangle = \psi_0(x)\,\Psi 0 + \psi_1(x)\,\Psi 1$$

The time dependence $\Psi(x,t)$ of the mixed wave is then

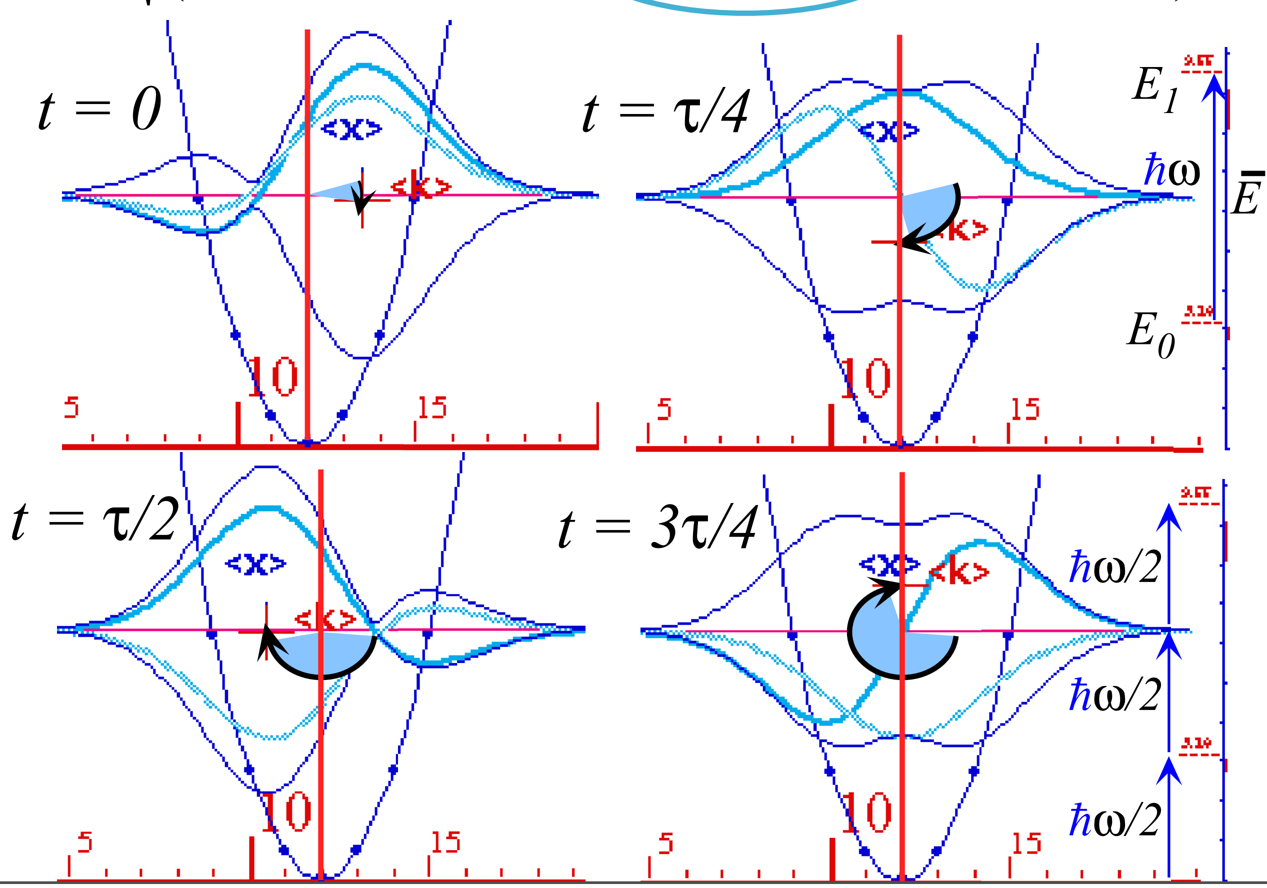
$$\Psi(x,t) = \psi_0(x)\, e^{-i\omega_0 t}\Psi_0 + \psi_1(x)\, e^{-i\omega_1 t}\Psi_1 = (\psi_0(x)\, e^{-i\omega_0 t} + \psi_1(x)\, e^{-i\omega_1 t})/\sqrt{2}$$

$$|\Psi(x,t)| = \sqrt{\Psi^*\Psi} = \sqrt{\left(e^{-i\omega_0 t}\psi_0(x) + e^{-i\omega_1 t}\psi_1(x)\right)^*\left(e^{-i\omega_0 t}\psi_0(x) + e^{-i\omega_1 t}\psi_1(x)\right)/2}$$

$$= \sqrt{\left(|\psi_0(x)|^2 + |\psi_1(x)|^2 + \psi_0(x)\psi_1(x)\left(e^{i(\omega_1-\omega_0)t} + e^{-i(\omega_1-\omega_0)t}\right)\right)/2}$$

$$= \sqrt{\left(|\psi_0(x)|^2 + |\psi_1(x)|^2 + 2\psi_0(x)\psi_1(x)\cos(\omega_1-\omega_0)t\right)/2}$$

Need some *overlap* somewhere to get some *wiggle*



Beat frequency is eigenfrequency difference

$$\omega_{beat} = \omega_1 - \omega_0 = \omega$$

Beat frequency ω = *Transition frequency* ω

Transition frequency is *transition energy*/$\hbar$

$$\Delta E = E_{1\leftarrow 0}\ transition = E_1 - E_0 = \hbar\omega$$

ω is frequency of radiating antenna of a transmitter or of a receiver, i.e., of an emitter or an absorber

(Usually of a dipole symmetry)

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Applying boost-translation combinations

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Translation operators and generators: (A "shove")

Translation operator $\mathbf{T}(a)$ shoves x-wavefunctions

$$\mathbf{T}(a)\cdot\psi(x) = \psi(x\text{-}a) = \langle x|\mathbf{T}(a)|\psi\rangle = \langle x\text{-}a\,|\psi\rangle$$

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Shoves ψ a-units to right or x-space a-units left

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1-D $a^\dagger a$ algebra of U(1) representations

Creation-Destruction $a^\dagger a$ algebra

Eigenstate creationism (and destruction)

Vacuum state

1st excited state

Normal ordering for matrix calculation

Commutator derivative identities

Binomial expansion identities

Matrix $\langle a^n a^{\dagger n} \rangle$ calculations

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Tiny translation $a \rightarrow da$ is identity $\mathbf{1}$ plus $\mathbf{G}\cdot da$

$$\mathbf{T}(da) = \mathbf{1} + \mathbf{G}\cdot da \quad \text{where: } \mathbf{G} = \left.\frac{\partial \mathbf{T}}{\partial a}\right|_{a=0}$$

is *generator* $\mathbf{G}$ *of translations*

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$$\mathbf{T}(a) = \left(\mathbf{T}\left(\frac{a}{N}\right)\right)^N = \lim_{N\to\infty}\left(1 + \frac{a}{N}\mathbf{G}\right)^N = e^{a\mathbf{G}}$$

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$$= \psi(x) - a\frac{\partial\psi(x)}{\partial x} + \frac{a^2}{2!}\frac{\partial^2\psi(x)}{\partial x^2} - \frac{a^3}{2!}\frac{\partial^3\psi(x)}{\partial x^3} + \ldots$$

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$$\mathbf{B}(b)\cdot\psi(p) = e^{b\mathbf{K}}\cdot\psi(p) = e^{-b\frac{\partial}{\partial p}}\cdot\psi(p)$$

$$= \psi(p) - b\frac{\partial\psi(p)}{\partial p} + \frac{b^2}{2!}\frac{\partial^2\psi(p)}{\partial p^2} - \frac{b^3}{2!}\frac{\partial^3\psi(p)}{\partial p^3} + \ldots$$

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Boost operators and generators: (A "kick")

Boost operator $\mathbf{B}(b)$ boosts p-wavefunctions

$$\mathbf{B}(b)\cdot\psi(p) = \psi(p-b) = \langle x|\mathbf{B}(b)|\psi\rangle = \langle p-b|\psi\rangle$$

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Oscillator coherent states ("Shoved" and "kicked" states)

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Translation operator $\mathbf{T}(a)$ **shoves** x**-wavefunctions**

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Check $\mathbf{T}(a)$ on plane-wave with $p=\hbar k$

Bottom Line

$$\mathbf{T}(a)e^{ikx} = e^{-ia\mathbf{p}/\hbar}e^{ikx} = e^{-iak}e^{ikx} = e^{ik(x-a)}$$

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Check $\mathbf{B}(b)$ on plane-wave with $p=\hbar k$

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1-D $a^{\dagger}a$ algebra of U(1) representations

Creation-Destruction $a^{\dagger}a$ algebra

Eigenstate creationism (and destruction)

Vacuum state

1st excited state

Normal ordering for matrix calculation

Commutator derivative identities

Binomial expansion identities

Matrix $\langle a^n a^{\dagger n}\rangle$ calculations

Number operator and Hamiltonian operator

Expectation values of position, momentum, and uncertainty for eigenstate $|n\rangle$

Harmonic oscillator beat dynamics of mixed states

Oscillator coherent states ("Shoved" and "kicked" states)

Translation operators vs. boost operators

Applying boost-translation combinations

Time evolution of coherent state

Properties of coherent state and "squeezed" states

2-D $a^{\dagger}a$ algebra of U(2) representations and R(3) angular momentum operators

Applying boost-translation combinations

T(a) and **B**(b) operations do not commute. Q. Which should come first? ??

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A. Neither and Both. Define a ***combined boost-translation operation:*** $\mathbf{C}(a,b)=e^{i(b\mathbf{x}-a\mathbf{p})/\hbar}$

(More like Darboux rotation $e^{-i\boldsymbol{\Theta}\cdot\mathbf{J}/\hbar}$ than Euler rotation with three factors $e^{-i\mathbf{J}_z\alpha/\hbar}e^{-i\mathbf{J}_y\beta/\hbar}e^{-i\mathbf{J}_z\gamma/\hbar}$)

May evaluate with *Baker-Campbell-Hausdorf identity* since $[\mathbf{x},\mathbf{p}]=i\hbar\mathbf{1}$ and $[[\mathbf{x},\mathbf{p}],\mathbf{x}]=[[\mathbf{x},\mathbf{p}],\mathbf{p}]=\mathbf{0}$.

$$e^{\mathbf{A}+\mathbf{B}}=e^{\mathbf{A}}e^{\mathbf{B}}e^{-[\mathbf{A},\mathbf{B}]/2}=e^{\mathbf{B}}e^{\mathbf{A}}e^{[\mathbf{A},\mathbf{B}]/2}\text{ , where: }\left[\mathbf{A},[\mathbf{A},\mathbf{B}]\right]=\mathbf{0}=\left[\mathbf{B},[\mathbf{A},\mathbf{B}]\right]$$ (left as an exercise)

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Reordering only affects the overall phase.

Complex *phasor coordinate* α*(a,b)* is defined by:

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$$=e^{\alpha\mathbf{a}^\dagger-\alpha^*\mathbf{a}}=e^{-|\alpha|^2/2}e^{\alpha\mathbf{a}^\dagger}e^{-\alpha^*\mathbf{a}}=e^{|\alpha|^2/2}e^{-\alpha^*\mathbf{a}}e^{\alpha\mathbf{a}^\dagger}$$

Coherent wavepacket state $|\alpha(x_0,p_0)\rangle$: $|\alpha_0(x_0,p_0)\rangle=\mathbf{C}(x_0,p_0)|0\rangle=e^{i(x_0\mathbf{x}-p_0\mathbf{p})/\hbar}|0\rangle$

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1-D **a**†**a** *algebra of U(1) representations*

Creation-Destruction **a**†**a** *algebra*

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1st excited state

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2-D **a**†**a** *algebra of U(2) representations and R(3) angular momentum operators*

Time evolution of coherent state: $\left|\alpha_0(x_0,p_0)\right\rangle = e^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\frac{(\alpha_0)^n}{\sqrt{n!}}\left|n\right\rangle$

Time evolution operator for constant **H** has general form : $\mathbf{U}(t,0)=e^{-i\mathbf{H}t/\hbar}$

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Evolution simplifies to a variable-α_0 coherent state with a *time dependent phasor coordinate* α_t:

$$\mathbf{U}(t,0)\left|\alpha_0(x_0,p_0)\right\rangle = e^{-i\omega t/2}\left|\alpha_t(x_t,p_t)\right\rangle$$

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$$\left[x_t + i\frac{p_t}{M\omega}\right] = e^{-i\omega t}\left[x_0 + i\frac{p_0}{M\omega}\right]$$

(x_t,p_t) mimics classical oscillator

$$x_t = x_0\cos\omega t + \frac{p_0}{M\omega}\sin\omega t$$

$$\frac{p_t}{M\omega} = -x_0\sin\omega t + \frac{p_0}{M\omega}\cos\omega t$$

Real and imaginary parts (x_t and $p_t/M\omega$) of α_t go clockwise on phasor circle

1-D $\mathbf{a}^\dagger\mathbf{a}$ algebra of U(1) representations

Creation-Destruction $\mathbf{a}^\dagger\mathbf{a}$ algebra

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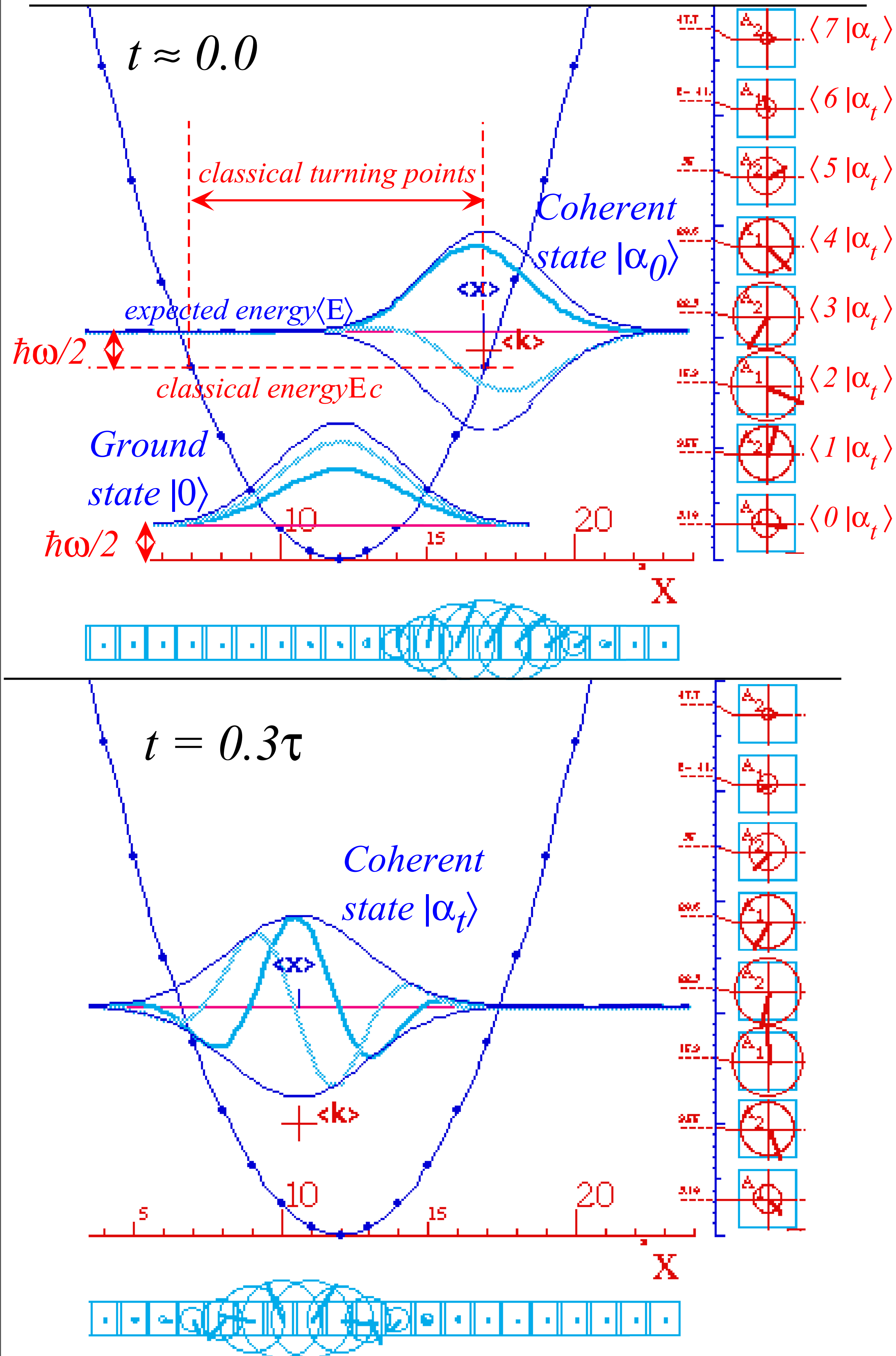
➡ *Properties of coherent state and "squeezed" states* ⬅

2-D $\mathbf{a}^\dagger\mathbf{a}$ algebra of U(2) representations and R(3) angular momentum operators

Properties of coherent state



Coherent ket $|\alpha(x0,p0)\rangle$ is eigenvector of destruct-op. **a**.

$$\mathbf{a}\left|\alpha_0(x_0,p_0)\right\rangle = e^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\frac{(\alpha_0)^n}{\sqrt{n!}}\mathbf{a}|n\rangle$$

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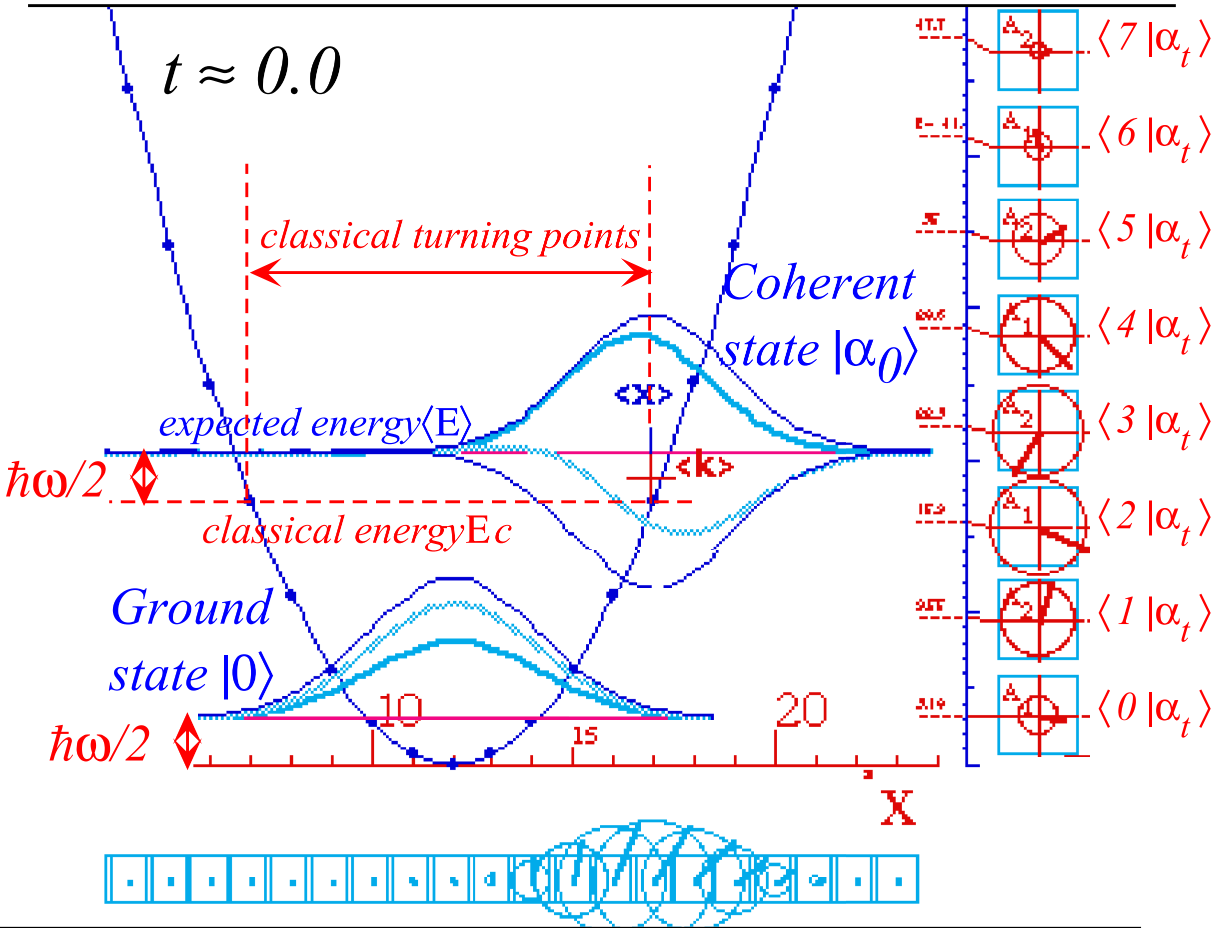

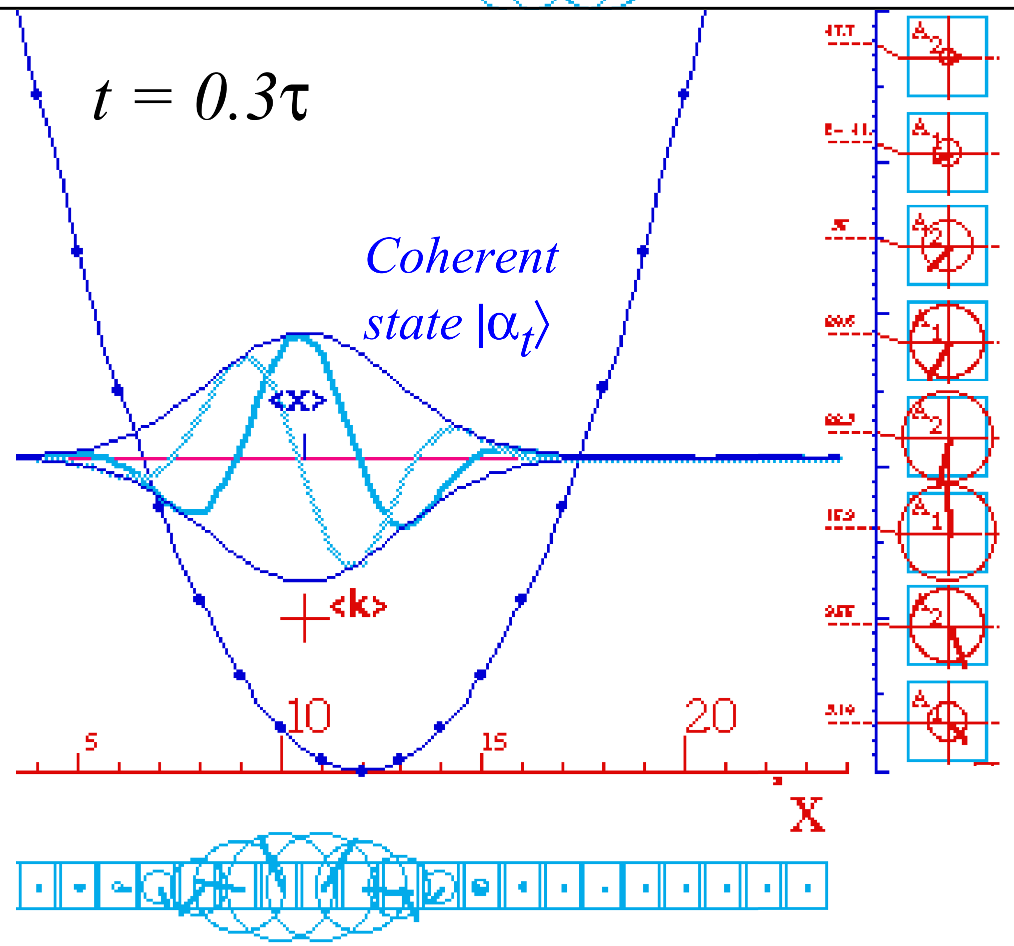

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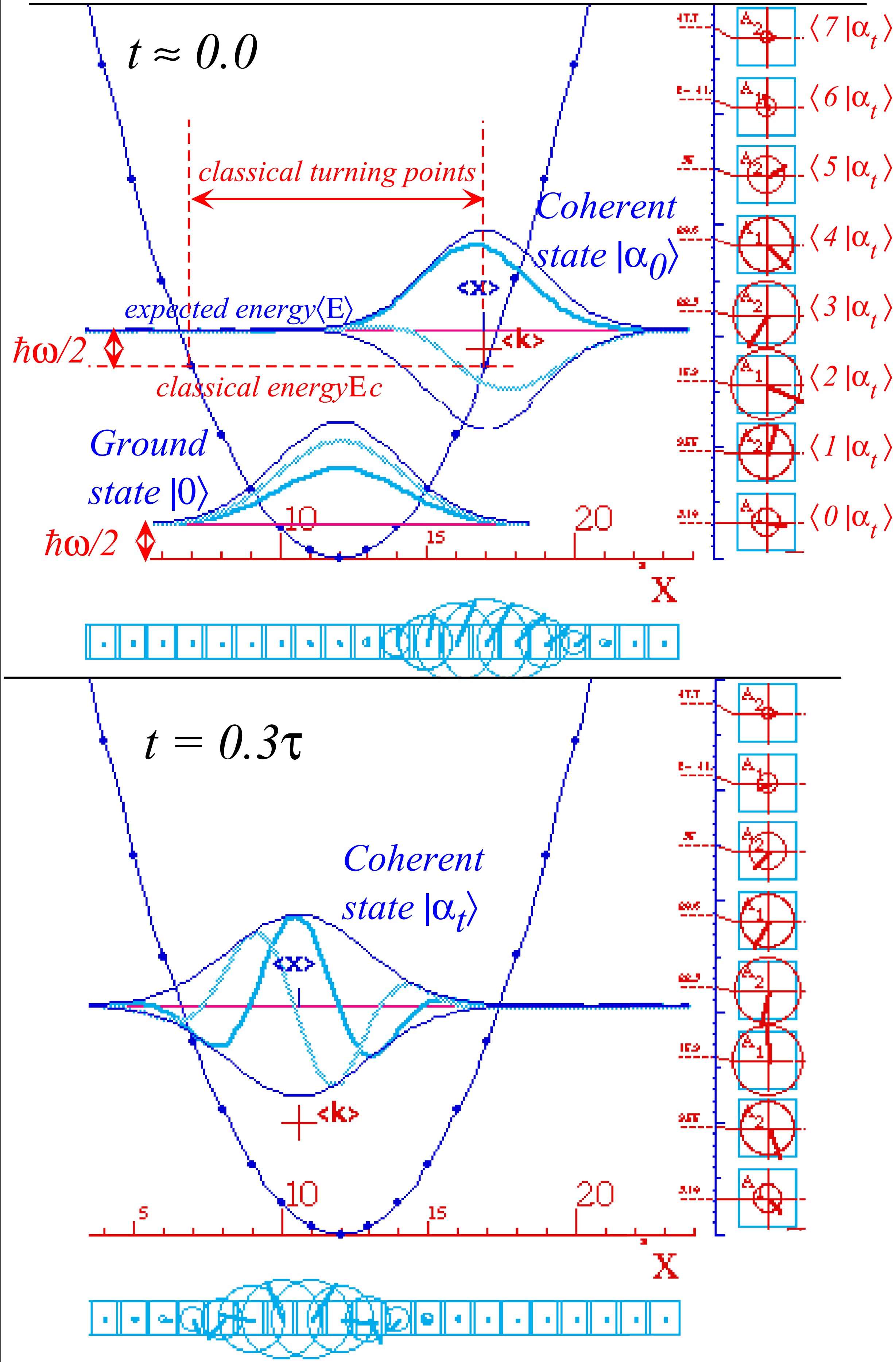

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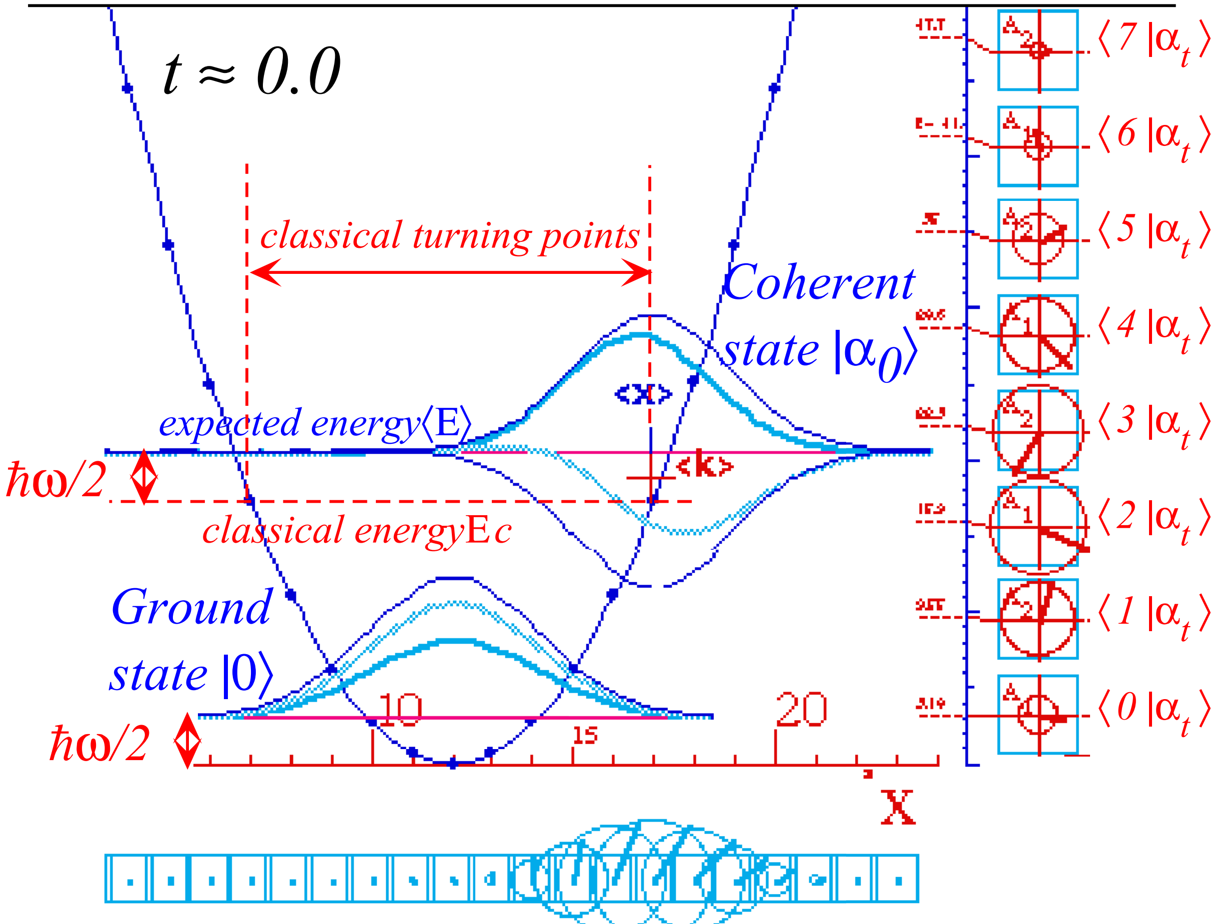

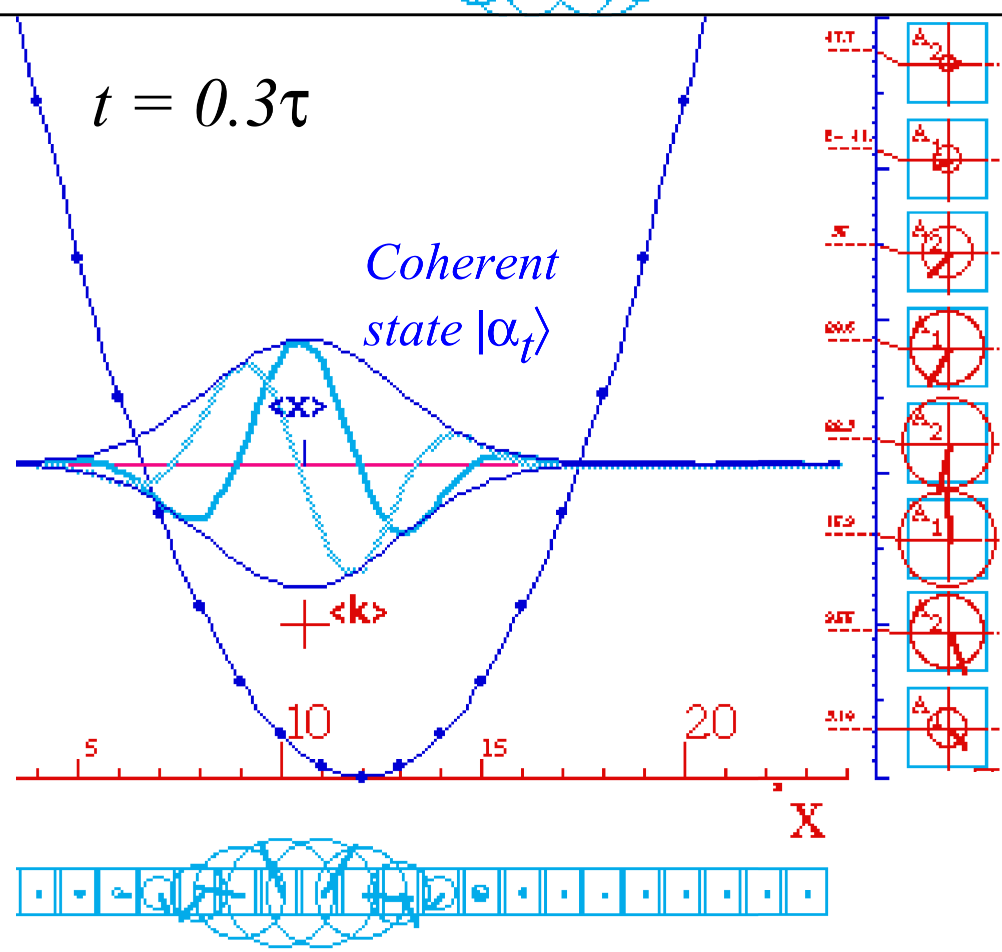

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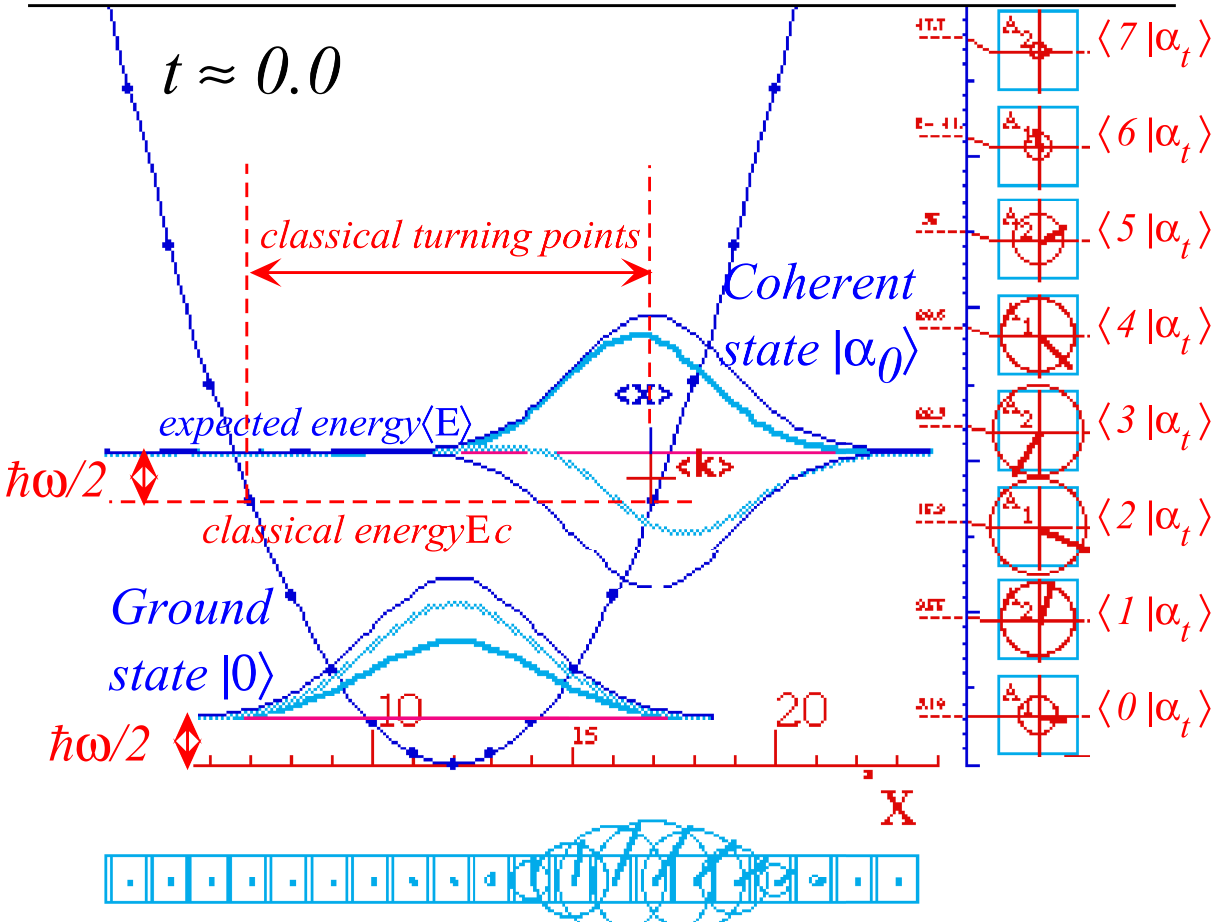

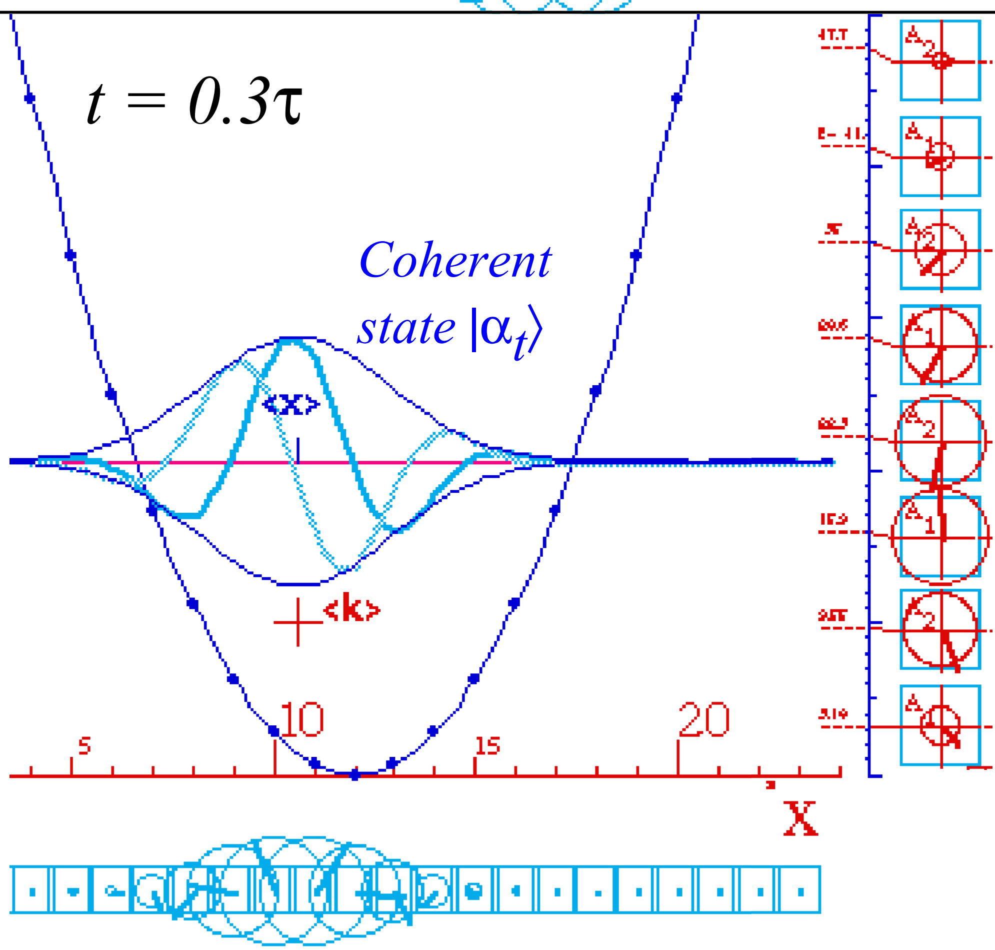

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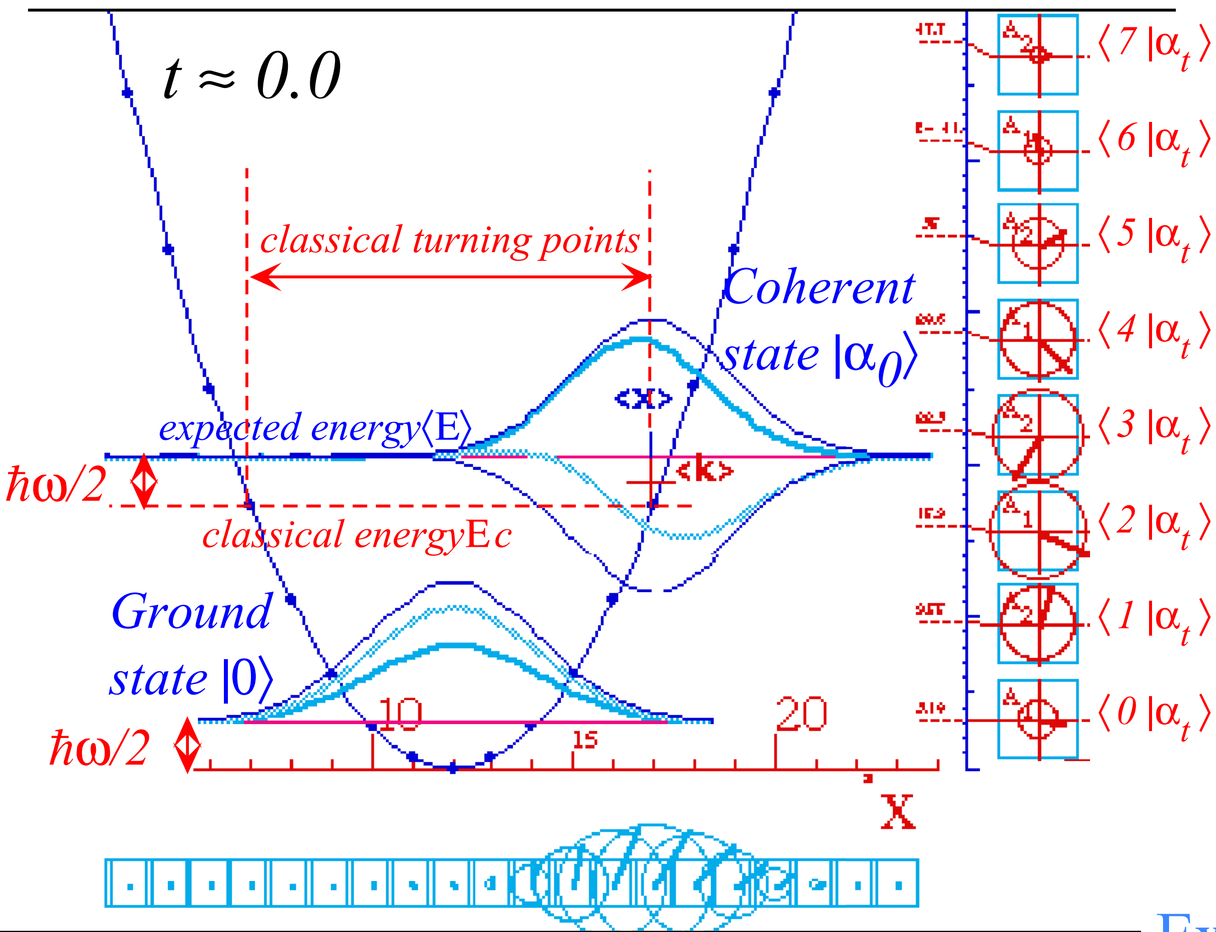
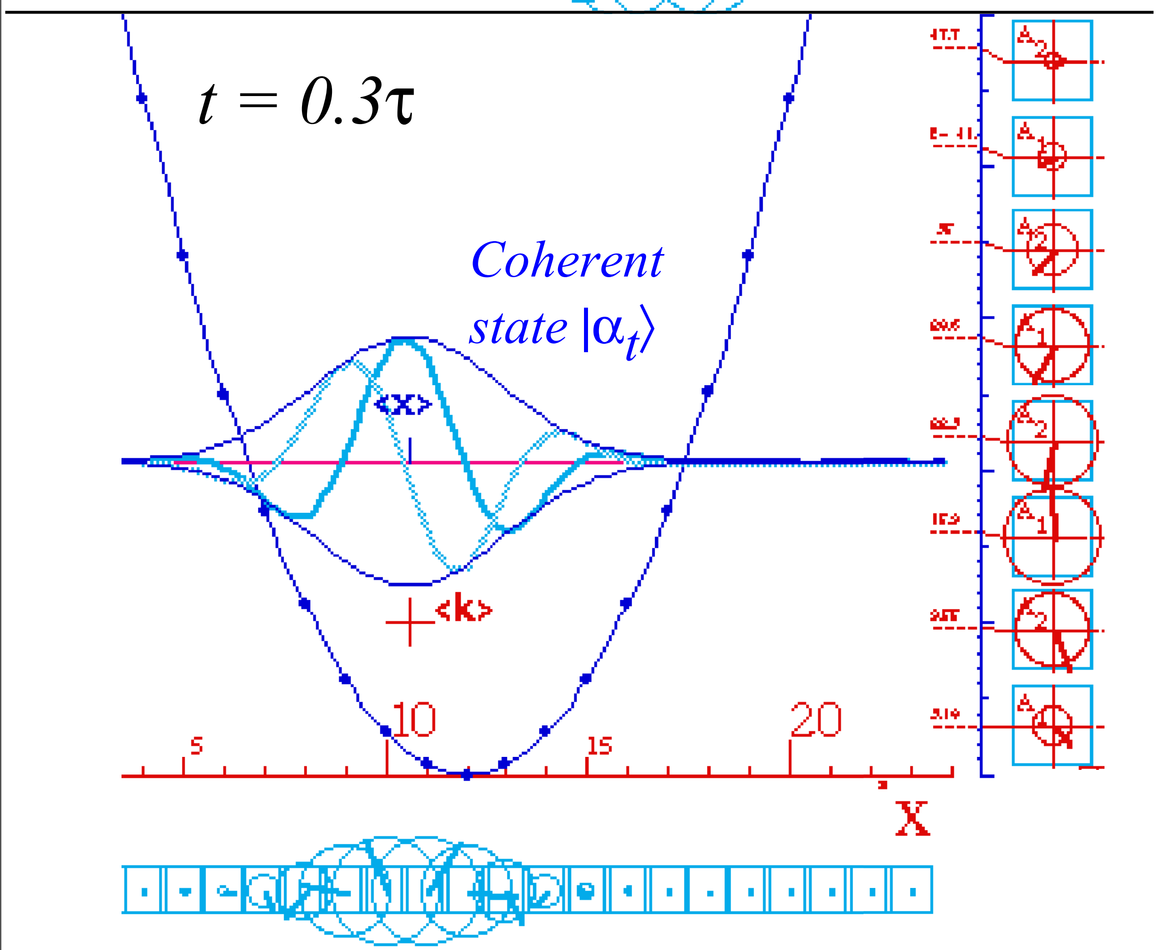
Properties of coherent state





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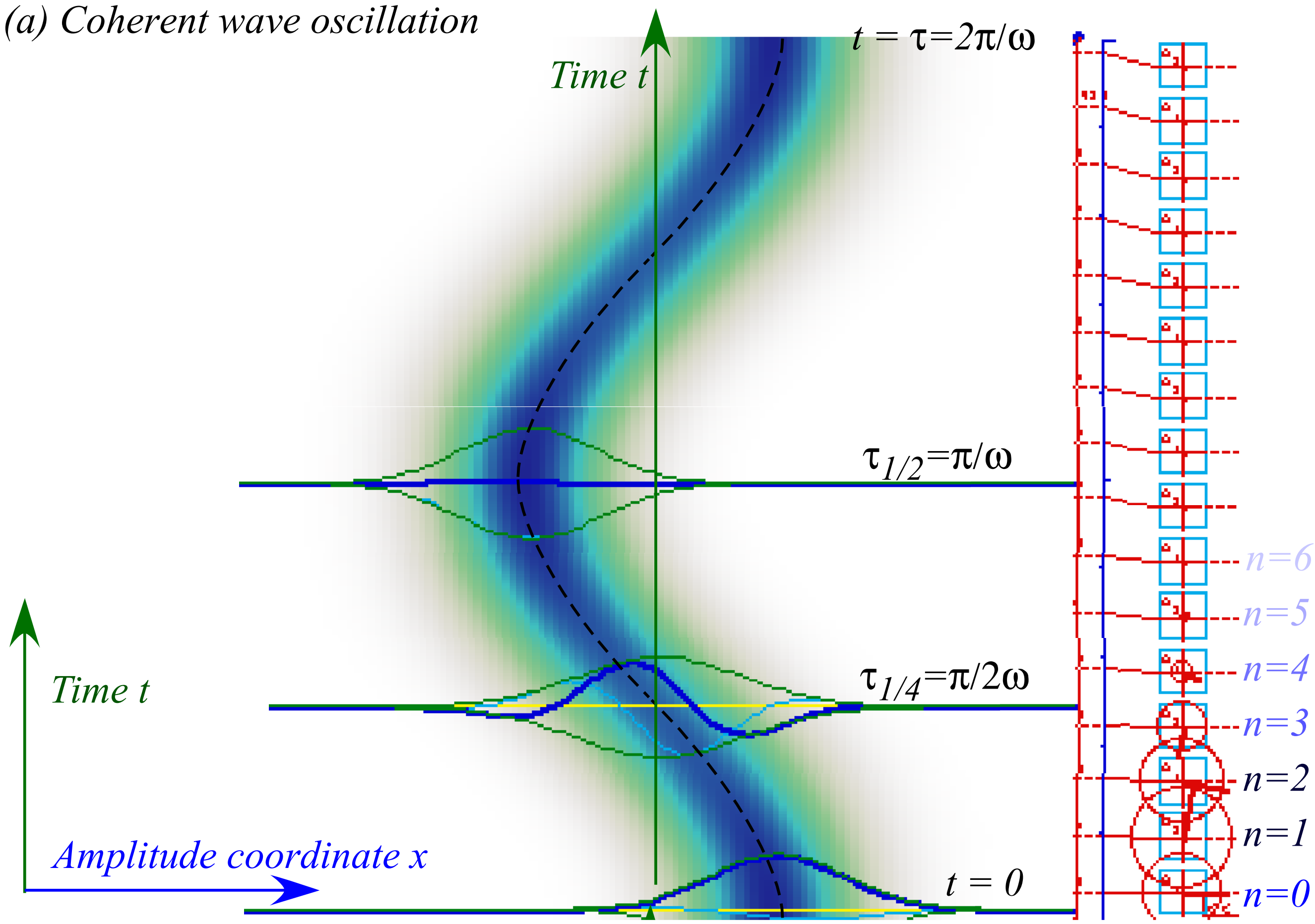
Expected quantum energy has simple time independent form.

$$\langle E\rangle\Big|_{\alpha_0} = \left\langle\alpha_0(x_0,p_0)\right|\mathbf{H}\left|\alpha_0(x_0,p_0)\right\rangle$$

$$= \left\langle\alpha_0(x_0,p_0)\right|\left(\hbar\omega\mathbf{a}^\dagger\mathbf{a} + \frac{\hbar\omega}{2}\mathbf{1}\right)\left|\alpha_0(x_0,p_0)\right\rangle$$

$$= \hbar\omega\alpha_0^*\alpha_0 + \frac{\hbar\omega}{2}$$

Properties of "squeezed" coherent states



Yeah! Cosine trajectory!

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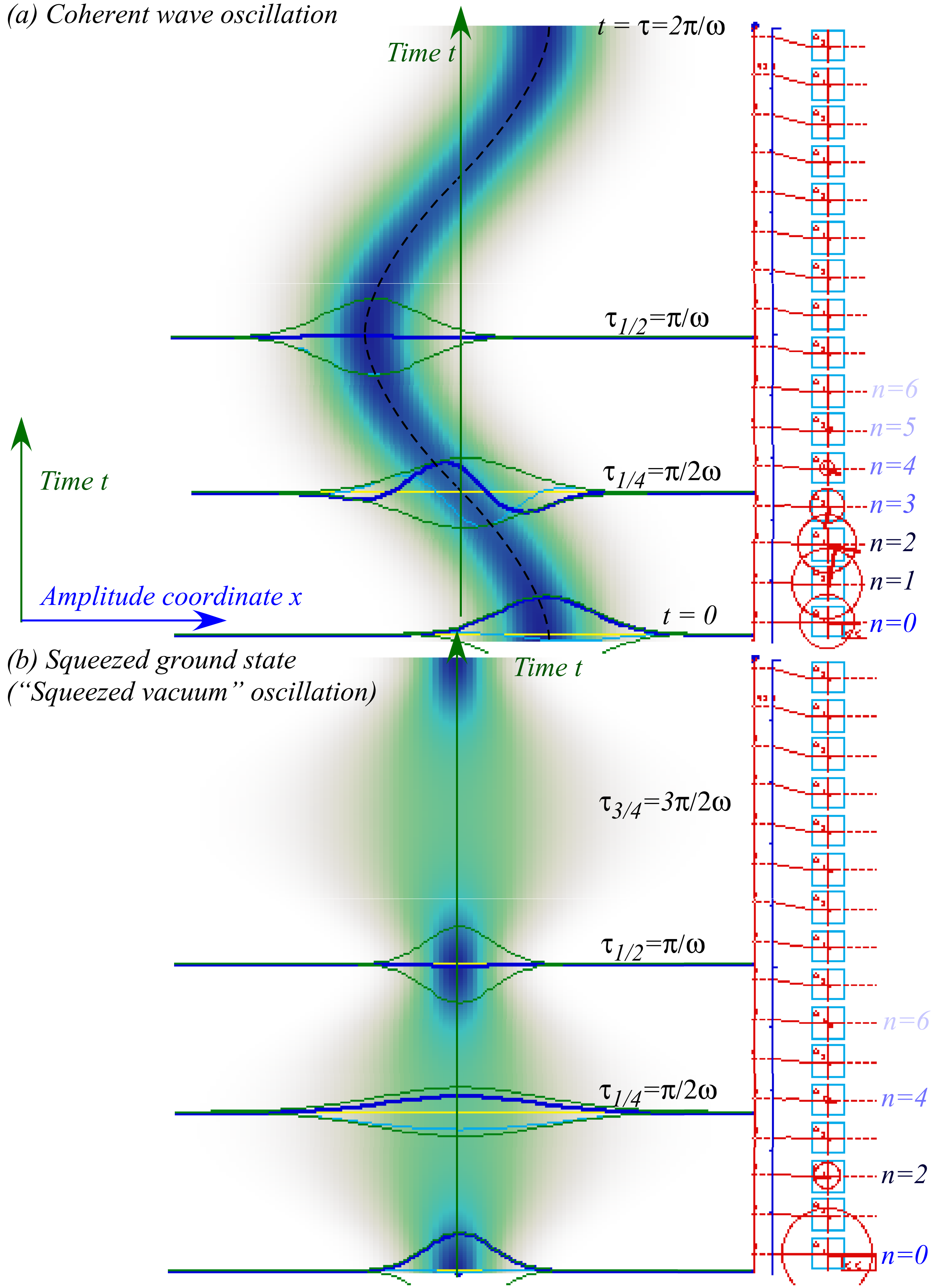
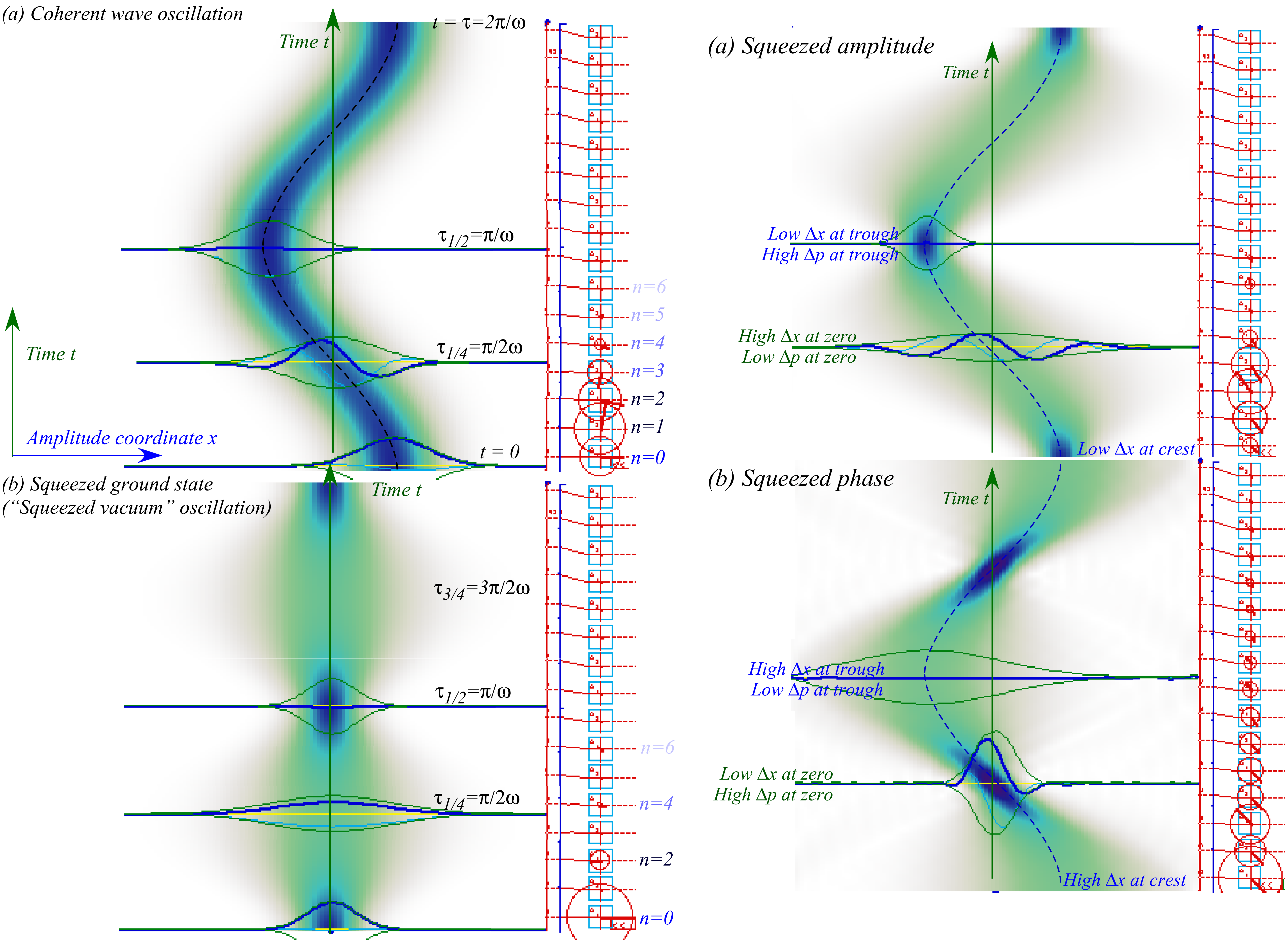


Yeah! Cosine trajectory!

what happens if you apply operators with non-linear "tensor" exponents $\exp(s\mathbf{x}^2)$, $\exp(f\,\mathbf{p}^2)$, *etc.*

Properties of "squeezed" coherent states



Review : *1-D* **a†a** *algebra of U(1) representations*

2-D **a†a** *algebra of U(2) representations and R(3) angular momentum operators*

2D-Oscillator basics

Commutation relations

Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry

Anti-commutation relations

Two-dimensional (or 2-particle) base states: ket-kets and bra-bras

Outer product arrays

Entangled 2-particle states

Two-particle (or 2-dimensional) matrix operators

U(2) Hamiltonian and irreducible representations

2D-Oscillator eigensolutions

U(1) Oscillator coherent states ("Shoved" and "kicked" states) ← Left from 4.23.13

Translation operators vs. boost operators

Applying boost-translation combinations

Time evolution of coherent state

Properties of coherent state and "squeezed" states

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Both are elementary "place-holders" for parameters H_{mn} or A, $B \pm iC$, and D.

$$|m\rangle\langle n| \to \left(\mathbf{a}_m^\dagger\mathbf{a}_n + \mathbf{a}_n\mathbf{a}_m^\dagger\right)/2 = \mathbf{a}_m^\dagger\mathbf{a}_n + \delta_{m,n}\mathbf{1}/2$$

Review : *1-D* **a†a** *algebra of U(1) representations*

2-D **a†a** *algebra of U(2) representations and R(3) angular momentum operators*

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Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry

Anti-commutation relations

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Left from 4.23.13

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Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry

Commutivity is known as *Bose symmetry.* Bose and Einstein discovered this symmetry of light quanta. $(\mathbf{a}_m, \mathbf{a}^\dagger_n)$ operators called *Boson operators* create or destroy *quanta* or "particles" known as *Bosons*.

If $\mathbf{a}^\dagger_m$ raises electromagnetic mode quantum number m to $m+1$ it is said to create a *photon*.

If $\mathbf{a}^\dagger_m$ raises crystal vibration mode quantum number m to $m+1$ it is said to create a *phonon*.

If $\mathbf{a}^\dagger_m$ raises liquid ^{4}He rotational quantum number m to $m+1$ it is said to create a *roton*.

Anti-commutivity is named *Fermi-Dirac symmetry* or *anti-symmetry*. It is found in electron waves.

Fermi operators $(\mathbf{c}_m,\mathbf{c}_n)$ are defined to create *Fermions* and use *anti-commutators* $\{\mathbf{A},\mathbf{B}\} = \mathbf{AB}+\mathbf{BA}$.

$$\{\mathbf{c}_m,\mathbf{c}_n\}=\mathbf{c}_m\mathbf{c}_n+\mathbf{c}_n\mathbf{c}_m=\mathbf{0} \qquad \{\mathbf{c}_m,\mathbf{c}^\dagger_n\}=\mathbf{c}_m\mathbf{c}^\dagger_n+\mathbf{c}^\dagger_n\mathbf{c}_m=\delta_{mn}\mathbf{1} \qquad \{\mathbf{c}^\dagger_m,\mathbf{c}^\dagger_n\}=\mathbf{c}^\dagger_m\mathbf{c}^\dagger_n+\mathbf{c}^\dagger_n\mathbf{c}^\dagger_m=\mathbf{0}$$

Fermi $\mathbf{c}^\dagger_n$ has a rigid birth-control policy; they are allowed just one Fermion or else, none at all.

Creating two Fermions of the same type is punished by death. This is because $x=-x$ implies $x=0$.

$$\mathbf{c}^\dagger_m\mathbf{c}^\dagger_m|0\rangle = -\mathbf{c}^\dagger_m\mathbf{c}^\dagger_m|0\rangle = \mathbf{0}$$

That no two indistinguishable Fermions can be in the same state, is called the *Pauli exclusion principle*.

Quantum numbers of $n=0$ and $n=1$ are the only allowed eigenvalues of the number operator $\mathbf{c}^\dagger_m\mathbf{c}_m$.

$$\mathbf{c}^\dagger_m\mathbf{c}_m|0\rangle = \mathbf{0}\ ,\ \mathbf{c}^\dagger_m\mathbf{c}_m|1\rangle = |1\rangle\ ,\ \mathbf{c}^\dagger_m\mathbf{c}_m|n\rangle = \mathbf{0}\ \text{ for: } n>1$$

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A state for a particle in two-dimensions (or two one-dimensional particles) is a ***"ket-ket"*** $|n_1\rangle|n_2\rangle$
It is outer product of the kets for each single dimension or particle.
The dual description is done similarly using ***"bra-bras"*** $\langle n_2|\langle n_1| = (|n_1\rangle|n_2\rangle)^\dagger$

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Must ask a perennial modern question: *"How are these structures stored in a computer program?"*
The usual answer is in *outer product* or *tensor arrays*. Next pages show sketches of these objects.

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Start with an elementary ket basis for each dimension or particle type-1 and type-2.

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Outer products are constructed for the states that might have non-negligible amplitudes.

$$
|0_1\rangle|0_2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ \hline 0 \\ 0 \\ 0 \\ \vdots \\ \hline 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, |0_1\rangle|1_2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ \hline 0 \\ 0 \\ 0 \\ \vdots \\ \hline 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \cdots |1_1\rangle|0_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \hline 1 \\ 0 \\ 0 \\ \vdots \\ \hline 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \cdots |1_1\rangle|2_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \hline 0 \\ 0 \\ 1 \\ \vdots \\ \hline 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix},
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Herein lies conflict between standard ∞-D analysis and finite computers

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A 2-wave state product has a lexicographic *(00, 01, 02, ...10, 11, 12,..., 20, 21, 22, ..)* array indexing.

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Outer products are constructed for the states that might have non-negligible amplitudes.

$$|0_1\rangle|0_2\rangle=\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}=\begin{pmatrix}1\\0\\0\\\vdots\\\hline 0\\0\\0\\\vdots\\\hline 0\\0\\0\\\vdots\end{pmatrix},\ |0_1\rangle|1_2\rangle=\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}=\begin{pmatrix}0\\1\\0\\\vdots\\\hline 0\\0\\0\\\vdots\\\hline 0\\0\\0\\\vdots\end{pmatrix},\ \cdots|1_1\rangle|0_2\rangle=\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}=\begin{pmatrix}0\\0\\0\\\vdots\\\hline 1\\0\\0\\\vdots\\\hline 0\\0\\0\\\vdots\end{pmatrix},\ \cdots|1_1\rangle|2_2\rangle=\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}\begin{pmatrix}0\\0\\1\\\vdots\end{pmatrix}=\begin{pmatrix}0\\0\\0\\\vdots\\\hline 0\\0\\1\\\vdots\\\hline 0\\0\\0\\\vdots\end{pmatrix},$$

Herein lies conflict between standard ∞-D analysis and finite computers

Make adjustable-size finite phasor arrays for each particle/dimension.

Convergence is achieved by orderly upgrades in the number of phasors to a point where results do not change.

A 2-wave state product has a lexicographic *(00, 01, 02, ...10, 11, 12,..., 20, 21, 22, ..)* array indexing.

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Outer product arrays

Start with an elementary ket basis for each dimension or particle type-1 and type-2.

$$\begin{array}{ccc} Type-1 & Type-2 & \cdots \end{array}$$

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shorthand
big-bra-big-ket
notation

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2-D **a†a** *algebra of U(2) representations and R(3) angular momentum operators*

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Left from 4.23.13

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	$\|00\rangle$	$\|01\rangle$	$\|02\rangle$	$\cdots$	$\|10\rangle$	$\|11\rangle$	$\|12\rangle$	$\cdots$	$\|20\rangle$	$\|21\rangle$	$\|22\rangle$	$\cdots$
$\langle 00\|$	0			$\cdots$	$\cdot$			$\cdots$				$\cdots$
$\langle 01\|$		D		$\cdots$	$B+iC$	$\cdot$		$\cdots$				$\cdots$
$\langle 02\|$			$2D$	$\cdots$		$\sqrt{2}(B+iC)$	$\cdot$	$\cdots$				$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$				$\cdots$
$\langle 10\|$	$\cdot$	$B-iC$		$\cdots$	A			$\cdots$	$\cdot$			$\cdots$
$\langle 11\|$		$\cdot$	$\sqrt{2}(B-iC)$	$\cdots$		$A+D$		$\cdots$	$\sqrt{2}(B+iC)$	$\cdot$		$\cdots$
$\langle 12\|$			$\cdot$	$\cdots$			$A+2D$	$\cdots$		$\sqrt{4}(B+iC)$	$\cdot$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\langle 20\|$					$\cdot$	$\sqrt{2}(B-iC)$		$\cdots$	$2A$			$\cdots$
$\langle 21\|$						$\cdot$	$\sqrt{4}(B-iC)$	$\cdots$		$2A+D$		$\cdots$
$\langle 22\|$							$\cdot$	$\cdots$			$2A+2D$	$\cdots$
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$$\mathbf{H} = H_{11}\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{1}/2\right) + H_{12}\mathbf{a}_1^\dagger\mathbf{a}_2 + H_{21}\mathbf{a}_2^\dagger\mathbf{a}_1 + H_{22}\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + \mathbf{1}/2\right)$$

The $\mathbf{a}_m{}^\dagger\mathbf{a}_n$ combinations in the $ABCD$ Hamiltonian $\mathbf{H}$ have fairly simple matrix elements.

$$\mathbf{H} = A\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{1}/2\right) + (B-iC)\mathbf{a}_1^\dagger\mathbf{a}_2 + (B+iC)\mathbf{a}_2^\dagger\mathbf{a}_1 + D\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + \mathbf{1}/2\right)$$

$$\mathbf{a}_1^\dagger\mathbf{a}_1|n_1 n_2\rangle = n_1|n_1\; n_2\rangle \qquad \mathbf{a}_1^\dagger\mathbf{a}_2|n_1 n_2\rangle = \sqrt{n_1+1}\sqrt{n_2}\,|n_1+1\; n_2-1\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1|n_1 n_2\rangle = \sqrt{n_1}\sqrt{n_2+1}\,|n_1-1\; n_2+1\rangle \qquad \mathbf{a}_2^\dagger\mathbf{a}_2|n_1 n_2\rangle = n_2|n_1\; n_2\rangle$$

$\langle\mathbf{H}\rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

	$\|00\rangle$	$\|01\rangle$	$\|02\rangle$	$\cdots$	$\|10\rangle$	$\|11\rangle$	$\|12\rangle$	$\cdots$	$\|20\rangle$	$\|21\rangle$	$\|22\rangle$	$\cdots$
$\langle 00\|$	0			$\cdots$	$\cdot$			$\cdots$				$\cdots$
$\langle 01\|$		D		$\cdots$	$B+iC$	$\cdot$		$\cdots$				$\cdots$
$\langle 02\|$			$2D$	$\cdots$		$\sqrt{2}(B+iC)$	$\cdot$	$\cdots$				$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$				$\cdots$
$\langle 10\|$	$\cdot$	$B-iC$		$\cdots$	A			$\cdots$	$\cdot$			$\cdots$
$\langle 11\|$		$\cdot$	$\sqrt{2}(B-iC)$	$\cdots$		$A+D$		$\cdots$	$\sqrt{2}(B+iC)$	$\cdot$		$\cdots$
$\langle 12\|$			$\cdot$	$\cdots$			$A+2D$	$\cdots$		$\sqrt{4}(B+iC)$	$\cdot$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\langle 20\|$					$\cdot$	$\sqrt{2}(B-iC)$		$\cdots$	$2A$			$\cdots$
$\langle 21\|$						$\cdot$	$\sqrt{4}(B-iC)$	$\cdots$		$2A+D$		$\cdots$
$\langle 22\|$							$\cdot$	$\cdots$			$2A+2D$	$\cdots$
$\vdots$					$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Two-particle (or 2-dimensional) matrix operators

When 2-particle operator $\mathbf{a}_k$ acts on a 2-particle state, $\mathbf{a}_k$ "finds" its type-k state but ignores the others.

$$\mathbf{a}_1^\dagger|n_1 n_2\rangle = \mathbf{a}_1^\dagger|n_1\rangle|n_2\rangle = \sqrt{n_1+1}\,|n_1+1\; n_2\rangle \qquad \mathbf{a}_2^\dagger|n_1 n_2\rangle = |n_1\rangle\mathbf{a}_2^\dagger|n_2\rangle = \sqrt{n_2+1}\,|n_1\; n_2+1\rangle$$

$$\mathbf{a}_1|n_1 n_2\rangle = \mathbf{a}_1|n_1\rangle|n_2\rangle = \sqrt{n_1}\,|n_1-1\; n_2\rangle \qquad \mathbf{a}_2|n_1 n_2\rangle = |n_1\rangle\mathbf{a}_2|n_2\rangle = \sqrt{n_2}\,|n_1\; n_2-1\rangle$$

$\mathbf{a}_1$"finds" its type-1 $\qquad$ $\mathbf{a}_2$"finds" its type-2

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$$\mathbf{H} = A\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{1}/2\right) + (B-iC)\mathbf{a}_1^\dagger\mathbf{a}_2 + (B+iC)\mathbf{a}_2^\dagger\mathbf{a}_1 + D\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + \mathbf{1}/2\right)$$

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$$\mathbf{a}_2^\dagger\mathbf{a}_1|n_1 n_2\rangle = \sqrt{n_1}\sqrt{n_2+1}\,|n_1-1\; n_2+1\rangle \qquad \mathbf{a}_2^\dagger\mathbf{a}_2|n_1 n_2\rangle = n_2|n_1\; n_2\rangle$$

$\langle\mathbf{H}\rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 02\rangle$	$\cdots$	$\vert 10\rangle$	$\vert 11\rangle$	$\vert 12\rangle$	$\cdots$	$\vert 20\rangle$	$\vert 21\rangle$	$\vert 22\rangle$	$\cdots$
$\langle 00\vert$	0			$\cdots$	$\cdot$			$\cdots$				$\cdots$
$\langle 01\vert$		D		$\cdots$	$B+iC$	$\cdot$		$\cdots$				$\cdots$
$\langle 02\vert$			$2D$	$\cdots$		$\sqrt{2}(B+iC)$	$\cdot$	$\cdots$				$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$				$\cdots$
$\langle 10\vert$	$\cdot$	$B-iC$		$\cdots$	A			$\cdots$	$\cdot$			$\cdots$
$\langle 11\vert$		$\cdot$	$\sqrt{2}(B-iC)$	$\cdots$		$A+D$		$\cdots$	$\sqrt{2}(B+iC)$	$\cdot$		$\cdots$
$\langle 12\vert$			$\cdot$	$\cdots$			$A+2D$	$\cdots$		$\sqrt{4}(B+iC)$	$\cdot$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\langle 20\vert$					$\cdot$	$\sqrt{2}(B-iC)$		$\cdots$	$2A$			$\cdots$
$\langle 21\vert$						$\cdot$	$\sqrt{4}(B-iC)$	$\cdots$		$2A+D$		$\cdots$
$\langle 22\vert$							$\cdot$	$\cdots$			$2A+2D$	$\cdots$
$\vdots$					$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Review : *1-D* **a†a** *algebra of U(1) representations*

2-D **a†a** *algebra of U(2) representations and R(3) angular momentum operators*

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U(2) Hamiltonian and irreducible representations

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U(1) Oscillator coherent states ("Shoved" and "kicked" states)

Left from 4.23.13

Translation operators vs. boost operators

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2-dimensional HO Hamiltonian matrices: U(2) irreducible representations

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	$\lvert 00\rangle$	$\lvert 01\rangle$	$\lvert 02\rangle$	$\cdots$	$\lvert 10\rangle$	$\lvert 11\rangle$	$\lvert 12\rangle$	$\cdots$	$\lvert 20\rangle$	$\lvert 21\rangle$	$\lvert 22\rangle$	$\cdots$
$\langle 00\rvert$	0			$\cdots$	$\cdot$			$\cdots$				$\cdots$
$\langle 01\rvert$		D		$\cdots$	$B+iC$	$\cdot$		$\cdots$				$\cdots$
$\langle 02\rvert$			$2D$	$\cdots$		$\sqrt{2}(B+iC)$	$\cdot$	$\cdots$				$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$				$\cdots$
$\langle 10\rvert$	$\cdot$	$B-iC$		$\cdots$	A			$\cdots$	$\cdot$			$\cdots$
$\langle 11\rvert$		$\cdot$	$\sqrt{2}(B-iC)$	$\cdots$		$A+D$		$\cdots$	$\sqrt{2}(B+iC)$	$\cdot$		$\cdots$
$\langle 12\rvert$			$\cdot$	$\cdots$			$A+2D$	$\cdots$		$\sqrt{4}(B+iC)$	$\cdot$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\langle 20\rvert$					$\cdot$	$\sqrt{2}(B-iC)$		$\cdots$	$2A$			$\cdots$
$\langle 21\rvert$						$\cdot$	$\sqrt{4}(B-iC)$	$\cdots$		$2A+D$		$\cdots$
$\langle 22\rvert$							$\cdot$	$\cdots$			$2A+2D$	$\cdots$
$\vdots$					$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

"Big-Endian" indexing
(...01,02,..10,11 ... 20,21...)

Rearrangement of rows and columns brings the matrix to a block-diagonal form.

2-dimensional HO Hamiltonian matrices: U(2) irreducible representations

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	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 02\rangle$	$\cdots$	$\vert 10\rangle$	$\vert 11\rangle$	$\vert 12\rangle$	$\cdots$	$\vert 20\rangle$	$\vert 21\rangle$	$\vert 22\rangle$	$\cdots$
$\langle 00\vert$	0			$\cdots$	$\cdot$			$\cdots$				$\cdots$
$\langle 01\vert$		D		$\cdots$	$B+iC$	$\cdot$		$\cdots$				$\cdots$
$\langle 02\vert$			$2D$	$\cdots$		$\sqrt{2}(B+iC)$	$\cdot$	$\cdots$				$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$				$\cdots$
$\langle 10\vert$	$\cdot$	$B-iC$		$\cdots$	A			$\cdots$	$\cdot$			$\cdots$
$\langle 11\vert$		$\cdot$	$\sqrt{2}(B-iC)$	$\cdots$		$A+D$		$\cdots$	$\sqrt{2}(B+iC)$	$\cdot$		$\cdots$
$\langle 12\vert$			$\cdot$	$\cdots$			$A+2D$	$\cdots$		$\sqrt{4}(B+iC)$	$\cdot$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\langle 20\vert$					$\cdot$	$\sqrt{2}(B-iC)$		$\cdots$	$2A$			$\cdots$
$\langle 21\vert$						$\cdot$	$\sqrt{4}(B-iC)$	$\cdots$		$2A+D$		$\cdots$
$\langle 22\vert$							$\cdot$	$\cdots$			$2A+2D$	$\cdots$
$\vdots$					$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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$\langle 00\vert$	0			$\cdots$	$\cdot$			$\cdots$				$\cdots$
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$\langle 02\vert$			$2D$	$\cdots$		$\sqrt{2}(B+iC)$	$\cdot$	$\cdots$				$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$				$\cdots$
$\langle 10\vert$	$\cdot$	$B-iC$		$\cdots$	A			$\cdots$	$\cdot$			$\cdots$
$\langle 11\vert$		$\cdot$	$\sqrt{2}(B-iC)$	$\cdots$		$A+D$		$\cdots$	$\sqrt{2}(B+iC)$	$\cdot$		$\cdots$
$\langle 12\vert$			$\cdot$	$\cdots$			$A+2D$	$\cdots$		$\sqrt{4}(B+iC)$	$\cdot$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\langle 20\vert$					$\cdot$	$\sqrt{2}(B-iC)$		$\cdots$	$2A$			$\cdots$
$\langle 21\vert$						$\cdot$	$\sqrt{4}(B-iC)$	$\cdots$		$2A+D$		$\cdots$
$\langle 22\vert$							$\cdot$	$\cdots$			$2A+2D$	$\cdots$
$\vdots$					$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 10\rangle$	$\vert 02\rangle$	$\vert 11\rangle$	$\vert 20\rangle$	$\vert 03\rangle$	$\vert 12\rangle$	$\vert 21\rangle$	$\vert 30\rangle$	$\cdots$
$\langle 00\vert$	0	*Vacuum* (ν=*0*)									
$\langle 01\vert$		D	$B+iC$	*Fundamental* (ν=*1*)							
$\langle 10\vert$		$B-iC$	A	vibrational sub-space							
$\langle 02\vert$				$2D$	$\sqrt{2}(B+iC)$						
$\langle 11\vert$				$\sqrt{2}(B-iC)$	$A+D$	$\sqrt{2}(B+iC)$	*Overtone* (ν=*2*) vibrational sub-space				
$\langle 20\vert$					$\sqrt{2}(B-iC)$	$2A$					
$\langle 03\vert$							$3D$	$\sqrt{3}(B+iC)$			
$\langle 12\vert$							$\sqrt{3}(B-iC)$	$A+2D$	$\sqrt{4}(B+iC)$		*Overtone* (ν=*3*) vibrational sub-space
$\langle 21\vert$								$\sqrt{4}(B-iC)$	$2A+D$	$\sqrt{3}(B+iC)$	
$\langle 30\vert$									$\sqrt{3}(B-iC)$	$3A$	
$\vdots$											

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$$\mathbf{H}^A = A\left(\mathbf{a}_1^\dagger \mathbf{a}_1 + \mathbf{1}/2\right) + D\left(\mathbf{a}_2^\dagger \mathbf{a}_2 + \mathbf{1}/2\right)$$

$$\varepsilon^A_{n_1 n_2} = A\left(n_1 + \frac{1}{2}\right) + D\left(n_2 + \frac{1}{2}\right) = \frac{A+D}{2}(n_1+n_2+1) + \frac{A-D}{2}(n_1 - n_2)$$

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➡ *2D-Oscillator eigensolutions* ⬅

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2D-Oscillator eigensolutions

"Little-Endian" indexing (*... 10, 01, ...20,11,21...*)

Fundamental eigenstates

The first step is to diagonalize the fundamental 2-by-2 matrix .

$$\langle \mathbf{H} \rangle_{\upsilon=1}^{Fundamental} = \begin{array}{c|cc} n_1,n_2 & |1,0\rangle & |0,1\rangle \\ \hline \langle 1,0| & A & B-iC \\ \langle 0,1| & B+iC & D \end{array} + \frac{A+D}{2}\mathbf{1}$$

2D-Oscillator eigensolutions

"Little-Endian" indexing (*... 10, 01, ...20,11,21...*)

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Recall decomposition of **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} + \frac{A+D}{2}\mathbf{1} = (A+D)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{2} + 2C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{1}{2} + (A-D)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2}$$

2D-Oscillator eigensolutions

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$\langle 0,1\vert$	$B+iC$	D

$+\frac{A+D}{2}\mathbf{1}$

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in terms of Jordan-Pauli spin operators.

$$\mathbf{H} = \Omega_0 \mathbf{1} + \boldsymbol{\Omega} \bullet \vec{\mathbf{S}} = \Omega_0 \mathbf{1} + \Omega_B \mathbf{S}_B + \Omega_C \mathbf{S}_C + \Omega_A \mathbf{S}_A \quad (ABC \text{ Optical vector notation})$$

$$= \Omega_0 \mathbf{1} + \Omega_X \mathbf{S}_X + \Omega_Y \mathbf{S}_Y + \Omega_Z \mathbf{S}_Z \quad (XYZ \text{ Electron spin notation})$$

2D-Oscillator eigensolutions

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$$= \Omega_0\mathbf{1} + \Omega_X\mathbf{S}_X + \Omega_Y\mathbf{S}_Y + \Omega_Z\mathbf{S}_Z \quad \text{(XYZ Electron spin notation)}$$

Frequency eigenvalues $\omega_\pm$ of $\mathbf{H}\text{-}\Omega_0\mathbf{1}/2$ and *fundamental transition frequency* $\Omega = \omega_+ - \omega_-$:

$$\omega_\pm = \frac{\Omega_0 \pm \Omega}{2} = \frac{A+D \pm \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}}{2} = \frac{A+D}{2} \pm \sqrt{\left(\frac{A-D}{2}\right)^2 + B^2 + C^2}$$

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$\langle 0,1\vert$	$B + iC$	D

$+ \frac{A+D}{2}\mathbf{1}$

Recall decomposition of **H**

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Polar angles (φ,ϑ) of $+\boldsymbol{\Omega}$-vector (or polar angles $(\varphi,\vartheta\pm\pi)$ of $-\boldsymbol{\Omega}$-vector) gives **H** eigenvectors.

$$|\omega_+\rangle = \begin{pmatrix} e^{-i\varphi/2}\cos\frac{\vartheta}{2} \\ e^{i\varphi/2}\sin\frac{\vartheta}{2} \end{pmatrix}, \quad |\omega_-\rangle = \begin{pmatrix} -e^{-i\varphi/2}\sin\frac{\vartheta}{2} \\ e^{i\varphi/2}\cos\frac{\vartheta}{2} \end{pmatrix} \quad \text{where: } \begin{cases} \cos\vartheta = \frac{A\text{-}D}{\Omega} \\ \tan\varphi = \frac{C}{B} \end{cases}$$

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Fundamental eigenstates

The first step is to diagonalize the fundamental 2-by-2 matrix .

$$\langle \mathbf{H} \rangle_{\upsilon=1}^{Fundamental} = \begin{array}{c|cc} n_1,n_2 & |1,0\rangle & |0,1\rangle \\ \hline \langle 1,0| & A & B-iC \\ \langle 0,1| & B+iC & D \end{array} + \frac{A+D}{2}\mathbf{1}$$

Recall decomposition of **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} + \frac{A+D}{2}\mathbf{1} = (A+D)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{2} + 2C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{1}{2} + (A-D)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2}$$

in terms of Jordan-Pauli spin operators.

$$\mathbf{H} = \Omega_0\mathbf{1} + \mathbf{\Omega}\bullet\vec{\mathbf{S}} = \Omega_0\mathbf{1} + \Omega_B\mathbf{S}_B + \Omega_C\mathbf{S}_C + \Omega_A\mathbf{S}_A \quad (ABC \text{ Optical vector notation})$$

$$= \Omega_0\mathbf{1} + \Omega_X\mathbf{S}_X + \Omega_Y\mathbf{S}_Y + \Omega_Z\mathbf{S}_Z \quad (XYZ \text{ Electron spin notation})$$

Frequency eigenvalues $\omega_\pm$ of $\mathbf{H}$-$\Omega_0\mathbf{1}/2$ and *fundamental transition frequency* $\Omega = \omega_+ - \omega_-$:

$$\omega_\pm = \frac{\Omega_0 \pm \Omega}{2} = \frac{A+D \pm \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}}{2} = \frac{A+D}{2} \pm \sqrt{\left(\frac{A-D}{2}\right)^2 + B^2 + C^2}$$

Polar angles (φ,ϑ) of $+\mathbf{\Omega}$-vector (or polar angles $(\varphi,\vartheta\pm\pi)$ of $-\mathbf{\Omega}$-vector) gives **H** eigenvectors.

$$|\omega_+\rangle = \begin{pmatrix} e^{-i\varphi/2}\cos\frac{\vartheta}{2} \\ e^{i\varphi/2}\sin\frac{\vartheta}{2} \end{pmatrix}, \quad |\omega_-\rangle = \begin{pmatrix} -e^{-i\varphi/2}\sin\frac{\vartheta}{2} \\ e^{i\varphi/2}\cos\frac{\vartheta}{2} \end{pmatrix} \quad \text{where: } \begin{cases} \cos\vartheta = \frac{A-D}{\Omega} \\ \tan\varphi = \frac{C}{B} \end{cases}$$

More important for the general solution, are the *eigen-creation operators* $\mathbf{a}^\dagger_+$ *and* $\mathbf{a}^\dagger_-$ defined by

$$\mathbf{a}_+^\dagger = e^{-i\varphi/2}\left(\cos\frac{\vartheta}{2}\mathbf{a}_1^\dagger + e^{i\varphi}\sin\frac{\vartheta}{2}\mathbf{a}_2^\dagger\right), \quad \mathbf{a}_-^\dagger = e^{-i\varphi/2}\left(-\sin\frac{\vartheta}{2}\mathbf{a}_1^\dagger + e^{i\varphi}\cos\frac{\vartheta}{2}\mathbf{a}_2^\dagger\right)$$

2D-Oscillator eigensolutions

"Little-Endian" indexing (... 10, 01, ...20,11,21...)

Fundamental eigenstates

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Recall decomposition of **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} + \frac{A+D}{2}\mathbf{1} = (A+D)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{2} + 2C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{1}{2} + (A-D)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2}$$

in terms of Jordan-Pauli spin operators.

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$$= \Omega_0\mathbf{1} + \Omega_X\mathbf{S}_X + \Omega_Y\mathbf{S}_Y + \Omega_Z\mathbf{S}_Z \quad \textit{(XYZ Electron spin notation)}$$

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More important for the general solution, are the *eigen-creation operators* $\mathbf{a}^\dagger_+$ *and* $\mathbf{a}^\dagger_-$ defined by

$$\mathbf{a}_+^\dagger = e^{-i\varphi/2}\left(\cos\frac{\vartheta}{2}\mathbf{a}_1^\dagger + e^{i\varphi}\sin\frac{\vartheta}{2}\mathbf{a}_2^\dagger\right), \quad \mathbf{a}_-^\dagger = e^{-i\varphi/2}\left(-\sin\frac{\vartheta}{2}\mathbf{a}_1^\dagger + e^{i\varphi}\cos\frac{\vartheta}{2}\mathbf{a}_2^\dagger\right)$$

$\mathbf{a}_\pm^\dagger$ create **H** eigenstates directly from the ground state.

$$\mathbf{a}_+^\dagger|0\rangle = |\omega_+\rangle, \quad \mathbf{a}_-^\dagger|0\rangle = |\omega_-\rangle$$

Setting $(B=0=C)$ and $(A=\omega_+)$ and $(D=\omega_-)$ gives diagonal block matrices.

$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

	$\lvert 00\rangle$	$\lvert 01\rangle$	$\lvert 10\rangle$	$\lvert 02\rangle$	$\lvert 11\rangle$	$\lvert 20\rangle$	$\lvert 03\rangle$	$\lvert 12\rangle$	$\lvert 21\rangle$	$\lvert 30\rangle$	$\cdots$
$\langle 00\rvert$	0										
$\langle 01\rvert$		ω_-									
$\langle 10\rvert$			ω_+								
$\langle 02\rvert$				$2\omega_-$							
$\langle 11\rvert$					$\omega_+ + \omega_-$						
$\langle 20\rvert$						$2\omega_+$					
$\langle 03\rvert$							$3\omega_-$				
$\langle 12\rvert$								$\omega_+ + 2\omega_-$			
$\langle 21\rvert$									$2\omega_+ + \omega_-$		
$\langle 30\rvert$										$3\omega_+$	
$\vdots$											

$$\omega_+ - \omega_- = \Omega$$
$$= \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}$$
$$= A - D$$

$$\mathbf{H}^A = A\left(\mathbf{a}_1^\dagger \mathbf{a}_1 + \mathbf{1}/2\right) + D\left(\mathbf{a}_2^\dagger \mathbf{a}_2 + \mathbf{1}/2\right)$$

Setting (B=0=C) and (A=ω_+) and (D=ω_-) gives diagonal block matrices.

$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 10\rangle$	$\vert 02\rangle$	$\vert 11\rangle$	$\vert 20\rangle$	$\vert 03\rangle$	$\vert 12\rangle$	$\vert 21\rangle$	$\vert 30\rangle$	$\cdots$
$\langle 00\vert$	0										
$\langle 01\vert$		ω_-									
$\langle 10\vert$			ω_+								
$\langle 02\vert$				$2\omega_-$							
$\langle 11\vert$					$\omega_+ + \omega_-$						
$\langle 20\vert$						$2\omega_+$					
$\langle 03\vert$							$3\omega_-$				
$\langle 12\vert$								$\omega_+ + 2\omega_-$			
$\langle 21\vert$									$2\omega_+ + \omega_-$		
$\langle 30\vert$										$3\omega_+$	
$\vdots$											

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$$= \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}$$
$$= A - D$$

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$$\varepsilon^A_{n_1 n_2} = A\left(n_1 + \frac{1}{2}\right) + D\left(n_2 + \frac{1}{2}\right) = \frac{A+D}{2}\left(n_1 + n_2 + 1\right) + \frac{A-D}{2}\left(n_1 - n_2\right)$$

Setting $(B=0=C)$ and $(A=\omega_+)$ and $(D=\omega_-)$ gives diagonal block matrices.

$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

	$\lvert 00\rangle$	$\lvert 01\rangle$	$\lvert 10\rangle$	$\lvert 02\rangle$	$\lvert 11\rangle$	$\lvert 20\rangle$	$\lvert 03\rangle$	$\lvert 12\rangle$	$\lvert 21\rangle$	$\lvert 30\rangle$	$\cdots$
$\langle 00\rvert$	0										
$\langle 01\rvert$		ω_-									
$\langle 10\rvert$			ω_+								
$\langle 02\rvert$				$2\omega_-$							
$\langle 11\rvert$					$\omega_+ + \omega_-$						
$\langle 20\rvert$						$2\omega_+$					
$\langle 03\rvert$							$3\omega_-$				
$\langle 12\rvert$								$\omega_+ + 2\omega_-$			
$\langle 21\rvert$									$2\omega_+ + \omega_-$		
$\langle 30\rvert$										$3\omega_+$	
$\vdots$											

$$\omega_+ - \omega_- = \Omega$$
$$= \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}$$
$$= A - D$$

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$$\varepsilon^A_{n_1 n_2} = A\left(n_1 + \frac{1}{2}\right) + D\left(n_2 + \frac{1}{2}\right) = \frac{A+D}{2}(n_1 + n_2 + 1) + \frac{A-D}{2}(n_1 - n_2)$$
$$= \Omega_0(n_1 + n_2 + 1) + \frac{\Omega}{2}(n_1 - n_2) = \Omega_0(\upsilon + 1) + \Omega\, m$$

Setting $(B=0=C)$ and $(A=\omega_+)$ and $(D=\omega_-)$ gives diagonal block matrices.

$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 10\rangle$	$\vert 02\rangle$	$\vert 11\rangle$	$\vert 20\rangle$	$\vert 03\rangle$	$\vert 12\rangle$	$\vert 21\rangle$	$\vert 30\rangle$	$\cdots$
$\langle 00\vert$	0										
$\langle 01\vert$		ω_-									
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$\langle 02\vert$				$2\omega_-$							
$\langle 11\vert$					$\omega_+ + \omega_-$						
$\langle 20\vert$						$2\omega_+$					
$\langle 03\vert$							$3\omega_-$				
$\langle 12\vert$								$\omega_+ + 2\omega_-$			
$\langle 21\vert$									$2\omega_+ + \omega_-$		
$\langle 30\vert$										$3\omega_+$	
$\vdots$											

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$$= \Omega_0 (n_1 + n_2 + 1) + \frac{\Omega}{2}(n_1 - n_2) = \Omega_0 (\upsilon + 1) + \Omega\, m$$

Define *total quantum number* $\upsilon = 2j$ and half-difference or *asymmetry quantum number m*

$$\upsilon = n_1 + n_2 = 2j \qquad j = \frac{n_1 + n_2}{2} = \frac{\upsilon}{2} \qquad m = \frac{n_1 - n_2}{2}$$

Setting (B=0=C) and (A=ω_+) and (D=ω_-) gives diagonal block matrices.

$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

	$\|00\rangle$	$\|01\rangle$	$\|10\rangle$	$\|02\rangle$	$\|11\rangle$	$\|20\rangle$	$\|03\rangle$	$\|12\rangle$	$\|21\rangle$	$\|30\rangle$	$\cdots$
$\langle 00\|$	0										
$\langle 01\|$		ω_-									
$\langle 10\|$			ω_+								
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$\langle 11\|$					$\omega_+ + \omega_-$						
$\langle 20\|$						$2\omega_+$					
$\langle 03\|$							$3\omega_-$				
$\langle 12\|$								$\omega_+ + 2\omega_-$			
$\langle 21\|$									$2\omega_+ + \omega_-$		
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$\vdots$											

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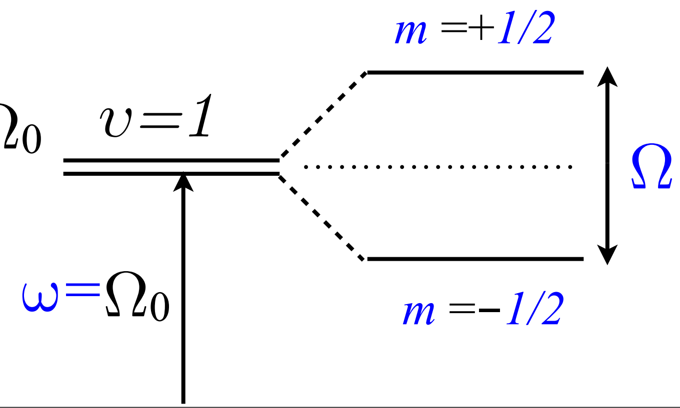
$\upsilon+1=2j+1$ multiplies *base frequency* $\omega=\Omega_0$

m multiplies *beat frequency* Ω



$$\omega_+ = \Omega_0 + \Omega\left(+\tfrac{1}{2}\right)$$

$$\omega_- = \Omega_0 + \Omega\left(-\tfrac{1}{2}\right)$$

Setting $(B=0=C)$ and $(A=\omega_+)$ and $(D=\omega_-)$ gives diagonal block matrices.

$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

	$\lvert 00\rangle$	$\lvert 01\rangle$	$\lvert 10\rangle$	$\lvert 02\rangle$	$\lvert 11\rangle$	$\lvert 20\rangle$	$\lvert 03\rangle$	$\lvert 12\rangle$	$\lvert 21\rangle$	$\lvert 30\rangle$	$\cdots$
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$\langle 21\rvert$									$2\omega_+ + \omega_-$		
$\langle 30\rvert$										$3\omega_+$	
$\vdots$											

$$\omega_+ - \omega_- = \Omega$$
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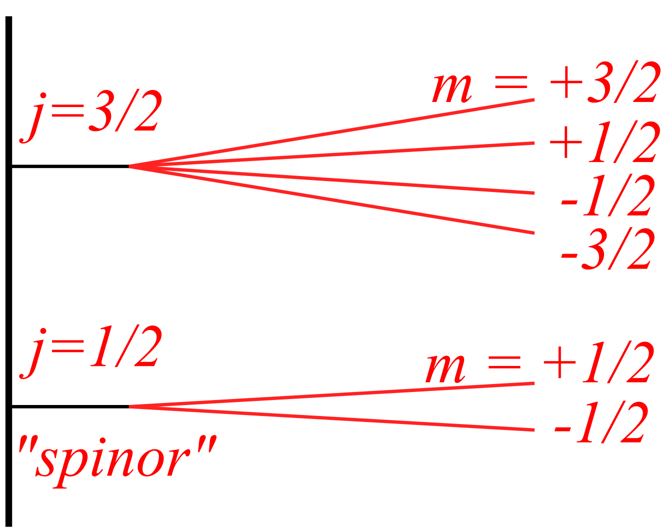
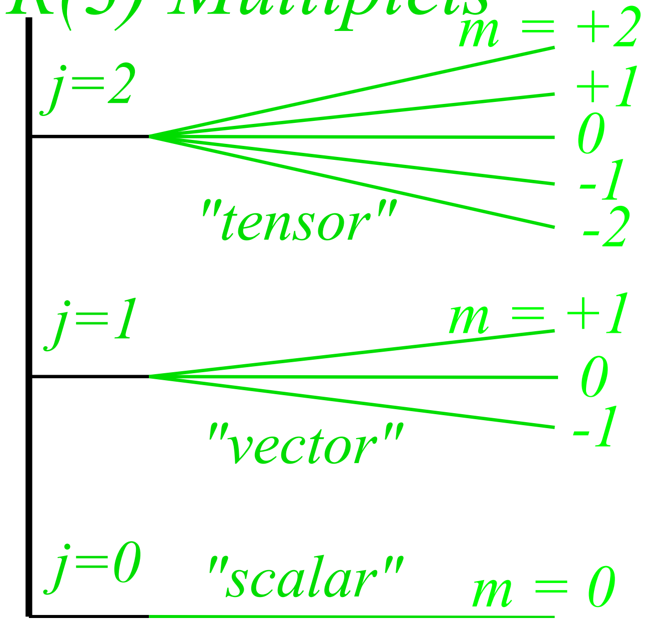
$$= A - D$$

SU(2) Multiplets



R(3) Multiplets



Setting $(B=0=C)$ and $(A=\omega_+)$ and $(D=\omega_-)$ gives diagonal block matrices.

$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

	$\vert 00\rangle$	$\vert 01\rangle$	$\vert 10\rangle$	$\vert 02\rangle$	$\vert 11\rangle$	$\vert 20\rangle$	$\vert 03\rangle$	$\vert 12\rangle$	$\vert 21\rangle$	$\vert 30\rangle$	$\cdots$
$\langle 00\vert$	0										
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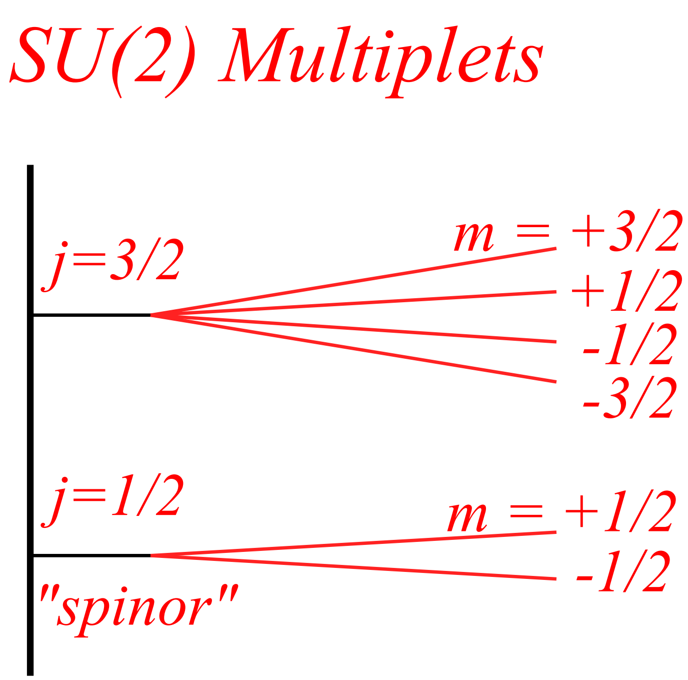
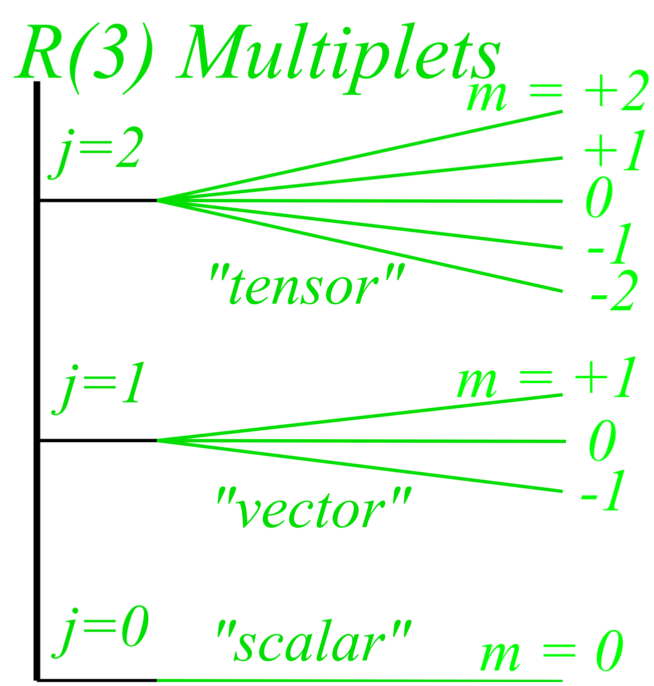
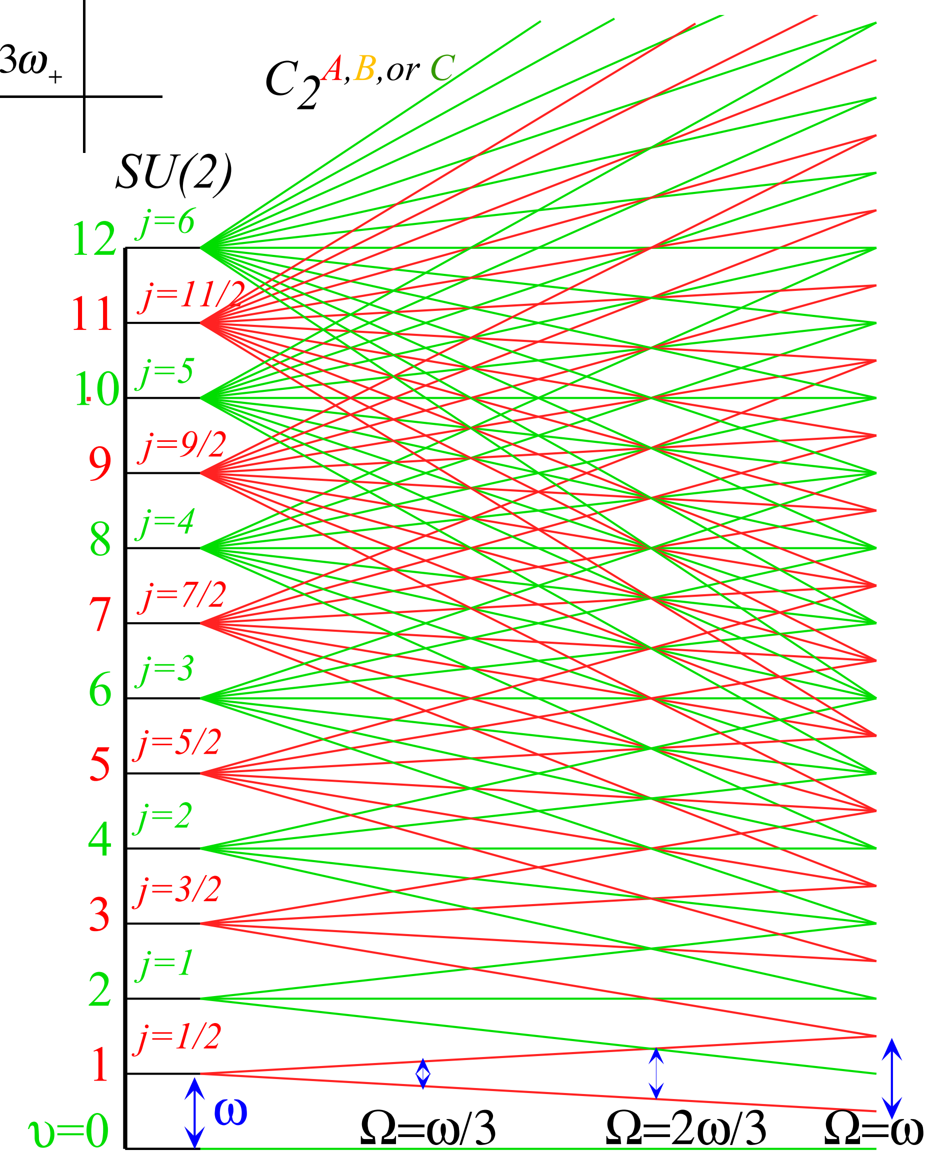
$\langle 21\vert$									$2\omega_+ + \omega_-$		
$\langle 30\vert$										$3\omega_+$	
$\vdots$											

$$\omega_+ - \omega_- = \Omega$$
$$= \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}$$
$$= A - D$$





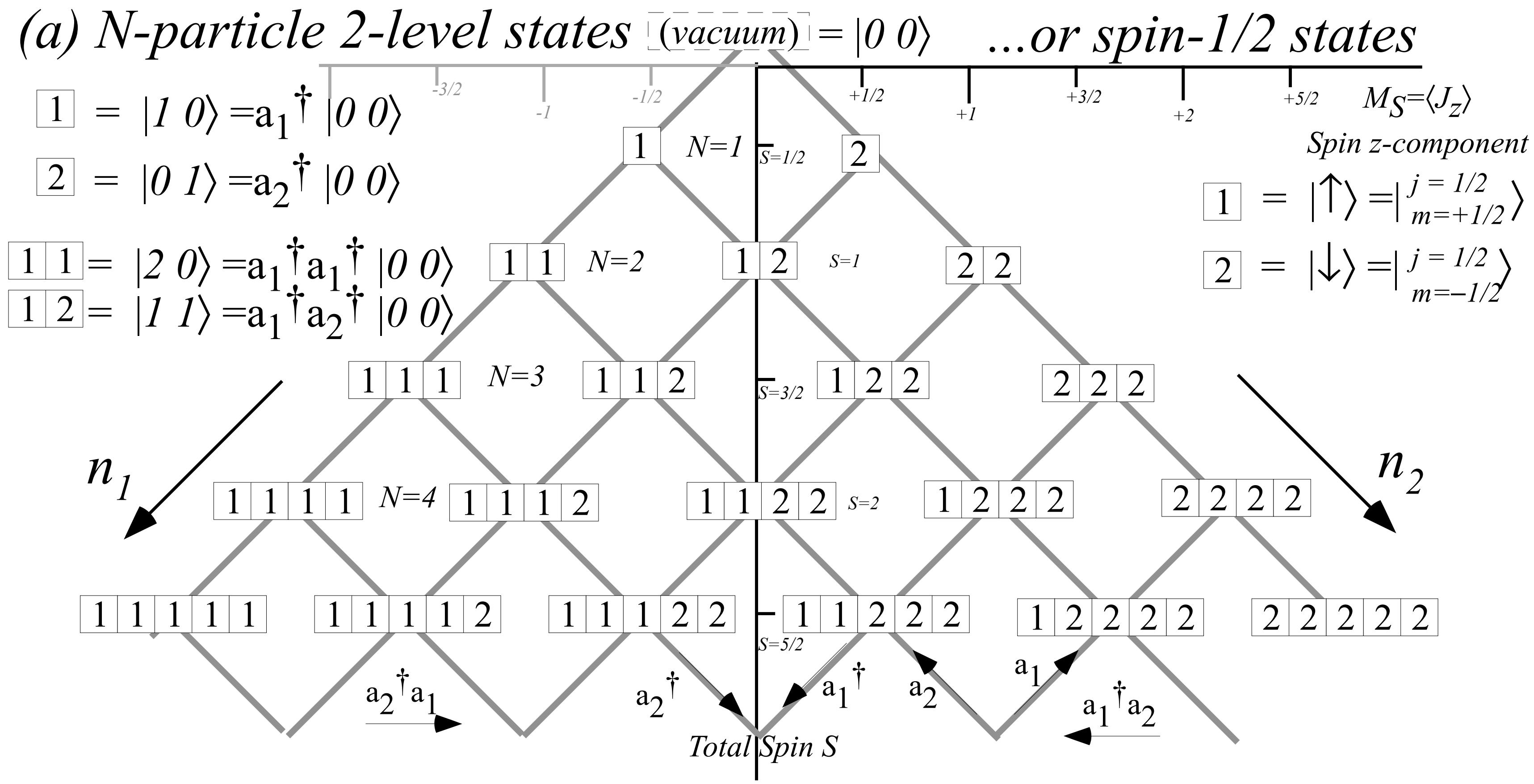


Structure of U(2)

$\left|{j \atop m}\right\rangle = |n_1 n_2\rangle$

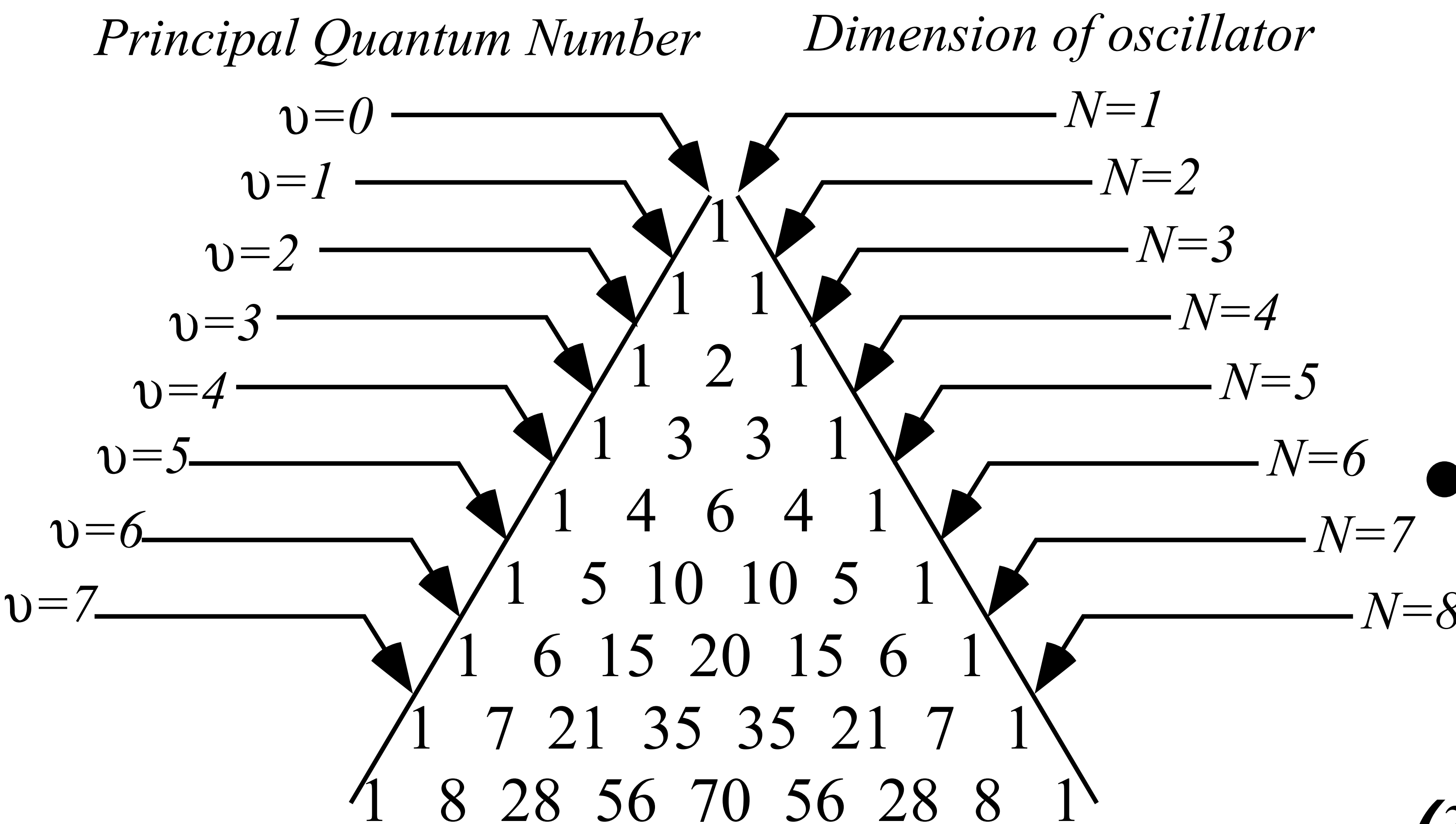
$j=0$	$\left|{0 \atop 0}\right\rangle = |00\rangle$	"scalar"
$j=\frac{1}{2}$	$\left|{1/2 \atop 1/2}\right\rangle = |10\rangle = |\uparrow\rangle$ $\left|{1/2 \atop -1/2}\right\rangle = |01\rangle = |\downarrow\rangle$	"spinor"
$j=1$	$\left|{1 \atop 1}\right\rangle = |20\rangle$ $\left|{1 \atop 0}\right\rangle = |11\rangle$ $\left|{1 \atop -1}\right\rangle = |02\rangle$	"3-vector"
$j=\frac{3}{2}$	$\left|{3/2 \atop 1/2}\right\rangle = |30\rangle$ $\left|{3/2 \atop 1/2}\right\rangle = |21\rangle$ $\left|{3/2 \atop -1/2}\right\rangle = |12\rangle$ $\left|{3/2 \atop -3/2}\right\rangle = |03\rangle$	"4-spinor"
$j=2$	$\left|{2 \atop 2}\right\rangle = |40\rangle$ $\left|{2 \atop 1}\right\rangle = |31\rangle$ $\left|{2 \atop 0}\right\rangle = |22\rangle$ $\left|{2 \atop -1}\right\rangle = |13\rangle$ $\left|{2 \atop -2}\right\rangle = |04\rangle$	"tensor"
	$\vdots$	

$$\begin{cases} j = \dfrac{\upsilon}{2} = \dfrac{n_1 + n_2}{2} \\ m = \dfrac{n_1 - n_2}{2} \end{cases} \qquad \begin{aligned} n_1 &= j + m = 2\upsilon + m \\ n_2 &= j - m = 2\upsilon - m \end{aligned}$$

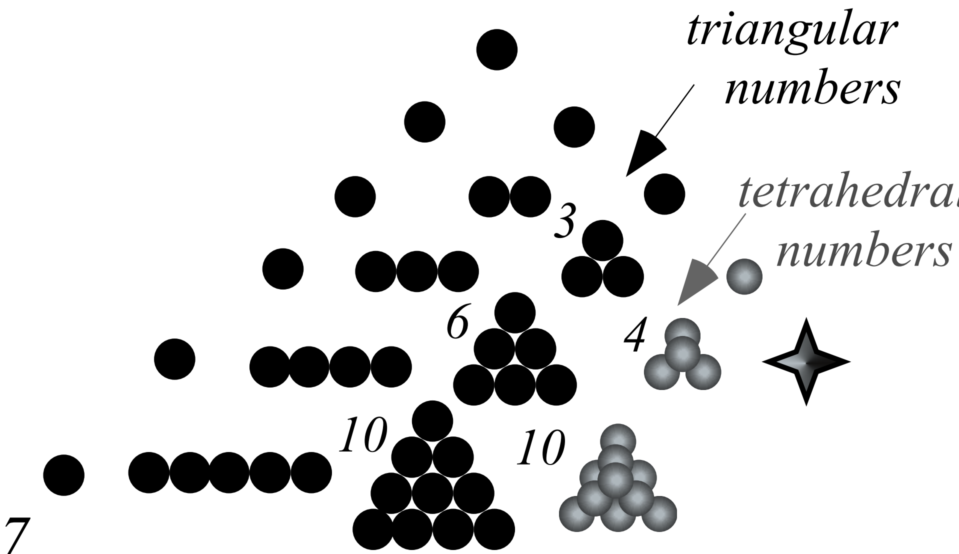


Introducing U(N)

(a) N-D Oscillator Degeneracy ℓ of quamtum level υ

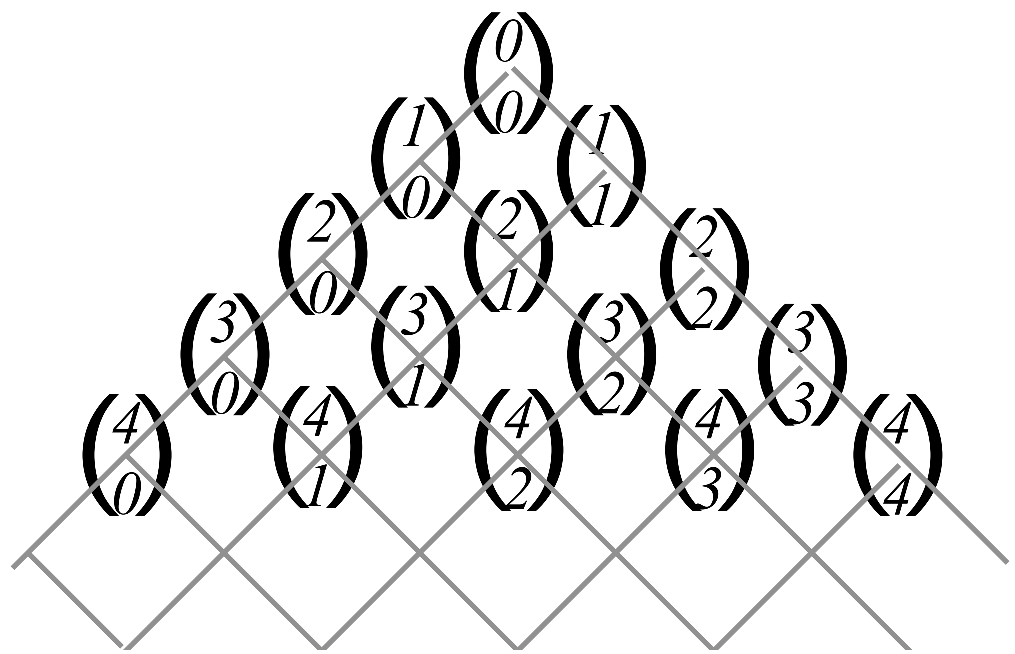


(b) Stacking numbers



(c) Binomial coefficients

$$\frac{(N-1+\upsilon)!}{(N-1)!\,\upsilon!} = \binom{N-1+\upsilon}{\upsilon} = \binom{N-1+\upsilon}{N-1}$$



Introducing U(3)

(b) N-particle 3-level states ...or spin-1 states

$\boxed{1} = |1\ 0\ 0\rangle = a_1^\dagger\,|0\ 0\ 0\rangle$

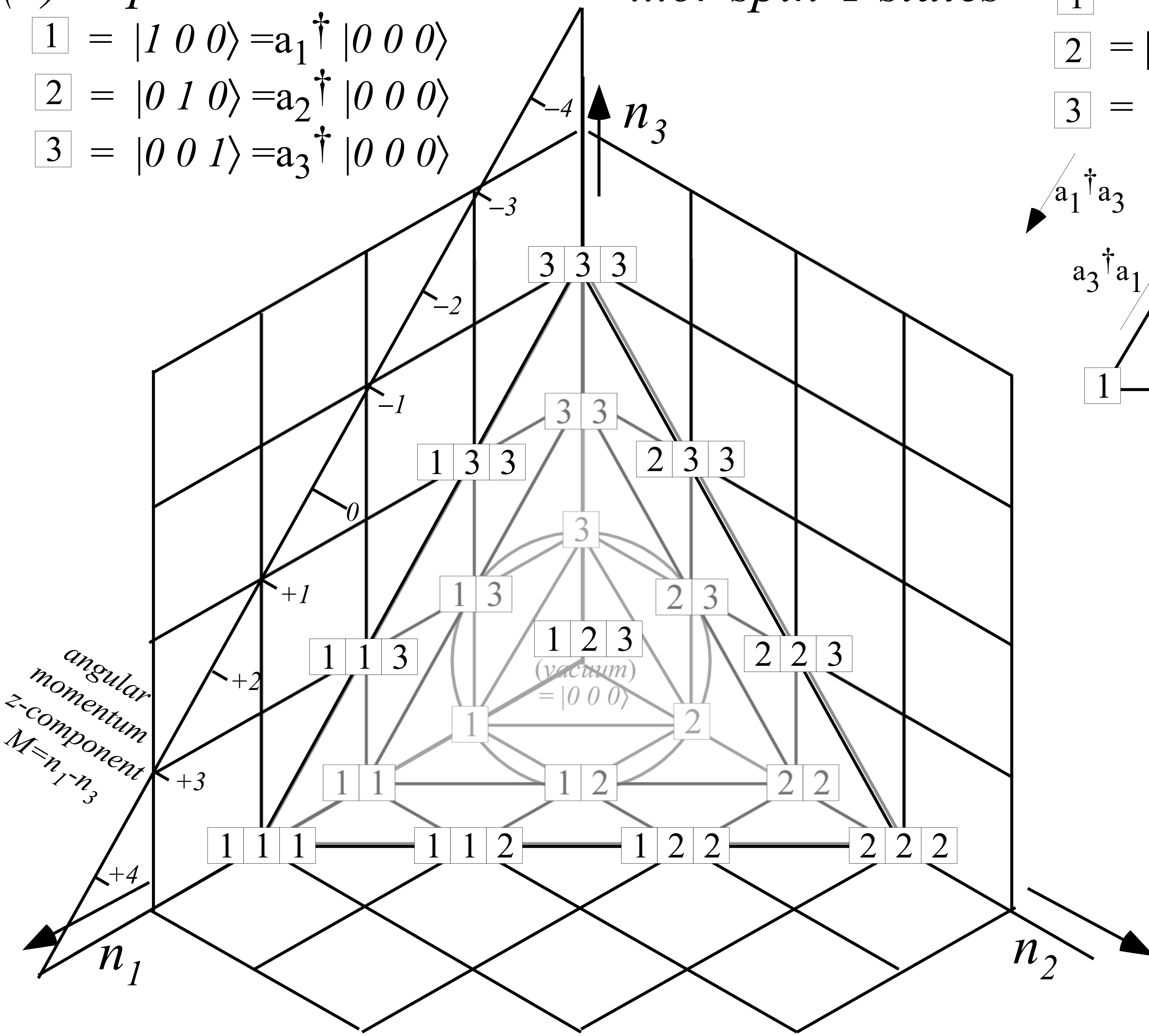
$\boxed{2} = |0\ 1\ 0\rangle = a_2^\dagger\,|0\ 0\ 0\rangle$

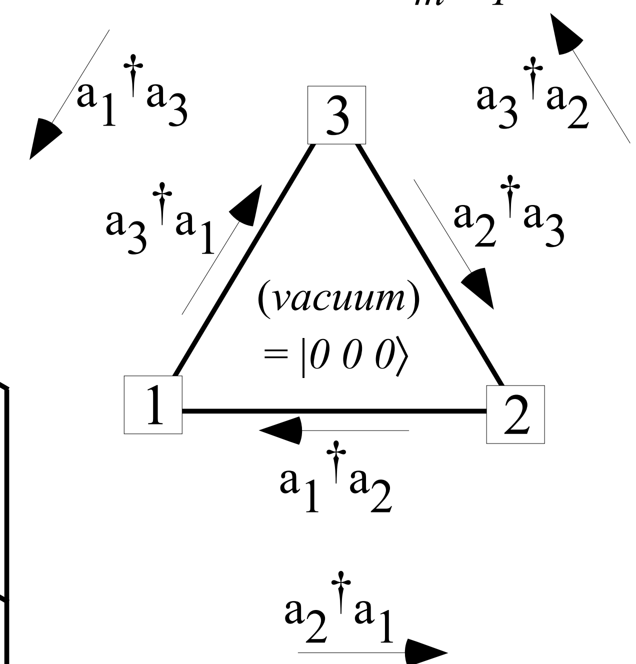
$\boxed{3} = |0\ 0\ 1\rangle = a_3^\dagger\,|0\ 0\ 0\rangle$

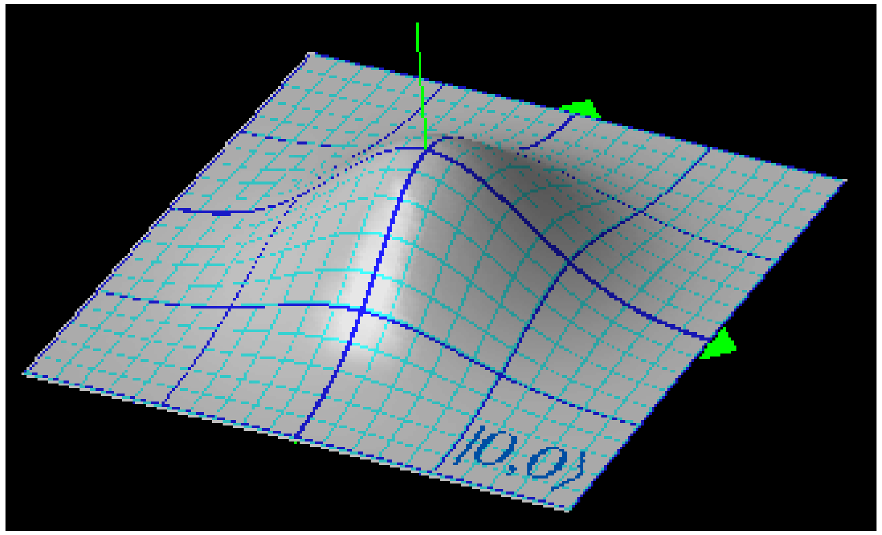
$\boxed{1} = |\uparrow\rangle = \left|{j=1 \atop m=+1}\right\rangle$

$\boxed{2} = |\leftrightarrow\rangle = \left|{j=1 \atop m=0}\right\rangle$

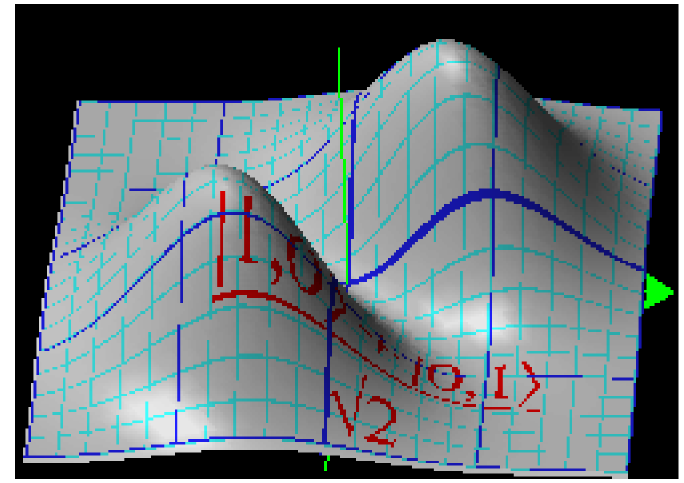
$\boxed{3} = |\downarrow\rangle = \left|{j=1 \atop m=-1}\right\rangle$



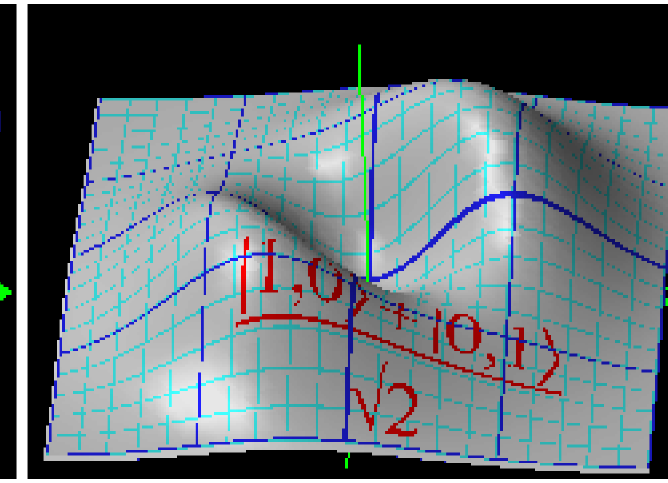


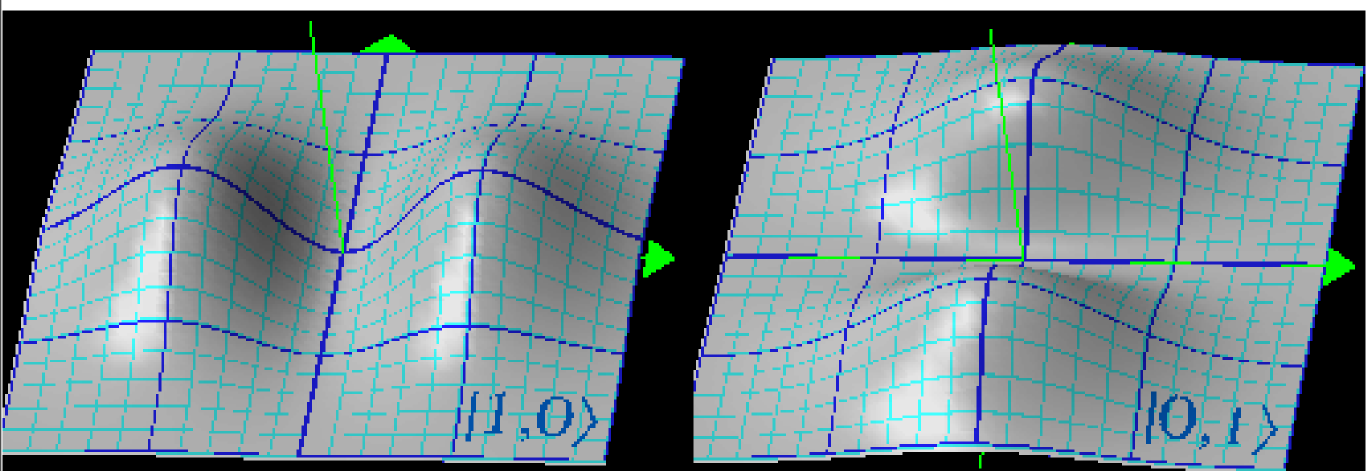


$t = 0$

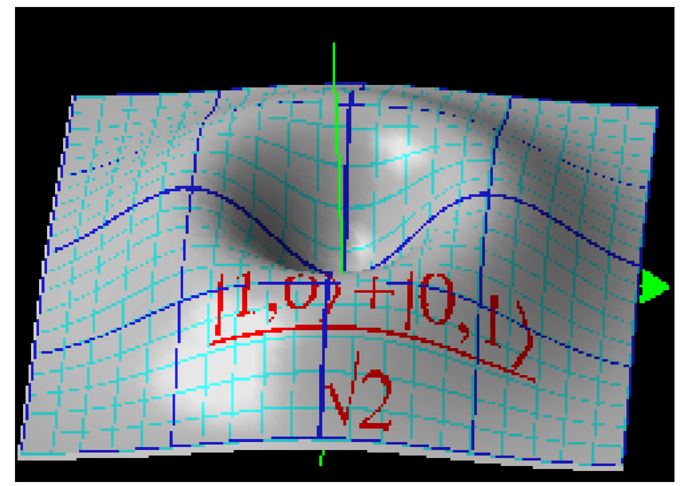


$t = \tau_{beat}/8$

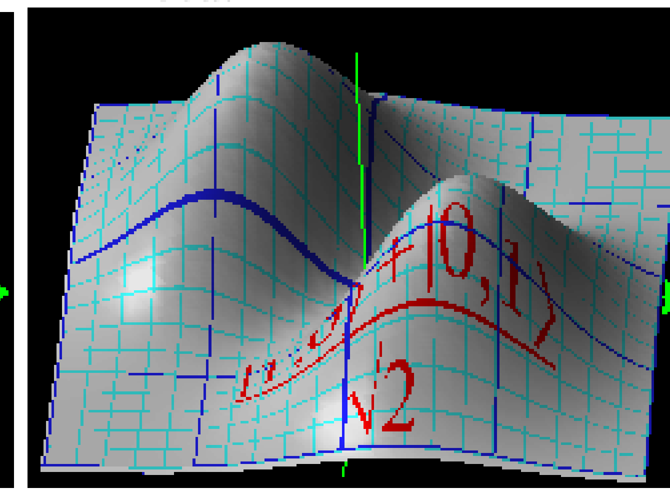




$t = \tau_{beat}/4$



$t = \tau_{beat}/2$



$$\Psi(x_1,x_2,t) = \frac{1}{2}\left|\psi_{10}(x_1,x_2)e^{-i\omega_{10}t} + \psi_{01}(x_1,x_2)e^{-i\omega_{01}t}\right|^2 e^{-\left(x_1^2+x_2^2\right)} = \frac{e^{-\left(x_1^2+x_2^2\right)}}{2\pi}\left|\sqrt{2}x_1 e^{-i\omega_{10}t} + \sqrt{2}x_1 e^{-i\omega_{01}t}\right|^2$$

$$= \frac{e^{-\left(x_1^2+x_2^2\right)}}{\pi}\left(x_1^2 + x_2^2 + 2x_1x_2\cos\left(\omega_{10}-\omega_{01}\right)t\right) = \frac{e^{-\left(x_1^2+x_2^2\right)}}{\pi}\begin{cases} \left|x_1 + x_2\right|^2 & \textit{for}: t{=}0 \\ x_1^2 + x_2^2 & \textit{for}: t{=}\tau_{\text{beat}}/4 \\ \left|x_1 - x_2\right|^2 & \textit{for}: t{=}\tau_{\text{beat}}/2 \end{cases} \qquad (21.1.30)$$