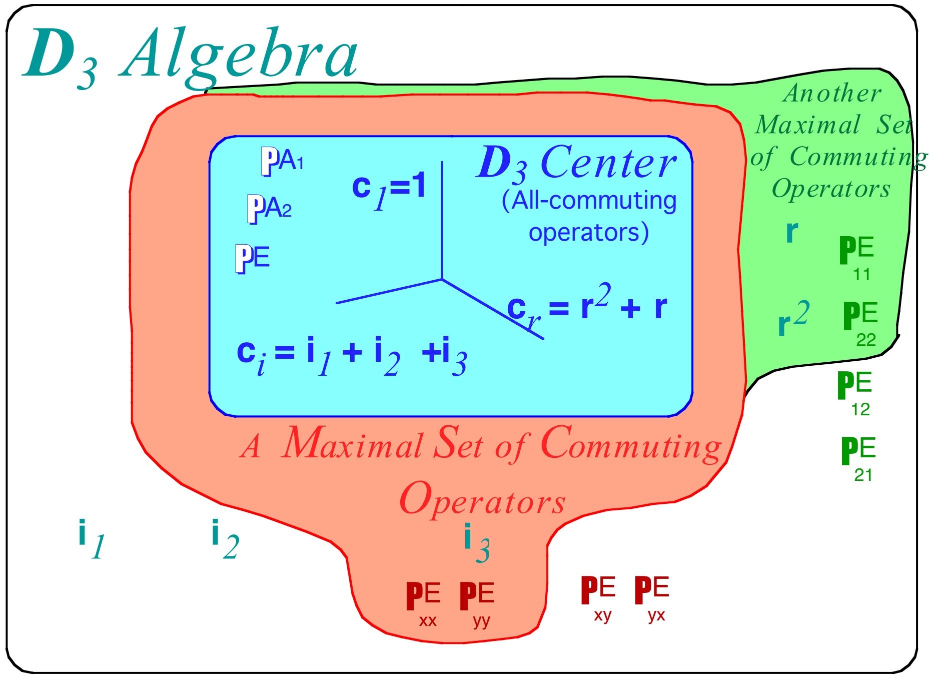
Group Theory in Quantum Mechanics

Lecture 13 *(3.12.13)*

Smallest non-Abelian isomorphic groups $D_3 \sim C_{3v}$

(Int.J.Mol.Sci, 14, 714(2013) p.755-774 , QTCA Unit 5 Ch. 15)
(PSDS - Ch. 3)



(Fig. 15.2.1 QTCA)

3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}
- *D_3 and C_{3v} are isomorphic ($D_3 \sim C_{3v}$ share product table)*
- *Deriving $D_3 \sim C_{3v}$ products:*
  - *By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$*
  - *By nomograms based on U(2) Hamilton-turns*
- *Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes*

Non-commutative symmetry expansion and Global-Local solution
- *Global vs Local symmetry and Mock-Mach principle*
- *Global vs Local matrix duality for D_3*
  - *Global vs Local symmetry expansion of D_3 Hamiltonian*

1st-Stage spectral decomposition of global/local D_3 Hamiltonian
- *All-commuting operators and D_3-invariant class algebra*
- *All-commuting projectors and D_3-invariant characters*
- *Group invariant numbers: Centrum, Rank, and Order*

*Spectral resolution to **irreducible representations** (or "**irreps**") foretold by **characters** or traces*
- *Crystal-field splitting: $O(3) \supset D_3$ symmetry reduction and $D^{\ell} \downarrow D_3$ splitting*

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Fig. 3.1.1 PSDS

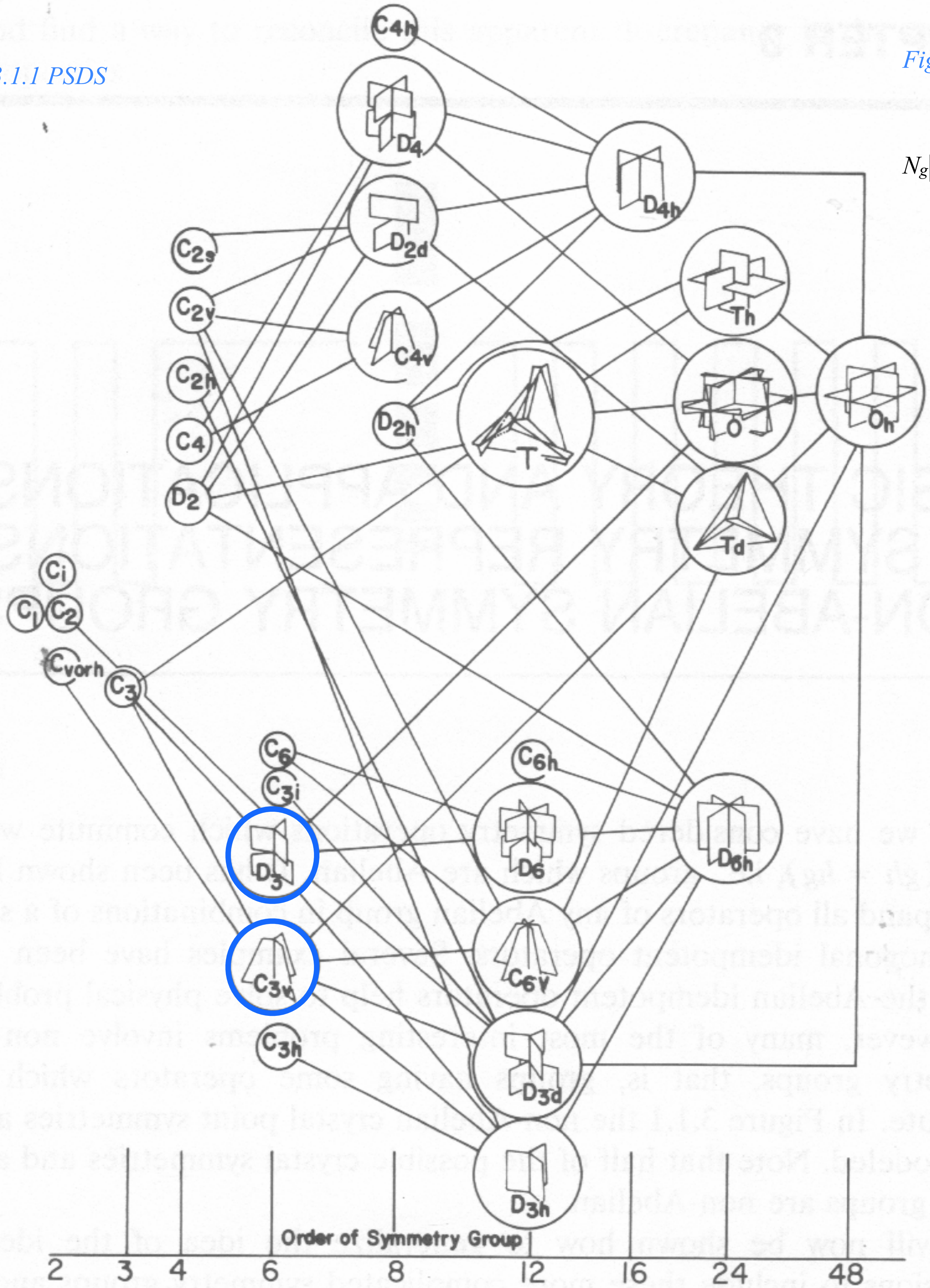


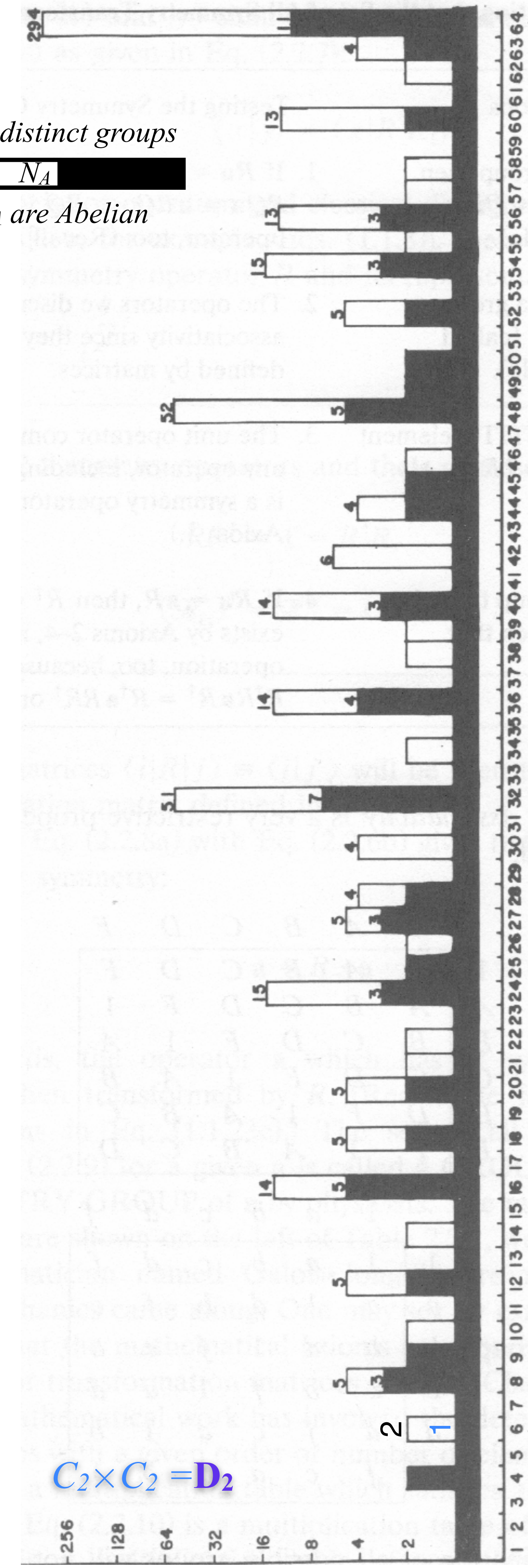
Figure 3.1.1 Crystal point symmetry groups. Models are sketched in circles for the 16 non-Abelian groups. (See also Figure 2.11.1.)

Fig. 2.2.2 PSDS

Total number N_g of distinct groups

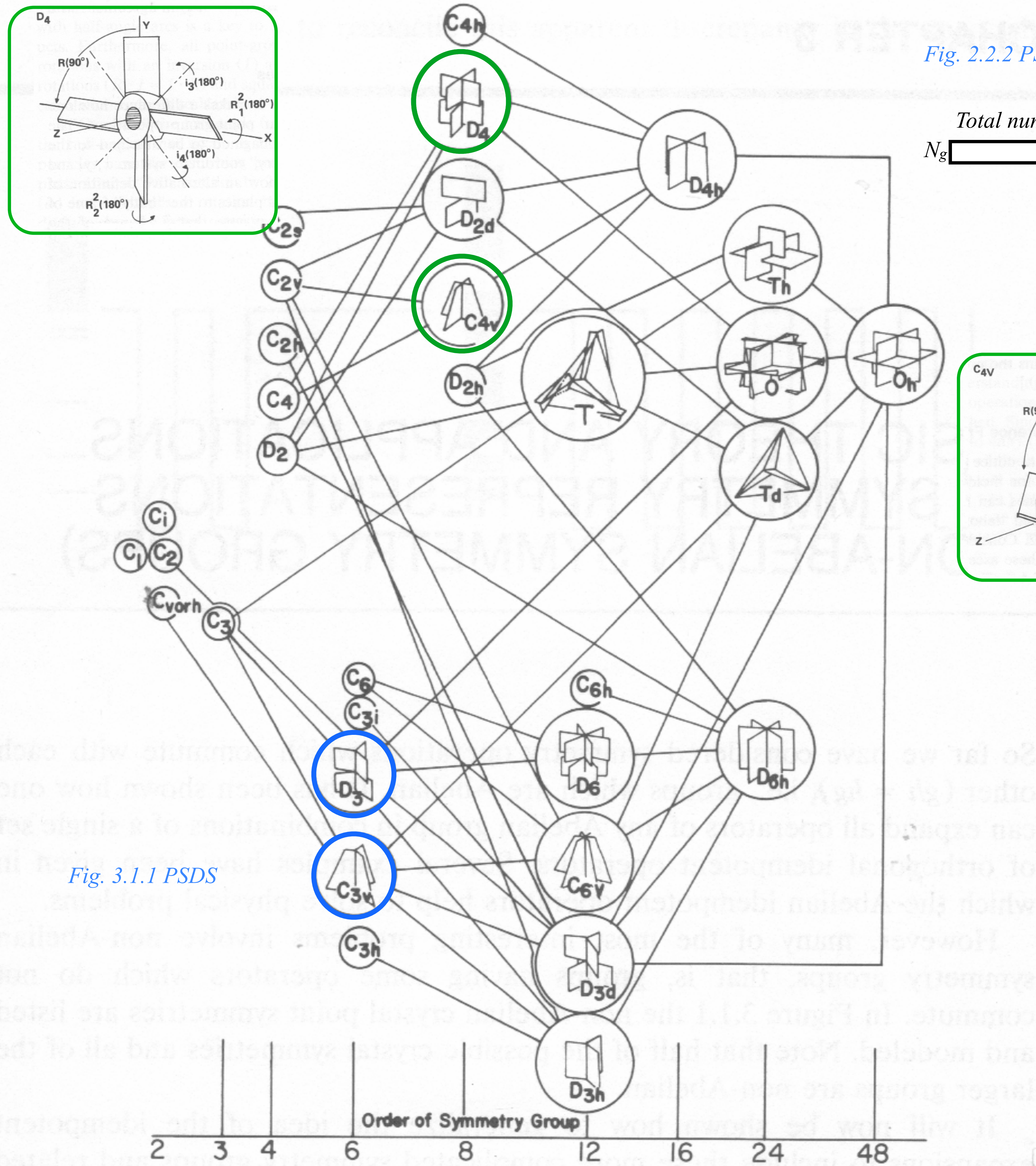
N_g N_A

number N_A are Abelian



$C_2 \times C_2 = \mathbf{D_2}$

C_6 C_5 C_4 C_3 C_2 C_1



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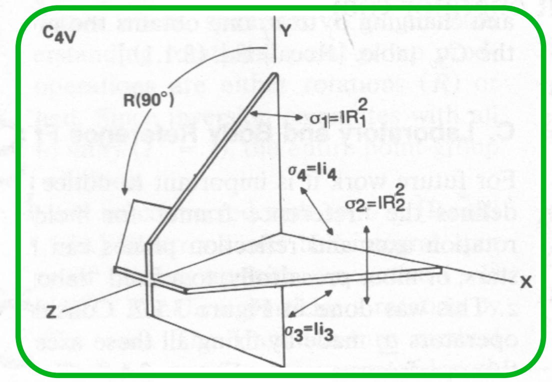

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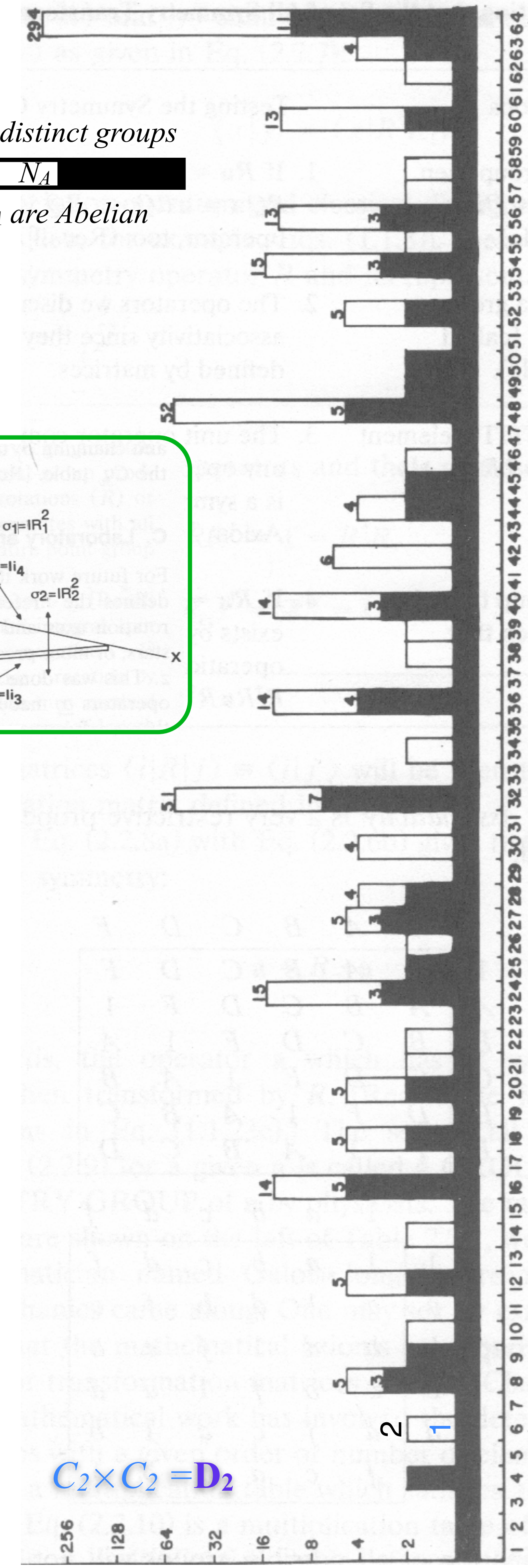

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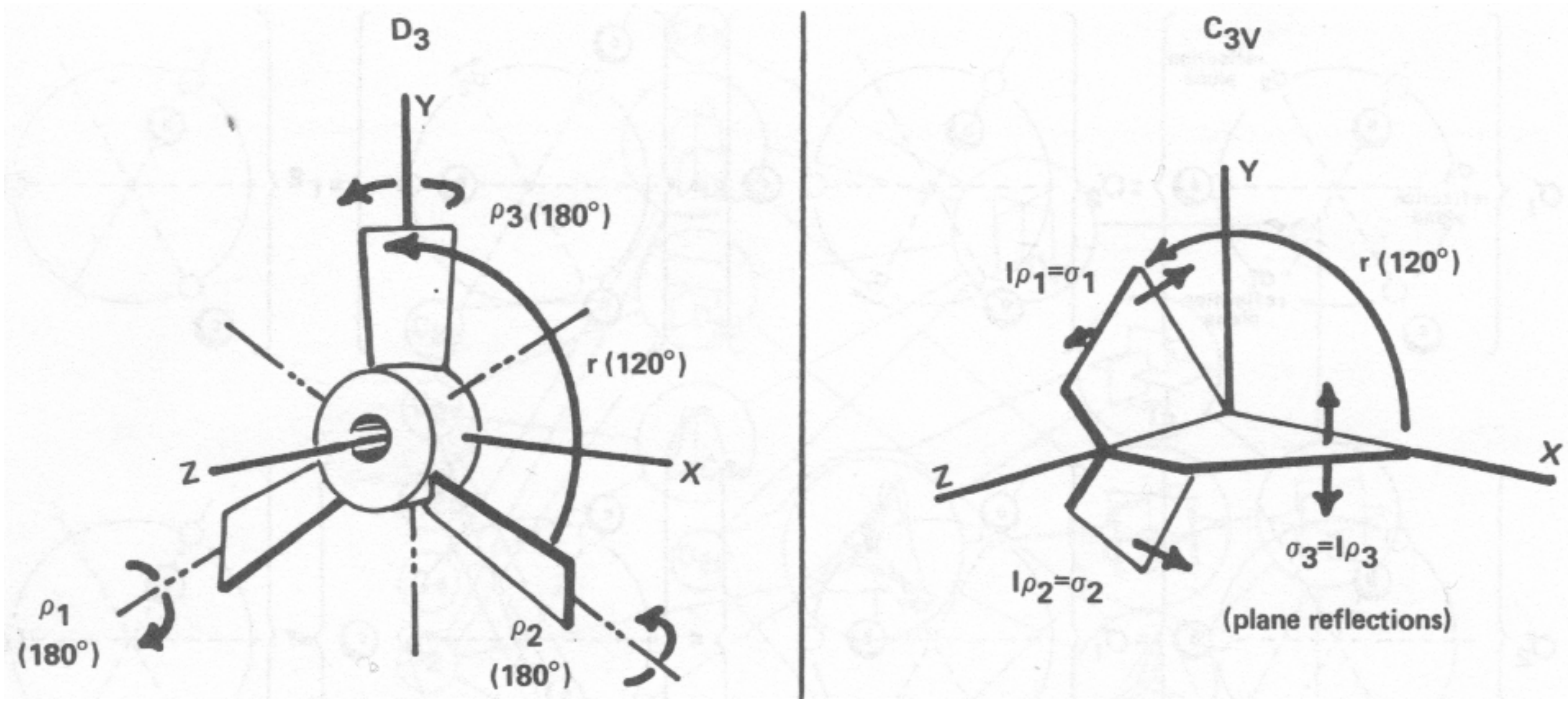

Figure 3.1.3 Pictorial comparison of D_3 and C_{3v} symmetry. A propeller having D_3 symmetry is shown next to a three-plane paddle having C_{3v} symmetry. The group operations are labeled by arrows, which indicate the effect they have. For example, ρ_3 is a 180° rotation around the y axis, while $I\rho_3 = \sigma_3$ is a reflection through the xz plane. (Here axes are fixed and the objects rotate.)

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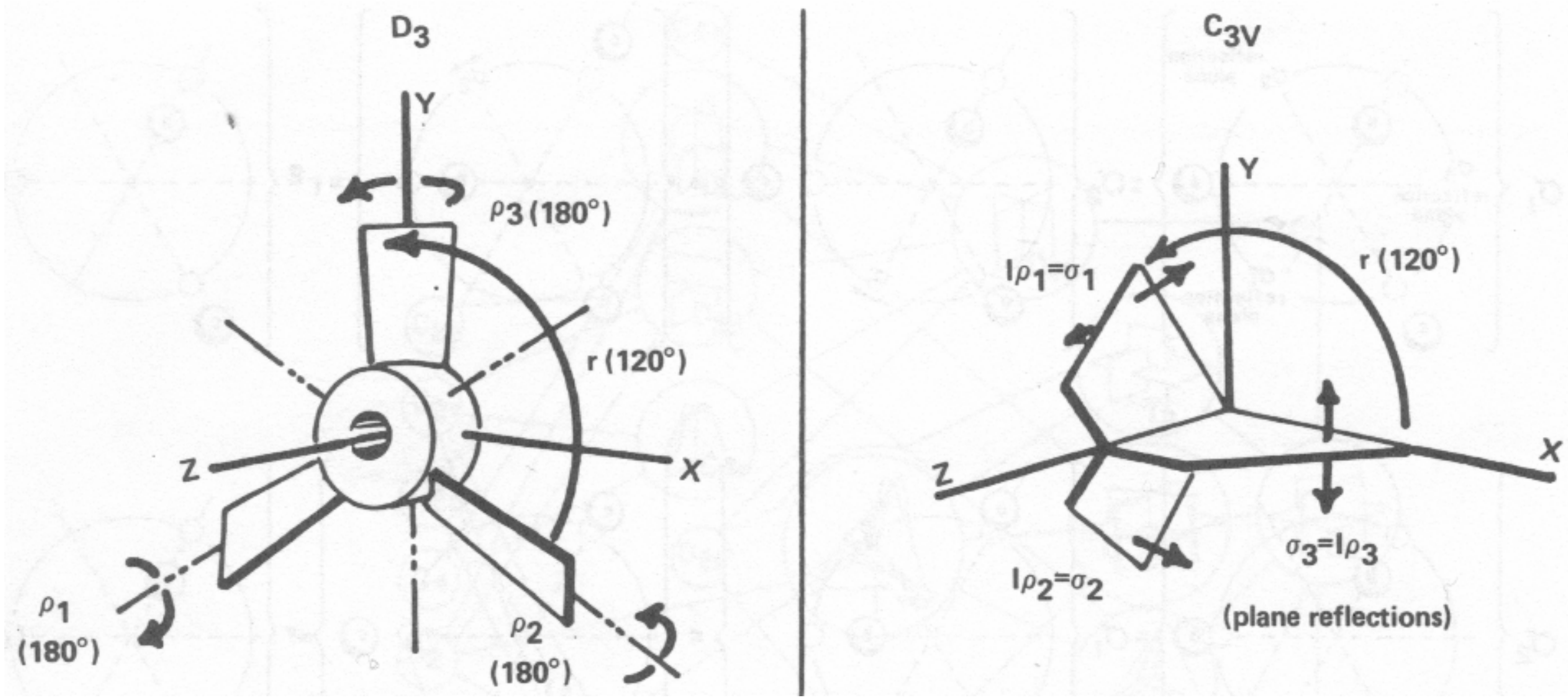

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$$180°\,D_3\text{-Y-axis-rotation: } \rho_3 = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & +1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix} \quad \textit{maps to}: \text{ XZ-mirror-plane reflection: } \sigma_3 = \begin{pmatrix} +1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & +1 \end{pmatrix}$$

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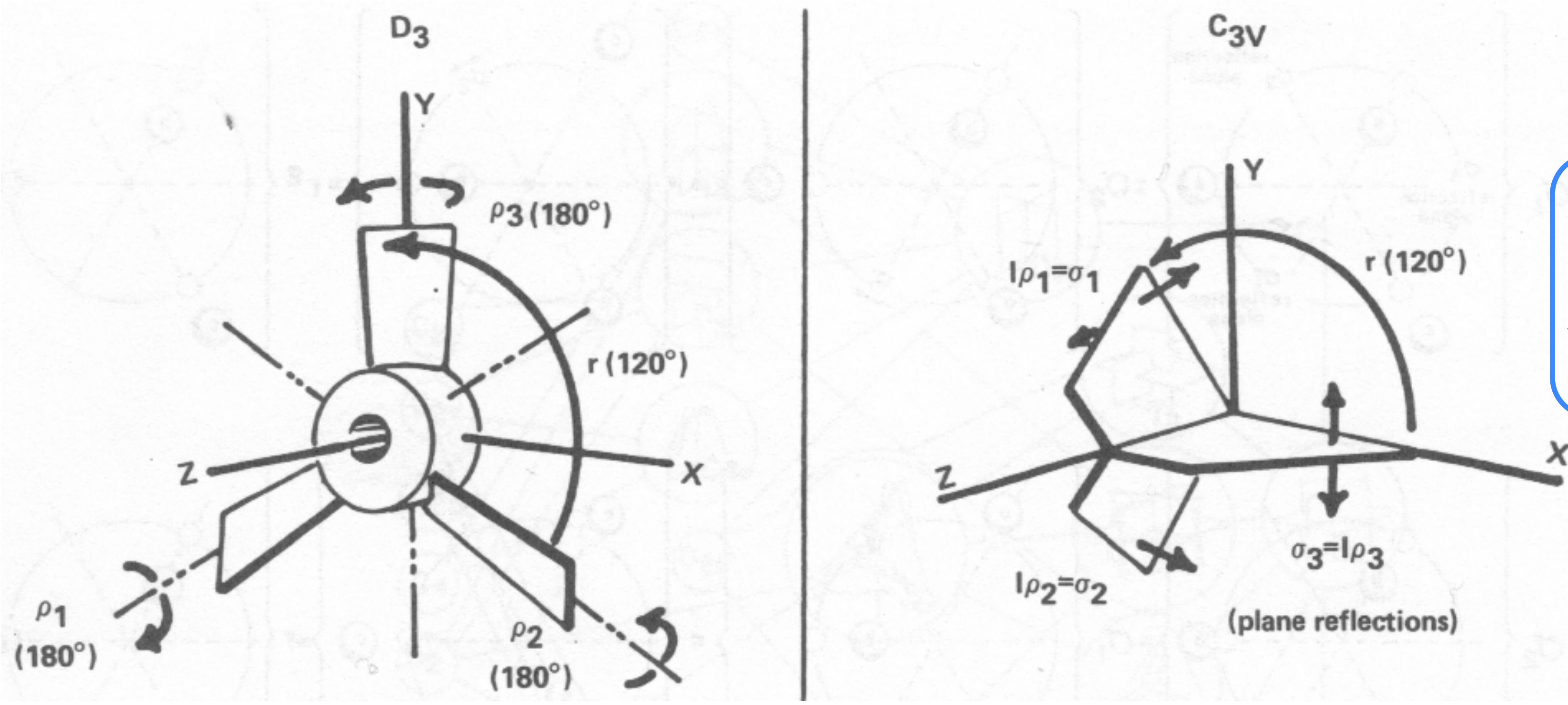

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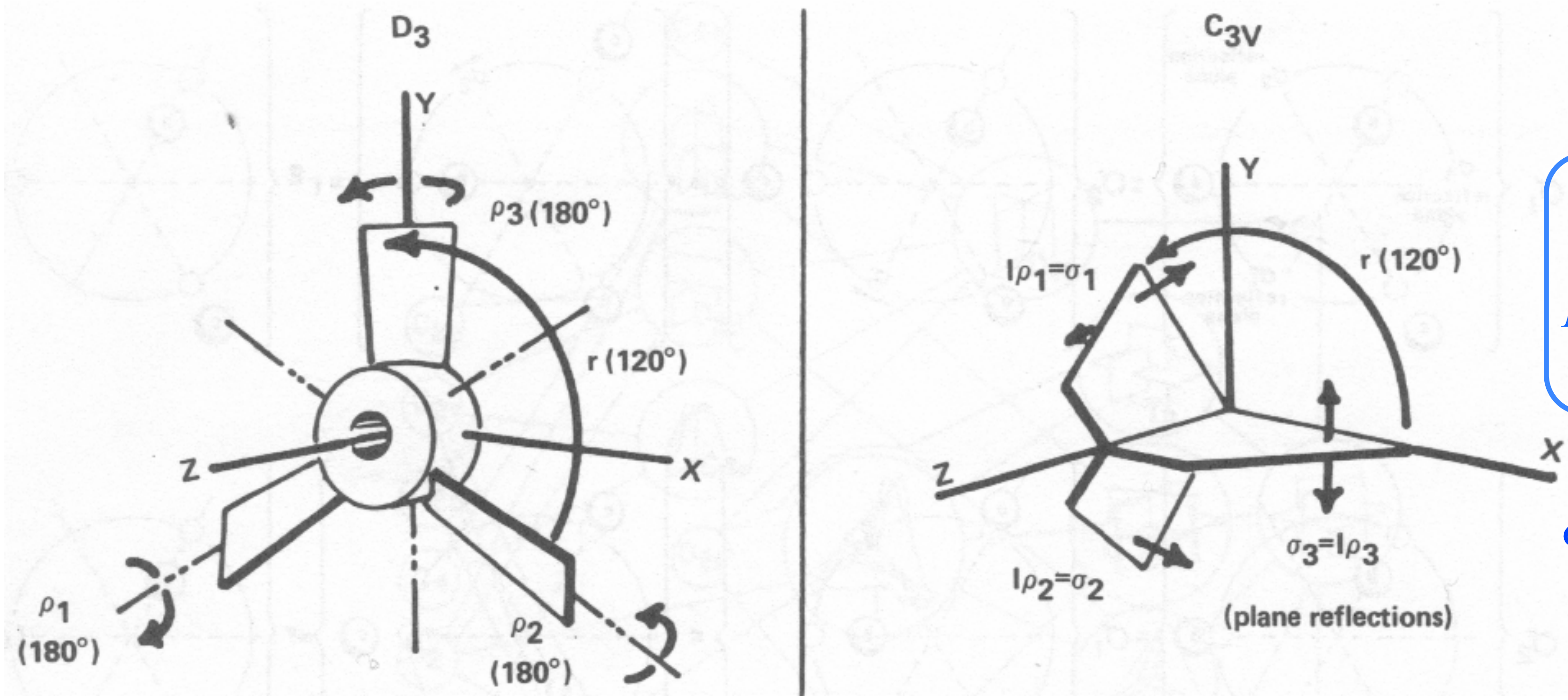

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Inversion
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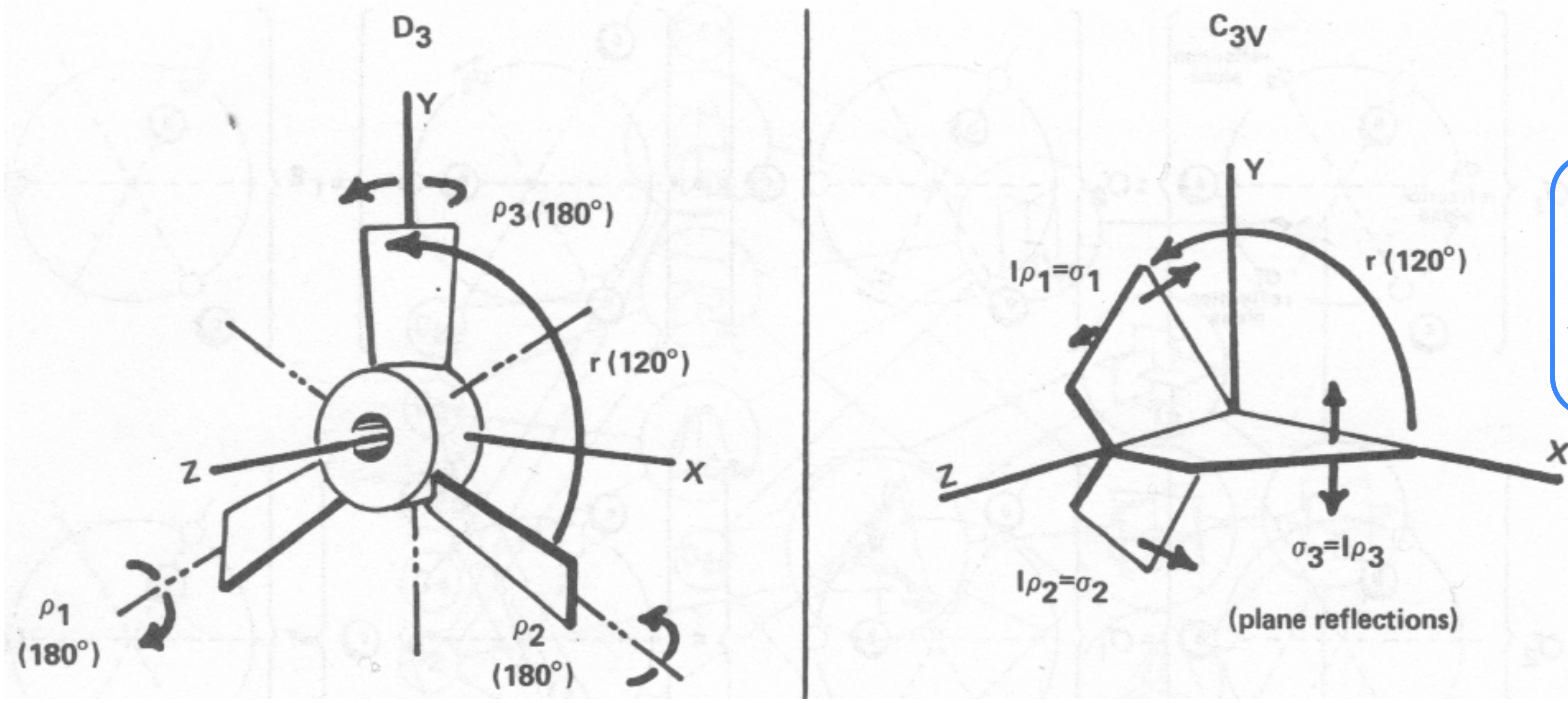

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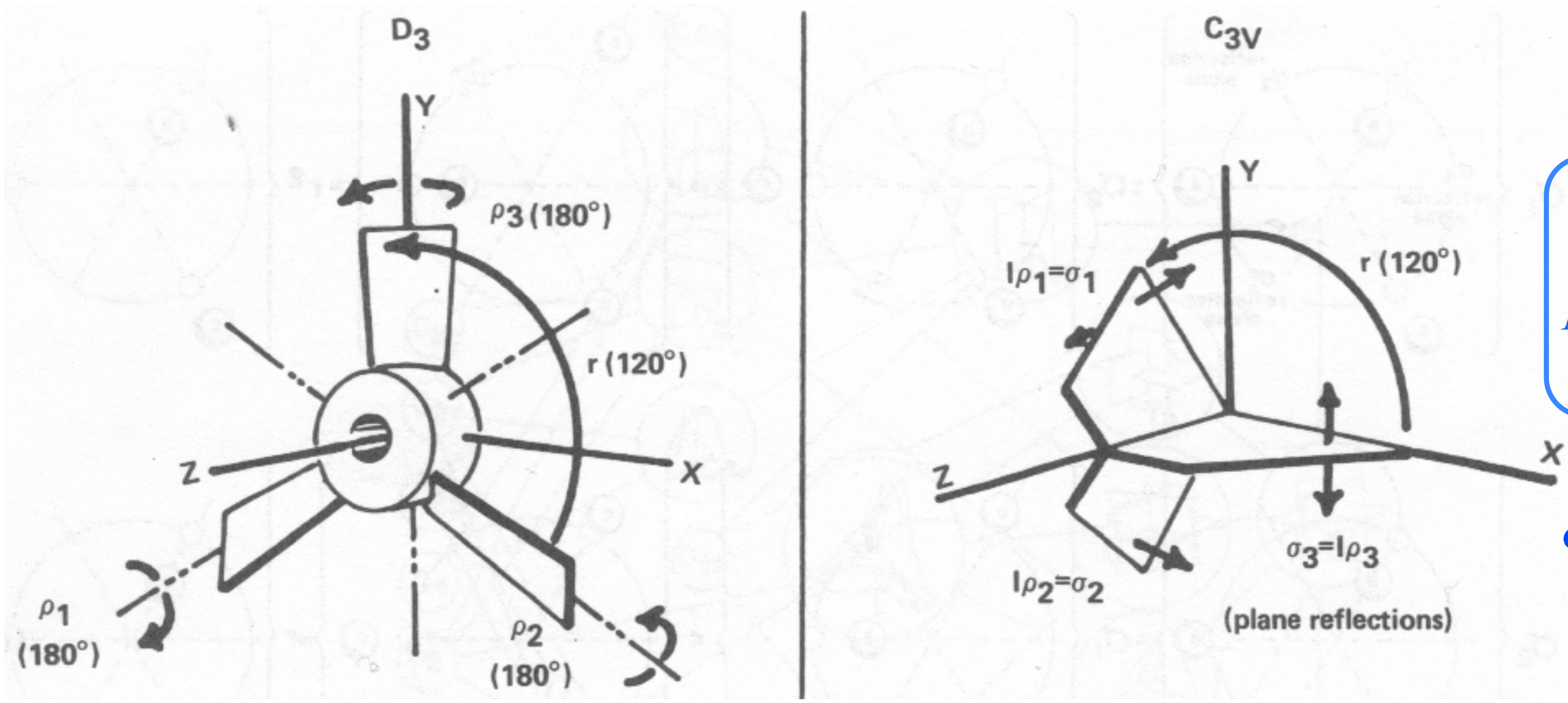

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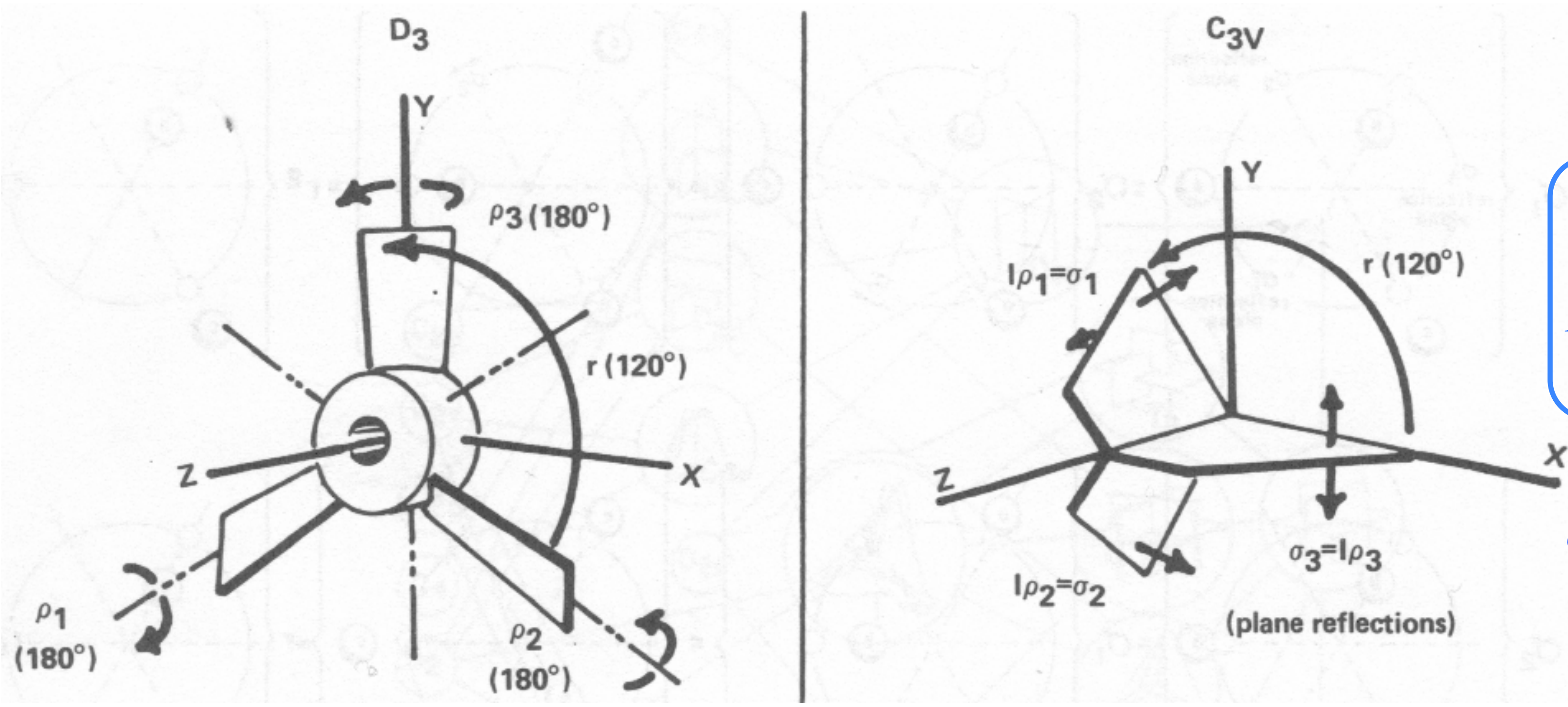

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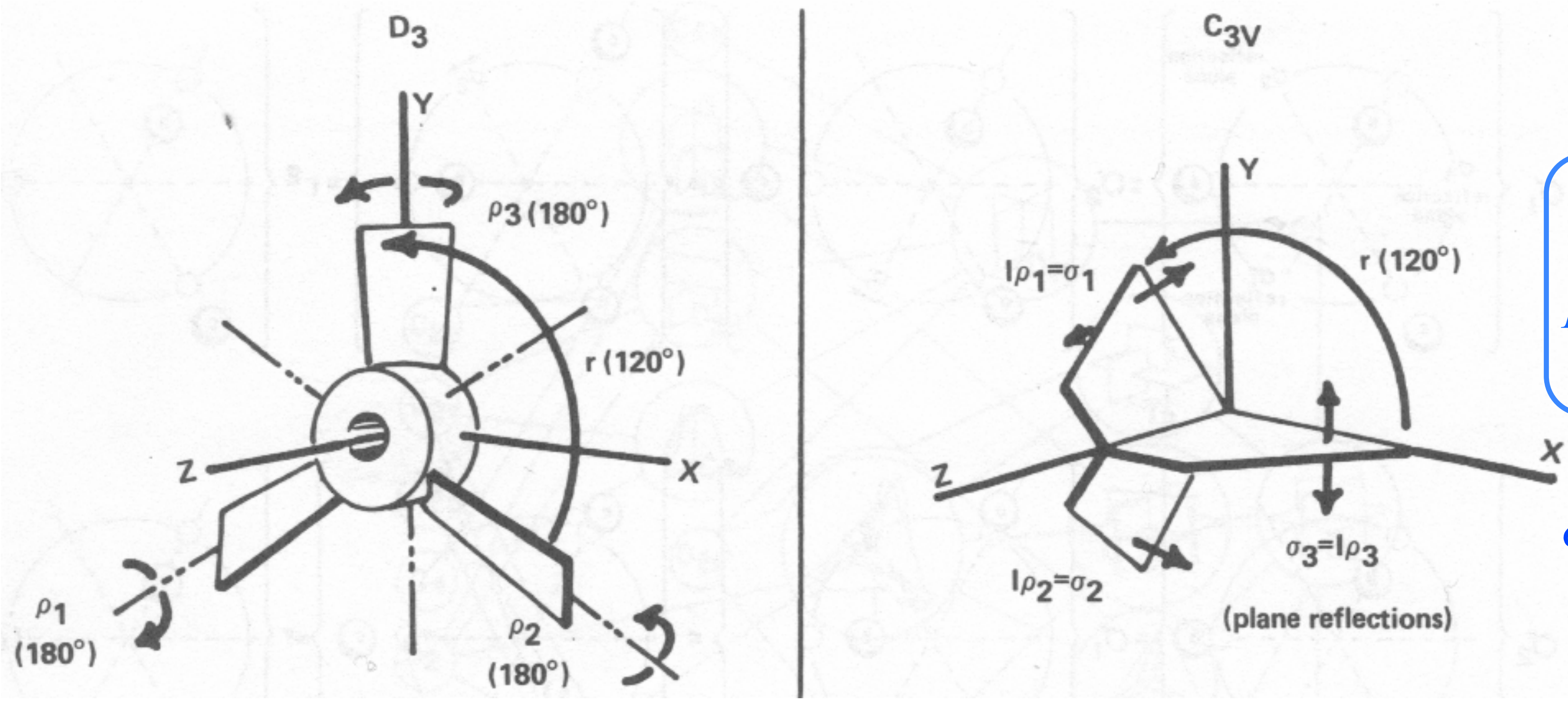

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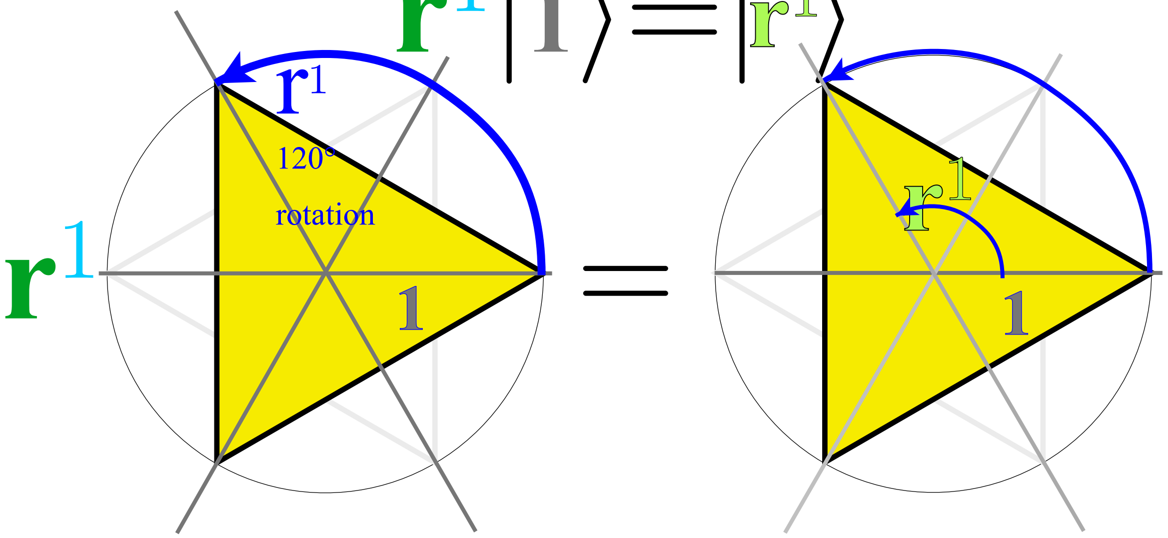

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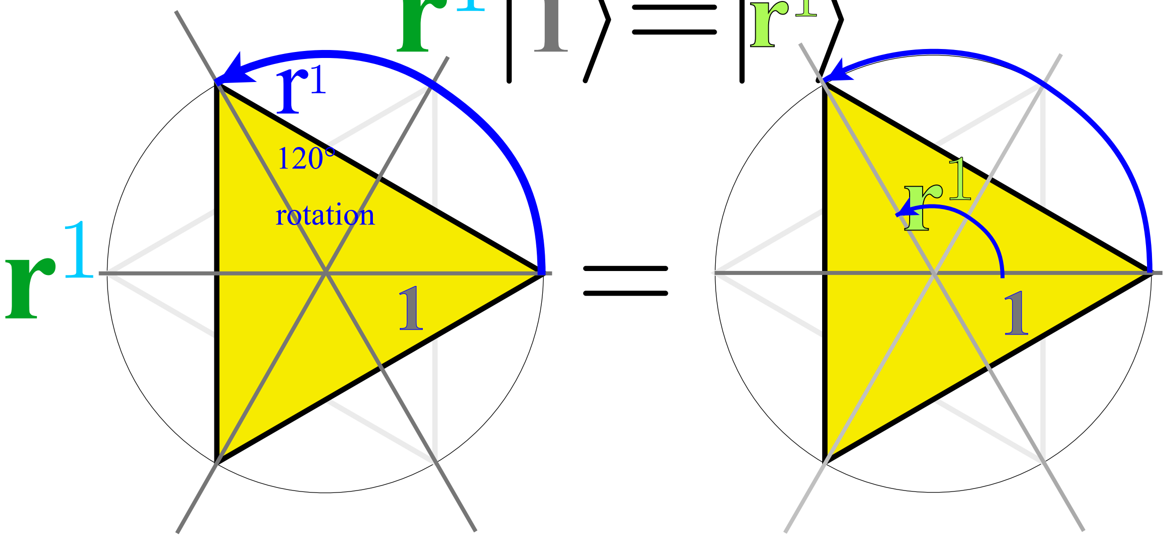

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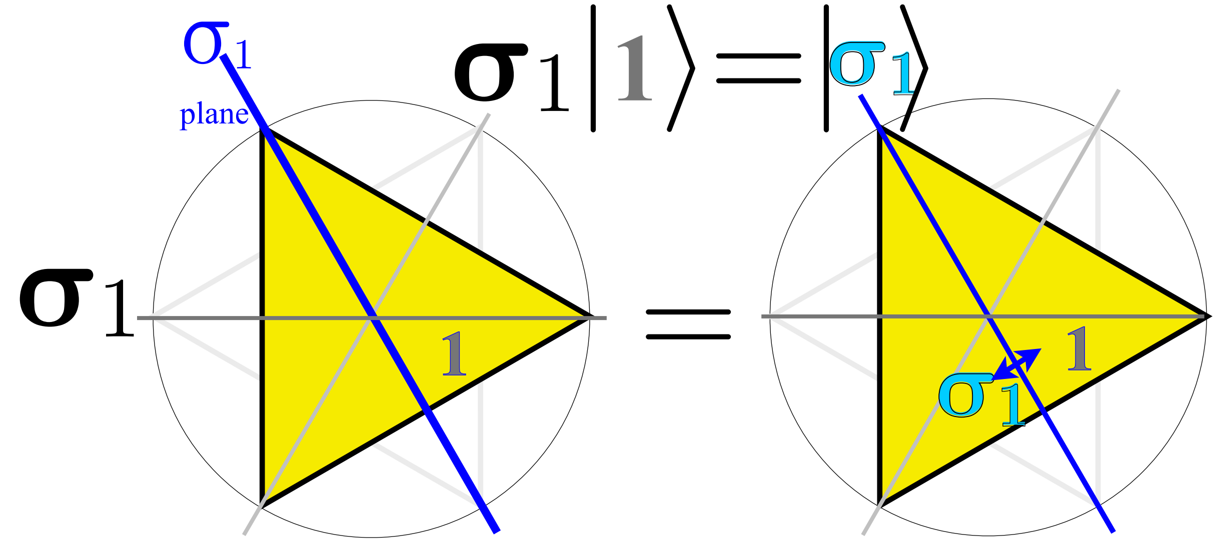
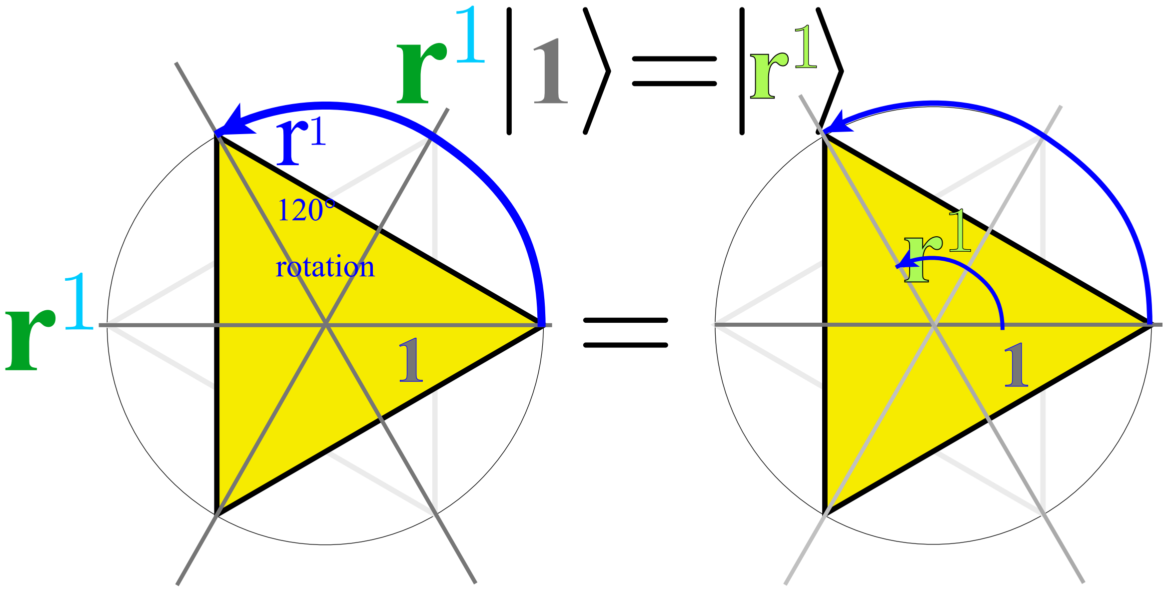
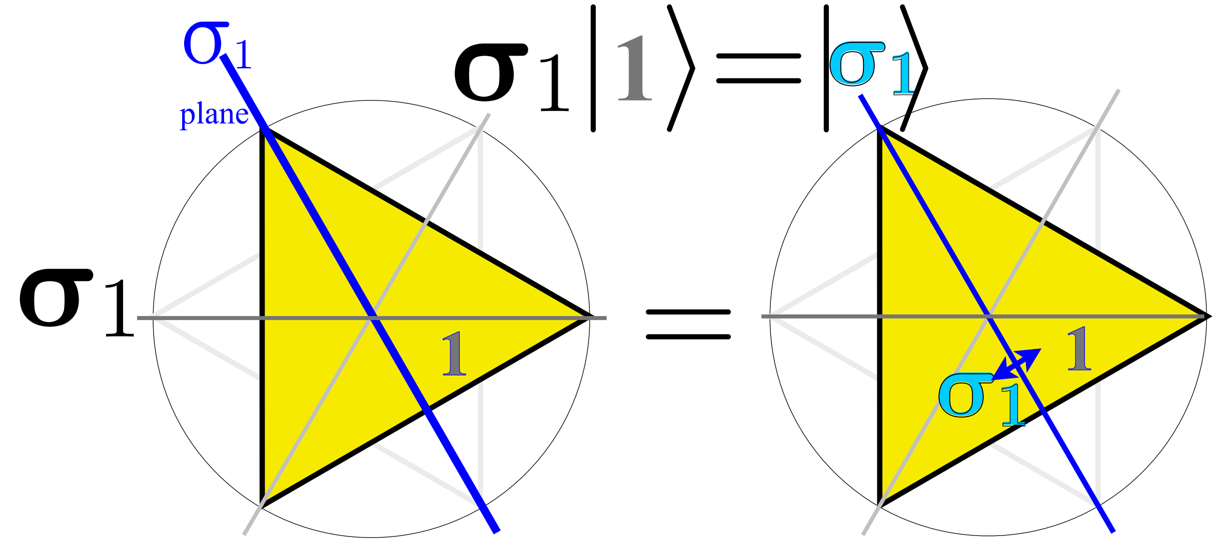
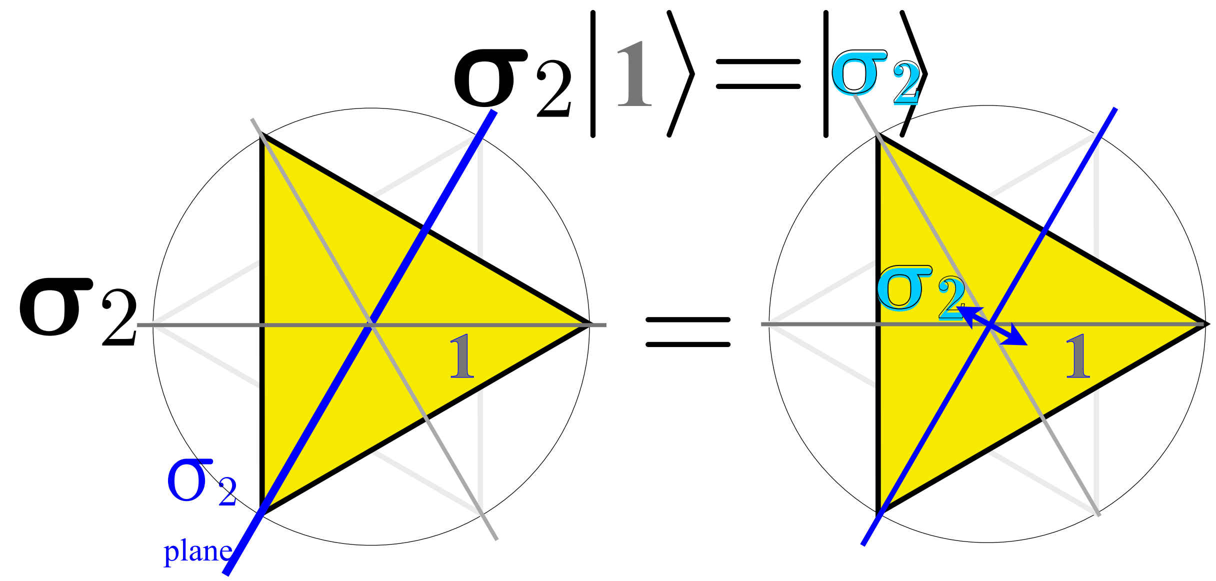


Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$

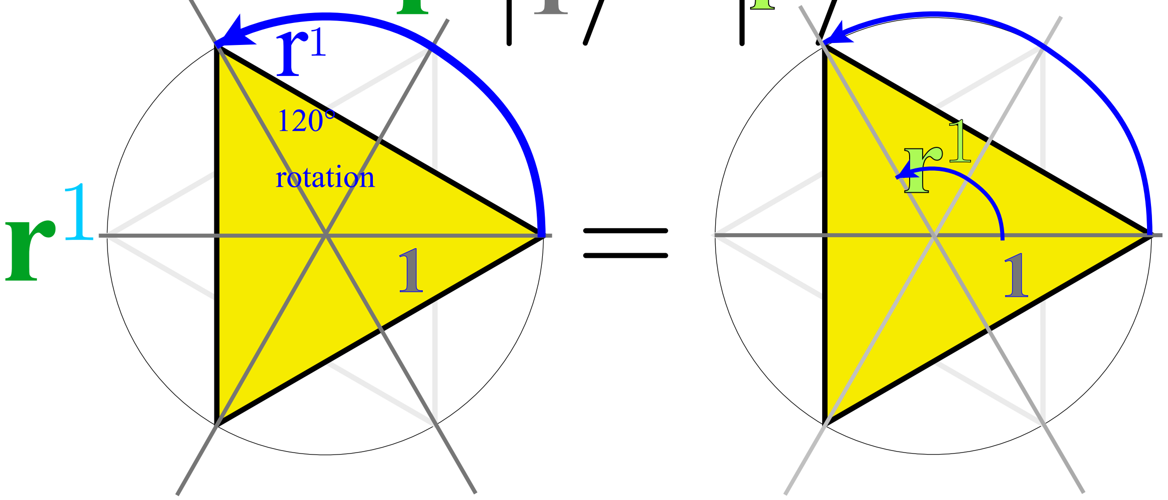




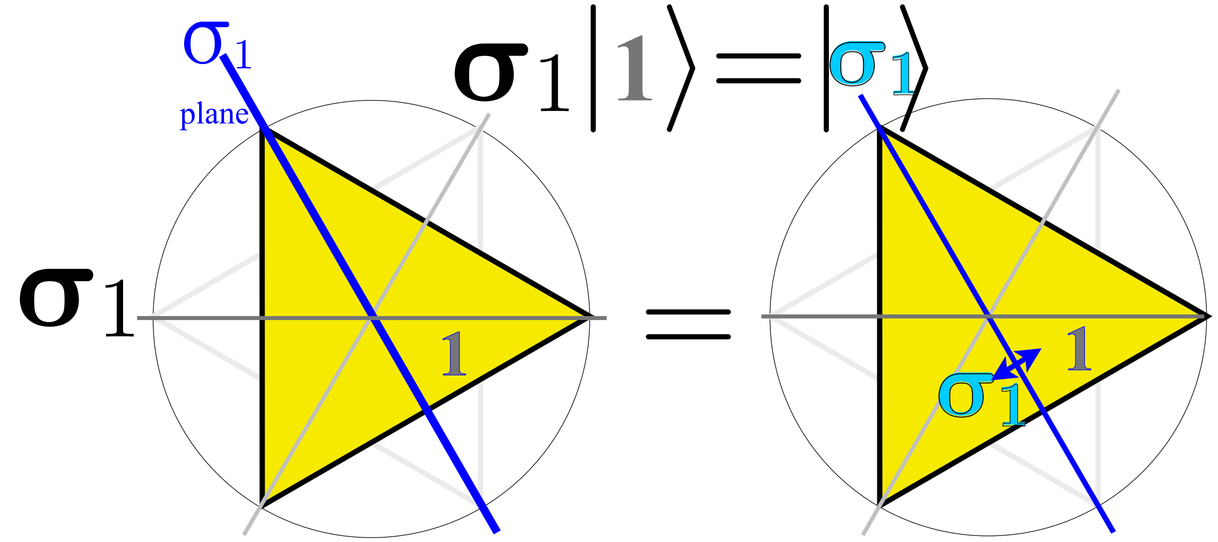


Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$

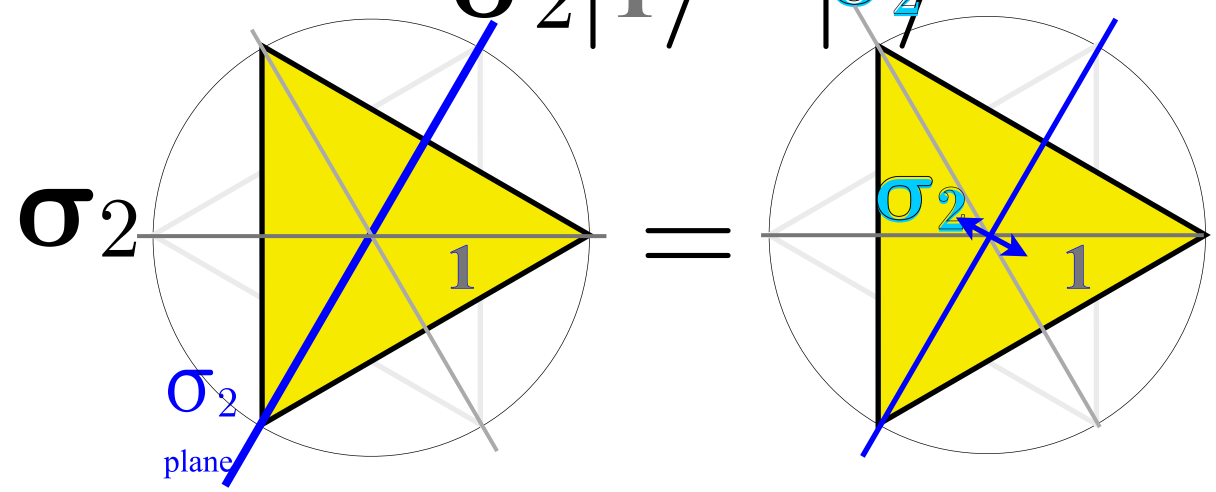
$$\mathbf{r}^1|1\rangle = |\mathbf{r}^1\rangle$$



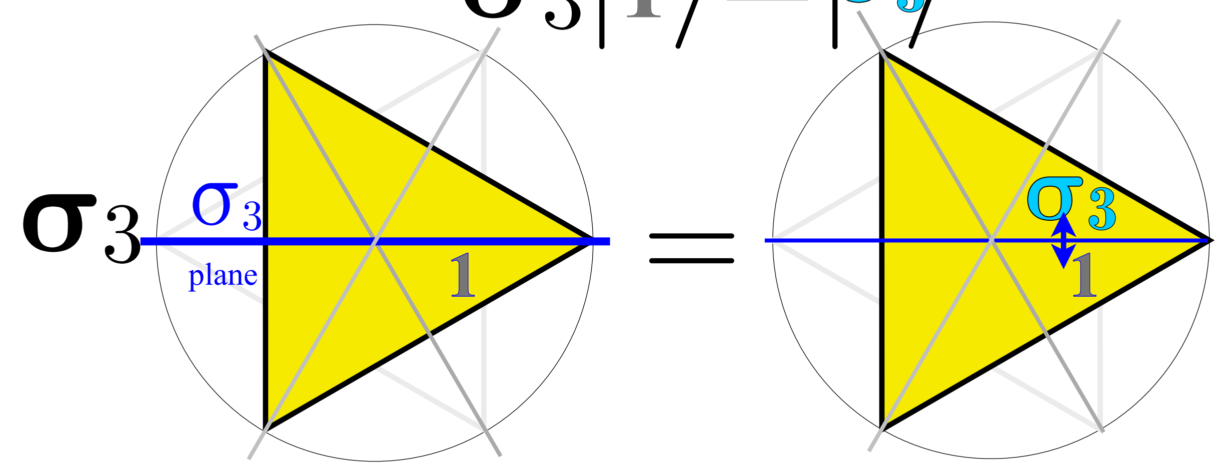
$$\boldsymbol{\sigma}_1|1\rangle = |\boldsymbol{\sigma}_1\rangle$$



$$\boldsymbol{\sigma}_2|1\rangle = |\boldsymbol{\sigma}_2\rangle$$



$$\boldsymbol{\sigma}_3|1\rangle = |\boldsymbol{\sigma}_3\rangle$$



3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

D_3 and C_{3v} are isomorphic ($D_3 \sim C_{3v}$ share product table)

Deriving $D_3 \sim C_{3v}$ products:

By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$

By nomograms based on U(2) Hamilton-turns

Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes

Non-commutative symmetry expansion and Global-Local solution

Global vs Local symmetry and Mock-Mach principle

Global vs Local matrix duality for D_3

Global vs Local symmetry expansion of D_3 Hamiltonian

1st-Stage spectral decomposition of global/local D_3 Hamiltonian

All-commuting operators and D_3-invariant class algebra

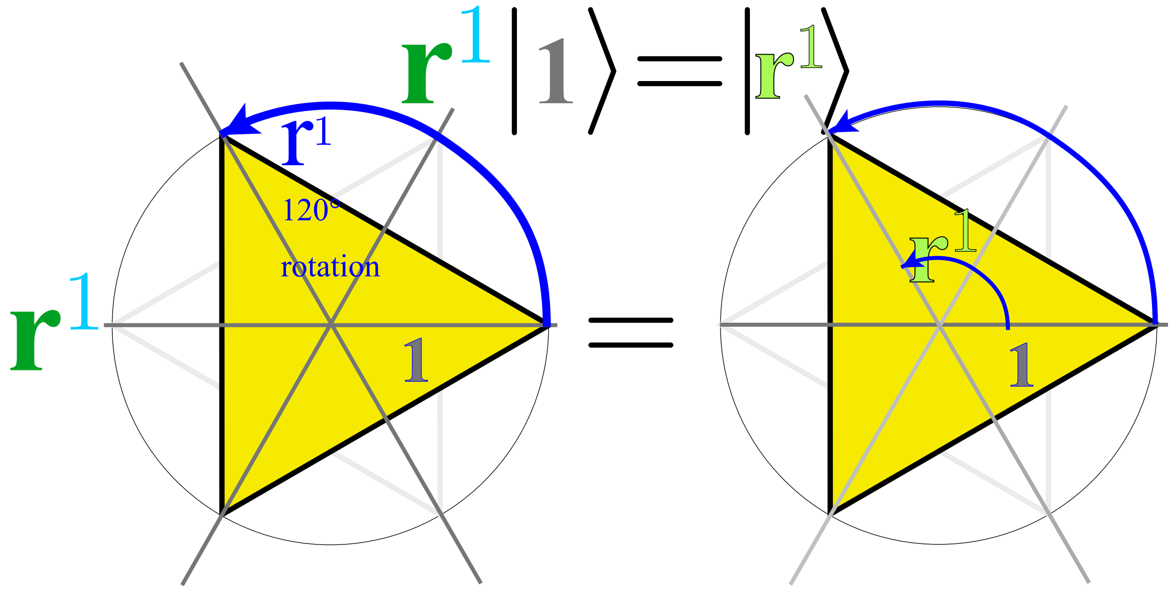
All-commuting projectors and D_3-invariant characters

Group invariant numbers: Centrum, Rank, and Order

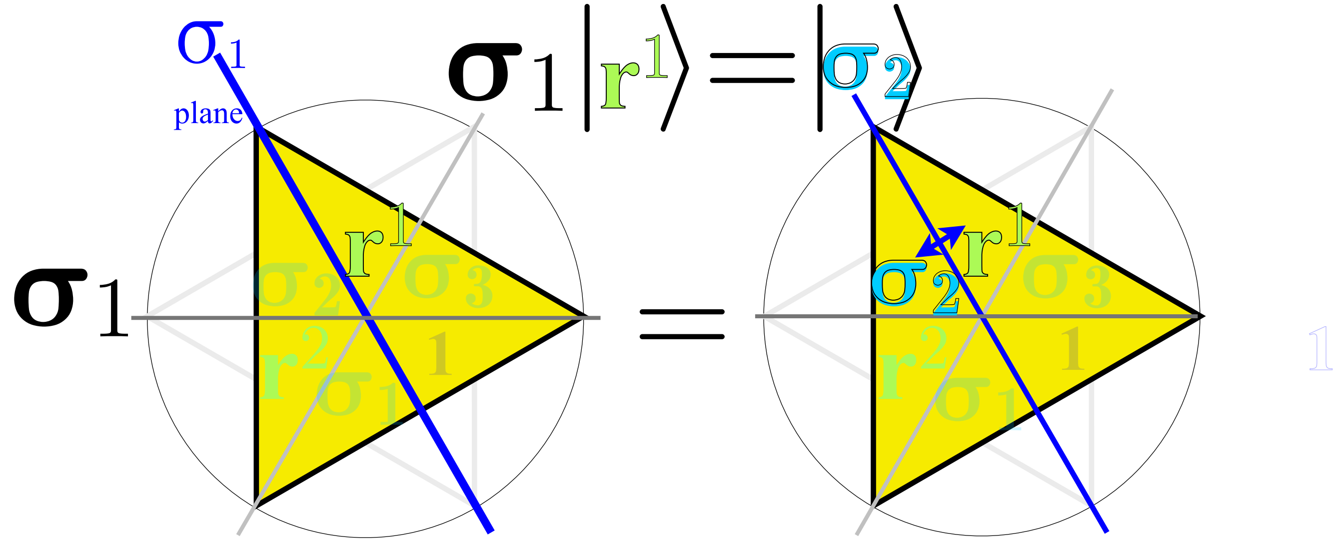
*Spectral resolution to **irreducible representations** (or "**irreps**") foretold by **characters** or traces*

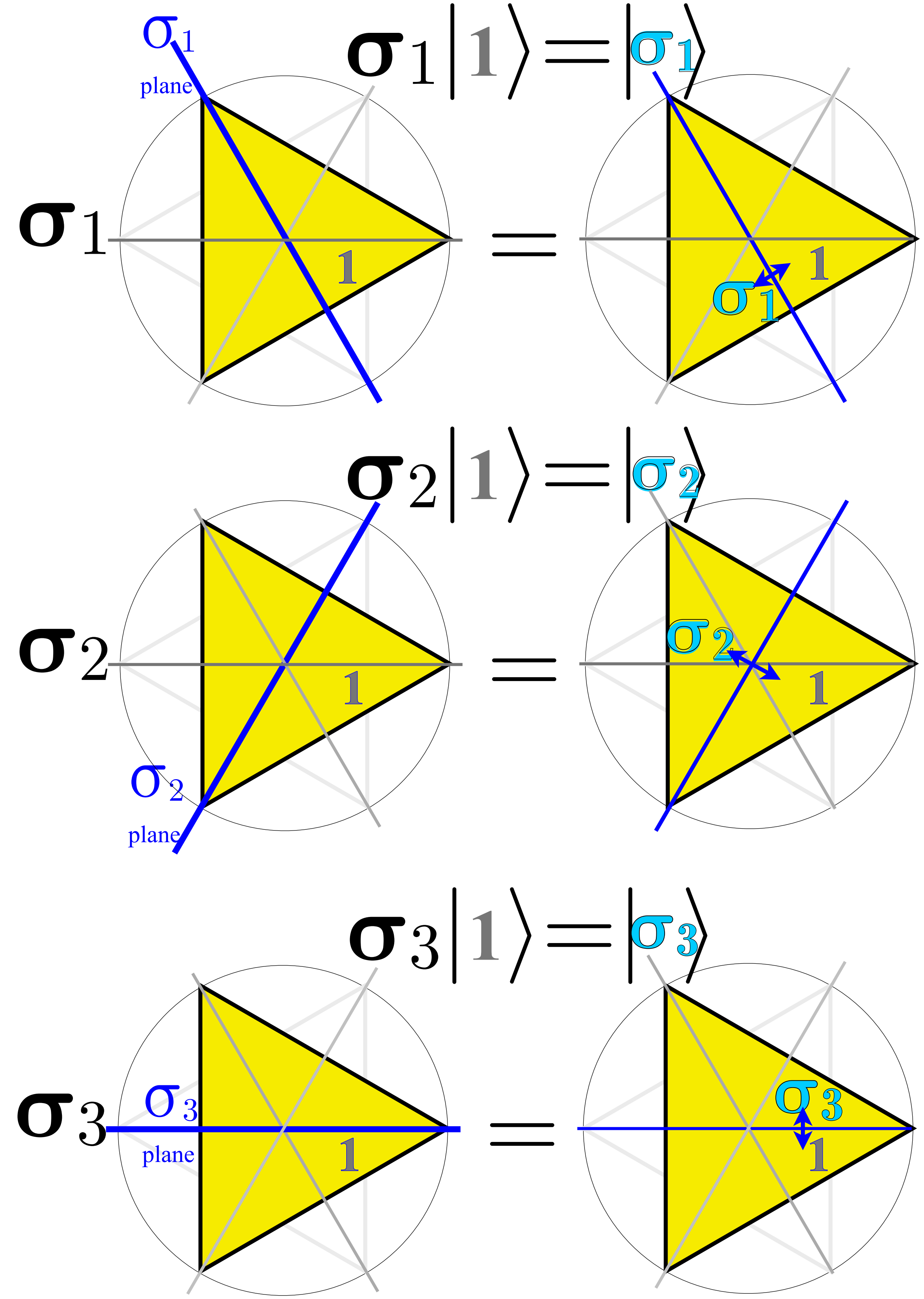
Crystal-field splitting: $O(3) \supset D_3$ symmetry reduction and $D^{\ell} \downarrow D_3$ splitting

Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$

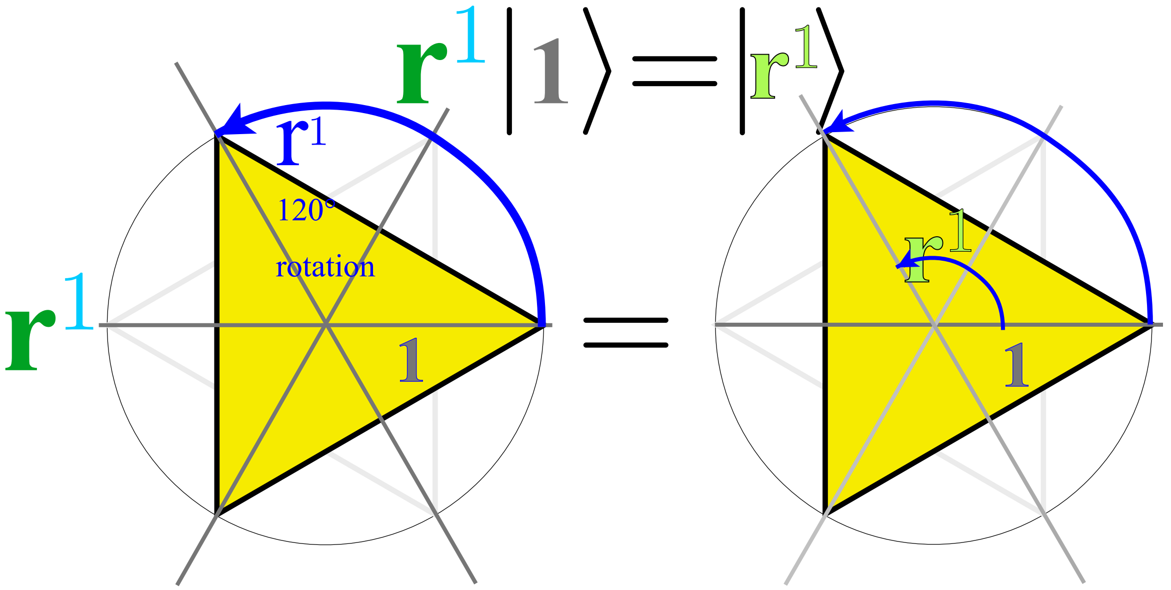


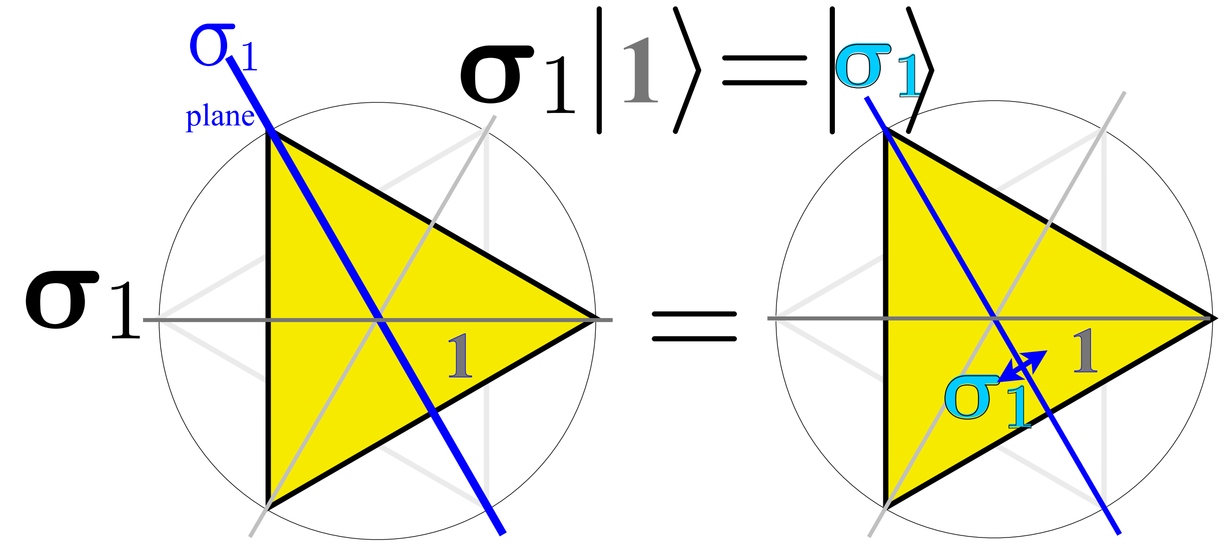
Example: Find C_{3v} product $\boldsymbol{\sigma}_1\mathbf{r}^1|1\rangle = \boldsymbol{\sigma}_1|\mathbf{r}^1\rangle$

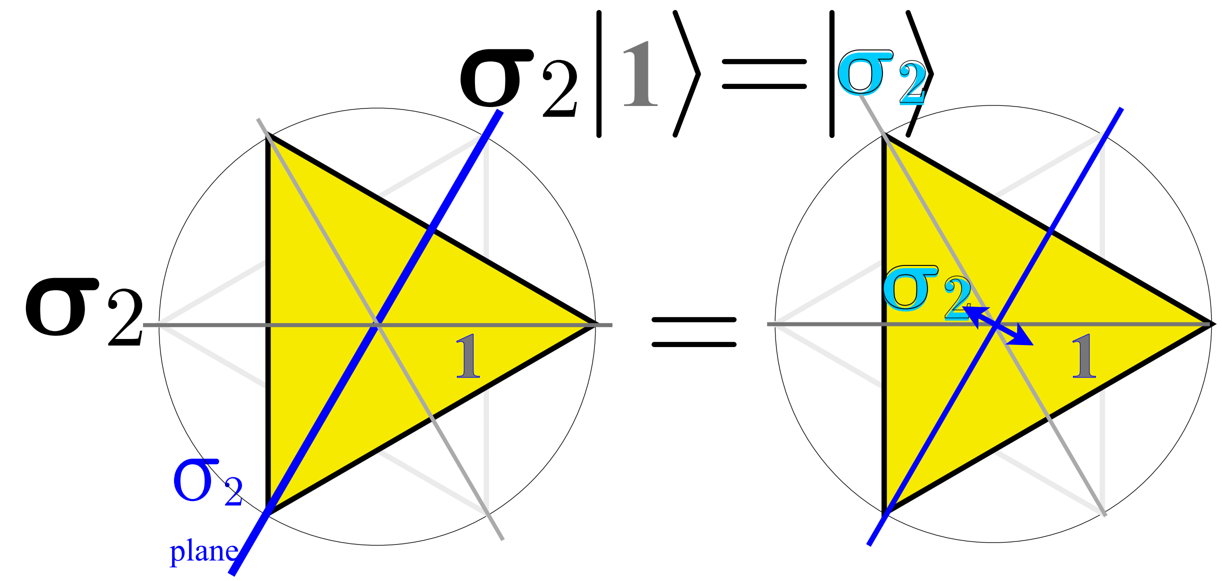




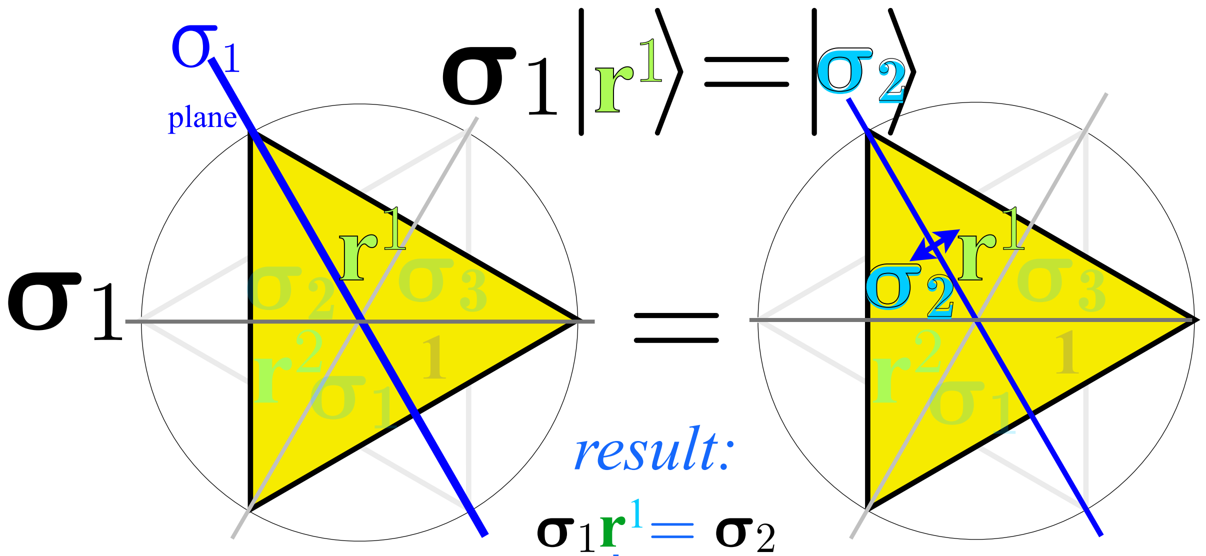
Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$





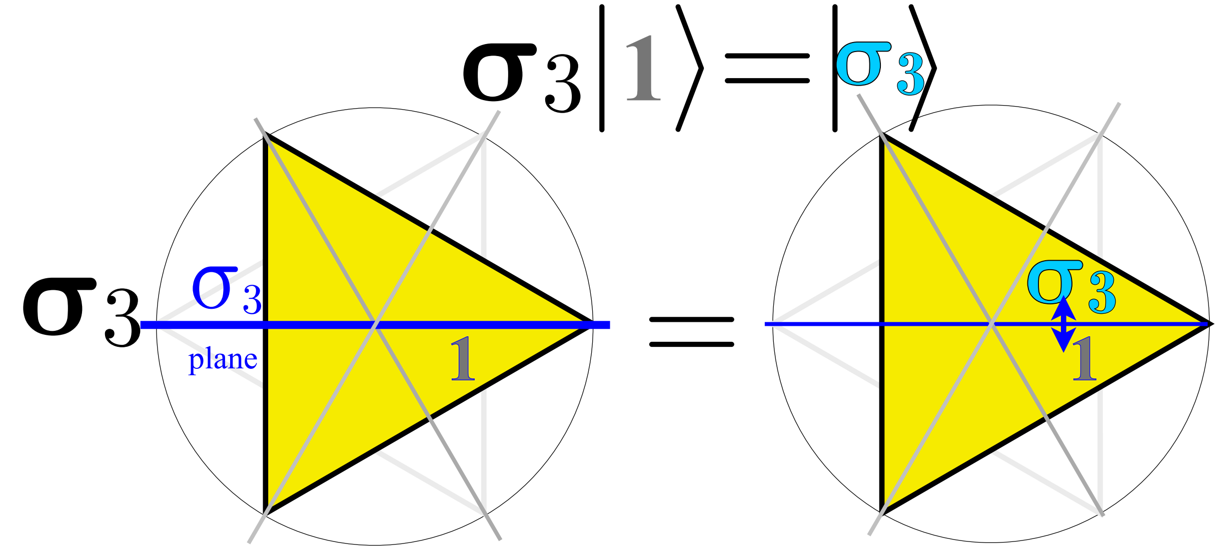


Example: Find C_{3v} product $\boldsymbol{\sigma}_1\mathbf{r}^1|1\rangle = \boldsymbol{\sigma}_1|\mathbf{r}^1\rangle$



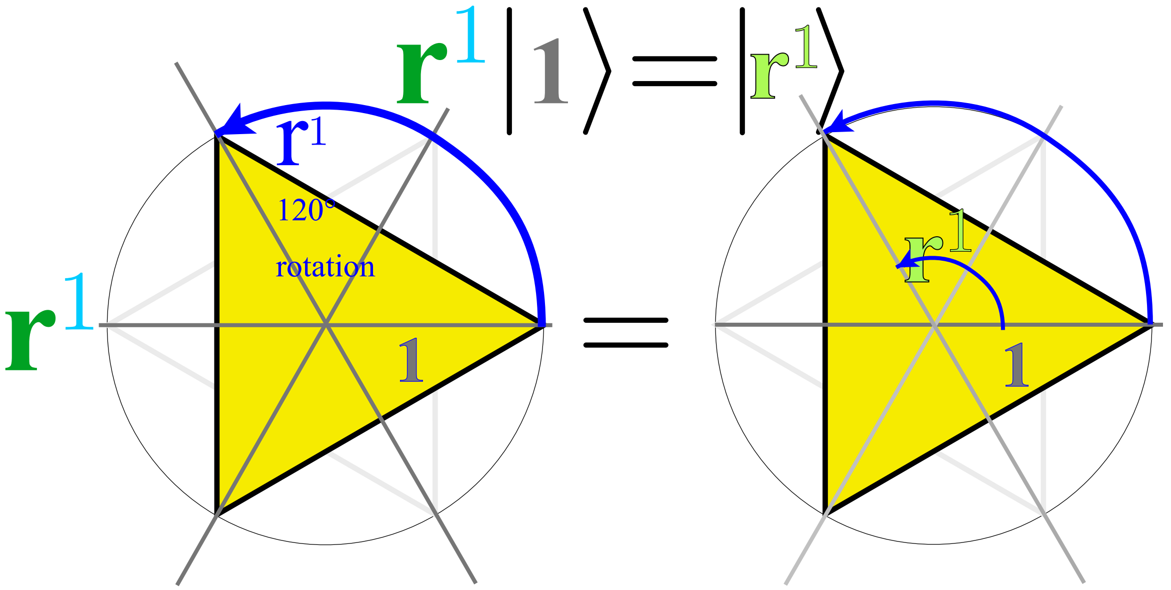
result:
$\boldsymbol{\sigma}_1\mathbf{r}^1 = \boldsymbol{\sigma}_2$

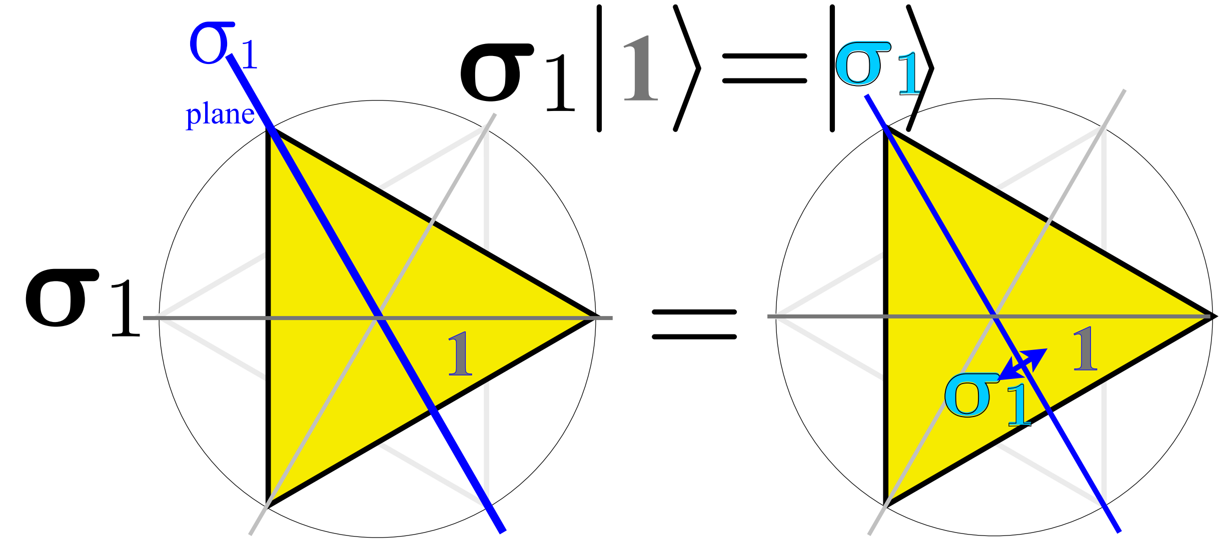
Factor $\mathbf{r}^1$
on right
acts first

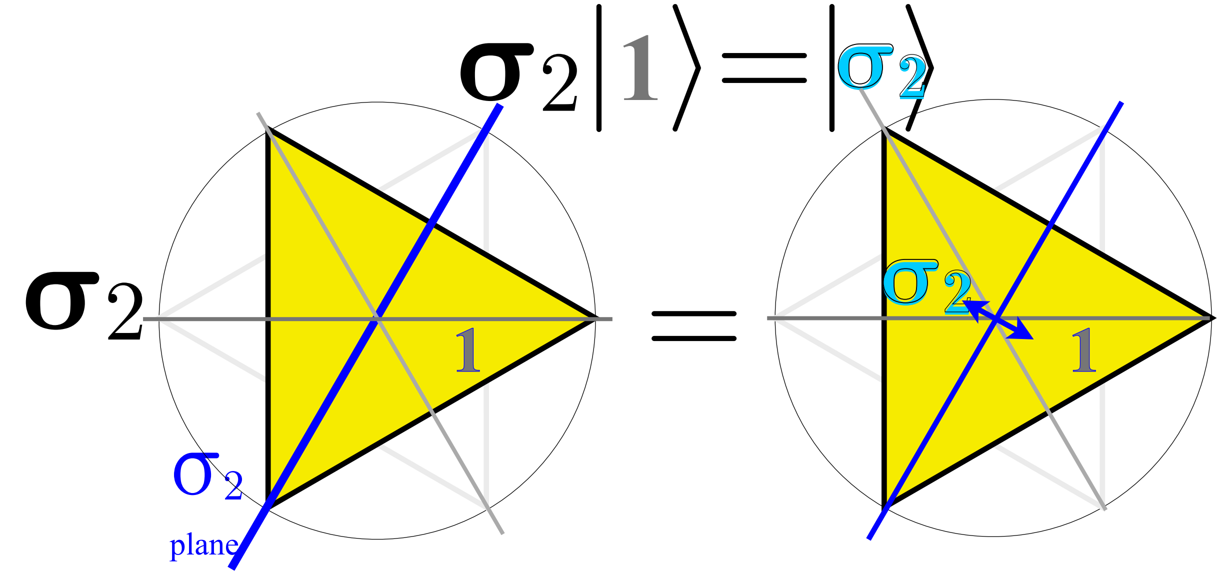


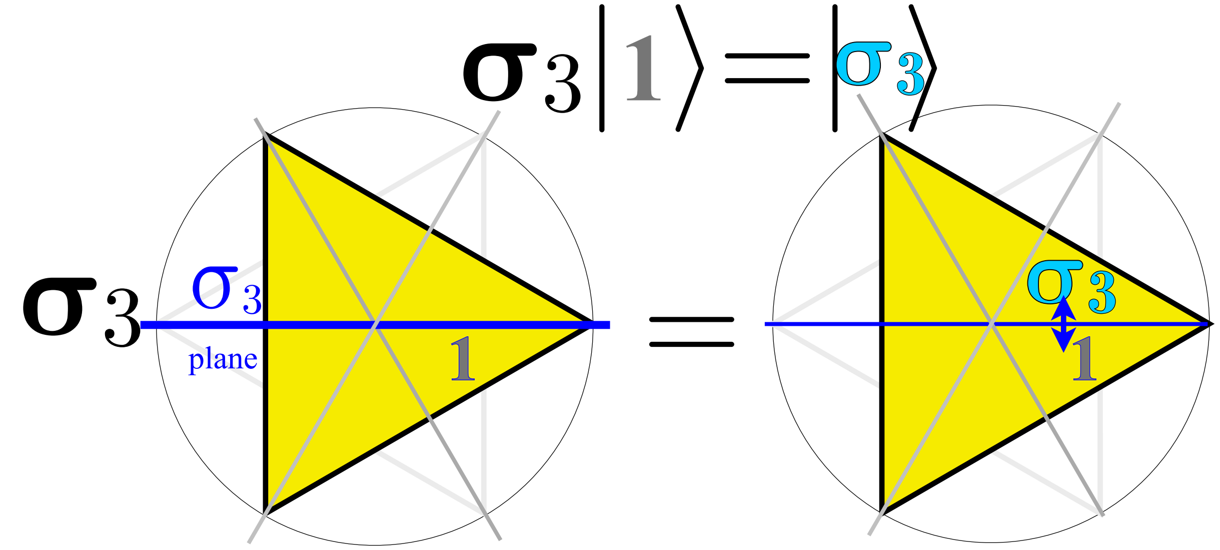
<u>left</u> is <u>last</u>
(like Hebrew)

Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$

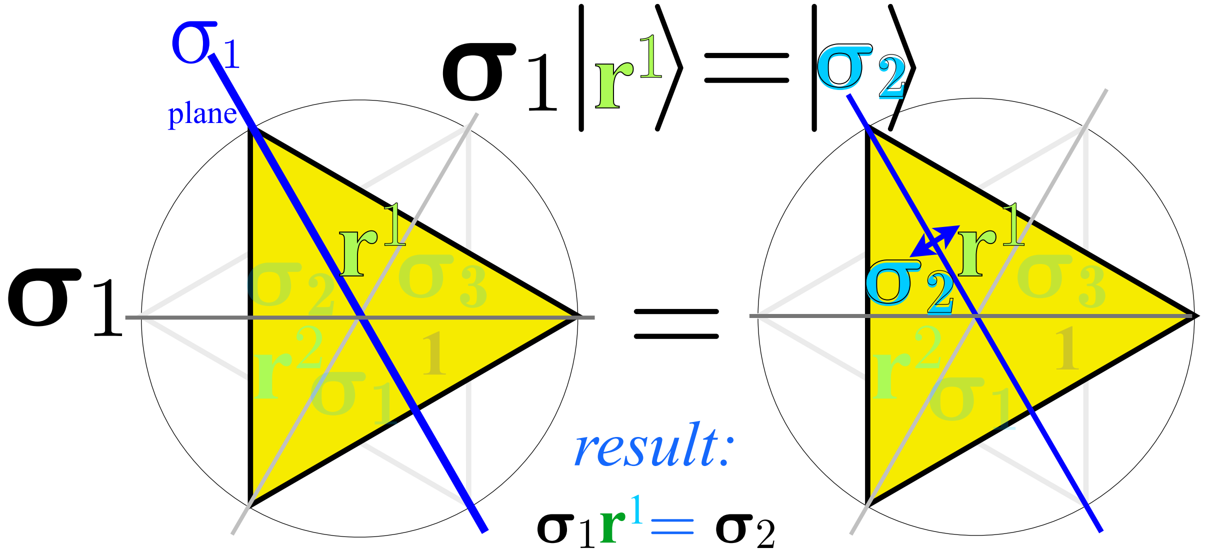








Example: Find C_{3v} product $\boldsymbol{\sigma}_1\mathbf{r}^1|1\rangle = \boldsymbol{\sigma}_1|\mathbf{r}^1\rangle$



result:

$\boldsymbol{\sigma}_1\mathbf{r}^1 = \boldsymbol{\sigma}_2$

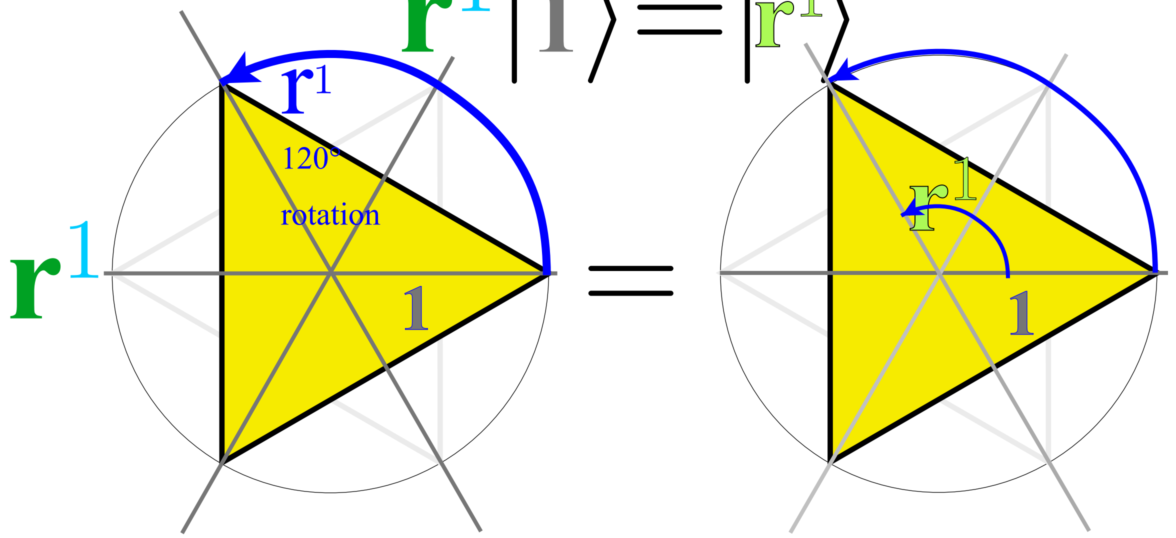
Other $\boldsymbol{\sigma}_1$ *results from graph:*

$\boldsymbol{\sigma}_1\{\mathbf{1}, \mathbf{r}^1, \mathbf{r}^2, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$

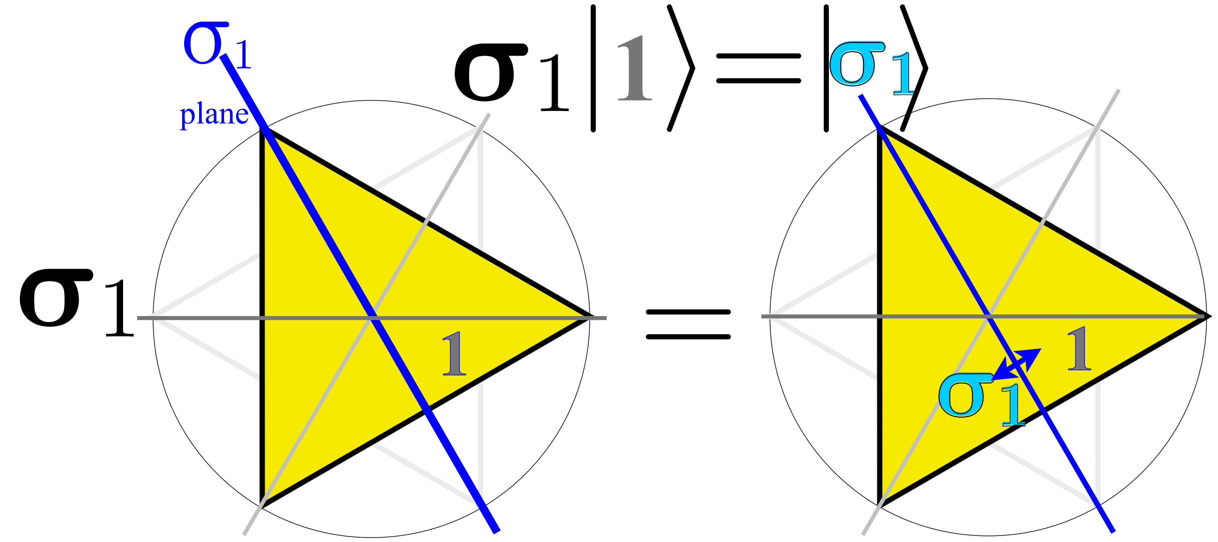
$= \{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3, \mathbf{1}, \mathbf{r}^1, \mathbf{r}^2\}$

Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$

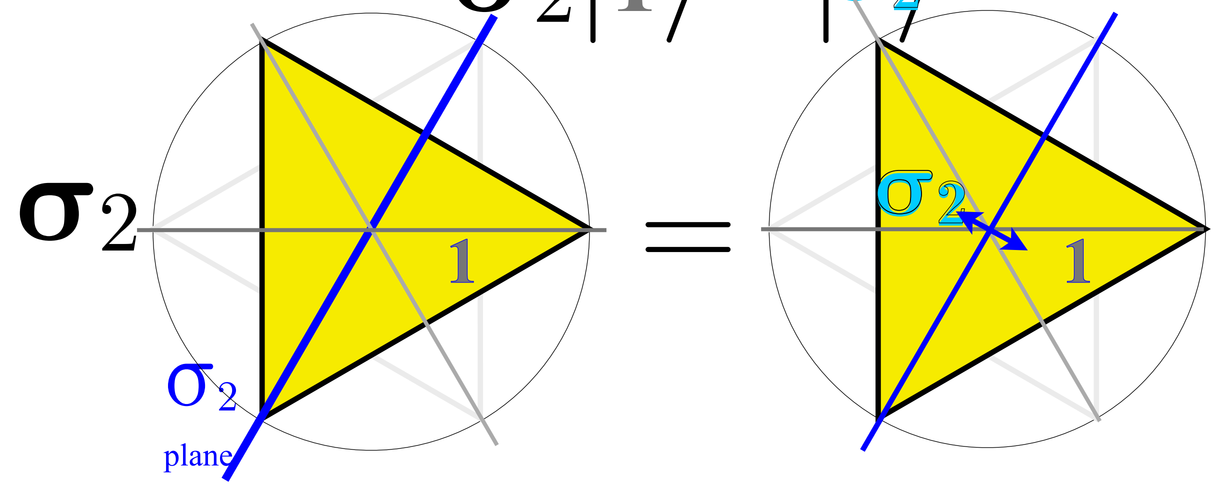
$$\mathbf{r}^1|1\rangle = |\mathbf{r}^1\rangle$$



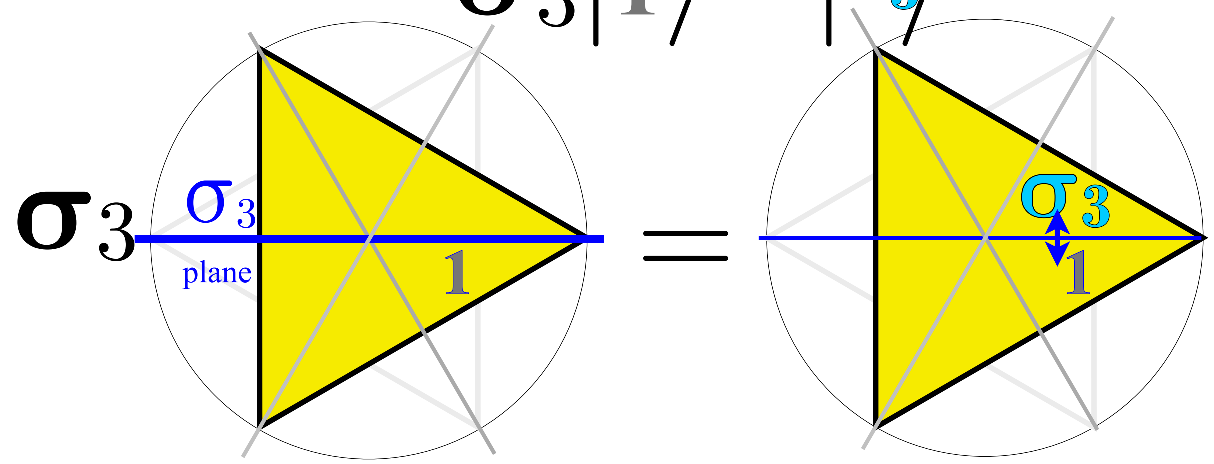
$$\boldsymbol{\sigma}_1|1\rangle = |\boldsymbol{\sigma}_1\rangle$$



$$\boldsymbol{\sigma}_2|1\rangle = |\boldsymbol{\sigma}_2\rangle$$

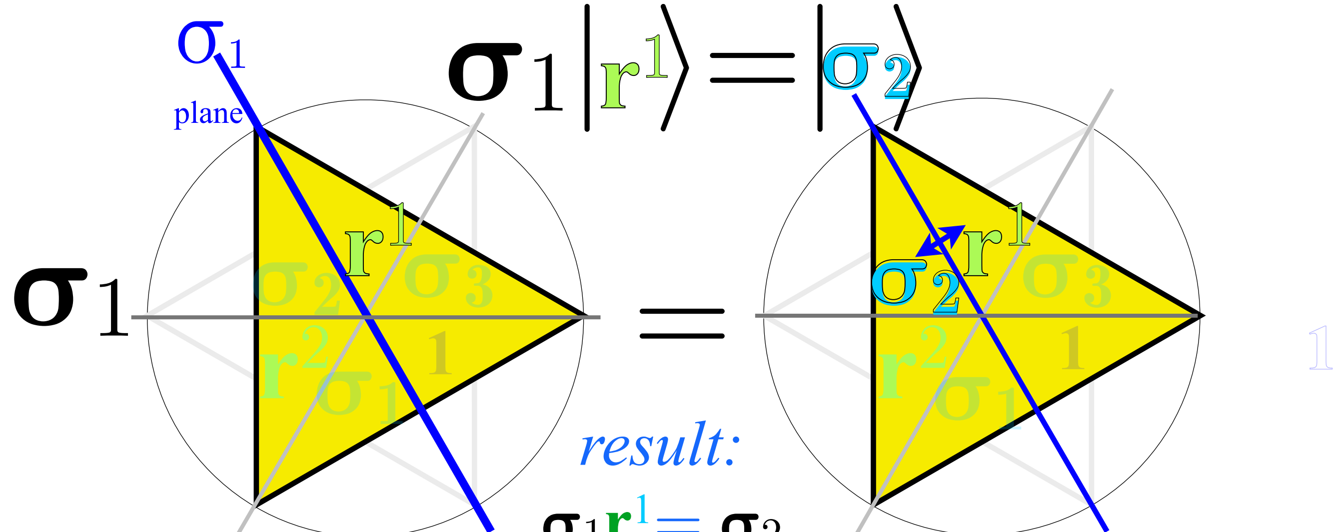


$$\boldsymbol{\sigma}_3|1\rangle = |\boldsymbol{\sigma}_3\rangle$$



Example: Find C_{3v} product $\boldsymbol{\sigma}_1\mathbf{r}^1|1\rangle = \boldsymbol{\sigma}_1|\mathbf{r}^1\rangle$

$$\boldsymbol{\sigma}_1|\mathbf{r}^1\rangle = |\boldsymbol{\sigma}_2\rangle$$



result:

$\boldsymbol{\sigma}_1\mathbf{r}^1 = \boldsymbol{\sigma}_2$

Other $\boldsymbol{\sigma}_1$ *results from graph:*

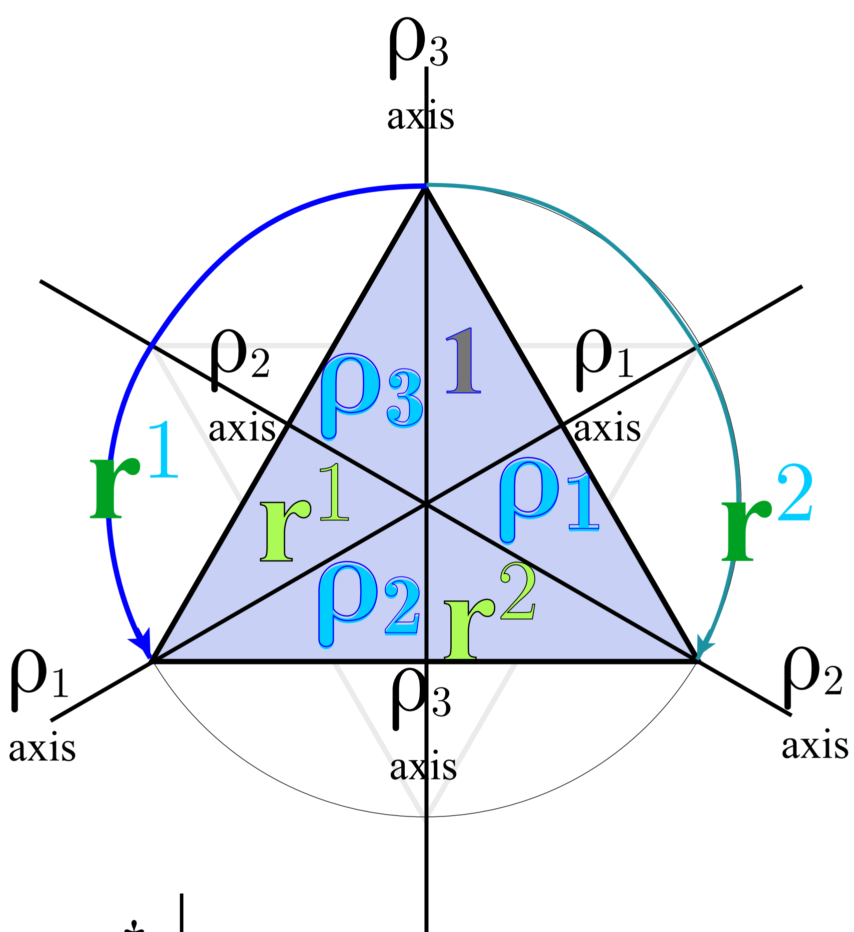
$\boldsymbol{\sigma}_1\{\mathbf{1}, \mathbf{r}^1, \mathbf{r}^2, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$

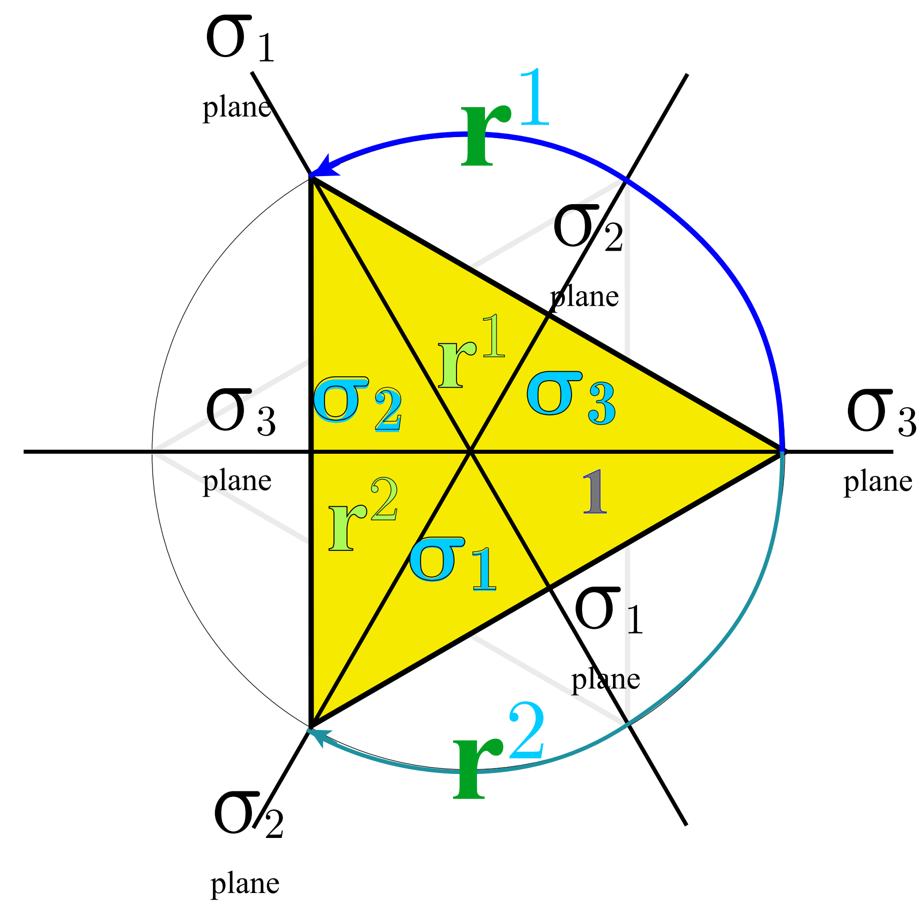
$= \{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3, \mathbf{1}, \mathbf{r}^1, \mathbf{r}^2\}$

....whole C_{3v} group table:

C_{3v} $\mathbf{gg}^\dagger$ *form*	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	$\boldsymbol{\sigma}_1$	$\boldsymbol{\sigma}_2$	$\boldsymbol{\sigma}_3$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	$\boldsymbol{\sigma}_1$	$\boldsymbol{\sigma}_2$	$\boldsymbol{\sigma}_3$
$\mathbf{r}^1$	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$	$\boldsymbol{\sigma}_3$	$\boldsymbol{\sigma}_1$	$\boldsymbol{\sigma}_2$
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$	$\boldsymbol{\sigma}_2$	$\boldsymbol{\sigma}_3$	$\boldsymbol{\sigma}_1$
$\boldsymbol{\sigma}_1$	$\boldsymbol{\sigma}_1$	$\boldsymbol{\sigma}_3$	$\boldsymbol{\sigma}_2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\boldsymbol{\sigma}_2$	$\boldsymbol{\sigma}_2$	$\boldsymbol{\sigma}_1$	$\boldsymbol{\sigma}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\boldsymbol{\sigma}_3$	$\boldsymbol{\sigma}_3$	$\boldsymbol{\sigma}_2$	$\boldsymbol{\sigma}_1$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$





D_3 $\mathbf{gg}^\dagger$ *form*	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{r}^1$	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$	ρ_3	ρ_1	ρ_2
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$	ρ_2	ρ_3	ρ_1
ρ_1	ρ_1	ρ_3	ρ_2	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
ρ_2	ρ_2	ρ_1	ρ_3	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
σ_3	ρ_3	ρ_2	ρ_1	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

D_3 and C_{3v} clearly are isomorphic $D_3 \sim C_{3v}$ share group table

...except for notation $\rho_k \leftrightarrow \sigma_k$

C_{3v} $\mathbf{gg}^\dagger$ *form*	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	σ_1	σ_2	σ_3
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	σ_1	σ_2	σ_3
$\mathbf{r}^1$	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$	σ_3	σ_1	σ_2
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$	σ_2	σ_3	σ_1
σ_1	σ_1	σ_3	σ_2	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
σ_2	σ_2	σ_1	σ_3	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
σ_3	σ_3	σ_2	σ_1	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

Fig. 3.1.1 PSDS

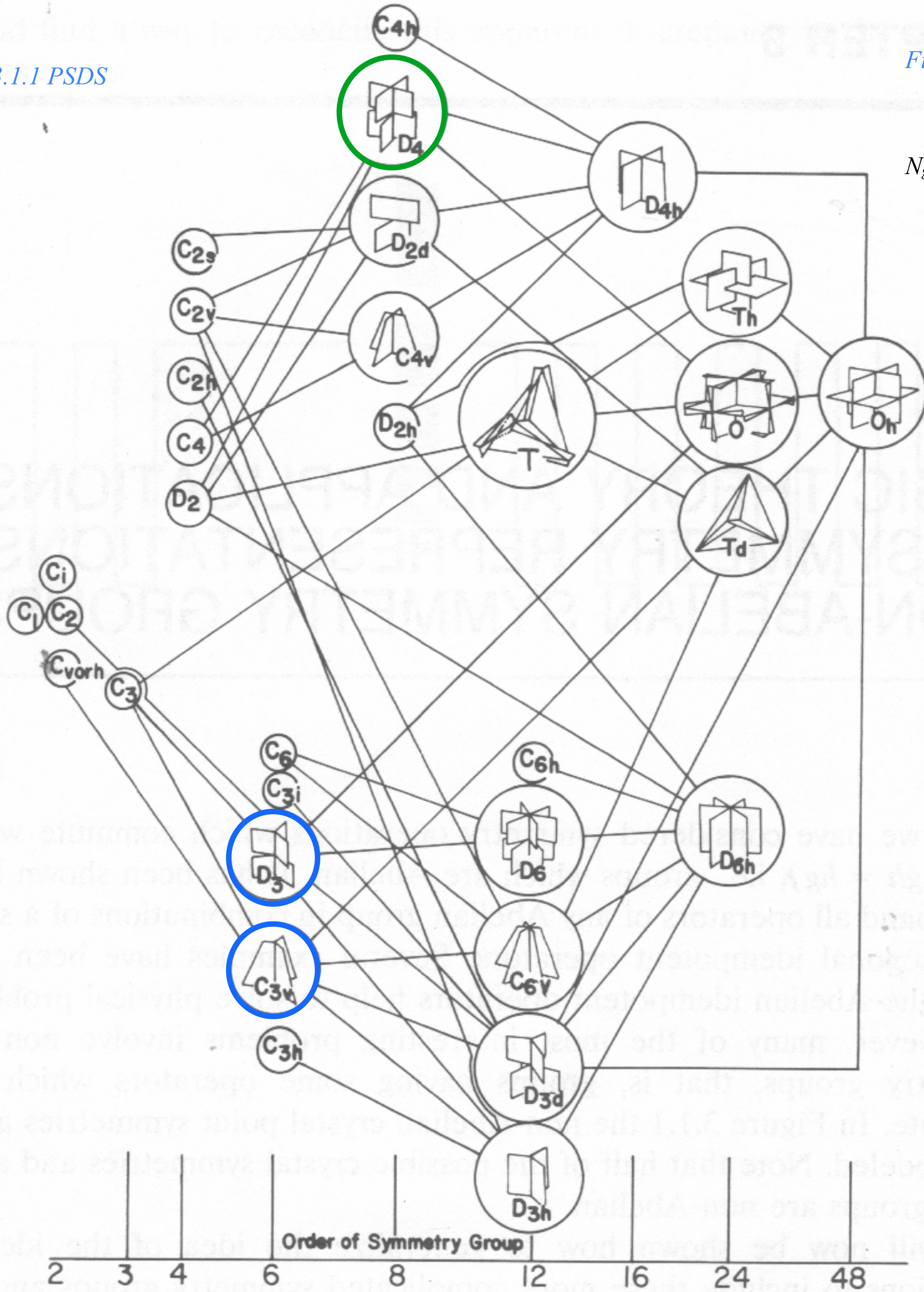


Figure 3.1.1 Crystal point symmetry groups. Models are sketched in circles for the 16 non-Abelian groups. (See also Figure 2.11.1.)

Fig. 2.2.2 PSDS

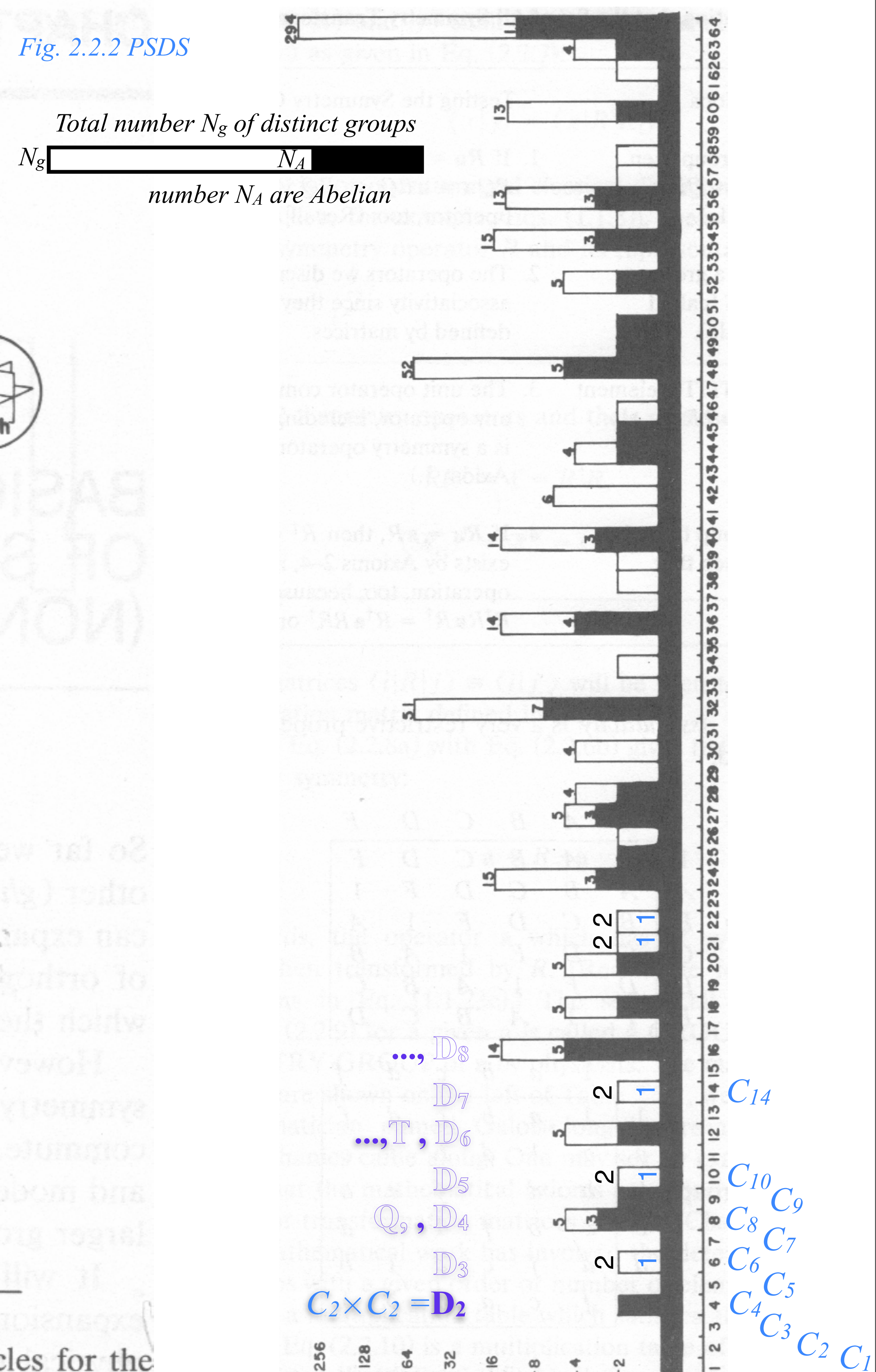
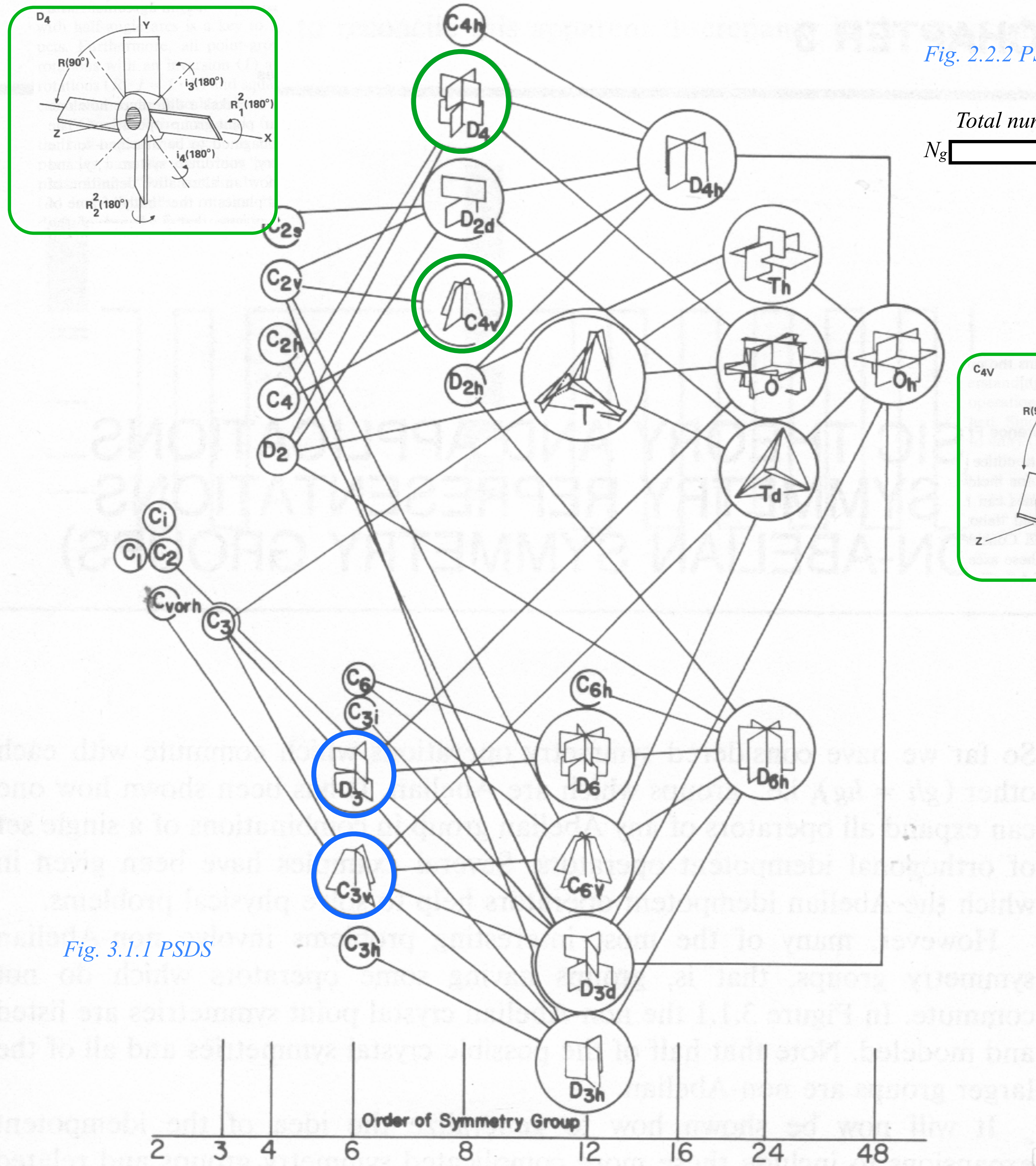


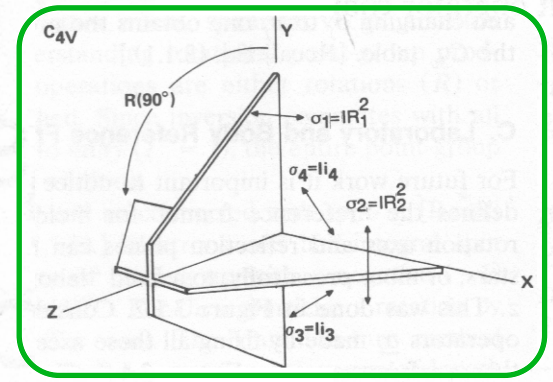
Fig. 2.2.2 PSDS

Total number N_g of distinct groups

N_g N_A

number N_A are Abelian





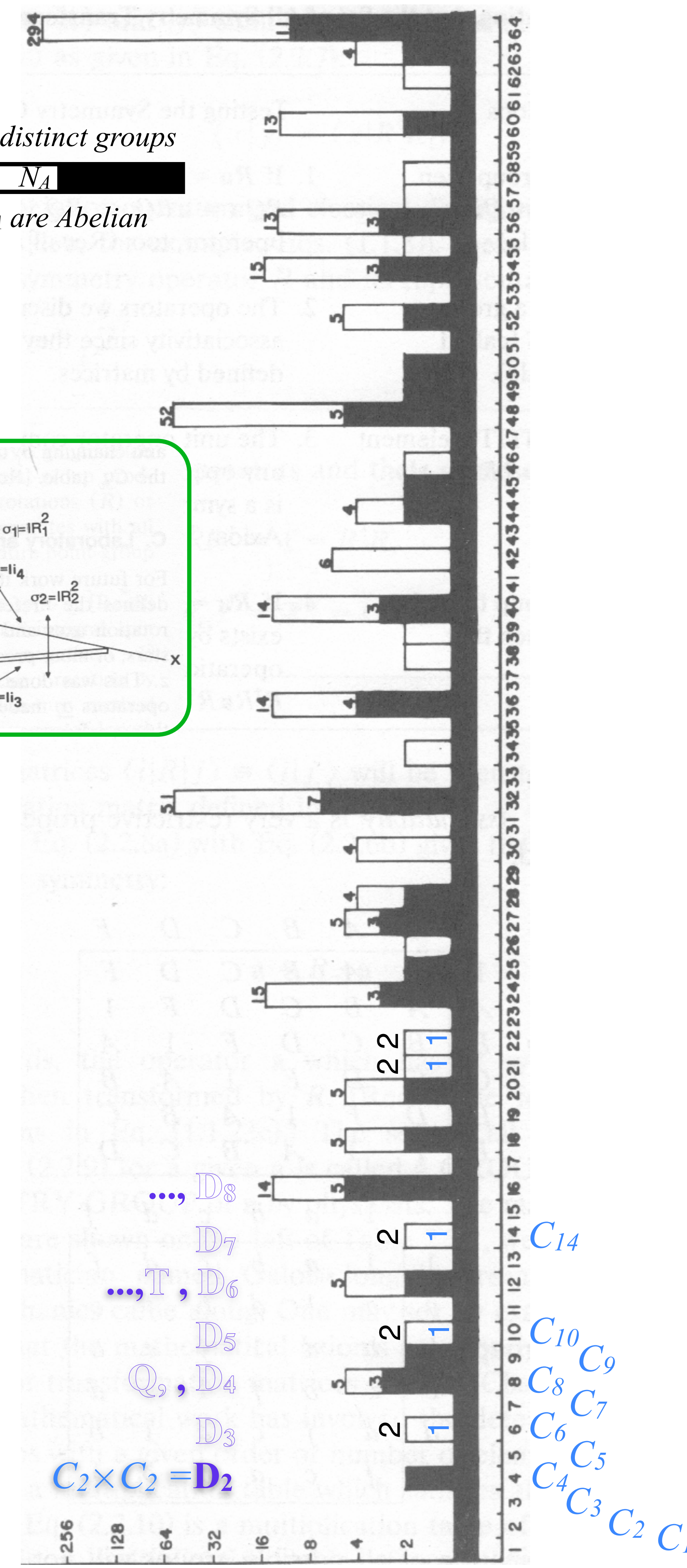


Fig. 3.1.1 PSDS

Figure 3.1.1 Crystal point symmetry groups. Models are sketched in circles for the 16 non-Abelian groups. (See also Figure 2.11.1.)

3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

D_3 and C_{3v} are isomorphic ($D_3 \sim C_{3v}$ share product table)

Deriving $D_3 \sim C_{3v}$ products:

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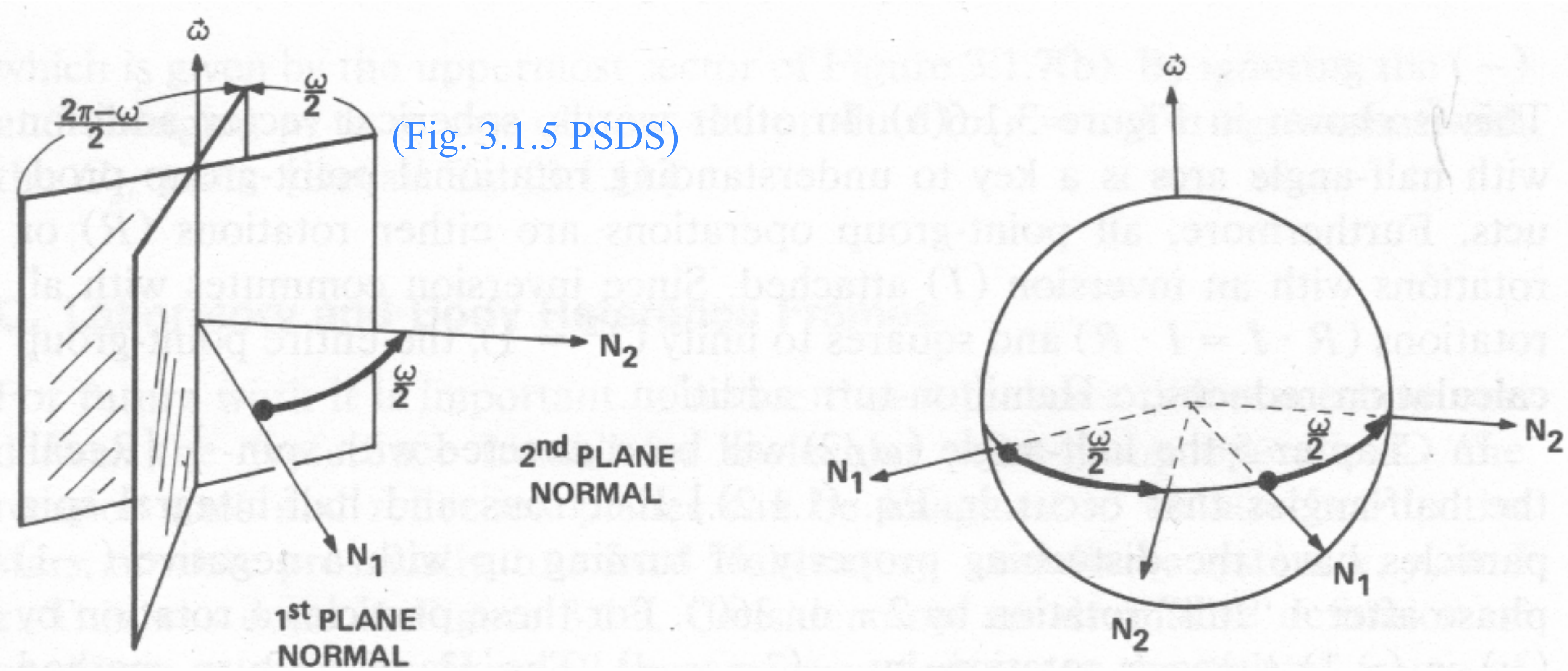
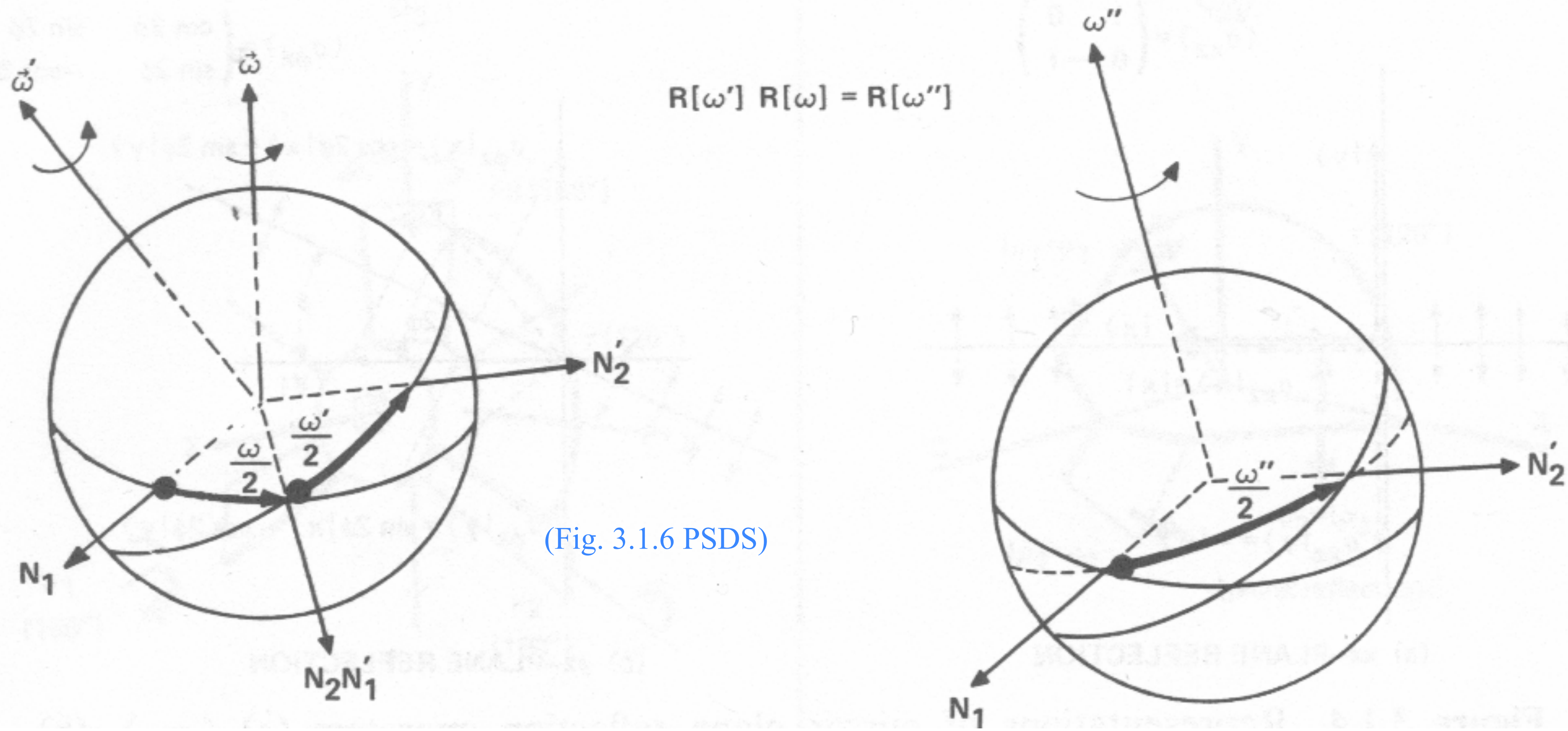
Deriving $D_3 \sim C_{3v}$ products by nomograms based on U(2) Hamilton-turns

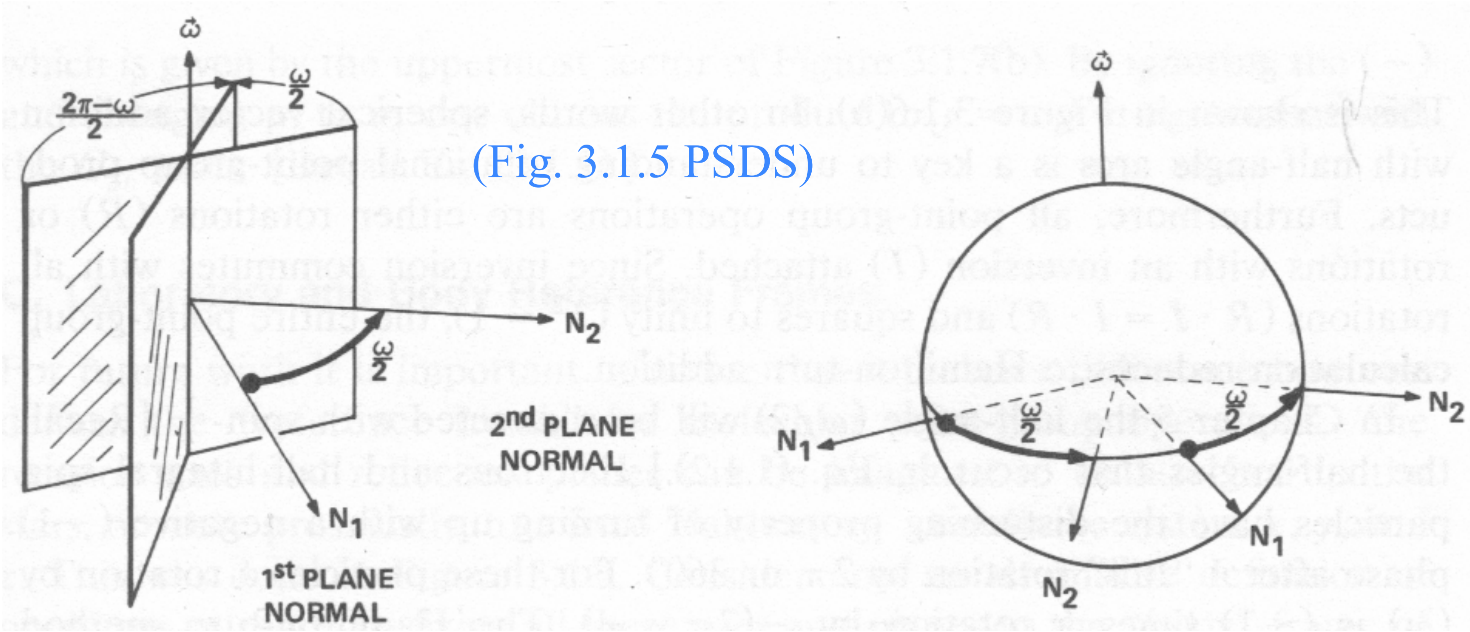


(Fig. 3.1.5 PSDS)

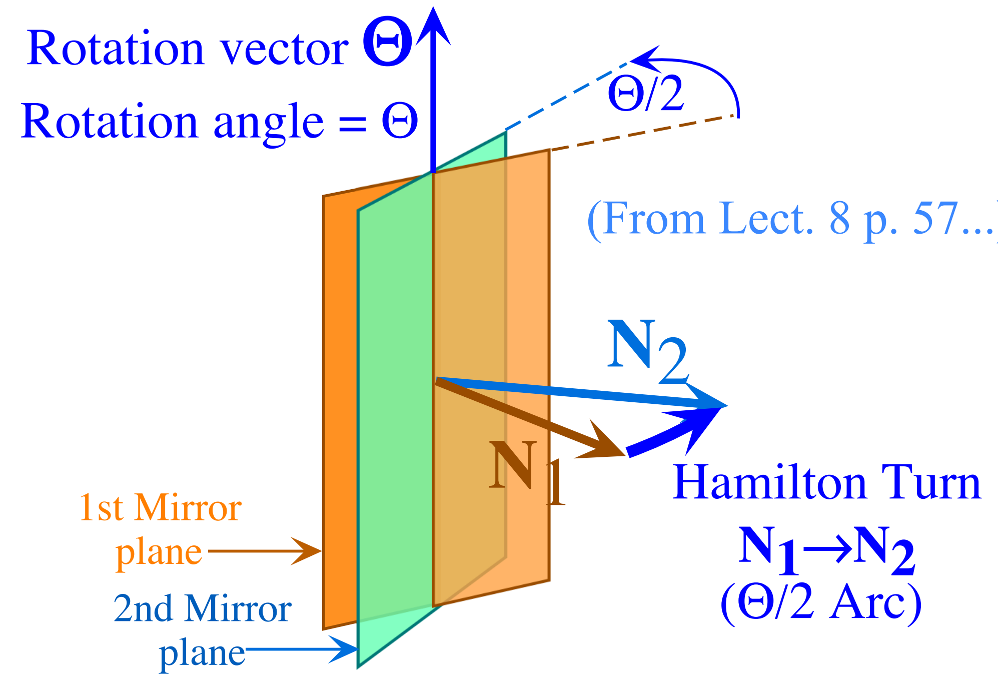


(Fig. 3.1.6 PSDS)

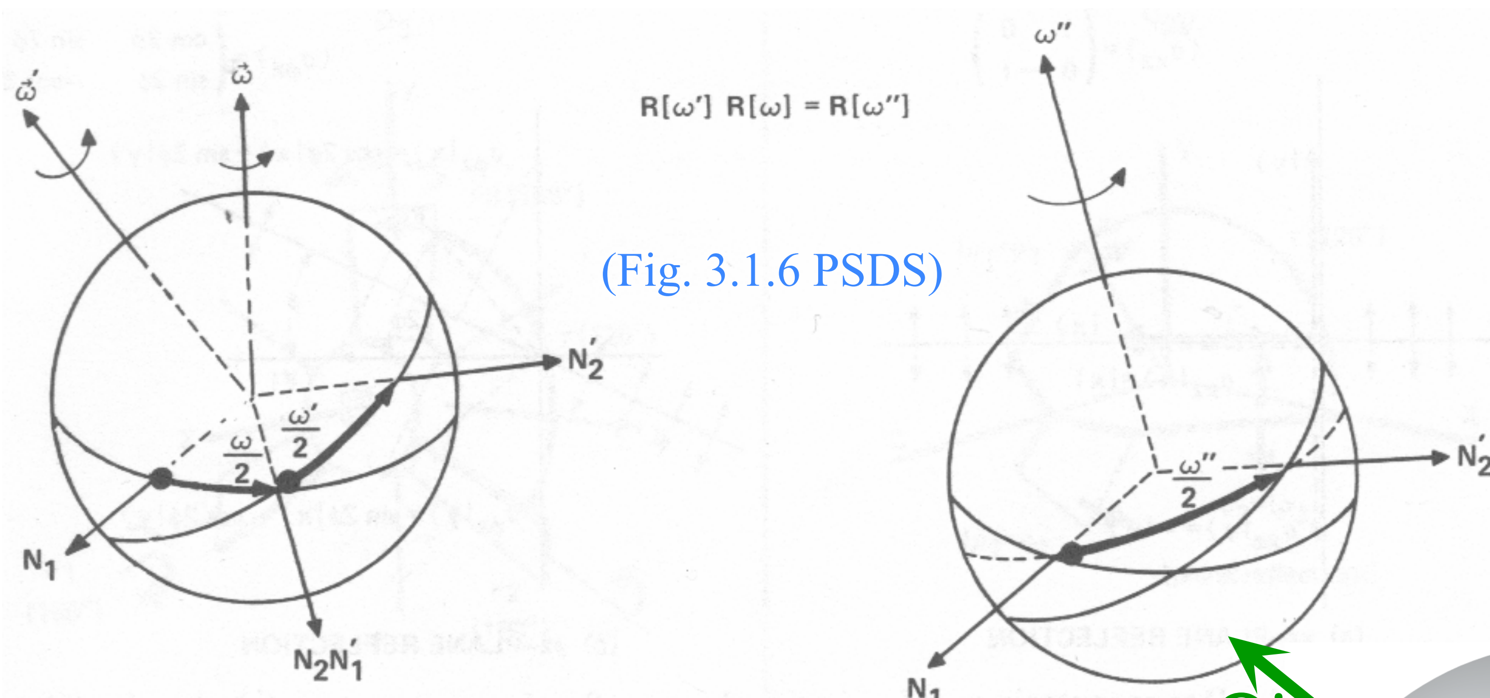
Deriving $D_3 \sim C_{3v}$ products by nomograms based on U(2) Hamilton-turns



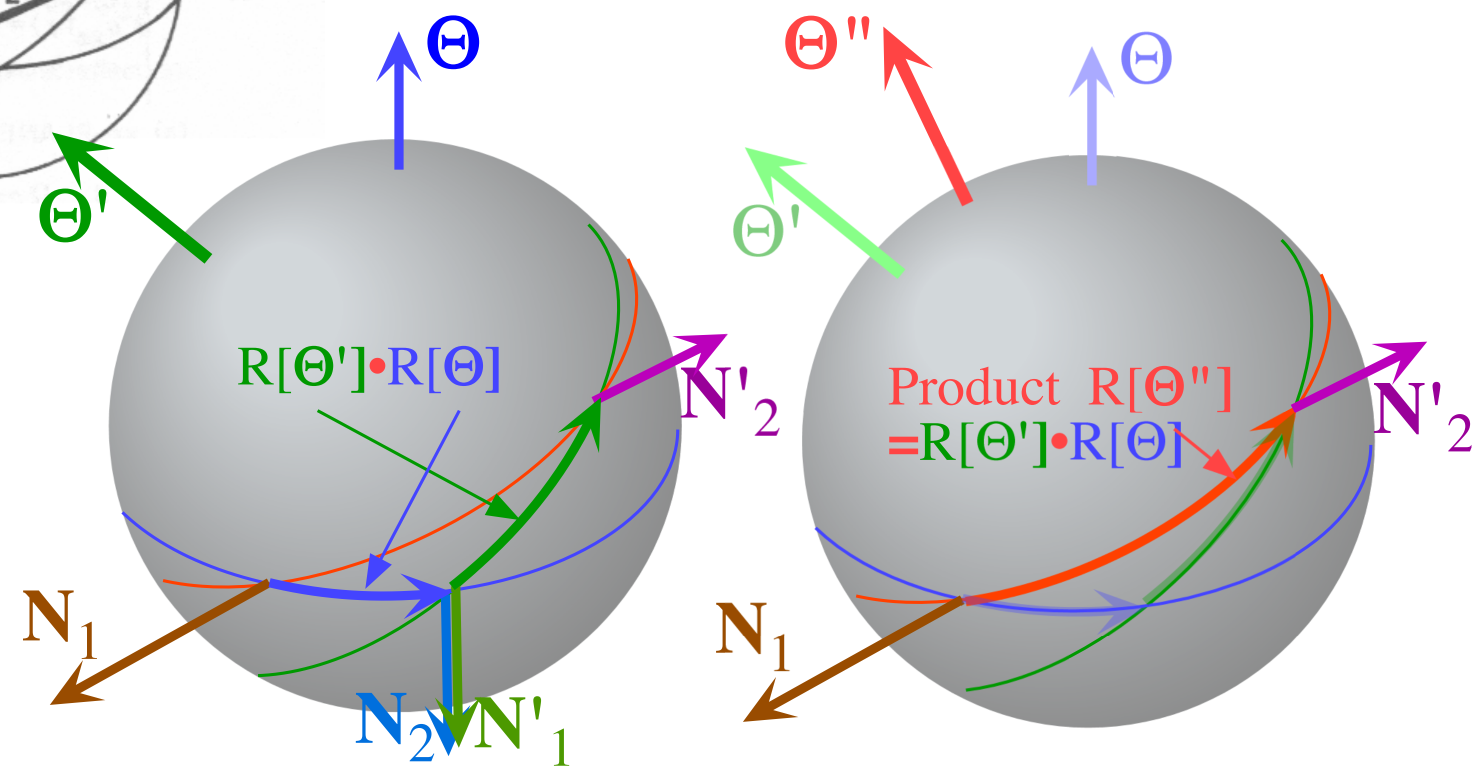
(Fig. 3.1.5 PSDS)



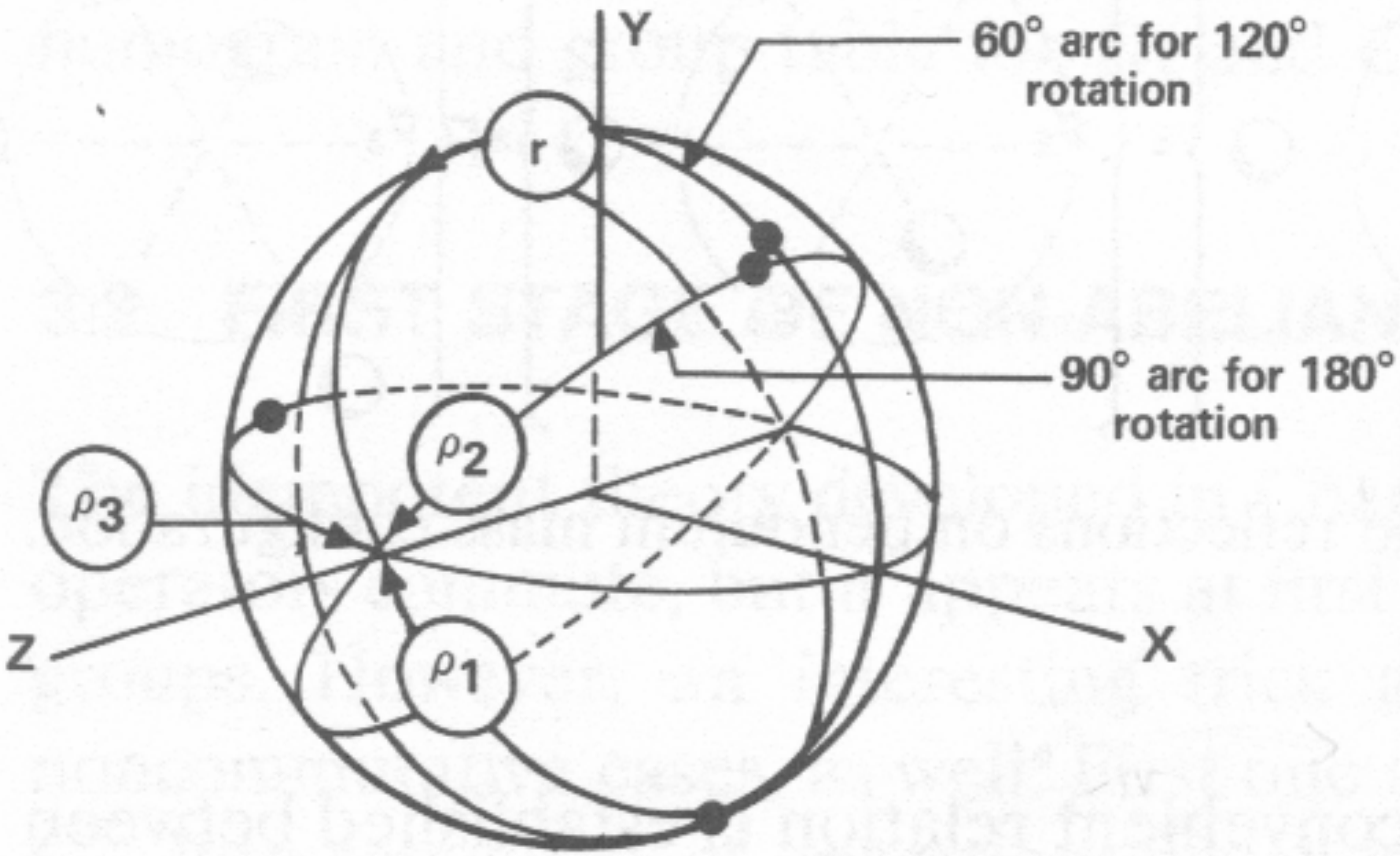
(Fig. 10.A.7 QTCA)



(Fig. 3.1.6 PSDS)



(Fig. 10.A.8 QTCA)



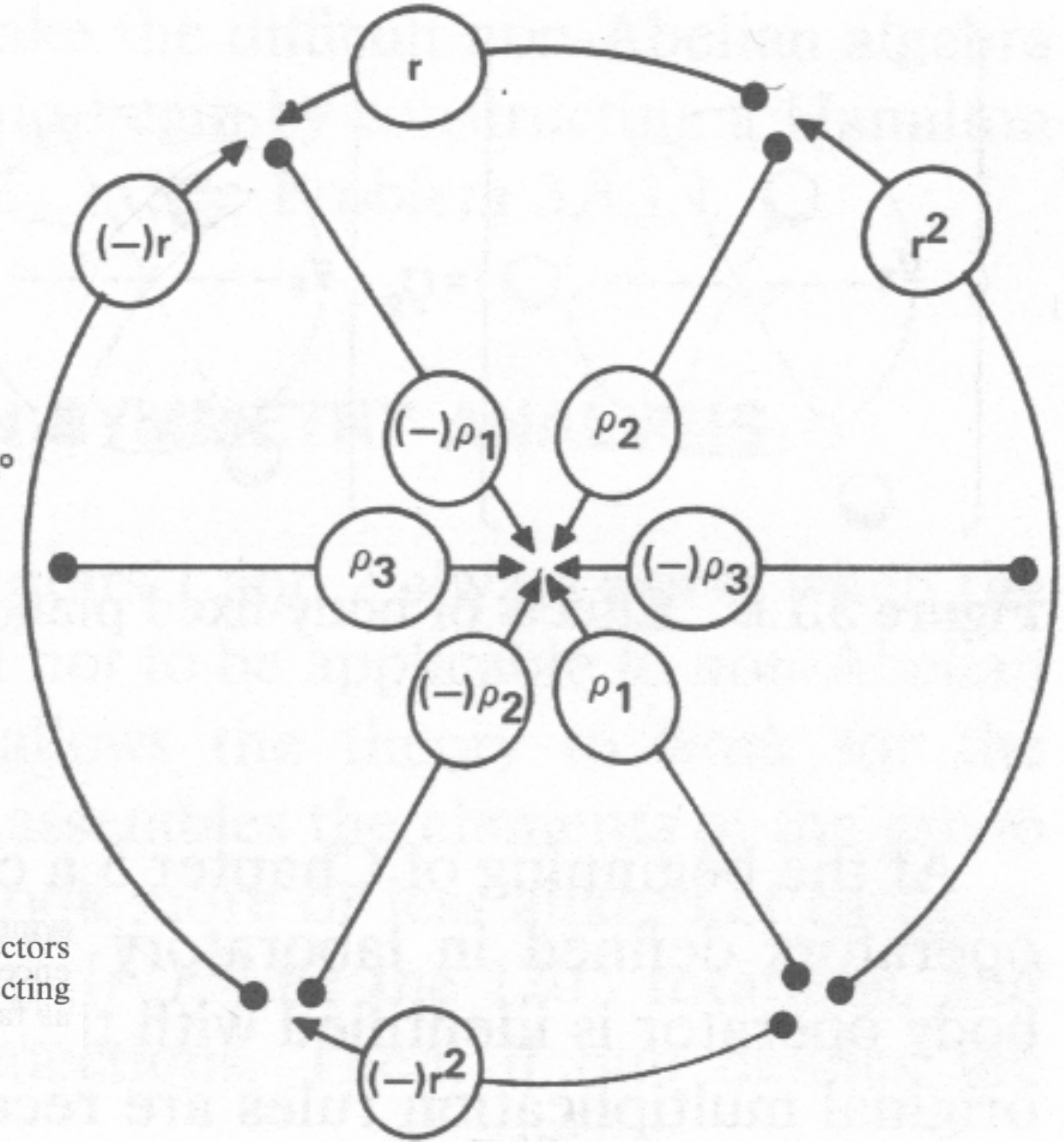


Figure 3.1.7 Geometrical definition of symmetry group D_3. (a) Hamilton arc vectors are drawn for rotations r, i_1, and i_3. (b) Group nomogram is obtained by projecting (a) onto the xy plane.

1	h^2	h^4	ρ_1	ρ_2	ρ_3
h^4	-1	$-h^2$	$-\rho_2$	$-\rho_3$	ρ_1
h^2	h^4	-1	$-\rho_3$	ρ_1	ρ_2
ρ_1	ρ_2	ρ_3	-1	$-h^2$	$-h^4$
ρ_2	ρ_3	$-\rho_1$	h^4	-1	$-h^2$
ρ_3	$-\rho_1$	$-\rho_2$	h^2	h^4	-1

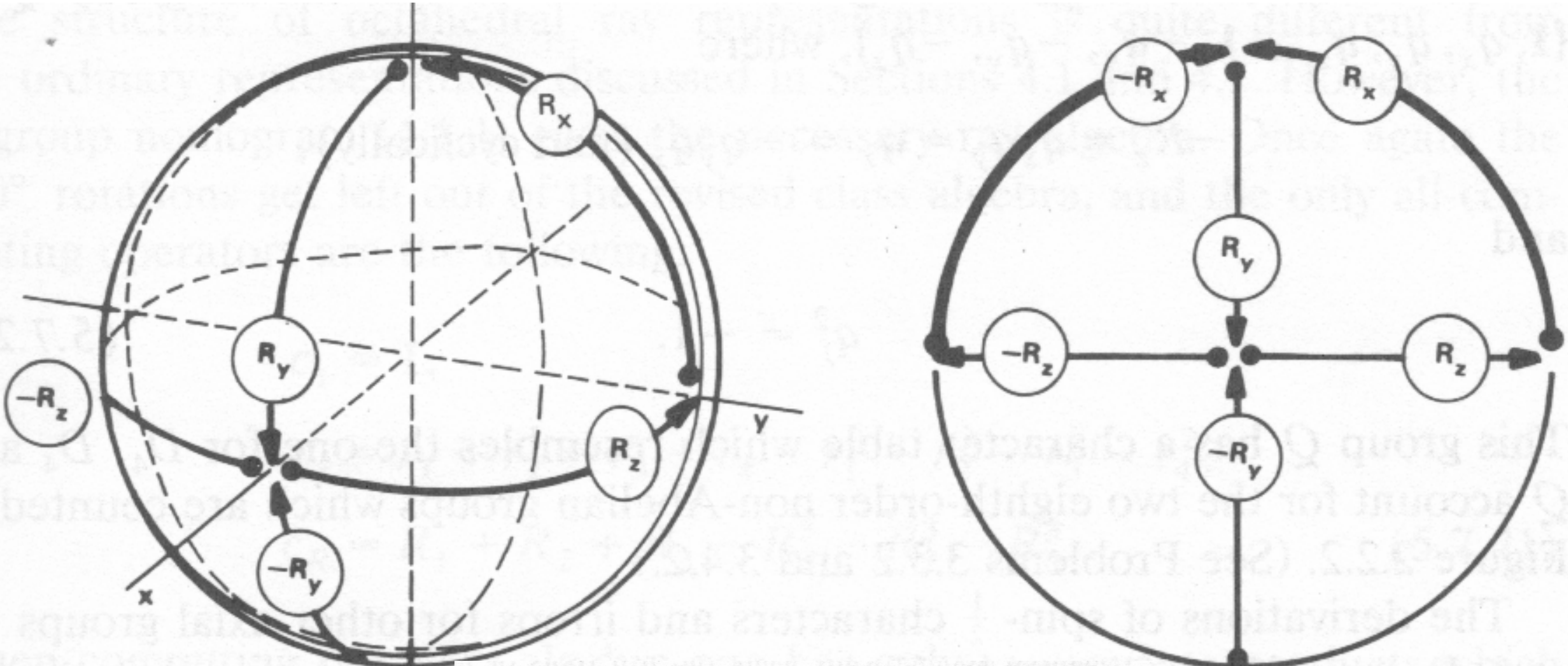
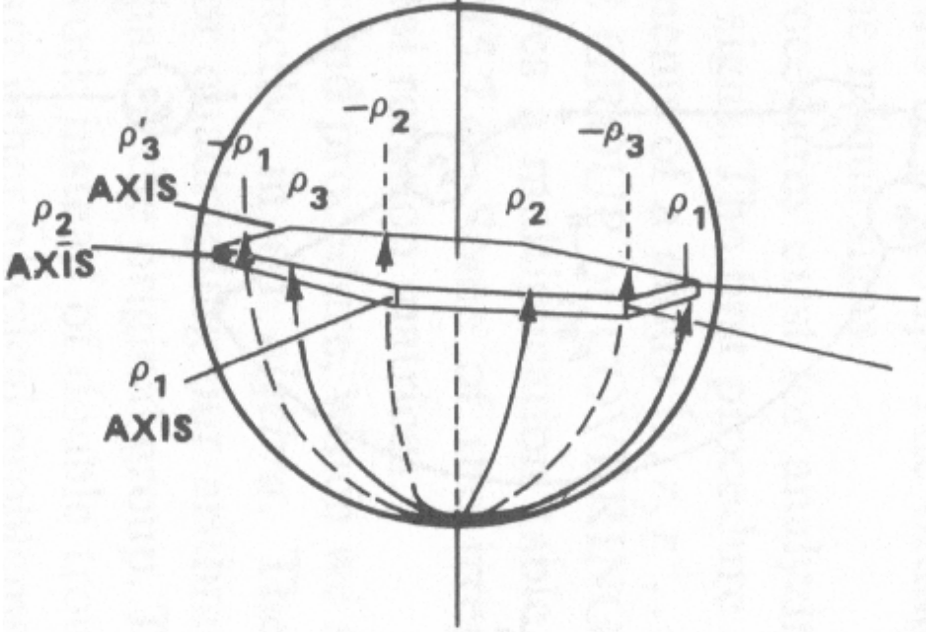


Figure 5.7.4 Hamilton arcs and vector nomogram for D_2

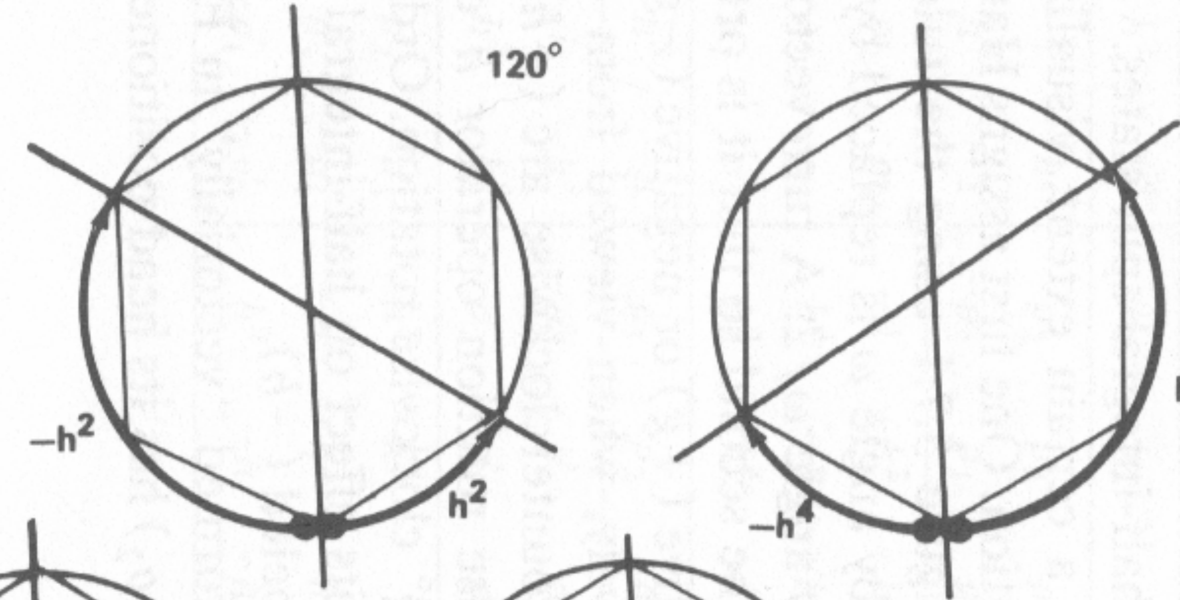
$\mathbf{1}$	R_x	R_y	R_z
R_x	$-\mathbf{1}$	R_z	$-R_y$
R_y	$-R_z$	$-\mathbf{1}$	R_x
R_z	R_y	$-R_x$	$-\mathbf{1}$

$$\mathscr{D}^E(R_x) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \mathscr{D}^E(R_y) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathscr{D}^E(R_z) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

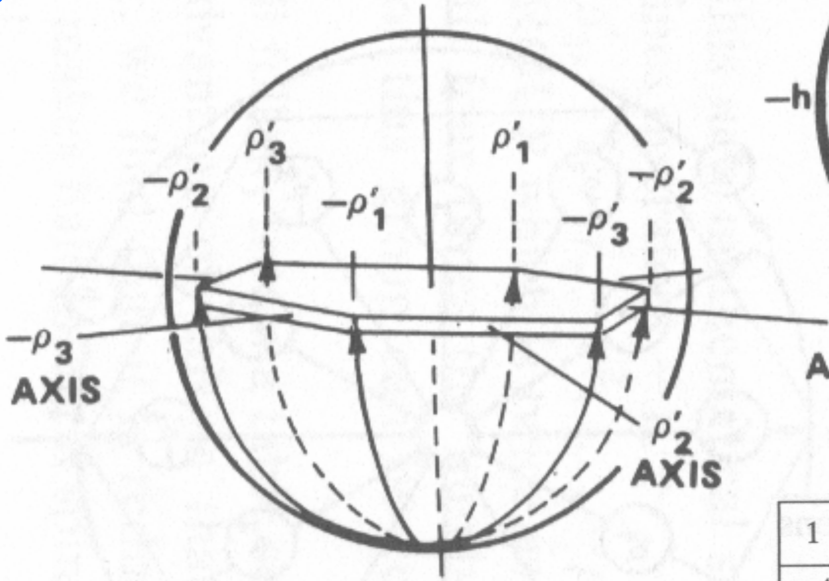
$D_3 \subset D_6$ Transverse 180° rotations

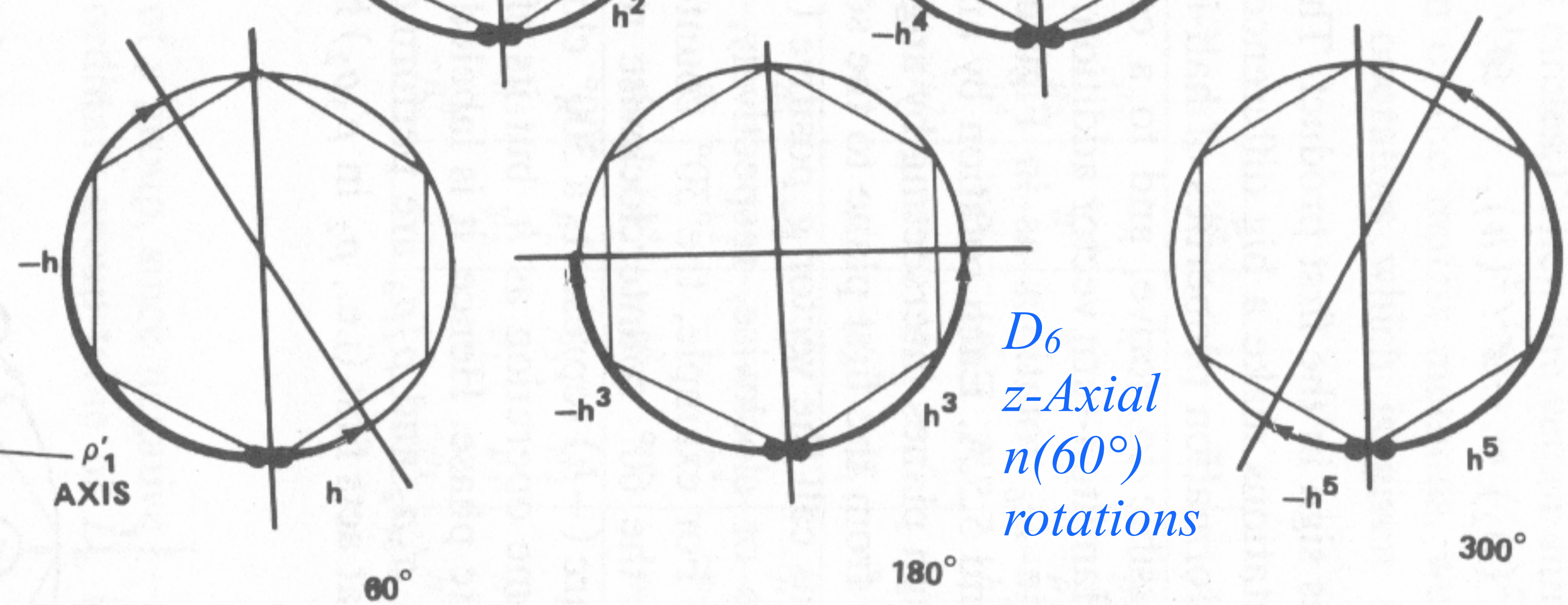


$D_3 \subset D_6$ z-Axial 120° rotations



D_6 Transverse 180° rotations





D_6 z-Axial $n(60°)$ rotations

1	h^2	h^4	ρ_1	ρ_2	ρ_3	h^3	h	h^5	ρ'_1	ρ'_2	ρ'_3
h^4	-1	$-h^2$	$-\rho_2$	$-\rho_3$	ρ_1	$-h$	h^5	$-h^3$	$-\rho'_2$	$-\rho'_3$	ρ'_1
h^2	h^4	-1	$-\rho_3$	ρ_1	ρ_2	h^5	h^3	$-h$	$-\rho'_3$	ρ'_1	ρ'_2
ρ_1	ρ_2	ρ_3	-1	$-h^2$	$-h^4$	$-\rho'_1$	ρ'_3	$-\rho'_2$	h^3	h^5	$-h$
ρ_2	ρ_3	$-\rho_1$	h^4	-1	$-h^2$	$-\rho'_2$	$-\rho'_1$	$-\rho'_3$	h	h^3	h^5
ρ_3	$-\rho_1$	$-\rho_2$	h^2	h^4	-1	$-\rho'_3$	$-\rho'_2$	ρ'_1	$-h^5$	h	h^3
h^3	h^5	$-h$	ρ'_1	ρ'_2	ρ'_3	-1	h^4	$-h^2$	$-\rho_1$	$-\rho_2$	$-\rho_3$
h^5	$-h$	$-h^3$	$-\rho'_3$	ρ'_1	ρ'_2	$-h^2$	-1	$-h^4$	ρ_3	$-\rho_1$	$-\rho_2$
h	h^3	h^5	ρ'_2	ρ'_3	$-\rho'_1$	h^4	h^2	-1	$-\rho_2$	$-\rho_3$	ρ_1
ρ'_1	ρ'_2	ρ'_3	$-h^3$	$-h^5$	h	ρ_1	$-\rho_3$	ρ_2	-1	$-h^2$	$-h^4$
ρ'_2	ρ'_3	$-\rho'_1$	$-h$	$-h^3$	$-h^5$	ρ_2	ρ_1	ρ_3	h^4	-1	$-h^2$
ρ'_3	$-\rho'_1$	$-\rho'_2$	h^5	$-h$	$-h^3$	ρ_3	ρ_2	$-\rho_1$	h^2	h^4	-1

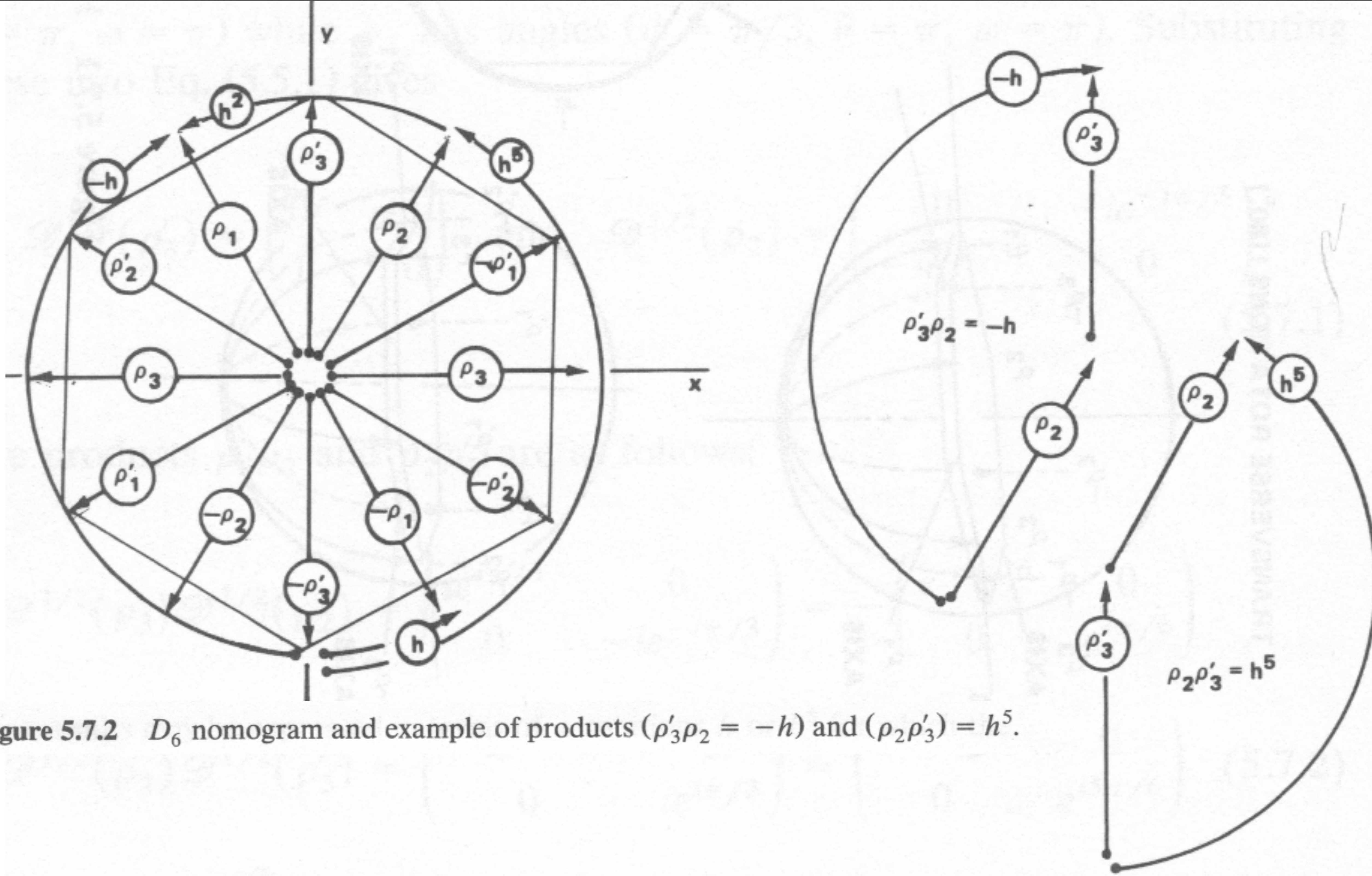


Figure 5.7.2 D_6 nomogram and example of products $(\rho'_3\rho_2 = -h)$ and $(\rho_2\rho'_3) = h^5$.

3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

D_3 and C_{3v} are isomorphic ($D_3 \sim C_{3v}$ share product table)

Deriving $D_3 \sim C_{3v}$ products:

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Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes

(Fig. 3.2.1 PSDS)

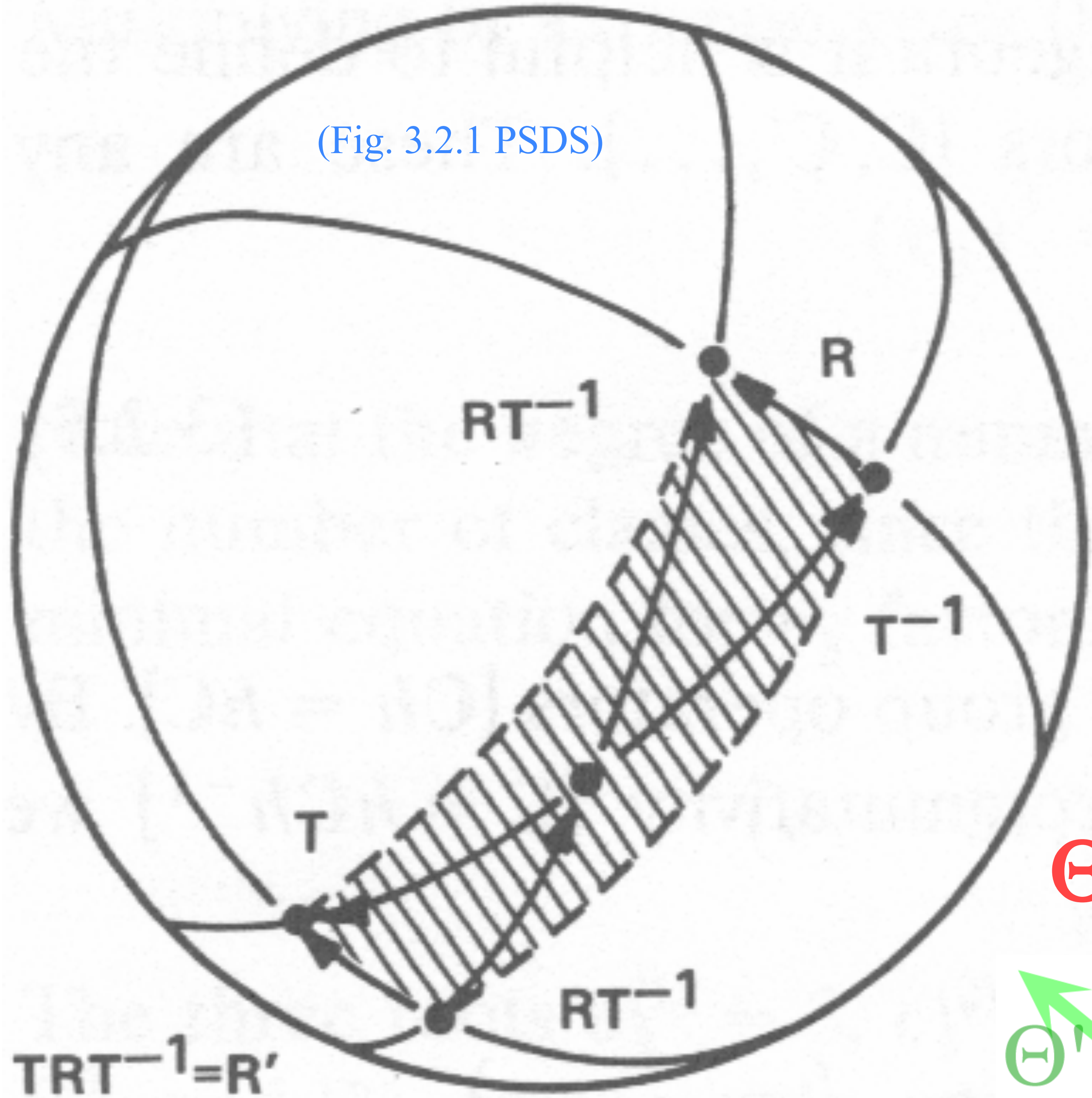
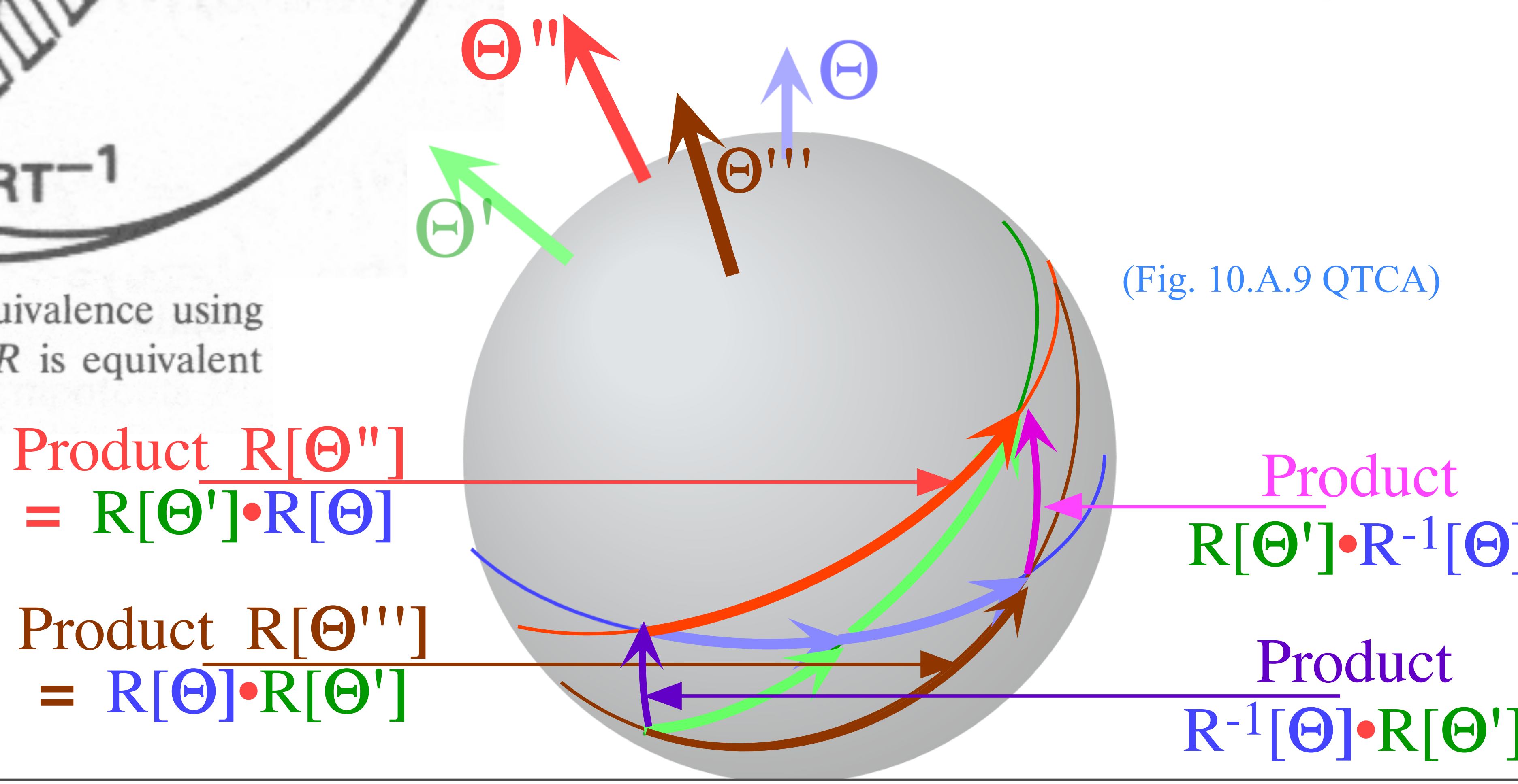


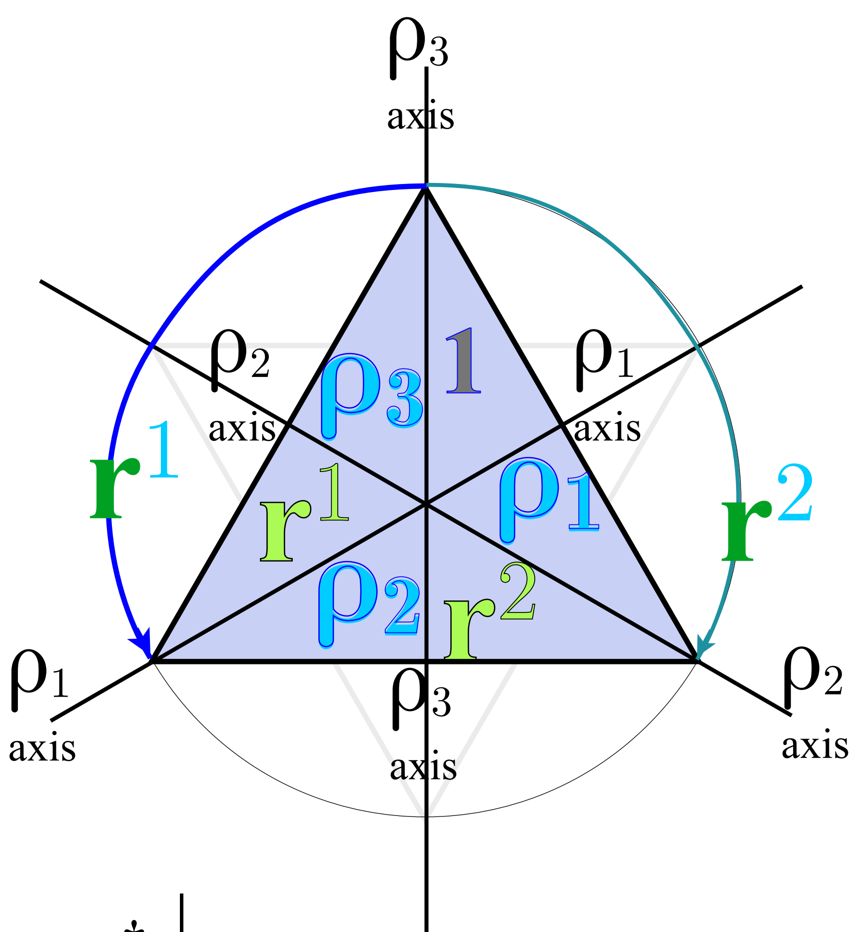
Figure 3.2.1 Showing class equivalence using Hamilton's vectors. Operation R is equivalent to $R' = TRT^{-1}$.

(From Lect. 8 p. 62...)



(Fig. 10.A.9 QTCA)

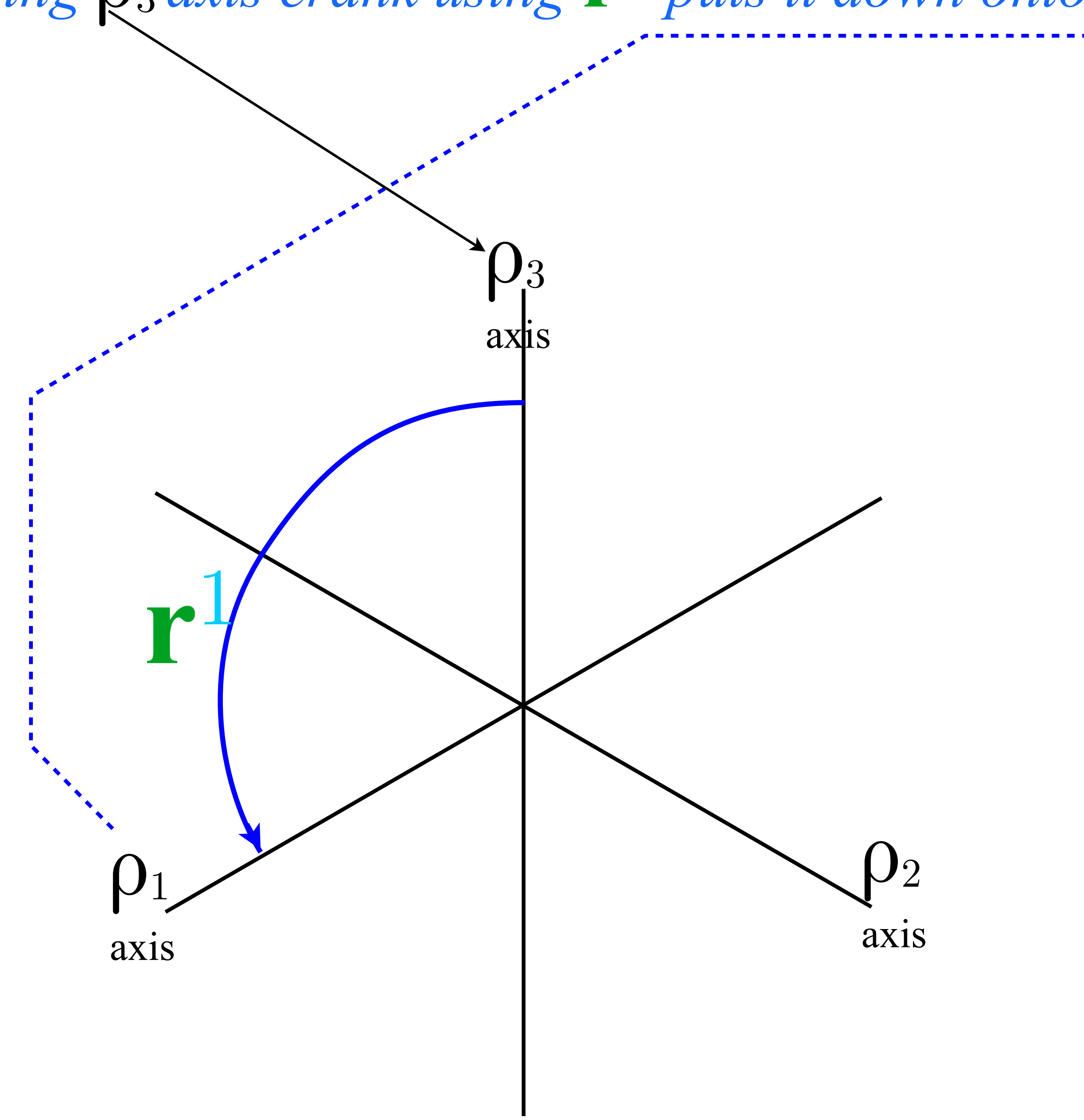
Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes



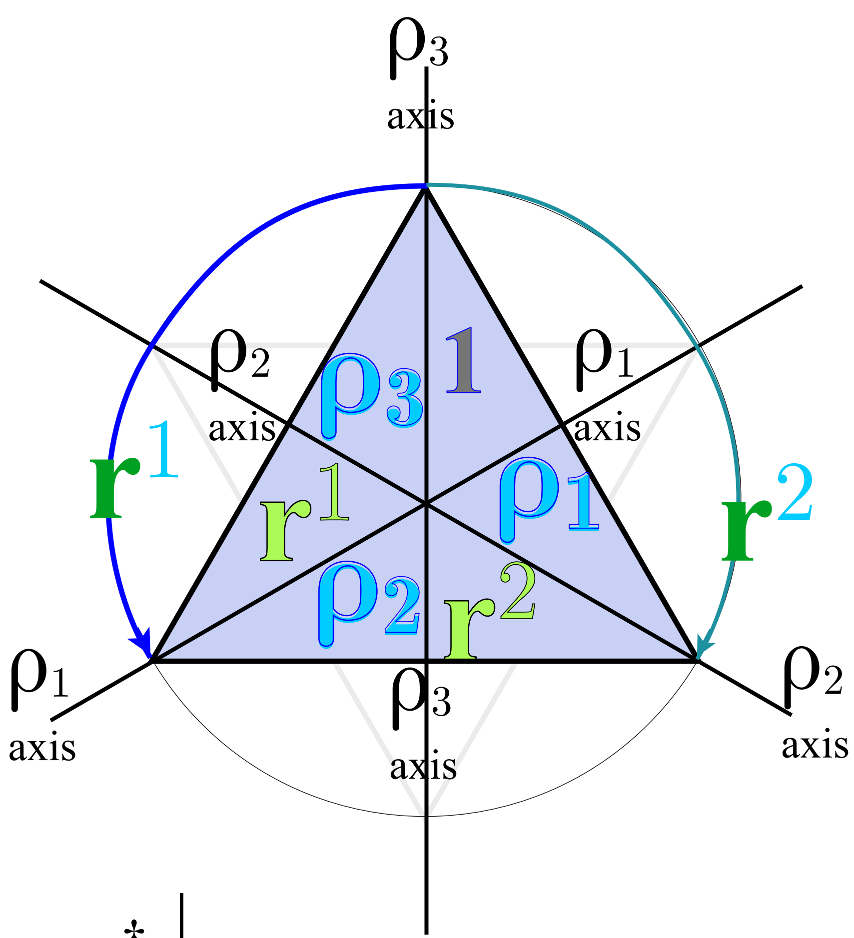
D_3 $\mathbf{gg}^\dagger$ *form*	**1**	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
1	**1**	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{r}^1$	$\mathbf{r}^1$	**1**	$\mathbf{r}^2$	ρ_3	ρ_1	ρ_2
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	**1**	ρ_2	ρ_3	ρ_1
ρ_1	ρ_1	ρ_3	ρ_2	**1**	$\mathbf{r}^1$	$\mathbf{r}^2$
ρ_2	ρ_2	ρ_1	ρ_3	$\mathbf{r}^2$	**1**	$\mathbf{r}^1$
σ_3	ρ_3	ρ_2	ρ_1	$\mathbf{r}^1$	$\mathbf{r}^2$	**1**

Transforming D_3 operators using D_3 operators

Example 1: Rotating ρ_3 axis crank using $\mathbf{r}^1$ puts it down onto ρ_1



Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes

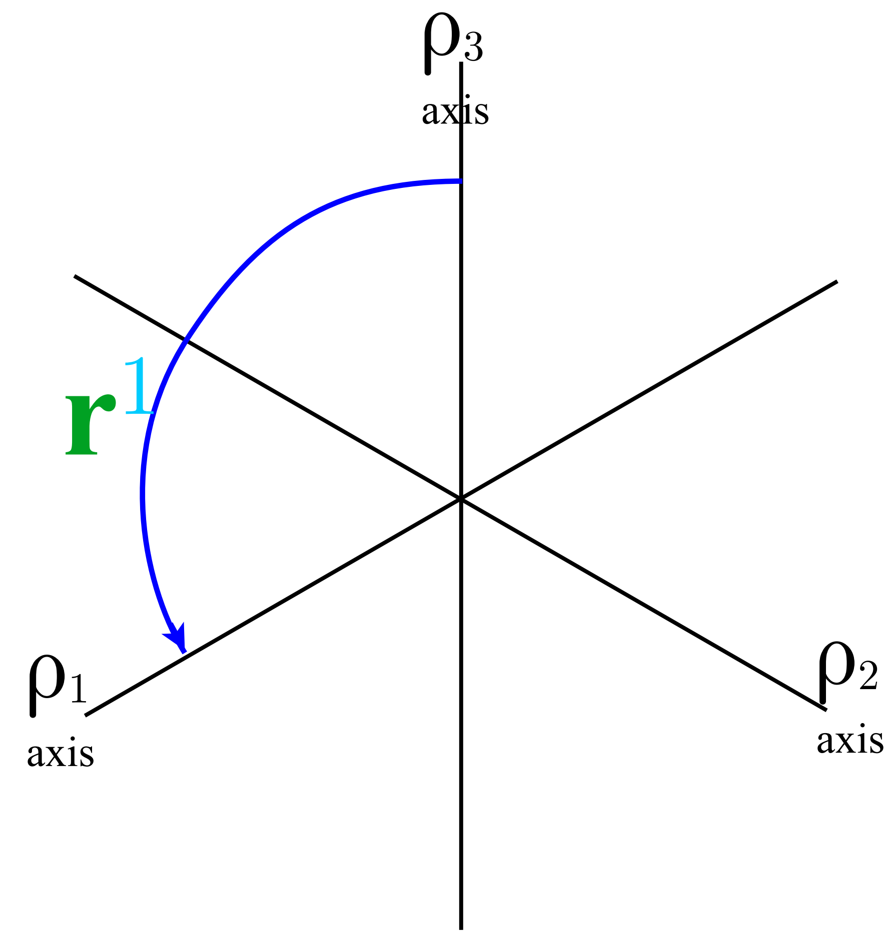


D_3 **gg**† *form*	**1**	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
1	**1**	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{r}^1$	$\mathbf{r}^1$	**1**	$\mathbf{r}^2$	ρ_3	ρ_1	ρ_2
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	**1**	ρ_2	ρ_3	ρ_1
ρ_1	ρ_1	ρ_3	ρ_2	**1**	$\mathbf{r}^1$	$\mathbf{r}^2$
ρ_2	ρ_2	ρ_1	ρ_3	$\mathbf{r}^2$	**1**	$\mathbf{r}^1$
σ_3	ρ_3	ρ_2	ρ_1	$\mathbf{r}^1$	$\mathbf{r}^2$	**1**

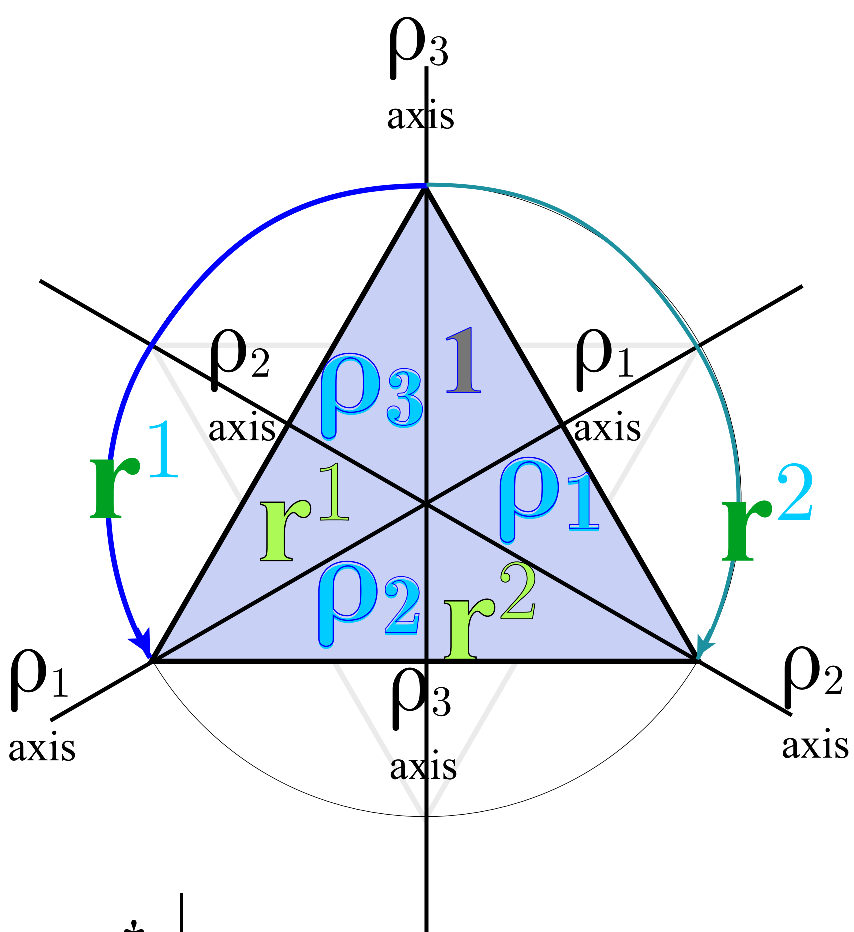
Transforming D_3 operators using D_3 operators

Example 1: Rotating ρ_3 *axis crank using* $\mathbf{r}^1$ *puts it down onto* ρ_1

Seems to imply: $\mathbf{r}^1\rho_3(\mathbf{r}^1)^{-1}=\mathbf{r}^1\rho_3\mathbf{r}^2=\rho_1$



Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes



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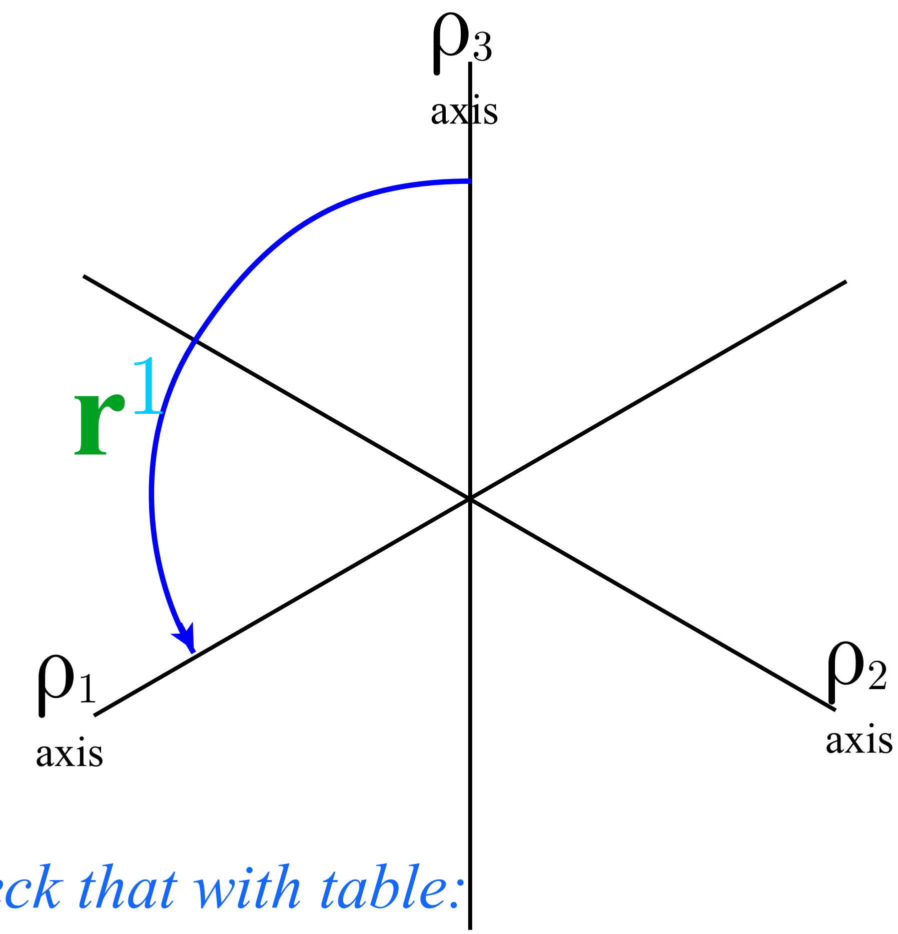
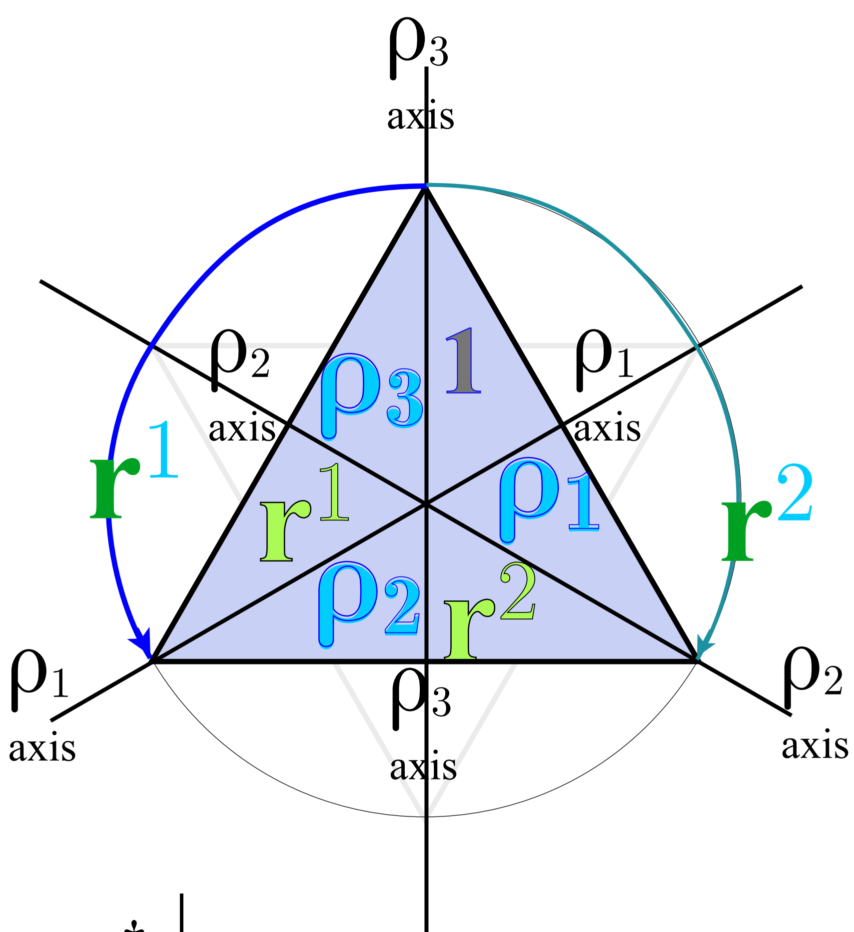


D_3 $\mathbf{gg}^\dagger$ *form*	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{r}^1$	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$	ρ_3	ρ_1	ρ_2
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$	ρ_2	ρ_3	ρ_1
ρ_1	ρ_1	ρ_3	ρ_2	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
ρ_2	ρ_2	ρ_1	ρ_3	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
σ_3	ρ_3	ρ_2	ρ_1	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

Need to check that with table:

$$\mathbf{r}^1\rho_3\mathbf{r}^2=\rho_2\,\mathbf{r}^2$$

Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes



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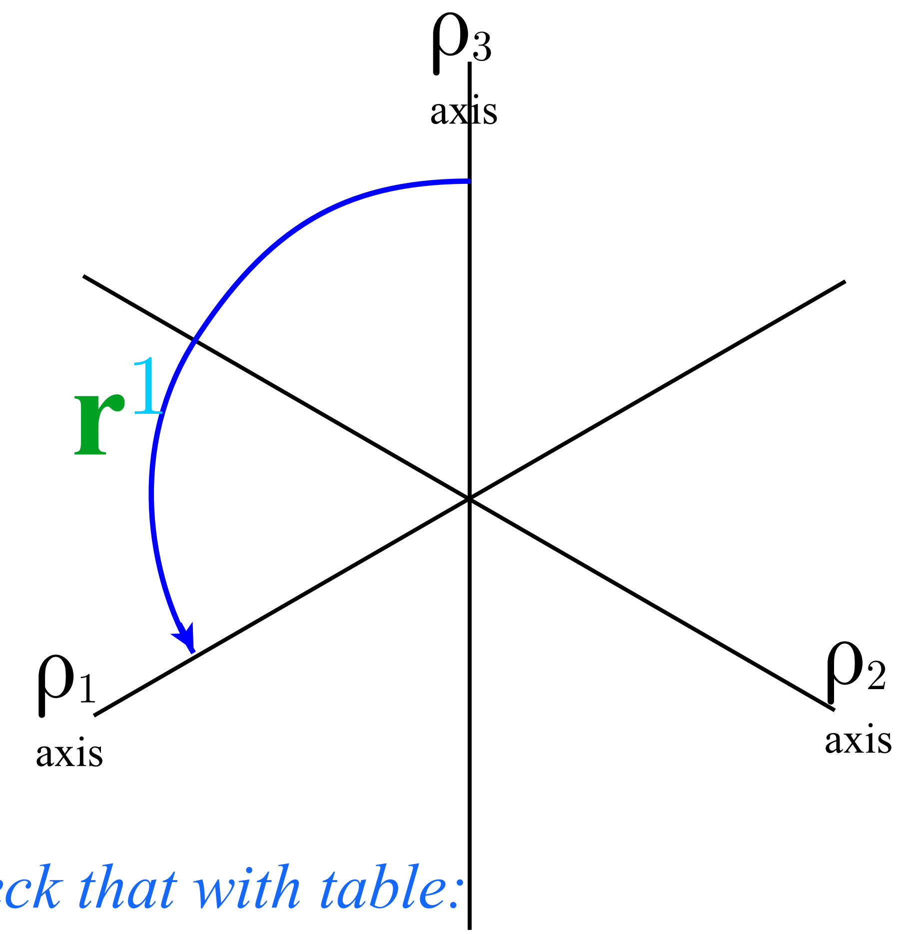
Example 1: Rotating ρ_3 *axis crank using* $\mathbf{r}^1$ *puts it down onto* ρ_1

Seems to imply: $\mathbf{r}^1\rho_3(\mathbf{r}^1)^{-1}=\mathbf{r}^1\rho_3\mathbf{r}^2=\rho_1$



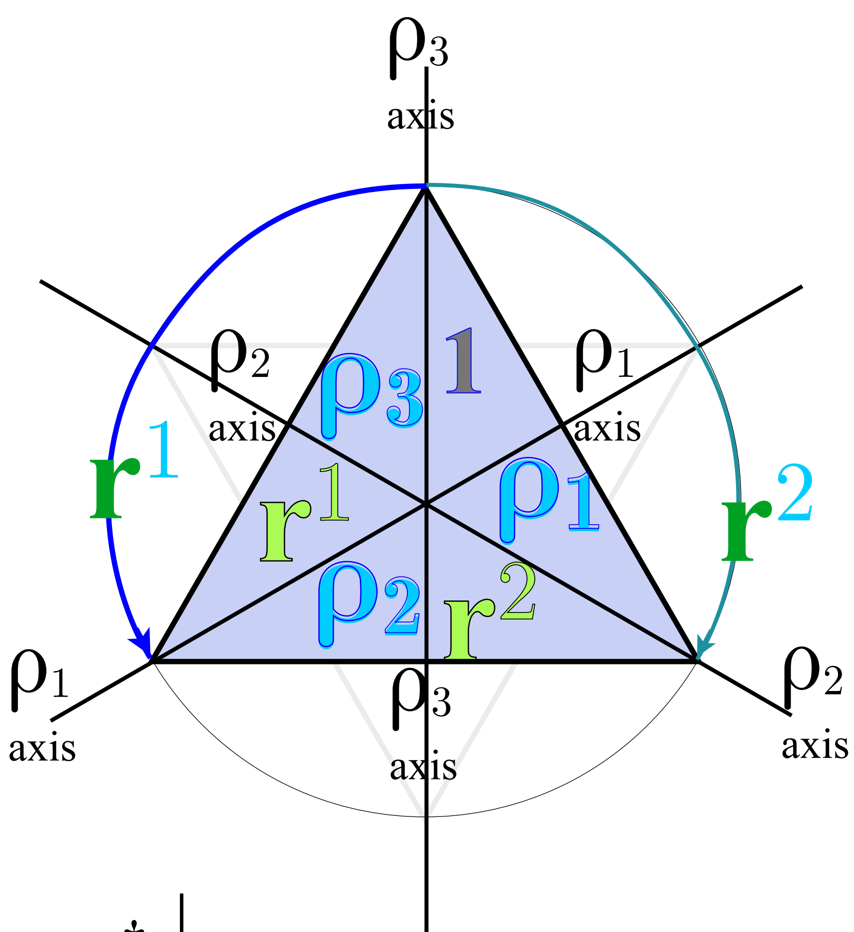
D_3 **gg**† *form*	**1**	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
1	**1**	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{r}^1$	$\mathbf{r}^1$	**1**	$\mathbf{r}^2$	ρ_3	ρ_1	ρ_2
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	**1**	ρ_2	ρ_3	ρ_1
ρ_1	ρ_1	ρ_3	ρ_2	**1**	$\mathbf{r}^1$	$\mathbf{r}^2$
ρ_2	ρ_2	$\boxed{\rho_1}$	ρ_3	$\mathbf{r}^2$	**1**	$\mathbf{r}^1$
σ_3	ρ_3	ρ_2	ρ_1	$\mathbf{r}^1$	$\mathbf{r}^2$	**1**

Need to check that with table:

$$\mathbf{r}^1\rho_3\mathbf{r}^2=\rho_2\mathbf{r}^2=\boxed{\rho_1}$$

Checks out!

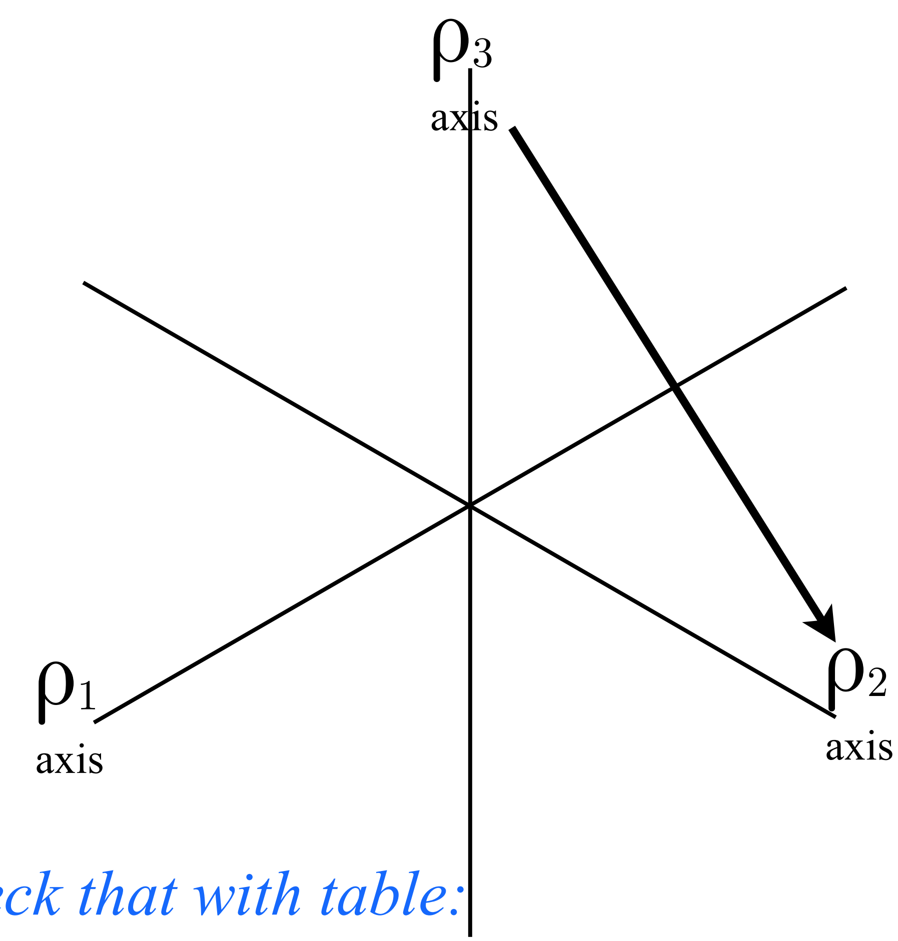
Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes



Transforming D_3 operators using D_3 operators

Example 2: Rotating ρ_3 axis crank using ρ_1 puts it down onto ρ_2

Seems to imply: $\rho_1 \rho_3 (\rho_1)^{-1} = \rho_1 \rho_3 \rho_1 = \rho_2$



D_3 $\mathbf{gg}^\dagger$ *form*	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{r}^1$	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$	ρ_3	ρ_1	ρ_2
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$	ρ_2	ρ_3	ρ_1
ρ_1	ρ_1	ρ_3	ρ_2	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
ρ_2	ρ_2	ρ_1	ρ_3	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
σ_3	ρ_3	ρ_2	ρ_1	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

Need to check that with table:

$\rho_1 \rho_3 \rho_1 = \mathbf{r}^2 \rho_1 = \rho_2$ *Checks out!*

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<u>Abelian</u> (Commutative) C_2, C_3, ...,C_6 ...

H diagonalized by r^p *symmetry operators that COMMUTE with H* $(r^p H = H r^p)$,

<u>and</u> with each other $(r^p r^q = r^{p+q} = r^q r^p)$.

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Non-Abelian (*do not commute*) D_3, O_h, ...

While all H symmetry operations COMMUTE with H ($\mathbf{U} H = H \mathbf{U}$)

most do not with each other ($\mathbf{U}\,\mathbf{V} \neq \mathbf{V}\mathbf{U}$).

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While all H symmetry operations COMMUTE with H $(\mathbf{U} H = H \mathbf{U})$

most do not with each other $(\mathbf{U}\,\mathbf{V} \neq \mathbf{V}\mathbf{U})$.

Q: *So how do we write* H *in terms of non-commutative* $\mathbf{U}$ *?*

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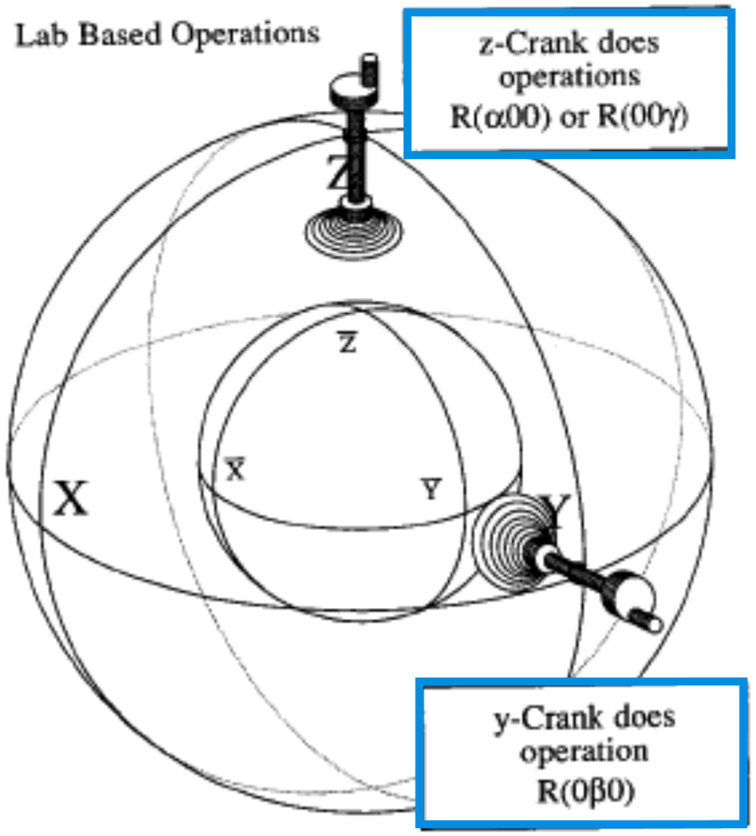
"Give me a place to stand...
and I will move the Earth"
Archimedes 287-212 B.C.E

Ideas of duality/relativity go *way* back (...VanVleck, Casimir..., Mach, Newton, Archimedes...)

Lab-fixed (Extrinsic-Global)**R**



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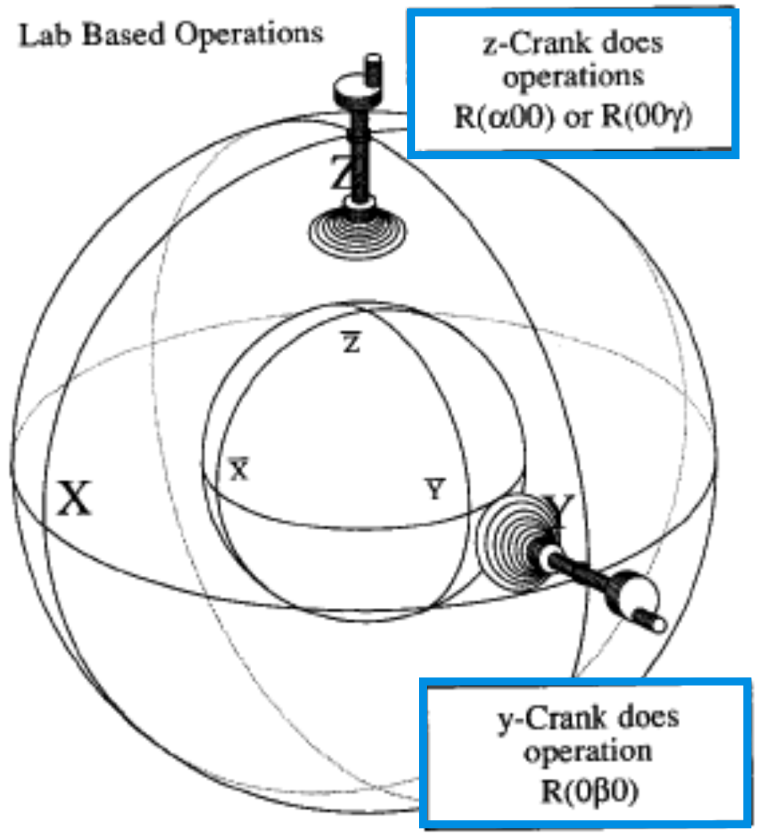
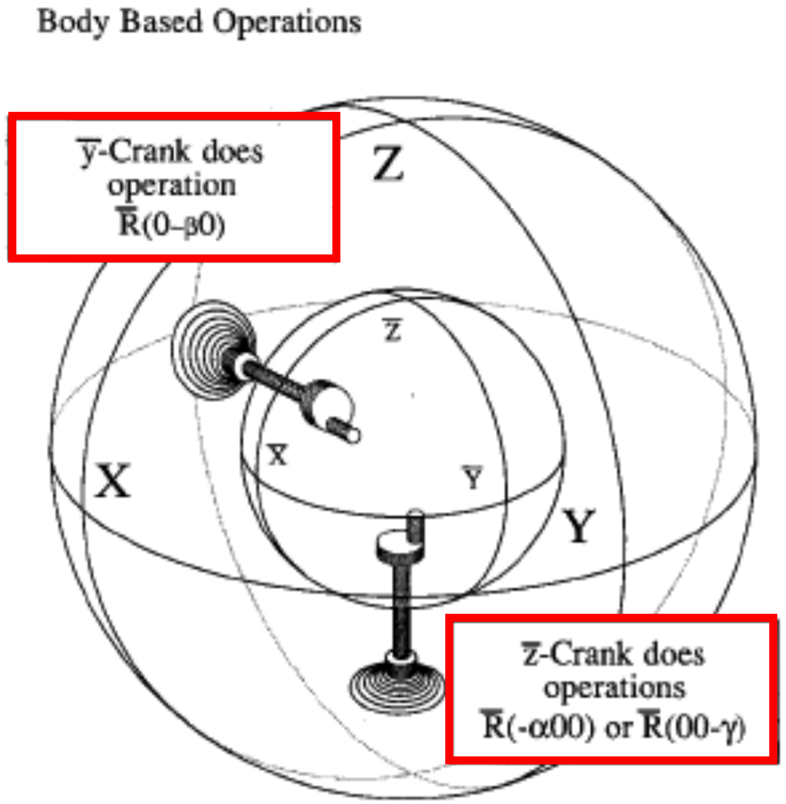
Lab-fixed (Extrinsic-Global)**R** *vs.* Body-fixed (Intrinsic-Local)$\bar{\mathbf{R}}$





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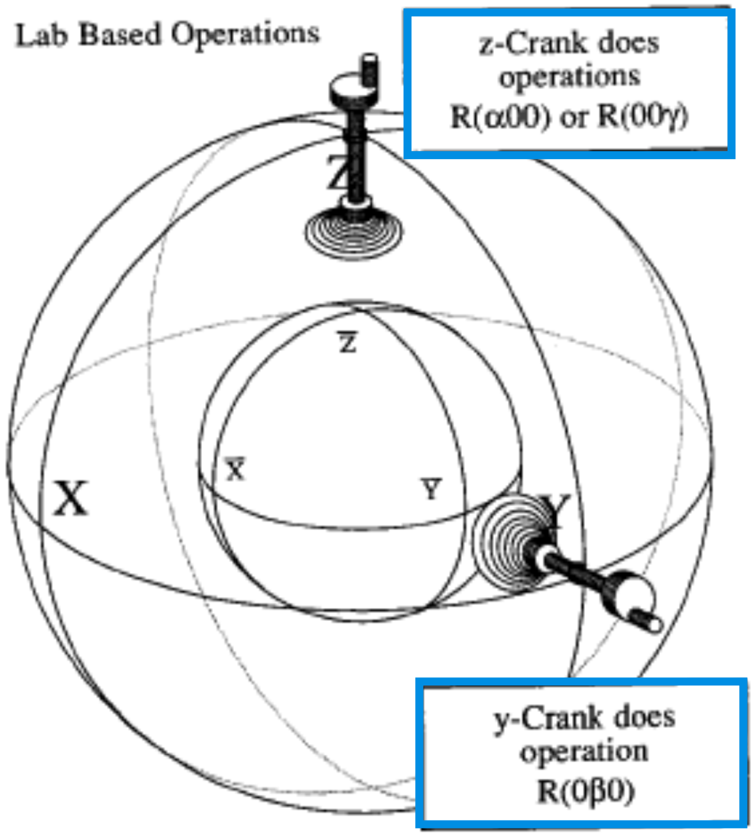
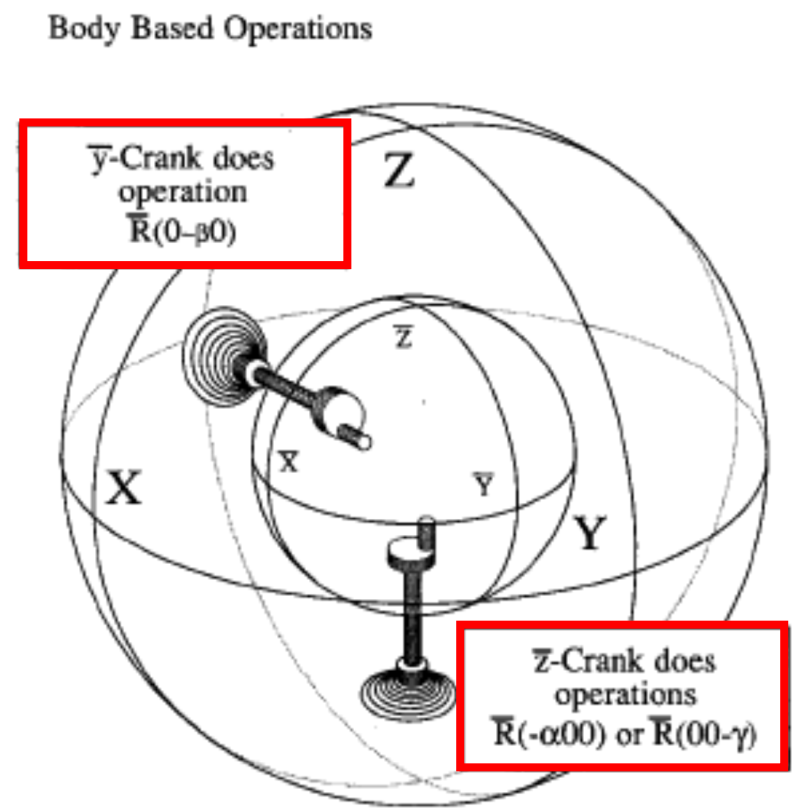


R *commutes*
with *all* $\bar{\mathbf{R}}$



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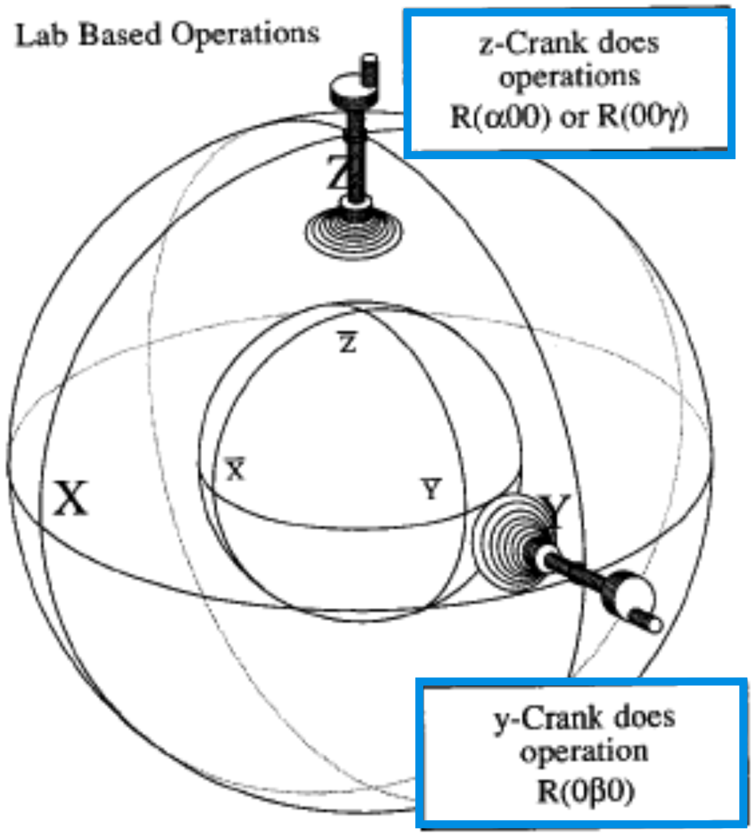


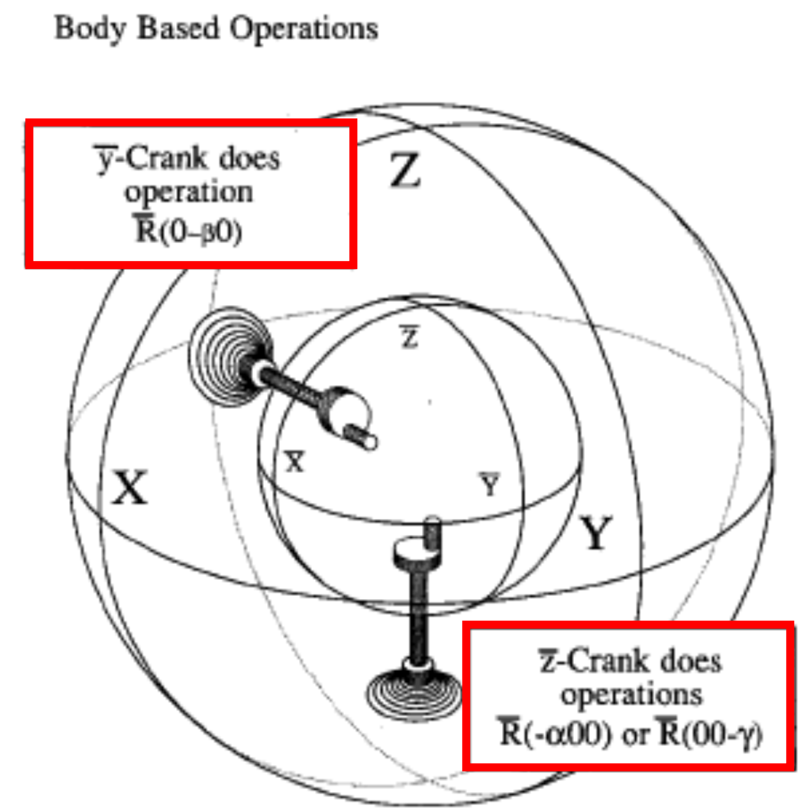
R *commutes*

with *all* $\bar{\mathbf{R}}$

Mock-Mach
relativity principle

$$\mathbf{R}|1\rangle=\bar{\mathbf{R}}^{-1}|1\rangle$$

...for *one* state |1) *only!*



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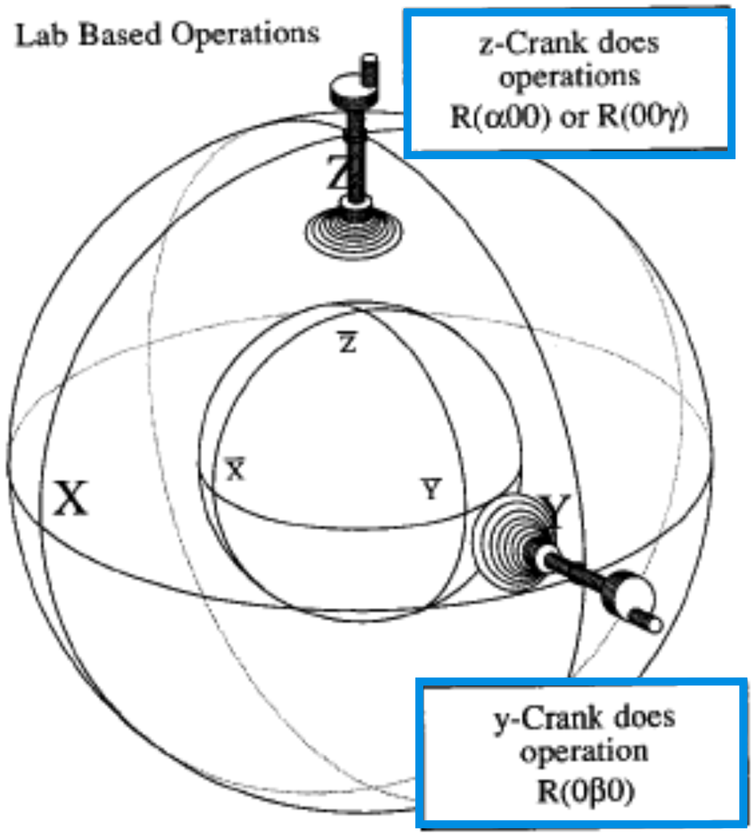

R *commutes*
with *all* $\bar{\mathbf{R}}$

Mock-Mach
relativity principle

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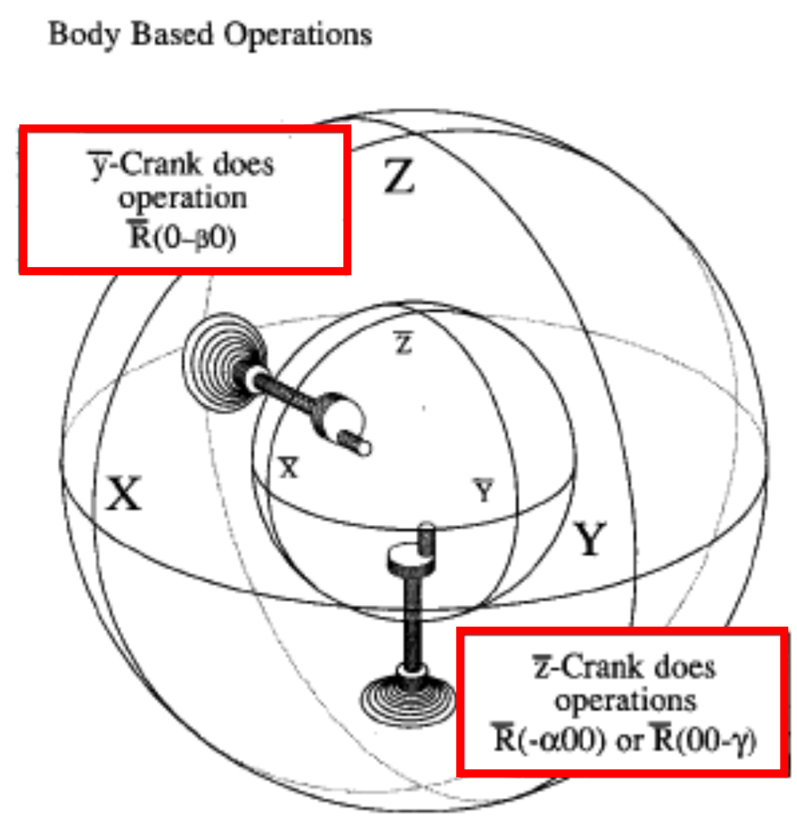

...But *how* do you actually *make* the **R** and $\bar{\mathbf{R}}$ operations?

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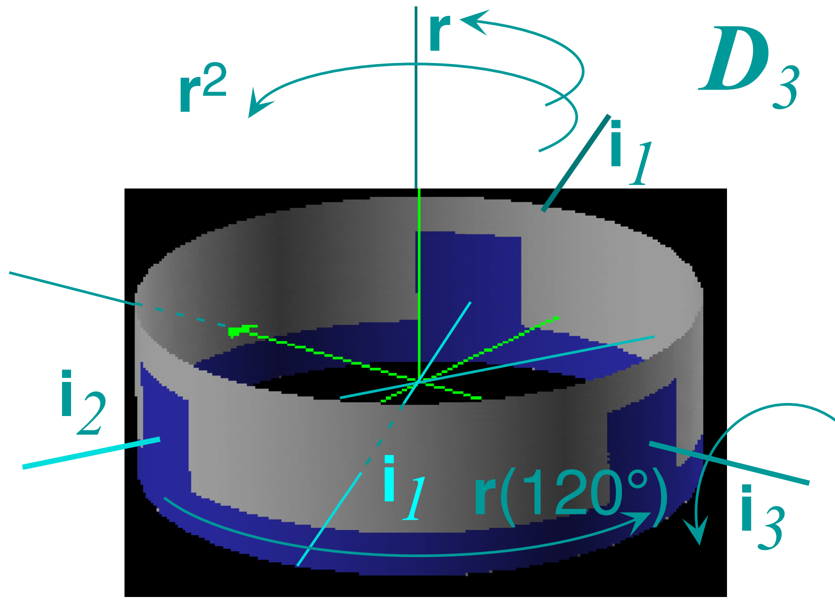
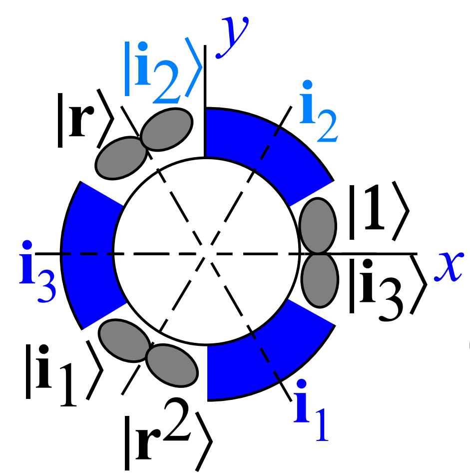
Example of GLOBAL vs LOCAL symmetry algebra for $D_3 \sim C_{3v}$

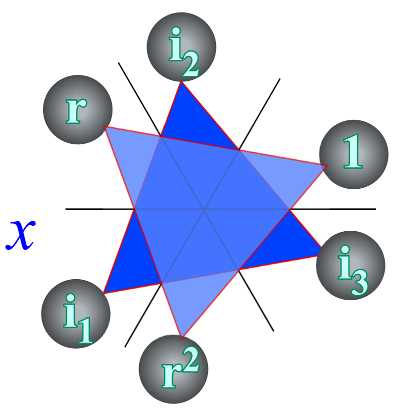


D_3

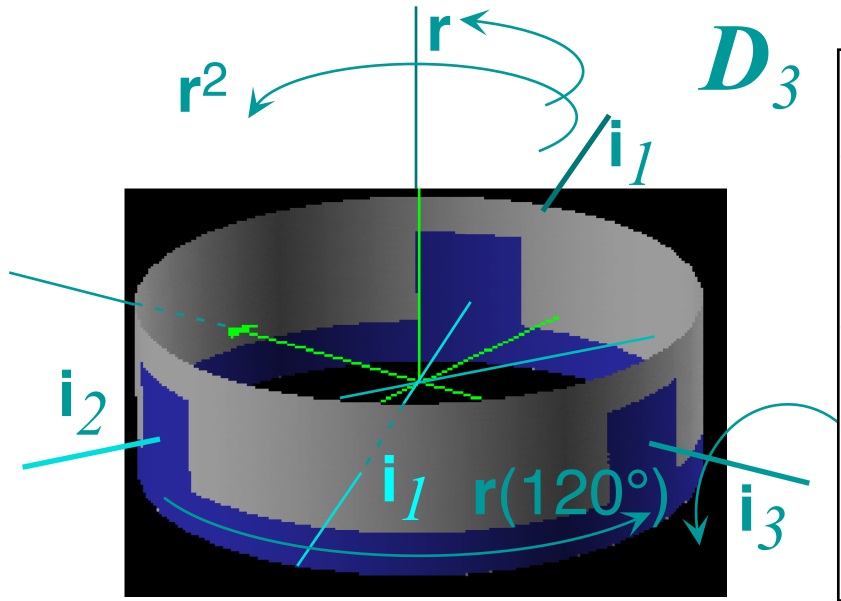
$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

D_3-defined local-wave bases





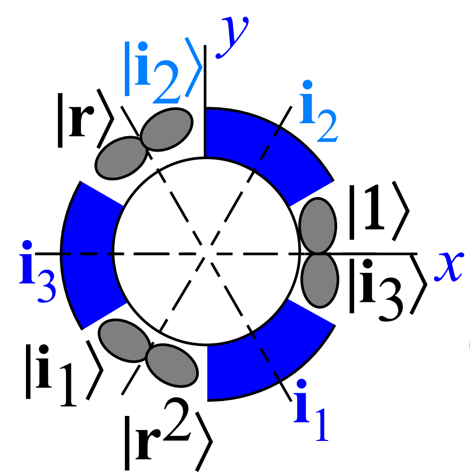
Example of GLOBAL vs LOCAL symmetry algebra for $D_3 \sim C_{3v}$

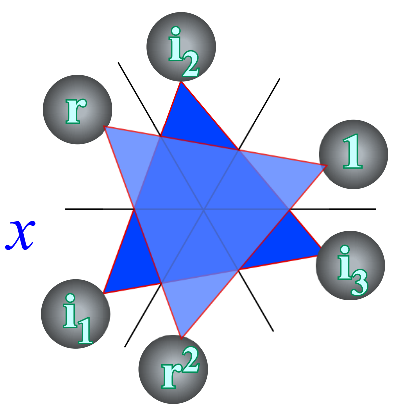


D_3

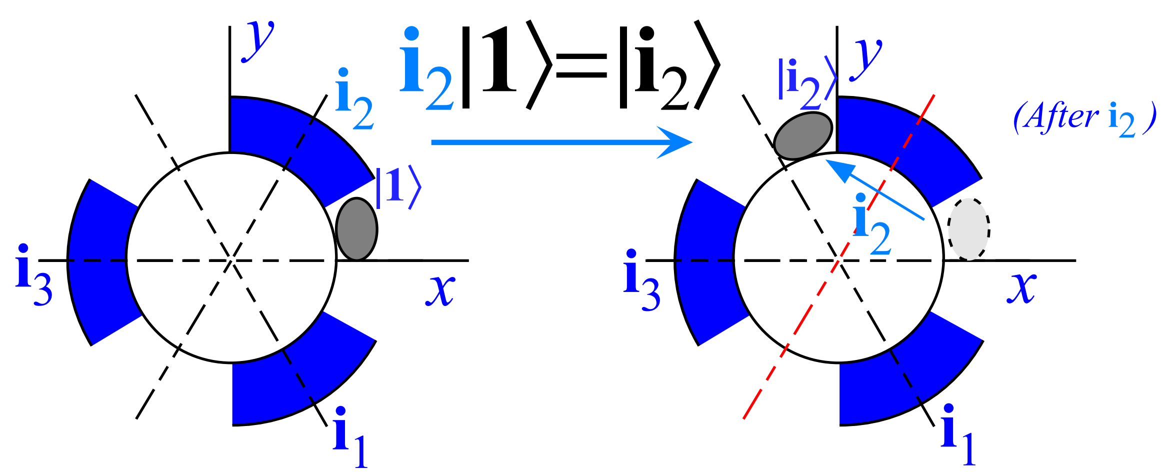
1	r^2	r	i_1	i_2	i_3
r	1	r^2	i_3	i_1	i_2
r^2	r	1	i_2	i_3	i_1
i_1	i_3	i_2	1	r	r^2
i_2	i_1	i_3	r^2	1	r
i_3	i_2	i_1	r	r^2	1

D_3-defined local-wave bases





Lab-fixed (Extrinsic-Global) operations and rotation axes



Example of RELATIVITY-DUALITY for $D_3 \sim C_{3v}$

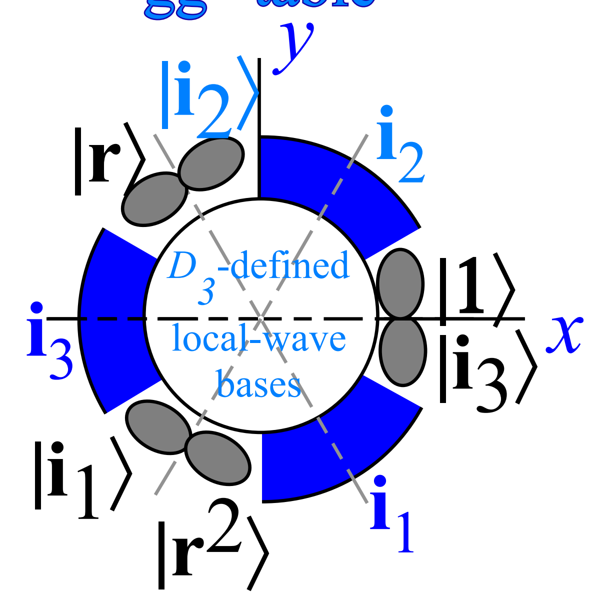
To represent *external* {..**T**,**U**,**V**,... }switch $\mathbf{g} \leftrightarrows \mathbf{g}^\dagger$ on top of group table

$$R^G(\mathbf{1})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\quad R^G(\mathbf{r})=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\quad R^G(\mathbf{r}^2)=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\mathbf{i}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix},\quad R^G(\mathbf{i}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\quad R^G(\mathbf{i}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

1	$\mathbf{r}^2$	**r**	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
r	**1**	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	**r**	**1**	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	**1**	**r**	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	**1**	**r**
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	**r**	$\mathbf{r}^2$	**1**

D_3 global $gg^\dagger$-table



Example of RELATIVITY-DUALITY for $D_3 \sim C_{3v}$

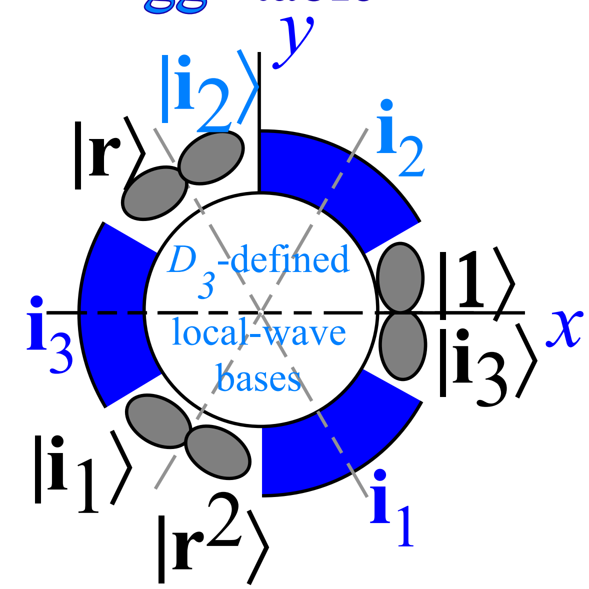
To represent *external* {..**T**,**U**,**V**,... }switch **g** ⇄ **g**† on top of group table

$$R^G(\mathbf{1})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\quad R^G(\mathbf{r})=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\quad R^G(\mathbf{r}^2)=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\mathbf{i}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix},\quad R^G(\mathbf{i}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\quad R^G(\mathbf{i}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

1	r^2	r	i_1	i_2	i_3
r	1	r^2	i_3	i_1	i_2
r^2	r	1	i_2	i_3	i_1
i_1	i_3	i_2	1	r	r^2
i_2	i_1	i_3	r^2	1	r
i_3	i_2	i_1	r	r^2	1

D_3 global
gg†-table



D_3 local
g†g-table

To represent *internal* {..$\bar{\mathbf{T}}$,$\bar{\mathbf{U}}$,$\bar{\mathbf{V}}$,... } switch **g** ⇄ **g**† on side of group table

$$R^G(\bar{\mathbf{1}})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\quad R^G(\bar{\mathbf{r}})=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\quad R^G(\bar{\mathbf{r}}^2)=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\bar{\mathbf{i}}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\quad R^G(\bar{\mathbf{i}}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix},\quad R^G(\bar{\mathbf{i}}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

1	r	r^2	i_1	i_2	i_3
r^2	1	r	i_2	i_3	i_1
r	r^2	1	i_3	i_1	i_2
i_1	i_2	i_3	1	r	r^2
i_2	i_3	i_2	r^2	1	r
i_3	i_1	i_2	r	r^2	1

Example of RELATIVITY-DUALITY for $D_3 \sim C_{3v}$

To represent *external* $\{..\mathbf{T},\mathbf{U},\mathbf{V},...\}$ switch $\mathbf{g} \leftrightarrows \mathbf{g}^\dagger$ on top of group table

$$R^G(\mathbf{1}) = \begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1 \end{pmatrix}, \quad R^G(\mathbf{r}) = \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot \end{pmatrix}, \quad R^G(\mathbf{r}^2) = \begin{pmatrix} \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot \end{pmatrix},$$

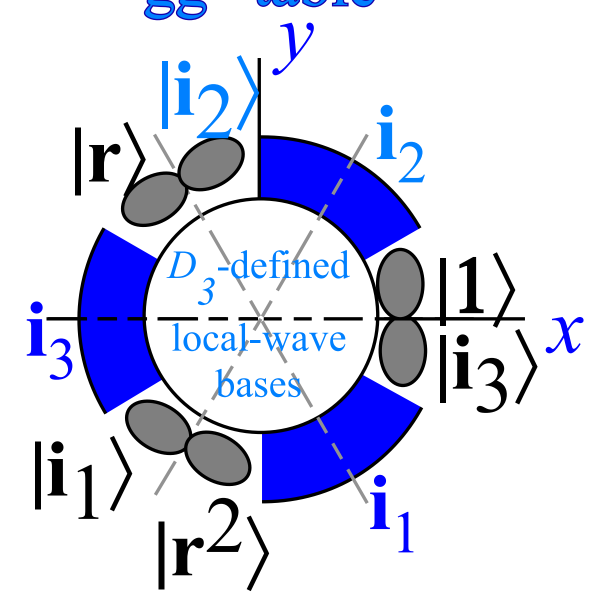
$$R^G(\mathbf{i}_1) = \begin{pmatrix} \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot \end{pmatrix}, \quad R^G(\mathbf{i}_2) = \begin{pmatrix} \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot \end{pmatrix}, \quad R^G(\mathbf{i}_3) = \begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}$$

1	**r²**	**r**	**i₁**	**i₂**	**i₃**
r	**1**	**r²**	**i₃**	**i₁**	**i₂**
r²	**r**	**1**	**i₂**	**i₃**	**i₁**
i₁	**i₃**	**i₂**	**1**	**r**	**r²**
i₂	**i₁**	**i₃**	**r²**	**1**	**r**
i₃	**i₂**	**i₁**	**r**	**r²**	**1**

D_3 global $gg^\dagger$-table

RESULT:
Any $R(\mathbf{T})$
commute (Even if $\mathbf{T}$ and $\mathbf{U}$ do not...)
with any $R(\bar{\mathbf{U}})$...

...and $\mathbf{T}\cdot\mathbf{U}=\mathbf{V}$ if & only if $\bar{\mathbf{T}}\cdot\bar{\mathbf{U}}=\bar{\mathbf{V}}$.



D_3 local $g^\dagger g$-table

To represent *internal* $\{..\bar{\mathbf{T}},\bar{\mathbf{U}},\bar{\mathbf{V}},...\}$ switch $\mathbf{g} \leftrightarrows \mathbf{g}^\dagger$ on side of group table

$$R^G(\bar{\mathbf{1}}) = \begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1 \end{pmatrix}, \quad R^G(\bar{\mathbf{r}}) = \begin{pmatrix} \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot \end{pmatrix}, \quad R^G(\bar{\mathbf{r}}^2) = \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot \end{pmatrix},$$

$$R^G(\bar{\mathbf{i}}_1) = \begin{pmatrix} \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot \end{pmatrix}, \quad R^G(\bar{\mathbf{i}}_2) = \begin{pmatrix} \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot \end{pmatrix}, \quad R^G(\bar{\mathbf{i}}_3) = \begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}$$

1	**r**	**r²**	**i₁**	**i₂**	**i₃**
r²	**1**	**r**	**i₂**	**i₃**	**i₁**
r	**r²**	**1**	**i₃**	**i₁**	**i₂**
i₁	**i₂**	**i₃**	**1**	**r**	**r²**
i₂	**i₃**	**i₂**	**r²**	**1**	**r**
i₃	**i₁**	**i₂**	**r**	**r²**	**1**

3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

D_3 and C_{3v} are isomorphic ($D_3 \sim C_{3v}$ share product table)

Deriving $D_3 \sim C_{3v}$ products:

By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$

By nomograms based on U(2) Hamilton-turns

Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes

Non-commutative symmetry expansion and Global-Local solution

Global vs Local symmetry and Mock-Mach principle

Global vs Local matrix duality for D_3

Global vs Local symmetry expansion of D_3 Hamiltonian

1st-Stage spectral decomposition of global/local D_3 Hamiltonian

All-commuting operators and D_3-invariant class algebra

All-commuting projectors and D_3-invariant characters

Group invariant numbers: Centrum, Rank, and Order

*Spectral resolution to **irreducible representations** (or "**irreps**") foretold by **characters** or traces*

Crystal-field splitting: $O(3) \supset D_3$ symmetry reduction and $D^{\ell} \downarrow D_3$ splitting

Example of RELATIVITY-DUALITY for $D_3 \sim C_{3v}$

To represent *external* $\{..\mathbf{T},\mathbf{U},\mathbf{V},...\}$ switch $\mathbf{g} \leftrightarrows \mathbf{g}^\dagger$ on top of group table

$$R^G(\mathbf{1})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\quad R^G(\mathbf{r})=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\quad R^G(\mathbf{r}^2)=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\mathbf{i}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix},\quad R^G(\mathbf{i}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\quad R^G(\mathbf{i}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

D_3 global $\mathbf{g}\mathbf{g}^\dagger$-table

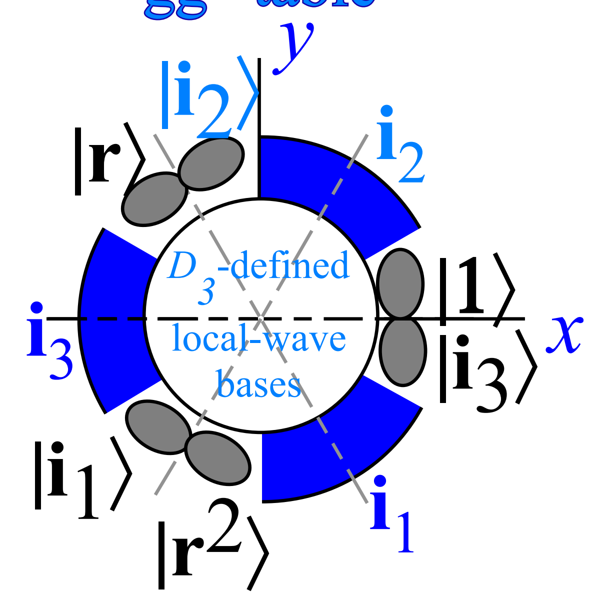
RESULT:
Any $R(\mathbf{T})$ *commute* (Even if $\mathbf{T}$ and $\mathbf{U}$ do not...)
with any $R(\bar{\mathbf{U}})$...

...and $\mathbf{T}\cdot\mathbf{U}=\mathbf{V}$ if & only if $\bar{\mathbf{T}}\cdot\bar{\mathbf{U}}=\bar{\mathbf{V}}$.

So an $\mathbb{H}$*-matrix having* ***Global*** *symmetry* D_3

$$\mathbb{H} = H\bar{\mathbf{1}}^0 + r_1\bar{\mathbf{r}}^1 + r_2\bar{\mathbf{r}}^2 + i_1\bar{\mathbf{i}}_1 + i_2\bar{\mathbf{i}}_2 + i_3\bar{\mathbf{i}}_3$$

is made from ***Local*** *symmetry matrices*



D_3 local $\mathbf{g}^\dagger\mathbf{g}$-table

To represent *internal* $\{..\bar{\mathbf{T}},\bar{\mathbf{U}},\bar{\mathbf{V}},...\}$ switch $\mathbf{g} \leftrightarrows \mathbf{g}^\dagger$ on side of group table

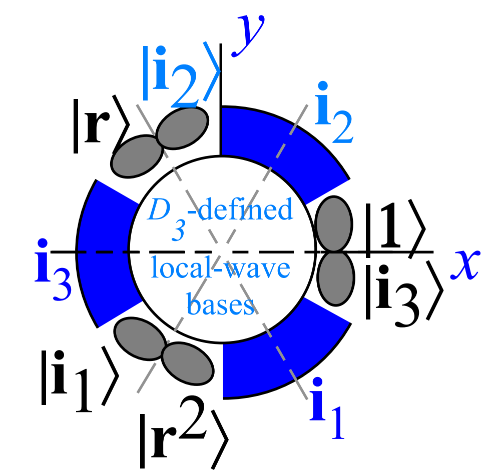
$$R^G(\bar{\mathbf{1}})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\quad R^G(\bar{\mathbf{r}})=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\quad R^G(\bar{\mathbf{r}}^2)=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\bar{\mathbf{i}}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\quad R^G(\bar{\mathbf{i}}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix},\quad R^G(\bar{\mathbf{i}}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

Example of RELATIVITY-DUALITY for $D_3 \sim C_{3v}$

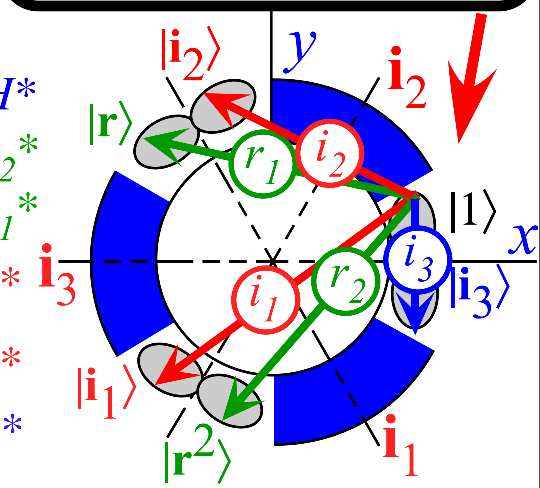
To represent *external* {..**T**,**U**,**V**,... }switch **g** ⇄ **g**† on top of group table



$$R^G(\mathbf{1})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\quad R^G(\mathbf{r})=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\quad R^G(\mathbf{r}^2)=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\mathbf{i}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix},\quad R^G(\mathbf{i}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\quad R^G(\mathbf{i}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

Local $\mathbb{H}$ *matrix parametrized by* $\bar{\mathbf{g}}$*'s*



$H = \langle 1|\mathbb{H}|1\rangle = H^*$

$r_1 = \langle r|\mathbb{H}|1\rangle = r_2^*$

$r_2 = \langle r^2|\mathbb{H}|1\rangle = r_1^*$

$i_1 = \langle i_1|\mathbb{H}|1\rangle = i_1^*$

$i_2 = \langle i_2|\mathbb{H}|1\rangle = i_2^*$

$i_3 = \langle i_3|\mathbb{H}|1\rangle = i_3^*$

RESULT:
Any $R(\mathbf{T})$ *commute* (Even if **T** and **U** do not...) with any $R(\bar{\mathbf{U}})$...

...and $\mathbf{T}\cdot\mathbf{U}=\mathbf{V}$ if & only if $\bar{\mathbf{T}}\cdot\bar{\mathbf{U}}=\bar{\mathbf{V}}$.

So an $\mathbb{H}$*-matrix having* ***Global*** *symmetry* D_3

$$\mathbb{H} = H\bar{\mathbf{1}}^0 + r_1\bar{\mathbf{r}}^1 + r_2\bar{\mathbf{r}}^2 + i_1\bar{\mathbf{i}}_1 + i_2\bar{\mathbf{i}}_2 + i_3\bar{\mathbf{i}}_3$$

is made from ***Local*** *symmetry matrices*

To represent *internal* {..$\bar{\mathbf{T}}$,$\bar{\mathbf{U}}$,$\bar{\mathbf{V}}$,... } switch **g** ⇄ **g**† on side of group table

$$R^G(\bar{\mathbf{1}})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\quad R^G(\bar{\mathbf{r}})=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\quad R^G(\bar{\mathbf{r}}^2)=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\bar{\mathbf{i}}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\quad R^G(\bar{\mathbf{i}}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix},\quad R^G(\bar{\mathbf{i}}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

local D_3 defined Hamiltonian matrix

$\mathbb{H} =$

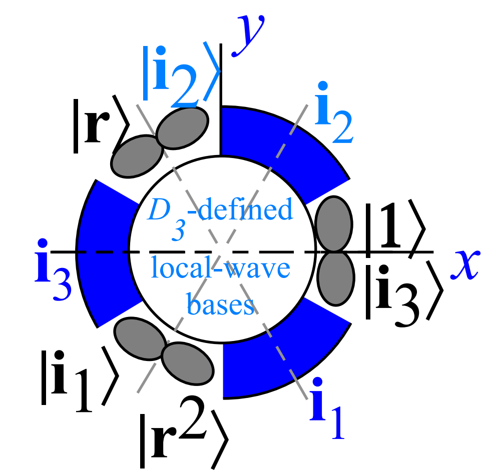
	$\|1)$	$\|r)$	$\|r^2)$	$\|i_1)$	$\|i_2)$	$\|i_3)$
$(1\|$	H	r_1	r_2	i_1	i_2	i_3
$(r\|$	r_2	H	r_1	i_2	i_3	i_1
$(r^2\|$	r_1	r_2	H	i_3	i_1	i_2
$(i_1\|$	i_1	i_2	i_3	H	r_1	r_2
$(i_2\|$	i_2	i_3	i_1	r_2	H	r_1
$(i_3\|$	i_3	i_1	i_2	r_1	r_2	H

Example of RELATIVITY-DUALITY for $D_3 \sim C_{3v}$

To represent *external* {..T,U,V,... }switch $\mathbf{g} \leftrightarrows \mathbf{g}^\dagger$ on top of group table

$$R^G(\mathbf{1})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\ R^G(\mathbf{r})=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\ R^G(\mathbf{r}^2)=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\mathbf{i}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix},\ R^G(\mathbf{i}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\ R^G(\mathbf{i}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$



Local $\mathbb{H}$ *matrix parametrized by* $\bar{\mathbf{g}}$*'s*

RESULT:
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with any $R(\bar{\mathbf{U}})$...

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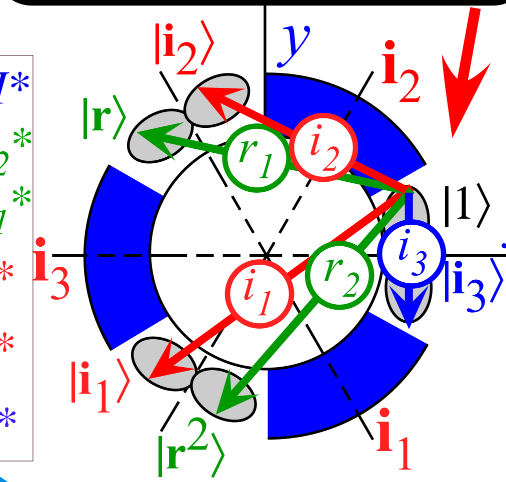
$$\mathbb{H}=H\bar{\mathbf{1}}^0+r_1\bar{\mathbf{r}}^1+r_2\bar{\mathbf{r}}^2+i_1\bar{\mathbf{i}}_1+i_2\bar{\mathbf{i}}_2+i_3\bar{\mathbf{i}}_3$$

is made from ***Local*** *symmetry matrices*

$H=\langle 1|\mathbb{H}|1\rangle=H^*$
$r_1=\langle \mathbf{r}|\mathbb{H}|1\rangle=r_2^*$
$r_2=\langle \mathbf{r}^2|\mathbb{H}|1\rangle=r_1^*$
$i_1=\langle \mathbf{i}_1|\mathbb{H}|1\rangle=i_1^*$
$i_2=\langle \mathbf{i}_2|\mathbb{H}|1\rangle=i_2^*$
$i_3=\langle \mathbf{i}_3|\mathbb{H}|1\rangle=i_3^*$



All the global $\mathbf{g}$ *commute with general local* $\mathbb{H}$ *matrix.*

local D_3 defined Hamiltonian matrix

$\mathbb{H}=$

	$\vert\mathbf{1})$	$\vert\mathbf{r})$	$\vert\mathbf{r}^2)$	$\vert\mathbf{i}_1)$	$\vert\mathbf{i}_2)$	$\vert\mathbf{i}_3)$
$(\mathbf{1}\vert$	H	r_1	r_2	i_1	i_2	i_3
$(\mathbf{r}\vert$	r_2	H	r_1	i_2	i_3	i_1
$(\mathbf{r}^2\vert$	r_1	r_2	H	i_3	i_1	i_2
$(\mathbf{i}_1\vert$	i_1	i_2	i_3	H	r_1	r_2
$(\mathbf{i}_2\vert$	i_2	i_3	i_1	r_2	H	r_1
$(\mathbf{i}_3\vert$	i_3	i_1	i_2	r_1	r_2	H

To represent *internal* {..$\bar{\mathbf{T}},\bar{\mathbf{U}},\bar{\mathbf{V}}$,... } switch $\mathbf{g} \leftrightarrows \mathbf{g}^\dagger$ on side of group table

$$R^G(\bar{\mathbf{1}})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\ R^G(\bar{\mathbf{r}})=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\ R^G(\bar{\mathbf{r}}^2)=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\bar{\mathbf{i}}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\ R^G(\bar{\mathbf{i}}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix},\ R^G(\bar{\mathbf{i}}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

Example of RELATIVITY-DUALITY for D

To represent *external* {..**T**,**U**,**V**,... }...

$$R^G(\mathbf{1})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\quad R^G(\mathbf{r})=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\quad R^G(\mathbf{r}^2)=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},\quad R^G(\mathbf{i}_1)=\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\1&\cdot&\cdot\\\cdot&1&\cdot\\\cdot&\cdot&1\end{pmatrix}$$

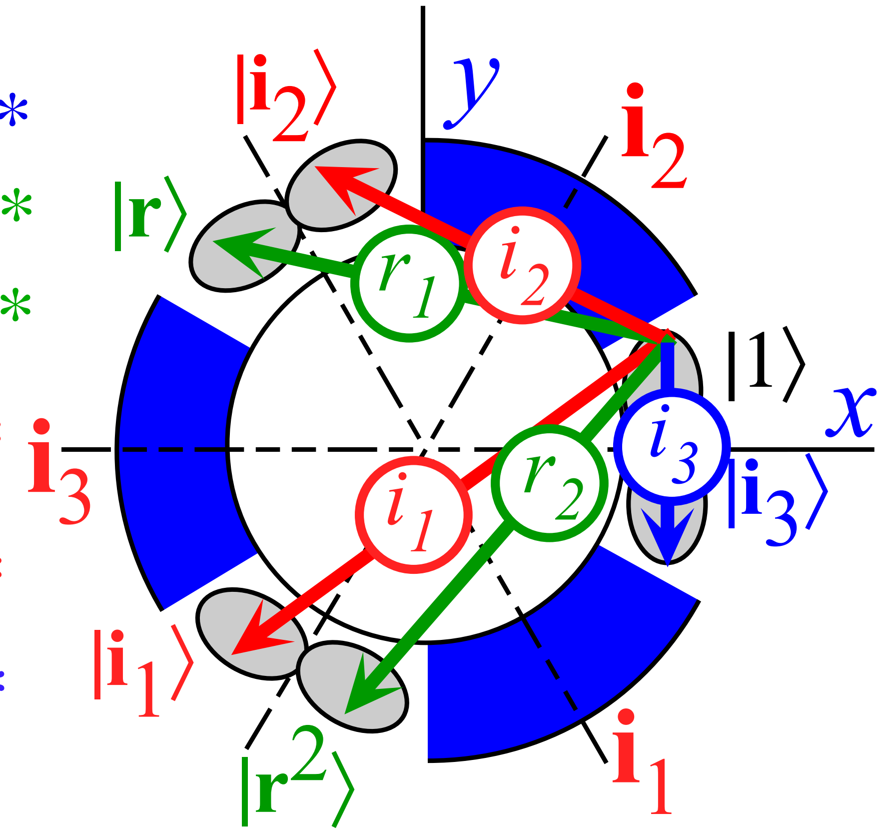
RESULT:
Any $R(\mathbf{T})$ *commute* (Even if **T** and **U** do not...)
with any $R(\bar{\mathbf{U}})$...

...and $\mathbf{T}\cdot\mathbf{U}=\mathbf{V}$ if & only if $\bar{\mathbf{T}}\cdot\bar{\mathbf{U}}=\bar{\mathbf{V}}$.

To represent *internal* {..$\bar{\mathbf{T}}$,$\bar{\mathbf{U}}$,$\bar{\mathbf{V}}$,... }....

$$R^G(\bar{\mathbf{1}})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\quad R^G(\bar{\mathbf{r}})=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\quad R^G(\bar{\mathbf{r}}^2)=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},\quad R^G(\bar{\mathbf{i}}_1)=\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\1&\cdot&\cdot\\\cdot&\cdot&1\\\cdot&1&\cdot\end{pmatrix}$$

$$H=\langle 1|\mathbb{H}|1\rangle=H^*$$
$$r_1=\langle \mathbf{r}|\mathbb{H}|1\rangle=r_2^*$$
$$r_2=\langle \mathbf{r}^2|\mathbb{H}|1\rangle=r_1^*$$
$$i_1=\langle \mathrm{i}_1|\mathbb{H}|1\rangle=i_1^*$$
$$i_2=\langle \mathrm{i}_2|\mathbb{H}|1\rangle=i_2^*$$
$$i_3=\langle \mathrm{i}_3|\mathbb{H}|1\rangle=i_3^*$$



So an $\mathbb{H}$*-matrix*
having ***Global*** *symmetry* D_3

local-D_3-defined
Hamiltonian matrix

$$\mathbb{H}=H\bar{\mathbf{1}}^0+r_1\bar{\mathbf{r}}^1+r_2\bar{\mathbf{r}}^2+i_1\bar{\mathbf{i}}_1+i_2\bar{\mathbf{i}}_2+i_3\bar{\mathbf{i}}_3$$

is made from
Local *symmetry matrices*

$\mathbb{H}=$

	$\vert\mathbf{1})$	$\vert\mathbf{r})$	$\vert\mathbf{r}^2)$	$\vert\mathbf{i}_1)$	$\vert\mathbf{i}_2)$	$\vert\mathbf{i}_3)$
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$(\mathbf{r}\vert$	r_2	H	r_1	i_2	i_3	i_1
$(\mathbf{r}^2\vert$	r_1	r_2	H	i_3	i_1	i_2
$(\mathbf{i}_1\vert$	i_1	i_2	i_3	H	r_1	r_2
$(\mathbf{i}_2\vert$	i_2	i_3	i_2	r_2	H	r_1
$(\mathbf{i}_3\vert$	i_3	i_1	i_2	r_1	r_2	H

3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

D_3 and C_{3v} are isomorphic ($D_3 \sim C_{3v}$ share product table)

Deriving $D_3 \sim C_{3v}$ products:

By group definition $|g\rangle=\mathbf{g}|1\rangle$ of position ket $|g\rangle$

By nomograms based on U(2) Hamilton-turns

Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes

Non-commutative symmetry expansion and Global-Local solution

Global vs Local symmetry and Mock-Mach principle

Global vs Local matrix duality for D_3

Global vs Local symmetry expansion of D_3 Hamiltonian

1st-Stage spectral decomposition of global/local D_3 Hamiltonian

All-commuting operators and D_3-invariant class algebra

All-commuting projectors and D_3-invariant characters

Group invariant numbers: Centrum, Rank, and Order

*Spectral resolution to **irreducible representations** (or "**irreps**") foretold by **characters** or traces*

Crystal-field splitting: $O(3)\supset D_3$ symmetry reduction and $D^\ell \downarrow D_3$ splitting

Spectral analysis of non-commutative "Group-table Hamiltonian"

1st Step: Spectral resolution of D_3-Center (Class algebra of D_3)

$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

Spectral analysis of non-commutative "Group-table Hamiltonian"

1st Step: Spectral resolution of D_3-Center (Class algebra of D_3)

$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

Each class-sum $\underline{\kappa}_k$ commutes with all of D_3.

$\kappa_1 = \mathbf{1}$	$\kappa_2 = \mathbf{r}^1 + \mathbf{r}^2$	$\kappa_3 = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_2	$2\kappa_1 + \kappa_2$	$2\kappa_3$
κ_3	$2\kappa_3$	$3\kappa_1 + 3\kappa_2$

$\boldsymbol{\kappa}_g$'s are *mutually commuting* with respect to themselves and *all-commuting* with respect to the whole group.

$$\mathbf{r}\,\boldsymbol{\kappa}_i\,\mathbf{r}^{-1} = \mathbf{i}_2 + \mathbf{i}_3 + \mathbf{i}_1 = \boldsymbol{\kappa}_i \quad \text{or:} \quad \mathbf{r}\,\boldsymbol{\kappa}_i = \boldsymbol{\kappa}_i\,\mathbf{r}$$

$$\sum_{\mathbf{h}=1}^{\circ G} \mathbf{hgh}^{-1} = \upsilon_g \boldsymbol{\kappa}_g \;, \qquad \text{where: } \upsilon_g = \frac{\circ G}{\circ \kappa_g} = integer$$

$^\circ\kappa g$ is order of class $\boldsymbol{\kappa} g$ and must evenly divide group order $^\circ G$.

Spectral analysis of non-commutative "Group-table Hamiltonian"

1st Step: Spectral resolution of D_3-Center (Class algebra of D_3)

$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

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κ_2	$2\kappa_1 + \kappa_2$	$2\kappa_3$
κ_3	$2\kappa_3$	$3\kappa_1 + 3\kappa_2$

$$\boldsymbol{\kappa}_3^2 = 3\cdot\boldsymbol{\kappa}_2 + 3\cdot\mathbf{1}$$

Note also:

$$\boldsymbol{\kappa}_2^2 - \boldsymbol{\kappa}_2 - 2\cdot\mathbf{1} = 0$$

Spectral analysis of non-commutative "Group-table Hamiltonian"

1st Step: Spectral resolution of D_3-Center (Class algebra of D_3)

$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

Each class-sum $\underline{\kappa}_k$ commutes with all of D_3.

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κ_2	$2\kappa_1 + \kappa_2$	$2\kappa_3$
κ_3	$2\kappa_3$	$3\kappa_1 + 3\kappa_2$

Class products give spectral polynomial and all-commuting projectors $\mathbf{P}^{(\alpha)}$

$\leftarrow \boldsymbol{\kappa}_3^2 = 3\cdot\boldsymbol{\kappa}_2 + 3\cdot\mathbf{1}$

$$0 = \kappa_3^3 - 9\kappa_3 = (\kappa_3 - 3\cdot\mathbf{1})(\kappa_3 + 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})$$

Note also:

$$\boldsymbol{\kappa}_2^2 - \boldsymbol{\kappa}_2 - 2\cdot\mathbf{1} = 0$$

$$0 = (\boldsymbol{\kappa}_2 - 2\cdot\mathbf{1})(\boldsymbol{\kappa}_2 + \mathbf{1})$$

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Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes

Non-commutative symmetry expansion and Global-Local solution

Global vs Local symmetry and Mock-Mach principle

Global vs Local matrix duality for D_3

Global vs Local symmetry expansion of D_3 Hamiltonian

1st-Stage spectral decomposition of global/local D_3 Hamiltonian

All-commuting operators and D_3-invariant class algebra

All-commuting projectors and D_3-invariant characters

Group invariant numbers: Centrum, Rank, and Order

*Spectral resolution to **irreducible representations** (or "**irreps**") foretold by **characters** or traces*

Crystal-field splitting: $O(3) \supset D_3$ symmetry reduction and $D^\ell \downarrow D_3$ splitting

Spectral analysis of non-commutative "Group-table Hamiltonian"

1st Step: Spectral resolution of D_3-Center (Class algebra of D_3)

$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

Each class-sum $\underline{\kappa}_k$ commutes with all of D_3.

$\kappa_1 = \mathbf{1}$	$\kappa_2 = \mathbf{r}^1 + \mathbf{r}^2$	$\kappa_3 = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
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Class products give spectral polynomial and all-commuting projectors $\mathbf{P}^{(\alpha)} = \mathbf{P}^{A_1}$, $\mathbf{P}^{A_2}$, and $\mathbf{P}^{E}$

Note also:

$$\kappa_2^2 - \kappa_2 - 2\cdot\mathbf{1} = 0$$

$$0 = (\kappa_2 - 2\cdot\mathbf{1})(\kappa_2 + \mathbf{1})$$

$$0 = \kappa_3^3 - 9\kappa_3 = (\kappa_3 - 3\cdot\mathbf{1})(\kappa_3 + 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})$$

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$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
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$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
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$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
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$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
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Inverse resolution gives D_3 Character Table

$$\mathbf{P}^{A_1} = (\kappa_1 + \kappa_2 + \kappa_3)/6 = (\mathbf{1} + \mathbf{r} + \mathbf{r}^2 + \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3)/6$$

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χ_k^α	χ_1^α	χ_2^α	χ_3^α
$\alpha = A_1$	1	1	1
$\alpha = A_2$	1	1	−1
$\alpha = E$	2	−1	0

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Irreducible ***characters*** *are traces* $\chi_\kappa^{(\alpha)} = Tr\, D^{(\alpha)}(\mathbf{r}_\kappa)$ *of irreducible representations* $D^{(\alpha)}(\mathbf{r}_\kappa)$

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3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

D_3 and C_{3v} are isomorphic ($D_3 \sim C_{3v}$ share product table)

Deriving $D_3 \sim C_{3v}$ products:

By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$

By nomograms based on U(2) Hamilton-turns

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Global vs Local symmetry and Mock-Mach principle

Global vs Local matrix duality for D_3

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1st-Step in spectral analysis of D_3 "group-table" Hamiltonian: Algebra of D_3 Center(Classes)

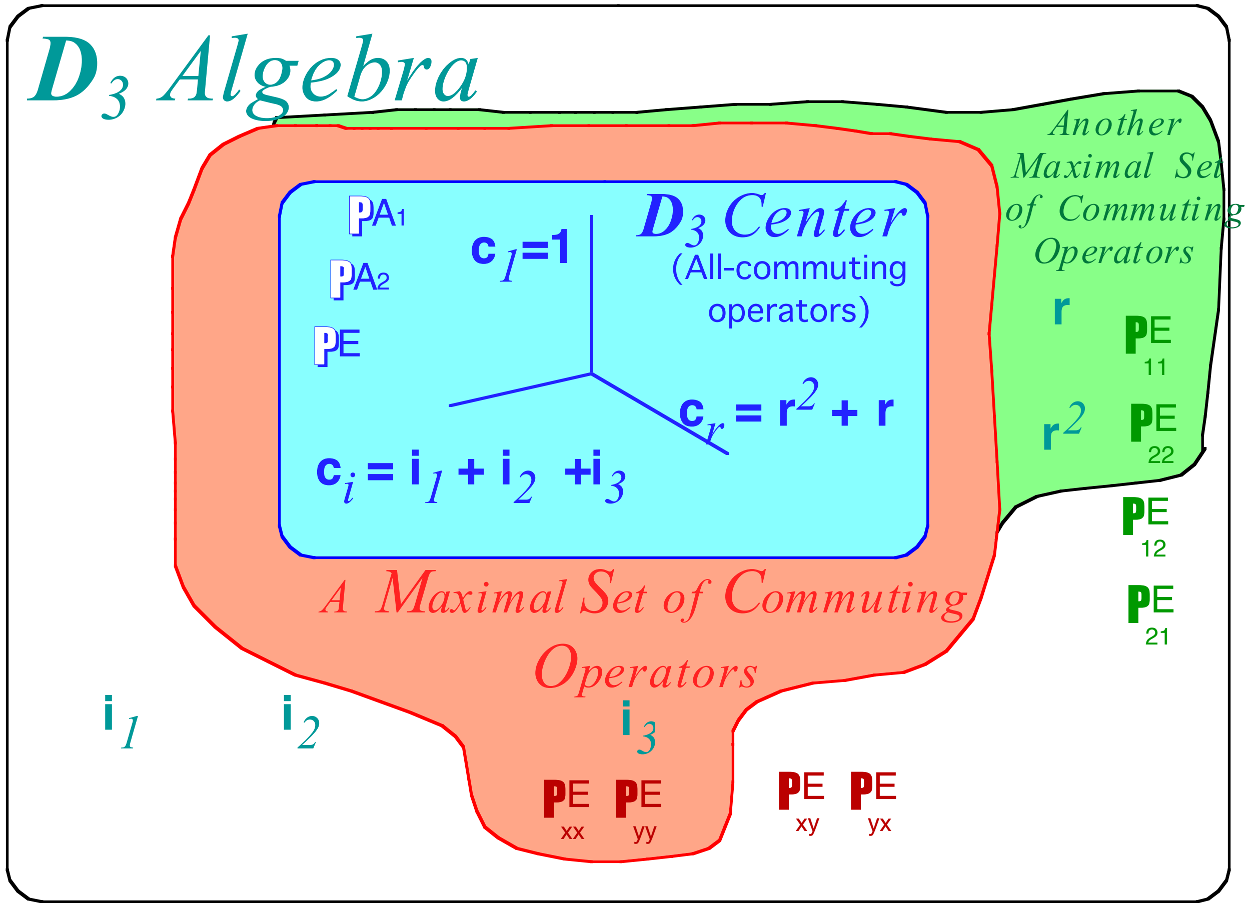
All-commuting operators and D_3-invariant class algebra

All-commuting projectors and D_3-invariant characters

→ *Group invariant numbers: Centrum, Rank, and Order* ←

*Spectral resolution to **irreducible representations** (or "**irreps**") foretold by **characters** or traces*

Crystal-field splitting: $O(3) \supset D_3$ symmetry reduction and $D^{\ell} \downarrow D_3$ splitting



(Fig. 15.2.1 QTCA)

D_3 $\kappa=$	$\mathbf{1}$	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
$\mathbf{P}^{A_1}=$	1	1	1	/6
$\mathbf{P}^{A_2}=$	1	1	-1	/6
$\mathbf{P}^{E}=$	2	-1	0	/3

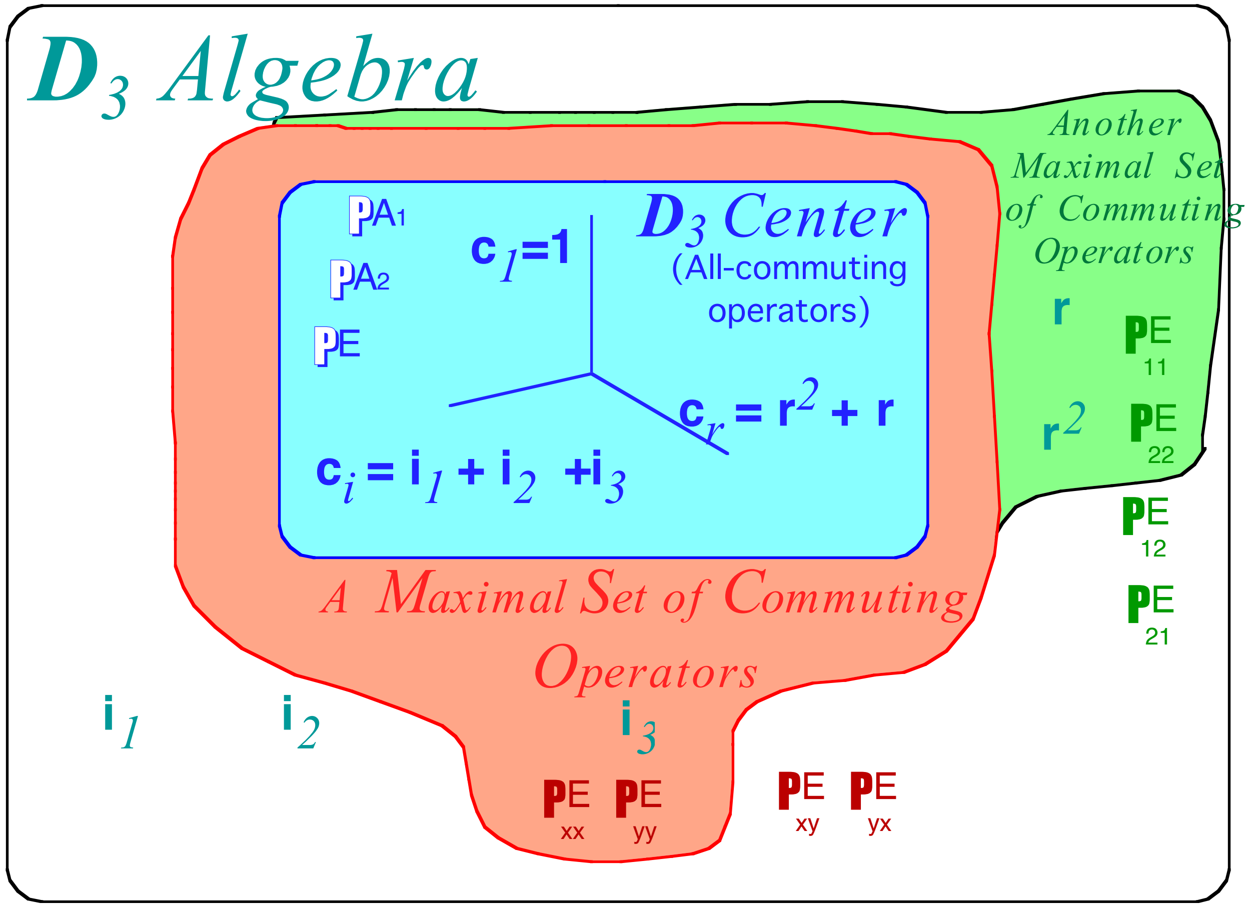
Important invariant numbers or "characters"

$\ell^\alpha=$ Irreducible representation (irrep) *dimension* or level *degeneracy*
For symmetry group or algebra G

Centrum: $\kappa(G)=\Sigma_{irrep(\alpha)}(\ell^\alpha)^0$ =Number of classes, invariants, irrep types, *all-commuting* ops

Rank: $\rho(G)=\Sigma_{irrep(\alpha)}(\ell^\alpha)^1$ =Number of irrep idempotents $\mathbf{P}^{(\alpha)}_{n,n}$, *mutually-commuting* ops

Order: $^o(G)=\Sigma_{irrep(\alpha)}(\ell^\alpha)^2$ =*Total* number of irrep projectors $\mathbf{P}^{(\alpha)}_{m,n}$ or symmetry ops



D_3 $\kappa=$	$\mathbf{1}$	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
$\mathbf{P}^{A_1}=$	1	1	1	/6
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$\mathbf{P}^{E}=$	2	-1	0	/3

Important invariant numbers or "characters"

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Centrum: $\kappa(G)=\Sigma_{irrep(\alpha)}\,(\ell^{\alpha})^{0}$ =Number of classes, invariants, irrep types, *all-commuting* ops

Rank: $\rho(G)=\Sigma_{irrep(\alpha)}\,(\ell^{\alpha})^{1}$ =Number of irrep idempotents $\mathbf{P}^{(\alpha)}_{n,n}$, *mutually-commuting* ops

Order: $^{\circ}(G)=\Sigma_{irrep(\alpha)}\,(\ell^{\alpha})^{2}$ =*Total* number of irrep projectors $\mathbf{P}^{(\alpha)}_{m,n}$ or symmetry ops

$$\kappa(D_3)=(1)^0+(1)^0+(2)^0=3$$
$$\rho(D_3)=(1)^1+(1)^1+(2)^1=4$$
$$^{\circ}(D_3)=(1)^2+(1)^2+(2)^2=6$$

3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

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*Spectral resolution to **irreducible representations** (or "**irreps**") foretold by **characters** or traces*

Crystal-field splitting: $O(3) \supset D_3$ symmetry reduction and $D^{\ell} \downarrow D_3$ splitting

Spectral resolution to ***irreducible representations*** *(or "****irreps****") foretold by* ***characters*** *or* *<u>traces</u>*

$$
\begin{array}{c}
1\\ r^1\\ r^2\\ i_1\\ i_2\\ i_3
\end{array}
\overset{R^G(\mathbf{1})=}{\begin{pmatrix}
1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1
\end{pmatrix}},
\overset{R^G(\mathbf{r})=}{\begin{pmatrix}
\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&1&\cdot&\cdot
\end{pmatrix}},
\overset{R^G(\mathbf{r}^2)=}{\begin{pmatrix}
\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot
\end{pmatrix}},
\overset{R^G(\mathbf{i}_1)=}{\begin{pmatrix}
\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1\\
1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot
\end{pmatrix}},
\overset{R^G(\mathbf{i}_2)=}{\begin{pmatrix}
\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}},
\overset{R^G(\mathbf{i}_3)=}{\begin{pmatrix}
\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}}
$$

$$
\mathbf{P}^{A_1}=(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3)/6=(\mathbf{1}+\mathbf{r}+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6\Rightarrow R(\mathbf{P}^{A_1})=\begin{pmatrix}
1&1&1&1&1&1\\
1&1&1&1&1&1\\
1&1&1&1&1&1\\
1&1&1&1&1&1\\
1&1&1&1&1&1\\
1&1&1&1&1&1
\end{pmatrix}/6 \qquad Trace R(\mathbf{P}^{A_1})=1
$$

Spectral resolution to ***irreducible representations*** *(or "****irreps****") is foretold by* ***characters*** *or* *<u>traces</u>*

$$
\begin{array}{c}
1\\ r^1\\ r^2\\ i_1\\ i_2\\ i_3
\end{array}
\overset{R^G(\mathbf{1})=}{\begin{pmatrix}
1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1
\end{pmatrix}},
\overset{R^G(\mathbf{r})=}{\begin{pmatrix}
\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&1&\cdot&\cdot
\end{pmatrix}},
\overset{R^G(\mathbf{r}^2)=}{\begin{pmatrix}
\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot
\end{pmatrix}},
\overset{R^G(\mathbf{i}_1)=}{\begin{pmatrix}
\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1\\
1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot
\end{pmatrix}},
\overset{R^G(\mathbf{i}_2)=}{\begin{pmatrix}
\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}},
\overset{R^G(\mathbf{i}_3)=}{\begin{pmatrix}
\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}}
$$

$$
\mathbf{P}^{A_1}=(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3)/6=(\mathbf{1}+\mathbf{r}+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6 \Rightarrow R(\mathbf{P}^{A_1})=\begin{pmatrix}
1&1&1&1&1&1\\
1&1&1&1&1&1\\
1&1&1&1&1&1\\
1&1&1&1&1&1\\
1&1&1&1&1&1\\
1&1&1&1&1&1
\end{pmatrix}/6 \qquad Trace R(\mathbf{P}^{A_1})=1
$$

So: $R(\mathbf{P}^{A_1})$ reduces to:
$$
\begin{pmatrix}
1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}
$$

Spectral resolution to ***irreducible representations*** *(or "****irreps****") is foretold by* ***characters*** *or* *<u>traces</u>*

$$
\begin{array}{c} 1 \\ r^1 \\ r^2 \\ i_1 \\ i_2 \\ i_3 \end{array}
\overset{R^G(\mathbf{1})=}{\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}},
\overset{R^G(\mathbf{r})=}{\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix}},
\overset{R^G(\mathbf{r}^2)=}{\begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix}},
\overset{R^G(\mathbf{i}_1)=}{\begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix}},
\overset{R^G(\mathbf{i}_2)=}{\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}},
\overset{R^G(\mathbf{i}_3)=}{\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}}
$$

$$
\mathbf{P}^{A_1}=(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3)/6=(\mathbf{1}+\mathbf{r}+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6 \Rightarrow R(\mathbf{P}^{A_1})=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}/6 \qquad TraceR(\mathbf{P}^{A_1})=1
$$

So: $R(\mathbf{P}^{A_1})$ reduces to:
$$
\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}
$$

$$
\mathbf{P}^{A_2}=(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2-\boldsymbol{\kappa}_3)/6=(\mathbf{1}+\mathbf{r}+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6 \Rightarrow R(\mathbf{P}^{A_2})=\begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \end{pmatrix}/6 \qquad TraceR(\mathbf{P}^{A_2})=1
$$

So: $R(\mathbf{P}^{A_2})$ reduces to:
$$
\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}
$$

*Spectral resolution to **irreducible representations** (or "**irreps**") is foretold by **characters** or <u>traces</u>*

$$
\begin{array}{c} 1 \\ r^1 \\ r^2 \\ i_1 \\ i_2 \\ i_3 \end{array}
\overset{R^G(\mathbf{1})=}{\begin{pmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}},
\overset{R^G(\mathbf{r})=}{\begin{pmatrix}
\cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot
\end{pmatrix}},
\overset{R^G(\mathbf{r}^2)=}{\begin{pmatrix}
\cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot
\end{pmatrix}},
\overset{R^G(\mathbf{i}_1)=}{\begin{pmatrix}
\cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot
\end{pmatrix}},
\overset{R^G(\mathbf{i}_2)=}{\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}},
\overset{R^G(\mathbf{i}_3)=}{\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}}
$$

$$
\mathbf{P}^{A_1}=(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3)/6=(\mathbf{1}+\mathbf{r}+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6 \Rightarrow R(\mathbf{P}^{A_1})=\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}/6 \qquad TraceR(\mathbf{P}^{A_1})=1
$$

So: $R(\mathbf{P}^{A_1})$ reduces to: $\begin{pmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}$

$$
\mathbf{P}^{A_2}=(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2-\boldsymbol{\kappa}_3)/6=(\mathbf{1}+\mathbf{r}+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6 \Rightarrow R(\mathbf{P}^{A_2})=\begin{pmatrix}
1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 \\
-1 & -1 & -1 & 1 & 1 & 1 \\
-1 & -1 & -1 & 1 & 1 & 1 \\
-1 & -1 & -1 & 1 & 1 & 1
\end{pmatrix}/6 \qquad TraceR(\mathbf{P}^{A_2})=1
$$

So: $R(\mathbf{P}^{A_2})$ reduces to: $\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}$

$$
\mathbf{P}^{E}=(2\boldsymbol{\kappa}_1-\boldsymbol{\kappa}_2+0)/3=(2\mathbf{1}-\mathbf{r}-\mathbf{r}^2+0+0+0)/3 \Rightarrow R(\mathbf{P}^{E})=\begin{pmatrix}
2 & -1 & -1 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 \\
-1 & -1 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & -1 & -1 \\
0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & -1 & -1 & 1
\end{pmatrix}/3 \qquad TraceR(\mathbf{P}^{E})=4
$$

So: $R(\mathbf{P}^{E})$ reduces to: $\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}$

3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

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Spectral resolution to ***irreducible representations*** *(or "**irreps**") foretold by* ***characters*** *or traces*

Crystal-field splitting: $O(3) \supset D_3$ symmetry reduction and $D^{\ell} \downarrow D_3$ splitting

Spectral resolution to ***irreducible representations*** *(or "****irreps****") is foretold by* ***characters*** *or* *<u>traces</u>*

$$
\begin{array}{c}
1\\ r^1\\ r^2\\ i_1\\ i_2\\ i_3
\end{array}
\quad
R^G(\mathbf{1})=\begin{pmatrix}
1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1
\end{pmatrix},
R^G(\mathbf{r})=\begin{pmatrix}
\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&1&\cdot&\cdot
\end{pmatrix},
R^G(\mathbf{r}^2)=\begin{pmatrix}
\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot
\end{pmatrix},
$$

$$
R^G(\mathbf{i}_1)=\begin{pmatrix}
\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1\\
1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot
\end{pmatrix},
R^G(\mathbf{i}_2)=\begin{pmatrix}
\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot
\end{pmatrix},
R^G(\mathbf{i}_3)=\begin{pmatrix}
\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}
$$

$$
R(\mathbf{P}^{A_1})=\begin{pmatrix}
1&1&1&1&1&1\\
1&1&1&1&1&1\\
1&1&1&1&1&1\\
1&1&1&1&1&1\\
1&1&1&1&1&1\\
1&1&1&1&1&1
\end{pmatrix}/6 \;\Rightarrow\; TraceR(\mathbf{P}^{A_1})=1 \qquad \text{So: } R(\mathbf{P}^{A_1}\mathbf{g}) \text{ reduces to:}
\begin{pmatrix}
D^{A_1}(\mathbf{g})&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}
$$

$$
R(\mathbf{P}^{A_2})=\begin{pmatrix}
1&1&1&-1&-1&-1\\
1&1&1&-1&-1&-1\\
1&1&1&-1&-1&-1\\
-1&-1&-1&1&1&1\\
-1&-1&-1&1&1&1\\
-1&-1&-1&1&1&1
\end{pmatrix}/6 \;\Rightarrow\; TraceR(\mathbf{P}^{A_2})=1
$$

$$
R(\mathbf{P}^{E})=\begin{pmatrix}
2&-1&-1&0&0&0\\
-1&2&-1&0&0&0\\
-1&-1&2&0&0&0\\
0&0&0&2&-1&-1\\
0&0&0&-1&2&-1\\
0&0&0&-1&-1&1
\end{pmatrix}/3 \;\Rightarrow\; TraceR(\mathbf{P}^{E})=4
$$

Spectral resolution to ***irreducible representations*** *(or "****irreps****") is foretold by* ***characters*** *or* *traces*

$$
\begin{array}{c}
\\ 1 \\ r^1 \\ r^2 \\ i_1 \\ i_2 \\ i_3
\end{array}
\begin{array}{c}
R^G(\mathbf{1}) = \\
\begin{pmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}
\end{array},
\begin{array}{c}
R^G(\mathbf{r}) = \\
\begin{pmatrix}
\cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot
\end{pmatrix}
\end{array},
\begin{array}{c}
R^G(\mathbf{r}^2) = \\
\begin{pmatrix}
\cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot
\end{pmatrix}
\end{array},
\begin{array}{c}
R^G(\mathbf{i}_1) = \\
\begin{pmatrix}
\cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot
\end{pmatrix}
\end{array},
\begin{array}{c}
R^G(\mathbf{i}_2) = \\
\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}
\end{array},
\begin{array}{c}
R^G(\mathbf{i}_3) = \\
\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}
\end{array}
$$

$$
R(\mathbf{P}^{A_1}) = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix} /6 \quad \Rightarrow \quad Trace R(\mathbf{P}^{A_1}) = 1 \qquad \text{So: } R(\mathbf{P}^{A_1}\mathbf{g}) \text{ reduces to: }
\begin{pmatrix}
D^{A_1}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}
$$

$$
R(\mathbf{P}^{A_2}) = \begin{pmatrix}
1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 \\
-1 & -1 & -1 & 1 & 1 & 1 \\
-1 & -1 & -1 & 1 & 1 & 1 \\
-1 & -1 & -1 & 1 & 1 & 1
\end{pmatrix} /6 \quad \Rightarrow \quad Trace R(\mathbf{P}^{A_2}) = 1 \qquad \text{So: } R(\mathbf{P}^{A_2}\mathbf{g}) \text{ reduces to: }
\begin{pmatrix}
D^{A_2}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}
$$

$$
R(\mathbf{P}^{E}) = \begin{pmatrix}
2 & -1 & -1 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 \\
-1 & -1 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & -1 & -1 \\
0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & -1 & -1 & 1
\end{pmatrix} /3 \quad \Rightarrow \quad Trace R(\mathbf{P}^{E}) = 4
$$

Spectral resolution to ***irreducible representations*** *(or "****irreps****") is foretold by* ***characters*** *or* <u>*traces*</u>

$$R^G(\mathbf{1})=\ R^G(\mathbf{r})=\ R^G(\mathbf{r}^2)=\ R^G(\mathbf{i}_1)=\ R^G(\mathbf{i}_2)=\ R^G(\mathbf{i}_3)=$$

$$\begin{array}{c} 1 \\ r^1 \\ r^2 \\ i_1 \\ i_2 \\ i_3 \end{array}
\begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1 \end{pmatrix},
\begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot \end{pmatrix},
\begin{pmatrix} \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot \end{pmatrix},
\begin{pmatrix} \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot \end{pmatrix},
\begin{pmatrix} \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot \end{pmatrix},
\begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}$$

$$R(\mathbf{P}^{A_1})=\begin{pmatrix} 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1 \end{pmatrix}/6 \Rightarrow \quad TraceR(\mathbf{P}^{A_1})=1 \quad \text{So: } R(\mathbf{P}^{A_1}\mathbf{g}) \text{ reduces to: } \begin{pmatrix} D^{A_1}(\mathbf{g})&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}$$

$$R(\mathbf{P}^{A_2})=\begin{pmatrix} 1&1&1&-1&-1&-1\\ 1&1&1&-1&-1&-1\\ 1&1&1&-1&-1&-1\\ -1&-1&-1&1&1&1\\ -1&-1&-1&1&1&1\\ -1&-1&-1&1&1&1 \end{pmatrix}/6 \Rightarrow \quad TraceR(\mathbf{P}^{A_2})=1 \quad \text{So: } R(\mathbf{P}^{A_2}\mathbf{g}) \text{ reduces to: } \begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&D^{A_2}(\mathbf{g})&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}$$

$$R(\mathbf{P}^{E})=\begin{pmatrix} 2&-1&-1&0&0&0\\ -1&2&-1&0&0&0\\ -1&-1&2&0&0&0\\ 0&0&0&2&-1&-1\\ 0&0&0&-1&2&-1\\ 0&0&0&-1&-1&1 \end{pmatrix}/3 \Rightarrow \quad TraceR(\mathbf{P}^{E})=4 \quad \text{So: } R(\mathbf{P}^{E}\mathbf{g}) \text{ reduces to: } \begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&D^E_{11}&D^E_{12}&\cdot&\cdot\\ \cdot&\cdot&D^E_{21}&D^E_{22}&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&D^E_{11}&D^E_{12}\\ \cdot&\cdot&\cdot&\cdot&D^E_{21}&D^E_{22} \end{pmatrix}$$

3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

D_3 and C_{3v} are isomorphic ($D_3 \sim C_{3v}$ share product table)

Deriving $D_3 \sim C_{3v}$ products:

By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$

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Non-commutative symmetry expansion and Global-Local solution

Global vs Local symmetry and Mock-Mach principle

Global vs Local matrix duality for D_3

Global vs Local symmetry expansion of D_3 Hamiltonian

1st-Step in spectral analysis of D_3 "group-table" Hamiltonian: Algebra of D_3 Center(Classes)

All-commuting operators and D_3-invariant class algebra

All-commuting projectors and D_3-invariant characters

Group invariant numbers: Centrum, Rank, and Order

*Spectral resolution to **irreducible representations** (or "**irreps**") foretold by **characters** or traces*

Crystal-field splitting: $O(3) \supset D_3$ symmetry reduction and $D^{\ell} \downarrow D_3$ splitting

Spectral resolution to ***irreducible representations*** *(or "****irreps****") foretold by* ***characters*** *or <u>traces</u>*

$$R^G(\mathbf{1})= \begin{array}{c} 1\\ r^1\\ r^2\\ i_1\\ i_2\\ i_3 \end{array}\begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1 \end{pmatrix},\quad R^G(\mathbf{r})=\begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot \end{pmatrix},\quad R^G(\mathbf{r}^2)=\begin{pmatrix} \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot \end{pmatrix},$$

$$R^G(\mathbf{i}_1)=\begin{pmatrix} \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot \end{pmatrix},\quad R^G(\mathbf{i}_2)=\begin{pmatrix} \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot \end{pmatrix},\quad R^G(\mathbf{i}_3)=\begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}$$

$Trace R(\mathbf{P}^{A_1})=1$ So: $R(\mathbf{P}^{A_1}\mathbf{g})$ reduces to:

$$\begin{pmatrix} D^{A_1}(\mathbf{g})&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}$$

$Trace R(\mathbf{P}^{A_2})=1$ So: $R(\mathbf{P}^{A_2}\mathbf{g})$ reduces to:

$$\begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&D^{A_2}(\mathbf{g})&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}$$

$Trace R(\mathbf{P}^{E})=4$ So: $R(\mathbf{P}^{E}\mathbf{g})$ reduces to:

$$\begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&D^E_{11}&D^E_{12}&\cdot&\cdot\\ \cdot&\cdot&D^E_{21}&D^E_{22}&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&D^E_{11}&D^E_{12}\\ \cdot&\cdot&\cdot&\cdot&D^E_{21}&D^E_{22} \end{pmatrix}$$

So: $R(\mathbf{g})$ reduces to:

$$\begin{pmatrix} D^{A_1}(\mathbf{g})&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&D^{A_2}(\mathbf{g})&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&D^E_{11}&D^E_{12}&\cdot&\cdot\\ \cdot&\cdot&D^E_{21}&D^E_{22}&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&D^E_{11}&D^E_{12}\\ \cdot&\cdot&\cdot&\cdot&D^E_{21}&D^E_{22} \end{pmatrix}$$

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Spectral resolution to ***irreducible representations*** *(or "****ireps****") foretold by* ***characters*** *or* *traces*

$$
\begin{array}{c} 1 \\ r^1 \\ r^2 \\ i_1 \\ i_2 \\ i_3 \end{array}
\overset{R^G(\mathbf{1})=}{\left(\begin{array}{cccccc}
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array}\right)},
\overset{R^G(\mathbf{r})=}{\left(\begin{array}{cccccc}
\cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot
\end{array}\right)},
\overset{R^G(\mathbf{r}^2)=}{\left(\begin{array}{cccccc}
\cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot
\end{array}\right)},
\overset{R^G(\mathbf{i}_1)=}{\left(\begin{array}{cccccc}
\cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot
\end{array}\right)},
\overset{R^G(\mathbf{i}_2)=}{\left(\begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot
\end{array}\right)},
\overset{R^G(\mathbf{i}_3)=}{\left(\begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)}
$$

$\{R^G(\mathbf{g})\}$ has lots of empty space and looks like it could be reduced.

But, $\{R^G(\mathbf{g})\}$ cannot be diagonalized all-at-once. (Not all $\mathbf{g}$ commute.)

Nevertheless, $\{R^G(\mathbf{g})\}$ can be *block-diagonalized*
all-at-once into ***"ireps"*** A_1, A_2, and E_1

$R(\mathbf{g})$ reduces to:

$$
\left(\begin{array}{cccccc}
D^{A_1}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & D^{A_2}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & D^E_{11} & D^E_{12} & \cdot & \cdot \\
\cdot & \cdot & D^E_{21} & D^E_{22} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & D^E_{11} & D^E_{12} \\
\cdot & \cdot & \cdot & \cdot & D^E_{21} & D^E_{22}
\end{array}\right)
$$

*Spectral resolution to **irreducible representations** (or "**ireps**") foretold by **characters** or traces*

$$
\begin{array}{c}
1 \\ r^1 \\ r^2 \\ i_1 \\ i_2 \\ i_3
\end{array}
\underbrace{\begin{pmatrix}
1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1
\end{pmatrix}}_{R^G(\mathbf{1})=},
\underbrace{\begin{pmatrix}
\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&1&\cdot&\cdot
\end{pmatrix}}_{R^G(\mathbf{r})=},
\underbrace{\begin{pmatrix}
\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot
\end{pmatrix}}_{R^G(\mathbf{r}^2)=},
\underbrace{\begin{pmatrix}
\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1\\
1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot
\end{pmatrix}}_{R^G(\mathbf{i}_1)=},
\underbrace{\begin{pmatrix}
\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}}_{R^G(\mathbf{i}_2)=},
\underbrace{\begin{pmatrix}
\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}}_{R^G(\mathbf{i}_3)=}
$$

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Nevertheless, $\{R^G(\mathbf{g})\}$ can be *block-diagonalized all-at-once* into *"**ireps**"* A_1, A_2, and E_1

We relate traces of $\{R^G(\mathbf{g})\}$:

$(\mathbf{g})=$	$\{\mathbf{1}\}$	$\{\mathbf{r}^1,\mathbf{r}^2\}$	$\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$
$Trace R^G(\mathbf{g})=$	6	0	0

to D_3 character table:

$(\mathbf{g})=$	$\{\mathbf{1}\}$	$\{\mathbf{r}^1,\mathbf{r}^2\}$	$\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$
$\chi^{A_1}(\mathbf{g})=$	1	1	1
$\chi^{A_2}(\mathbf{g})=$	1	1	-1
$\chi^{E_1}(\mathbf{g})=$	2	-1	0

$\chi^{A_1}(\mathbf{g})$	1	1	1
$+\chi^{A_2}(\mathbf{g})$	1	1	-1
$+2\chi^{E_1}(\mathbf{g})$	$2\cdot 2$	$-2\cdot 1$	0
	6	0	0

$R(\mathbf{g})$ reduces to:

$$
\begin{pmatrix}
D^{A_1}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot & \cdot\\
\cdot & D^{A_2}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot\\
\cdot & \cdot & D^E_{11} & D^E_{12} & \cdot & \cdot\\
\cdot & \cdot & D^E_{21} & D^E_{22} & \cdot & \cdot\\
\cdot & \cdot & \cdot & \cdot & D^E_{11} & D^E_{12}\\
\cdot & \cdot & \cdot & \cdot & D^E_{21} & D^E_{22}
\end{pmatrix}
$$

Spectral resolution to ***irreducible representations*** *(or "****ireps****") foretold by* ***characters*** *or* *traces*

$$R^G(\mathbf{1})=\quad R^G(\mathbf{r})=\quad R^G(\mathbf{r}^2)=\quad R^G(\mathbf{i}_1)=\quad R^G(\mathbf{i}_2)=\quad R^G(\mathbf{i}_3)=$$

$$\begin{array}{c} 1 \\ r^1 \\ r^2 \\ i_1 \\ i_2 \\ i_3 \end{array}
\begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&1&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1 \end{pmatrix},
\begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&1&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1 \\ \cdot&\cdot&\cdot&1&\cdot&\cdot \end{pmatrix},
\begin{pmatrix} \cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1 \\ \cdot&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&1&\cdot \end{pmatrix},
\begin{pmatrix} \cdot&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&1&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1 \\ 1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot \end{pmatrix},
\begin{pmatrix} \cdot&\cdot&\cdot&\cdot&1&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1 \\ \cdot&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&1&\cdot&\cdot&\cdot&\cdot \end{pmatrix},
\begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot&1 \\ \cdot&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&1&\cdot \\ \cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}$$

$\{R^G(\mathbf{g})\}$ has lots of empty space and looks like it could be reduced.

But, $\{R^G(\mathbf{g})\}$ cannot be diagonalized all-at-once. (Not all $\mathbf{g}$ commute.)

Nevertheless, $\{R^G(\mathbf{g})\}$ can be *block-diagonalized all-at-once* into ***"ireps"*** A_1, A_2, and E_1

We relate traces of $\{R^G(\mathbf{g})\}$:

$(\mathbf{g})=$	$\{\mathbf{1}\}$	$\{\mathbf{r}^1,\mathbf{r}^2\}$	$\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$
$Trace R^G(\mathbf{g})=$	6	0	0

to D_3 character table:

$(\mathbf{g})=$	$\{\mathbf{1}\}$	$\{\mathbf{r}^1,\mathbf{r}^2\}$	$\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$
$\chi^{A_1}(\mathbf{g})=$	1	1	1
$\chi^{A_2}(\mathbf{g})=$	1	1	-1
$\chi^{E_1}(\mathbf{g})=$	2	-1	0

$\chi^{A_1}(\mathbf{g})$	1	1	1
$+\chi^{A_2}(\mathbf{g})$	1	1	-1
$+2\chi^{E_1}(\mathbf{g})$	$2\cdot 2$	$-2\cdot 1$	0
	6	0	0

$R(\mathbf{g})$ reduces to:

$$\begin{pmatrix} D^{A_1}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & D^{A_2}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & D^{E}_{11} & D^{E}_{12} & \cdot & \cdot \\ \cdot & \cdot & D^{E}_{21} & D^{E}_{22} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & D^{E}_{11} & D^{E}_{12} \\ \cdot & \cdot & \cdot & \cdot & D^{E}_{21} & D^{E}_{22} \end{pmatrix}$$

So $\{R^G(\mathbf{g})\}$ can be *block-diagonalized* into a ***direct sum*** $\oplus$ of ***"ireps"*** $R^G(\mathbf{g})=D^{A_1}(\mathbf{g})\oplus D^{A_2}(\mathbf{g})\oplus 2D^{E_1}(\mathbf{g})$

3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

D_3 and C_{3v} are isomorphic ($D_3 \sim C_{3v}$ share product table)

Deriving $D_3 \sim C_{3v}$ products:

By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$

By nomograms based on U(2) Hamilton-turns

Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes

Non-commutative symmetry expansion and Global-Local solution

Global vs Local symmetry and Mock-Mach principle

Global vs Local matrix duality for D_3

Global vs Local symmetry expansion of D_3 Hamiltonian

1st-Step in spectral analysis of D_3 "group-table" Hamiltonian: Algebra of D_3 Center(Classes)

All-commuting operators and D_3-invariant class algebra

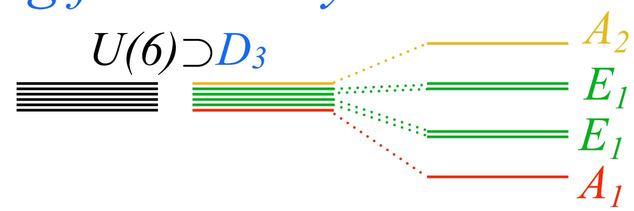
All-commuting projectors and D_3-invariant characters

Group invariant numbers: Centrum, Rank, and Order

*Spectral resolution to **irreducible representations** (or "**irreps**") foretold by **characters** or <u>traces</u>*

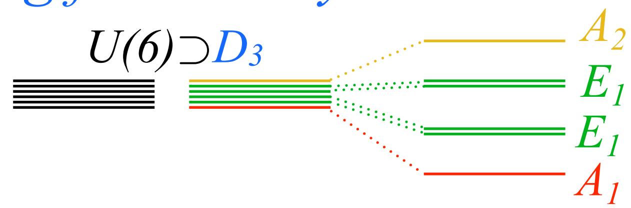
→ *Crystal-field splitting: $O(3) \supset D_3$ symmetry reduction and $D^{\ell} \downarrow D_3$ splitting*

Spectral splitting in symmetry breaking foretold by character analysis



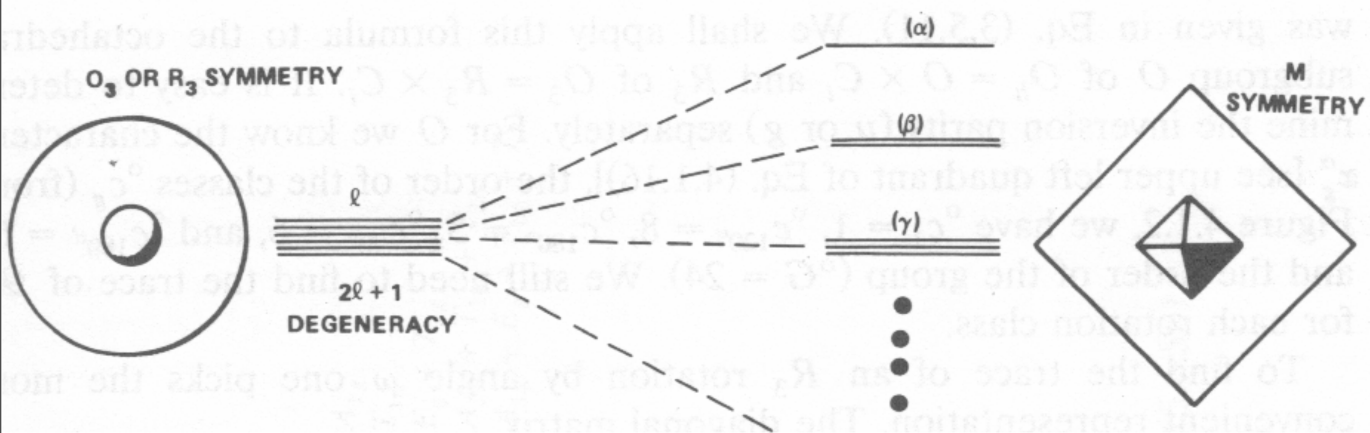
$$R^G(U(6))\downarrow D_3 = D^{A_1}(\mathbf{g}) \oplus D^{A_2}(\mathbf{g}) \oplus 2D^{E_1}(\mathbf{g})$$

Spectral splitting in symmetry breaking foretold by character analysis



$$R^G(U(6))\downarrow D_3 = D^{A_1}(\mathbf{g})\oplus D^{A_2}(\mathbf{g})\oplus 2D^{E_1}(\mathbf{g})$$

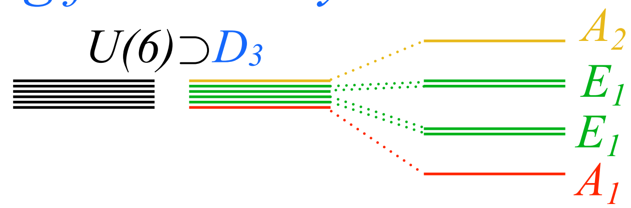
Crystal-field splitting: O(3)⊃D₃ symmetry reduction and $D^{\ell}\downarrow D_3$ splitting



(Fig. 5.6.1 PSDS)

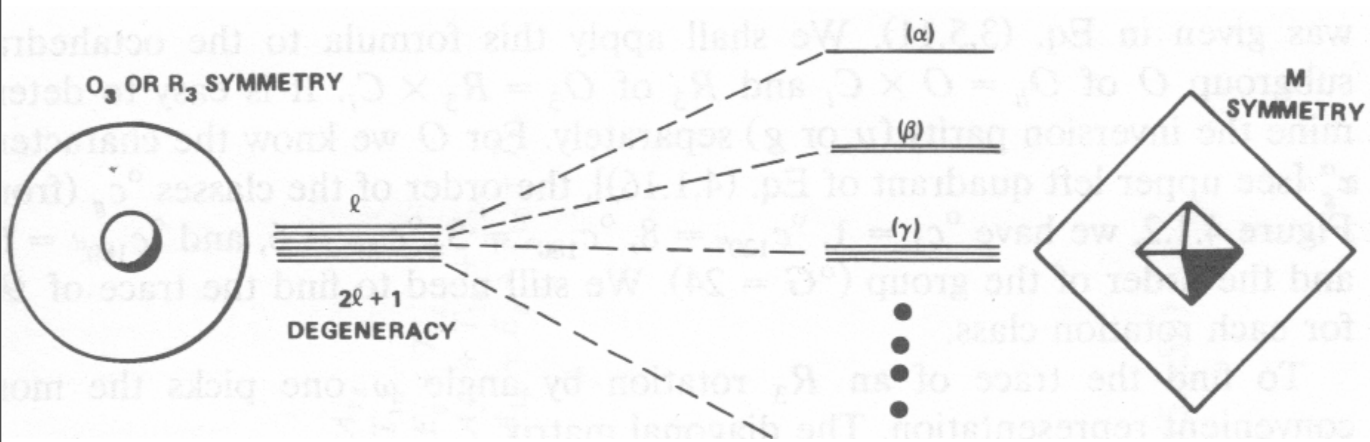
$$D^{\ell}(R) = \begin{pmatrix} D^{\ell}_{\ell,\ell}(R) & \cdots & D_{\ell,-\ell} \\ D_{\ell-1,\ell} & & \\ D_{-\ell,\ell} & \cdots & D_{-\ell,-\ell} \end{pmatrix} \downarrow M \cong \begin{pmatrix} D^{\alpha}(R) & & \\ & D^{\beta}(R) & \\ & & D^{\gamma}(R) \end{pmatrix}$$

Spectral splitting in symmetry breaking foretold by character analysis

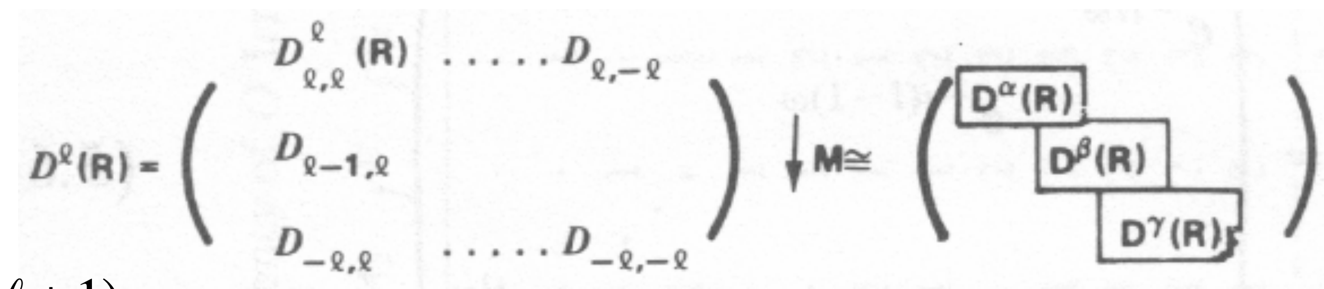


$R^G(U(6))\downarrow D_3 = D^{A_1}(\mathbf{g})\oplus D^{A_2}(\mathbf{g})\oplus 2D^{E_1}(\mathbf{g})$

Crystal-field splitting: $O(3)\supset D_3$ symmetry reduction and $D^\ell\downarrow D_3$ splitting



(Fig. 5.6.1 PSDS)

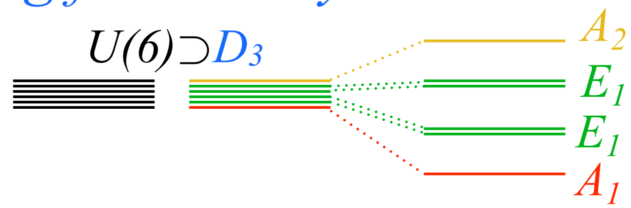


Use $R(3)\sim U(2)$ character formula: $$\chi^\ell\left(\frac{2\pi}{n}\right)=\frac{\sin\frac{(2\ell+1)\pi}{n}}{\sin\frac{\pi}{n}}$$

$R(3)$ character where: $2\ell+1$ is ℓ-orbital dimension

(From Lect. 11 p. 11)

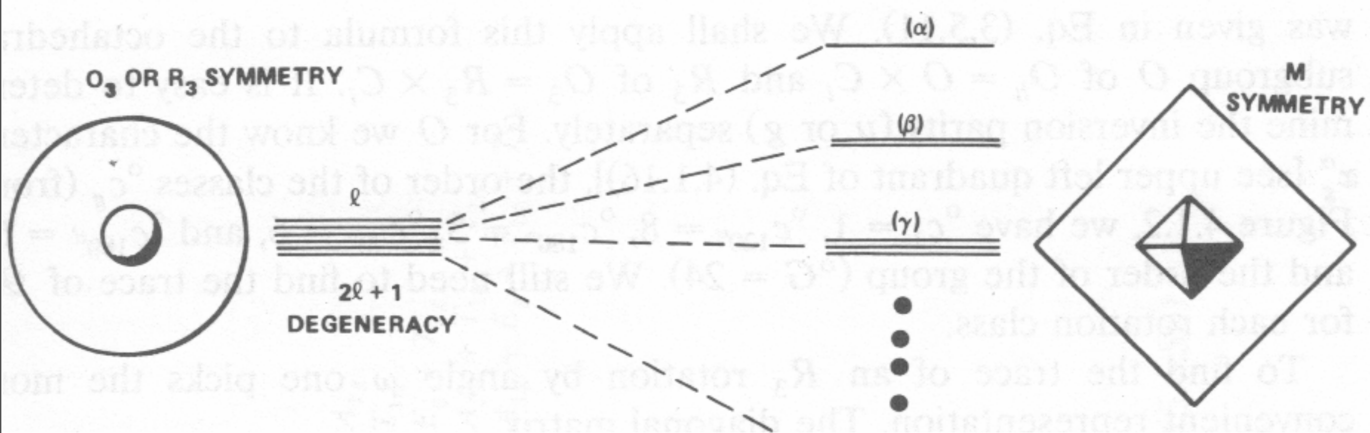
Spectral splitting in symmetry breaking foretold by character analysis

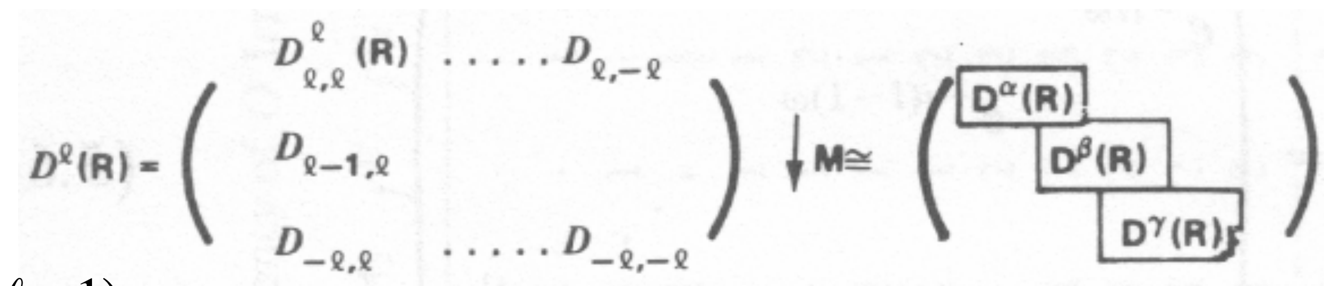


$R^G(U(6))\downarrow D_3 = D^{A_1}(\mathbf{g}) \oplus D^{A_2}(\mathbf{g}) \oplus 2D^{E_1}(\mathbf{g})$

Crystal-field splitting: O(3)⊃D₃ symmetry reduction and $D^\ell \downarrow D_3$ splitting

(Fig. 5.6.1 PSDS)





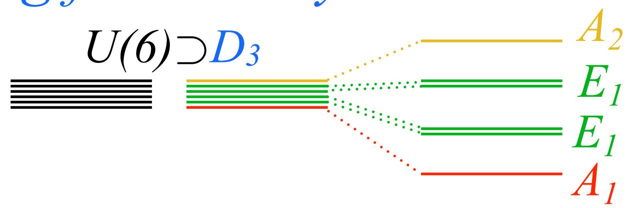
Use R(3)~U(2) character formula: $\chi^\ell\left(\frac{2\pi}{n}\right) = \frac{\sin\frac{(2\ell+1)\pi}{n}}{\sin\frac{\pi}{n}}$

R(3) character where: $2\ell+1$ is ℓ-orbital dimension

(From Lect. 11 p. 11)

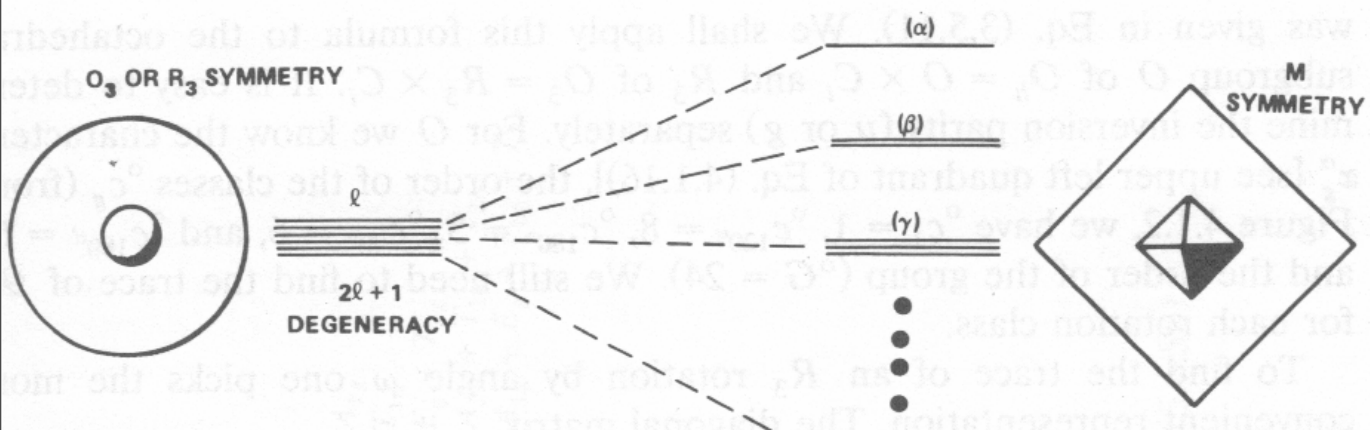
$$\chi^\ell(\Theta) = \frac{\sin(\ell+\frac{1}{2})\Theta}{\sin\frac{\Theta}{2}}$$

Spectral splitting in symmetry breaking foretold by character analysis



$R^G(U(6))\downarrow D_3 = D^{A_1}(\mathbf{g}) \oplus D^{A_2}(\mathbf{g}) \oplus 2D^{E_1}(\mathbf{g})$

Crystal-field splitting: $O(3)\supset D_3$ symmetry reduction and $D^\ell \downarrow D_3$ splitting



(Fig. 5.6.1 PSDS)

$$D^\ell(R) = \begin{pmatrix} D^\ell_{\ell,\ell}(R) & \cdots\cdots & D_{\ell,-\ell} \\ D_{\ell-1,\ell} & & \\ D_{-\ell,\ell} & \cdots\cdots & D_{-\ell,-\ell} \end{pmatrix} \Big\downarrow M \cong \begin{pmatrix} \boxed{D^\alpha(R)} & & \\ & \boxed{D^\beta(R)} & \\ & & \boxed{D^\gamma(R)} \end{pmatrix}$$

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R(3) character where: $2\ell+1$ is ℓ-orbital dimension

(From Lect. 11 p. 11)

$$\chi^\ell(\Theta) = \frac{\sin(\ell+\frac{1}{2})\Theta}{\sin\frac{\Theta}{2}}$$

$\chi^\ell(\Theta)$	$\Theta=0$	$\frac{2\pi}{3}$	π
$\ell=0$	1	1	1
1	3	0	−1
2	5	−1	1
3	7	1	−1
4	9	0	1
5	11	−1	−1
6	13	1	1
7	15	0	−1

...and D_3 character table:

$(\mathbf{g})=$	$\{\mathbf{1}\}$	$\{\mathbf{r}^1,\mathbf{r}^2\}$	$\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$
$\chi^{A_1}(\mathbf{g})=$	1	1	1
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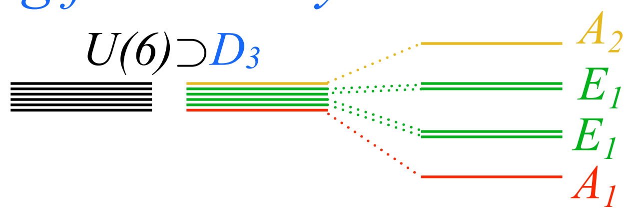
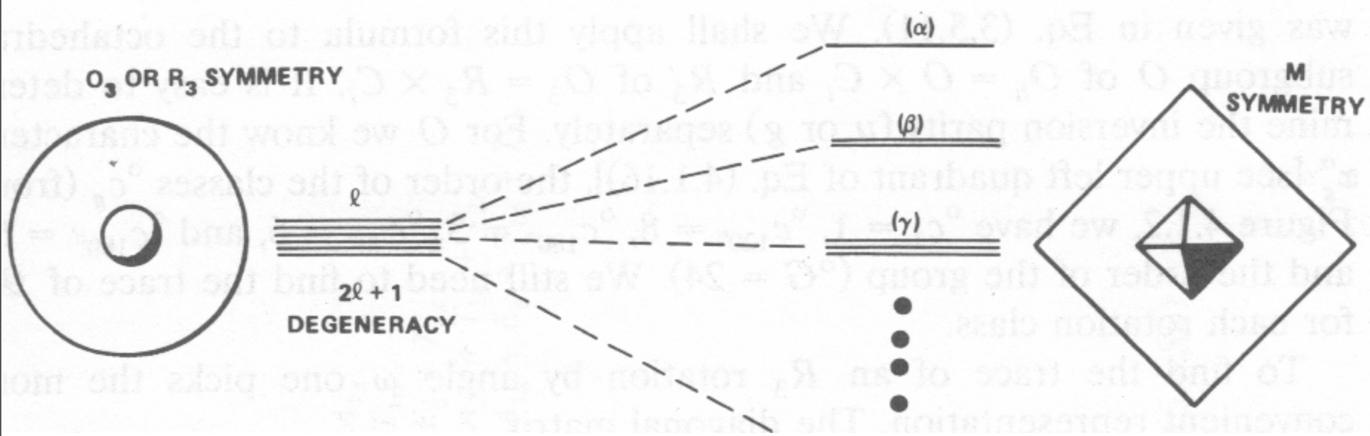
$\chi^{E_1}(\mathbf{g})=$	2	−1	0

Spectral splitting in symmetry breaking foretold by character analysis

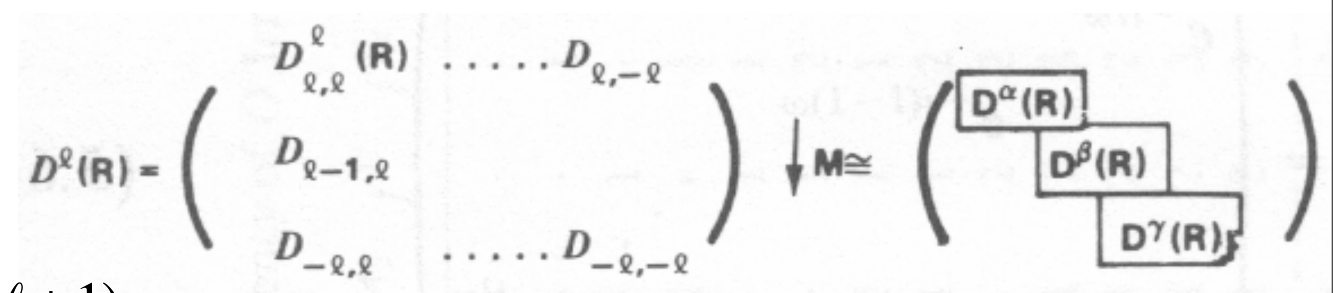


$$R^G(U(6))\downarrow D_3 = D^{A_1}(\mathbf{g}) \oplus D^{A_2}(\mathbf{g}) \oplus 2D^{E_1}(\mathbf{g})$$

Crystal-field splitting: O(3)⊃D₃ symmetry reduction and $D^\ell\downarrow D_3$ splitting



(Fig. 5.6.1 PSDS)



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(From Lect. 11 p. 11)

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$\ell=0$	1	1	1
1	3	0	−1
2	5	−1	1
3	7	1	−1
4	9	0	1
5	11	−1	−1
6	13	1	1
7	15	0	−1

...and D_3 character table:

$(\mathbf{g})=$	$\{\mathbf{1}\}$	$\{\mathbf{r}^1,\mathbf{r}^2\}$	$\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$
$\chi^{A_1}(\mathbf{g})=$	1	1	1
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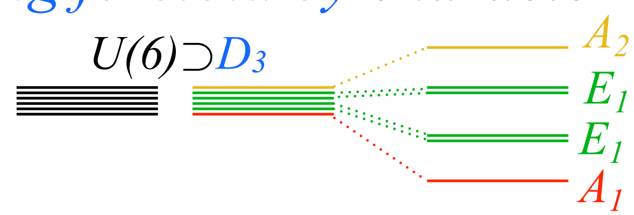
$\chi^{E_1}(\mathbf{g})=$	2	−1	0

$f^{(\alpha)}(\ell)$	f^{A_1}	f^{A_2}	f^{E_1}	
$\ell=0$	1	·	·	$1A_1$
1	·	1	1	$0A_1\oplus A_2\oplus E_1$
2	1	·	2	$1A_1 \quad \oplus 2E_1$
3	1	2	2	$1A_1\oplus 2A_2\oplus 2E_1$
4	1	2	3	$1A_1\oplus 2A_2\oplus 3E_1$
5	2	1	3	$2A_1\oplus \ A_2\oplus 3E_1$
6	3	2	4	$3A_1\oplus 2A_2\oplus 4E_1$
7	2	3	5	$2A_1\oplus 3A_2\oplus 5E_1$

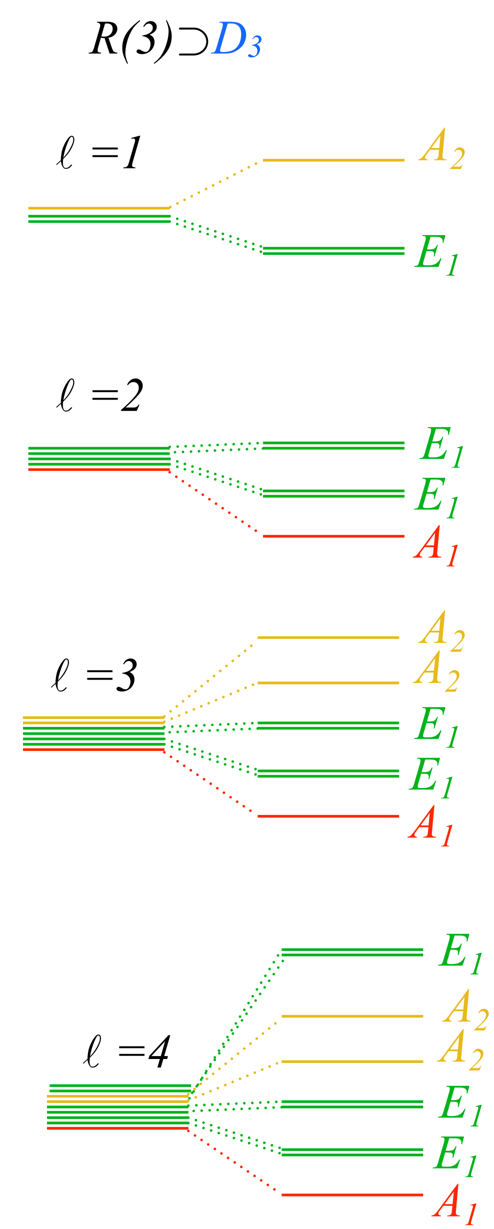
Spectral splitting in symmetry breaking foretold by character analysis



$R^G(U(6))\downarrow D_3 = D^{A_1}(\mathbf{g})\oplus D^{A_2}(\mathbf{g})\oplus 2D^{E_1}(\mathbf{g})$

Crystal-field splitting: O(3)⊃D₃ symmetry reduction and $D^{\ell}\downarrow D_3$ splitting

$f^{(\alpha)}(\ell)$	f^{A_1}	f^{A_2}	f^{E_1}	
$\ell=0$	1	·	·	$1A_1$
1	·	1	1	$0A_1\oplus A_2\oplus E_1$
2	1	·	2	$1A_1 \quad \oplus 2E_1$
3	1	2	2	$1A_1\oplus 2A_2\oplus 2E_1$
4	1	2	3	$1A_1\oplus 2A_2\oplus 3E_1$
5	2	1	3	$2A_1\oplus \ A_2\oplus 3E_1$
6	3	2	4	$3A_1\oplus 2A_2\oplus 4E_1$
7	2	3	5	$2A_1\oplus 3A_2\oplus 5E_1$



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$\chi^{A_1}(\mathbf{g})=$	1	1	1
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$\chi^{E_1}(\mathbf{g})=$	2	−1	0