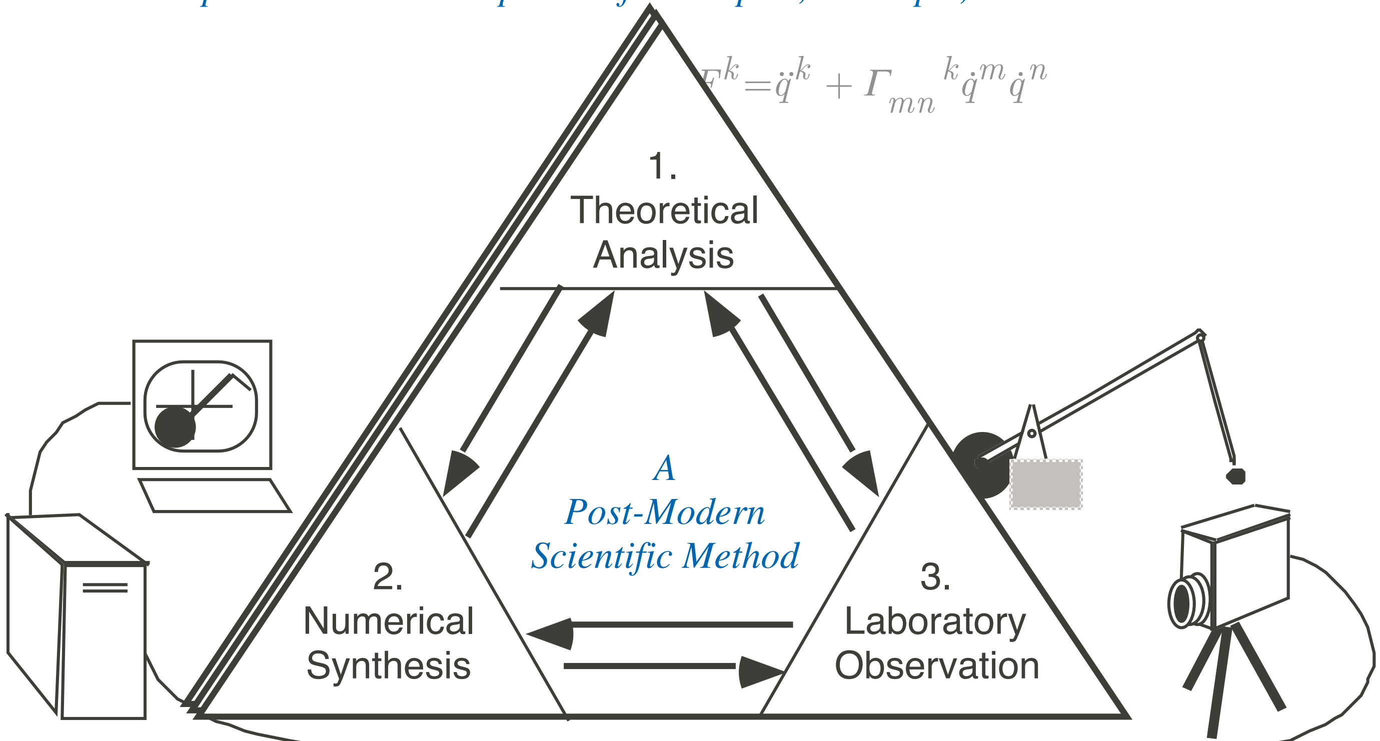
# Trebuchets, SuperBall Missles, and Related Multi-Frame Mechanics

*A millenial embarrassment*
*(and redemption)*
*for Physics*

*Computer Aided Development of Principles, Concepts, and Connections*

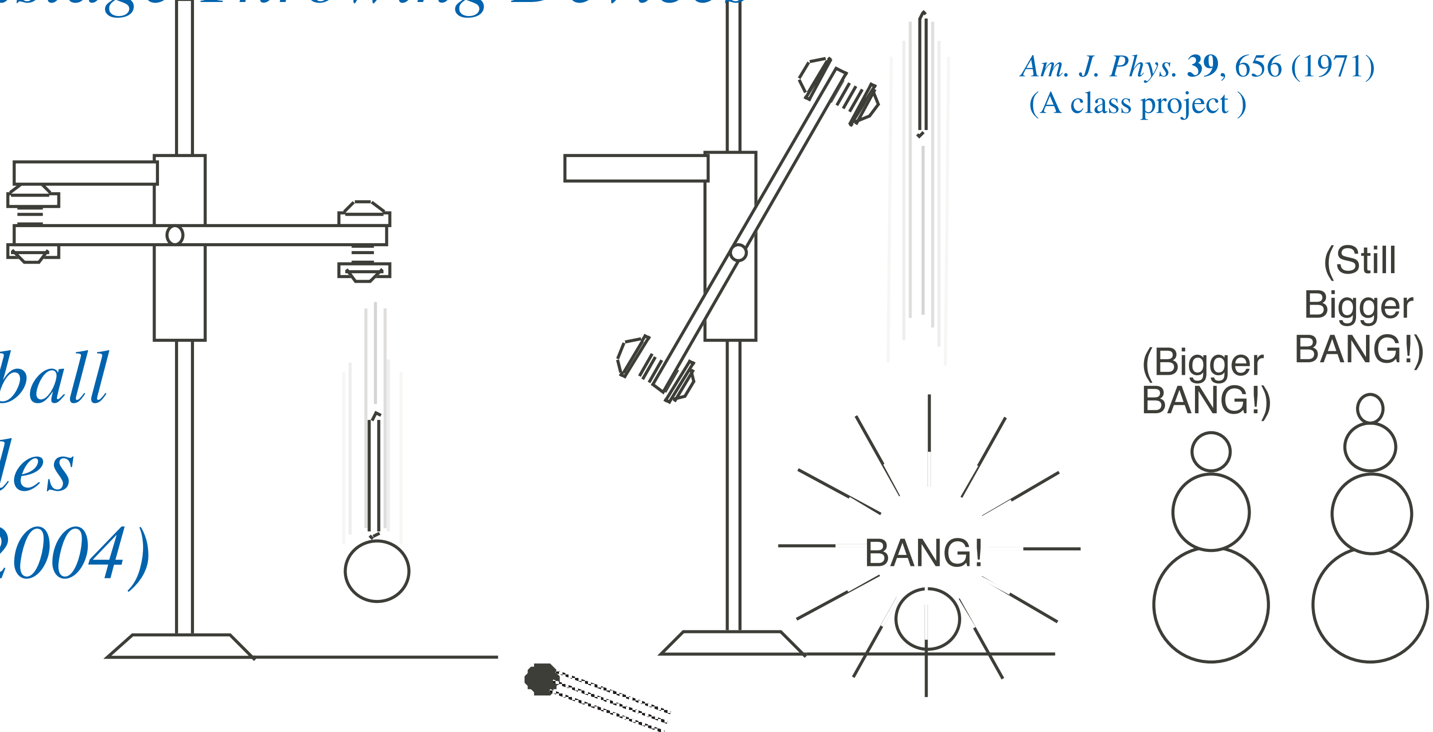



*Bill Harter and Dave Wall*
*University of Arkansas*
*and*
HARTER-*Soft*
*Elegant Educational Tools Since 2001*

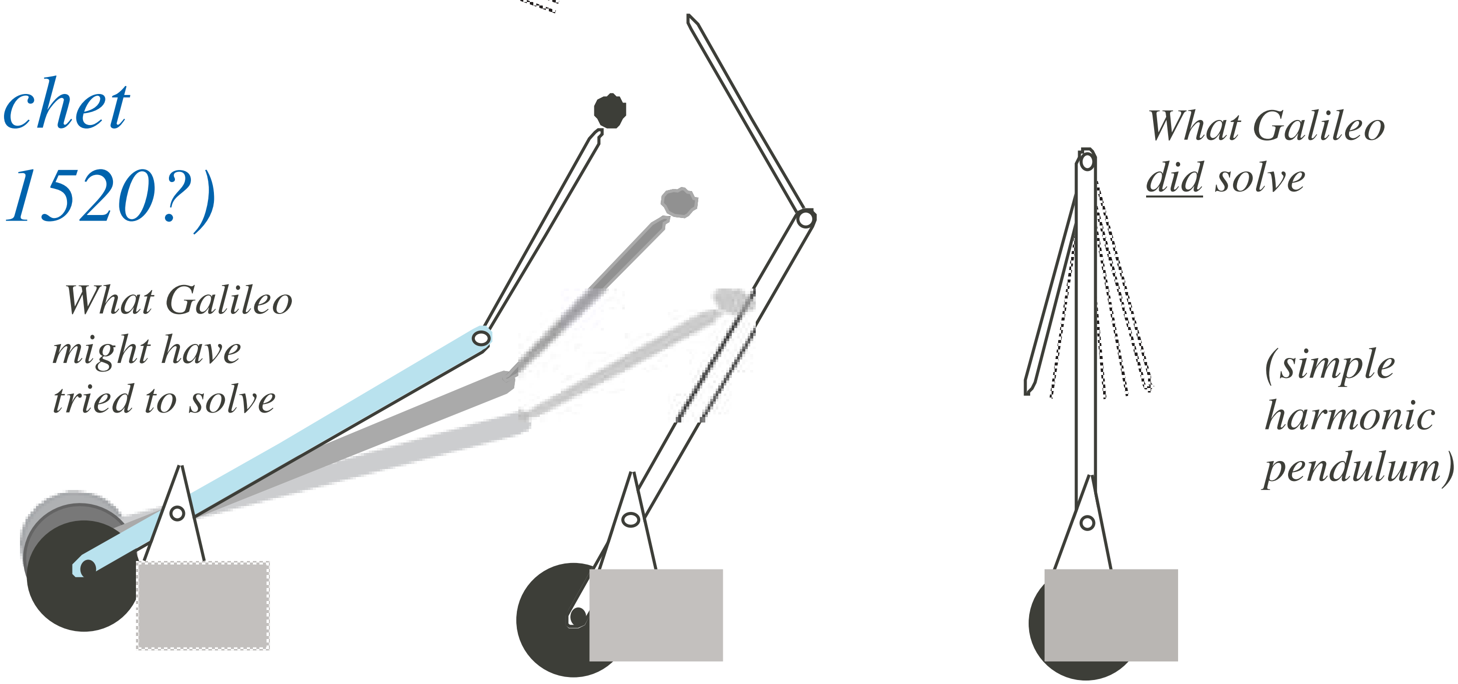
# *Multistage Throwing Devices*

*Am. J. Phys.* **39**, 656 (1971)
(A class project )

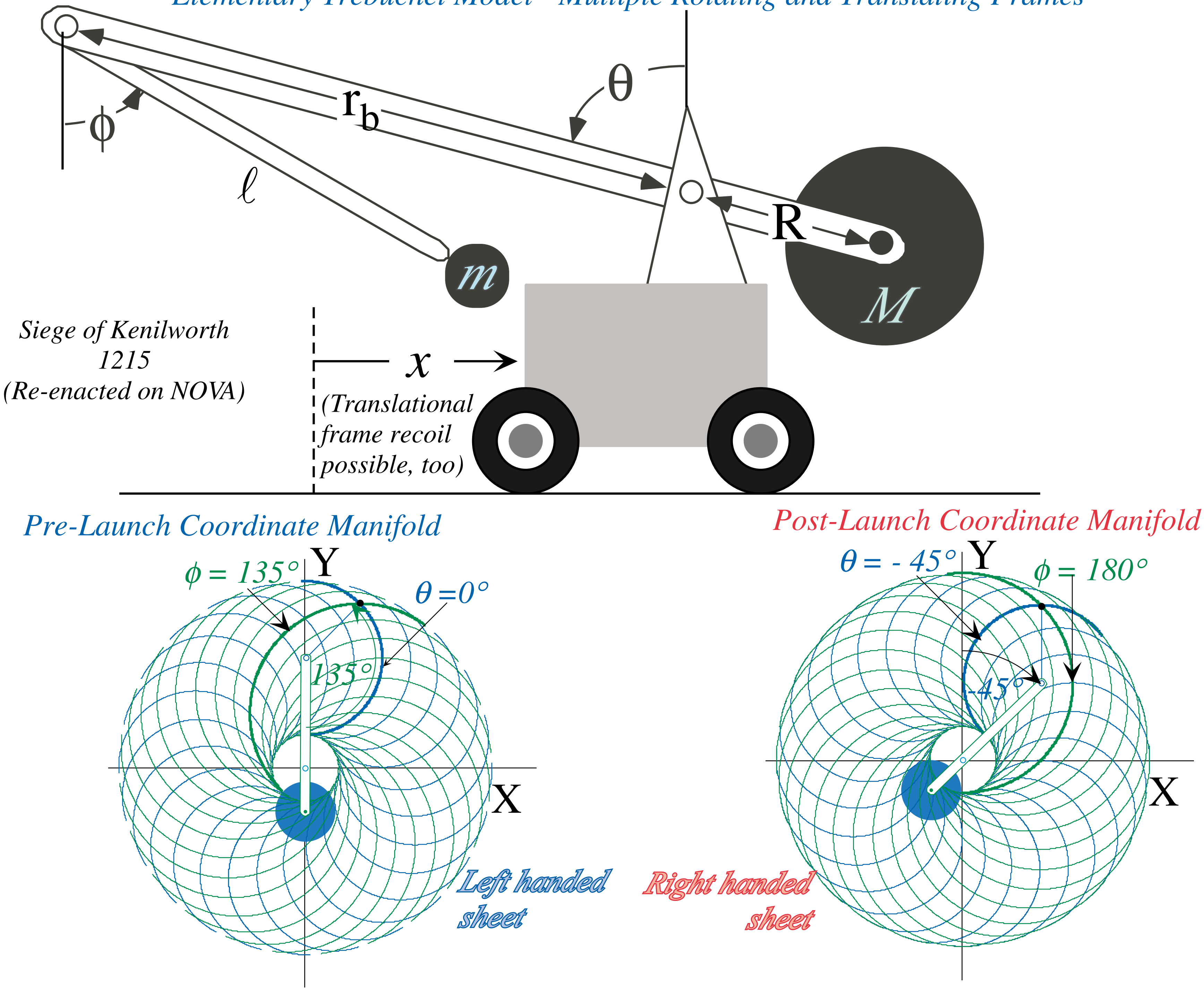
## *Superball Missles (1965-2004)*



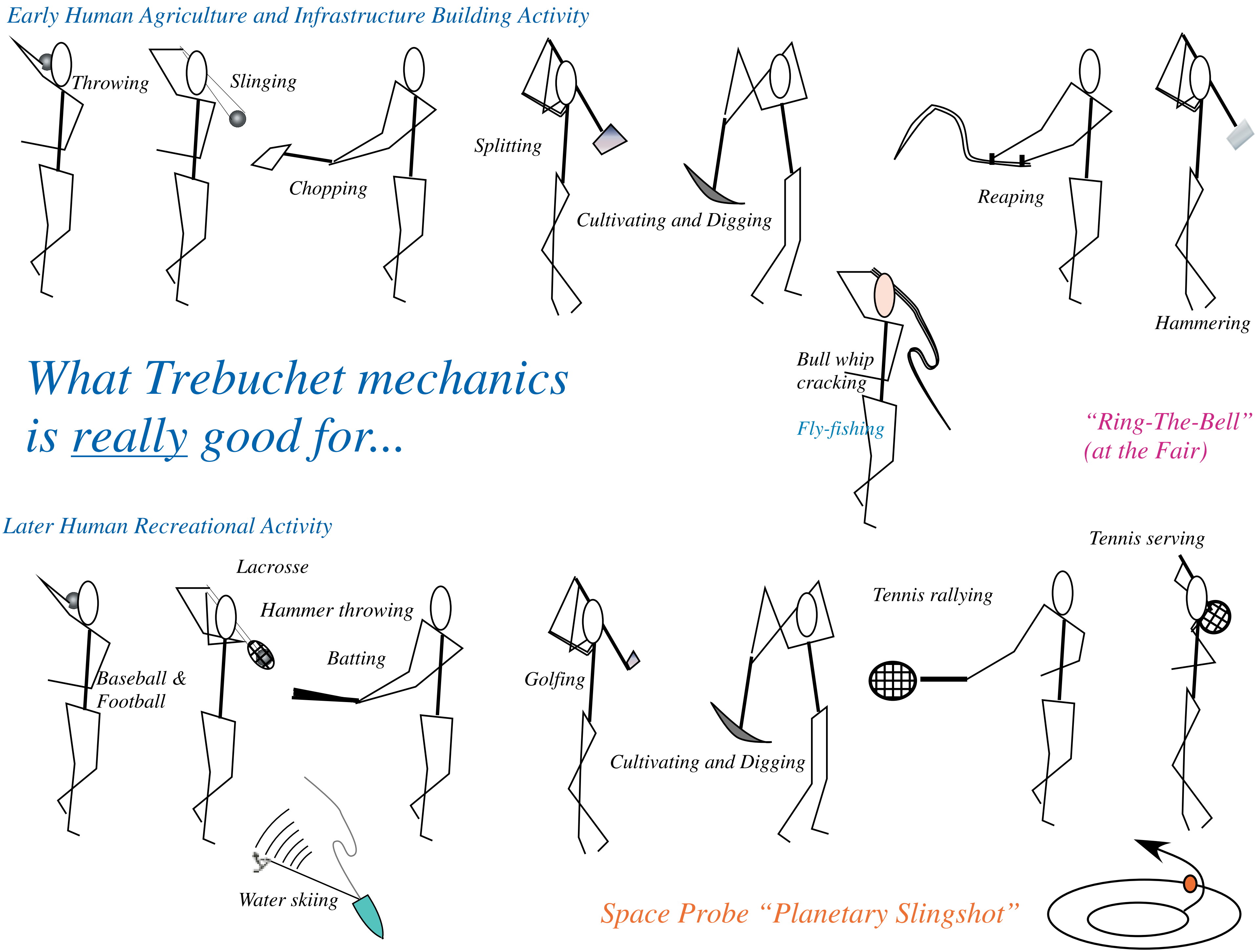

## *The Trebuchet ($\sim 10^3$ BC-1520?)*



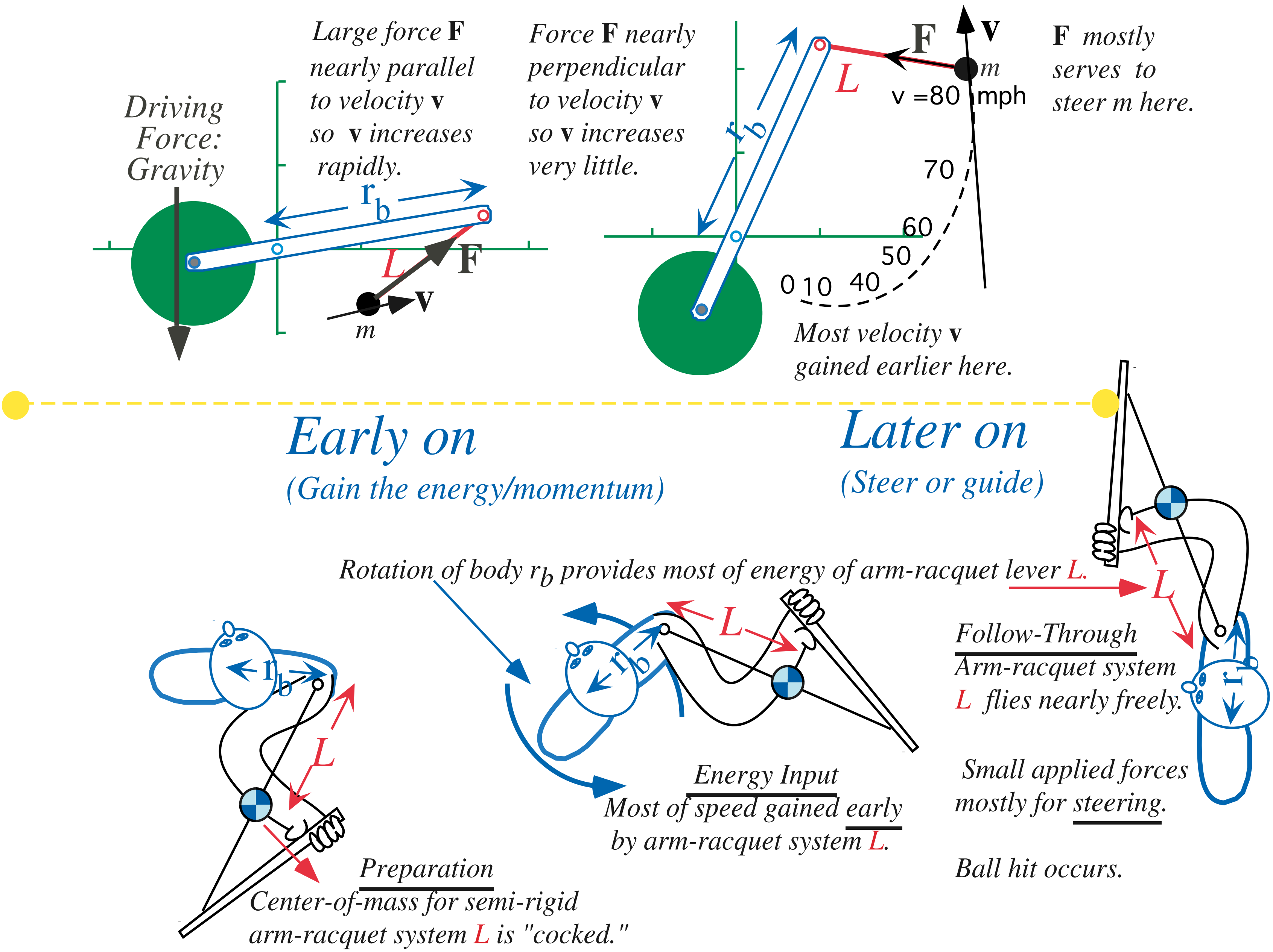
## Elementary Trebuchet Model - Multiple Rotating and Translating Frames



# What Trebuchet mechanics is *really* good for...



# *Trebuchet analogy with racquet swing - What we learn*

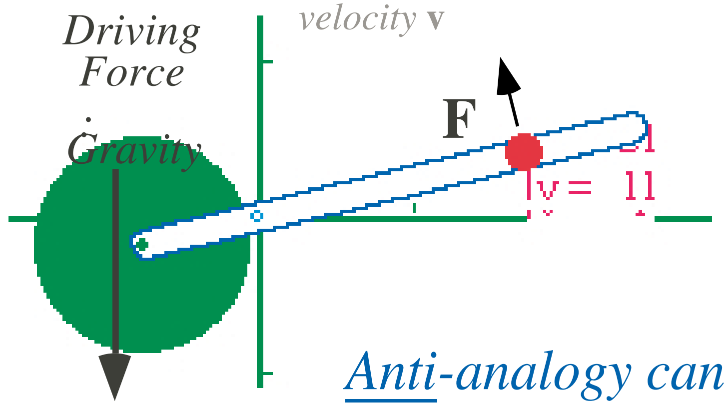


# An *Opposite* to Trebuchet Mechanics- The "Flinger"

## Early on

*(Not much happening)*

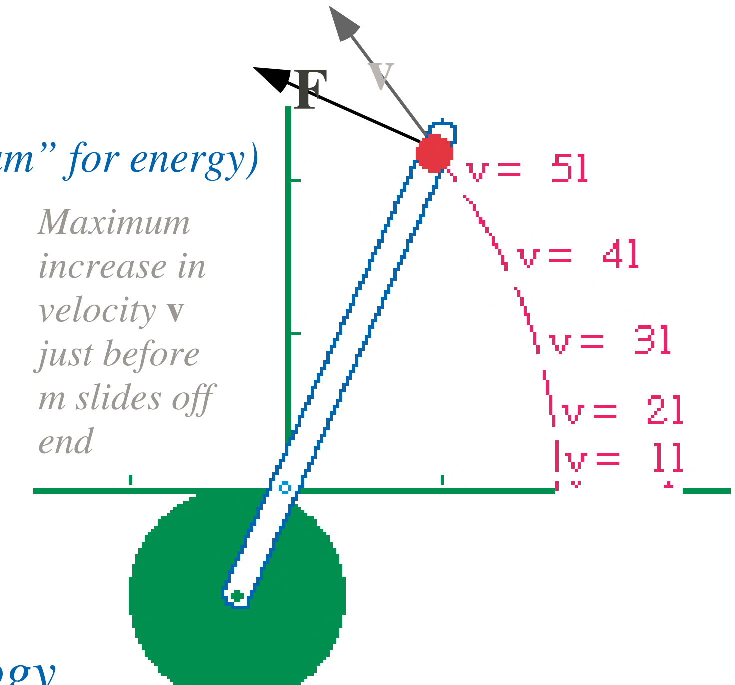
*Not much increase in velocity* **v**




## Later on

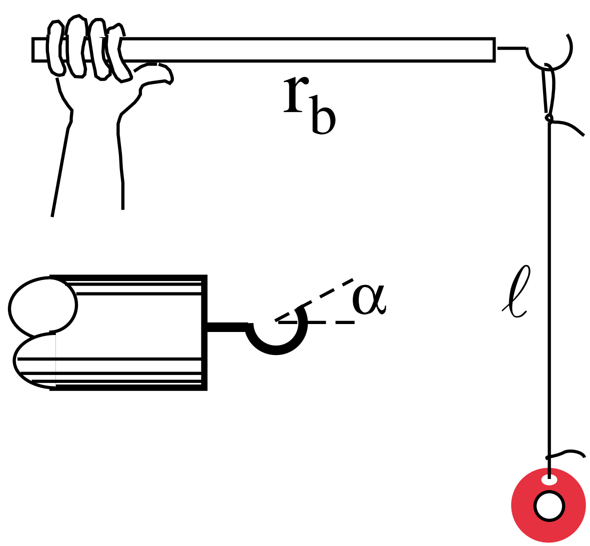
*(Last-minute "cram" for energy)*

*Maximum increase in velocity* **v** *just before m slides off end*

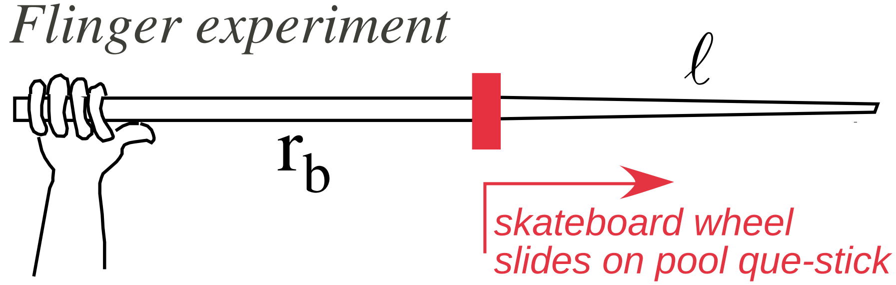



*Anti-analogy can be useful pedagogy*

### Trebuchet-like experiment




### Flinger experiment

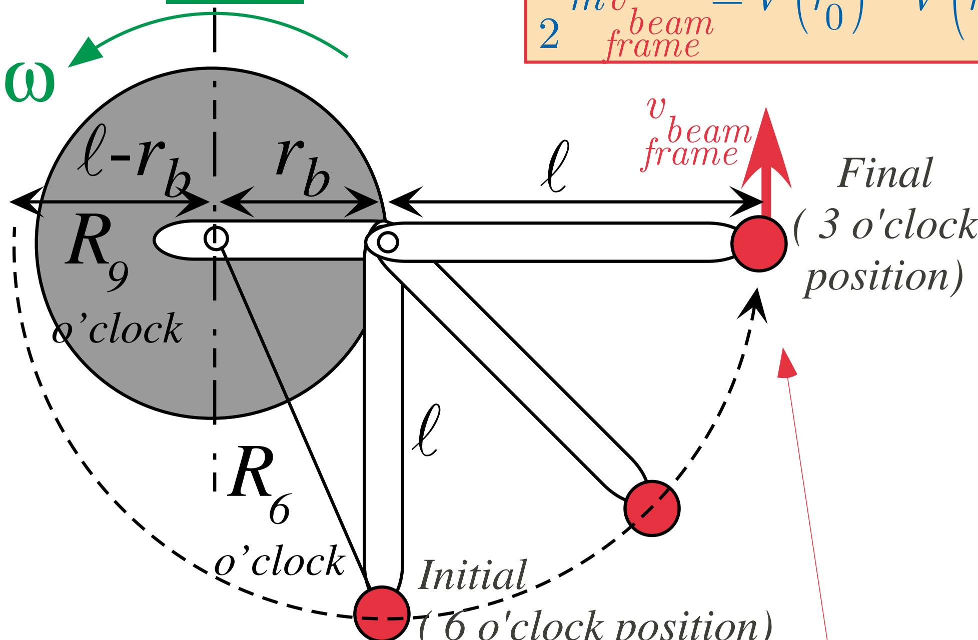


## Trebuchet model in rotating beam frame

## Flinger model in rotating beam frame

$$\frac{1}{2}mv^2_{\substack{beam\\frame}} = V(r_0) - V(r_f) = \frac{1}{2}m\omega^2 r_f^2 - \frac{1}{2}m\omega^2 r_0^2$$

Assume: Constant beam ω




$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{trebuchet}) = \textit{Final Trebuchet } KE_{\substack{beam\\frame}}$$

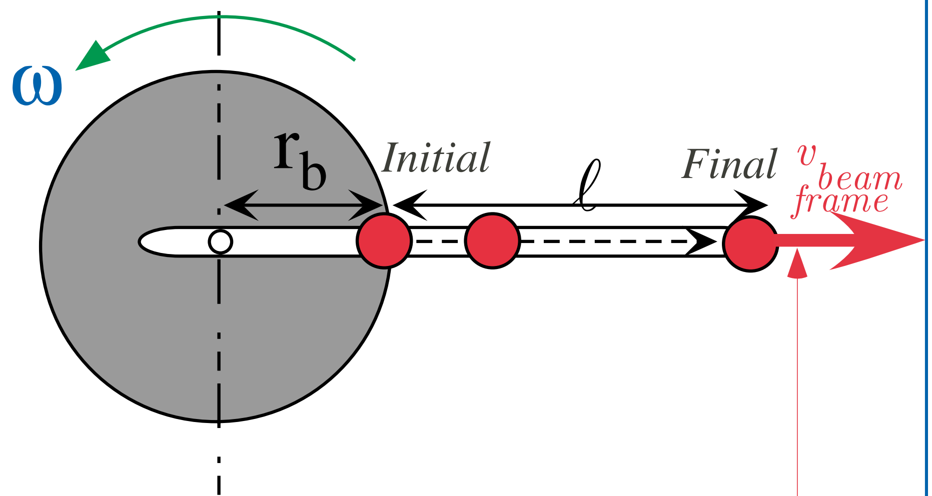
$$\underbrace{\frac{1}{2}m\omega^2\left(r_b+\ell\right)^2}_{\substack{\text{Final}\\ \text{3 o'clock}}} - \underbrace{\frac{1}{2}m\omega^2\left(r_b^2+\ell^2\right)}_{\substack{\text{Initial}\\ \text{6 o'clock}}} = \frac{1}{2}m\omega^2\left(2r_b\,\ell\right)$$

$$R^2_{\substack{6\\o'clock}} = r_b^2+\ell^2$$

Initial 9 o'clock $= \frac{1}{2}m\omega^2\left(4r_b\,\ell\right)$

$$R^2_{\substack{9\\o'clock}} = r_b^2+\ell^2-2r_b\ell$$

Assume: Constant beam ω



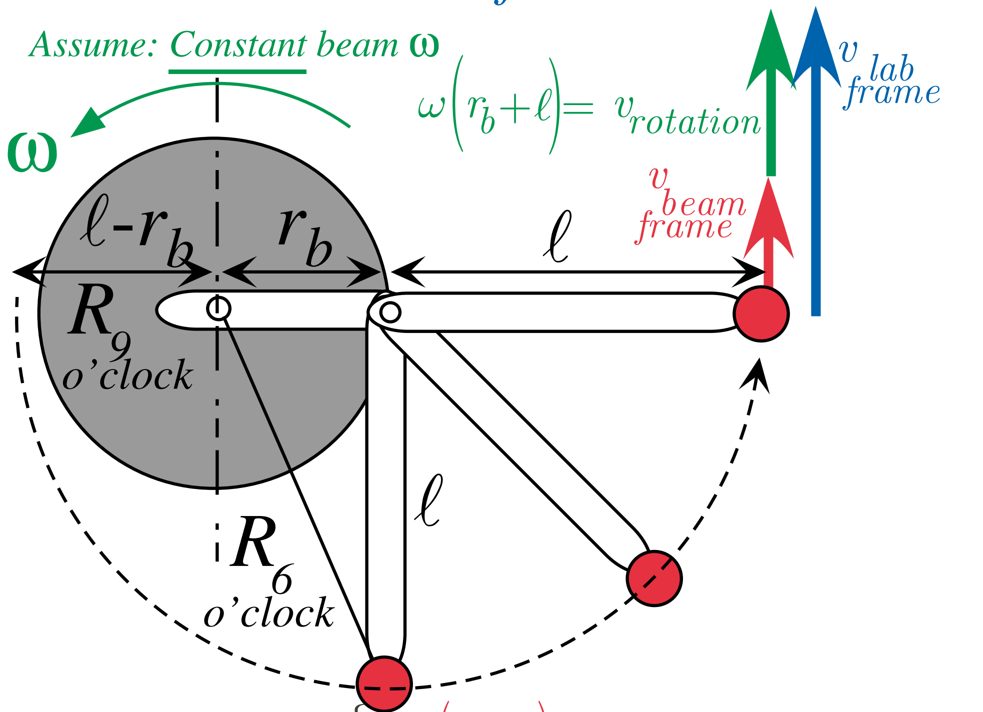

$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{flinger}) = \textit{Final Flinger } KE_{\substack{beam\\frame}}$$

$$\underbrace{\frac{1}{2}m\omega^2\left(r_b+\ell\right)^2}_{\substack{\text{Final}\\ \text{3 o'clock}}} - \underbrace{\frac{1}{2}m\omega^2 r_b^2}_{\substack{\text{Initial}\\ \text{3 o'clock}}} = \frac{1}{2}m\omega^2\ell\left(2r_b+\ell\right)$$

*Flinger KE is* $\frac{m\omega^2}{2}\ell^2$ *more than 6 o'clock trebuchet but misdirected*

*Flinger KE is* $\frac{m\omega^2}{2}\left(2r_b\ell-\ell^2\right)$ *less than 9 o'clock trebuchet and misdirected*

# Trebuchet model in lab frame




$$v^2_{beam\ frame}(trebuchet) = \begin{cases} \omega^2\left(2r_b\,\ell\right) & \textit{half-cocked 6 o'clock} \\ \omega^2\left(4r_b\,\ell\right) & \textit{fully-cocked 9 o'clock} \end{cases}$$

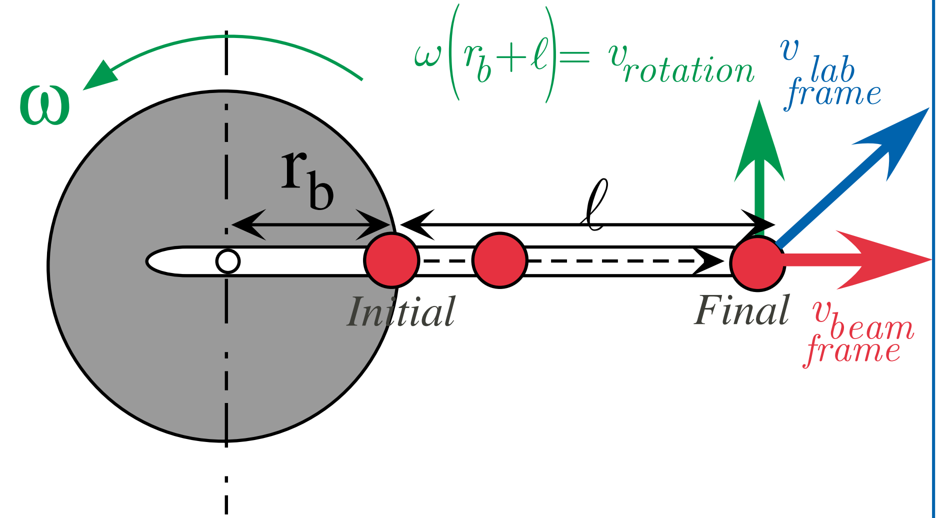
$$v_{lab\ frame}\left(trebuchet\right) =$$

$$\begin{cases} \omega\left(r_b + \ell + \sqrt{2\ell r_b}\right) & \textit{half-cocked 6 o'clock} \\ \omega\left(r_b + \ell + 2\sqrt{\ell r_b}\right) & \textit{fully-cocked 9 o'clock} \end{cases}$$

$$= \begin{cases} 5.00\omega \\ 5.82\omega \end{cases} \qquad = \begin{cases} 5.16\omega \\ 6.00\omega \end{cases} \qquad = \begin{cases} 5.00\omega \\ 5.82\omega \end{cases}$$

$$\left(r_b = 2,\ \ell = 1\right),\ \left(r_b = 1.5,\ \ell = 1.5\right),\left(r_b = 1,\ \ell = 2\right)$$

# Flinger model in lab frame




$$v^2_{beam\ frame}(flinger) = \omega^2\ell\left(2r_b + \ell\right)$$

$$v_{lab\ frame}\left(flinger\right) =$$

$$= \omega\sqrt{\left(r_b + \ell\right)^2 + \ell\left(2r_b + \ell\right)} = \omega\sqrt{2\left(r_b + \ell\right)^2 - r_b^2}$$

(compare)

$$= 3.74\omega \qquad = 3.96\omega \qquad = 4.12\omega$$

$$\left(r_b = 2,\ \ell = 1\right),\ \left(r_b = 1.5,\ \ell = 1.5\right),\left(r_b = 1,\ \ell = 2\right)$$

# *Many Approaches to Mechanics (Trebuchet Equations)*

*Each has advantages and disadvantages (Trebuchet exposes them)*

- U.S. Approach

*Quick'n dirty*

Newton F=Ma Equations

Cartesian coordinates

- French Approach

*Tres elegant*

Lagrange Equations

in Generalized Coordinates

$$F_\ell = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell}$$

- German Approach

*Pride and Precision*

Riemann Christoffel Equations

in Differential Manifolds

$$F^k = \ddot{q}^k + \Gamma_{mn}{}^k \dot{q}^m \dot{q}^n$$

- Anglo-Irish Appproach

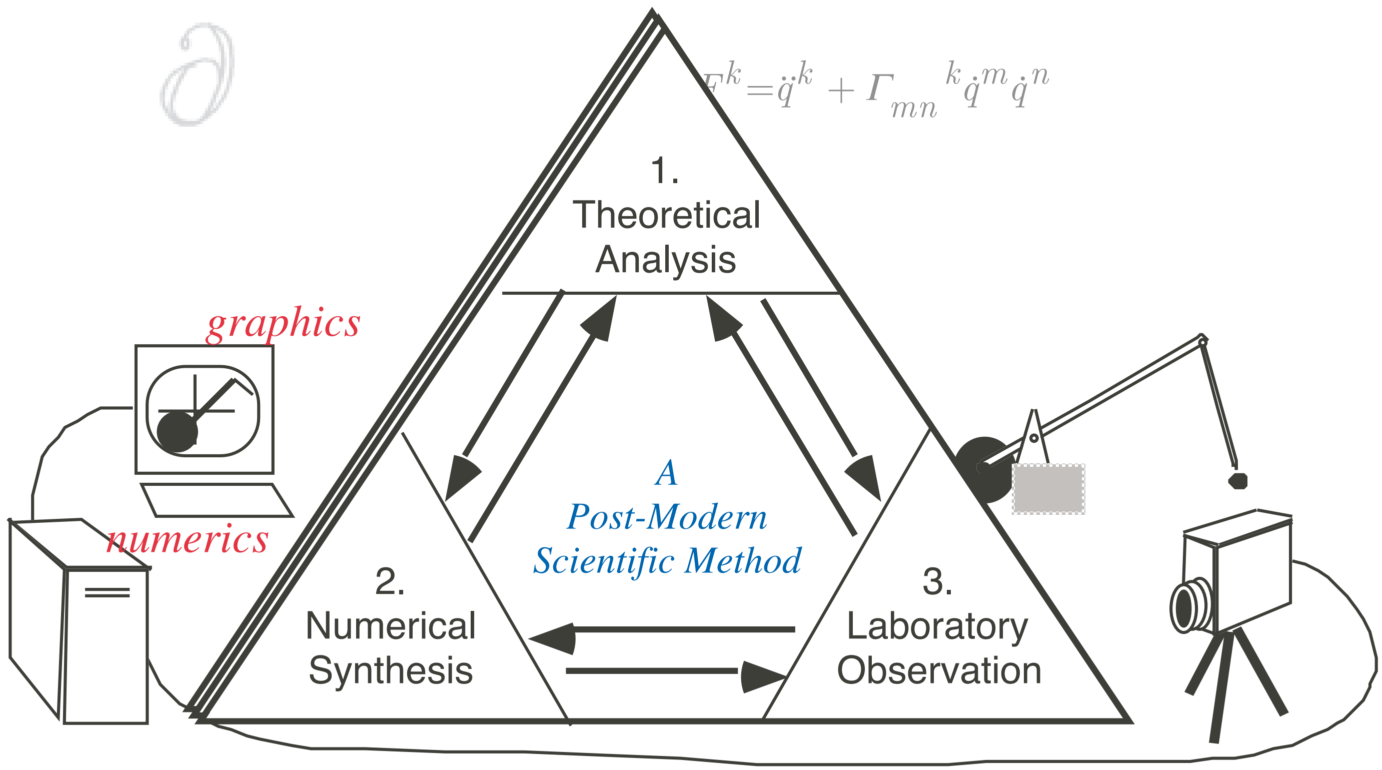
*Powerfully Creative*

Hamilton's Equations

Phase Space

$$\dot{p}_j = -\frac{\partial H}{\partial q^j}, \qquad \dot{q}^k = \frac{\partial H}{\partial p^k}.$$

- Unified Approach




All approaches have one thing in common:

The *Art of Approximation*

Physics lives and dies by the art of approximate models and analogs.

Another thing in common:
Equations Require *Kinetic Energy* $T = \frac{1}{2}\gamma_{\mu\nu}\dot{q}^{\mu}\dot{q}^{\nu}$ (sum, sum)
in terms of coordinates and derivitives.

It helps to use *Covariant Metric* $\gamma_{\mu\nu}$ matrix:

$$T = \frac{1}{2}\left(MR^2 + mr^2\right)\dot{\theta}^2 \quad - \quad \frac{1}{2}mr\ell\dot{\theta}\dot{\phi}\cos(\theta-\phi) = \frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix}$$

$$-\frac{1}{2}mr\ell\dot{\phi}\dot{\theta}\cos(\theta-\phi) \quad + \quad \frac{1}{2}m\ell^2\dot{\phi}^2$$

The $\gamma_{\mu\nu}$ give
*Covariant Momentum* $p_{\mu} = \gamma_{\mu\nu}\dot{q}^{\nu}$
(a.k.a. "canonical" momentum)

$$\begin{pmatrix}p_{\theta} \\ p_{\phi}\end{pmatrix} = \begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix}$$

The inverse $\gamma^{\mu\nu}$ give
*Contravariant Momentum* $\dot{q}^{\nu}$ $\quad p^{\nu} = \gamma^{\nu\mu}p_{\mu}$ (sum)
(a.k.a. "generalized" velocity)

$$\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \begin{pmatrix}\gamma^{\theta,\theta} & \gamma^{\theta,\phi} \\ \gamma^{\phi,\theta} & \gamma^{\phi,\phi}\end{pmatrix}\begin{pmatrix}p_{\theta} \\ p_{\phi}\end{pmatrix}$$

Trebuchet equations nonlinear and *Lagrange-Hamilton* methods are a bit messy..

*Lagrangian*
*L=T-V*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} + F_\theta \qquad \dot{p}_\theta - \frac{\partial L}{\partial \theta} = F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} + F_\phi \qquad \dot{p}_\phi - \frac{\partial L}{\partial \phi} = F_\phi = -mg\ell \;\; \sin\phi$$

Lagrange quations need rearrangement to solve numerically

$$-MgR\sin\theta + mgr\sin\theta = \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi)$$

$$-mg\ell\sin\phi = \;\; m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi)$$

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*Riemann Christofffel Equations* give less mess.. $T = \gamma_{mn} \;\; \dot{q}^m \dot{q}^n$

$$F^k = \ddot{q}^k + \Gamma_{mn}{}^k \dot{q}^m \dot{q}^n \qquad \text{where} : \Gamma_{mn;\ell} \qquad \frac{1}{2}\left[\frac{\partial \gamma_{n\ell}}{\partial q^m} + \frac{\partial \gamma_{\ell m}}{\partial q^n} - \frac{\partial \gamma_{mn}}{\partial q^\ell}\right]$$

...they are *immediately computer integrable*. (..and help with qualitative analysis..)

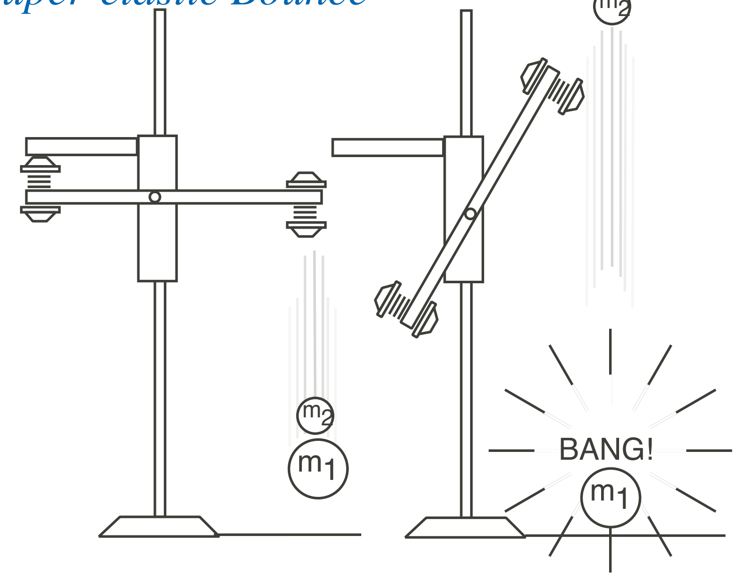
$$\begin{bmatrix}\ddot{\theta}\\ \ddot{\phi}\end{bmatrix} = \frac{1}{\mu}\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2\end{pmatrix}\begin{bmatrix} -mr\ell\dot{\phi}^2\sin(\theta-\phi) + \left(mr - MR\right)g\sin\theta \\ mr\ell\dot{\theta}^2\sin(\theta-\phi) - mg\ell\sin\phi\end{bmatrix}$$

where: $\mu = m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]$

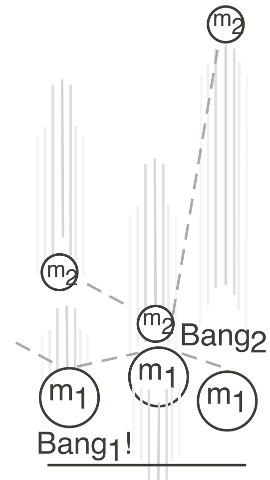
# Analogous Superball Models

*Class of W. G. Harter,*
*"Velocity Amplification in Collision Experiments Involving Superballs,"*
*Am. J. Phys.*
**39**, 656 (1971)
(A class project )

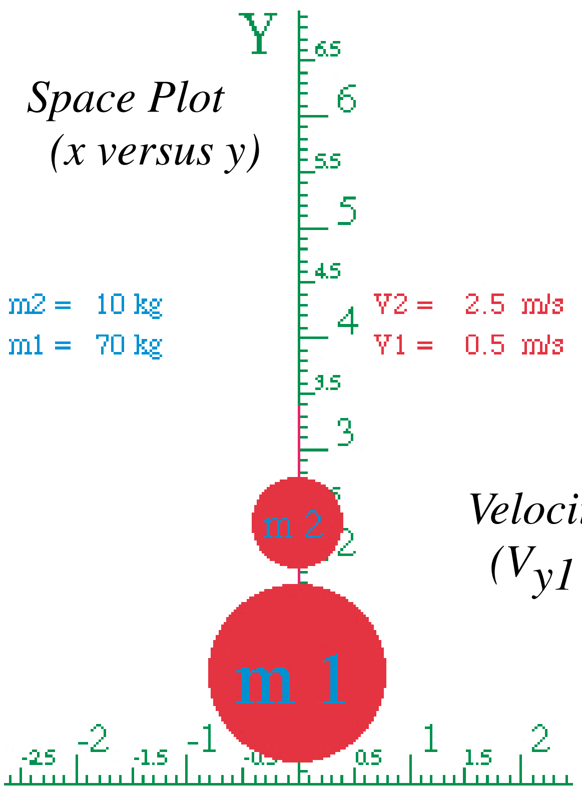
## *Super-elastic Bounce*




## *2-Bang Model*

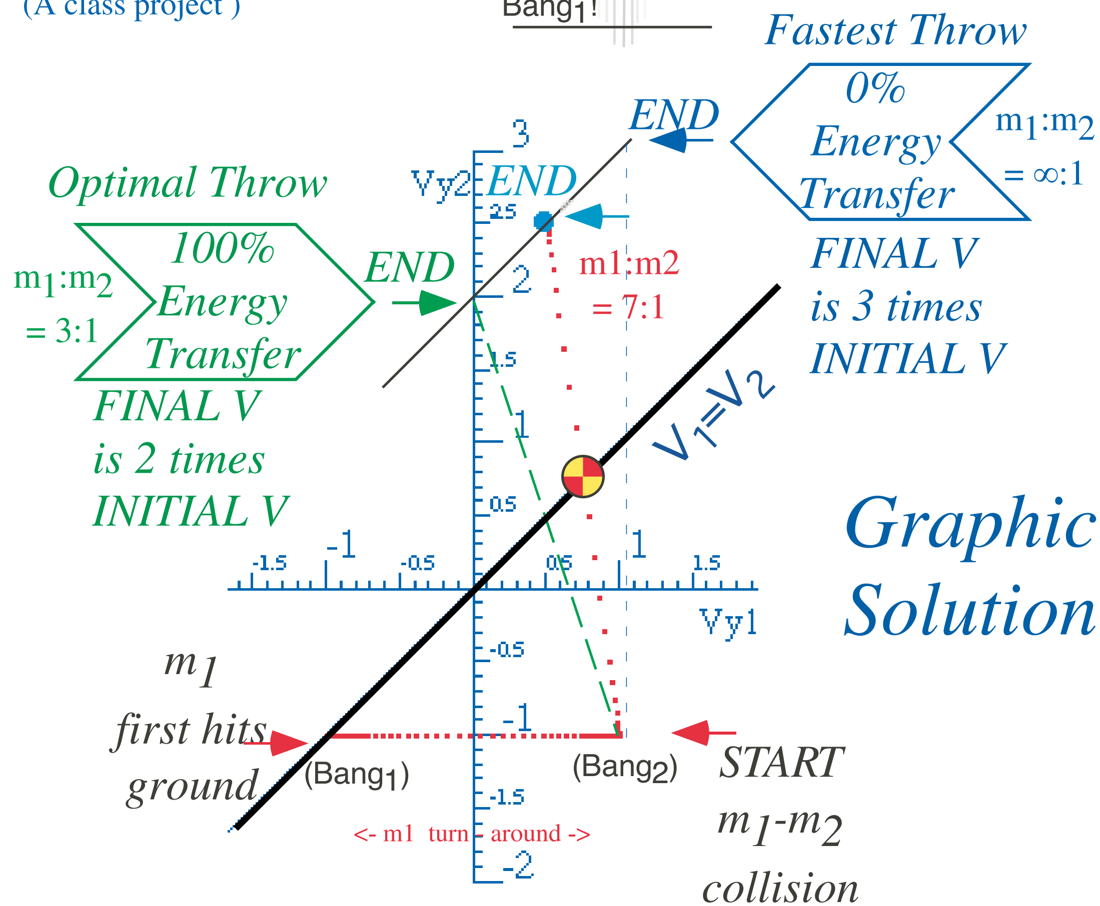


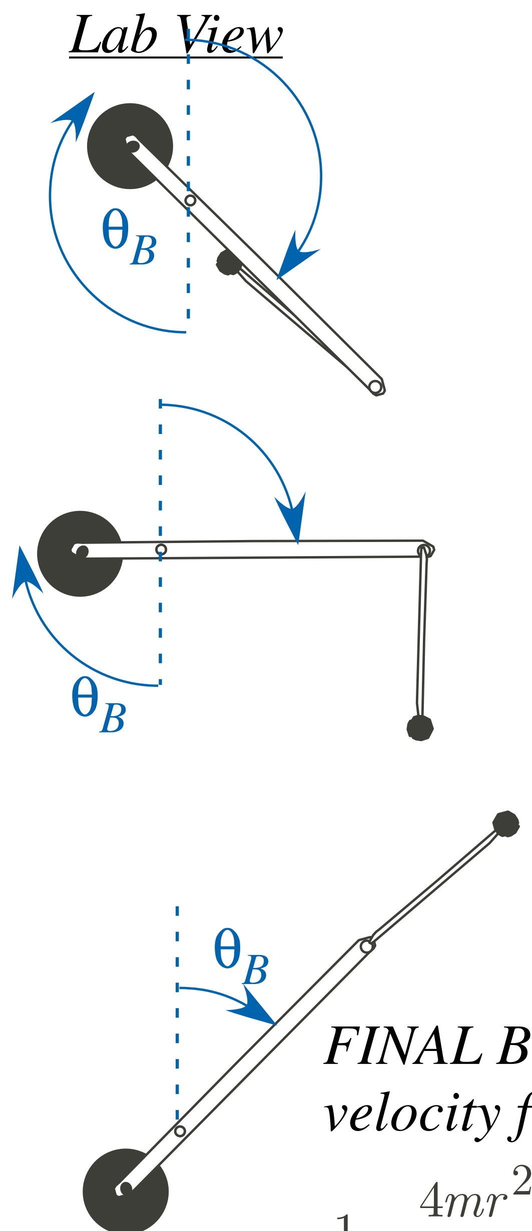

*Space Plot*
*(x versus y)*

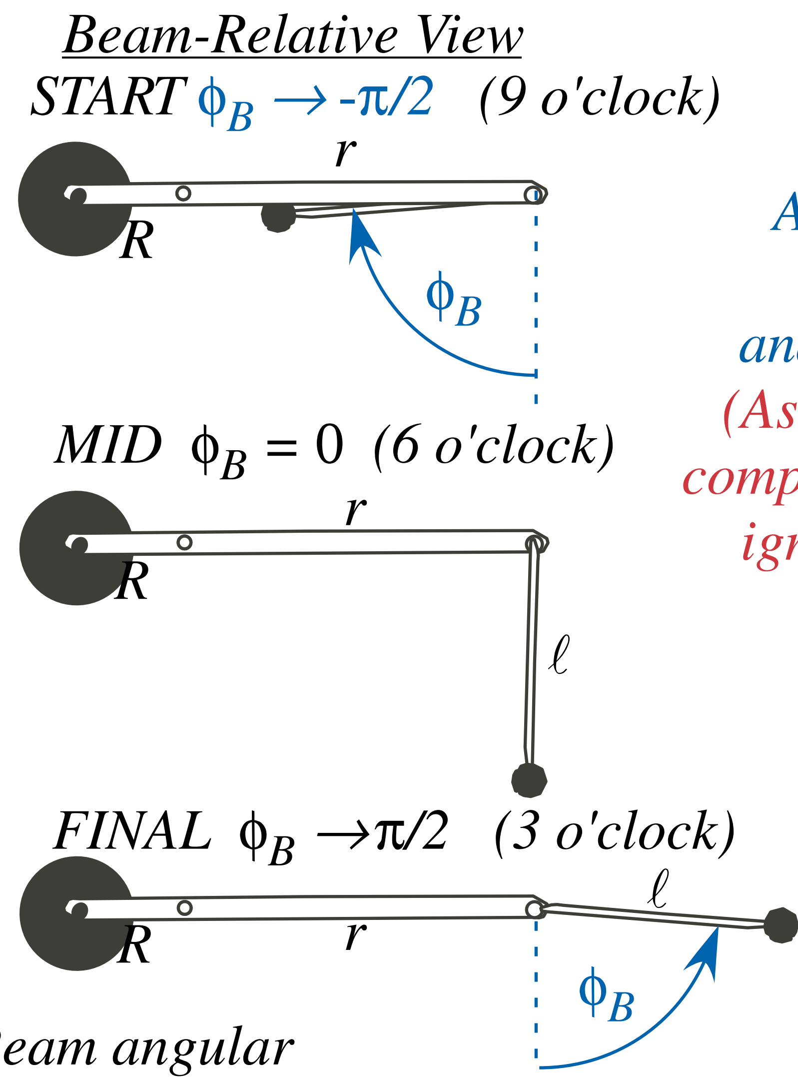



*Velocity Plot*
*($V_{y1}$ versus $V_{y2}$)*

## *Graphic Solution*






# Hamiltonian Model

*Approximation conserves total energy and total angular momentum*

*(Assumes internal forces large compared to gravity which is then ignored after initial impulse)*

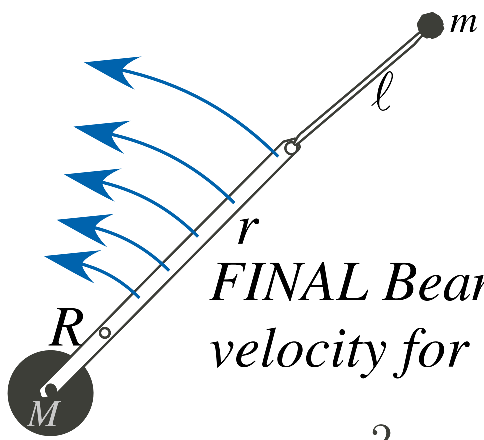
*FINAL beam-relative lever angular velocity for r=ℓ*

$$\dot{\phi}_{FINAL} = \dot{\theta}_{FINAL} + 2\dot{\theta}_{INITIAL}$$

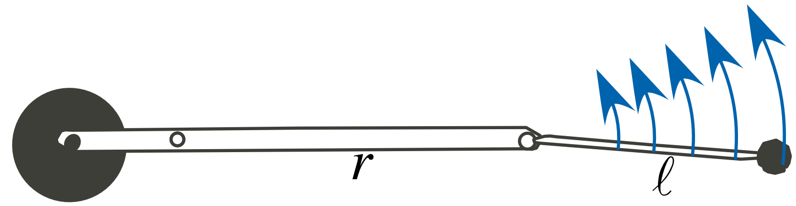
$$\begin{matrix} \left(\dot{\theta}_{FINAL} = 0\right) \\ \left(\dot{\theta}_{FINAL} = \dot{\theta}_{INITIAL}\right) \end{matrix} = \begin{cases} 2\dot{\theta}_{INITIAL} & \text{Optimal Throw} \\ 3\dot{\theta}_{INITIAL} & \text{Quickest Throw} \end{cases}$$

*FINAL Beam angular velocity for r=ℓ*

$$\dot{\theta}_{FINAL} = \frac{1 - \dfrac{4mr^2}{MR^2}}{1 + \dfrac{4mr^2}{MR^2}} \dot{\theta}_{INITIAL}$$




*FINAL Beam angular velocity for r=ℓ*




*FINAL beam-relative lever angular velocity for r=ℓ*

$$\dot{\theta}_{FINAL} = \frac{1 - \dfrac{4mr^2}{MR^2}}{1 + \dfrac{4mr^2}{MR^2}}\,\dot{\theta}_{INITIAL} = \begin{cases} 0 & \text{Optimal Throw} \\ \dot{\theta}_{INITIAL} & \text{Quickest Throw} \end{cases}$$

$$\dot{\phi}_{FINAL} = \dot{\theta}_{FINAL} + 2\dot{\theta}_{INITIAL} = \begin{cases} 2\dot{\theta}_{INITIAL} & \text{Optimal Throw} \\ 3\dot{\theta}_{INITIAL} & \text{Quickest Throw} \end{cases}$$

*FINAL "Bottom line" lab velocity for r=ℓ*

$$KE_{FINAL}^{mass\ \mathrm{m}} = \frac{1}{2}mr^2\left(\dot{\phi}_{FINAL} + \dot{\theta}_{FINAL}\right)^2 = \frac{1}{2}mr^2 \begin{cases} \left(2\dot{\theta}_{INITIAL}\right)^2 & \left(\dot{\theta}_{FINAL} = 0\right) \\ \left(4\dot{\theta}_{INITIAL}\right)^2 & \left(\dot{\theta}_{FINAL} = \dot{\theta}_{INITIAL}\right) \end{cases}$$

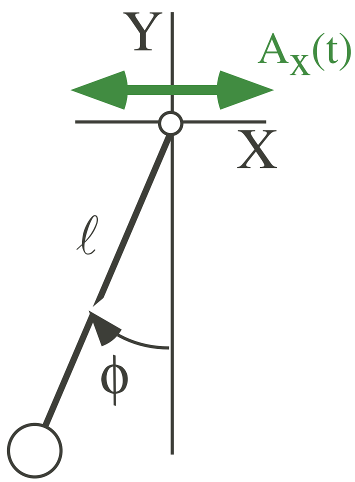
*Consistent with fully-cocked 9 o'clock velocity*

$$\omega\left(r_b + \ell + 2\sqrt{\ell r_b}\right)$$

# Coupled Rotation and Translation (Throwing)

## Early non-human (or in-human) machines: trebuchets, whips..     (3000 BC-1542 AD)

*X-stimulated pendulum:*
*(Quasi-Linear Resonance)*



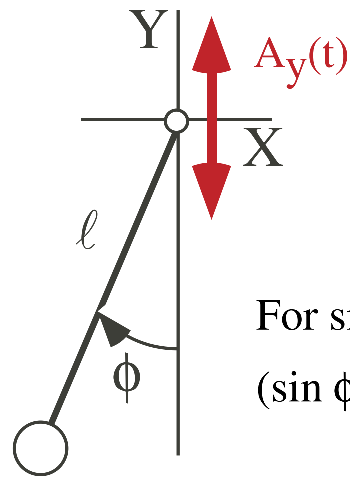

For small $\phi$
($\cos\phi \sim 1$ ) :

Forced Harmonic Resonance

$$\frac{d^2\phi}{dt^2} + \frac{g}{\ell}\phi = \frac{A_x(t)}{\ell}$$

A Newtonian F=Ma equation
Lorentz equation (with $\Gamma$=0)

*Y-stimulated pendulum:*
*(Non-Linear Resonance)*

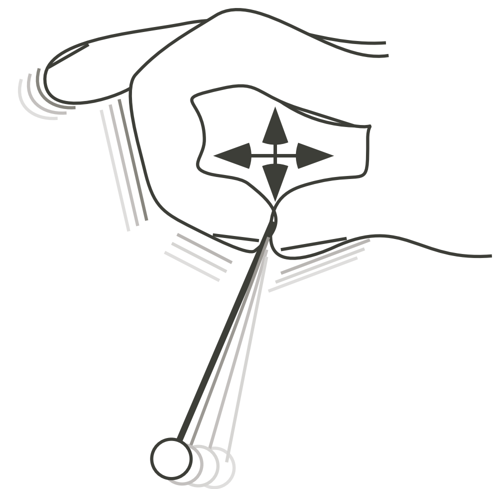



For small $\phi$
($\sin\phi \sim \phi$ ) :

Parametric Resonance

$$\frac{d^2\phi}{dt^2} + \left(\frac{g}{\ell} + \frac{A_y(t)}{\ell}\right)\phi = 0$$

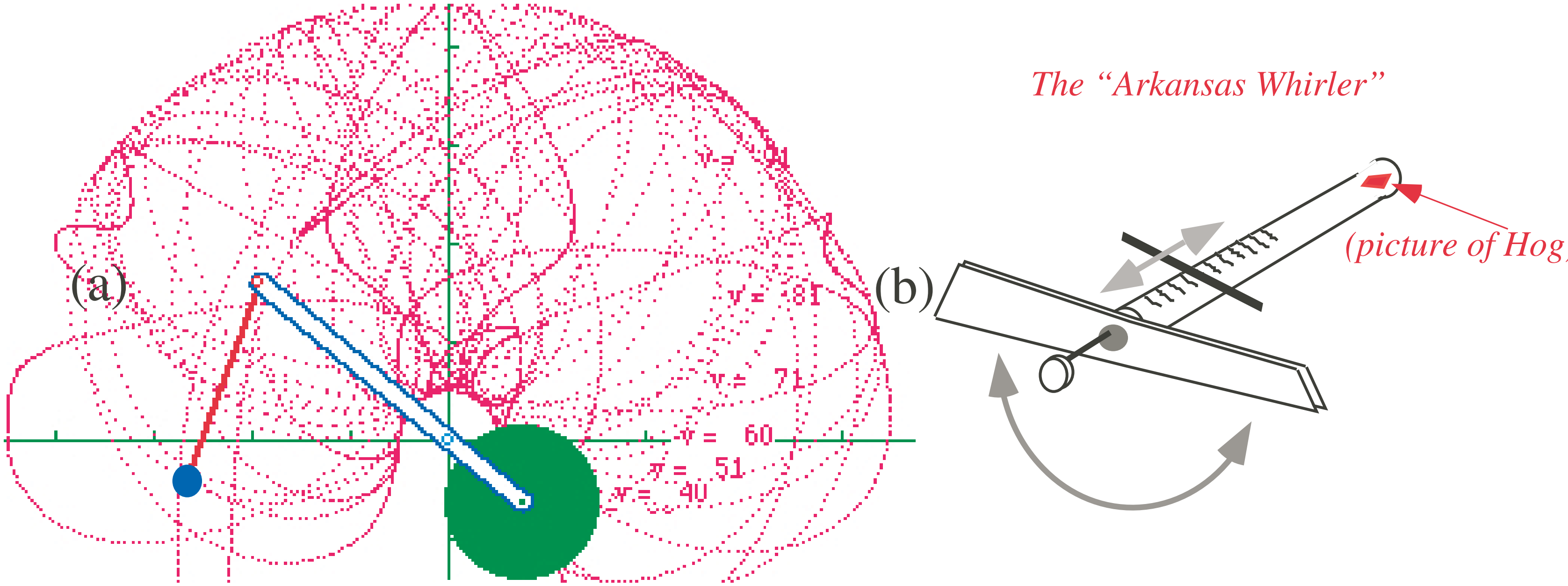
A Schrodinger-like equation
(Time *t* replaces coord. *x*)



General $\phi$ :

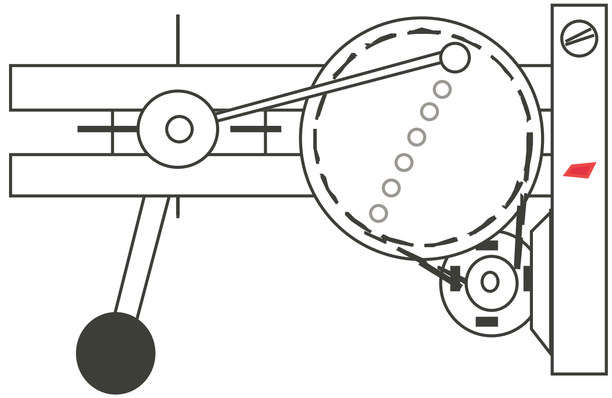
(1542-2004 AD)

General case: A Nasty equation! $$\frac{d^2\phi}{dt^2} + \frac{g+A_y(t)}{\ell}\sin\phi + \frac{A_x(t)}{\ell}\cos\phi = 0$$

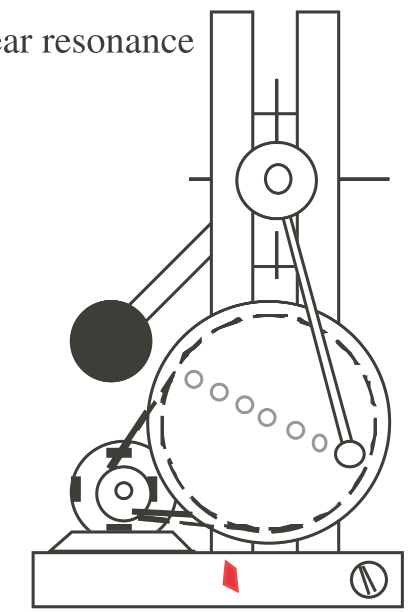



*Chaotic motion from both linear and non-linear resonance (a) Trebuchet, (b) Whirler .*

Positioned for linear resonance



Positioned for nonlinear resonance



# Schrodinger Equation Parametric Resonance — Related to — Jerked-Pendulum Trebuchet Dynamics

*Schrodinger Wave Equation*

$$\frac{d^2\phi}{dx^2} + \big(E - V(x)\big)\phi = 0$$

*With periodic potential*

$$V(x) = -V_0 \cos(Nx)$$

*Mathieu Equation*

$$\frac{d^2\phi}{dx^2} + \big(E + V_0 \cos(Nx)\big)\phi = 0$$

*Jerked Pendulum Equation*

$$\frac{d^2\phi}{dt^2} + \left(\frac{g}{\ell} + \frac{A_y(t)}{\ell}\right)\phi = 0$$

*On periodic roller coaster: $y=-A_y \cos w_y t$*

$$A_y(t) = \omega_y{}^2 A_y \cos(\omega_y t)$$

$$\frac{d^2\phi}{dt^2} + \left(\frac{g}{\ell} + \frac{\omega_y{}^2 A_y}{\ell}\cos(\omega_y t)\right)\phi = 0$$

$$Nx = \omega_y t$$

**Connection Relations**

$$\frac{N}{\omega_y}dx = dt$$

$$\frac{N^2}{\omega_y{}^2}dx^2 = dt^2$$

*(Let N=2 to get edge modes)*

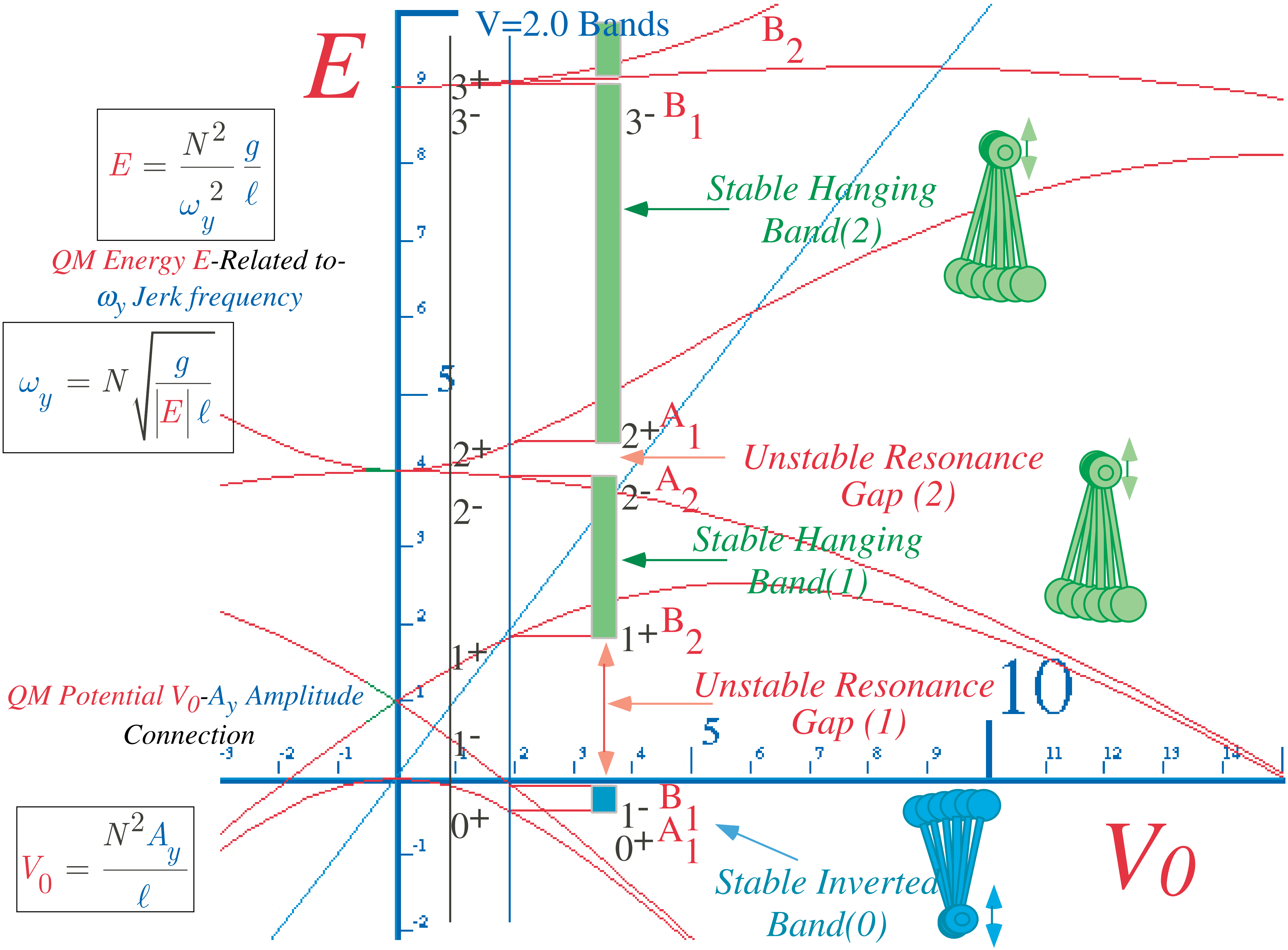
$$\frac{d^2\phi}{dx^2} + \frac{N^2}{\omega_y{}^2}\left(\frac{g}{\ell} + \frac{\omega_y{}^2 A_y}{\ell}\cos(Nx)\right)\phi = 0$$

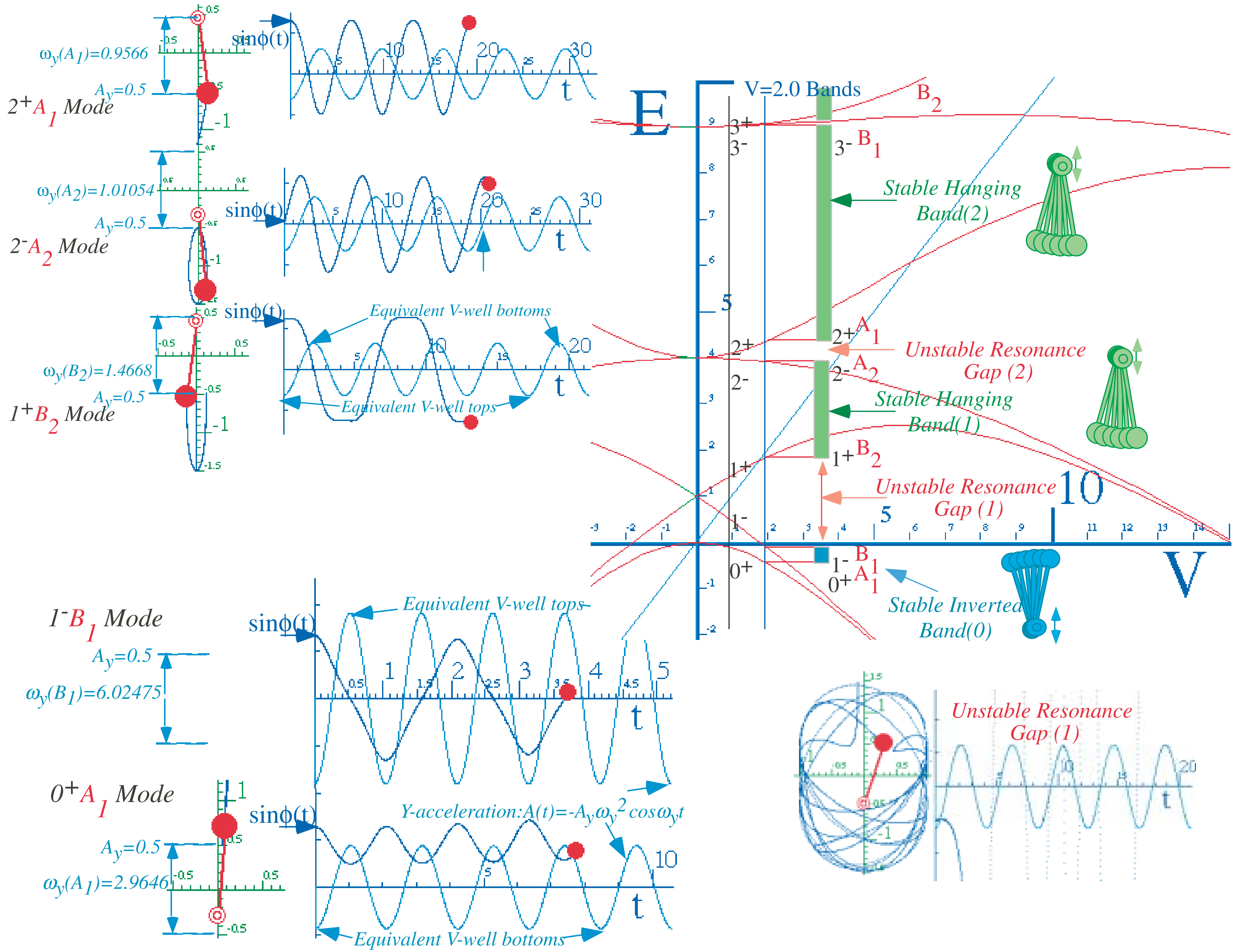
$$E = \frac{N^2}{\omega_y{}^2}\frac{g}{\ell}$$

*QM Energy $E$-to-$\omega_y$ Jerk frequency* Connection

$$V_0 = \frac{N^2 A_y}{\ell}$$

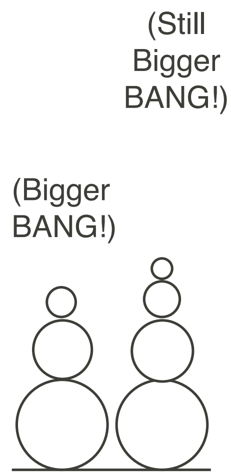
*QM Potential $V_0$-$A_y$ Amplitude* Connection


V=2.0 Bands
E
$E = \frac{N^2}{\omega_y^2}\frac{g}{\ell}$
QM Energy E-Related to-
$\omega_y$ Jerk frequency
$\omega_y = N\sqrt{\frac{g}{|E|\,\ell}}$
QM Potential $V_0$-$A_y$ Amplitude
Connection
$V_0 = \frac{N^2 A_y}{\ell}$
$B_2$
3+
3-
3- $B_1$
Stable Hanging
Band(2)
2+ $A_1$
Unstable Resonance
Gap (2)
2+
2- $A_2$
2-
Stable Hanging
Band(1)
1+ $B_2$
1+
Unstable Resonance
Gap (1)
1-
0+
1- $B_1$
0+ $A_1$
Stable Inverted
Band(0)
$V_0$



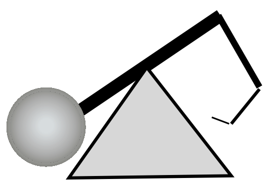
2+A1 Mode
ωy(A1)=0.9566
Ay=0.5
sinϕ(t)
t
2-A2 Mode
ωy(A2)=1.01054
Ay=0.5
sinϕ(t)
t
1+B2 Mode
ωy(B2)=1.4668
Ay=0.5
sinϕ(t)
Equivalent V-well bottoms
Equivalent V-well tops
t
E
V=2.0 Bands
B2
3+
3-
3- B1
Stable Hanging Band(2)
2+ A1
2+
Unstable Resonance Gap (2)
2- A2
2-
Stable Hanging Band(1)
1+ B2
1+
Unstable Resonance Gap (1)
1-
1- B1
0+ A1
0+
Stable Inverted Band(0)
V
1-B1 Mode
Ay=0.5
ωy(B1)=6.02475
sinϕ(t)
Equivalent V-well tops
t
0+A1 Mode
Ay=0.5
ωy(A1)=2.9646
sinϕ(t)
Y-acceleration:A(t)=-Ayωy2 cosωyt
Equivalent V-well bottoms
t
Unstable Resonance Gap (1)
t

*Supernova Superballs*




*Class of W. G. Harter,*
*"Velocity Amplification*
*in Collision Experiments*
*Involving Superballs,"*
*Am. J. Phys.*
**39**, 656 (1971)
(A class project )

# *Coming Next to Theaters Near You??!!*



*Super Trebuchet?*

*(Multi-frame)*

*Supersonic?*

*Most important: Quantum multiframe trebuchets...they're already inside you! (Proteins RNA)*