Equations of Lagrange and Hamilton mechanics in Generalized Curvilinear Coordinates (GCC)
(Ch. 12 of Unit 1 and Ch. 1-5 of Unit 2 and Ch. 1-5 of Unit 3)

Quick Review of Lagrange Relations in Lectures 7-8

Using differential chain-rules for coordinate transformations
Polar coordinate example of Generalized Curvilinear Coordinates (GCC)
Getting the GCC ready for mechanics: Generalized velocity and Jacobian Lemma 1
Getting the GCC ready for mechanics: Generalized acceleration and Lemma 2

How to say Newton’s “F=ma” in Generalized Curvilinear Coords.
Use Cartesian KE quadratic form $KE=T=\frac{1}{2}v\cdot M\cdot v$ and $F=M\cdot a$ to get GCC force
Lagrange GCC trickery gives Lagrange force equations
Lagrange GCC trickery gives Lagrange potential equations (Lagrange 1 and 2)

GCC Cells, base vectors, and metric tensors
Polar coordinate examples: Covariant $E_m$ vs. Contravariant $E^m$
Covariant $g_{mn}$ vs. Invariant $\delta_{m}^{n}$ vs. Contravariant $g^{mn}$

Lagrange prefers Covariant $g_{mn}$ with Contravariant velocity
GCC Lagrangian definition
GCC “canonical” momentum $p_{m}$ definition
GCC “canonical” force $F_{m}$ definition
Coriolis “fictitious” forces (... and weather effects)
This Lecture’s Reference Link Listing

Web Resources - front page
UAF Physics UTube channel
Quantum Theory for the Computer Age
Principles of Symmetry, Dynamics, and Spectroscopy
Classical Mechanics with a Bang!
Modern Physics and its Classical Foundations
2017 Group Theory for QM
2018 Adv CM
2018 AMOP
2019 Advanced Mechanics

CMwithBang Lecture 8, page=20
WWW.sciencenewsforstudents.org: Cassini - Saturnian polar vortex

Select, exciting, and related Research & Articles of Interest:
These are hot off the presses. Out in MISC for quick reference.

- Burning a hole in reality—design for a new laser may be powerful enough to pierce space-time - Sumner-Daily KOS-2019
- Trampoline mirror may push laser pulse through fabric of the Universe - Lee-ArsTechnica-2019
- Achieving Extreme Light Intensities using Optically Curved Relativistic Plasma Mirrors - Vincenti-prl-2019
- Correlated Insulator Behaviour at Half-Filling in Magic-Angle Graphene Superlattices - cao-n-2018
- Sorting ultracold atoms in a three-dimensional optical lattice in a realization of Maxwell’s Demon - Kumar-n-2018
- Synthetic three-dimensional atomic structures assembled atom by atom - Barredo-n-2018

Older ones:
- Wave-particle duality of C60 molecules - Arndt-ltn-1999
- Optical Vortex Knots - One Photon At A Time - Tempone-Wiltshire-Sr-2018
- Hadronic Molecules - Guo-x-2017
- Hidden-charm pentaquark and tetraquark states - Chen-pr-2016
"RelaWavity" Web Simulations:
   2-CW laser wave, Lagrangian vs Hamiltonian,
   Physical Terms Lagrangian L(u) vs Hamiltonian H(p)
CoulIt Web Simulation of the Volcanoes of Io
BohrIt Multi-Panel Plot:
   Relativistically shifted Time-Space plots of 2 CW light waves

RelaWavity Web Elliptical Motion Simulations:
   Orbits with b/a=0.125
   Orbits with b/a=0.5
   Orbits with b/a=0.7
   Exegesis with b/a=0.125
   Exegesis with b/a=0.5
   Exegesis with b/a=0.7
   Contact Ellipsometry

AMOP Ch 0 Space-Time Symmetry - 2019
Seminar at Rochester Institute of Optics, Aux. slides-2018

BoxIt Web Simulations:
   Generic/Default
   Most Basic A-Type
   Basic A-Type w/reference lines
   Basic A-Type A-Type with Potential energy
   A-Type with Potential energy and Stokes Plot
   A-Type w/3 time rates of change
   A-Type w/3 time rates of change with Stokes Plot
   B-Type (A=1.0, B=-0.05, C=0.0, D=1.0)

Pirelli Site: Phasors animation
CMwithBang Lecture #6, page=70 (9.10.18)
BounceIt Web Animation - Scenarios:

**Generic Scenario:** 2-Balls dropped no Gravity (7:1) - V vs V Plot (Power=4)
- 1-Ball dropped w/Gravity=0.5 w/Potential Plot: Power=1, Power=4
- 7:1 - V vs V Plot: Power=1
- 3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=4
- 3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=1 w/Gaps
- 4-Ball Stack (27:9:3:1) w/Newton plot (y vs t) - Power=4
- 4-Ball Stack (27:9:3:1) w/Newton plot (y vs t) - Power=1 w/Gaps
- 3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=1 w/Gaps
- 6-Ball Totally Inelastic (1:1:1:1:1:1) w/Gaps: Newtonian plot (t vs x)
- 5-Ball Totally Inelastic Pile-up w/ 5-Stationary-Balls - Minkowski plot (t vs x1) w/Gaps
- 1-Ball Totally Inelastic Pile-up w/ 5-Stationary-Balls - Vx2 vs Vx1 plot w/Gaps

**BounceIt Dual plots**

- $m_1:m_2=3:1$
  - $v_2$ vs $v_1$ and $V_2$ vs $V_1$: $(v_1, v_2)=(1, 0.1), (v_1, v_2)=(1, 0)$
  - $v_2$ vs $y_1$ plots: $(v_1, v_2)=(1, 0.1), (v_1, v_2)=(1, 0), (v_1, v_2)=(1, -1)$
  - Estrangian plot $V_2$ vs $V_1$: $(v_1, v_2)=(0, 1), (v_1, v_2)=(1, -1)$

- $m_1:m_2=4:1$
  - $v_2$ vs $v_1$, $y_2$ vs $v_1$

- $m_1:m_2=100:1$, $(v_1, v_2)=(1, 0)$: V2 vs V1 Estrangian plot, $y_2$ vs $y_1$ plot

With g=0 and 70:10 mass ratio
- With non zero g, velocity dependent damping and mass ratio of 70:35
- $M_1=49, M_2=1$ with Newtonian time plot
- $M_1=49, M_2=1$ with $V_2$ vs $V_1$ plot

**Example with friction**
- Low force constant with drag displaying a Pass-thru, Fall-Thru, Bounce-Off
- $m_1:m_2=3:1$ and $(v_1, v_2)=(1, 0)$ Comparison with Estrangian

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AJP article on superball dynamics

AAPT Summer Reading List

Scitation.org - AIP publications

HarterSoft Youtube Channel

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X2 paper: Velocity Amplification in Collision Experiments Involving Superballs - Harter et al. 1971 (pdf)

Car Collision Web Simulator: https://modphys.hosted.uark.edu/markup/CMMotionWeb.html

Superball Collision Web Simulator: https://modphys.hosted.uark.edu/markup/BounceItWeb.html; with Scenarios: 1007

BounceIt web simulation with g=0 and 70:10 mass ratio
- With non zero g, velocity dependent damping and mass ratio of 70:35
- Elastic Collision Dual Panel Space vs Space: Space vs Time (Newton), Time vs. Space(Minkowski)
- Inelastic Collision Dual Panel Space vs Space: Space vs Time (Newton), Time vs. Space(Minkowski)
- Matrix Collision Simulator: $M_1=49, M_2=1$ V2 vs V1 plot

More Advanced QM and classical references at the end of this Lecture
Quick Review of Lagrange Relations in Lectures 7-8

0th and 1st equations of Lagrange and Hamilton
Quick Review of Lagrange Relations in Lectures 7-8

0th and 1st equations of Lagrange and Hamilton

Starts out with simple demands for explicit-dependence, “loyalty” or “fealty to the colors”

Lagrangian and Estrangian have no explicit dependence on momentum \( p \)

\[
\frac{\partial L}{\partial p_k} \equiv 0 \equiv \frac{\partial E}{\partial p_k}
\]

Hamiltonian and Estrangian have no explicit dependence on velocity \( v \)

\[
\frac{\partial H}{\partial v_k} \equiv 0 \equiv \frac{\partial E}{\partial v_k}
\]

Lagrangian and Hamiltonian have no explicit dependence on speedinum \( V \)

\[
\frac{\partial L}{\partial V_k} \equiv 0 \equiv \frac{\partial H}{\partial V_k}
\]

Such non-dependencies hold in spite of “under-the-table” matrix and partial-differential connections

\[
\nabla_v L = \frac{\partial L}{\partial v} = \frac{\partial}{\partial v} \left( \mathbf{v} \cdot \mathbf{M} \cdot \mathbf{v} \right) = \mathbf{M} \cdot \mathbf{v} = \mathbf{p}
\]

\[
\begin{bmatrix}
\frac{\partial L}{\partial v_1} \\
\frac{\partial L}{\partial v_2}
\end{bmatrix} = \begin{bmatrix} m_1 & 0 \\
0 & m_2
\end{bmatrix} \begin{bmatrix} v_1 \\
v_2
\end{bmatrix} = \begin{bmatrix} p_1 \\
p_2
\end{bmatrix}
\]

Lagrange’s 1st equation(s)

\[
\frac{\partial L}{\partial v_k} = p_k \quad \text{or:} \quad \frac{\partial L}{\partial v} = \mathbf{p}
\]

\[
\nabla_p H = \mathbf{v} = \frac{\partial H}{\partial p} = \frac{\partial}{\partial p} \left( \mathbf{p} \cdot \mathbf{M}^{-1} \cdot \mathbf{p} \right) = \mathbf{M}^{-1} \cdot \mathbf{p} = \mathbf{v}
\]

(Forget Estrangian for now)

\[
\begin{bmatrix}
\frac{\partial H}{\partial p_1} \\
\frac{\partial H}{\partial p_2}
\end{bmatrix} = \begin{bmatrix} m_1^{-1} & 0 \\
0 & m_2^{-1}
\end{bmatrix} \begin{bmatrix} p_1 \\
p_2
\end{bmatrix} = \begin{bmatrix} v_1 \\
v_2
\end{bmatrix}
\]

Hamilton’s 1st equation(s)

\[
\frac{\partial H}{\partial p_k} = v_k \quad \text{or:} \quad \frac{\partial H}{\partial \mathbf{p}} = \mathbf{v}
\]

p. 28 of Lecture 8
(a) Lagrangian plot
\[ L(v) = \text{const.} = v \cdot M \cdot v / 2 \]

(b) Hamiltonian plot
\[ H(p) = \text{const.} = p \cdot M^{-1} \cdot p / 2 \]

(c) Overlapping plots
Lagrangian tangent at velocity \( v \)
is normal to momentum \( p \)

(d) Less mass

(e) More mass

\[ p_2 = m_2 v_2 \]
\[ p_1 = m_1 v_1 \]

\[ H = \text{const} = E \]

\[ L = \text{const} = E \]

\[ v_1 = p_1 / m_1 \]
\[ v_2 = p_2 / m_2 \]

\[ a_x = \sqrt{2E/m_1} \]
\[ a_y = \sqrt{2E/m_2} \]
Unit 1
Fig. 12.2

(a) Lagrangian plot
$L(v) = \text{const.} = v \cdot M \cdot v / 2$

(b) Hamiltonian plot
$H(p) = \text{const.} = p \cdot M^{-1} \cdot p / 2$

(c) Overlapping plots

1st equation of Lagrange
$L = \text{const.} = E$

1st equation of Hamilton
$H = \text{const.} = E$

(d) Less mass

(e) More mass
Using differential chain-rules for coordinate transformations

Polar coordinate example of Generalized Curvilinear Coordinates (GCC)
Getting the GCC ready for mechanics: Generalized velocity and Jacobian Lemma 1
Getting the GCC ready for mechanics: Generalized acceleration and Lemma 2
Using differential chain-rules for coordinate transformations

A pair of 2-variable functions \( f(x,y) \) and \( g(x,y) \) can define a coordinate system on \((x,y)\)-space

\[
\begin{align*}
    df(x,y) &= \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy \\
    dg(x,y) &= \frac{\partial g}{\partial x} \, dx + \frac{\partial g}{\partial y} \, dy
\end{align*}
\]

for example: polar coordinates

\[
\begin{align*}
    r^2(x,y) &= x^2 + y^2 \quad \text{and} \quad \theta(x,y) = \text{atan2}(y,x) \\
    dr(x,y) &= \frac{\partial r}{\partial x} \, dx + \frac{\partial r}{\partial y} \, dy \\
    d\theta(x,y) &= \frac{\partial \theta}{\partial x} \, dx + \frac{\partial \theta}{\partial y} \, dy
\end{align*}
\]

(Not in text. Recall Lecture 8 p. 6-22)
Using differential chain-rules for coordinate transformations

A pair of 2-variable functions \( f(x,y) \) and \( g(x,y) \) can define a coordinate system on \((x,y)\)-space for example: polar coordinates

\[
\begin{align*}
\frac{df(x,y)}{dx} &= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \\
\frac{dg(x,y)}{dx} &= \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy
\end{align*}
\]

\[
\begin{align*}
r^2(x,y) &= x^2 + y^2 \quad \text{and} \quad \theta(x,y) = \text{atan2}(y,x) \\
dr(x,y) &= \frac{\partial r}{\partial x} dx + \frac{\partial r}{\partial y} dy \\
d\theta(x,y) &= \frac{\partial \theta}{\partial x} dx + \frac{\partial \theta}{\partial y} dy
\end{align*}
\]

Easy to invert differential chain relations (even if functions are not easily inverted)

\[
\begin{align*}
\frac{dx}{df} &= \frac{\partial x}{\partial f} df + \frac{\partial y}{\partial f} dg \\
\frac{dy}{df} &= \frac{\partial y}{\partial f} df + \frac{\partial y}{\partial g} dg
\end{align*}
\]

\[
\begin{align*}
x &= r \cos \theta \\
y &= r \sin \theta
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
dx \\
\ dy
\end{bmatrix}
&= \begin{bmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}
\end{bmatrix}
\begin{bmatrix}
dr \\
\ d\theta
\end{bmatrix}
= \begin{bmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta
\end{bmatrix}
\begin{bmatrix}
dr \\
\ d\theta
\end{bmatrix}
\end{align*}
\]
Using differential chain-rules for coordinate transformations

A pair of 2-variable functions \( f(x,y) \) and \( g(x,y) \) can define a coordinate system on \((x,y)\)-space for example: polar coordinates 
\[
\begin{align*}
  r^2(x,y) &= x^2 + y^2 \quad \text{and} \quad \theta(x,y) = \text{atan2}(y,x) \\
  dr(x,y) &= \frac{\partial r}{\partial x} dx + \frac{\partial r}{\partial y} dy \\
  d\theta(x,y) &= \frac{\partial \theta}{\partial x} dx + \frac{\partial \theta}{\partial y} dy
\end{align*}
\]

Easy to invert differential chain relations (even if functions are not easily inverted)
\[
\begin{align*}
  dx &= \frac{\partial x}{\partial f} df + \frac{\partial x}{\partial g} dg \\
  dy &= \frac{\partial y}{\partial f} df + \frac{\partial y}{\partial g} dg \\
  x &= r \cos \theta \\
  y &= r \sin \theta \\
  dx &= \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta \\
  dy &= \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta
\end{align*}
\]

(Not in text. Recall Lecture 8 p. 6-22)

Notation for differential GCC (Generalized Curvilinear Coordinates \( \{q^1, q^2, q^3,\ldots\} \))
\[
dx^j = \frac{\partial x^j}{\partial q^m} dq^m \equiv \sum_{m=1}^{N} \frac{\partial x^j}{\partial q^m} dq^m \quad \{ \text{Defining a shorthand dummy-index } m\text{-sum} \}
\]

What does "q" stand for? One guess: "Queer" And they do get pretty queer!
Using differential chain-rules for coordinate transformations

A pair of 2-variable functions \( f(x,y) \) and \( g(x,y) \) can define a coordinate system on \((x,y)\)-space for example: polar coordinates

\[
\begin{align*}
\text{df}(x,y) &= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \\
\text{dg}(x,y) &= \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy
\end{align*}
\]

\( r^2(x,y) = x^2 + y^2 \) and \( \theta(x,y) = \text{atan2}(y,x) \)

\[
\begin{align*}
\text{dr}(x,y) &= \frac{\partial r}{\partial x} dx + \frac{\partial r}{\partial y} dy \\
\text{d} \theta(x,y) &= \frac{\partial \theta}{\partial x} dx + \frac{\partial \theta}{\partial y} dy
\end{align*}
\]

Easy to invert differential chain relations (even if functions are not easily inverted)

\[
\begin{align*}
\text{dx} &= \frac{\partial x}{\partial f} df + \frac{\partial x}{\partial g} dg \\
\text{dy} &= \frac{\partial y}{\partial f} df + \frac{\partial y}{\partial g} dg
\end{align*}
\]

\[
\begin{align*}
x &= r \cos \theta \\
y &= r \sin \theta
\end{align*}
\]

\[
\begin{align*}
\left(\begin{array}{c}
\text{dx} \\
\text{dy}
\end{array}\right) &= \frac{\partial x}{\partial r} \frac{\partial x}{\partial \theta} \begin{pmatrix}
\frac{\partial r}{\partial x} \\
\frac{\partial \theta}{\partial x}
\end{pmatrix} \begin{pmatrix}
\frac{\partial r}{\partial \theta} \\
\frac{\partial \theta}{\partial \theta}
\end{pmatrix} \begin{pmatrix}
\text{dr} \\
\text{d} \theta
\end{pmatrix} \\
&= \begin{pmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta
\end{pmatrix} \begin{pmatrix}
\text{dr} \\
\text{d} \theta
\end{pmatrix}
\end{align*}
\]

Index \( m \) REPEATED on SAME side of \( = \) is SUMMED

Notation for differential GCC (Generalized Curvilinear Coordinates \( \{q^1, q^2, q^3,\ldots\} \))

\[
\begin{align*}
\text{dx}^j &= \frac{\partial x^j}{\partial q^m} dq^m \\
&\equiv \sum_{m=1}^{N} \frac{\partial x^j}{\partial q^m} dq^m
\end{align*}
\]

\[
\begin{align*}
\left\{ \text{Defining a shorthand} \right. \\
\text{dummy-index} \ m \text{-sum} \left. \right\}
\end{align*}
\]

What does “q” stand for? One guess: “Queer” And they do get pretty queer!

Connection lines may help to indicate summation (OK on scratch paper...Difficult in text)

These \( x \) are plain old CC (Cartesian Coordinates \( \{dx^1=dx, dx^2=dy, dx^3=dx, dx^4=dt\} \) )
Using differential chain-rules for coordinate transformations
Polar coordinate example of Generalized Curvilinear Coordinates (GCC)
Getting the GCC ready for mechanics: Generalized velocity and Jacobian Lemma 1
Getting the GCC ready for mechanics: Generalized acceleration and Lemma 2
Getting the GCC ready for mechanics:
Generalized velocity relation follows from GCC chain rule

\[
\begin{align*}
   dx^j &= \frac{\partial x^j}{\partial q^m} dq^m \\
   \dot{x}^j &= \frac{\partial x^j}{\partial q^m} \dot{q}^m 
\end{align*}
\]

Same kind of linear relation exists between CC velocity \( \dot{v}^j \equiv \frac{dx^j}{dt} \) and GCC velocity \( \dot{v}^m \equiv \frac{dq^m}{dt} \)
Getting the GCC ready for mechanics:

*Generalized velocity relation follows from GCC chain rule*

\[
dx^j = \frac{\partial x^j}{\partial q^m} dq^m
\]

Same kind of linear relation exists between CC velocity \( v^j \equiv \dot{x}^j \equiv \frac{dx^j}{dt} \) and GCC velocity \( v^m \equiv q^m \equiv \frac{dq^m}{dt} \):

\[
\dot{x}^j = \frac{\partial x^j}{\partial q^m} q^m
\]

This is a key “lemma-1” for setting up mechanics:

\[
\frac{\partial \dot{x}^j}{\partial q^m} = \frac{\partial x^j}{\partial q^m} \quad \text{or:} \quad \frac{\partial \dot{x}^j}{\partial q^m} = \frac{\partial x^j}{\partial q^m}
\]
Getting the GCC ready for mechanics:

Generalized velocity relation follows from GCC chain rule:

\[
\begin{align*}
\dot{x}^j &= \frac{\partial x^j}{\partial q^m} \dot{q}^m \\
\end{align*}
\]

Same kind of linear relation exists between CC velocity \( v^j \equiv \dot{x}^j \equiv \frac{dx^j}{dt} \) and GCC velocity \( v^m \equiv \dot{q}^m \equiv \frac{dq^m}{dt} \):

\[
\begin{align*}
\dot{x}^j &= \frac{\partial x^j}{\partial q^m} \dot{q}^m \\
\end{align*}
\]

This is a key “lemma-1” for setting up mechanics:

\[
\begin{align*}
\frac{\partial \dot{x}^j}{\partial \dot{q}^m} &= \frac{\partial x^j}{\partial q^m} \\
\end{align*}
\]

or:

\[
\begin{align*}
\frac{\partial \dot{x}^j}{\partial q^m} &= \frac{\partial x^j}{\partial q^m} \\
\end{align*}
\]

Jacobian \( J_{mj} \) matrix gives each CCC differential \( dx^j \) or velocity \( \dot{x}^j \) in terms of GCC \( dq^m \) or \( \dot{q}^m \).

\[
J_{mj} \equiv \frac{\partial x^j}{\partial q^m} = \frac{\partial \dot{x}^j}{\partial q^m} \quad \text{Defining Jacobian matrix component} \\
\]

Recall polar coordinate transformation matrix:

\[
\begin{pmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}
\end{pmatrix} = \begin{pmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta
\end{pmatrix}
\]
Getting the GCC ready for mechanics:

Generalized velocity relation follows from GCC chain rule

\[ dx^j = \frac{\partial x^j}{\partial q^m} dq^m \]

Same kind of linear relation exists between CC velocity \( v^j \equiv \dot{x}^j \equiv \frac{dx^j}{dt} \) and GCC velocity \( v^m \equiv q^m \equiv \frac{dq^m}{dt} \)

\[ \dot{x}^j = \frac{\partial x^j}{\partial q^m} q^m \]

\[ \frac{\partial x^j}{\partial q^m} = \frac{\partial x^j}{\partial q^m} \]

This is a key "lemma-1" for setting up mechanics:

Jacobian \( J^m_j \) matrix gives each CCC differential \( dx^j \) or velocity \( \dot{x}^j \) in terms of GCC \( dq^m \) or \( q^m \).

\[ J^m_j = \frac{\partial x^j}{\partial q^m} = \frac{\partial x^j}{\partial q^m} \]

[Defining Jacobian matrix component]

Recall polar coordinate transformation matrix:

\[ \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} \]

Inverse (so-called) \( K^m_j \) matrix is flipped partial derivatives of \( J^m_j \).

\[ K^m_j = \frac{\partial q^m}{\partial x^j} = \frac{\partial q^m}{\partial \dot{x}^j} \]

[Defining "Kajobian" (inverse to Jacobian)]

Polar coordinate inverse transformation matrix:

\[ \begin{pmatrix} \frac{\partial r}{\partial r} & \frac{\partial r}{\partial \theta} \\ \frac{\partial \theta}{\partial r} & \frac{\partial \theta}{\partial \theta} \end{pmatrix}^{-1} = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} \]

Defining 2x2 matrix inverse: (always test inverse matrices!)

\[ \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} \]

\[ \frac{1}{AD - BC} \]
Getting the GCC ready for mechanics:

Generalized velocity relation follows from GCC chain rule

\[ dx^j = \frac{\partial x^j}{\partial q^m} dq^m \]

Same kind of linear relation exists between CC velocity \( v^i \equiv \dot{x}^j \equiv \frac{dx^j}{dt} \) and GCC velocity \( \nu^m \equiv q^m \equiv \frac{dq^m}{dt} \)

\[ \dot{x}^j = \frac{\partial x^j}{\partial q^m} \dot{q}^m \]

This is a key “lemma-1” for setting up mechanics:

Jacobian \( J^j_m \) matrix gives each CCC differential \( dx^j \) or velocity \( \dot{x}^j \) in terms of GCC \( dq^m \) or \( \dot{q}^m \).

\[
J^j_m \equiv \frac{\partial x^j}{\partial q^m} = \frac{\partial x^j}{\partial q^m} \quad \{ \text{Defining Jacobian matrix component} \}
\]

Recall polar coordinate transformation matrix:

\[
\begin{pmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}
\end{pmatrix}
= \begin{pmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta
\end{pmatrix}
\]

Inverse (so-called) Kajobian \( K^m_j \) matrix is flipped partial derivatives of \( J^j_m \).

\[
K^m_j \equiv \frac{\partial q^m}{\partial x^j} = \frac{\partial q^m}{\partial x^j} \quad \{ \text{Defining "Kajobian"} \}
\]

Polar coordinate inverse transformation matrix:

\[
\begin{pmatrix}
\frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial r} \\
\frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial r}
\end{pmatrix}^{-1}
= \begin{pmatrix}
\frac{\partial r}{\partial \theta} & \frac{\partial r}{\partial \theta} \\
\frac{\partial \theta}{\partial r} & \frac{\partial \theta}{\partial r}
\end{pmatrix}
= \begin{pmatrix}
\cos \theta & \frac{\partial \theta}{\partial \theta} \\
\sin \theta & \frac{\partial \theta}{\partial \theta}
\end{pmatrix}
= \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\]

Defining 2x2 matrix inverse: (always test inverse matrices!)

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}^{-1}
= \frac{1}{AD - BC}
\begin{pmatrix}
D & -B \\
-C & A
\end{pmatrix}
= \frac{1}{AD - BC}
\begin{pmatrix}
D & -B \\
-C & A
\end{pmatrix}
= \frac{1}{AD - BC}
\begin{pmatrix}
A & -B \\
0 & AD - BC
\end{pmatrix}
\]

Defining 2x2 matrix inverse: (always test inverse matrices!)

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
= \frac{1}{AD - BC}
\begin{pmatrix}
D & -B \\
-C & A
\end{pmatrix}
= \frac{1}{AD - BC}
\begin{pmatrix}
A & -B \\
0 & AD - BC
\end{pmatrix}
\]
Getting the GCC ready for mechanics:

Generalized velocity relation follows from GCC chain rule:

\[ dx^j = \frac{\partial x^j}{\partial q^m} dq^m \]

Same kind of linear relation exists between CC velocity \( v^j \equiv \dot{x}^j = \frac{dx^j}{dt} \) and GCC velocity \( v^m \equiv \dot{q}^m = \frac{dq^m}{dt} \):

\[ \dot{x}^j = \frac{\partial x^j}{\partial q^m} \dot{q}^m \]

This is a key “lemma-1” for setting up mechanics:

\[ \frac{\partial \dot{x}^j}{\partial \dot{q}^m} = \frac{\partial x^j}{\partial q^m} \]  \( \text{lemma-1} \)

Jacobian \( J^m_j \) matrix gives each CCC differential \( dx^j \) or velocity \( \dot{x}^j \) in terms of GCC \( dq^m \) or \( \dot{q}^m \).

\[ J^m_j \equiv \frac{\partial x^j}{\partial q^m} = \frac{\partial \dot{x}^j}{\partial \dot{q}^m} \quad \{ \text{Defining Jacobian matrix component} \} \]

Recall polar coordinate transformation matrix:

\[ \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} \]

Inverse (so-called) Kajobian \( K^m_j \) matrix is flipped partial derivatives of \( J^m_j \).

\[ K^m_j \equiv \frac{\partial q^m}{\partial x^j} = \frac{\partial \dot{q}^m}{\partial \dot{x}^j} \quad \{ \text{Defining "Kajobian" (inverse to Jacobian)} \} \]

\[ \frac{\partial q^m}{\partial x^j} = \frac{\partial \dot{q}^m}{\partial \dot{x}^j} = \frac{1}{(\det J = r)} \begin{pmatrix} r \cos \theta & r \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} \cos \theta & -\frac{\sin \theta}{r} \\ \frac{\sin \theta}{r} & \frac{\cos \theta}{r} \end{pmatrix} \]

Product of matrix \( J^m_j \) and \( K^m_j \) is a unit matrix by definition of partial derivatives. (always test inverse matrices!)

\[ K^m_j \cdot J^j_n \equiv \frac{\partial q^m}{\partial x^j} \cdot \frac{\partial x^j}{\partial q^n} = \frac{\partial q^m}{\partial q^n} = \delta_n^m = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \]

\[ \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -r \cos \theta & r \sin \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]
Using differential chain-rules for coordinate transformations

Polar coordinate example of Generalized Curvilinear Coordinates (GCC)

Getting the GCC ready for mechanics: Generalized velocity and Jacobian Lemma 1

Getting the GCC ready for mechanics: Generalized acceleration and Lemma 2
Getting the GCC ready for mechanics (2nd part)

Generalized acceleration relations are a little more complicated (It’s curved coords, after all!)

First apply $\frac{d}{dt}$ to velocity $\dot{x}^j$ and use product rule: $\frac{d}{dt}(u \cdot v) = \frac{du}{dt} \cdot v + u \cdot \frac{dv}{dt}$

$$
\ddot{x}^j \equiv \frac{d}{dt} \dot{x}^j = \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \dot{q}^m \right) = \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) \ddot{q}^m + \frac{\partial x^j}{\partial q^m} \dddot{q}^m
$$
Getting the GCC ready for mechanics (2\textsuperscript{nd} part)

Generalized acceleration relations are a little more complicated (It’s curved coords, after all!)

First apply $\frac{d}{dt}$ to velocity $\dot{x}^j$ and use product rule: $\frac{d}{dt}(u \cdot v) = \frac{du}{dt} \cdot v + u \cdot \frac{dv}{dt}$

$$\ddot{x}^j \equiv \frac{d}{dt} \dot{x}^j = \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \dot{q}^m \right) = \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) \ddot{q}^m + \frac{\partial x^j}{\partial q^m} \dddot{q}^m$$

Apply derivative chain sum to Jacobian.

$$\frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) = \frac{\partial}{\partial q^n} \left( \frac{\partial x^j}{\partial q^m} \right) \frac{dq^n}{dt} = \left( \frac{\partial^2 x^j}{\partial q^n \partial q^m} \right) \frac{dq^n}{dt}$$
Getting the GCC ready for mechanics (2nd part)

Generalized acceleration relations are a little more complicated (It’s curved coords, after all!)

First apply $\frac{d}{dt}$ to velocity $\dot{x}^j$ and use product rule: $\frac{d}{dt}(u \cdot v) = \frac{du}{dt} \cdot v + u \cdot \frac{dv}{dt}$

$$\ddot{x}^j \equiv \frac{d}{dt} \dot{x}^j = \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \dot{q}^m \right) = \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) \dot{q}^m + \frac{\partial x^j}{\partial q^m} \ddot{q}^m$$

Apply derivative chain sum to Jacobian. Partial derivatives are reversible. $\partial_m \partial_n = \partial_n \partial_m$

$$\frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) = \frac{\partial}{\partial q^n} \left( \frac{\partial x^j}{\partial q^m} \right) \frac{dq^n}{dt} = \left( \frac{\partial^2 x^j}{\partial q^n \partial q^m} \right) \frac{dq^n}{dt} = \frac{\partial}{\partial q^m} \left( \frac{\partial x^j}{\partial q^n} \frac{dq^n}{dt} \right)$$

Important thing about mechanics to recall:
coordinates $q^n$

velocities $\frac{dq^m}{dt} = \dot{q}^m$
Getting the GCC ready for mechanics (2\textsuperscript{nd} part)

Generalized acceleration relations are a little more complicated (It’s curved coords, after all!)

First apply $\frac{d}{dt}$ to velocity $\dot{x}^j$ and use product rule: 

$$
\ddot{x}^j \equiv \frac{d}{dt} \dot{x}^j = \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \dot{q}^m \right) = \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) \dot{q}^m + \frac{\partial x^j}{\partial q^m} \ddot{q}^m
$$

Apply derivative chain sum to Jacobian. Partial derivatives are reversible. $\frac{\partial}{\partial q^n} \frac{\partial}{\partial q^m} = \frac{\partial}{\partial q^m} \frac{\partial}{\partial q^n}$

$$
\frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) = \frac{\partial}{\partial q^n} \left( \frac{\partial x^j}{\partial q^m} \right) \frac{dq^n}{dt} = \left( \frac{\partial}{\partial q^n} \frac{\partial x^j}{\partial q^m} \right) \frac{dq^n}{dt} = \frac{\partial}{\partial q^m} \left( \frac{\partial x^j}{\partial q^n} \frac{dq^n}{dt} \right)
$$

Important thing about mechanics to recall:

coordinates $q^n$ independent of velocities $\frac{dq^m}{dt} = \dot{q}^m$

By chain-rule def. of CC velocity:

$$
= \frac{\partial}{\partial q^m} \left( \dot{x}^j \right)
$$
Getting the GCC ready for mechanics (2nd part)

Generalized acceleration relations are a little more complicated (It’s curved coords, after all!)

First apply $\frac{d}{dt}$ to velocity $\dot{x}^j$ and use product rule: $\frac{d}{dt}(u \cdot v) = \frac{du}{dt} \cdot v + u \cdot \frac{dv}{dt}$

$$\ddot{x}^j \equiv \frac{d}{dt} \dot{x}^j = \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \dot{q}^m \right) = \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) \dot{q}^m + \frac{\partial x^j}{\partial q^m} \ddot{q}^m$$

(Not in text. Recall Lecture 9 p. 15-19)$^\dagger$

Apply derivative chain sum to Jacobian. Partial derivatives are reversible. $\partial_m \partial_n = \partial_n \partial_m$

$$\frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) = \frac{\partial}{\partial q^n} \left( \frac{\partial x^j}{\partial q^m} \right) \frac{dq^n}{dt} = \left( \frac{\partial^2 x^j}{\partial q^n \partial q^m} \right) \frac{dq^n}{dt} = \frac{\partial}{\partial q^m} \left( \frac{\partial x^j}{\partial q^n} \frac{dq^n}{dt} \right)$$

Important thing about mechanics to recall:
coordinates $q^n$

independent of

velocities $\frac{dq^m}{dt} = \dot{q}^m$

By chain-rule def. of CC velocity:

$$= \frac{\partial}{\partial q^m} (\dot{x}^j)$$

This is the key “lemma-2” for setting up Lagrangian mechanics.

$$\frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) = \frac{\partial \dot{x}^j}{\partial q^m}$$

lemma $^2$
Getting the GCC ready for mechanics (2nd part)

Generalized acceleration relations are a little more complicated (It's curved coords, after all!)

First apply $\frac{d}{dt}$ to velocity $\dot{x}^j$ and use product rule: $\frac{d}{dt}(u \cdot v) = \frac{du}{dt} \cdot v + u \cdot \frac{dv}{dt}$

$$\ddot{x}^j \equiv \frac{d}{dt} \dot{x}^j = \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \dot{q}^m \right) = \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) \dot{q}^m + \frac{\partial x^j}{\partial q^m} \ddot{q}^m$$

(Not in text. Recall Lecture 9 p. 15-19)†

Apply derivative chain sum to Jacobian. Partial derivatives are reversible. $\partial_m \partial_n = \partial_n \partial_m$

$$\frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) = \frac{\partial}{\partial q^n} \left( \frac{\partial x^j}{\partial q^m} \right) \frac{dq^n}{dt} = \left( \frac{\partial^2 x^j}{\partial q^n \partial q^m} \right) \frac{dq^n}{dt} = \frac{\partial}{\partial q^m} \left( \frac{\partial x^j}{\partial q^n} \frac{dq^n}{dt} \right)$$

By chain-rule def. of CC velocity:

$$= \frac{\partial}{\partial q^m} (\dot{x}^j)$$

The “lemma-1” was in the GCC velocity analysis just before this one for acceleration.

This is the key “lemma-2” for setting up Lagrangian mechanics.

$$\frac{\partial x^j}{\partial \dot{q}^m} = \frac{\partial x^j}{\partial q^m} \quad \text{lemma 1}$$

$$\frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) = \frac{\partial \dot{x}^j}{\partial q^m} \quad \text{lemma 2}$$
How to say Newton’s “F=ma” in Generalized Curvilinear Coords.

Use Cartesian KE quadratic form $KE = T = \frac{1}{2}v \cdot M \cdot v$ and $F = M \cdot a$ to get GCC force.
Lagrange GCC trickery gives Lagrange force equations.
Lagrange GCC trickery gives Lagrange potential equations (Lagrange 1 and 2).
Deriving GCC mechanics from Cartesian Coord. (CC) Newton I-II
Start with stuff we know...(sort of)

Multidimensional CC version of kinetic energy $\frac{1}{2} \mathbf{v} \cdot \mathbf{M} \cdot \mathbf{v}$

$T = \frac{1}{2} M_{jk} \, v^j v^k = \frac{1}{2} M_{jk} \, \dot{x}^j \dot{x}^k \quad \text{where: } M_{jk} \text{ are CC inertia constants}$

Multidimensional CC version of Newt-II ($\mathbf{F} = \mathbf{M} \cdot \mathbf{a}$) using $M_{jk}$ constants

$f_j = M_{jk} \, a^k = M_{jk} \, \ddot{x}^k$
Deriving GCC mechanics from Cartesian Coord. (CC) Newton I-II

Start with stuff we know...(sort of)

Multidimensional CC version of kinetic energy $\frac{1}{2} \mathbf{v} \cdot \mathbf{M} \cdot \mathbf{v}$

$$T = \frac{1}{2} M_{jk} v^j v^k = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k$$

where: $M_{jk}$ are inertia constants that are symmetric: $M_{jk}=M_{kj}$

Multidimensional CC version of Newt-II ($\mathbf{F} = \mathbf{M} \cdot \mathbf{a}$) using $M_{jk}$ constants

$$f_j = M_{jk} a^k = M_{jk} \ddot{x}^k$$

Multidimensional CC version of work-energy differential ($dW = \mathbf{F} \cdot d\mathbf{x}$). Insert GCC differentials $dq^m$

$$dW = f_j dx^j = f_j \left( \frac{\partial x^j}{\partial q^m} dq^m \right) = M_{jk} \ddot{x}^k \left( \frac{\partial x^j}{\partial q^m} dq^m \right)$$

(It’s time to bring in the queer $q^m$ !)
Deriving GCC mechanics from Cartesian Coord. (CC) Newton I-II

Start with stuff we know...(sort of)

Multidimensional CC version of kinetic energy \( \frac{1}{2} \mathbf{v} \cdot \mathbf{M} \cdot \mathbf{v} \)

\[
T = \frac{1}{2} M_{jk} v^j v^k = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k \quad \text{where: } M_{jk} \text{ are inertia constants that are symmetric: } M_{jk} = M_{kj}
\]

Multidimensional CC version of Newt-II (\( \mathbf{F} = \mathbf{M} \cdot \mathbf{a} \)) using \( M_{jk} \) constants

\[
f_j = M_{jk} a^k = M_{jk} \ddot{x}^k
\]

Multidimensional CC version of work-energy differential (\( dW = \mathbf{F} \cdot d\mathbf{x} \)). Insert GCC differentials \( dq^m \)

\[
dW = f_j dx^j = f_j \left( \frac{\partial x^j}{\partial q^m} dq^m \right) = M_{jk} \ddot{x}^k \left( \frac{\partial x^j}{\partial q^m} dq^m \right)
\]

\( dq^m \) are independent so \( dq^m \)-sum is true term-by-term.

\[
dW = f_j dx^j = F_m dq^m = f_j \frac{\partial x^j}{\partial q^m} dq^m = M_{jk} \ddot{x}^k \frac{\partial x^j}{\partial q^m} dq^m
\]
Deriving GCC mechanics from Cartesian Coord. (CC) Newton I-II
Start with stuff we know...(sort of)

Multidimensional CC version of kinetic energy \( T = \frac{1}{2} v \cdot M \cdot v \)

\[
T = \frac{1}{2} M_{jk} \ v^j v^k = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k \quad \text{where: } M_{jk} \text{ are inertia constants}
\]

Multidimensional CC version of Newt-II (\( F = M \cdot a \)) using \( M_{jk} \) constants

\[
f_j = M_{jk} \ a^k = M_{jk} \dot{x}^k
\]

Multidimensional CC version of work-energy differential (\( dW = F \cdot dx \)). Insert GCC differentials \( dq^m \)

\[
dW = f_j dx^j = f_j \left( \frac{\partial x^j}{\partial q^m} dq^m \right) = M_{jk} \dot{x}^k \left( \frac{\partial x^j}{\partial q^m} dq^m \right)
\]

(\( dq^m \) are independent so \( dq^m \)-sum is true term-by-term. (Still holds if all \( dq^m \) are zero but one.)

\[
dW = f_j dx^j = F_m dq^m = f_j \frac{\partial x^j}{\partial q^m} dq^m = M_{jk} \dot{x}^k \frac{\partial x^j}{\partial q^m} dq^m \quad \Rightarrow \quad F_m = f_j \frac{\partial x^j}{\partial q^m} = M_{jk} \dot{x}^k \frac{\partial x^j}{\partial q^m}
\]
Deriving GCC mechanics from Cartesian Coord. (CC) Newton I-II
Start with stuff we know...(sort of)

Multidimensional CC version of kinetic energy \( \frac{1}{2} \mathbf{v} \cdot \mathbf{M} \cdot \mathbf{v} \)

\[
T = \frac{1}{2} M_{jk} v^j v^k = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k
\]
where: \( M_{jk} \) are inertia constants

Multidimensional CC version of Newt-II (\( \mathbf{F} = \mathbf{M} \cdot \mathbf{a} \)) using \( M_{jk} \) constants

\[
f_j = M_{jk} a^k = M_{jk} \ddot{x}^k
\]

Multidimensional CC version of work-energy differential (\( dW = \mathbf{F} \cdot d\mathbf{x} \)). Insert GCC differentials \( dq^m \)

\[
dW = f_j dx^j = f_j \left( \frac{\partial x^j}{\partial q^m} dq^m \right) = M_{jk} \ddot{x}^k \left( \frac{\partial x^j}{\partial q^m} dq^m \right)
\]

(\( dq^m \) are independent so \( dq^m \)-sum is true term-by-term. (Still holds if all \( dq^m \) are zero but one.)

\[
dW = f_j dx^j = F_m dq^m = f_j \frac{\partial x^j}{\partial q^m} dq^m = M_{jk} \ddot{x}^k \frac{\partial x^j}{\partial q^m} dq^m \Rightarrow F_m = f_j \frac{\partial x^j}{\partial q^m} = M_{jk} \ddot{x}^k \frac{\partial x^j}{\partial q^m}
\]

Here generalized GCC force component \( F_m \) is defined:

\[
where: \quad F_m = f_j \frac{\partial x^j}{\partial q^m} = M_{jk} \ddot{x}^k \frac{\partial x^j}{\partial q^m}
\]
How to say Newton’s “F=ma” in Generalized Curvilinear Coords.

Use Cartesian KE quadratic form $KE = T = \frac{1}{2}v \cdot M \cdot v$ and $F = M \cdot a$ to get GCC force

Lagrange GCC trickery gives Lagrange force equations
Lagrange GCC trickery gives Lagrange potential equations (Lagrange 1 and 2)
Now Lagrange GCC trickery begins

Obvious stuff...(sort of, if you’ve looked at it for a century!)

Lagrange’s clever end game: First set $A = M_{jk} \dddot{x}^k$ and $B = \frac{\partial x^j}{\partial q^m}$ with calc. formula: $[\dddot{A}B = \frac{d}{dt}(\dddot{A}B) - \dddot{A}\dddot{B}]$

$$F_m = f_j \frac{\partial x^j}{\partial q^m} = M_{jk} \dddot{x}^k \frac{\partial x^j}{\partial q^m} = \frac{d}{dt}\left(M_{jk} \dddot{x}^k \frac{\partial x^j}{\partial q^m}\right) - M_{jk} \dddot{x}^k \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^m}\right)$$
Now Lagrange GCC trickery begins
Obvious stuff...(sort of, if you’ve looked at it for a century!)

Lagrange’s clever end game:  First set $A = M_{jk} \dot{x}^k$ and $B = \frac{\partial x^j}{\partial q^m}$ with calc. formula:

$$\ddot{A}B = \frac{d}{dt}(\dot{A}B) - \dot{A}\dot{B}$$

$$F_m = f_j \frac{\partial x^j}{\partial q^m} = M_{jk} \dot{x}^k \frac{\partial x^j}{\partial q^m} = \frac{d}{dt} \left( M_{jk} \dot{x}^k \frac{\partial x^j}{\partial q^m} \right) - M_{jk} \dot{x}^k \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right)$$

Cartesian $M_{jk}$ must be constant for this to work
(Bye, Bye relativistic mechanics or QM!)
Now Lagrange GCC trickery begins
Obvious stuff...(sort of, if you’ve looked at it for a century!)

Lagrange’s clever end game: First set $A = M_{jk} \dot{x}^k$ and $B = \frac{\partial x^j}{\partial q^m}$ with calc. formula: $\ddot{A}B = \frac{d}{dt}(\dot{A}B) - \dot{A}\dot{B}$

$$F_m = f_j \frac{\partial x^j}{\partial q^m} = M_{jk} \dot{x}^k \frac{\partial x^j}{\partial q^m} = \frac{d}{dt} \left( M_{jk} \dot{x}^k \frac{\partial x^j}{\partial q^m} \right) - M_{jk} \dot{x}^k \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right)$$

Then convert $\partial x^j$ to $\dot{x}^j$ by Lemma 1 and Lemma 2 on 2nd term.

$F_m = \frac{d}{dt} \left( M_{jk} \dot{x}^k \frac{\partial \dot{x}^j}{\partial q^m} \right) - M_{jk} \dot{x}^k \left( \frac{\partial \dot{x}^j}{\partial q^m} \right)$

Cartesian $M_{jk}$ must be constant for this to work
(Bye, Bye relativistic mechanics or QM!)

\[
\frac{\partial \dot{x}^j}{\partial \dot{q}^m} = \frac{\partial x^j}{\partial q^m} \quad \text{lemma 1}
\]

\[
\frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) = \frac{\partial \dot{x}^j}{\partial q^m} \quad \text{lemma 2}
\]
Now Lagrange GCC trickery begins

Lagrange’s clever end game: First set $A = M_{jk} \ddot{x}^k$ and $B = \frac{\partial x^j}{\partial q^m}$ with calc. formula: $\ddot{A}B = \frac{d}{dt}(\dot{A}B) - \dot{A} \dot{B}$

$$F_m = f_j \frac{\partial x^j}{\partial q^m} = M_{jk} \ddot{x}^k \frac{\partial x^j}{\partial q^m} = \frac{d}{dt} \left( M_{jk} \dot{x}^k \frac{\partial x^j}{\partial q^m} \right) - M_{jk} \dot{x}^k \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right)$$

Then convert $\partial x^j$ to $\dot{x}^j$ by Lemma 1 and Lemma 2 on 2nd term.

$$F_m = \frac{d}{dt} \left( M_{jk} \dot{x}^k \frac{\partial \dot{x}^j}{\partial \dot{q}^m} \right) - M_{jk} \dot{x}^k \left( \frac{\partial \dot{x}^j}{\partial q^m} \right)$$

Simplify using: $\left[ M_{ij} v_i \frac{\partial v^j}{\partial q} = M_{ij} \frac{\partial}{\partial q} \frac{v^i v^j}{2} \right]$ where $q$ may be $\dot{q}^m$ or $q^m$

$$F_m = \frac{d}{dt} \frac{\partial}{\partial \dot{q}^m} \left( \frac{M_{jk} \dot{x}^k \dot{x}^j}{2} \right) - \frac{\partial}{\partial q^m} \left( \frac{M_{jk} \dot{x}^k \dot{x}^j}{2} \right)$$

$$\frac{\partial \dot{x}^j}{\partial \dot{q}^m} = \frac{\partial x^j}{\partial q^m} \text{ lemma 1}$$

$$\frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right) = \frac{\partial \dot{x}^j}{\partial q^m} \text{ lemma 2}$$
Now Lagrange GCC trickery begins
Obvious stuff...(sort of, if you’ve looked at it for a century!)

Lagrange’s clever end game: First set $A = M_{jk} \dot{x}^k$ and $B = \frac{\partial x^j}{\partial q^m}$ with calc. formula: $\ddot{AB} = \frac{d}{dt} (\dot{AB}) - \dot{AB}$

$$F_m = f_j \frac{\partial x^j}{\partial q^m} = M_{jk} \dot{x}^k \frac{\partial x^j}{\partial q^m} = \frac{d}{dt} \left( M_{jk} \dot{x}^k \frac{\partial x^j}{\partial q^m} \right) - M_{jk} \dot{x}^k \frac{d}{dt} \left( \frac{\partial x^j}{\partial q^m} \right)$$

Then convert $\partial x^j$ to $\partial \dot{x}^j$ by Lemma 1 and Lemma 2 on 2nd term.

Simplify using: $M_{ij} v^i \frac{\partial v^j}{\partial q} = M_{ij} \frac{\partial}{\partial q} \frac{v^i v^j}{2}$ where $q$ may be $\dot{q}^m$ or $q^m$

$$F_m = \frac{d}{dt} \frac{\partial}{\partial q^m} \left( \frac{M_{jk} \dot{x}^k \dot{x}^j}{2} \right) - \frac{\partial}{\partial q^m} \left( \frac{M_{jk} \dot{x}^k \dot{x}^j}{2} \right)$$

The result is Lagrange’s GCC force equation in terms of kinetic energy $T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k$

$$F_m = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^m} - \frac{\partial T}{\partial q^m}$$

or: $F = \frac{d}{dt} \frac{\partial T}{\partial v} - \frac{\partial T}{\partial r}$
How to say Newton’s “$F=ma$” in Generalized Curvilinear Coords.

Use Cartesian KE quadratic form $KE = T = \frac{1}{2}v \cdot M \cdot v$ and $F = M \cdot a$ to get GCC force.

Lagrange GCC trickery gives Lagrange force equations.

Lagrange GCC trickery gives Lagrange potential equations (Lagrange 1 and 2).
But, Lagrange GCC trickery is not yet done...
(Still another trick-up-the-sleeve!)

If the force is conservative it’s a gradient $\mathbf{F} = -\nabla U$

In GCC: $F_m = -\frac{\partial U}{\partial q^m}$

$$F_m = -\frac{\partial U}{\partial q^m} = \frac{dT}{dt} \frac{\partial T}{\partial q^m} - \frac{\partial T}{\partial q^m}$$
But, Lagrange GCC trickery is not yet done...
(Still another trick-up-the-sleeve!)
If the force is conservative it’s a gradient $\mathbf{F} = -\nabla U$
In GCC: $F_m = -\frac{\partial U}{\partial q^m}$

$$F_m = -\frac{\partial U}{\partial q^m} = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^m} - \frac{\partial T}{\partial q^m}$$

Becomes Lagrange’s GCC potential equation with a new definition for the Lagrangian: $L=T-U$.

$$0 = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^m} - \frac{\partial L}{\partial q^m}$$

$L(\dot{q}^m, q^m) = T(\dot{q}^m, q^m) - U(q^m)$

This trick requires: $\frac{\partial U}{\partial \dot{q}^m} \equiv 0$  
$U(r)$ has NO explicit velocity dependence!
If the force is conservative it’s a gradient $\mathbf{F} = -\nabla U$

In GCC: $F_m = -\frac{\partial U}{\partial q^m}$

$F_m = -\frac{\partial U}{\partial q^m} = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^m} - \frac{\partial T}{\partial q^m}$

Becomes Lagrange’s GCC potential equation with a new definition for the Lagrangian: $L=T-U$.

$0 = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^m} - \frac{\partial L}{\partial q^m}$

$L(\dot{q}^m, q^m) = T(\dot{q}^m, q^m) - U(q^m)$

This trick requires: $\frac{\partial U}{\partial \dot{q}^m} \equiv 0$

$U(r)$ has NO explicit velocity dependence!

Lagrange’s 1st GCC equation (Defining GCC momentum)

$p_m = \frac{\partial L}{\partial \dot{q}^m}$

Recall: $p = \frac{\partial L}{\partial \dot{q}^m}$

Lagrange’s 2nd GCC equation (Change of GCC momentum)

$\frac{dp_m}{dt} \equiv \dot{p}_m = \frac{\partial L}{\partial q^m}$
But, Lagrange GCC trickery is not yet done...
(Still another trick-up-the-sleeve!)
If the force is conservative it’s a gradient $\mathbf{F} = -\nabla U$

In GCC: $F_m = -\frac{\partial U}{\partial q^m}$

$F_m = -\frac{\partial U}{\partial q^m} = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^m} - \frac{\partial T}{\partial q^m}$

Becomes Lagrange’s GCC potential equation with a new definition for the Lagrangian: $L = T-U$.

$0 = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^m} - \frac{\partial L}{\partial q^m}$

$L(\dot{q}^m, q^m) = T(\dot{q}^m, q^m) - U(q^m)$

This trick requires: $\frac{\partial U}{\partial \dot{q}^m} \equiv 0$

U(r) has NO explicit velocity dependence!

If $L$ has no explicit $q^m$ dependence then:
$\dot{p}_m = 0$
or:
$p_m = \text{const.}$

Lagrange’s 1$^{st}$ GCC equation
(Defining GCC momentum)

$p_m = \frac{\partial L}{\partial \dot{q}^m}$

Recall:
$p = \frac{\partial L}{\partial \dot{v}}$

Lagrange’s 2$^{nd}$ GCC equation
(Change of GCC momentum)

$\frac{dp_m}{dt} \equiv \dot{p}_m = \frac{\partial L}{\partial q^m}$
GCC Cells, base vectors, and metric tensors

- Polar coordinate examples: **Covariant** $E_m$ vs. **Contravariant** $E^m$
- **Covariant** $g_{mn}$ vs. **Invariant** $\delta_{mn}$ vs. **Contravariant** $g^{mn}$
A dual set of quasi-unit vectors show up in Jacobian J and Kajobian K.

J-Columns are covariant vectors \( \{E_1 = E_r, E_2 = E_\phi\} \)  
K-Rows are contravariant vectors \( \{E^1 = E^r, E^2 = E^\phi\} \)

\[
\begin{bmatrix}
\frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} \\
\frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2}
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial x}{\partial r} = \cos \phi & \frac{\partial x}{\partial \phi} = -r \sin \phi \\
\frac{\partial y}{\partial r} = \sin \phi & \frac{\partial y}{\partial \phi} = r \cos \phi
\end{bmatrix}
\]

\[
K = J^{-1} = \begin{bmatrix}
\frac{\partial r}{\partial x} = \cos \phi & \frac{\partial r}{\partial y} = \sin \phi \\
\frac{\partial \phi}{\partial x} = -\frac{\sin \phi}{r} & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{r}
\end{bmatrix}
\]

Derived from polar definition: \( x = r \cos \phi \) and \( y = r \sin \phi \)

\[
\begin{bmatrix}
E^1 = E^r \\
E^2 = E^\phi
\end{bmatrix}
\]

\[
\begin{bmatrix}
E_1 = E_r \\
E_2 = E_\phi
\end{bmatrix}
\]

Inverse polar definition:
\( r^2 = x^2 + y^2 \) and \( \phi = \text{atan2}(y,x) \)

(a) Polar coordinate bases

Unit 1
Fig. 12.10
A dual set of quasi-unit vectors show up in Jacobian $J$ and Kajobian $K$.

J-Columns are covariant vectors $\{E_1 = E_r, E_2 = E_\phi\}$

K-Rows are contravariant vectors $\{E^1 = E^r, E^2 = E^\phi\}$

\[
(J) = \begin{pmatrix}
\frac{\partial x}{\partial q^1} & \frac{\partial x}{\partial q^2} \\
\frac{\partial x}{\partial q^1} & \frac{\partial x}{\partial q^2}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial x}{\partial r} = \cos \phi & \frac{\partial x}{\partial \phi} = -r \sin \phi \\
\frac{\partial y}{\partial r} = \sin \phi & \frac{\partial y}{\partial \phi} = r \cos \phi
\end{pmatrix}
\]

\[
(K) = \langle J^{-1} \rangle = \begin{pmatrix}
\frac{\partial r}{\partial x} = \cos \phi & \frac{\partial r}{\partial y} = \sin \phi \\
\frac{\partial \phi}{\partial x} = -\frac{\sin \phi}{r} & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{r}
\end{pmatrix} \leftarrow E^r = E_1
\]

\[
\leftarrow E^\phi = E_2
\]

Inverse polar definition:
$r^2 = x^2 + y^2$ and $\phi = \text{atan2}(y,x)$

\[
\text{Derived from polar definition: } x = r \cos \phi \text{ and } y = r \sin \phi
\]

(a) Polar coordinate bases

(b) Covariant bases $\{E_1 E_2\}$
(Tangent)

\[
dr = E_1 dq^1 + E_2 dq^2
\]

(c) Contra\text{variant bases} $\{E^1 E^2\}$
(Normal)

\[
F = F_1 E^1 + F_2 E^2
\]

NOTE: These are 2D drawings! No 3D perspective.
**Comparison:** **Covariant** \( E_m = \frac{\partial r}{\partial q^m} \) vs. **Contravariant** \( E^m = \frac{\partial q^m}{\partial r} = \nabla q^m \)

**Covariant bases** \( \{ E_1, E_2 \} \) match cell walls

\[
\Delta r = E_1 \Delta q^1 + E_2 \Delta q^2
\]

is based on chain rule:

\[
dr = \frac{\partial r}{\partial q^1} dq^1 + \frac{\partial r}{\partial q^2} dq^2 = E_1 dq^1 + E_2 dq^2
\]

**NOTE:** These are 2D drawings! **No 3D perspective**
Comparison: **Covariant** $E_m = \frac{\partial r}{\partial q^m}$ vs. **Contravariant** $E^m = \frac{\partial q^m}{\partial r} = \nabla q^m$

**Covariant bases** $\{E_1, E_2\}$ match cell walls

$$\Delta r = E_1 \Delta q^1 + E_2 \Delta q^2$$

is based on chain rule: 

$$dr = \frac{\partial r}{\partial q^1} dq^1 + \frac{\partial r}{\partial q^2} dq^2 = E_1 dq^1 + E_2 dq^2$$

$E_1$ follows tangent to $q^2 = \text{const.}$ ... since only $q^1$ varies in $\frac{\partial r}{\partial q^1}$ while $q^2, q^3, ...$ remain constant

NOTE: These are 2D drawings! **No 3D perspective**
Comparison: **Covariant** $E_m = \frac{\partial r}{\partial q^m}$ vs. **Contravariant** $E^m = \frac{\partial q^m}{\partial r} = \nabla q^m$

Covariant bases $\{E_1, E_2\}$ match cell walls

$\Delta r = E_1 \Delta q^1 + E_2 \Delta q^2$

is based on chain rule: $d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^1} dq^1 + \frac{\partial \mathbf{r}}{\partial q^2} dq^2 = E_1 dq^1 + E_2 dq^2$

$E_1$ follows tangent to $q^2 = \text{const.}$ ...

since only $q^1$ varies in $\frac{\partial r}{\partial q^1}$

while $q^2$, $q^3$, ... remain constant

$E_m$ are convenient bases for *extensive* quantities like distance and velocity.

$$V = V^1 E_1 + V^2 E_2 = V^1 \frac{\partial \mathbf{r}}{\partial q^1} + V^2 \frac{\partial \mathbf{r}}{\partial q^2}$$

**NOTE:** These are 2D drawings!

*No 3D perspective*
Comparison: **Covariant** \( \mathbf{E}_m = \frac{\partial \mathbf{r}}{\partial q^m} \) vs. **Contravariant** \( \mathbf{E}^m = \frac{\partial q^m}{\partial \mathbf{r}} = \nabla q^m \)

Covariant bases \( \{ \mathbf{E}_1, \mathbf{E}_2 \} \) match cell walls

\[ \Delta \mathbf{r} = \mathbf{E}_1 \Delta q^1 + \mathbf{E}_2 \Delta q^2 \]

is based on chain rule:

\[ d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^1} dq^1 + \frac{\partial \mathbf{r}}{\partial q^2} dq^2 = \mathbf{E}_1 dq^1 + \mathbf{E}_2 dq^2 \]

\( \mathbf{E}_1 \) follows tangent to \( q^2 = \text{const.} \) ...

since only \( q^1 \) varies in \( \frac{\partial \mathbf{r}}{\partial q^1} \)

while \( q^2, q^3, \ldots \) remain constant

\( \mathbf{E}_m \) are convenient bases for extensive quantities like distance and velocity.

\[ \mathbf{V} = V^1 \mathbf{E}_1 + V^2 \mathbf{E}_2 = V^1 \frac{\partial \mathbf{r}}{\partial q^1} + V^2 \frac{\partial \mathbf{r}}{\partial q^2} \]

Contravariant \( \{ \mathbf{E}^1, \mathbf{E}^2 \} \) match reciprocal cells

\[ \frac{\partial q^2}{\partial \mathbf{r}} = \nabla q^2 = \mathbf{E}^2 \]

\[ \mathbf{F} = F^1 \mathbf{E}^1 + F^2 \mathbf{E}^2 \]

\( \mathbf{E}^1 \) is normal to \( q^1 = \text{const.} \) since gradient of \( q^1 \) is vector sum \( \nabla q^1 = \frac{\partial q^1}{\partial x} \hat{x} + \frac{\partial q^1}{\partial y} \hat{y} \)

**NOTE:** These are 2D drawings! No 3D perspective
Comparison: **Covariant** \( E_m = \frac{\partial r}{\partial q_m} \) vs. **Contravariant** \( E^m = \frac{\partial q^m}{\partial r} = \nabla q^m \)

*Covariant bases* \( \{E_1, E_2\} \) match cell walls

\[
\Delta r = E_1 \Delta q^1 + E_2 \Delta q^2
\]

is based on chain rule:

\[
dr = \frac{\partial r}{\partial q^1} dq^1 + \frac{\partial r}{\partial q^2} dq^2 = E_1 dq^1 + E_2 dq^2
\]

\( E_1 \) follows *tangent* to \( q^2 = \text{const.} \) ...

since only \( q^1 \) varies in \( \frac{\partial r}{\partial q^1} \)

while \( q^2, q^3, \ldots \) remain constant

\( E_m \) are convenient bases for *extensive* quantities like distance and velocity.

\[
V = V^1 E_1 + V^2 E_2 = V^1 \frac{\partial r}{\partial q^1} + V^2 \frac{\partial r}{\partial q^2}
\]

*Contravariant* \( \{E^1, E^2\} \) match reciprocal cells

\[
\frac{\partial q^2}{\partial r} = \nabla q^2 = E^2
\]

\[
F = F_1 E^1 + F_2 E^2
\]

\( E^1 \) is *normal* to \( q^1 = \text{const.} \) since

gradient of \( q^1 \) is vector sum \( \nabla q^1 = \left[ \frac{\partial q^1}{\partial x}, \frac{\partial q^1}{\partial y} \right] \)

\( E^m \) are convenient bases for *intensive* quantities like force and momentum.

\[
F = F_1 E^1 + F_2 E^2 = F_1 \frac{\partial q^1}{\partial r} + F_2 \frac{\partial q^2}{\partial r} = F_1 \nabla q^1 + F_2 \nabla q^2
\]
Comparison: **Covariant** \( E_m = \frac{\partial r}{\partial q^m} \) vs. **Contravariant** \( E^n = \frac{\partial q^n}{\partial r} = \nabla q^n \)

**Covariant bases** \( \{E_1, E_2\} \) match **cell walls** (Tangent)

\[
\Delta r = E_1 \Delta q^1 + E_2 \Delta q^2
\]

**Contravariant bases** \( \{E^1, E^2\} \) match **reciprocal cells** (Normal)

\[
E^1 = \frac{\partial q^1}{\partial r} = \nabla q^1
\]

\[
E^2 = \frac{\partial q^2}{\partial r} = \nabla q^2
\]

\( E^m \) are convenient bases for **extensive quantities** like distance and velocity.

\[
V = V^1 E_1 + V^2 E_2 = V^1 \frac{\partial r}{\partial q^1} + V^2 \frac{\partial r}{\partial q^2}
\]

\( E_1 \) follows **tangent** to \( q^2 = \text{const.} \) ...

since only \( q^1 \) varies in \( \frac{\partial r}{\partial q^1} \)

while \( q^2, q^3, \ldots \) remain constant

\( E^m \) are convenient bases for **intensive quantities** like force and momentum.

\[
F = F_1 E^1 + F_2 E^2 = F_1 \frac{\partial q^1}{\partial r} + F_2 \frac{\partial q^2}{\partial r} = F_1 \nabla q^1 + F_2 \nabla q^2
\]

By chain rule:

\[
\frac{\partial q^n}{\partial q^m} = \delta^n_m
\]
GCC Cells, base vectors, and metric tensors

Polar coordinate examples: Covariant \( E_m \) vs. Contravariant \( E^m \)

\( \text{Covariant } g_{mn} \) vs. Invariant \( \delta_{m}^{n} \) vs. Contravariant \( g^{mn} \)
**Covariant** $g_{mn}$ vs. **Invariant** $\delta^m_n$ vs. **Contravariant** $g^{mn}$

\[
E_m \cdot E_n = \frac{\partial r}{\partial q^m} \cdot \frac{\partial r}{\partial q^n} \equiv g_{mn}
\]

**Covariant metric tensor** $g_{mn}$

\[
E_m \cdot E^m = \frac{\partial r}{\partial q^m} \cdot \frac{\partial q^n}{\partial r} = \delta^m_n
\]

**Invariant** Kroneker unit tensor $\delta^m_n$

\[
\delta^m_n \equiv \begin{cases} 
1 & \text{if } m = n \\
0 & \text{if } m \neq n 
\end{cases}
\]

\[
E^m \cdot E^n = \frac{\partial q^m}{\partial r} \cdot \frac{\partial q^n}{\partial r} \equiv g^{mn}
\]

**Contravariant metric tensor** $g^{mn}$
**Covariant** $g_{mn}$ **vs.** **Invariant** $\delta_m^n$ **vs.** **Contravariant** $g^{mn}$

\[
E_m \cdot E_n = \frac{\partial r}{\partial q^m} \cdot \frac{\partial r}{\partial q^n} \equiv g_{mn}
\]

\[
E_m \cdot E^n = \frac{\partial r}{\partial q^m} \cdot \frac{\partial q^n}{\partial r} = \delta^m_n
\]

\[
E^m \cdot E^n = \frac{\partial q^m}{\partial r} \cdot \frac{\partial q^n}{\partial r} \equiv g^{mn}
\]

**Covariant** metric tensor \[ g_{mn} \]

**Invariant** Kroneker unit tensor \[ \delta_m^n \equiv \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \]

**Contravariant** metric tensor \[ g^{mn} \]

Polar coordinate examples (again):

\[
\langle J \rangle = \begin{pmatrix}
\frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} \\
\frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial x}{\partial r} = \cos \phi & \frac{\partial x}{\partial \phi} = -r \sin \phi \\
\frac{\partial y}{\partial r} = \sin \phi & \frac{\partial y}{\partial \phi} = r \cos \phi
\end{pmatrix}
\]

\[
\langle K \rangle = \langle J^{-1} \rangle = \begin{pmatrix}
\frac{\partial r}{\partial x} = \cos \phi & \frac{\partial r}{\partial y} = \sin \phi \\
\frac{\partial \phi}{\partial x} = -\sin \phi & \frac{\partial \phi}{\partial y} = \cos \phi
\end{pmatrix} \leftarrow E^r = E^1
\]

\[
\langle J \rangle = \begin{pmatrix}
\frac{\partial x}{\partial x} = \cos \phi & \frac{\partial x}{\partial y} = -\sin \phi \\
\frac{\partial y}{\partial x} = 1/r & \frac{\partial y}{\partial y} = 1/r
\end{pmatrix} \leftarrow E^\phi = E^2
\]
**Covariant** $g_{mn}$ vs. **Invariant** $\delta_m^n$ vs. **Contravariant** $g^{mn}$

\[
E_m \cdot E_n = \frac{\partial r}{\partial q^m} \frac{\partial r}{\partial q^n} \equiv g_{mn}
\]

\[
E_m \cdot E^n = \frac{\partial r}{\partial q^m} \frac{\partial q^n}{\partial r} = \delta_m^n
\]

\[
E^m \cdot E^n = \frac{\partial q^m}{\partial r} \frac{\partial q^n}{\partial r} \equiv g^{mn}
\]

**Covariant** metric tensor $g_{mn}$

**Invariant** Kronecker unit tensor $\delta_m^n$

**Contravariant** metric tensor $g^{mn}$

\[
\delta_m^n \equiv \begin{cases} 
1 & \text{if } m = n \\
0 & \text{if } m \neq n
\end{cases}
\]

Polar coordinate examples (again):

\[
\begin{pmatrix}
\frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} \\
\frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2}
\end{pmatrix}
= \begin{pmatrix}
\frac{\partial x}{\partial r} = \cos \phi & \frac{\partial x}{\partial \phi} = -r \sin \phi \\
\frac{\partial y}{\partial r} = \sin \phi & \frac{\partial y}{\partial \phi} = r \cos \phi
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{\partial x}{\partial r} = \cos \phi & \frac{\partial y}{\partial r} = \sin \phi \\
\frac{\partial \phi}{\partial x} = -r \sin \phi & \frac{\partial \phi}{\partial y} = r \cos \phi
\end{pmatrix}
= \begin{pmatrix}
E^r & E^1 \\
E^\phi & E^2
\end{pmatrix}
\]

**Covariant** $g_{mn}$

\[
\begin{pmatrix}
g_{rr} & g_{r\phi} \\
g_{\phi r} & g_{\phi\phi}
\end{pmatrix}
= \begin{pmatrix}
E_r \cdot E_r & E_r \cdot E_\phi \\
E_\phi \cdot E_r & E_\phi \cdot E_\phi
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & r^2
\end{pmatrix}
\]

**Invariant** $\delta_{r\phi}$

\[
\begin{pmatrix}
\delta_r^r & \delta_r^\phi \\
\delta_\phi^r & \delta_\phi^\phi
\end{pmatrix}
= \begin{pmatrix}
E_r \cdot E_r & E_r \cdot E_\phi \\
E_\phi \cdot E_r & E_\phi \cdot E_\phi
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

**Contravariant** $g^{mn}$

\[
\begin{pmatrix}
g^{rr} & g^{r\phi} \\
g^{\phi r} & g^{\phi\phi}
\end{pmatrix}
= \begin{pmatrix}
E^r \cdot E^r & E^r \cdot E^\phi \\
E^\phi \cdot E^r & E^\phi \cdot E^\phi
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & 1/r^2
\end{pmatrix}
\]
Lagrange prefers Covariant $g_{mn}$ with Contravariant velocity $\dot{q}^m$.  
GCC Lagrangian definition
GCC “canonical” momentum $p_m$ definition
GCC “canonical” force $F_m$ definition
Coriolis “fictitious” forces (… and weather effects)
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity.

Lagrangian $L=KE-U$ is supposed to be explicit function of velocity.

$$L(v) = \frac{1}{2} M v \cdot v - U = \frac{1}{2} M \dot{r} \cdot \dot{r} - U = \frac{1}{2} M (E_m \dot{q}^m) \cdot (E_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})$$
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity.

Lagrangian KE-$U$ is supposed to be explicit function of velocity.

$$L(v) = \frac{1}{2} Mv \cdot v - U = \frac{1}{2} M\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - U = \frac{1}{2} M (E_m \dot{q}^m) \cdot (E_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})$$

Use polar coordinate **Covariant** $g_{nn}$ metric (page 53)

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi \phi} \end{pmatrix} \begin{pmatrix} E_r \cdot E_r & E_r \cdot E_\phi \\ E_\phi \cdot E_r & E_\phi \cdot E_\phi \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ r^2 \end{pmatrix}$$
Lagrange prefers \textbf{Covariant} $g_{mn}$ with \textbf{Contravariant} velocity.

Lagrangian KE-$U$ is supposed to be explicit function of velocity.

$$L(v) = \frac{1}{2} M v \cdot v - U = \frac{1}{2} M \dot{r} \cdot \dot{r} - U = \frac{1}{2} M (E_m \dot{q}^m) \cdot (E_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})$$

Use polar coordinate \textbf{Covariant} $g_{mn}$ metric (page 53)

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi \phi} \end{pmatrix} = \begin{pmatrix} E_r \cdot E_r & E_r \cdot E_\phi \\ E_\phi \cdot E_r & E_\phi \cdot E_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

This gives polar GCC form (Actually it's an OCC or Orthogonal Curvilinear Coordinate form)

$$L(\dot{r}, \dot{\phi}) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi \phi} \dot{\phi}^2) - U(r, \phi) = \frac{1}{2} M (1 \cdot \dot{r}^2 + r^2 \dot{\phi}^2) - U(r, \phi)$$
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity $\dot{q}^m$

GCC Lagrangian definition
GCC “canonical” momentum $p_m$ definition
GCC “canonical” force $F_m$ definition
Coriolis “fictitious” forces (... and weather effects)
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity.

Lagrangian KE-$U$ is supposed to be explicit function of velocity.

$$L(v) = \frac{1}{2}Mv \cdot v - U = \frac{1}{2}M\dot{r} \cdot \dot{r} - U = \frac{1}{2}M(E_m \dot{q}^m)(E_n \dot{q}^n) - U = \frac{1}{2}M(g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})$$

Use polar coordinate **Covariant** $g_{mn}$ metric (page 53)

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi \phi} \end{pmatrix} = \begin{pmatrix} E_r \cdot E_r & E_r \cdot E_\phi \\ E_\phi \cdot E_r & E_\phi \cdot E_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

This gives polar GCC form (Actually it’s an OCC or Orthogonal Curvilinear Coordinate form)

$$L(\dot{r},\dot{\phi}) = \frac{1}{2}M(g_{rr} \dot{r}^2 + g_{\phi \phi} \dot{\phi}^2) - U(r,\phi) = \frac{1}{2}M(1 \cdot \dot{r}^2 + r^2 \dot{\phi}^2) - U(r,\phi)$$

(From preceding page)
Lagrange prefers **Covariant** \( g_{mn} \) with **Contravariant** velocity.

Lagrangian KE-U is supposed to be explicit function of velocity.

\[
L(v) = \frac{1}{2} M v \cdot v - U = \frac{1}{2} M \dot{r} \cdot \dot{r} - U = \frac{1}{2} M (E_m \dot{q}^m) \cdot (E_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})
\]

Use polar coordinate **Covariant** \( g_{mn} \) metric (page 53)

\[
\begin{pmatrix}
g_{rr} & g_{r\phi} \\
g_{\phi r} & g_{\phi\phi}
\end{pmatrix} = \begin{pmatrix} E_r \cdot E_r & E_r \cdot E_\phi \\
E_\phi \cdot E_r & E_\phi \cdot E_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\
0 & r^2 \end{pmatrix}
\]

This gives polar GCC form (Actually it’s an OCC or Orthogonal Curvilinear Coordinate form)

\[
L(\dot{r}, \dot{\phi}) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r, \phi) = \frac{1}{2} M (1 \cdot \dot{r}^2 + r^2 \dot{\phi}^2) - U(r, \phi)
\]

GCC Lagrange equations follow. 1st L-equation is momentum \( p_m \) definition for each coordinate \( q^m \):

\[
p_r = \frac{\partial L}{\partial \dot{r}} = M \ g_{rr} \dot{r} = M \dot{r} \quad \text{Nothing too surprising;}
\quad \text{radial momentum } p_r \text{ has the usual linear } M \cdot v \text{ form}
\]
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity

Lagrangian KE-U is supposed to be explicit function of velocity.

$$L(\mathbf{v}) = \frac{1}{2} M \mathbf{v} \cdot \mathbf{v} - U = \frac{1}{2} M \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - U = \frac{1}{2} M (E_m \dot{q}^m) \cdot (E_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})$$

Use polar coordinate **Covariant** $g_{mn}$ metric (page 53)

$$\begin{pmatrix}
g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi}
\end{pmatrix} = \begin{pmatrix}
E_r \cdot E_r & E_r \cdot E_\phi \\ E_\phi \cdot E_r & E_\phi \cdot E_\phi
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\ 0 & r^2
\end{pmatrix}$$

This gives polar GCC form (Actually it’s an OCC or Orthogonal Curvilinear Coordinate form)

$$L(\dot{r}, \dot{\phi}) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r, \phi) = \frac{1}{2} M (1 \cdot \dot{r}^2 + r^2 \dot{\phi}^2) - U(r, \phi)$$

GCC Lagrange equations follow. **1st** L-equation is momentum $p_m$ definition for each coordinate $q^m$:

$$p_r = \frac{\partial L}{\partial \dot{r}} = M g_{rr} \dot{r} = M \dot{r}$$

Nothing too surprising; radial momentum $p_r$ has the usual linear $M \cdot \mathbf{v}$ form

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = M g_{\phi\phi} \dot{\phi} = M r^2 \dot{\phi}$$

Wow! $g_{\phi\phi}$ gives moment-of-inertia factor $Mr^2$ automatically for the angular momentum $p_\phi = Mr^2 \omega$.

---

Lagrange’s 1\textsuperscript{st} GCC equation
(Defining GCC momentum)

$$p_m = \frac{\partial L}{\partial \dot{q}^m}$$

Recall:

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}}$$

Lagrange’s 2\textsuperscript{nd} GCC equation
(Change of GCC momentum)

$$\frac{dp_m}{dt} \equiv \dot{p}_m = \frac{\partial L}{\partial q^m}$$
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant velocity** $q^m$

GCC Lagrangian definition
GCC “canonical” momentum $p_m$ definition
GCC “canonical” force $F_m$ definition
Coriolis “fictitious” forces (... and weather effects)
Lagrange prefers \textbf{Covariant} $g_{mn}$ with \textbf{Contravariant} velocity

Lagrangian KE-$U$ is supposed to be explicit function of \textbf{velocity}. 

$L(\mathbf{v}) = \frac{1}{2} M \mathbf{v} \cdot \mathbf{v} - U = \frac{1}{2} M \mathbf{\dot{r}} \cdot \mathbf{\dot{r}} - U = \frac{1}{2} M (\mathbf{E}_m \mathbf{\dot{q}}^m) \cdot (\mathbf{E}_n \mathbf{\dot{q}}^n) - U = \frac{1}{2} M (g_{mn} \mathbf{\dot{q}}^m \mathbf{\dot{q}}^n) - U = L(\mathbf{\dot{q}})$

Use polar coordinate \textbf{Covariant} $g_{mn}$ metric (page 53)

$$
\begin{pmatrix}
g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi}
\end{pmatrix} =
\begin{pmatrix}
\mathbf{E}_r \cdot \mathbf{E}_r & \mathbf{E}_r \cdot \mathbf{E}_\phi \\ \mathbf{E}_\phi \cdot \mathbf{E}_r & \mathbf{E}_\phi \cdot \mathbf{E}_\phi
\end{pmatrix} =
\begin{pmatrix}
1 & 0 \\ 0 & r^2
\end{pmatrix}
$$

This gives polar GCC form (Actually it’s an OCC or Orthogonal Curvilinear Coordinate form)

$L(\mathbf{\dot{r}}, \mathbf{\dot{\phi}}) = \frac{1}{2} M (g_{rr} \mathbf{\dot{r}}^2 + g_{\phi\phi} \mathbf{\dot{\phi}}^2) - U(r, \phi) = \frac{1}{2} M (1 \cdot \mathbf{\dot{r}}^2 + r^2 \mathbf{\dot{\phi}}^2) - U(r, \phi)$

GCC Lagrange equations follow. \textit{1st} $L$-equation is momentum $p_m$ definition for each coordinate $q^m$:

$$p_r = \frac{\partial L}{\partial \dot{r}} = M \ g_{rr} \dot{r} = M \dot{r} \quad \text{Nothing too surprising; radial momentum } p_r \text{ has the usual linear } M \cdot \mathbf{v} \text{ form}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = M g_{\phi\phi} \dot{\phi} = M r^2 \dot{\phi} \quad \text{Wow! } g_{\phi\phi} \text{ gives moment-of-inertia factor } M r^2 \text{ automatically for the angular momentum } p_\phi = M r^2 \omega.$$ 

\textit{(From preceding page)}
Lagrange prefers **Covariant** \( g_{mn} \) with **Contravariant** velocity.

Lagrangian KE-U is supposed to be explicit function of velocity.

\[
L(v) = \frac{1}{2} M v \cdot v - U = \frac{1}{2} M \dot{r} \cdot \dot{r} - U = \frac{1}{2} M \left( E_m \dot{q}_m \right) \cdot \left( E_n \dot{q}_n \right) - U = \frac{1}{2} M \left( g_{mn} \dot{q}_m \dot{q}_n \right) - U = L(\dot{q})
\]

Use polar coordinate **Covariant** \( g_{mn} \) metric (page 53)

\[
\begin{pmatrix}
g_{rr} & g_{r\phi} \\
g_{\phi r} & g_{\phi \phi}
\end{pmatrix} = \begin{pmatrix} E_r \cdot E_r & E_r \cdot E_\phi \\
E_\phi \cdot E_r & E_\phi \cdot E_\phi
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\
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\end{pmatrix}
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\[
L(\dot{r}, \dot{\phi}) = \frac{1}{2} M \left( g_{rr} \dot{r}^2 + g_{\phi \phi} \dot{\phi}^2 \right) - U(r, \phi) = \frac{1}{2} M \left( 1 \cdot \dot{r}^2 + r^2 \dot{\phi}^2 \right) - U(r, \phi)
\]

**GCC Lagrange equations follow. 1st L-equation is momentum** \( p_m \) **definition for each coordinate** \( q^m \):

\[
p_r = \frac{\partial L}{\partial \dot{r}} = M g_{rr} \dot{r} = M \dot{r}
\]

Nothing too surprising; radial momentum \( p_r \) has the usual linear \( M \cdot v \) form.

\[
p_\phi = \frac{\partial L}{\partial \dot{\phi}} = M g_{\phi \phi} \dot{\phi} = M r^2 \dot{\phi}
\]

Wow! \( g_{\phi \phi} \) gives moment-of-inertia factor \( M r^2 \) automatically for the angular momentum \( p_\phi = M r^2 \omega \).

**2nd L-equation involves total time derivative** \( \dot{p}_m \) **for each momentum** \( p_m \):

\[
\dot{p}_r = \frac{\partial L}{\partial r} = M \frac{d}{dr} \frac{\partial g_{\phi \phi}}{\partial \dot{\phi}} \dot{\phi}^2 - \frac{\partial U}{\partial \dot{r}} = M r \dot{\phi}^2 - \frac{\partial U}{\partial \dot{r}}
\]

Centrifugal force \( M r \omega^2 \)

\[
\dot{p}_\phi = \frac{\partial L}{\partial \phi} = 0 - \frac{\partial U}{\partial \phi}
\]

Angular momentum \( p_\phi \) is conserved if potential \( U \) has no explicit \( \phi \)-dependence.

---

**Lagrange's 1st GCC equation** (Defining GCC momentum)

\[
p_m = \frac{\partial L}{\partial \dot{q}^m}
\]

Recall:

\[
p = \frac{\partial L}{\partial v}
\]

**Lagrange's 2nd GCC equation** (Change of GCC momentum)

\[
\frac{dp_m}{dt} \equiv \dot{p}_m = \frac{\partial L}{\partial q^m}
\]
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity

Lagrangian KE-U is supposed to be explicit function of **velocity**.

$$L(\mathbf{v}) = \frac{1}{2} M \mathbf{v} \cdot \mathbf{v} - U = \frac{1}{2} M \mathbf{\dot{r}} \cdot \mathbf{\dot{r}} - U = \frac{1}{2} M (E_m \dot{q}^m) \cdot (E_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})$$

Use polar coordinate **Covariant** $g_{mn}$ metric (page 53)

$$\begin{pmatrix}
g_{rr} & g_{r \phi} \\
g_{\phi r} & g_{\phi \phi}
\end{pmatrix} = \begin{pmatrix}
E_r \cdot E_r & E_r \cdot E_\phi \\
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$$L(\dot{r}, \dot{\phi}) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi \phi} \dot{\phi}^2) - U(r, \phi) = \frac{1}{2} M (1 \cdot \dot{r}^2 + r^2 \dot{\phi}^2) - U(r, \phi)$$

GCC Lagrange equations follow. **1st** L-equation is momentum $p_m$ definition for each coordinate $q^m$:

$$p_r = \frac{\partial L}{\partial \dot{r}} = M g_{rr} \dot{r} = M \dot{r}$$

Nothing too surprising; radial momentum $p_r$ has the usual linear $M \cdot \mathbf{v}$ form

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Wow! $g_{\phi \phi}$ gives moment-of-inertia factor $Mr^2$ automatically for the angular momentum $p_\phi = Mr^2 \omega$.

**2nd** L-equation involves total time derivative $\dot{p}_m$ for each momentum $p_m$:

$$\dot{p}_r = \frac{\partial L}{\partial \dot{r}} = \frac{M}{2} \frac{\partial g_{\phi \phi}}{\partial r} \dot{\phi}^2 - \frac{\partial U}{\partial r} = M r \dot{\phi}^2 - \frac{\partial U}{\partial r}$$

Centrifugal force $Mr\omega^2$

$$\dot{p}_\phi = \frac{\partial L}{\partial \dot{\phi}} = 0 - \frac{\partial U}{\partial \phi}$$

Angular momentum $p_\phi$ is conserved if potential $U$ has no explicit $\phi$-dependence

Find $\dot{p}_m$ directly from **1st** L-equation: $\dot{p}_m = \frac{dp_m}{dt} = \frac{d}{dt} M (g_{mn} \dot{q}^n) = M (g_{mn} \ddot{q}^n + g_{mn} \dot{q}^n) \quad \text{Equate it to } \dot{p}_m \text{ in } \text{2nd} \text{ L-equation}:
Lagrange prefers Covariant $g_{mn}$ with Contravariant velocity $\dot{q}^m$

GCC Lagrangian definition
GCC “canonical” momentum $p_m$ definition
GCC “canonical” force $F_m$ definition
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$$L(v) = \frac{1}{2} Mv \cdot v - U = \frac{1}{2} M(\dot{r} \cdot \dot{r} - U = \frac{1}{2} M(E_m \dot{q}^m) \cdot (E_n \dot{q}^n) - U = \frac{1}{2} M(g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})$$

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\begin{pmatrix}
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E_\phi \cdot E_r & E_\phi \cdot E_\phi
\end{pmatrix} =
\begin{pmatrix}
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$$L(\dot{r}, \dot{\phi}) = \frac{1}{2} M(g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r, \phi) = \frac{1}{2} M(1 \cdot \dot{r}^2 + r^2 \dot{\phi}^2) - U(r, \phi)$$

GCC Lagrange equations follow. 1st L-equation is momentum $p_m$ definition for each coordinate $q^m$:

$$p_r = \frac{\partial L}{\partial \dot{r}} = M g_{rr} \dot{r} = M \dot{r}$$

Nothing too surprising; radial momentum $p_r$ has the usual linear $M \cdot v$ form

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Wow! $g_{\phi\phi}$ gives moment-of-inertia factor $Mr^2$ automatically for the angular momentum $p_\phi = Mr^2 \omega$.

2nd L-equation involves total time derivative $\dot{p}_m$ for each momentum $p_m$:

$$\dot{p}_r = \frac{\partial L}{\partial \ddot{r}} = \frac{M}{2} \frac{\partial g_{\phi\phi}}{\partial r} \ddot{\phi}^2 - \frac{\partial U}{\partial r} = M r \ddot{\phi}^2 - \frac{\partial U}{\partial r}$$

Centrifugal force $M r \omega^2$

$$\dot{p}_\phi = \frac{\partial L}{\partial \ddot{\phi}} = 0 - \frac{\partial U}{\partial \phi}$$

Angular momentum $p_\phi$ is conserved if potential $U$ has no explicit $\phi$-dependence

Find $\dot{p}_m$ directly from 1st L-equation:

$$\dot{p}_m = \frac{d}{dt} M(g_{mn} \dot{q}^n) = M(\dot{g}_{mn} \dot{q}^n + g_{mn} \ddot{q}^n)$$

Equate it to $\dot{p}_m$ in 2nd L-equation:

$$\text{(From preceding page)}$$
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant** velocity

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$$L(\mathbf{v}) = \frac{1}{2} M \mathbf{v} \cdot \mathbf{v} - U = \frac{1}{2} M \mathbf{\dot{r}} \cdot \mathbf{\dot{r}} - U = \frac{1}{2} M (E_m \dot{q}^m) \cdot (E_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})$$

Use polar coordinate **Covariant** $g_{mn}$ metric (page 53) $$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} E_r \cdot E_r & E_r \cdot E_\phi \\ E_\phi \cdot E_r & E_\phi \cdot E_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

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$$L(\mathbf{\dot{r}}, \mathbf{\dot{\phi}}) = \frac{1}{2} M (g_{rr} \mathbf{\dot{r}}^2 + g_{\phi\phi} \mathbf{\dot{\phi}}^2) - U(r, \phi) = \frac{1}{2} M (1 \mathbf{\dot{r}}^2 + r^2 \mathbf{\dot{\phi}}^2) - U(r, \phi)$$

GCC Lagrange equations follow. 1st L-equation is momentum $p_m$ definition for each coordinate $q^m$:

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Wow! $g_{\phi\phi}$ gives moment-of-inertia factor $Mr^2$ automatically for the angular momentum $p_\phi = Mr^2 \omega$.

2nd L-equation involves total time derivative $\mathbf{\dot{p}}_m$ for each momentum $p_m$:

$$\mathbf{\dot{p}}_r = \frac{\partial L}{\partial r} = \frac{M}{2} \frac{\partial g_{rr}}{\partial r} \mathbf{\dot{r}}^2 - \frac{\partial U}{\partial r} = M r \mathbf{\dot{\phi}}^2 - \frac{\partial U}{\partial r}$$

Centrifugal force $Mr^2 \omega^2$

$$\mathbf{\dot{p}}_\phi = \frac{\partial L}{\partial \phi} = 0 - \frac{\partial U}{\partial \phi}$$

Angular momentum $p_\phi$ is conserved if potential $U$ has no explicit $\phi$-dependence

Find $\mathbf{\dot{p}}_m$ directly from 1st L-equation:

$$\mathbf{\dot{p}}_m = \frac{d p_m}{dt} = \frac{d}{dt} M (g_{mn} \dot{q}^n) = M (\dot{g}_{mn} \dot{q}^n + g_{mn} \ddot{q}^n)$$

Equate it to $\mathbf{\dot{p}}_m$ in 2nd L-equation:

$$\mathbf{\dot{p}}_r \equiv \frac{dp_r}{dt} = M \mathbf{\dot{r}}$$

Centrifugal (center-fleeing) force equals total Centripetal (center-pulling) force
Lagrange prefers **Covariant** $g_{mn}$ with **Contravariant velocity**

Lagrangian KE-U is supposed to be explicit function of velocity.

$$L(v) = \frac{1}{2} Mv \cdot v - U = \frac{1}{2} M \dot{r} \cdot \dot{r} - U = \frac{1}{2} M (E_m \dot{q}^m) \cdot (E_n \dot{q}^n) - U = \frac{1}{2} M (g_{mn} \dot{q}^m \dot{q}^n) - U = L(\dot{q})$$

Use polar coordinate **Covariant** $g_{mn}$ metric (page 53)

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} E_r \cdot E_r & E_r \cdot E_\phi \\ E_\phi \cdot E_r & E_\phi \cdot E_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

This gives polar GCC form (Actually it’s an OCC or Orthogonal Curvilinear Coordinate form)

$$L(\dot{r}, \dot{\phi}) = \frac{1}{2} M (g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2) - U(r, \phi) = \frac{1}{2} M (1 \cdot \dot{r}^2 + r^2 \dot{\phi}^2) - U(r, \phi)$$

**GCC Lagrange equations follow.** 1st $L$-equation is momentum $p_m$ definition for each coordinate $q^m$:

$$p_r = \frac{\partial L}{\partial \dot{r}} = M g_{rr} \dot{r} = Mr \dot{r}$$

Centripetal (center-pulling) force equals total radial momentum $p_r$ has the usual linear $M \cdot v$ form

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = Mg_{\phi\phi} \dot{\phi} = Mr^2 \dot{\phi}$$

Wow! $g_{\phi\phi}$ gives moment-of-inertia factor $Mr^2$ automatically for the angular momentum $p_\phi = Mr^2 \omega$.

2nd $L$-equation involves total time derivative $\dot{p}_m$ for each momentum $p_m$:

$$\dot{p}_r = \frac{\partial L}{\partial r} = \frac{M}{2} \frac{\partial g_{\phi\phi}}{\partial r} \dot{\phi}^2 - \frac{\partial U}{\partial r} = Mr^2 \dot{\phi} - \frac{\partial U}{\partial r}$$

Centrifugal (center-fleeing) force $Mr^2 \dot{\phi}$

$$\begin{align*}
\dot{p}_r &= \frac{dp_r}{dt} = M \ddot{r} \\
&= Mr^2 \ddot{\phi} - \frac{\partial U}{\partial r}
\end{align*}$$

Centripetal (center-pulling) force equals total

$$\dot{p}_\phi = \frac{\partial L}{\partial \phi} = 0 - \frac{\partial U}{\partial \phi}$$

Angular momentum $p_\phi$ is conserved if potential $U$ has no explicit $\phi$-dependence

Find $\dot{p}_m$ directly from 1st $L$-equation: $\dot{p}_m = \frac{dp_m}{dt} = \frac{d}{dt} M (g_{mn} \dot{q}^n) = M (g_{mn} \dot{q}^n + g_{mn} \ddot{q}^n)$

Equate it to $\dot{p}_m$ in 2nd $L$-equation:

$$\dot{p}_r = 2Mr \ddot{\phi} + Mr^2 \dot{\phi}$$

Torque relates to two distinct parts: Coriolis and angular acceleration

$$\dot{p}_\phi = \frac{dp_\phi}{dt} = 2Mr \ddot{\phi} + Mr^2 \dot{\phi}$$

Angular momentum $p_\phi$ is conserved if potential $U$ has no explicit $\phi$-dependence
Rewriting GCC Lagrange equations:

\[ \dot{p}_r = \frac{dp_r}{dt} = M \ddot{r} \]
\[ = M r \ddot{\phi} - \frac{\partial U}{\partial r} \]

Centrifugal (center-fleeing) force equals total
Centripetal (center-pulling) force

Angular momentum \( p_\phi \) is conserved if
potential \( U \) has no explicit \( \phi \)-dependence

Torque relates to two distinct parts:
Coriolis and angular acceleration

\[ \dot{p}_\phi = \frac{dp_\phi}{dt} = 2 M r \dot{\phi} + M r^2 \ddot{\phi} \]
\[ = 0 - \frac{\partial U}{\partial \phi} \]

Coriolis acceleration with \( \ddot{\phi} > 0 \) and \( \dot{r} < 0 \)
\[ \dddot{\phi} = -2 \frac{\dot{r} \ddot{\phi}}{r} \]
(makes \( \phi \) positive)

Inward flow to pressure Low
\[ \dot{r} < 0 \]
..makes wind turn to the right

Effect on Northern Hemisphere local weather
Cyclonic flow around lows

Field-free (\( U=0 \))

radial force:
\[ M \ddot{r} = M r \ddot{\phi} - \frac{\partial U}{\partial r} \]

radial acceleration:
\[ \ddot{r} = r \dddot{\phi} \]

angular acceleration:
\[ \dddot{\phi} = -2 \frac{\dot{r} \ddot{\phi}}{r} \]

Northern hemisphere rotation
\[ \phi > 0 \]

(with \( \ddot{\phi} = 0 \))
Rewriting GCC Lagrange equations:

\[ \dot{p}_r = \frac{dp_r}{dt} = M \ddot{r} \]

Centrifugal (center-fleeing) force equals total

\[ = M r \dot{\phi}^2 - \frac{\partial U}{\partial r} \]

Centripetal (center-pulling) force

\[ \dot{p}_\phi \equiv \frac{dp_\phi}{dt} = 2 M r \dot{r} \dot{\phi} + M r^2 \ddot{\phi} \]

Torque relates to two distinct parts: Coriolis and angular acceleration

\[ = 0 - \frac{\partial U}{\partial \phi} \]

Angular momentum \( p_\phi \) is conserved if potential \( U \) has no explicit \( \phi \)-dependence

Conventional forms

radial force: \( M \ddot{r} = M r \dot{\phi}^2 - \frac{\partial U}{\partial r} \)

angular force or torque: \( M r^2 \ddot{\phi} = -2 M r \dot{r} \dot{\phi} - \frac{\partial U}{\partial \phi} \)

Field-free (\( U=0 \))

radial acceleration: \( \ddot{r} = r \dot{\phi}^2 \)

angular acceleration: \( \ddot{\phi} = -2 \frac{r \dot{\phi}}{r} \)

Coriolis acceleration with \( \dot{\phi} > 0 \) and \( \ddot{r} < 0 \)

\( \dot{\phi} = -2 \frac{r \dot{\phi}}{r} \) (makes \( \phi \) positive)

Effect on Northern Hemisphere local weather

Cyclonic flow around lows

Cool North winds follow storms

Warm South winds precede storms

Effect on Northern Hemisphere local weather

Cyclonic flow around lows

Northern hemisphere rotation

\( \phi > 0 \)

(with \( \dot{\phi} = 0 \))
Rewriting GCC Lagrange equations:

Centrifugal (center-fleeing) force equals total Centripetal (center-pulling) force

\[ \dot{p}_r = \frac{dp_r}{dt} = M \ddot{r} \]
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Conventional forms

radial force: \( M \ddot{r} = M r \dot{\phi}^2 - \frac{\partial U}{\partial r} \)

Field-free (\( U=0 \))

radial acceleration: \( \ddot{r} = r \dddot{\phi} \)

Angular momentum \( p_\phi \) is conserved if potential \( U \) has no explicit \( \phi \)-dependence

\[ \dot{p}_\phi = \frac{dp_\phi}{dt} = 2Mr\ddot{\phi} + Mr^2\dddot{\phi} \]

Torque relates to two distinct parts: Coriolis and angular acceleration

\[ = 0 - \frac{\partial U}{\partial \phi} \]

Angular momentum \( p_\phi \) is conserved if potential \( U \) has no explicit \( \phi \)-dependence

Effect on Northern Hemisphere local weather

Cyclonic flow around lows

Deep quantum rule: Flow tries to mimic the external rotation (least relative \( v \))

Northern hemisphere systems drift West to East

Coriolis acceleration with \( \dddot{\phi} > 0 \) and \( \ddot{r} < 0 \)

\( \dddot{\phi} = -2 \frac{\dot{r} \dot{\phi}}{r} \) (makes \( \phi \) positive)

Inward flow to pressure Low \( \ddot{r} < 0 \)

...makes wind turn to the right

Cool North winds follow storms

Warm South winds precede storms

Northern hemisphere rotation

(with \( \dot{\phi} = 0 \))

Easterly winds

(Warm North winds follow storms)

Northern hemisphere systems drift West to East
GOES-16 captured this geocolor image of Hurricane Irma approaching Anguilla at about 7:15 am (eastern), September 6, 2017. Irma's maximum sustained winds remain near 185 mph with higher gusts, making it a category 5 hurricane on the Saffir-Simpson Hurricane Wind Scale. According to the latest information from NOAA's National Hurricane Center (issued at 8:00 am eastern), Irma was located about 15 miles west-southwest of Anguilla and moving toward the west-northwest near 16 miles per hour.
Saturn's north pole was dark when Cassini arrived in 2004. But as the seasons changed, light illuminated a bizarre six-sided swirl of gases at the pole (shown here in false color). The hexagon has been known since the 1980s. It is about 30,000 kilometers (18,600 miles) wide with a massive hurricane centered on the north pole.

JPL-CALTECH/NASA, SPACE SCIENCE INSTITUTE
Representaions Of Multidimensional Symmetries In Networks - harter-imp-1973

**Alternative Basis for the Theory of Complex Spectra**
- Alternative Basis for the Theory of Complex Spectra I - harter-pra-1973
- Alternative Basis for the Theory of Complex Spectra II - harter-patterson-pra-1976
- Alternative Basis for the Theory of Complex Spectra III - patterson-harter-pra-1977

Frame Transformation Relations And Multipole Transitions In Symmetric Polyatomic Molecules - RMP-1978

Asymptotic eigensolutions of fourth and sixth rank octahedral tensor operators - Harter-Patterson-JMP-1979

Rotational energy surfaces and high- J eigenvalue structure of polyatomic molecules - Harter - Patterson - 1984

Galloping waves and their relativistic properties - ajp-1985-Harter

Rovibrational Spectral Fine Structure Of Icosahedral Molecules - Cpl 1986 (Alt Scan)

**Theory of hyperfine and superfine levels in symmetric polyatomic molecules.**
- I) Trigonal and tetrahedral molecules: Elementary spin-1/2 cases in vibronic ground states - PRA-1979-Harter-Patterson (Alt scan)
- II) Elementary cases in octahedral hexafluoride molecules - Harter-PRA-1981 (Alt scan)

**Rotation–vibration spectra of icosahedral molecules.**
- I) Icosahedral symmetry analysis and fine structure - harter-weeks-jcp-1989 (Alt scan)
- II) Icosahedral symmetry, vibrational eigenfrequencies, and normal modes of buckminsterfullerene - weeks-harter-jcp-1989 (Alt scan)
- III) Half-integral angular momentum - harter-reimer-jcp-1991


Nuclear spin weights and gas phase spectral structure of 12C60 and 13C60 buckminsterfullerene - Harter-Reimer-Cpl-1992 - (Alt1, Alt2 Erratum)

Gas Phase Level Structure of C60 Buckyball and Derivatives Exhibiting Broken Icosahedral Symmetry - reimer-diss-1996

Fullerene symmetry reduction and rotational level fine structure/ the Buckyball isotopomer 12C 13C59 - icp-Reimer-Harter-1997 (HiRez)

Wave Node Dynamics and Revival Symmetry in Quantum Rotors - harter - jms - 2001

Molecular Symmetry and Dynamics - Ch32-Springer Handbooks of Atomic, Molecular, and Optical Physics - Harter-2006

**Resonance and Revivals**
- I) QUANTUM ROTOR AND INFINITE-WELL DYNAMICS - ISMSLi2012 (Talk) OSU knowledge Bank
- II) Comparing Half-integer Spin and Integer Spin - Alva-ISMS-Ohio2013-R777 (Talks)
- III) Quantum Resonant Beats and Revivals in the Morse Oscillators and Rotors - (2013-Li-Diss)

Resonance and Revivals in Quantum Rotors - Comparing Half-integer Spin and Integer Spin - Alva-ISMS-Ohio2013-R777 (Talk)

Molecular Eigensolution Symmetry Analysis and Fine Structure - IJMS-harter-mitchell-2013

Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-cpl-2013

QTCA Unit 10 Ch 30 - 2013

*AMOP Ch 0 Space-Time Symmetry - 2019*
### AMOP reference links

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Predrag Cvitanovic's: Birdtrack Notation, Calculations, and Simplification
- Chaos_Classical_and_Quantum - 2018-Cvitanovic-ChaosBook
- Group Theory - PUP_Lucy_Day - Diagrammatic_notation - Ch4
- Group Theory - Birdtracks_Lies_and_Exceptional_Groups - Cvitanovic-2011
- Simplification_rules_for_birdtrack_operators - imp-alcock-zeilinger-2017
- Birdtracks for SU(N) - 2017-Keppeler

Frank Rioux's: UMA method of vibrational induction
- Quantum_Mechanics_Group_Theory_and_C60 - Frank_Rioux - Department_of_Chemistry_Saint_Johns_U
- Symmetry_Analysis_for_H2O - H2OGrpTheory_Rioux
- Group_Theory_Problems - Rioux - SymmetryProblemsX
- Comment_on_the_Vibrational_Analysis_for_C60_and_Other_Fullerenes_Rioux-RSP

Supplemental AMOP Techniques & Experiment
- Many Correlation Tables are Molien Sequences - Klee (Draft 2016)
- Symmetry and Chirality - Continuous_Measures - Avnir

Special Topics & Colloquial References
- r-process_nucleosynthesis_from_matter_ejected_in_binary_neutron_star_mergers-PhysRevD-Bovard-2017

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