Lecture 28
Mon. 12.02.2019

Multi-particle orbits and rotating body dynamics

(Ch. 2-7 of Unit 6 12.07.17)

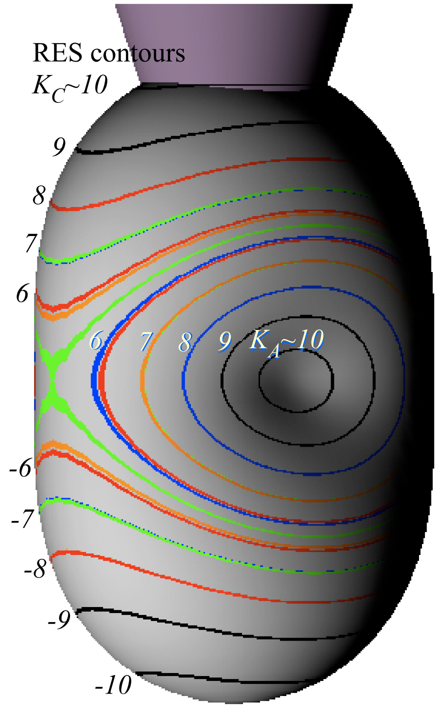
2-Particle orbits
- *Ptolemetric or LAB view and reduced mass*
- *Copernican or COM view and reduced coupling*

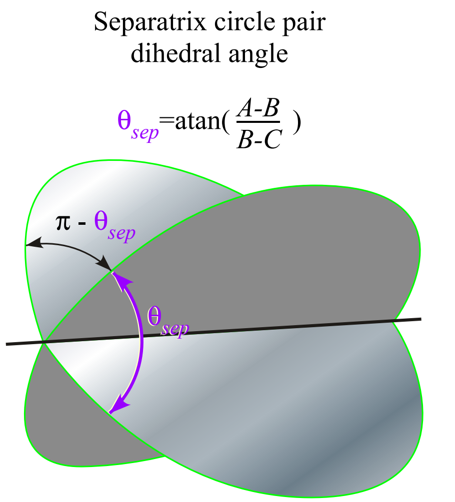
2-Particle orbits and scattering: LAB-vs.-COM frame views
- *Ruler & compass construction (or not)*

Rotational equivalent of Newton's **F**=*d***p**/*dt equations:* **N**=*d***L**/*dt*
- *How to make my boomerang come back*
- *The gyrocompass and mechanical spin analogy*

Rotational momentum and velocity tensor relations
- *Quadratic form geometry and duality (again)*
  - *angular velocity ω-ellipsoid vs. angular momentum* **L**-*ellipsoid*
  - *Lagrangian ω-equations vs. Hamiltonian momentum* **L**-*equation*

Rotational Energy Surfaces (RES) and Constant Energy Surfaces (CES)
- *Symmetric, asymmetric, and spherical-top dynamics (Constant* **L**)
  - *BOD-frame cone rolling on LAB frame cone*
- *Deformable spherical rotor RES and semi-classical rotational states and spectra*
- *Cycloidal geometry of flying levers and Practical poolhall application*





Asym. Rotor AJP 44,11 1976

This Lecture's Reference Link Listing

Web Resources - front page | Quantum Theory for the Computer Age | 2017 Group Theory for QM
UAF Physics UTube channel | Principles of Symmetry, Dynamics, and Spectroscopy | 2018 Adv CM
Classical Mechanics with a Bang! | 2018 AMOP
Modern Physics and its Classical Foundations | **2019 Advanced Mechanics**

Lecture #22-28

In reverse order

CMwBang Text 2012 Unit 6 page=5
BounceIt Web App/Scenarios: 5002, 5003
CoulIt Web App/Scenarios:
TwoParticleCollision_LToR, TwoParticleCollision_LToR_CM, TwoParticleOrbit_Coulomb, TwoParticleOrbit_Coulomb_CM, TwoParticleOrbit_Hooke, TwoParticleOrbit_Hooke_CM
Singular Motion of Asymetric Rotators AJP 44, 11 p1080 Harter-Kim-1976
Molecular Eigensolution Symmetry Analysis and Fine Structure - Harter-IJMS-2013
Lenz Vector and Orbital Analog Computers - AJP 44 p348 1976
CoulIt Web App Simulations: p19, p32, p72, p73, p92, R=-0.375, R=+0.5, Rutherford
OscillatorPE Web App: IHO Scenario 2, Coulomb Scenario 3
RelaWavity Web App/Simulator/Calculator: Elliptical - IHO orbits
JerkIt Web App: 2-, 2+, Amp50Omega147-, Amp50Omega296, Amp50Omega602, Gap(1)
MolVibes Web App: C3vN3
WaveIt Web App:
Dim = 3 w/Wave Components;
Static Char Table: 6, 12, 12(b), 16, 36, 256
Quantum Carpet with N=20: Gaussian, Boxcar
Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-CPL-2015
QTCA Unit_5 Ch14 2013
Lester. R. Ford, Am. Math. Monthly 45,586(1938)
John Farey, Phil. Mag.(1816) Wolfram
Harter, J. Mol. Spec. 210, 166-182 (2001)
Harter, Li IMSS (2013)
Li, Harter, Chem.Phys.Letters (2015)
Advanced Atomic and Molecular Optical Physics 2018 Class #9, pages: 5, 61
BoxIt Web Simulations
Pure A-Type A=4.9, B=0 ,C=0, & D=4.0
Pure B-Type: A=4.0, B=-0.2, C=0, & D=4.0
Pure C-Type A,D=4.055, B=0, C=0.1
Mixed AB-Type w/Cosine
Mixed AB Type A=4.0, BU2=0.866..., CU2=0, & D=1.0 w/Stokes & Freq rats
Mixed AB Type A=5.086 B=-0.27 C=0 D=2.024 w/Stokes plot
Mixed ABC Type A=4.833 B=0.2403 C=0.4162 D=4.277 w/Stokes plot
Recent mixed ABC Type A=0.325 B=0.375 C=0.825 D=0.05 w/Stokes plot

Select, exciting, and/or related Research

***This Indestructible NASA Camera Revealed Hidden Patterns on Jupiter** - seeker-yt-2019*
What did NASA's New Horizons discover around Pluto? - Astrum-yt-2018
Synthetic_Chiral_Light_for_Efficient_Control_of_Chiral_Light–Matter_Interaction_-_Ayuso-np-2019

Continued for 3 more pages

Some Geometric Aspects of Classical Coulomb Scattering AJP 40 4 p1852 1972
How Molecules do Self-NMR - Harter-Mitchell-Columbus-2009
Classical Mechanics with a Bang! - Asymmetric Top Demo
Allbookstores.com - Compare for Heller's SemiClassical Way - 0691163731
"My Bomerang Won't Come Back" (YouTube: Playlist)
Rotating Solid Bodies in Microgravity (YouTube)
Dancing T-handle in zero-g (YouTube)

Classical Mechanics with a Bang! 2018
Lectures 8, 9, 23 page 93
Text Unit 6, page=27
ColorU2 for the Web - in development
Group Theory for Quantum Mechanics - 2017 Lectures: 6, 7, 8, and the combined 9-10
Quantum Theory for the Computer Age Unit 3 Ch.7-10, page=90
Spectral Decomposition with Repeated Eigenvalues - 2017 GTQM - Lecture 5

Web based 3D & XR ($x \in \{A,M,V\}$, R=Reality) https://www.babylonjs.com/
Web based 3D graphics WebGL API (Graphics Layer modeled after OpenGL)

Recent In-House draft Articles:

Springer handbook on Molecular Symmetry and Dynamics - Ch_32 - Molecular Symmetry
AMOP Ch 0 Space-Time Symmetry - 2019
Seminar at Rochester Institute of Optics, Auxiliary slides, June 19, 2018

Quantum_Computing - (Current) State of the Art - Reimer-www-2019
Geometric Algebra- A Guided Tour through Space and Time - Reimer-www-2019
Wildlife Monitoring Identification and Behavioral Study - Section 1 - Reimer-www-2019
Wildlife Monitoring Identification and Behavioral Study - Section 2 - Reimer-www-2019

Quantum Computing (*QC*) and Geometric Algebra (*GA*) references:

Quantum_Supremacy_Using_a_Programmable_Superconducting_Processor_-_Arute-n-2019
Quantum Computing for Computer Scientists - Helwer-mr-yt-2018, Slides
Quantum Computing and Workforce, Curriculum, and App Devel - Roetteler-MS-2019

Quantum_Computing - (Current) State of the Art - Reimer-www-2019
Excerpts (Page 44-47 in *Preliminary Draft*) for a GA take on the Complex Numbers
Geometric Algebra- A Guided Tour through Space and Time - Reimer-www-2019
GA & QC references (Page 11-16 in Preliminary Draft)

In development, but close to role out.

More Advanced QM and classical references will *soon* be available through our: ***References Page***

Would be great to have our Apache SOLR *Search & Index system up for a bigger* Bang!*)*

This Lecture's Reference Link Listing

Web Resources - front page
UAF Physics UTube channel

Quantum Theory for the Computer Age
Principles of Symmetry, Dynamics, and Spectroscopy
Classical Mechanics with a Bang!
Modern Physics and its Classical Foundations

2017 Group Theory for QM
2018 Adv CM
2018 AMOP
2019 Advanced Mechanics

Lectures #12 through #21

In reverse order

Wiki on Pafnuty Chebyshev
Nobelprize.org
    2005 Physics Award

***BoxIt* Web Simulations:**

A-Type w/Cosine, A-Type w/Freq ratios,
AB-Type w/Cosine, AB-Type 2:1 Freq ratio

***OscillIt* Web Simulations:**

Default/Generic, Weakly Damped #18,
Forced : Way below resonance,On resonance
    Way above resonance,Underdamped
    Complex Response Plot

***CoulIt* Web Simulations:**

Stark-Coulomb : Bound-state motion in parabolic coordinates
Molecular Ion : Bound-state motion in hyperbolic coordinates
Synchrotron Motion, Synchrotron Motion #2
Mechanical Analog to EM Motion (YouTube video)
iBall demo - Quasi-periodicity (YouTube video)

***Trebuchet* Web Simulations:**

Default/Generic URL, Montezuma's Revenge, Seige of Kenilworth,
"Flinger",
Position Space (Course), Position Space (Fine)

Wacky Waving Solid Metal Arm Flailing Chaos Pendulum - Scooba_Steeve-yt-2015
Triple Double-Pendulum - Cohen-yt-2008
Punkin Chunkin - TheArmchairCritic-2011
Jersey Team Claims Title in Punkin Chunkin - sussexcountyonline-1999
Shooting range for medieval siege weapons. Anybody knows? - twcenter.net/forums
The Trebuchet - Chevedden-SciAm-1995
NOVA Builds a Trebuchet

Recent Articles of Interest:

A_Semi-Classical_Approach_to_the_Calculation_of_Highly_Excited_Rotational_Energies for …
    Asymmetric-Top_Molecules_-_Schmiedt-pccp-2017
Tunable and broadband coherent perfect absorption by ultrathin blk phos metasurfaces - Guo-josab-2019
Vortex Detection in Vector Fields Using Geometric Algebra - Pollock-aaca-2013.pdf

Pirelli Relativity Challenge (Introduction level) - *Visualizing Waves:*

Using Earth as a clock,
Tesla's AC Phasors ,
Phasors using complex numbers.

CM wBang Unit 1 - Chapter 10, pdf_page=135
    Calculus of exponentials, logarithms, and complex fields,
RelaWavity Web Simulation - Unit Circle and Hyperbola (Mixed labeling)
Smith Chart, Invented by Phillip H. Smith (1905-1987)

Select, exciting, and related Research

Clifford_Algebra_And_The_Projective_Model_Of_Homogeneous_Metric_Spaces -
    Foundations - Sokolov-x-2013
Geometric Algebra 3 - Complex Numbers - MacDonald-yt-2015
Biquaternion -Complexified Quaternion- Roots of -1 - Sangwine-x-2015
An_Introduction_to_Clifford_Algebras_and_Spinors_-_Vaz-Rocha-op-2016
Unified View on Complex Numbers and Quaternions- Bongardt-wcmms-2015
Complex Functions and the Cauchy-Riemann Equations - complex2 - Friedman-columbia-2019
An_sp-hybridized_Molecular_Carbon_Allotrope-_cyclo-18-carbon_-_Kaiser-s-2019
An_Atomic-Scale_View_of_Cyclocarbon_Synthesis_-_Maier-s-2019
Discovery_Of_Topological_Weyl_Fermion_Lines_And_Drumhead_Surface_States_in_a_
    Room_Temperature_Magnet_-_Belopolski-s-2019
"Weyl"ing_away_Time-reversal_Symmetry_-_Neto-s-2019
Non-Abelian_Band_Topology_in_Noninteracting_Metals_-_Wu-s-2019
What_Industry_Can_Teach_Academia_-_Mao-s-2019
RoVib-_quantum_state_resolution_of_the_C60_fullerene_-_Changala-Ye-s-2019 (Alt)
A_Degenerate_Fermi_Gas_of_Polar_molecules_-_DeMarco-s-2019

An assist from *Physics Girl* (YouTube Channel):

How to Make VORTEX RINGS in a Pool
Crazy pool vortex - pg-yt-2014
Fun with Vortex Rings in the Pool - pg-yt-2014

Running Reference Link Listing

Lectures #11 through #7

In reverse order

Eric J Heller Gallery:
Main portal, Consonance and Dissonance II, Bessel 21, Chladni

The Semiclassical Way to Molecular Spectroscopy - Heller-acs-1981
Quantum_dynamical_tunneling_in_bound_states_-_Davis-Heller-jcp-1981

Pendulum Web Simulation
Cycloidulum Web Simulation

Links to previous lecture: *Page=74, Page=75, Page=79*
Pendulum Web Sim
Cycloidulum Web Sim
***JerkIt* Web Simulations:** *Basic/Generic: Inverted, FVPlot*
CMwithBang Lecture 8, page=20
WWW.sciencenewsforstudents.org: Cassini - Saturnian polar vortex

"RelaWavity" Web Simulations:
2-CW laser wave, Lagrangian vs Hamiltonian,
Physical Terms Lagrangian L(u) vs Hamiltonian H(p)
CoulIt Web Simulation of the Volcanoes of Io
BohrIt Multi-Panel Plot:
Relativistically shifted Time-Space plots of 2 CW light waves

BoxIt Web Simulations:
Generic/Default
Most Basic A-Type
Basic A-Type w/reference lines
Basic A-Type A-Type with Potential energy
A-Type with Potential energy and Stokes Plot
A-Type w/3 time rates of change
A-Type w/3 time rates of change with Stokes Plot
B-Type (A=1.0, B=-0.05, C=0.0, D=1.0)

RelaWavity Web Elliptical Motion Simulations:
Orbits with b/a=0.125
Orbits with b/a=0.5
Orbits with b/a=0.7
Exegesis with b/a=0.125
Exegesis with b/a=0.5
Exegesis with b/a=0.7
Contact Ellipsometry

***CoulIt* Web Simulations:**
Basic/Generic
Exploding Starlet
Volcanoes of Io (Color Quantized)

***JerkIt* Web Simulations:**
Basic/Generic
Catcher in the Eye - IHO with Linear Hooke perturbation - Force-potential-Velocity Plot

***OscillatorPE* Web Simulation:**
Coulomb-Newton-Inverse_Square,
Hooke-Isotropic Harmonic,
Pendulum-Circular_Constraint

AMOP Ch 0 Space-Time Symmetry - 2019
Seminar at Rochester Institute of Optics, Aux. slides-2018

NASA Astronomy Picture of the Day -
Io: The Prometheus Plume *(Just Image)*
NASA Galileo - *Io's Alien Volcanoes*
New Horizons - *Volcanic Eruption Plume on Jupiter's moon IO*
NASA Galileo - A Hawaiian-Style Volcano on Io
Pirelli Site: Phasors animimation
CMwithBang Lecture #6, page=70 (9.10.18)

Select, exciting, and related Research & Articles of Interest:
Burning a hole in reality—design for a new laser may be powerful enough to pierce space-time - Sumner-KOS-2019
Trampoline mirror may push laser pulse through fabric of the Universe - Lee-ArsTechnica-2019
Achieving_Extreme_Light_Intensities_using_Optically_Curved_Relativistic_Plasma_Mirrors_-_Vincenti-prl-2019
A_Soft_Matter_Computer_for_Soft_Robots_-_Garrad-sr-2019
Correlated_Insulator_Behaviour_at_Half-Filling_in_Magic-Angle_Graphene_Superlattices_-_cao-n-2018
*Sorting_ultracold_atoms_in_a_three-dimensional_optical_lattice_in_a_*
realization_of_Maxwell's_Demon - Kumar-n-2018
Synthetic_three-dimensional_atomic_structures_assembled_atom_by_atom - Barredo-n-2018

Older ones:
Wave-particle_duality_of_C60_molecules - Arndt-ltn-1999
Optical_Vortex_Knots_-_One_Photon__At_A_Time - Tempone-Wiltshire-Sr-2018
Baryon_Deceleration_by_Strong_Chromofields_in_Ultrarelativistic_,
Nuclear_Collisions - Mishustin-PhysRevC-2007, APS Link & Abstract
Hadronic Molecules - Guo-x-2017
Hidden-charm_pentaquark_and_tetraquark_states - Chen-pr-2016

Running Reference Link Listing

Lectures #6 through #1

In reverse order

RelaWavity Web Simulation: Contact Ellipsometry
BoxIt Web Simulation: Elliptical Motion (A-Type)
CMwBang Course: Site Title Page
Pirelli Relativity Challenge: Describing Wave Motion With Complex Phasors
UAF Physics UTube channel

Velocity Amplification in Collision Experiments Involving Superballs - Harter, 1971
MIT OpenCourseWare: High School/Physics/Impulse and Momentum
Hubble Site: Supernova - SN 1987A

BounceItIt Web Animation - Scenarios:
49:1 y vs t, *49:1 V2 vs V1*, *1:500:1 - 1D Gas Model w/ faux restorative force (Cool)*,
1:500:1 - 1D Gas (Warm), *1:500:1 - 1D Gas Model (Cool, Zoomed in)*,
Farey Sequence - Wolfram
Fractions - Ford-AMM-1938
Monstermash BounceItIt Animations:
1000:1 - V2 vs V1, *1000:1 with t vs x - Minkowski Plot*
Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-2013
Quantum_Revivals_of_Morse_Oscillators_and_Farey-Ford_Geometry - Li-Harter-cpl-2015
Quant. Revivals of Morse Oscillators and Farey-Ford Geom. - Harter-Li-CPL-2015 (Publ.)
Velocity_Amplification_in_Collision_Experiments_Involving_Superballs-Harter-1971
WaveIt Web Animation - Scenarios:
Quantum_Carpet, *Quantum_Carpet_wMBars*,
Quantum_Carpet_BCar, *Quantum_Carpet_BCar_wMBars*
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - Harter-JMS-2001
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - Harter-jms-2001 (Publ.)

BounceIt Web Animation - Scenarios:
Generic Scenario: 2-Balls dropped no Gravity (7:1) - V vs V Plot (Power=4)
1-Ball dropped w/Gravity=0.5 w/Potential Plot: Power=1, Power=4
7:1 - V vs V Plot: Power=1
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=4
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=1
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=1 w/Gaps
4-Ball Stack (27:9:3:1) w/Newton plot (y vs t) - Power=4
4-Newton's Balls (1:1:1:1) w/Newtonian plot (y vs t) - Power=4 w/Gaps
6-Ball Totally Inelastic (1:1:1:1:1:1) w/Gaps: Newtonian plot (t vs x), V6 vs V5 plot
5-Ball Totally Inelastic Pile-up w/ 5-Stationary-Balls - Minkowski plot (t vs x1) w/Gaps
1-Ball Totally Inelastic Pile-up w/ 5-Stationary-Balls - Vx2 vs Vx1 plot w/Gaps

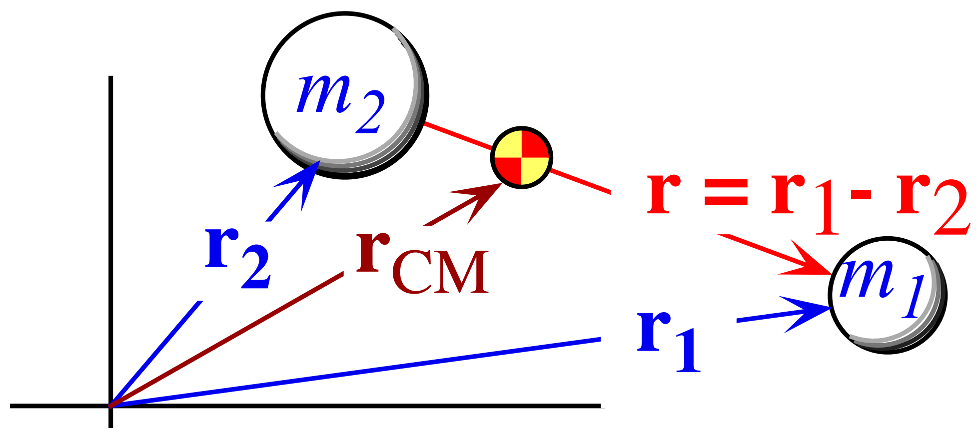
BounceIt Dual plots
m_1:m_2 = 3:1
v_2 vs v_1 and V_2 vs V_1, $(v_1, v_2)=(1, 0.1)$, $(v_1, v_2)=(1, 0)$
y_2 vs y_1 plots: $(v_1, v_2)=(1, 0.1)$, $(v_1, v_2)=(1, 0)$, $(v_1, v_2)=(1, -1)$
Estrangian plot *V_2 vs V_1: $(v_1, v_2)=(0, 1)$, $(v_1, v_2)=(1, -1)$*
m_1:m_2 = 4:1
v2 vs v1, *y2 vs y1*
m_1:m_2 = 100:1, *$(v_1, v_2)=(1, 0)$: V2 vs V1 Estrangian plot, y2 vs y1 plot*

With g=0 and 70:10 mass ratio
With non zero g, velocity dependent damping and mass ratio of 70:35
M_1=49, M_2=1 with Newtonian time plot
M_1=49, M_2=1 with V_2 vs V_1 plot
Example with friction
Low force constant with drag displaying a Pass-thru, Fall-Thru, Bounce-Off
m1:m2= 3:1 and (v1, v2) = (1, 0) Comparison with Estrangian

AJP article on superball dynamics
AAPT Summer Reading List
Scitation.org - AIP publications
HarterSoft Youtube Channel

X2 paper: *Velocity Amplification in Collision Experiments Involving Superballs - Harter, et. al. 1971 (pdf)*
Car Collision Web Simulator: https://modphys.hosted.uark.edu/markup/CMMotionWeb.html
Superball Collision Web Simulator: https://modphys.hosted.uark.edu/markup/BounceItWeb.html; with Scenarios: 1007
BounceIt web simulation with g=0 and 70:10 mass ratio
With non zero g, velocity dependent damping and mass ratio of 70:35
Elastic Collision Dual Panel Space vs Space: Space vs Time (Newton), Time vs. Space(Minkowski)
Inelastic Collision Dual Panel Space vs Space: Space vs Time (Newton), Time vs. Space(Minkowski)
Matrix Collision Simulator: M_1=49, M_2=1 V_2 vs V_1 plot <<Under Construction>>

2-Particle orbits and center-of-mass (CM) coordinate frame



$$\mathbf{r}_{CM} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

Defining *relative coordinate vector*

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

and *mass-weighted-average* or *center-of-mass coordinate vector* $\mathbf{r}_{CM}$

$$\bar{\mathbf{r}} = \mathbf{r}_{\mathbf{CM}} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

The inverse coordinate transformation.

$$\mathbf{r}_1 = \mathbf{r}_{\mathbf{CM}} + \frac{m_2\mathbf{r}}{m_1 + m_2} \;, \qquad \mathbf{r}_2 = \mathbf{r}_{\mathbf{CM}} - \frac{m_1\mathbf{r}}{m_1 + m_2}$$

2-Particle orbits

Ptolemetric or LAB view and reduced mass

Copernican or COM view and reduced coupling

Reduced mass: Ptolemetric views

Radial inter-particle force $\mathbf{F}_{12}$ is on m_1 due to m_2 and $\mathbf{F}_{21} = -\mathbf{F}_{12}$ is on m_2 due to m_1

$$\mathbf{F}_{12} = \mathbf{F}(\mathrm{r})\mathbf{e_r} = -\mathbf{F}_{21} \quad = F(r)\hat{\mathbf{r}} = \quad F(r)\frac{\mathbf{r}}{r} = \quad \frac{F(r)}{r}\left(\mathbf{r_1} - \mathbf{r_2}\right)$$

$\mathbf{F}_{12}$ acts along relative coordinate vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

Depends only upon the relative distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$

$$\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_1 = \quad F(r)\hat{\mathbf{r}} = \quad F(r)\frac{\mathbf{r}}{r} = \quad \frac{F(r)}{r}\left(\mathbf{r_1} - \mathbf{r_2}\right)$$

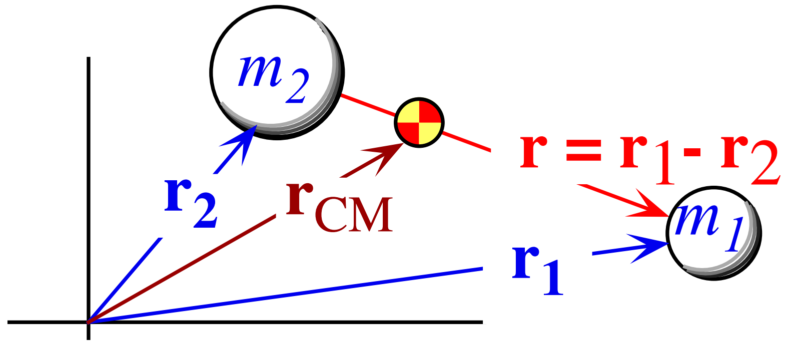
$$\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_2 = -F(r)\hat{\mathbf{r}} = -F(r)\frac{\mathbf{r}}{r} = -\frac{F(r)}{r}\left(\mathbf{r_1} - \mathbf{r_2}\right)$$

Re-scaled force: A Copernican view

$$\mathbf{r}_1 = \frac{m_2\mathbf{r}}{m_1 + m_2} = \frac{\mu}{m_1}\mathbf{r}\ , \qquad \mathbf{r}_2 = \frac{-m_1\mathbf{r}}{m_1 + m_2} = \frac{-\mu}{m_2}\mathbf{r}$$

relative radius vector

$$\frac{m_1}{\mu}\mathbf{r}_1 = \mathbf{r} = \frac{-m_2}{\mu}\mathbf{r}_2$$



$$\mathbf{r}_{\mathrm{CM}} = \frac{m_1\mathbf{r_1} + m_2\mathbf{r_2}}{m_1 + m_2}$$

$$\mathbf{r_1} = \mathbf{r_{CM}} + \frac{m_2\mathbf{r}}{m_1 + m_2}\ , \qquad \mathbf{r}_2 = \mathbf{r_{CM}} - \frac{m_1\mathbf{r}}{m_1 + m_2}$$

Reduced mass: Ptolemetric views

Radial inter-particle force $\mathbf{F}_{12}$ is on m_1 due to m_2 and $\mathbf{F}_{21} = -\mathbf{F}_{12}$ is on m_2 due to m_1

$$\mathbf{F}_{12} = \mathbf{F}(r)\mathbf{e_r} = -\mathbf{F}_{21} = F(r)\hat{\mathbf{r}} = F(r)\frac{\mathbf{r}}{r} = \frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$\mathbf{F}_{12}$ acts along relative coordinate vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

Depends only upon the relative distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$

$$\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_1 = F(r)\hat{\mathbf{r}} = F(r)\frac{\mathbf{r}}{r} = \frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_2 = -F(r)\hat{\mathbf{r}} = -F(r)\frac{\mathbf{r}}{r} = -\frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

Sum $\mathbf{F}_{12}+\mathbf{F}_{21}$ yields zero because of Newton's 3rd -law action-reaction cancellation.

$$(m_1 + m_2)\ddot{\mathbf{r}}_{\mathbf{CM}} = m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = \mathbf{0}$$

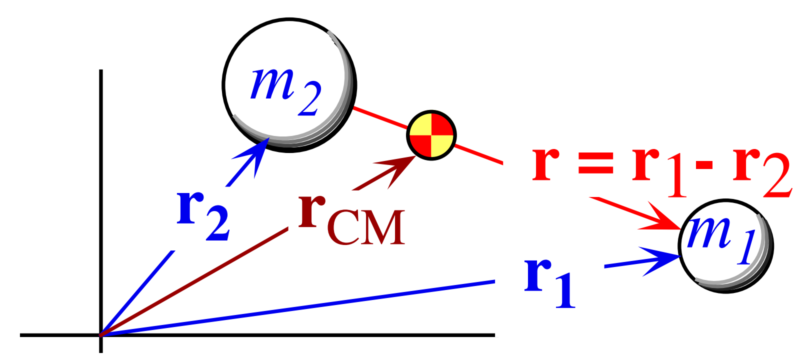
(a.k.a: Momentm Conservation)

Re-scaled force: A Copernican view

$$\mathbf{r}_1 = \frac{m_2\mathbf{r}}{m_1 + m_2} = \frac{\mu}{m_1}\mathbf{r}, \qquad \mathbf{r}_2 = \frac{-m_1\mathbf{r}}{m_1 + m_2} = \frac{-\mu}{m_2}\mathbf{r}$$

relative radius vector

$$\frac{m_1}{\mu}\mathbf{r}_1 = \mathbf{r} = \frac{-m_2}{\mu}\mathbf{r}_2$$



$$\mathbf{r}_{CM} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

$$\mathbf{r}_1 = \mathbf{r}_{\mathbf{CM}} + \frac{m_2\mathbf{r}}{m_1 + m_2}, \qquad \mathbf{r}_2 = \mathbf{r}_{\mathbf{CM}} - \frac{m_1\mathbf{r}}{m_1 + m_2}$$

Reduced mass: Ptolemetric views

Radial inter-particle force $\mathbf{F}_{12}$ is on m_1 due to m_2 and $\mathbf{F}_{21} = -\mathbf{F}_{12}$ is on m_2 due to m_1

$$\mathbf{F}_{12} = \mathbf{F}(r)\mathbf{e_r} = -\mathbf{F}_{21} \quad = F(r)\hat{\mathbf{r}} = \quad F(r)\frac{\mathbf{r}}{r} = \quad \frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$\mathbf{F}_{12}$ acts along relative coordinate vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

Depends only upon the relative distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$

$$\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_1 = \quad F(r)\hat{\mathbf{r}} = \quad F(r)\frac{\mathbf{r}}{r} = \quad \frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_2 = -F(r)\hat{\mathbf{r}} = -F(r)\frac{\mathbf{r}}{r} = -\frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

Sum $\mathbf{F}_{12}+\mathbf{F}_{21}$ yields zero because of Newton's 3rd -law action-reaction cancellation.

$$(m_1 + m_2)\ddot{\mathbf{r}}_{\mathbf{CM}} = m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = \mathbf{0}$$

(a.k.a: Momentm Conservation)

Difference $\mathbf{F}_{12}-\mathbf{F}_{21}$ reduces to $\mu\ddot{\mathbf{r}} = \mathbf{F}(r)$ using $\boxed{\textit{reduced mass}: \quad \mu = \frac{m_2 m_1}{m_1 + m_2}}$ $\quad \ddot{\mathbf{r}}_{\mathbf{CM}} = \mathbf{0}$

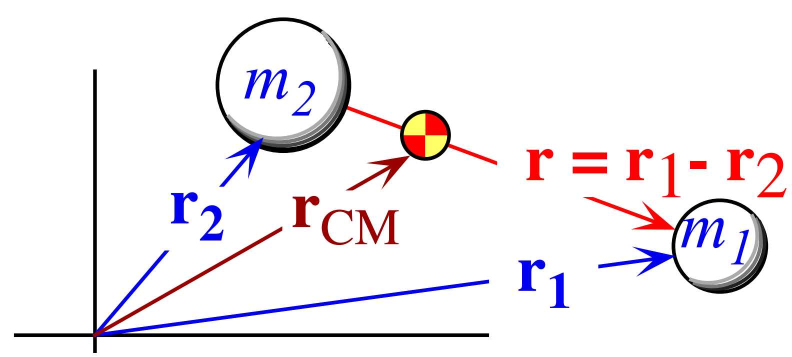
Re-scaled force: A Copernican view

$$\mathbf{r}_1 = \frac{m_2\mathbf{r}}{m_1 + m_2} = \frac{\mu}{m_1}\mathbf{r}\ , \qquad \mathbf{r}_2 = \frac{-m_1\mathbf{r}}{m_1 + m_2} = \frac{-\mu}{m_2}\mathbf{r}$$

relative radius vector

$$\frac{m_1}{\mu}\mathbf{r}_1 = \mathbf{r} = \frac{-m_2}{\mu}\mathbf{r}_2$$

$$\mathbf{r}_1 = \mathbf{r}_{\mathbf{CM}} + \frac{m_2\mathbf{r}}{m_1 + m_2}\ , \qquad \mathbf{r}_2 = \mathbf{r}_{\mathbf{CM}} - \frac{m_1\mathbf{r}}{m_1 + m_2}$$



$$\mathbf{r}_{\mathrm{CM}} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

Reduced mass: Ptolemetric views

Radial inter-particle force $\mathbf{F}_{12}$ is on m_1 due to m_2 and $\mathbf{F}_{21} = -\mathbf{F}_{12}$ is on m_2 due to m_1

$$\mathbf{F}_{12} = \mathbf{F}(r)\mathbf{e_r} = -\mathbf{F}_{21} = F(r)\hat{\mathbf{r}} = F(r)\frac{\mathbf{r}}{r} = \frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$\mathbf{F}_{12}$ acts along relative coordinate vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

Depends only upon the relative distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$

$$\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_1 = F(r)\hat{\mathbf{r}} = F(r)\frac{\mathbf{r}}{r} = \frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_2 = -F(r)\hat{\mathbf{r}} = -F(r)\frac{\mathbf{r}}{r} = -\frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

Sum $\mathbf{F}_{12}+\mathbf{F}_{21}$ yields zero because of Newton's 3rd -law action-reaction cancellation.

$$(m_1 + m_2)\ddot{\mathbf{r}}_{\mathbf{CM}} = m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = \mathbf{0}$$

Difference $\mathbf{F}_{12}-\mathbf{F}_{21}$ reduces to $\mu\ddot{\mathbf{r}} = \mathbf{F}(r)$ using *reduced mass*: $\boxed{\mu = \frac{m_2 m_1}{m_1 + m_2}}$ and: $\ddot{\mathbf{r}}_{\mathbf{CM}} = \mathbf{0}$

$$[\quad m_1\ddot{\mathbf{r}}_1 \quad] - [\quad m_2\ddot{\mathbf{r}}_2 \quad] = \frac{2F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

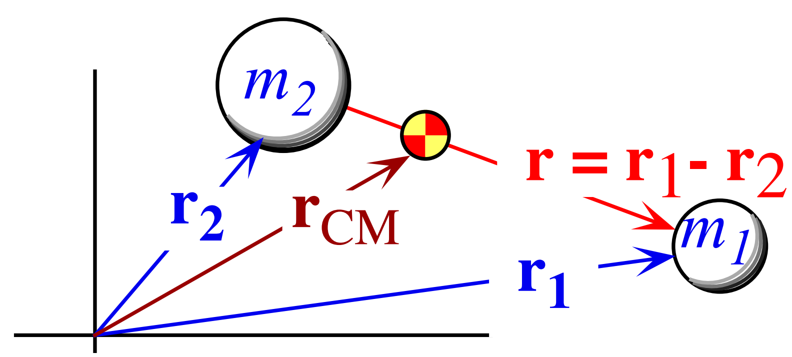
Re-scaled force: A Copernican view

$$\mathbf{r}_1 = \frac{m_2\mathbf{r}}{m_1 + m_2} = \frac{\mu}{m_1}\mathbf{r}, \qquad \mathbf{r}_2 = \frac{-m_1\mathbf{r}}{m_1 + m_2} = \frac{-\mu}{m_2}\mathbf{r}$$

relative radius vector

$$\frac{m_1}{\mu}\mathbf{r}_1 = \mathbf{r} = \frac{-m_2}{\mu}\mathbf{r}_2$$

$$\mathbf{r}_1 = \mathbf{r}_{\mathbf{CM}} + \frac{m_2\mathbf{r}}{m_1 + m_2}, \qquad \mathbf{r}_2 = \mathbf{r}_{\mathbf{CM}} - \frac{m_1\mathbf{r}}{m_1 + m_2}$$



$$\mathbf{r}_{CM} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

Reduced mass: Ptolemetric views

Radial inter-particle force $\mathbf{F}_{12}$ is on m_1 due to m_2 and $\mathbf{F}_{21} = -\mathbf{F}_{12}$ is on m_2 due to m_1

$$\mathbf{F}_{12} = \mathbf{F}(\mathrm{r})\mathbf{e_r} = -\mathbf{F}_{21} \quad = F(r)\hat{\mathbf{r}} = \quad F(r)\frac{\mathbf{r}}{r} = \quad \frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$\mathbf{F}_{12}$ acts along relative coordinate vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

Depends only upon the relative distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$

$$\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_1 = \quad F(r)\hat{\mathbf{r}} = \quad F(r)\frac{\mathbf{r}}{r} = \quad \frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_2 = -F(r)\hat{\mathbf{r}} = -F(r)\frac{\mathbf{r}}{r} = -\frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

Sum $\mathbf{F}_{12}+\mathbf{F}_{21}$ yields zero because of Newton's 3rd -law action-reaction cancellation.

$$(m_1+m_2)\ddot{\mathbf{r}}_{\mathbf{CM}} = m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = \mathbf{0}$$

Difference $\mathbf{F}_{12}-\mathbf{F}_{21}$ reduces to $\mu\ddot{\mathbf{r}} = \mathbf{F}(r)$ using *reduced mass*: $\boxed{\mu = \dfrac{m_2 m_1}{m_1+m_2}}$ and: $\ddot{\mathbf{r}}_{\mathbf{CM}} = \mathbf{0}$

$$[\quad m_1\ddot{\mathbf{r}}_1 \quad] - [\quad m_2\ddot{\mathbf{r}}_2 \quad] = \frac{2F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\left[m_1\ddot{\mathbf{r}}_{\mathbf{CM}} + \frac{m_1 m_2\ddot{\mathbf{r}}}{m_1+m_2}\right] - \left[m_2\ddot{\mathbf{r}}_{\mathbf{CM}} - \frac{m_2 m_1\ddot{\mathbf{r}}}{m_1+m_2}\right] = \frac{2F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

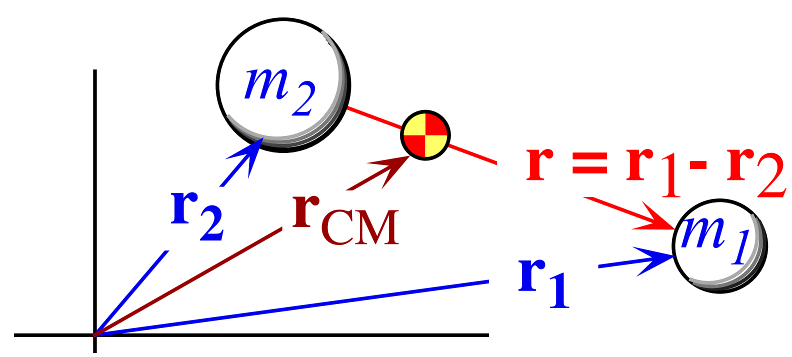
gives: $\mu\ddot{\mathbf{r}} = F(r)\hat{\mathbf{r}} = F(r)\mathbf{e_r} = \mathbf{F}(r)$

Re-scaled force: A Copernican view

$$\mathbf{r}_1 = \frac{m_2\mathbf{r}}{m_1+m_2} = \frac{\mu}{m_1}\mathbf{r}\,, \qquad \mathbf{r}_2 = \frac{-m_1\mathbf{r}}{m_1+m_2} = \frac{-\mu}{m_2}\mathbf{r}$$

relative radius vector

$$\frac{m_1}{\mu}\mathbf{r}_1 = \mathbf{r} = \frac{-m_2}{\mu}\mathbf{r}_2$$



$$\mathbf{r}_{\mathrm{CM}} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1+m_2}$$

$$\mathbf{r}_1 = \mathbf{r}_{\mathbf{CM}} + \frac{m_2\mathbf{r}}{m_1+m_2}\,, \qquad \mathbf{r}_2 = \mathbf{r}_{\mathbf{CM}} - \frac{m_1\mathbf{r}}{m_1+m_2}$$

Reduced mass: Ptolemetric views

Radial inter-particle force $\mathbf{F}_{12}$ is on m_1 due to m_2 and $\mathbf{F}_{21} = -\mathbf{F}_{12}$ is on m_2 due to m_1

$$\mathbf{F}_{12} = \mathbf{F}(r)\mathbf{e_r} = -\mathbf{F}_{21} \quad = F(r)\hat{\mathbf{r}} = \quad F(r)\frac{\mathbf{r}}{r} = \quad \frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$\mathbf{F}_{12}$ acts along relative coordinate vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

Depends only upon the relative distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$

$$\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_1 = \quad F(r)\hat{\mathbf{r}} = \quad F(r)\frac{\mathbf{r}}{r} = \quad \frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_2 = -F(r)\hat{\mathbf{r}} = -F(r)\frac{\mathbf{r}}{r} = -\frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

Sum $\mathbf{F}_{12}+\mathbf{F}_{21}$ yields zero because of Newton's 3rd -law action-reaction cancellation.

$$(m_1+m_2)\ddot{\mathbf{r}}_{\mathbf{CM}} = m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = \mathbf{0}$$

Difference $\mathbf{F}_{12}-\mathbf{F}_{21}$ reduces to $\mu\ddot{\mathbf{r}} = \mathbf{F}(r)$ using $\boxed{\textit{reduced mass}: \quad \mu = \frac{m_2 m_1}{m_1+m_2}}$ and: $\ddot{\mathbf{r}}_{\mathbf{CM}} = \mathbf{0}$

$$[\quad m_1\ddot{\mathbf{r}}_1 \quad] - [\quad m_2\ddot{\mathbf{r}}_2 \quad] = \frac{2F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\left[\cancel{m_1\ddot{\mathbf{r}}_{\mathbf{CM}}} + \frac{m_1 m_2\ddot{\mathbf{r}}}{m_1+m_2}\right] - \left[\cancel{m_2\ddot{\mathbf{r}}_{\mathbf{CM}}} - \frac{m_2 m_1\ddot{\mathbf{r}}}{m_1+m_2}\right] = \frac{2F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\frac{2m_1 m_2\ddot{\mathbf{r}}}{m_1+m_2} = \frac{2F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

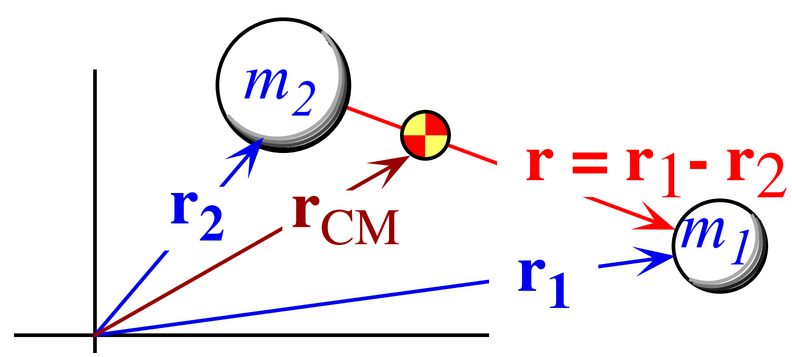
gives: $\mu\ddot{\mathbf{r}} = F(r)\hat{\mathbf{r}} = F(r)\mathbf{e_r} = \mathbf{F}(r)$

Re-scaled force: A Copernican view

$$\mathbf{r}_1 = \frac{m_2\mathbf{r}}{m_1+m_2} = \frac{\mu}{m_1}\mathbf{r}, \qquad \mathbf{r}_2 = \frac{-m_1\mathbf{r}}{m_1+m_2} = \frac{-\mu}{m_2}\mathbf{r}$$

relative radius vector

$$\frac{m_1}{\mu}\mathbf{r}_1 = \mathbf{r} = \frac{-m_2}{\mu}\mathbf{r}_2$$



$$\mathbf{r}_{\mathrm{CM}} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1+m_2}$$

$$\mathbf{r}_1 = \mathbf{r}_{\mathbf{CM}} + \frac{m_2\mathbf{r}}{m_1+m_2}, \qquad \mathbf{r}_2 = \mathbf{r}_{\mathbf{CM}} - \frac{m_1\mathbf{r}}{m_1+m_2}$$

Reduced mass: Ptolemetric views

Radial inter-particle force $\mathbf{F}_{12}$ is on m_1 due to m_2 and $\mathbf{F}_{21} = -\mathbf{F}_{12}$ is on m_2 due to m_1

$$\mathbf{F}_{12} = \mathbf{F}(\mathrm{r})\mathbf{e_r} = -\mathbf{F}_{21} \quad = F(r)\hat{\mathbf{r}} = \quad F(r)\frac{\mathbf{r}}{r} = \quad \frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$\mathbf{F}_{12}$ acts along relative coordinate vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

Depends only upon the relative distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$

$$\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_\mathbf{1} = \quad F(r)\hat{\mathbf{r}} = \quad F(r)\frac{\mathbf{r}}{r} = \quad \frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_\mathbf{2} = -F(r)\hat{\mathbf{r}} = -F(r)\frac{\mathbf{r}}{r} = -\frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

Sum $\mathbf{F}_{12}+\mathbf{F}_{21}$ yields zero because of Newton's 3rd -law action-reaction cancellation.

$$(m_1 + m_2)\ddot{\mathbf{r}}_\mathbf{CM} = m_1\ddot{\mathbf{r}}_\mathbf{1} + m_2\ddot{\mathbf{r}}_\mathbf{2} = \mathbf{0}$$

Difference $\mathbf{F}_{12}-\mathbf{F}_{21}$ reduces to $\mu\ddot{\mathbf{r}} = \mathbf{F}(r)$ using $\boxed{\textit{reduced mass}: \quad \mu = \frac{m_2 m_1}{m_1 + m_2}}$ and: $\ddot{\mathbf{r}}_\mathbf{CM} = \mathbf{0}$

$$[\quad m_1\ddot{\mathbf{r}}_\mathbf{1} \quad] - [\quad m_2\ddot{\mathbf{r}}_\mathbf{2} \quad] = \frac{2F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\left[\cancel{m_1\ddot{\mathbf{r}}_\mathbf{CM}} + \frac{m_1 m_2\ddot{\mathbf{r}}}{m_1 + m_2}\right] - \left[\cancel{m_2\ddot{\mathbf{r}}_\mathbf{CM}} - \frac{m_2 m_1\ddot{\mathbf{r}}}{m_1 + m_2}\right] = \frac{2F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\boxed{\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} = \frac{m_1 + m_2}{m_1 m_2}}$$

$$\mu = \frac{m_2}{1 + \frac{m_2}{m_1}} = m_2\left(1 - \frac{m_2}{m_1}\dots\right) \quad (m_1 >> m_2)$$

(Why it's reduced)

$$\mu = \frac{m_1}{1 + \frac{m_1}{m_2}} = m_1\left(1 - \frac{m_1}{m_2}\dots\right) \quad (m_2 >> m_1)$$

$$\frac{2m_1 m_2\ddot{\mathbf{r}}}{m_1 + m_2} = \frac{2F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

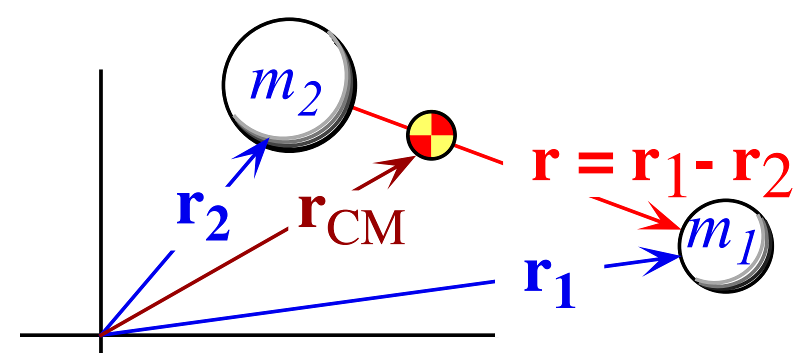
gives: $\mu\ddot{\mathbf{r}} = F(r)\hat{\mathbf{r}} = F(r)\mathbf{e_r} = \mathbf{F}(r)$

Re-scaled force: A Copernican view

$$\mathbf{r}_1 = \frac{m_2\mathbf{r}}{m_1 + m_2} = \frac{\mu}{m_1}\mathbf{r}, \qquad \mathbf{r}_2 = \frac{-m_1\mathbf{r}}{m_1 + m_2} = \frac{-\mu}{m_2}\mathbf{r}$$

relative radius vector

$$\frac{m_1}{\mu}\mathbf{r}_1 = \mathbf{r} = \frac{-m_2}{\mu}\mathbf{r}_2$$



$$\mathbf{r}_{\mathrm{CM}} = \frac{m_1\mathbf{r_1} + m_2\mathbf{r_2}}{m_1 + m_2}$$

$$\mathbf{r}_1 = \mathbf{r}_\mathbf{CM} + \frac{m_2\mathbf{r}}{m_1 + m_2}, \qquad \mathbf{r}_2 = \mathbf{r}_\mathbf{CM} - \frac{m_1\mathbf{r}}{m_1 + m_2}$$

2-Particle orbits

Ptolemetric view and reduced mass

➤ *Copernican view and reduced coupling*

Reduced mass: Ptolemetric views

Radial inter-particle force $\mathbf{F}_{12}$ is on m_1 due to m_2 and $\mathbf{F}_{21} = -\mathbf{F}_{12}$ is on m_2 due to m_1

$$\mathbf{F}_{12} = \mathbf{F}(r)\mathbf{e_r} = -\mathbf{F}_{21} = F(r)\hat{\mathbf{r}} = F(r)\frac{\mathbf{r}}{r} = \frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$\mathbf{F}_{12}$ acts along relative coordinate vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

Depends only upon the relative distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$

$$\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_1 = F(r)\hat{\mathbf{r}} = F(r)\frac{\mathbf{r}}{r} = \frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_2 = -F(r)\hat{\mathbf{r}} = -F(r)\frac{\mathbf{r}}{r} = -\frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

Sum $\mathbf{F}_{12}+\mathbf{F}_{21}$ yields zero because of Newton's 3rd -law action-reaction cancellation.

$$(m_1 + m_2)\ddot{\mathbf{r}}_{\mathbf{CM}} = m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = \mathbf{0}$$

Difference $\mathbf{F}_{12}-\mathbf{F}_{21}$ reduces to $\mu\ddot{\mathbf{r}} = \mathbf{F}(r)$ using *reduced mass*: $\boxed{\mu = \frac{m_2 m_1}{m_1 + m_2}}$ and: $\ddot{\mathbf{r}}_{\mathbf{CM}} = \mathbf{0}$

$$[\quad m_1\ddot{\mathbf{r}}_1 \quad] - [\quad m_2\ddot{\mathbf{r}}_2 \quad] = \frac{2F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\left[\cancel{m_1\ddot{\mathbf{r}}_{\mathbf{CM}}} + \frac{m_1 m_2\ddot{\mathbf{r}}}{m_1 + m_2}\right] - \left[\cancel{m_2\ddot{\mathbf{r}}_{\mathbf{CM}}} - \frac{m_2 m_1\ddot{\mathbf{r}}}{m_1 + m_2}\right] = \frac{2F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\frac{2m_1 m_2\ddot{\mathbf{r}}}{m_1 + m_2} = \frac{2F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\boxed{\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} = \frac{m_1 + m_2}{m_1 m_2}}$$

$$\mu = \frac{m_2}{1 + \frac{m_2}{m_1}} = m_2\left(1 - \frac{m_2}{m_1}\dots\right) \quad (m_1 >> m_2)$$

(Why it's reduced)

$$\mu = \frac{m_1}{1 + \frac{m_1}{m_2}} = m_1\left(1 - \frac{m_1}{m_2}\dots\right) \quad (m_2 >> m_1)$$

gives: $\mu\ddot{\mathbf{r}} = F(r)\hat{\mathbf{r}} = F(r)\mathbf{e_r} = \mathbf{F}(r)$

Re-scaled force: A Copernican view

$$\mathbf{r_1} = \frac{m_2\mathbf{r}}{m_1 + m_2} = \frac{\mu}{m_1}\mathbf{r}, \qquad \mathbf{r}_2 = \frac{-m_1\mathbf{r}}{m_1 + m_2} = \frac{-\mu}{m_2}\mathbf{r}$$

relative radius vector

$$\frac{m_1}{\mu}\mathbf{r_1} = \mathbf{r} = \frac{-m_2}{\mu}\mathbf{r}_2$$

(Here we get "reduced" coupling constants)

each particle keeps it original mass m_1 or m_2, but feels

coordinate-re-scaled force field $F(m_1 r_1/\mu)$ or *$F(m_2 r_2/\mu)$* field

$$\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_1 = F\left(\frac{m_1}{\mu}r_1\right)\hat{\mathbf{r}}_1 = -\mathbf{F}_{21}$$

$$\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_2 = F\left(\frac{m_2}{\mu}r_2\right)\hat{\mathbf{r}}_2 = -\mathbf{F}_{12}$$

Reduced mass: Ptolemetric views

Radial inter-particle force $\mathbf{F}_{12}$ is on m_1 due to m_2 and $\mathbf{F}_{21} = -\mathbf{F}_{12}$ is on m_2 due to m_1

$$\mathbf{F}_{12} = \mathbf{F}(r)\mathbf{e_r} = -\mathbf{F}_{21} = F(r)\hat{\mathbf{r}} = F(r)\frac{\mathbf{r}}{r} = \frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$\mathbf{F}_{12}$ acts along relative coordinate vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

Depends only upon the relative distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$

$$\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_1 = F(r)\hat{\mathbf{r}} = F(r)\frac{\mathbf{r}}{r} = \frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_2 = -F(r)\hat{\mathbf{r}} = -F(r)\frac{\mathbf{r}}{r} = -\frac{F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

Sum $\mathbf{F}_{12}+\mathbf{F}_{21}$ yields zero because of Newton's 3rd -law action-reaction cancellation.

$$(m_1+m_2)\ddot{\mathbf{r}}_{\mathbf{CM}} = m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = \mathbf{0}$$

Difference $\mathbf{F}_{12}-\mathbf{F}_{21}$ reduces to $\mu\ddot{\mathbf{r}} = \mathbf{F}(r)$ using *reduced mass*: $\boxed{\mu = \frac{m_2 m_1}{m_1+m_2}}$ and: $\ddot{\mathbf{r}}_{\mathbf{CM}} = \mathbf{0}$

$$[\quad m_1\ddot{\mathbf{r}}_1 \quad] - [\quad m_2\ddot{\mathbf{r}}_2 \quad] = \frac{2F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\left[\cancel{m_1\ddot{\mathbf{r}}_{\mathbf{CM}}} + \frac{m_1 m_2\ddot{\mathbf{r}}}{m_1+m_2}\right] - \left[\cancel{m_2\ddot{\mathbf{r}}_{\mathbf{CM}}} - \frac{m_2 m_1\ddot{\mathbf{r}}}{m_1+m_2}\right] = \frac{2F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

$$\boxed{\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} = \frac{m_1+m_2}{m_1 m_2}}$$

$$\mu = \frac{m_2}{1+\frac{m_2}{m_1}} = m_2\left(1 - \frac{m_2}{m_1}\dots\right) \quad (m_1 >> m_2)$$

(Why it's reduced)

$$\mu = \frac{m_1}{1+\frac{m_1}{m_2}} = m_1\left(1 - \frac{m_1}{m_2}\dots\right) \quad (m_2 >> m_1)$$

$$\frac{2m_1 m_2\ddot{\mathbf{r}}}{m_1+m_2} = \frac{2F(r)}{r}(\mathbf{r_1} - \mathbf{r_2})$$

gives: $\mu\ddot{\mathbf{r}} = F(r)\hat{\mathbf{r}} = F(r)\mathbf{e_r} = \mathbf{F}(r)$

Re-scaled force: A Copernican view

$$\mathbf{r}_1 = \frac{m_2\mathbf{r}}{m_1+m_2} = \frac{\mu}{m_1}\mathbf{r}, \qquad \mathbf{r}_2 = \frac{-m_1\mathbf{r}}{m_1+m_2} = \frac{-\mu}{m_2}\mathbf{r}$$

relative radius vector

$$\frac{m_1}{\mu}\mathbf{r}_1 = \mathbf{r} = \frac{-m_2}{\mu}\mathbf{r}_2$$

(Here we get "reduced" coupling constants)

each particle keeps it original mass m_1 or m_2, but feels
coordinate-re-scaled force field $F(m_1 r_1/\mu)$ or $F(m_2 r_2/\mu)$ field

$$\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_1 = F\left(\frac{m_1}{\mu}r_1\right)\hat{\mathbf{r}}_1 = -\mathbf{F}_{21}$$

$$\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_2 = F\left(\frac{m_2}{\mu}r_2\right)\hat{\mathbf{r}}_2 = -\mathbf{F}_{12}$$

$F(r) = \frac{k}{r^2}$ becomes: $F\left(\frac{m_1}{\mu}r_1\right) = \frac{\mu^2}{m_1^2}\frac{k}{r_1^2}$

(Coulomb)

$$k \to k_1 = k\,\mu^2/m_1^2, \qquad k \to k_2 = k\,\mu^2/m_2^2$$

Reduced mass: Ptolemetric views

Radial inter-particle force $\mathbf{F}_{12}$ is on m_1 due to m_2 and $\mathbf{F}_{21} = -\mathbf{F}_{12}$ is on m_2 due to m_1

$$\mathbf{F}_{12} = \mathbf{F}(\mathrm{r})\mathbf{e_r} = -\mathbf{F}_{21} \quad = F(r)\hat{\mathbf{r}} = \quad F(r)\frac{\mathbf{r}}{r} = \quad \frac{F(r)}{r}(\mathbf{r_1}-\mathbf{r_2})$$

$\mathbf{F}_{12}$ acts along relative coordinate vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

Depends only upon the relative distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$

$$\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_1 = \quad F(r)\hat{\mathbf{r}} = \quad F(r)\frac{\mathbf{r}}{r} = \quad \frac{F(r)}{r}(\mathbf{r_1}-\mathbf{r_2})$$

$$\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_2 = -F(r)\hat{\mathbf{r}} = -F(r)\frac{\mathbf{r}}{r} = -\frac{F(r)}{r}(\mathbf{r_1}-\mathbf{r_2})$$

Sum $\mathbf{F}_{12}+\mathbf{F}_{21}$ yields zero because of Newton's 3rd -law action-reaction cancellation.

$$(m_1+m_2)\ddot{\mathbf{r}}_{\mathbf{CM}} = m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = \mathbf{0}$$

Difference $\mathbf{F}_{12}-\mathbf{F}_{21}$ reduces to $\mu\ddot{\mathbf{r}} = \mathbf{F}(r)$ using *reduced mass*: $\boxed{\mu = \frac{m_2 m_1}{m_1+m_2}}$ and: $\ddot{\mathbf{r}}_{\mathbf{CM}} = \mathbf{0}$

$$[\qquad m_1\ddot{\mathbf{r}}_1 \qquad] - [\qquad m_2\ddot{\mathbf{r}}_2 \qquad] = \frac{2F(r)}{r}(\mathbf{r_1}-\mathbf{r_2})$$

$$\left[m_1\cancel{\ddot{\mathbf{r}}_{\mathbf{CM}}} + \frac{m_1 m_2\ddot{\mathbf{r}}}{m_1+m_2}\right] - \left[m_2\cancel{\ddot{\mathbf{r}}_{\mathbf{CM}}} - \frac{m_2 m_1\ddot{\mathbf{r}}}{m_1+m_2}\right] = \frac{2F(r)}{r}(\mathbf{r_1}-\mathbf{r_2})$$

$$\frac{2m_1 m_2\ddot{\mathbf{r}}}{m_1+m_2} = \frac{2F(r)}{r}(\mathbf{r_1}-\mathbf{r_2})$$

$$\boxed{\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} = \frac{m_1+m_2}{m_1 m_2}}$$

$$\mu = \frac{m_2}{1+\frac{m_2}{m_1}} = m_2\left(1-\frac{m_2}{m_1}\ldots\right) \quad (m_1 >> m_2)$$

(Why it's reduced)

$$\mu = \frac{m_1}{1+\frac{m_1}{m_2}} = m_1\left(1-\frac{m_1}{m_2}\ldots\right) \quad (m_2 >> m_1)$$

gives: $\mu\ddot{\mathbf{r}} = F(r)\hat{\mathbf{r}} = F(r)\mathbf{e_r} = \mathbf{F}(r)$

Re-scaled force: A Copernican view

$$\mathbf{r}_1 = \frac{m_2\mathbf{r}}{m_1+m_2} = \frac{\mu}{m_1}\mathbf{r}\ , \qquad \mathbf{r}_2 = \frac{-m_1\mathbf{r}}{m_1+m_2} = \frac{-\mu}{m_2}\mathbf{r}$$

relative radius vector

$$\frac{m_1}{\mu}\mathbf{r}_1 = \mathbf{r} = \frac{-m_2}{\mu}\mathbf{r}_2$$

(Here we get "reduced" coupling constants)

each particle keeps it original mass m_1 or m_2, but feels

coordinate-re-scaled force field $F(m_1 r_1/\mu)$ or *$F(m_2 r_2/\mu)$* field

$$\mathbf{F}_{12} = m_1\ddot{\mathbf{r}}_1 = F\left(\frac{m_1}{\mu}r_1\right)\hat{\mathbf{r}}_1 = -\mathbf{F}_{21}$$

$$\mathbf{F}_{21} = m_2\ddot{\mathbf{r}}_2 = F\left(\frac{m_2}{\mu}r_2\right)\hat{\mathbf{r}}_2 = -\mathbf{F}_{12}$$

$F(r) = \frac{k}{r^2}$ becomes: $F\left(\frac{m_1}{\mu}r_1\right) = \frac{\mu^2}{m_1^2}\frac{k}{r_1^2}$

(Coulomb)

$k \to k_1 = k\mu^2/m_1^2\ , \qquad k \to k_2 = k\mu^2/m_2^2$

$F(r) = -kr$ becomes: $F\left(\frac{m_1}{\mu}r_1\right) = -\frac{m_1}{\mu}k\,r_1$

(Harmonic Oscillator)

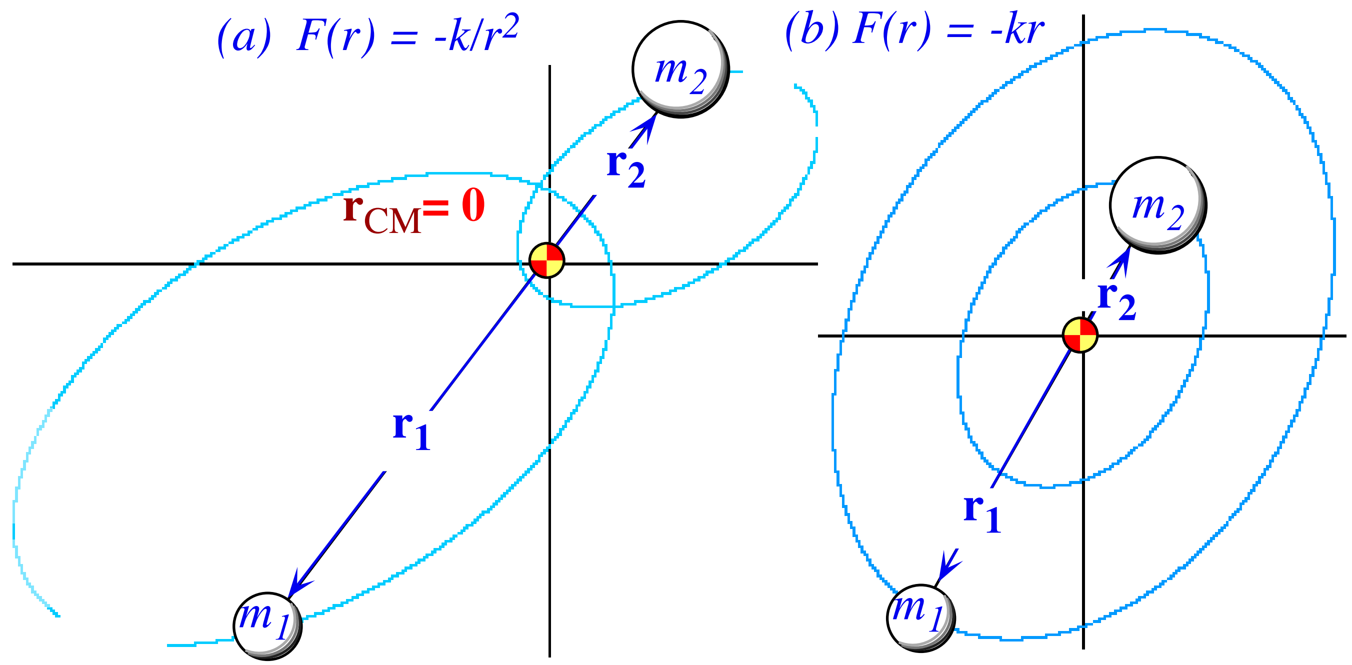
$k \to k_1 = km_1/\mu\ , \quad k \to k_2 = km_2/\mu$

2-Particle orbits and scattering: LAB-vs.-COM frame views
Ruler & compass construction (or not)

Examples of Coulomb and harmonic oscillator 2-particle "Copernican" orbits in CM system.

[CoulIt Web Simulations](#)
[Coulombic Orbit (CM Frame)](#)

[Coulombic Orbit (Lab Frame)](#)



[CoulIt Web Simulations](#)
[Hooke Orbit (CM Frame)](#)

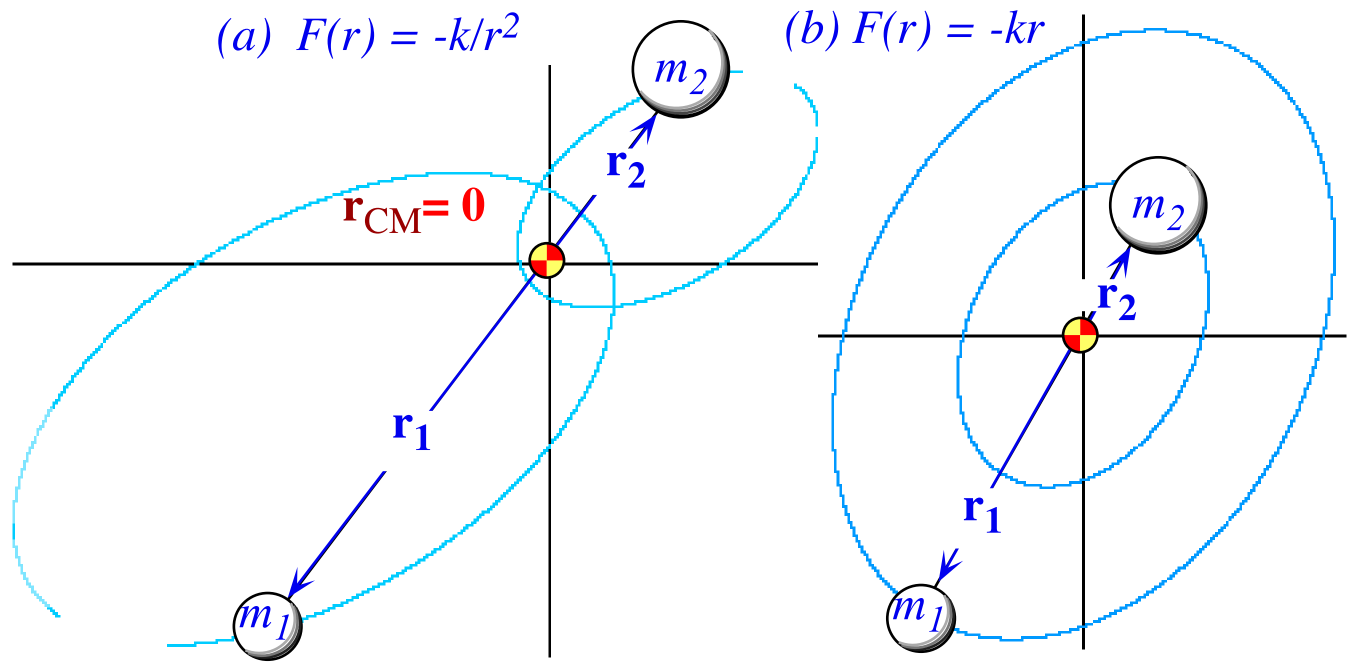
[Hooke Orbit (Lab Frame)](#)

Two particles are in synchronous motion around fixed CM origin.
Orbit periods are identical to each other.
Orbits are mass-scaled copies with equal aspect ratio (a/b), eccentricity, and orientation.

Examples of Coulomb and harmonic oscillator 2-particle "Copernican" orbits in CM system.

CoulIt Web Simulations
Coulombic Orbit (CM Frame)

Coulombic Orbit (Lab Frame)



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Orbits differ in placement of center (for the Coulomb case) or foci (for the oscillator).

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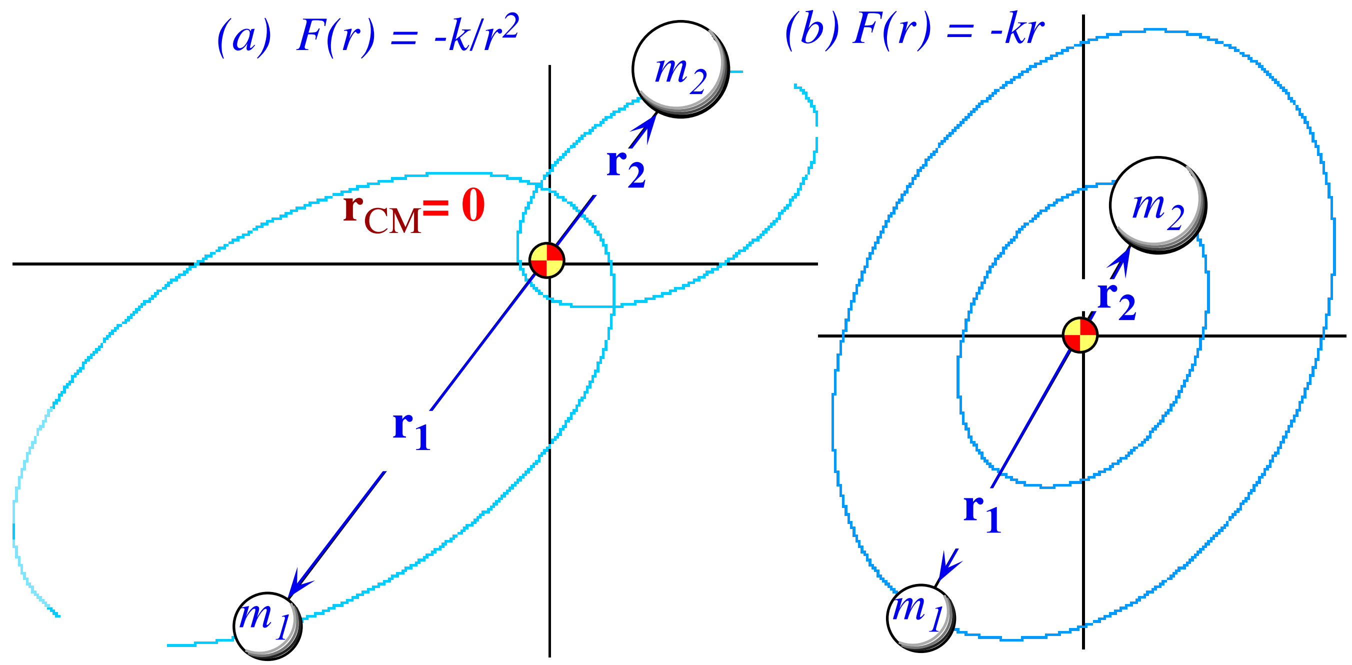

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Orbit axial dimensions (a_k , b_k) and λ_k are in inverse proportion to mass values.

$$a_1 m_1 = a_2 m_2 = a\mu \ , \qquad b_1 m_1 = b_2 m_2 = b\mu \qquad \lambda_1 m_1 = \lambda_2 m_2 = \lambda\mu$$

Examples of Coulomb and harmonic oscillator 2-particle "Copernican" orbits in CM system.

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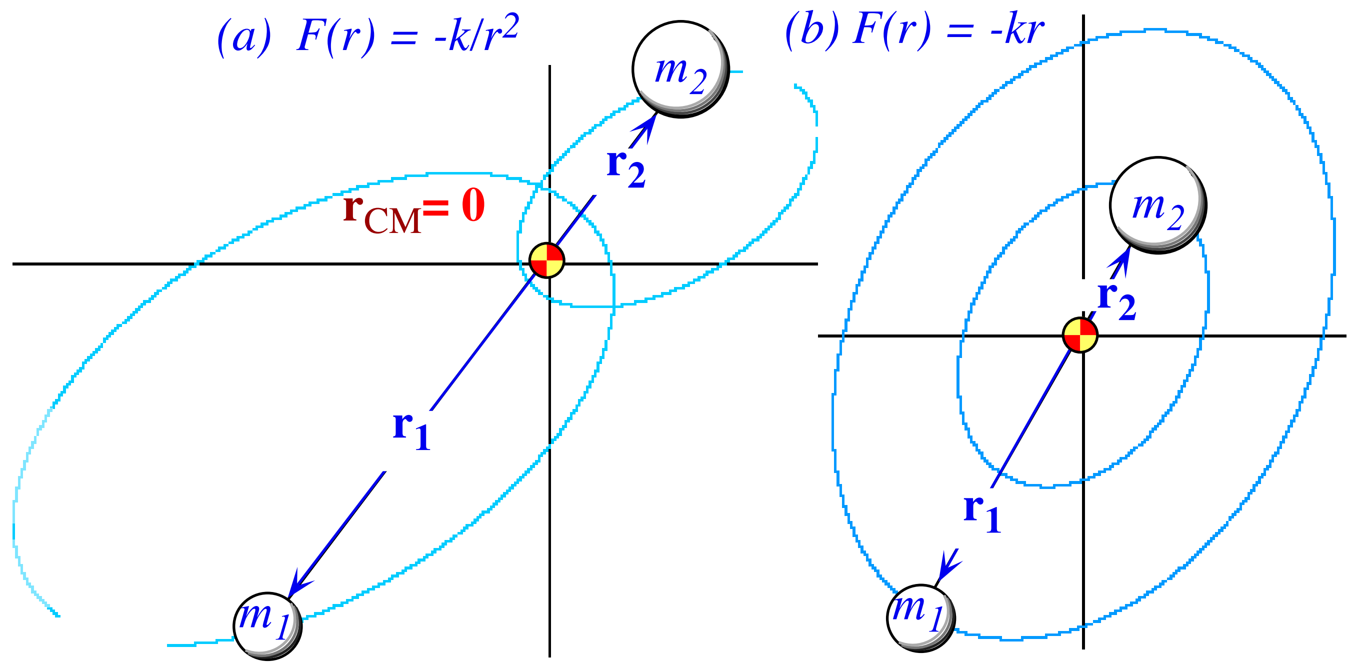
Coulombic Orbit (Lab Frame)



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$$a_1 m_1 = a_2 m_2 = a\mu \ , \qquad b_1 m_1 = b_2 m_2 = b\mu \qquad \lambda_1 m_1 = \lambda_2 m_2 = \lambda\mu$$

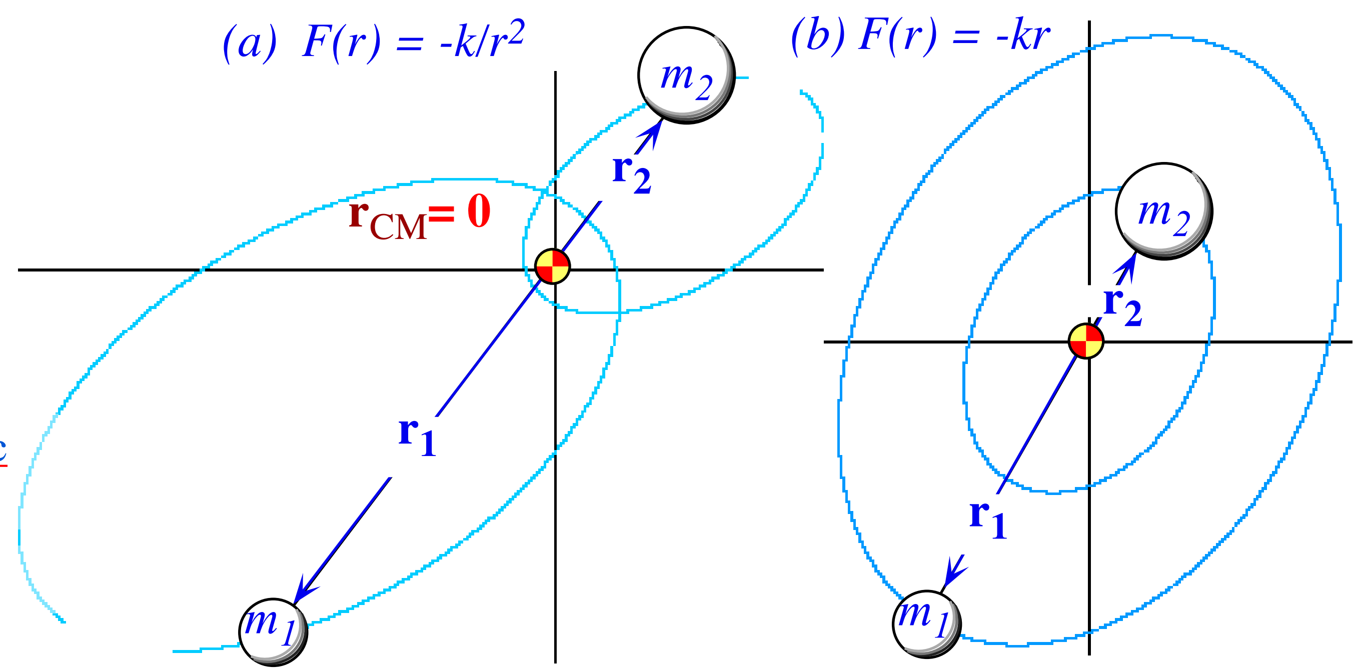
Harmonic oscillator periods and Coulomb orbit periods and eccentricity must match

$$T_{IHO} = 2\pi\sqrt{\frac{\mu}{k}} = 2\pi\sqrt{\frac{m_1}{k_1}} = 2\pi\sqrt{\frac{m_2}{k_2}} \qquad T_{Coul} = 2\pi\sqrt{\frac{\mu\, a^3}{k}} = 2\pi\sqrt{\frac{m_1 a_1{}^3}{k_1}} = 2\pi\sqrt{\frac{m_2 a_2{}^3}{k_2}} \qquad \varepsilon_1 = \varepsilon_2 = \varepsilon$$

Examples of Coulomb and harmonic oscillator 2-particle "Copernican" orbits in CM system.

[CoulIt Web Simulations Coulombic Orbit (CM Frame)]

[Coulombic Orbit (Lab Frame)]



[CoulIt Web Simulations Hooke Orbit (CM Frame)]

[Hooke Orbit (Lab Frame)]

Two particles are in synchronous motion around fixed CM origin.
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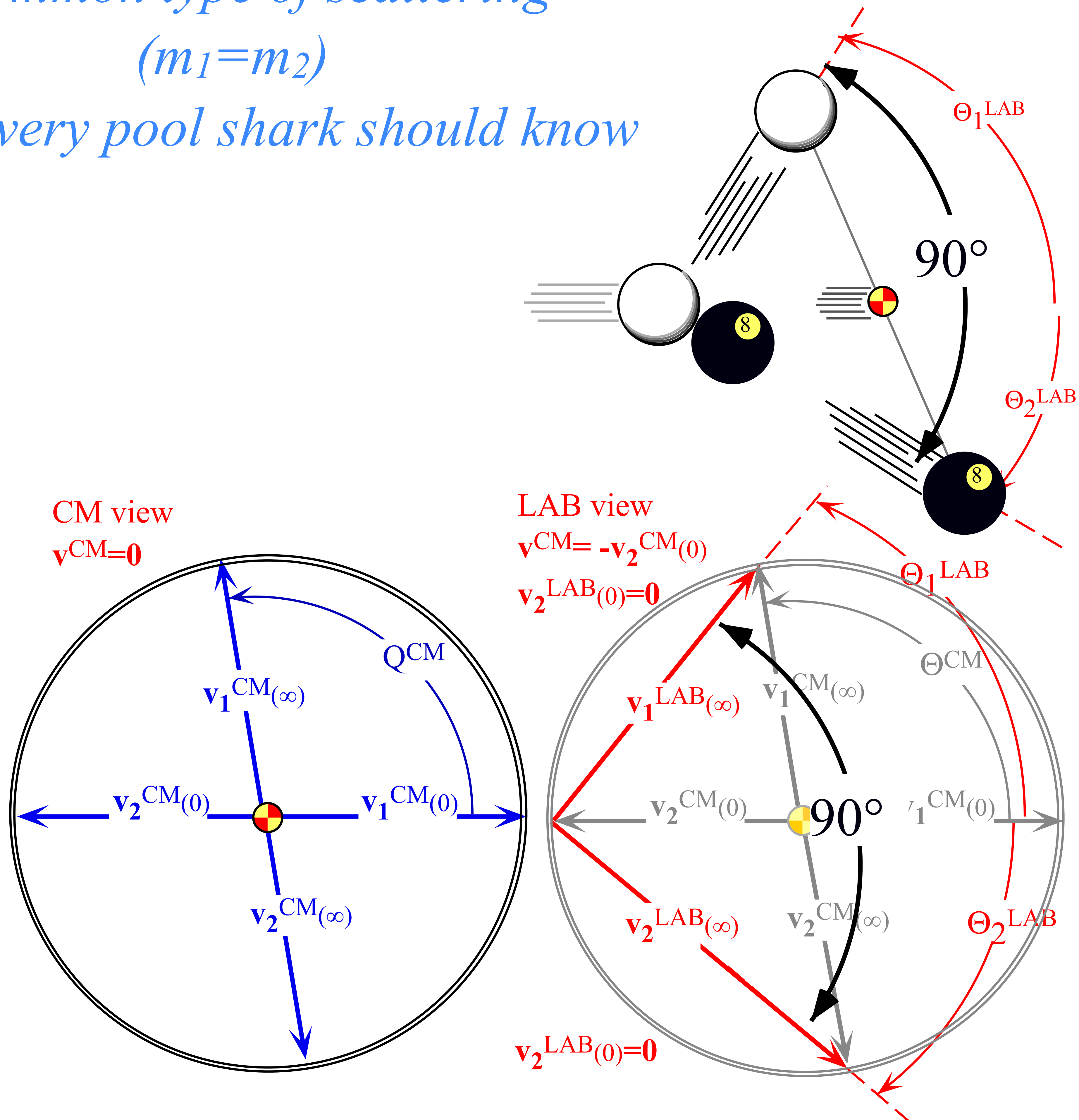
Three Coulomb orbit energy values satisfy the same proportion relation as their axes

$$E_1 m_1 = E_2 m_2 = E\mu \;, \quad \text{where:} \quad |E_1| = \frac{|k_1|}{2a_1} \;, \quad |E_2| = \frac{|k_2|}{2a_2} \;, \quad |E| = \frac{|k|}{2a} \;.$$

Energy values and axes satisfy similar sum relations

$$E_1 + E_2 = \frac{m_1}{\mu}E + \frac{m_2}{\mu}E = E \;, \quad \text{and:} \quad a_1 + a_2 = \frac{m_1}{\mu}a + \frac{m_2}{\mu}a = a$$

A common type of scattering
$(m_1 = m_2)$
...that every pool shark should know



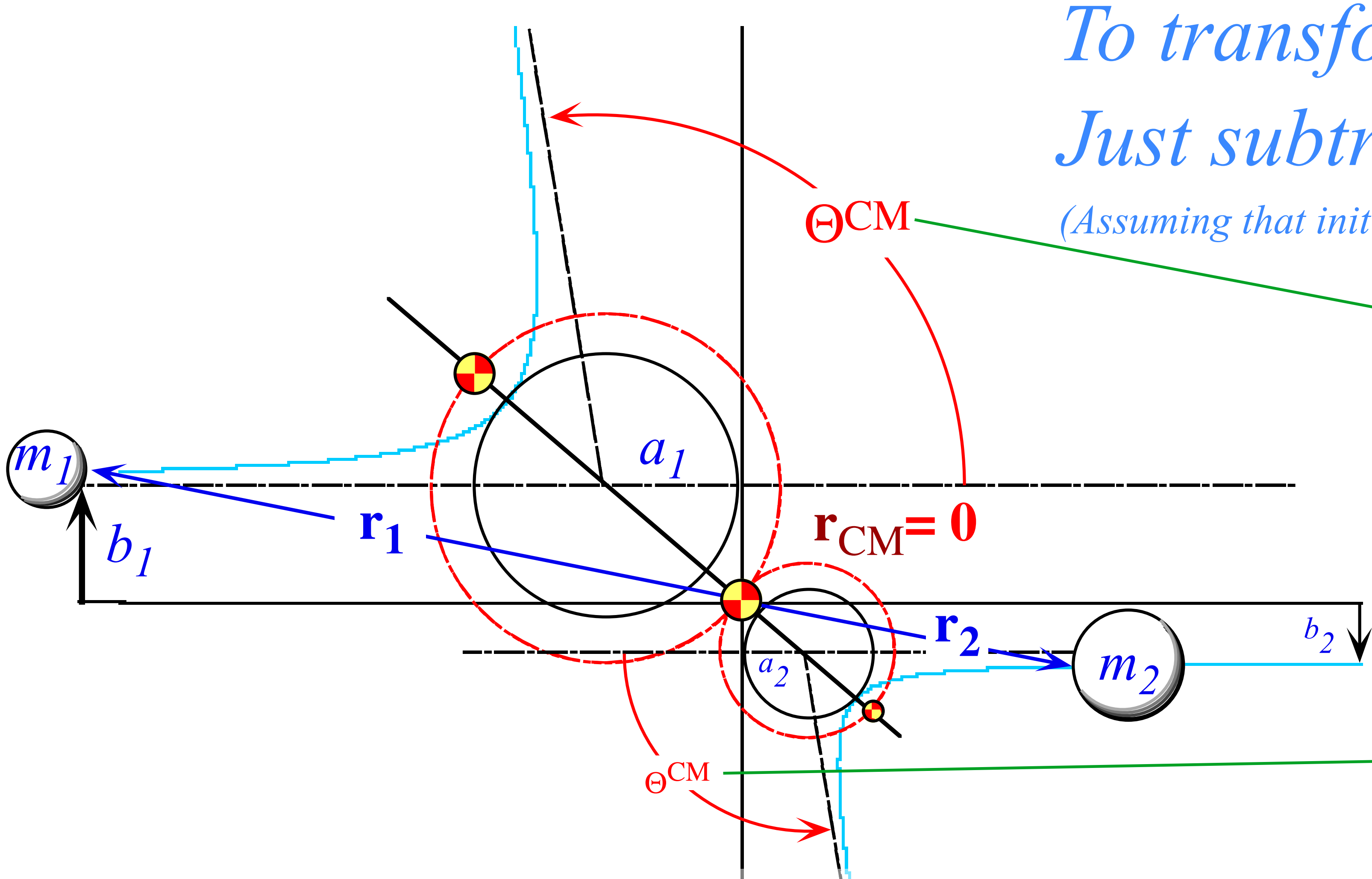
[BounceIt Web Simulations Hard Collision (CM Frame)]

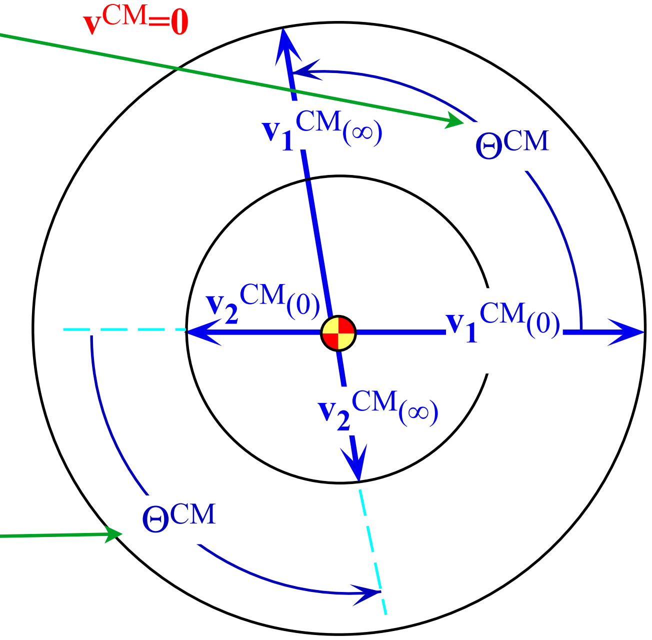
[Hard Collision (Lab Frame)]

To transform CM to LAB frame
Just subtract $\mathbf{v}_2^{CM}(0)$ from all $\mathbf{v}$

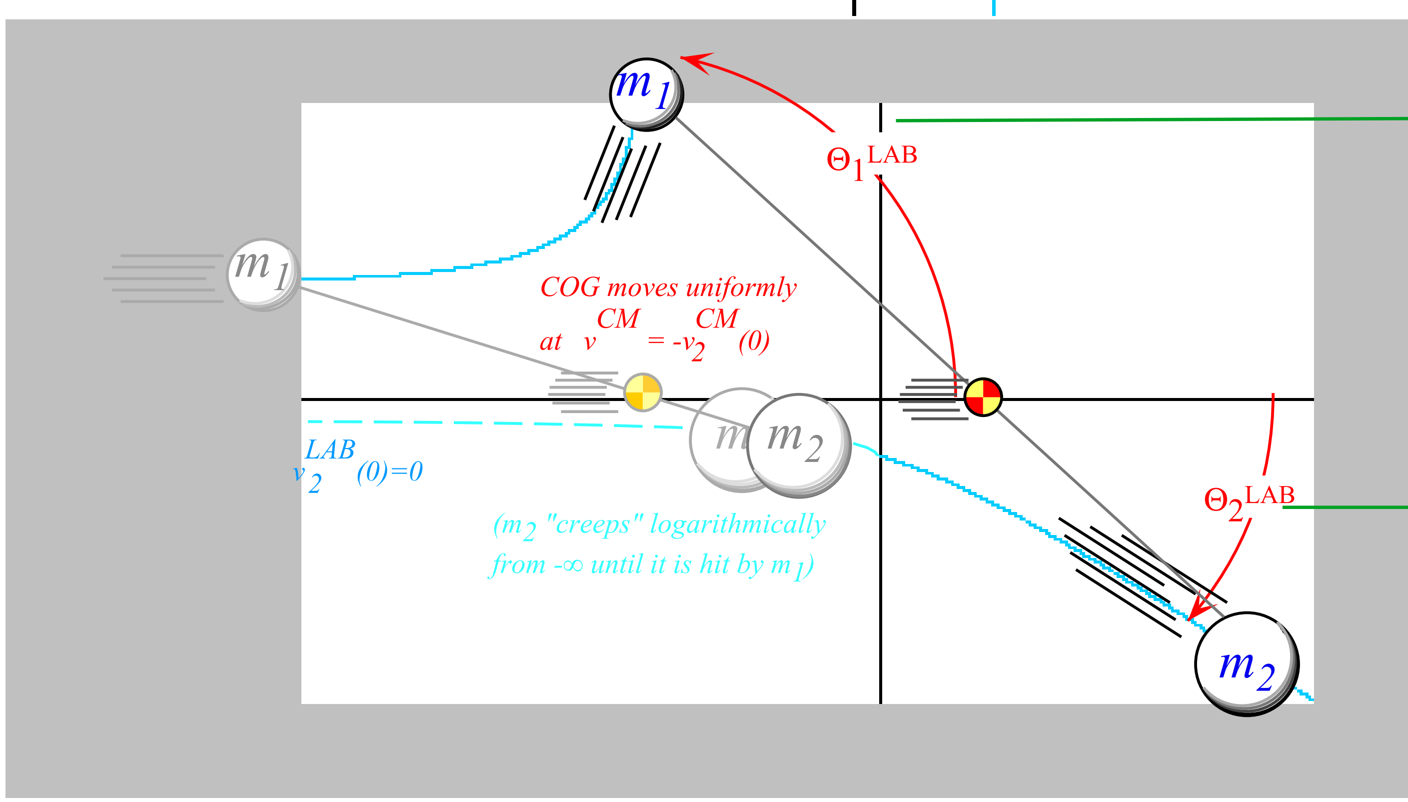
(Said: Galileo)

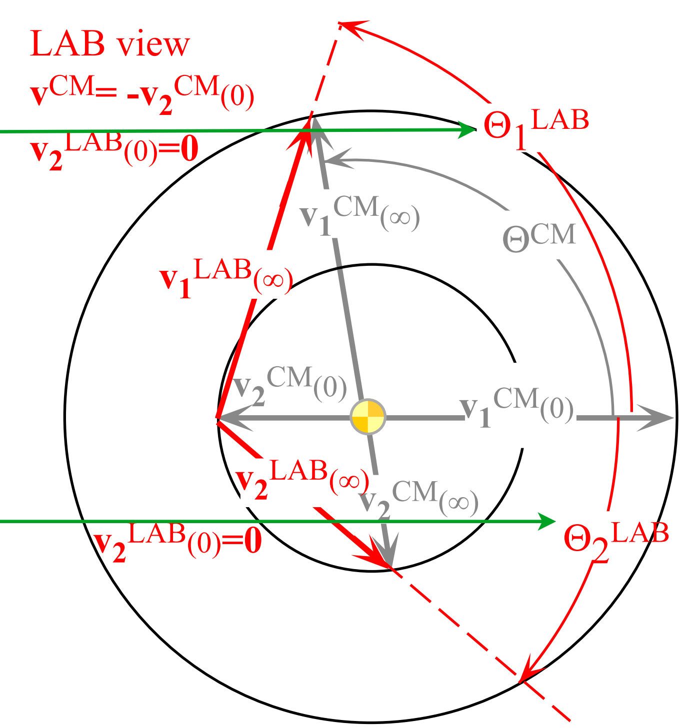
(Assuming that initial $\mathbf{v}_2^{LAB}(0)$ is zero so $\mathbf{v}_2^{CM}(0)$ is CM velocity in LAB)





[CoulIt Web Simulation - Coulombic Collision (CM Frame)]





[CoulIt Web Simulation - Coulombic Collision (LAB Frame)]

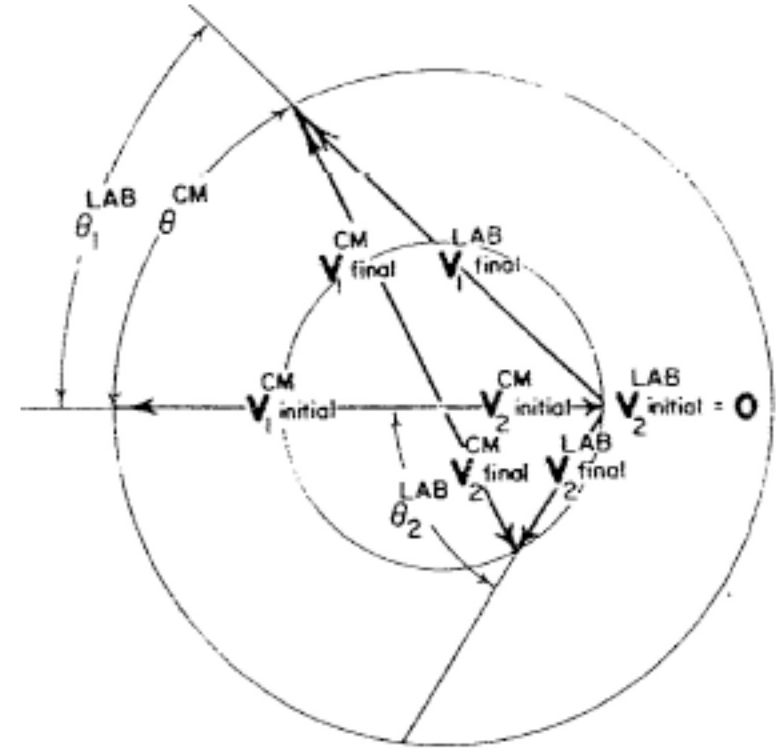


FIG. 4. Given the center of mass scattering angle θ^{CM} (from Fig. 3) and the mass ratio (2:1 in this case) a vector addition construction produces angles θ_1^{LAB} and θ_2^{LAB} shown here.

From:

[Geometric aspects of classical Coulomb scattering](#)

[American Journal of Physics 40,1852-1856 (1972)](#)

Class project when I taught Jr. CM at Georgia Tech

(Just 5 students)

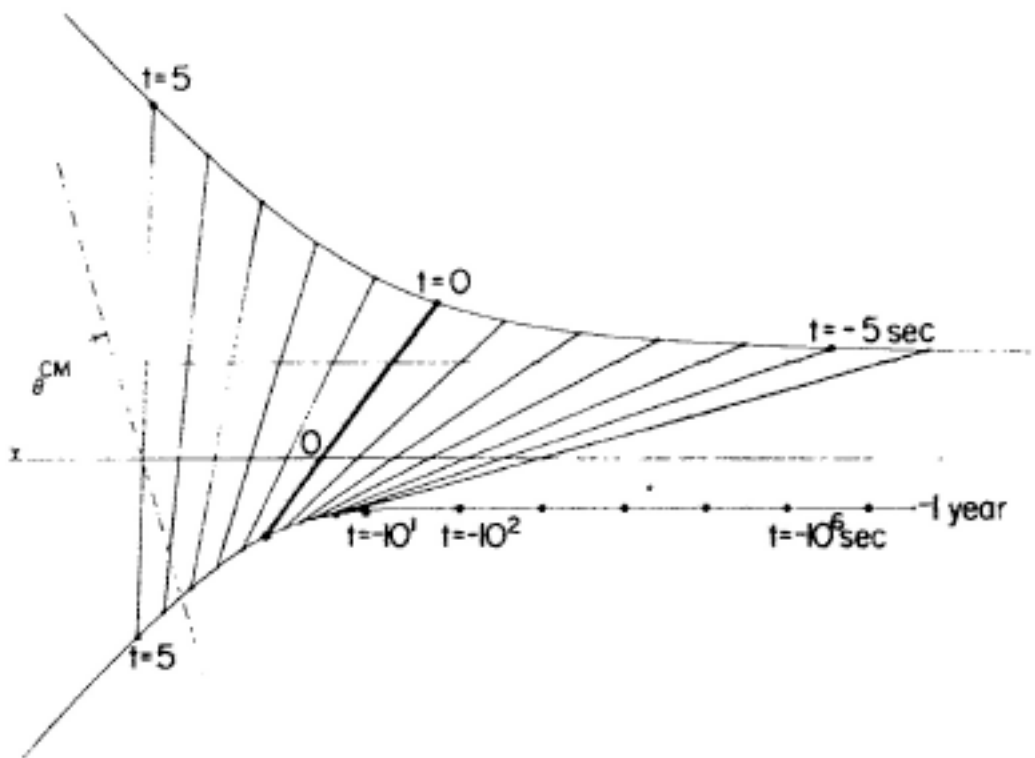
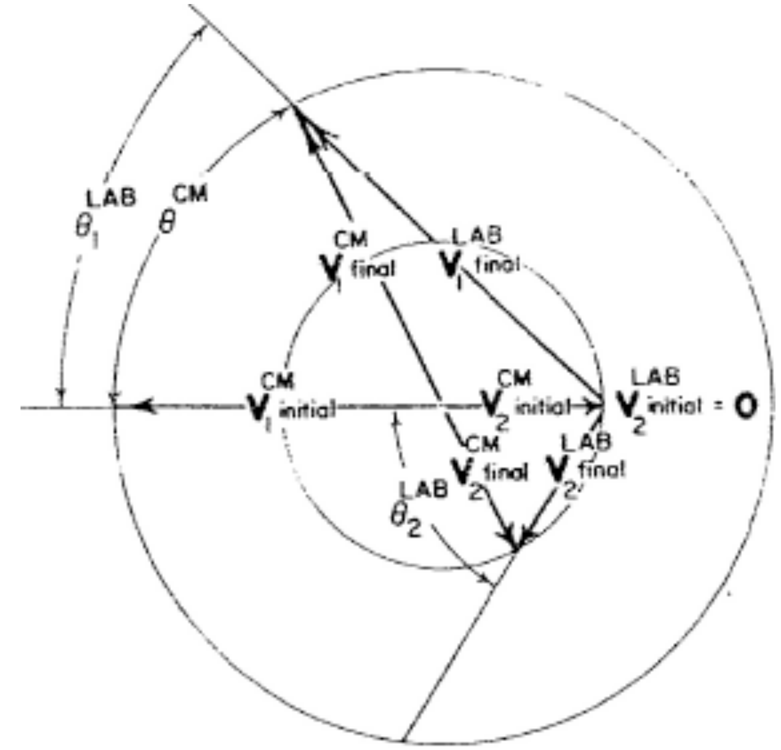


FIG. 5. The laboratory picture of Fig. 3. The scattering begins with both particles infinitely far to the right. The heavier particle is at rest and the lighter particle is moving left about 0.3 mile per day in the scale of this drawing. When the heavier particle first appears on this picture, one or two years before the "collision," it is creeping extremely slowly leftward, while the lighter particle is still over a hundred miles off to the right. The heavier particle continues creeping until finally the lighter particle arrives in the picture and moves through in about 12 sec. Most of the momentum is transferred in 3 or 4 sec.



The trouble with the Coulomb field is...

$$\int t^{-1}dt = \ln t + C$$

$$v_2^{LAB}(t) = \int dt \frac{|F_{Coulomb}|}{m_2}$$

$$\cong \int dt \frac{|k_{Coulomb}|}{m_2\left[v_1^{CM}(initial)\cdot t\right]^2}$$

$$\cong \int dt \frac{-|k_{Coulomb}|}{m_2\left[v_1^{CM}(initial)\right]^2}\frac{1}{t} \sim -\ln(t)$$

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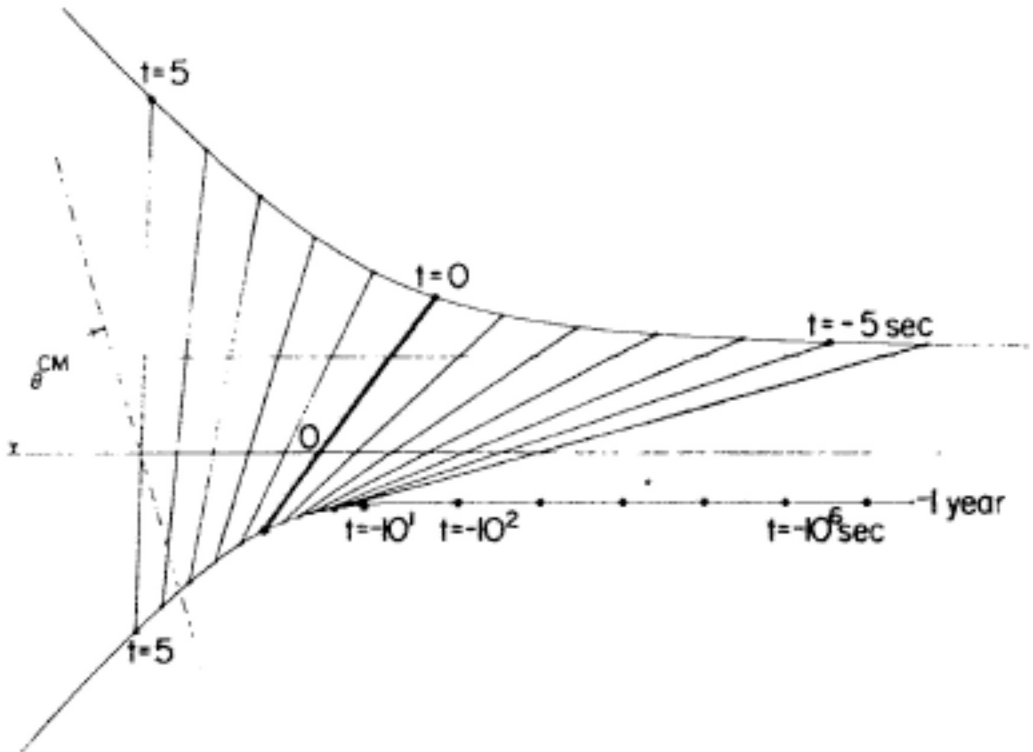
[American Journal of Physics 40,1852-1856 (1972)](#)

Class project when I taught Jr. CM at Georgia Tech
(Just 5 students)



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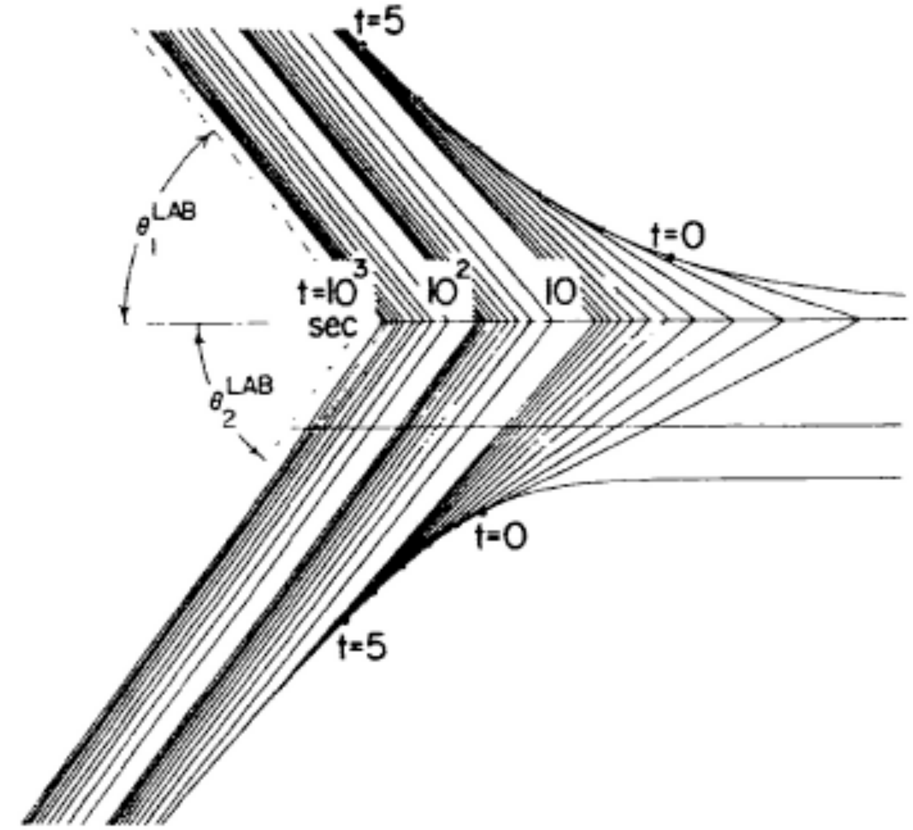

FIG. 6. Logarithmic recession of tangents demonstrates the nonexistence of asymptotes, for pure Coulomb scattering in laboratory system. At $t=10^3$ the slopes of the tangents are shy of θ_1^{LAB} and θ_2^{LAB} by only 0.02° and 0.04°, respectively.

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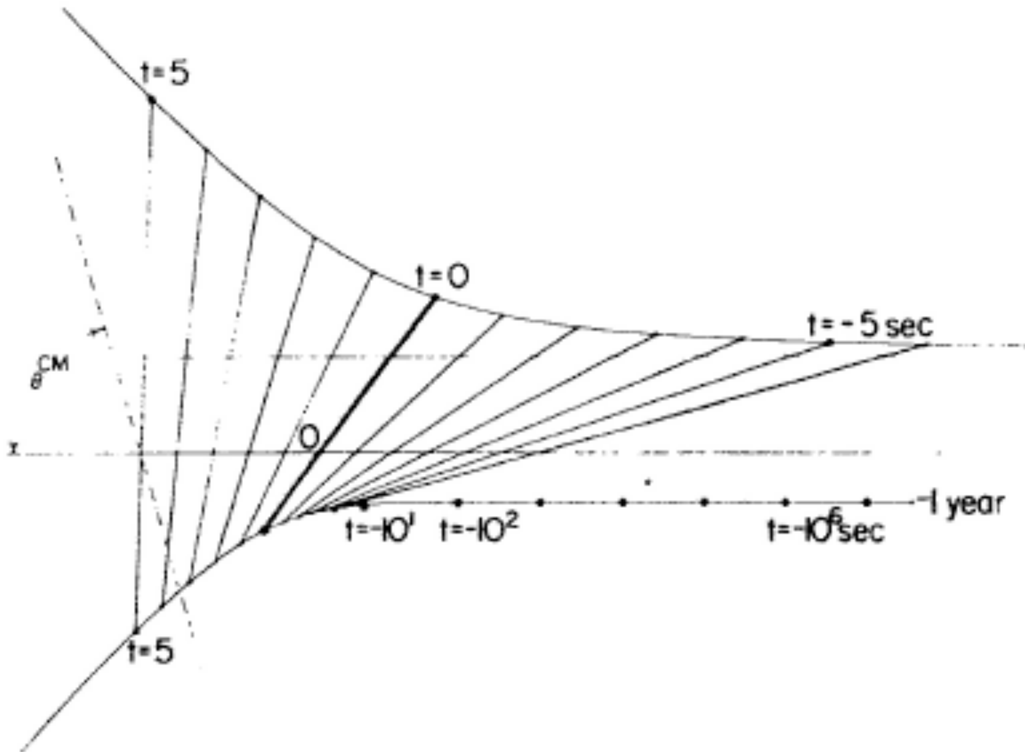
(Just 5 students)



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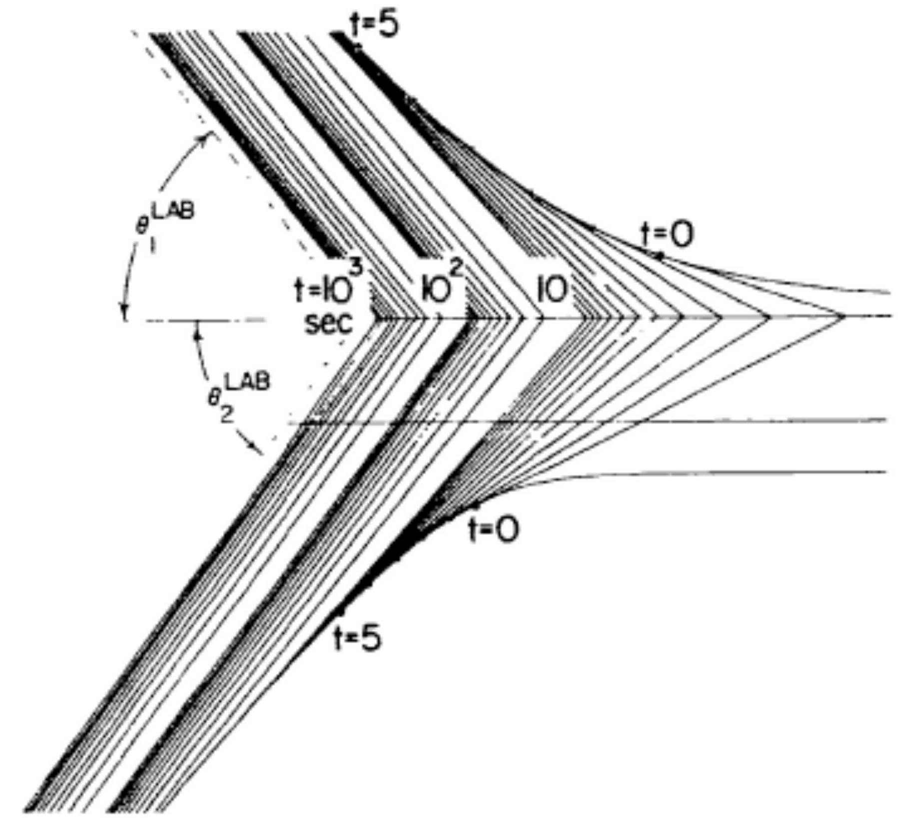

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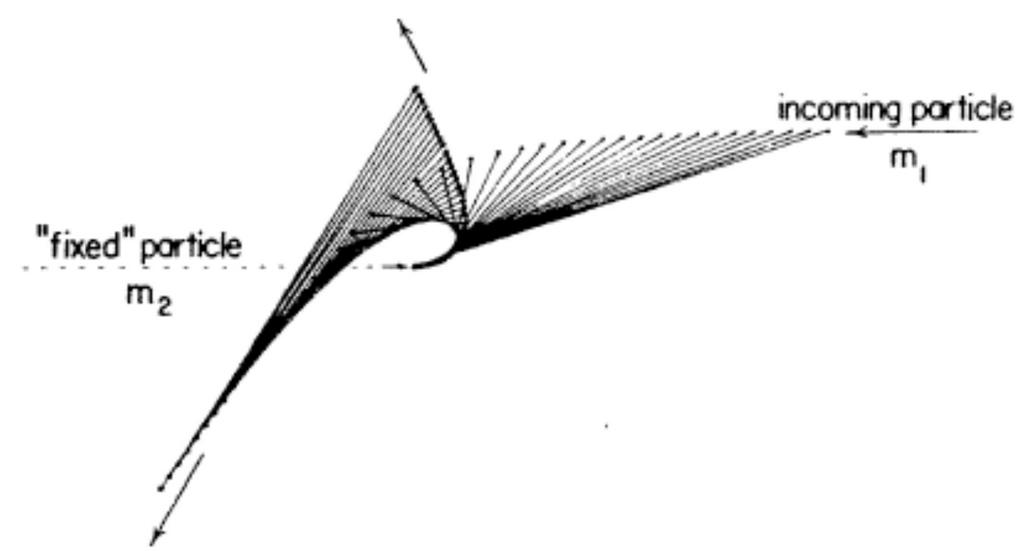

FIG. 7. Attractive Coulomb scattering in laboratory system. This has the same "anomalies" as the repulsive case.

From:

[Geometric aspects of classical Coulomb scattering](#)

[American Journal of Physics 40,1852-1856 (1972)](#)

Class project when I taught Jr. CM at Georgia Tech

(Just 5 students)

Rotational equivalent of Newton's **F**=*d***p**/*dt equations:* **N**=*d***L**/*dt*

How to make my boomerang come back

The gyrocompass and mechanical spin analogy

Rotational equivalent of Newton's $\mathbf{F}=d\mathbf{p}/dt$ *equations:* $\mathbf{N}=d\mathbf{L}/dt$

Angular momentum vector $\mathbf{L}_j$ of a mass m_j is its linear momentum $\mathbf{p}_j$ times its lever arm as given by the *angular momentum cross-product relation* $\mathbf{L}_j=\mathbf{r}_j \times m_j \dot{\mathbf{r}}_j \equiv \mathbf{r}_j \times \mathbf{p}_j$

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The sum-total angular momentum is $\mathbf{L}=\mathbf{L}^{total}=\sum_{j=1}^{3} \mathbf{L}_{\mathrm{j}}=\sum_{j=1}^{3} \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j$

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The sum-total angular momentum is $\mathbf{L}=\mathbf{L}^{total}=\sum_{j=1}^{3}\mathbf{L}_j=\sum_{j=1}^{3}\mathbf{r}_j\times m_j\dot{\mathbf{r}}_j$

d**L** /dt gives a rotor Newton equation relating rotor momentum **r**X**p** to rotor force or *torque* **r**X**F**.

$$\frac{d\mathbf{L}}{dt}=\sum_{j=1}^{3}\mathbf{r}_j\times m_j\ddot{\mathbf{r}}_j=\sum_{j=1}^{3}\mathbf{r}_j\times\mathbf{F}_j^{total}$$

$$=\sum_{j=1}^{3}\mathbf{r}_j\times\mathbf{F}_j^{applied}+\sum_{j=1}^{3}\mathbf{r}_j\times\left(\sum_{k=1(k\neq j)}^{3}\mathbf{F}_{jk}^{constraint}\right)$$

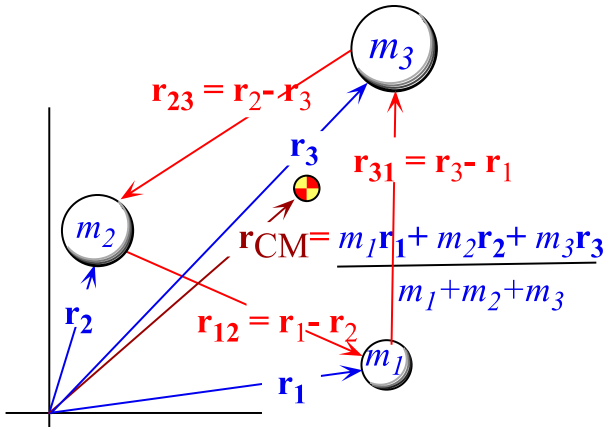

Fig. 6.4.1 Three-particle coordinate vectors

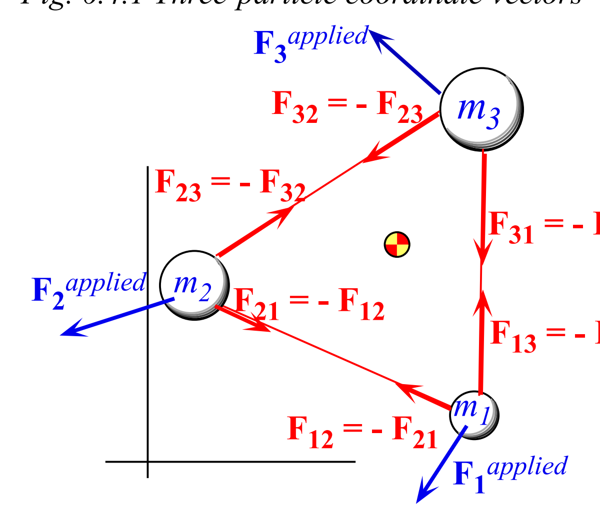

Fig. 6.4.2 Three-particle force vectors

Rotational equivalent of Newton's $\mathbf{F}=d\mathbf{p}/dt$ *equations:* $\mathbf{N}=d\mathbf{L}/dt$

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The sum-total angular momentum is $$\mathbf{L} = \mathbf{L}^{total} = \sum_{j=1}^{3} \mathbf{L}_j = \sum_{j=1}^{3} \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j$$

$d\mathbf{L}/dt$ gives a rotor Newton equation relating rotor momentum $\mathbf{r}\times\mathbf{p}$ to rotor force or *torque* $\mathbf{r}\times\mathbf{F}$.

$$\frac{d\mathbf{L}}{dt} = \sum_{j=1}^{3} \mathbf{r}_j \times m_j \ddot{\mathbf{r}}_j = \sum_{j=1}^{3} \mathbf{r}_j \times \mathbf{F}_j^{total}$$

$$= \sum_{j=1}^{3} \mathbf{r}_j \times \mathbf{F}_j^{applied} + \sum_{j=1}^{3} \mathbf{r}_j \times \left(\sum_{k=1(k\neq j)}^{3} \mathbf{F}_{jk}^{constraint} \right)$$

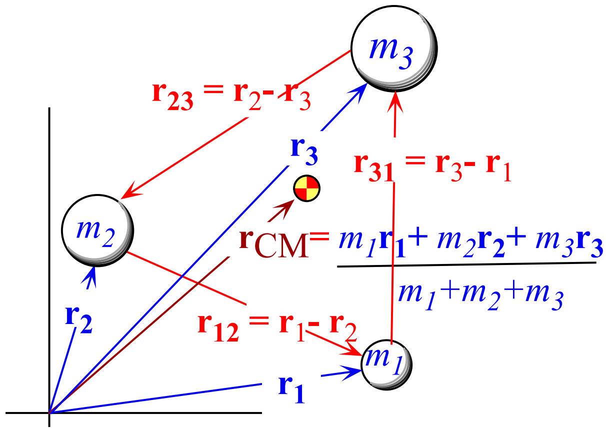
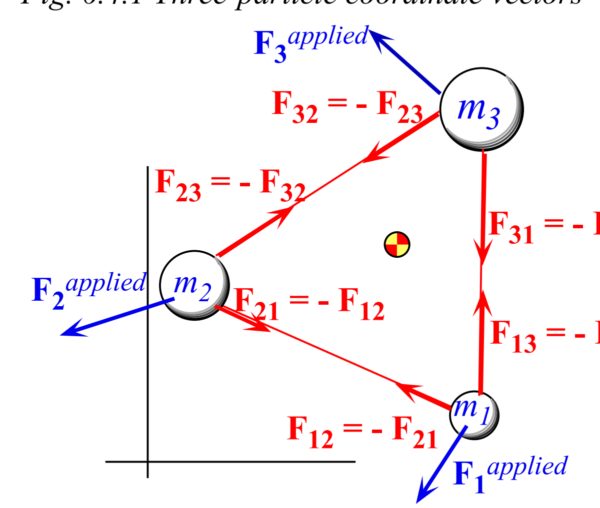


Fig. 6.4.1 Three-particle coordinate vectors

Internal constraint or coupling force terms appear at first to be a nuisance.

$$\sum_{j=1}^{3} \sum_{k=1(k\neq j)}^{3} \mathbf{r}_j \times \mathbf{F}_{jk}^{constraint} = \mathbf{r}_1 \times \left(\mathbf{F}_{12} + \mathbf{F}_{13}^{constraint}\right) + \mathbf{r}_2 \times \left(\mathbf{F}_{21} + \mathbf{F}_{23}^{constraint}\right) + \mathbf{r}_3 \times \left(\mathbf{F}_{31} + \mathbf{F}_{32}^{constraint}\right)$$

$$= \left(\mathbf{r}_1 - \mathbf{r}_2\right) \times \mathbf{F}_{12}^{constraint} + \left(\mathbf{r}_1 - \mathbf{r}_3\right) \times \mathbf{F}_{13}^{constraint} + \left(\mathbf{r}_2 - \mathbf{r}_3\right) \times \mathbf{F}_{23}^{constraint} = \mathbf{0}$$



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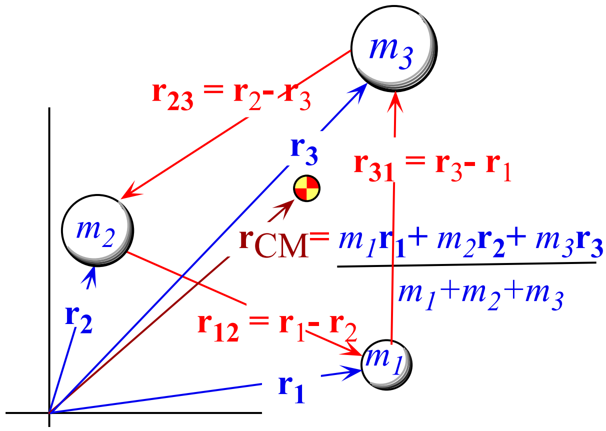

Fig. 6.4.1 Three-particle coordinate vectors

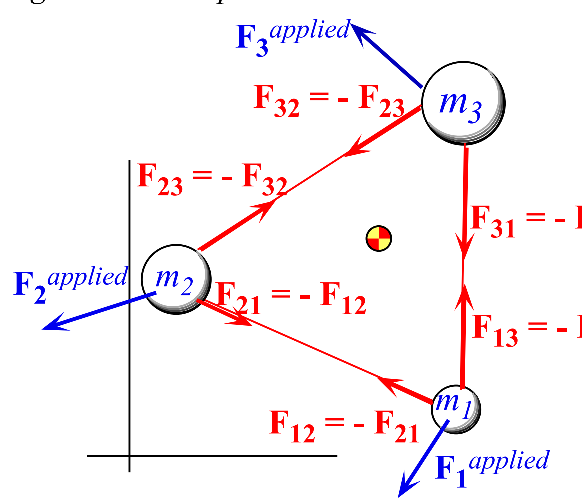

Fig. 6.4.2 Three-particle force vectors

Rotational equivalent of Newton's $\mathbf{F}=d\mathbf{p}/dt$ *equations:* $\mathbf{N}=d\mathbf{L}/dt$

Angular momentum vector $\mathbf{L}_j$ of a mass m_j is its linear momentum $\mathbf{p}_j$ times its lever arm as given by the *angular momentum cross-product relation* $\mathbf{L}_j=\mathbf{r}_j\times m_j\dot{\mathbf{r}}_j\equiv\mathbf{r}_j\times\mathbf{p}_j$

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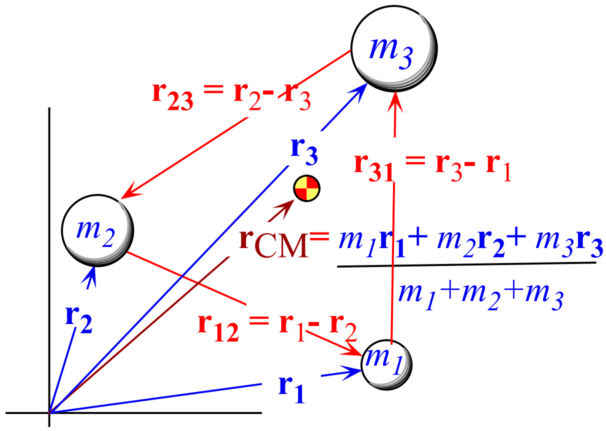

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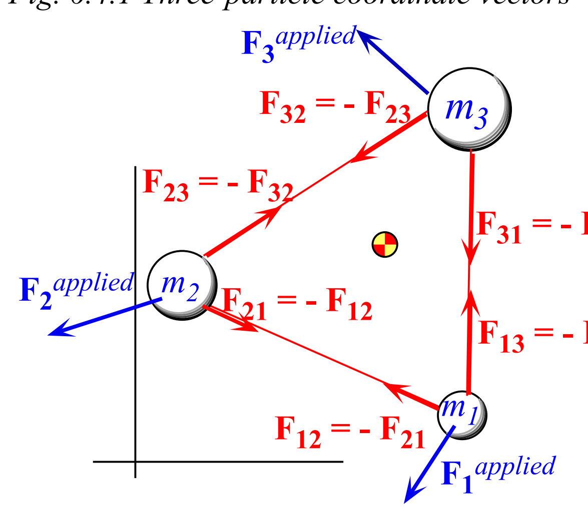

Fig. 6.4.2 Three-particle force vectors

The results are the *rotational Newton's equation*.

$$\frac{d\mathbf{L}}{dt}=\mathbf{N}\ ,\ \text{where: }\mathbf{N}=\sum_{j=1}^{3}\mathbf{N}_j\quad\text{and:}\quad\mathbf{N}_j=\sum_{j=1}^{3}\mathbf{r}_j\times\mathbf{F}_j^{applied}$$

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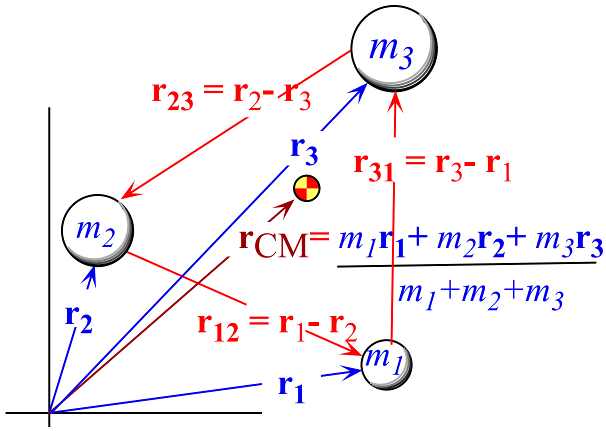

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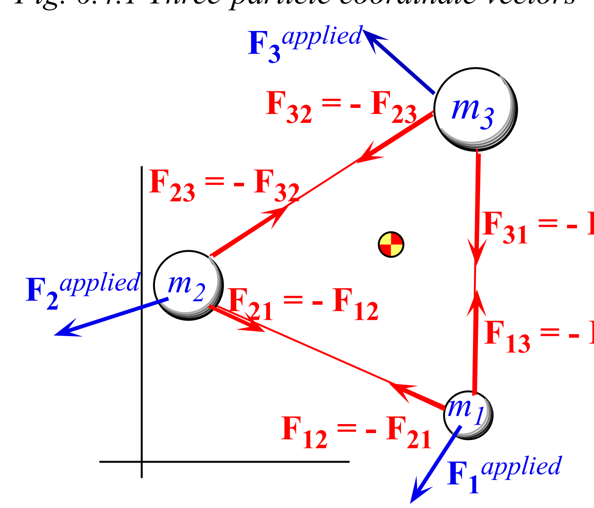

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Taken together with *translational Newton's equation* the six equations describe rigid body mechanics.

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Taken together with *translational Newton's equation* the six equations describe rigid body mechanics.

$$\frac{d\mathbf{P}}{dt}=\mathbf{F}\ ,\ \text{where: }\mathbf{F}=\sum_{j=1}^{3}\mathbf{F}_j^{applied}$$

Remaining 3N-6 equations consist of normal mode or GCC equations of some kind.

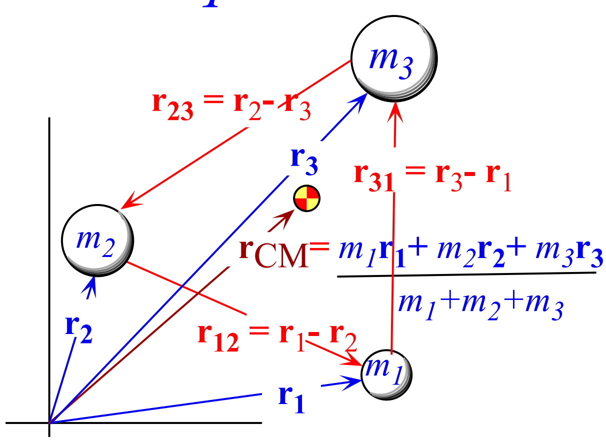


Fig. 6.4.1 Three-particle coordinate vectors

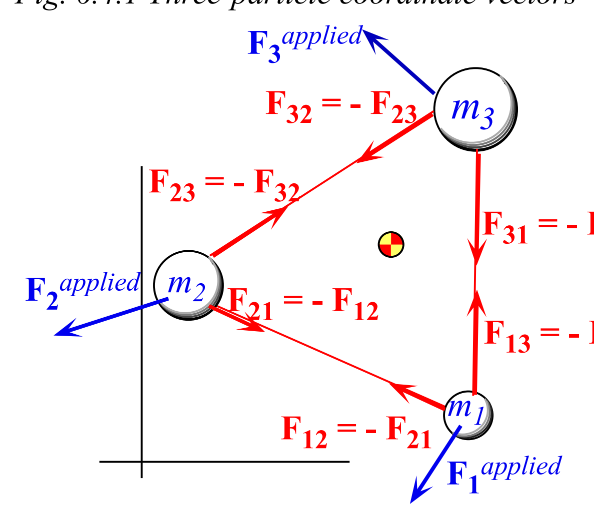


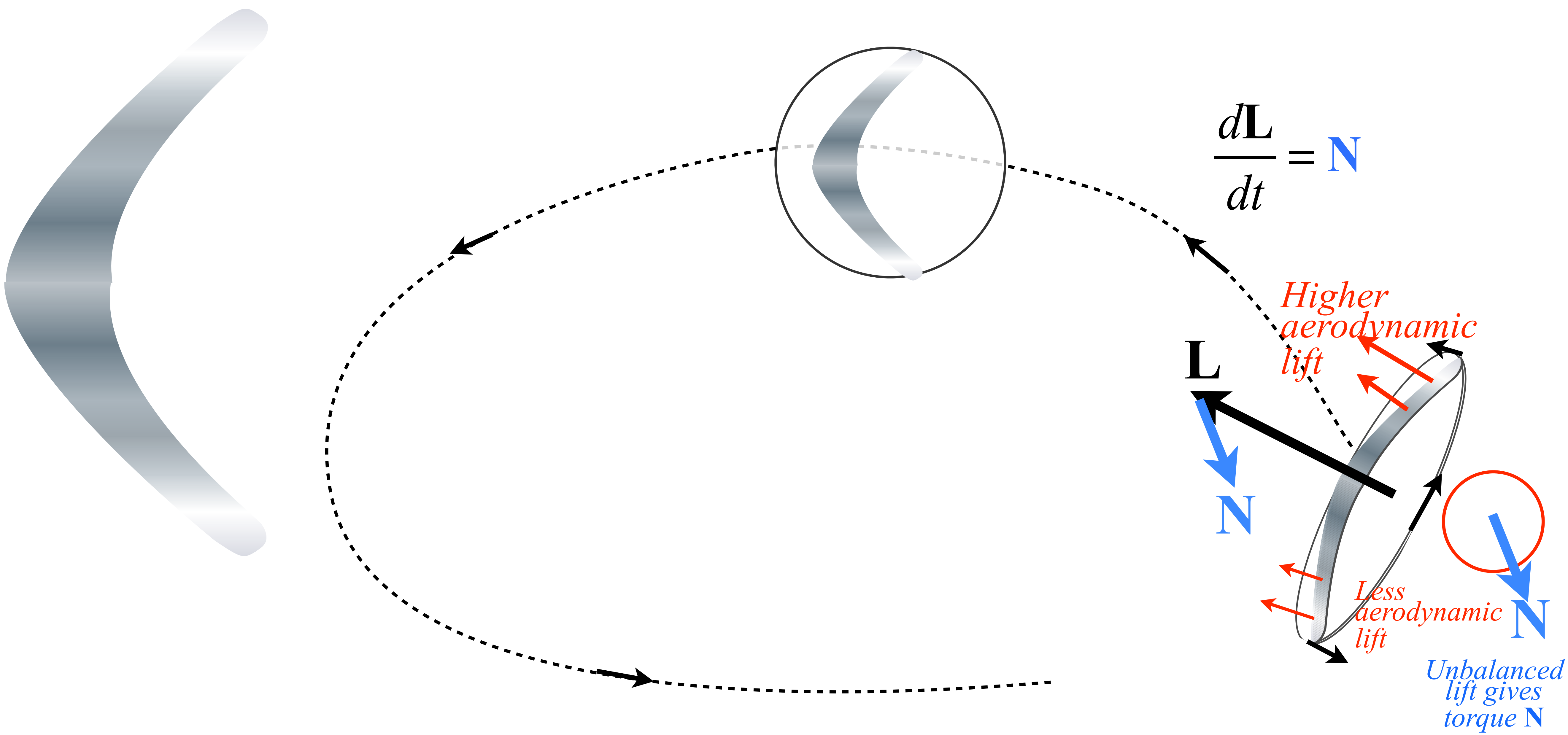
Fig. 6.4.2 Three-particle force vectors

Rotational equivalent of Newton's $\mathbf{F}=d\mathbf{p}/dt$ *equations:* $\mathbf{N}=d\mathbf{L}/dt$

How to make my boomerang come back

The gyrocompass and mechanical spin analogy

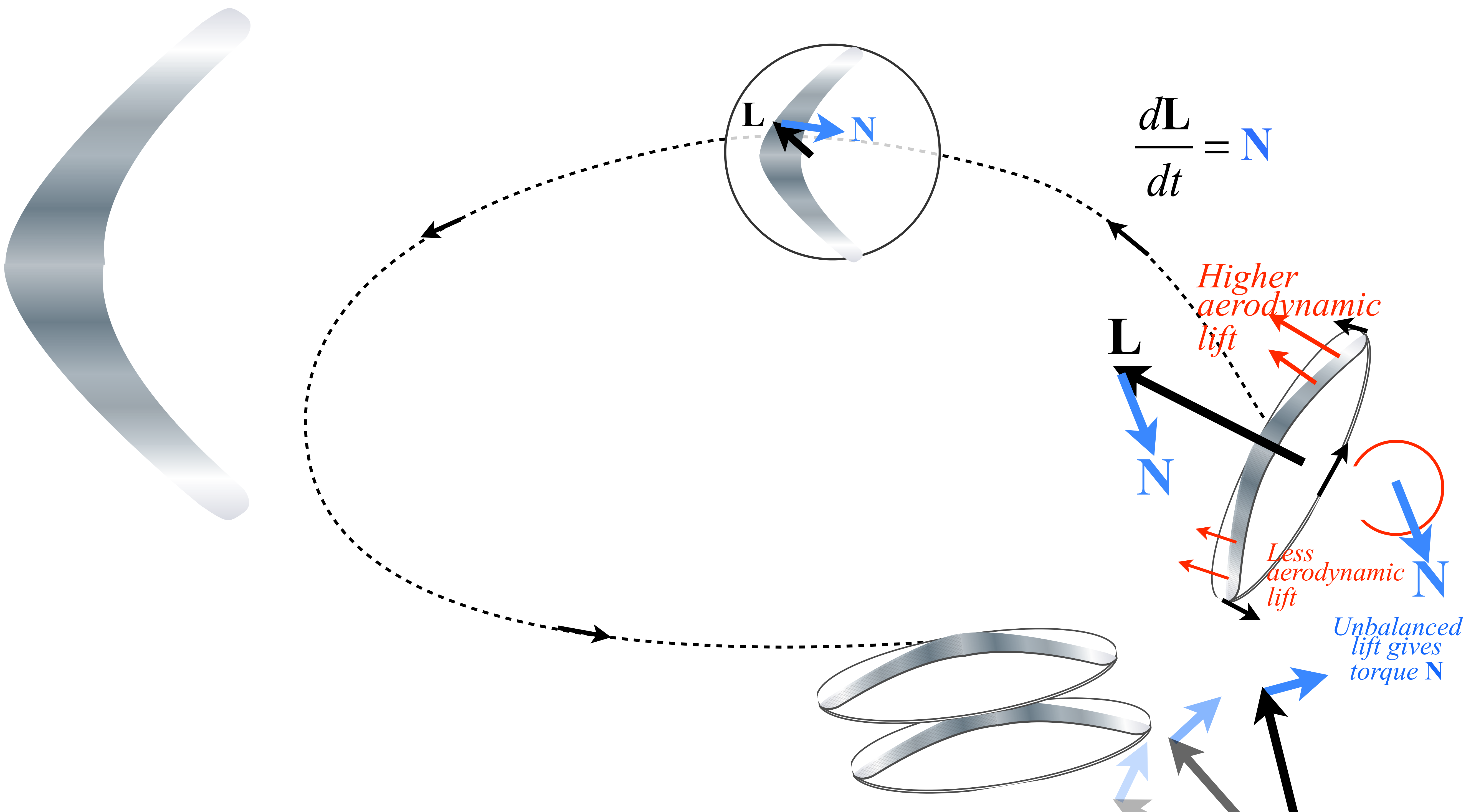
The Australian Boomerang (that comes back!)

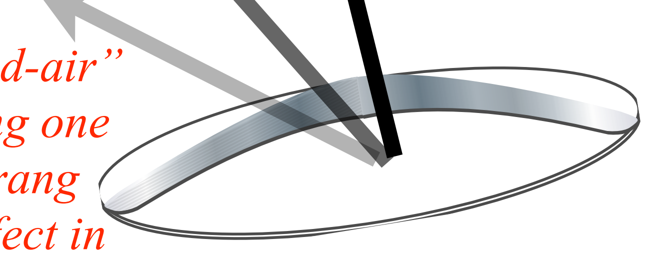






The Australian Boomerang (that comes back and hovers down!)





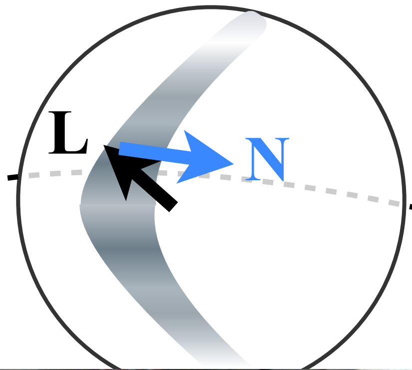
The Australian Boomerang (that comes back and hovers down!)

Charlie Drake's famous 1961 song:

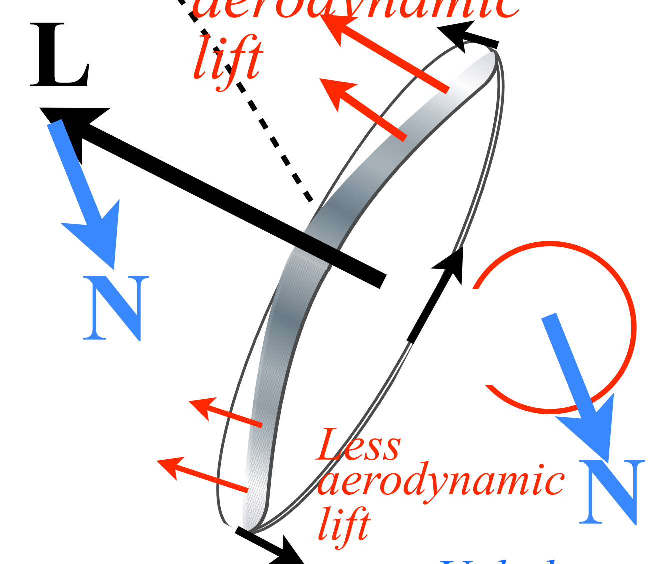
My boomerang won't come back!

My boomerang won't come back!
My boomerang won't come back!
I've waved the thing all over the place
Practiced til' I was blue in the face*
I'm a big disgrace
to the Aborigine Race
My boomerang won't come back!

**blue later replaced black*



$$\frac{d\mathbf{L}}{dt} = \mathbf{N}$$

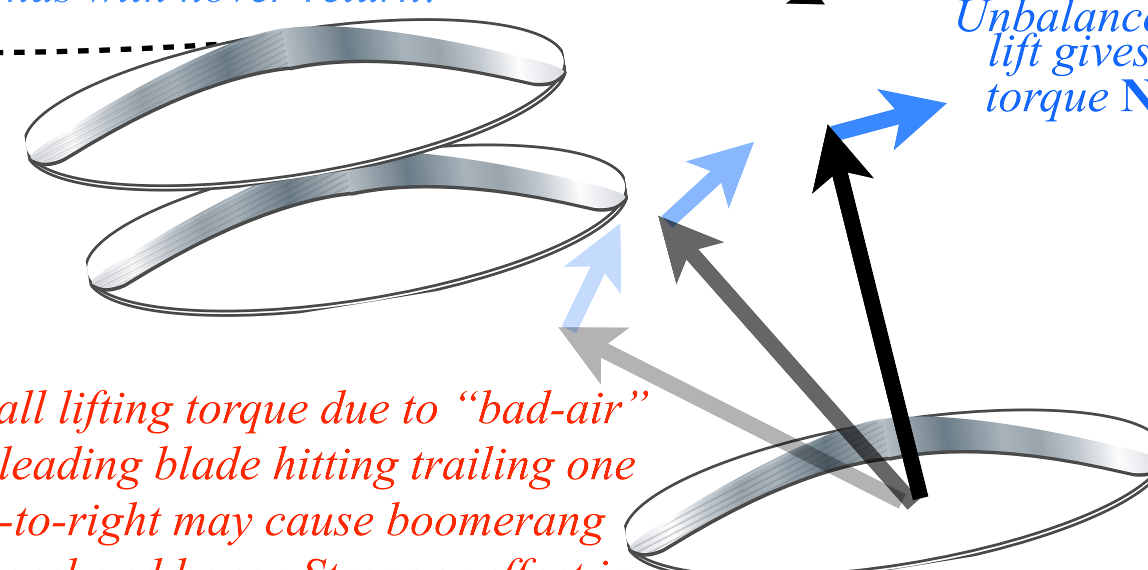


Aluminum boomerang I made in 1965.
It once flew over 18 seconds with hover-return!

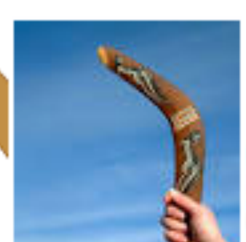
Song at end of Lect.28

p.90

Unbalanced lift gives torque **N**



https://www.youtube.com/playlist?list=PLGwmGldCxzLxbPlFVG8Z89WZIBuT4m0Ii



Small lifting torque due to "bad-air" of leading blade hitting trailing one left-to-right may cause boomerang to level and hover. Stronger effect in 3-blade boomers causes figure-8 paths.

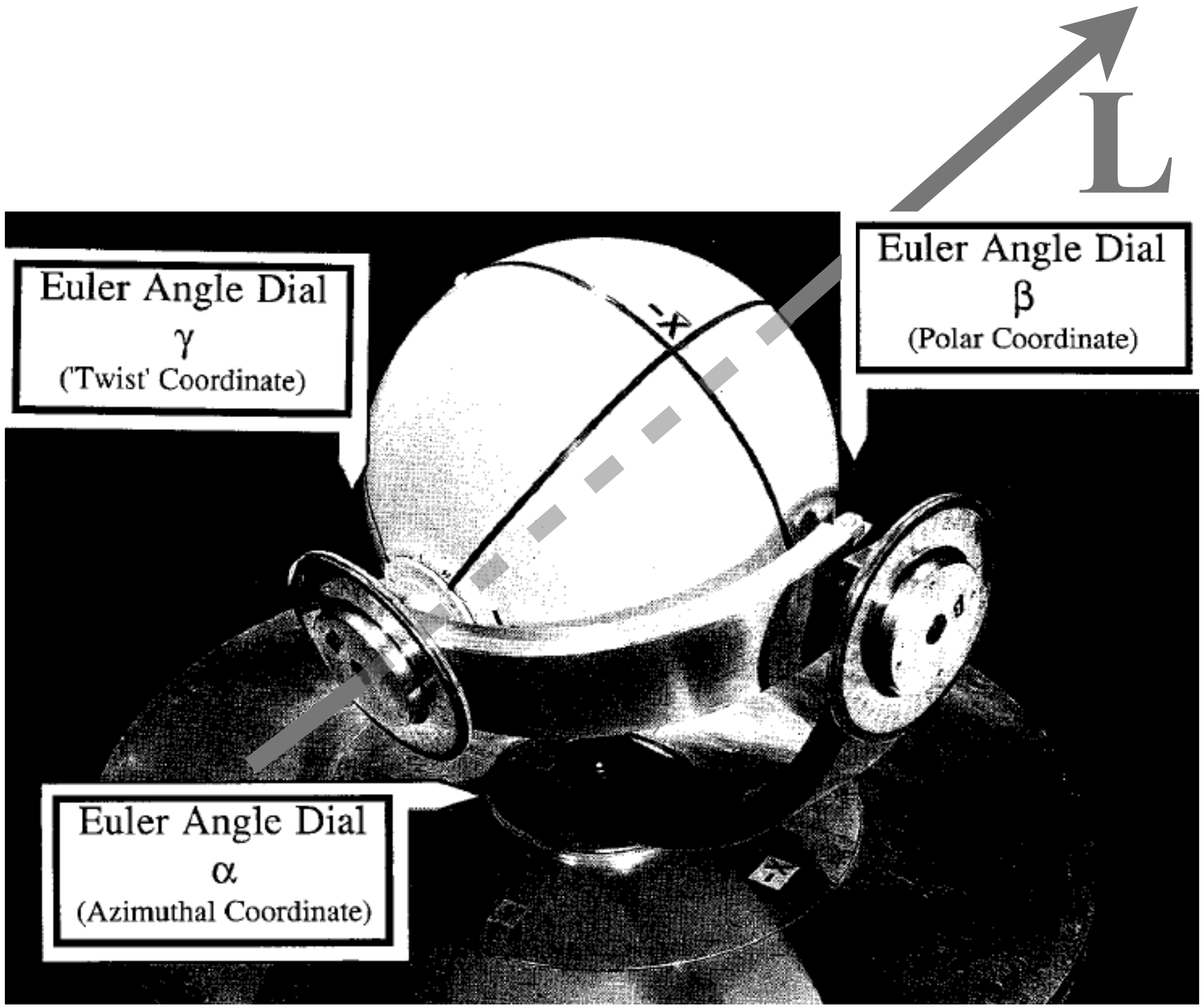
Rotational equivalent of Newton's $\mathbf{F}=d\mathbf{p}/dt$ *equations:* $\mathbf{N}=d\mathbf{L}/dt$

How to make my boomerang come back

The gyrocompass and mechanical spin analogy

The gyrocompass and mechanical spin analogy

Suppose Euler ball has right-hand rotation with angular momentum **L**



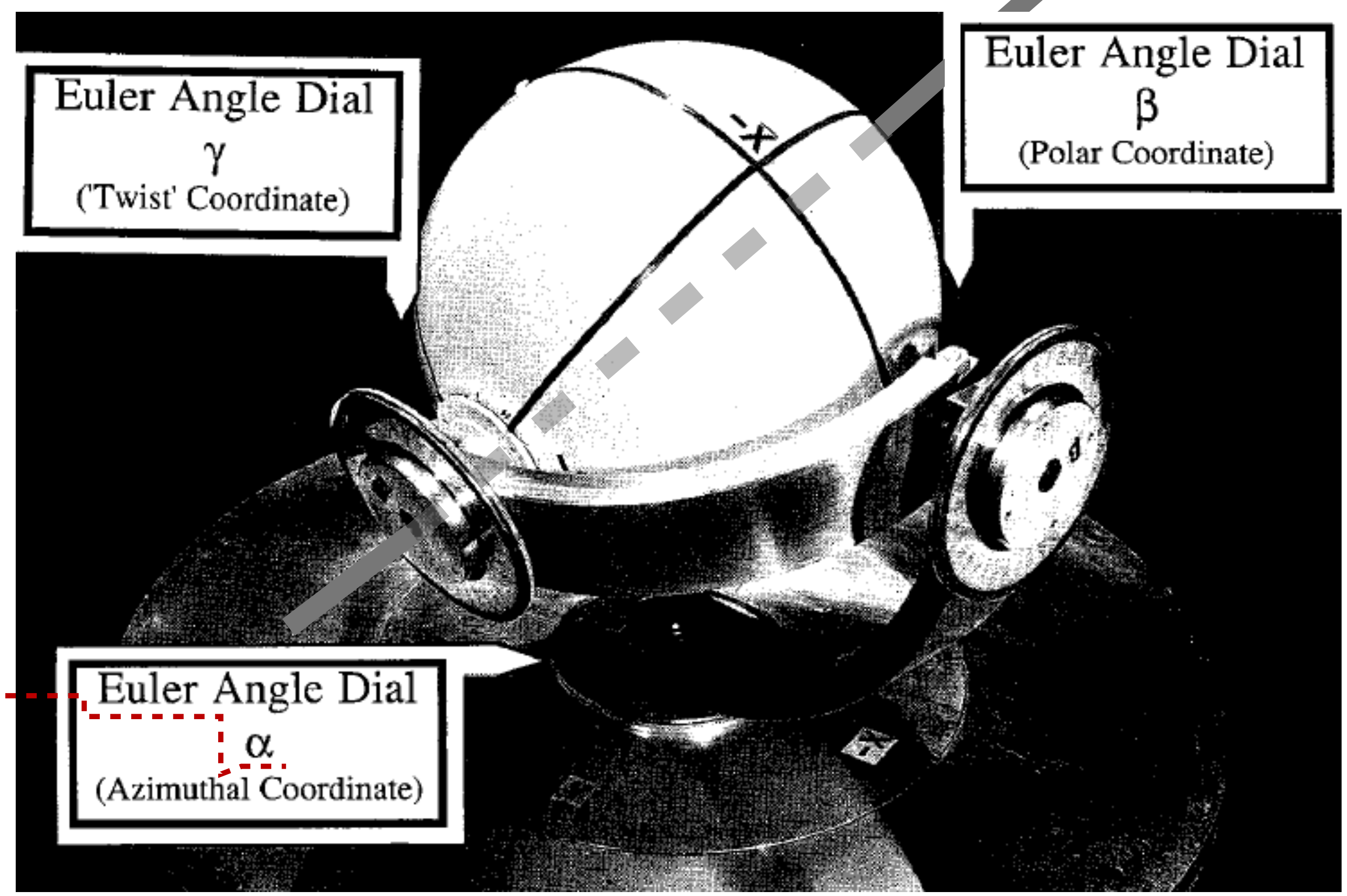
The gyrocompass and mechanical spin analogy

Suppose Euler ball has right-hand rotation with angular momentum **L**

If the α-dial for z-rotation is turning left-to-right this applies righthand "thumbs-up" torque **N**

N

L



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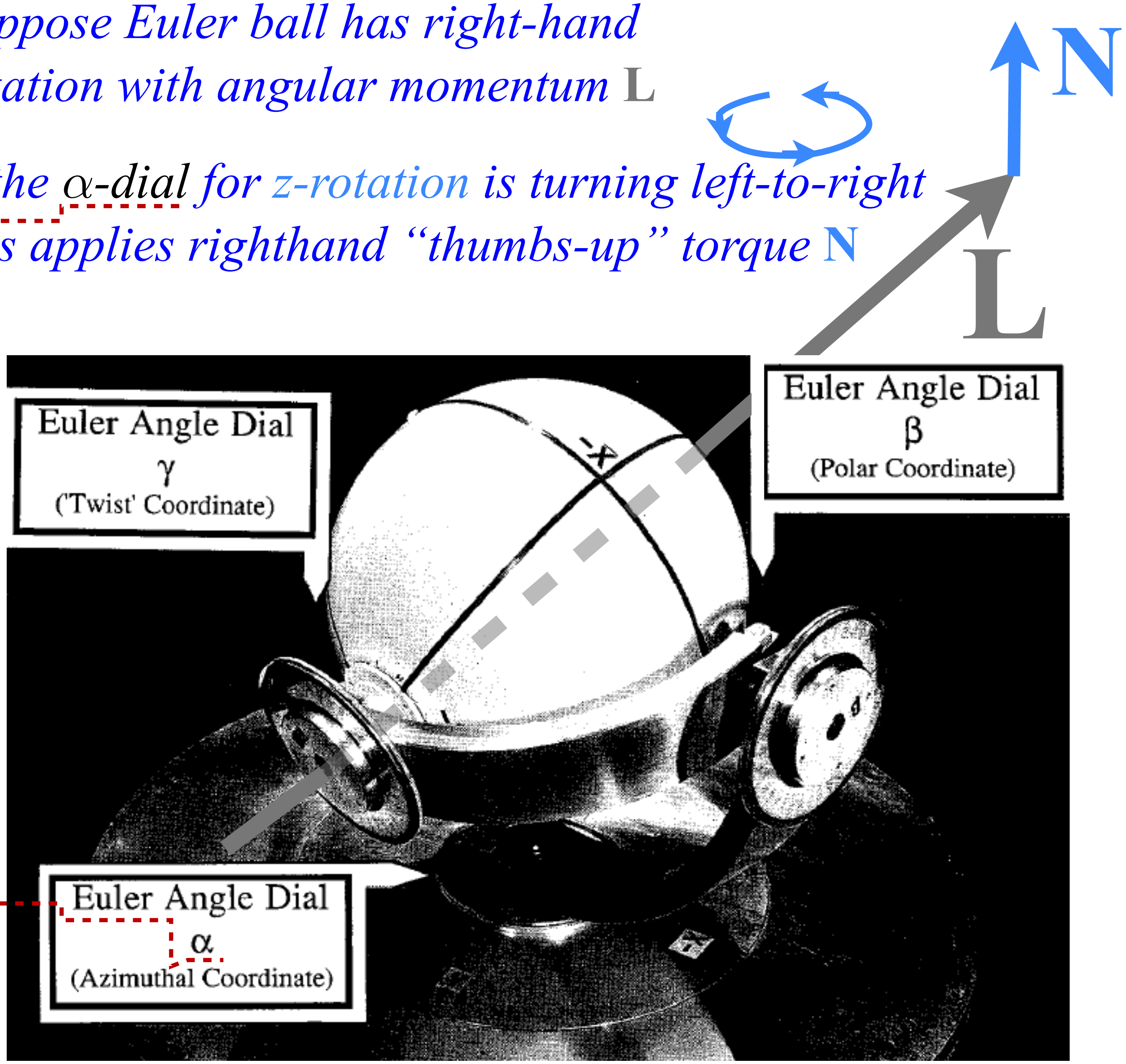

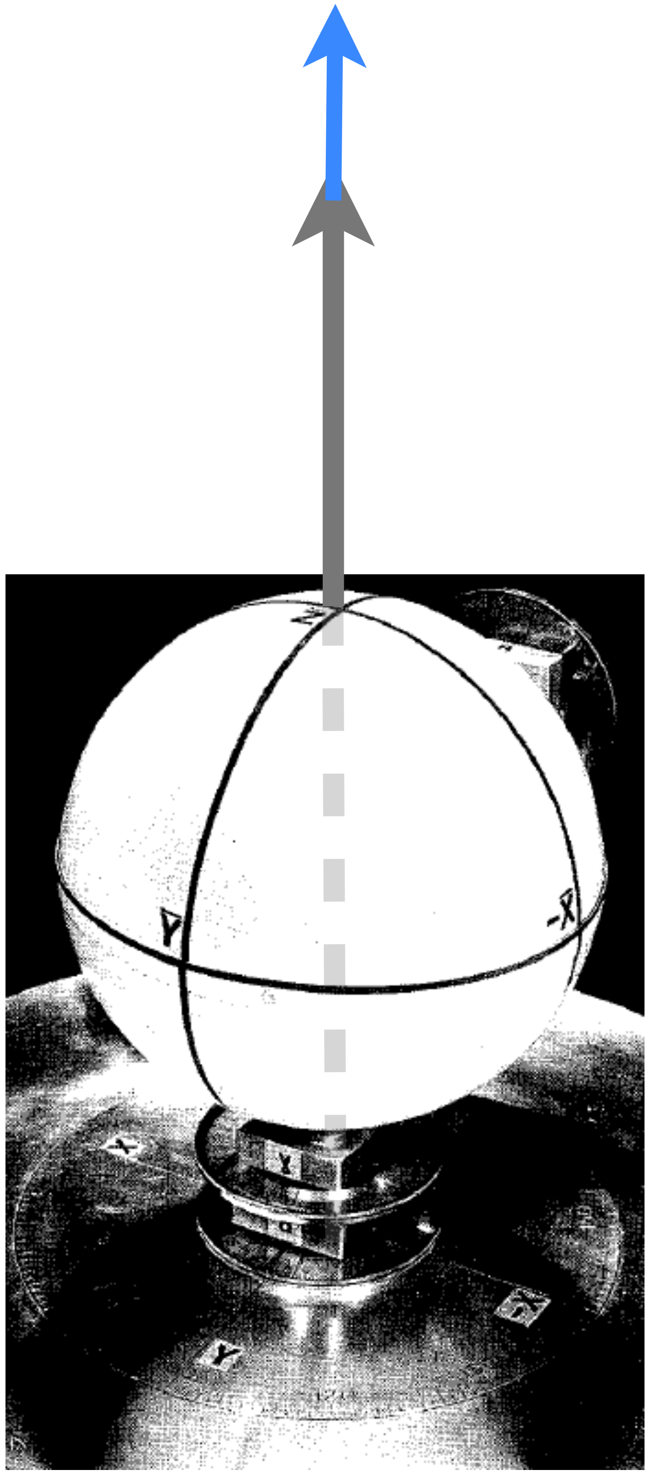

Then the ball tends to line-up with z-axis (and may go past z, then come back, etc. in a precessional or "hunting" motion)

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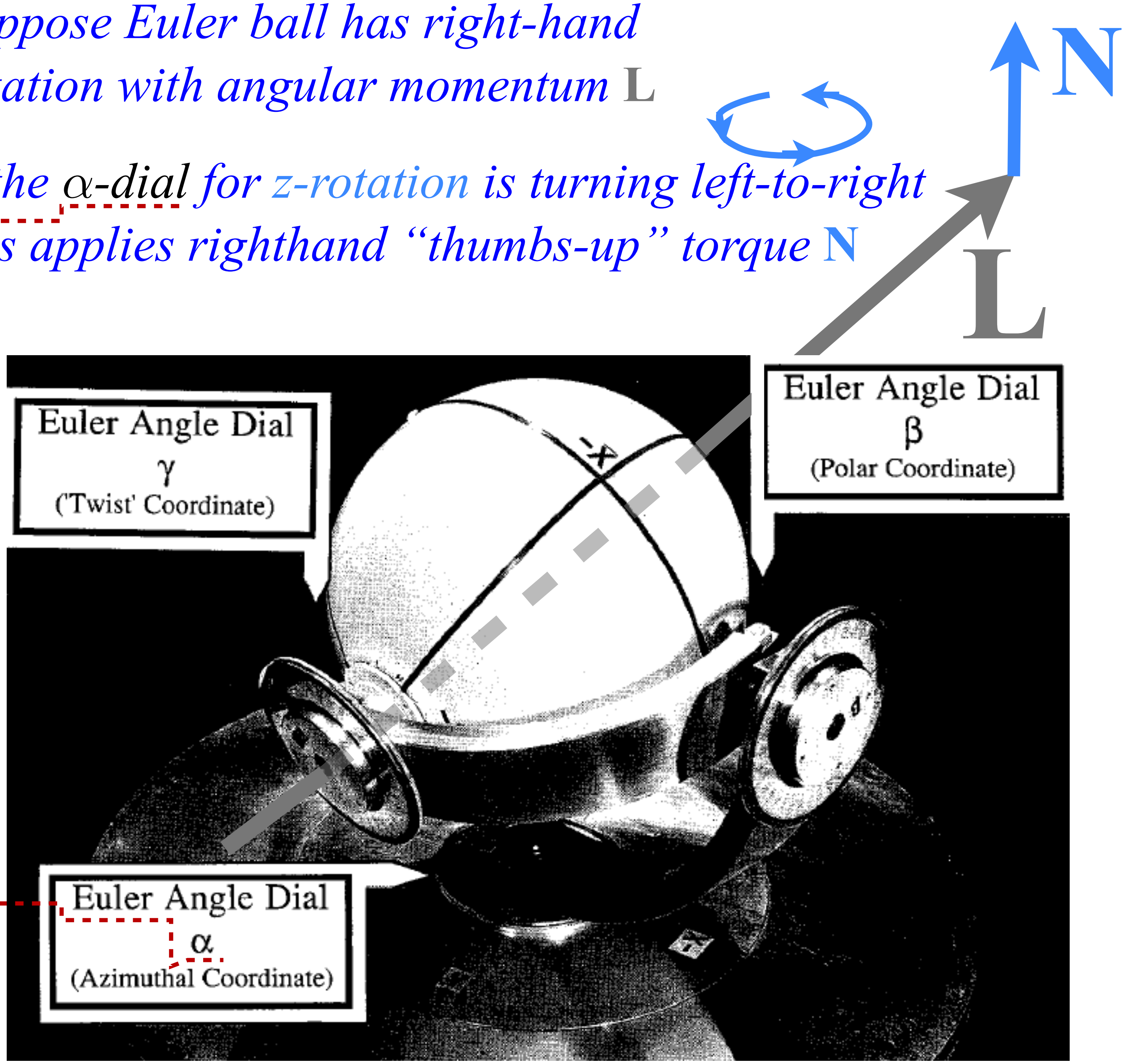
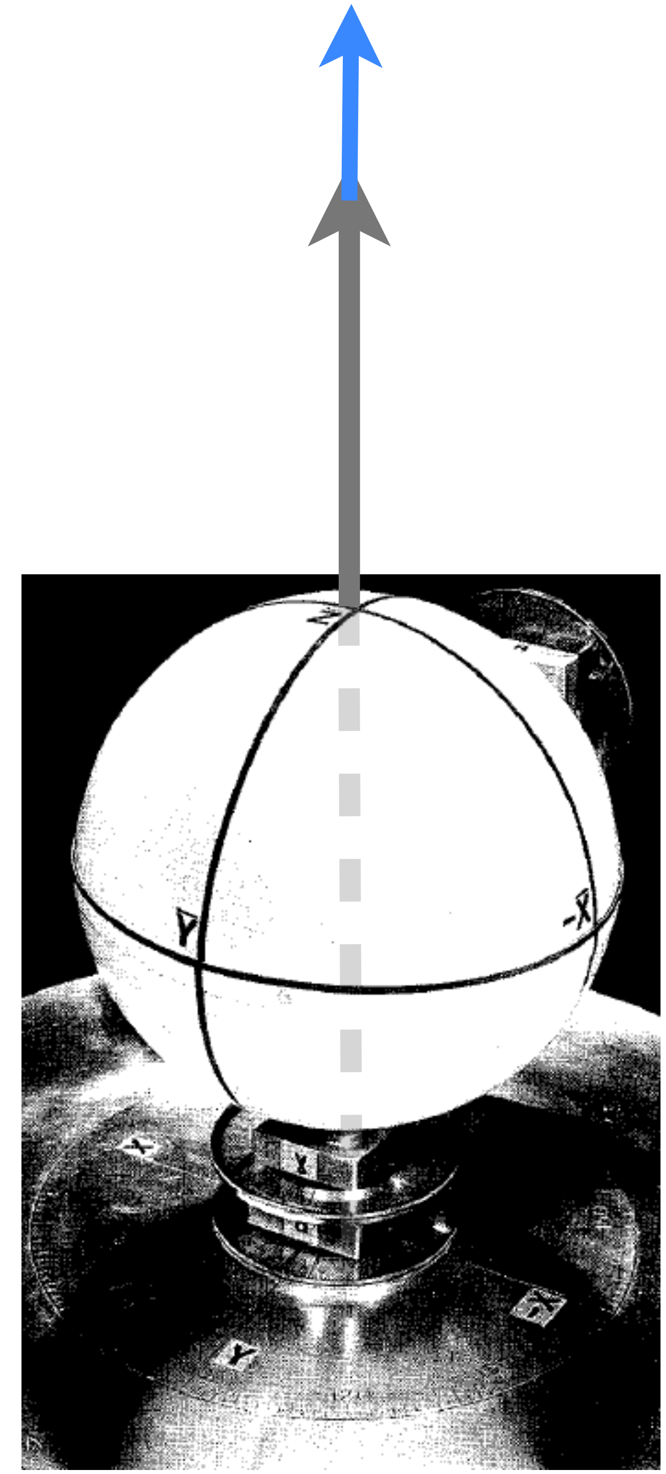


A very high speed ball in a gyro-compass will similarly seek true North due to Earth rotation.



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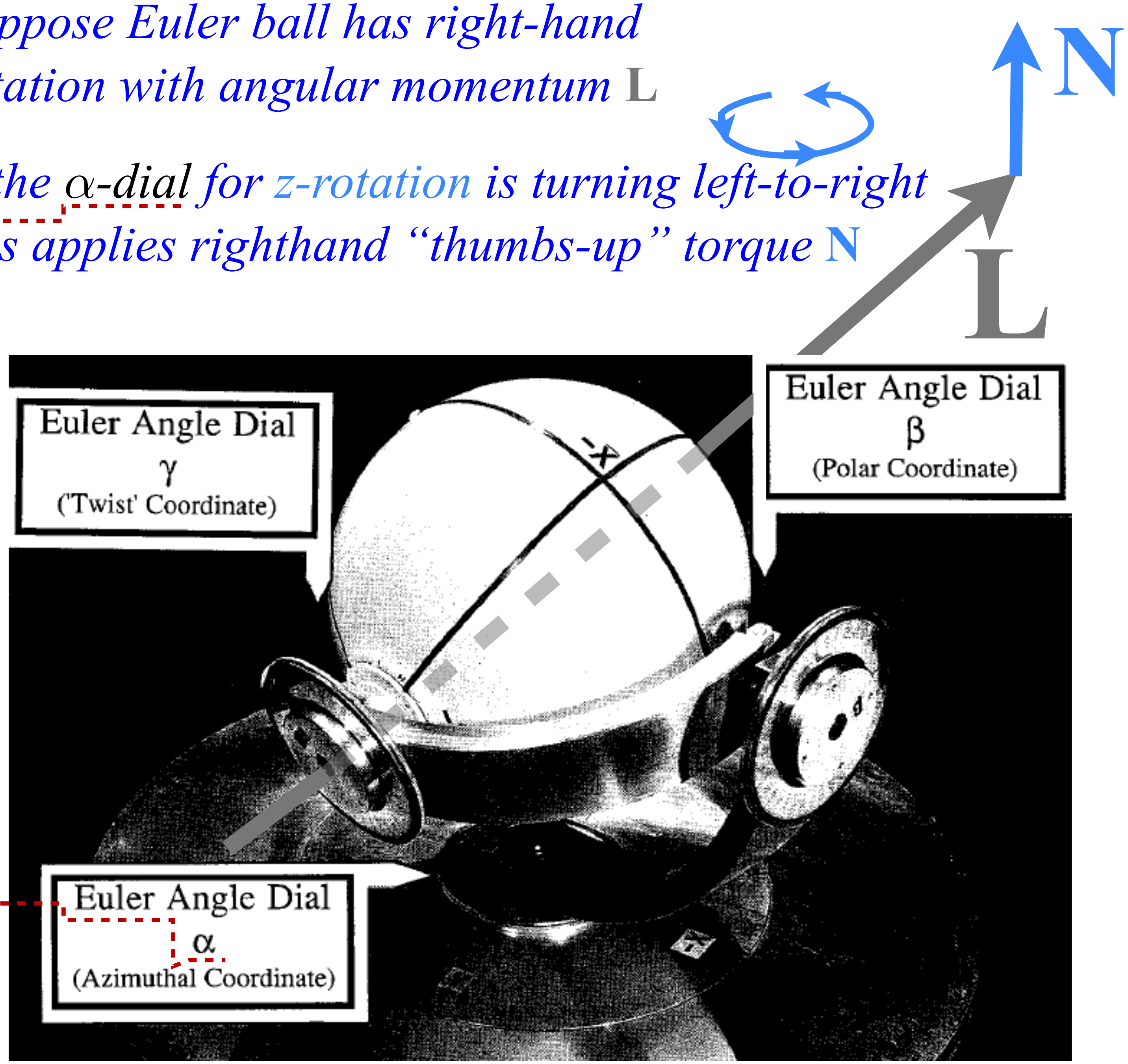

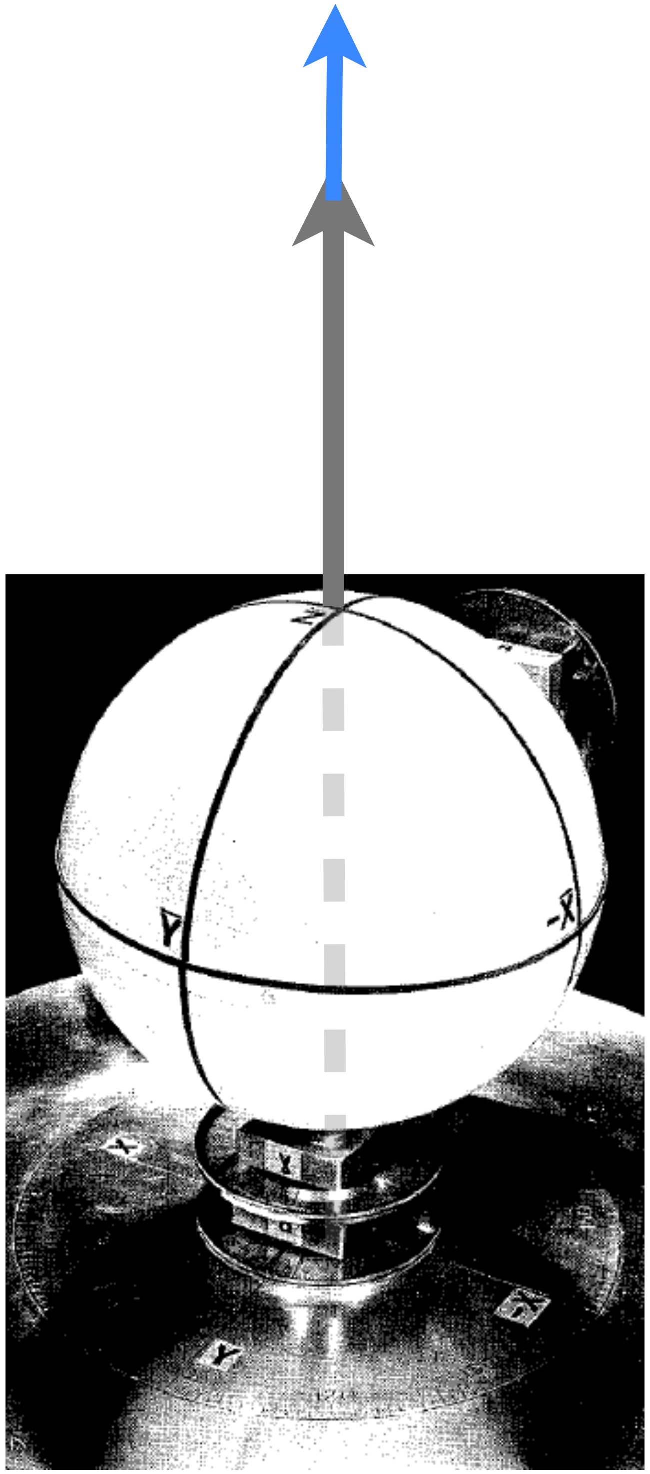

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Then the ball tends to line-up with z-axis (and may go past z, then come back, etc. in a precessional or "hunting" motion)

This is analogous to the tendency for spin magnetic moments to align (or precess about) the B-direction of a magnetic field

Recall **S**-precession discussion in CMwB Unit 4 Ch.4 and Lect.26

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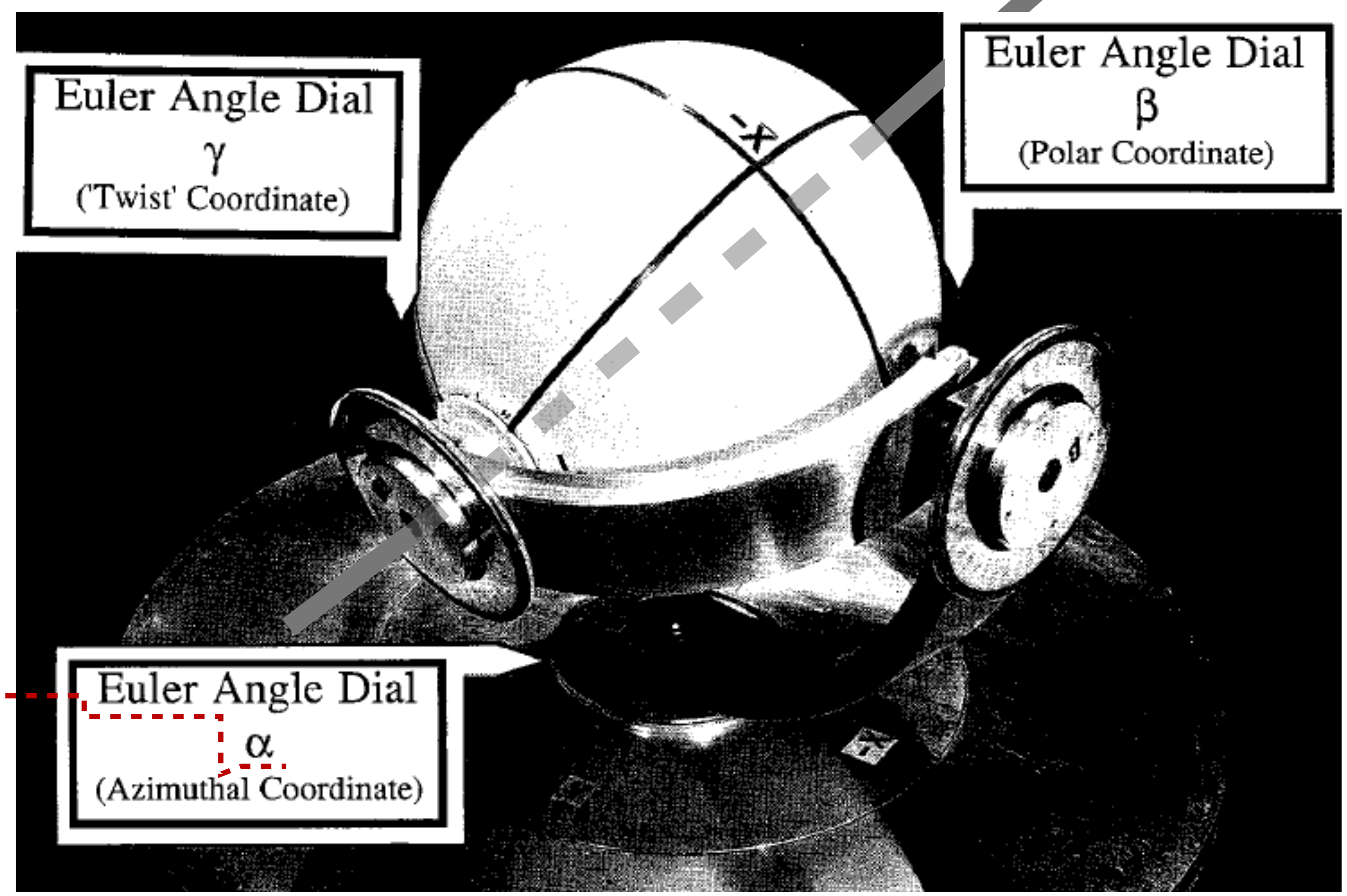

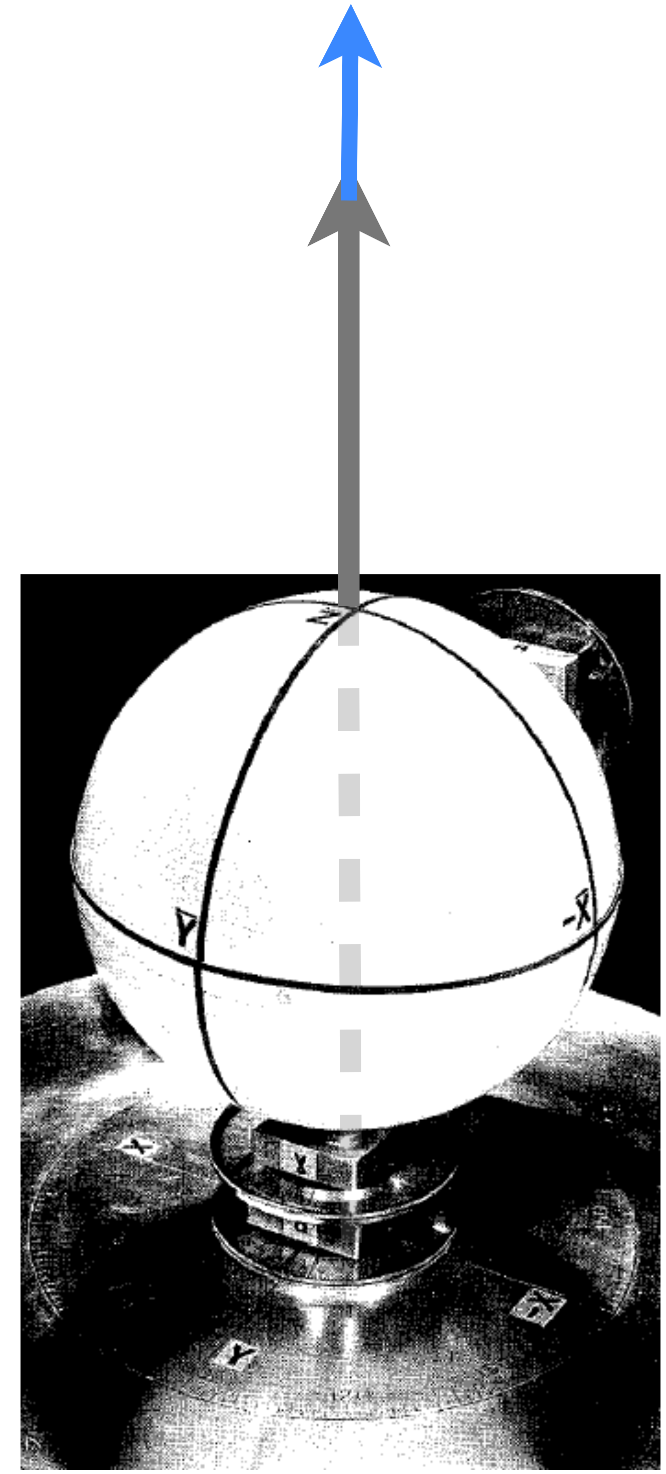

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General Rule: Gyros tend to "line-up" so they are rotating with whatever is most closely coupled to them.

This is analogous to the tendency for spin magnetic moments to align (or precess about) the B-direction of a magnetic field

Recall **S***-precession discussion in CMwB Unit 4 Ch.4 and Lect.26*

Rotational momentum and velocity tensor relations

Quadratic form geometry and duality (again)

angular velocity $\boldsymbol{\omega}$*-ellipsoid vs. angular momentum* $\mathbf{L}$*-ellipsoid*

Lagrangian $\boldsymbol{\omega}$*-equations vs. Hamiltonian momentum* $\mathbf{L}$*-equation*

Inertia tensors

Consider N-body *angular velocity* $\boldsymbol{\omega}$ and *angular momentum* $\mathbf{L}$ relations with Levi-Civita analysis

$$\dot{\mathbf{r}}_j = \boldsymbol{\omega} \times \mathbf{r}_j \qquad \text{and} \qquad \mathbf{L} = \sum_{j=1}^{N} \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j = \sum_{j=1}^{N} m_j \mathbf{r}_j \times \left(\boldsymbol{\omega} \times \mathbf{r}_j \right) \qquad \text{with} \qquad \mathbf{A} \times \left(\mathbf{B} \times \mathbf{C} \right) = \left(\mathbf{A} \bullet \mathbf{C} \right) \mathbf{B} - \left(\mathbf{A} \bullet \mathbf{B} \right) \mathbf{C}$$

Consider mass m instantaneously at $\mathbf{r}_m = (x_m, y_m, z_m) = r(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ on a bent axle rotating in a fixed bearing:

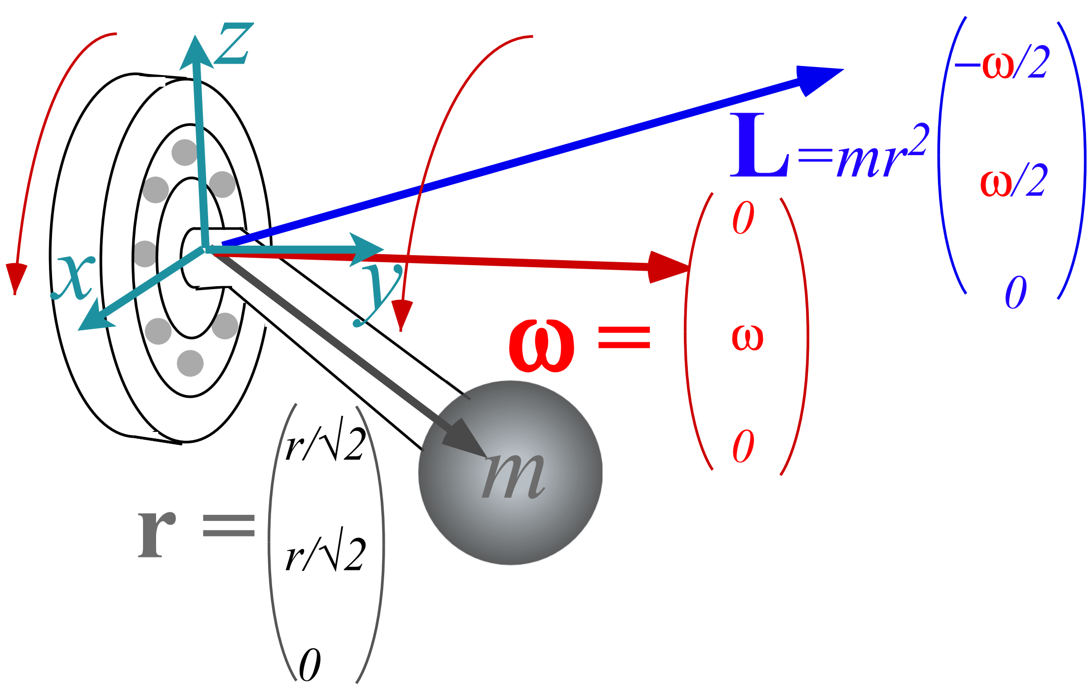

Fig. 6.5.1 Angular momentum for mass rotating on bent axle.

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This produces the *rotational inertia tensor* $\mathbf{I}$:

$$\overleftrightarrow{\mathbf{I}} = \sum_{j=1}^{N} \overleftrightarrow{\mathbf{I}}_j = \sum_{j=1}^{N} m_j \left[\left(\mathbf{r}_j \bullet \mathbf{r}_j\right)\mathbf{1} - \mathbf{r}_j \mathbf{r}_j\right]$$

in the $\boldsymbol{\omega}$-to-$\mathbf{L}$ relation:

$$\mathbf{L} = \sum_{j=1}^{N} m_j \left[\left(\mathbf{r}_j \bullet \mathbf{r}_j\right)\boldsymbol{\omega} - \left(\mathbf{r}_j \bullet \boldsymbol{\omega}\right)\mathbf{r}_j\right] = \sum_{j=1}^{N} m_j \left[\left(\mathbf{r}_j \bullet \mathbf{r}_j\right)\mathbf{1} - \mathbf{r}_j \mathbf{r}_j\right] \bullet \boldsymbol{\omega} = \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega}$$

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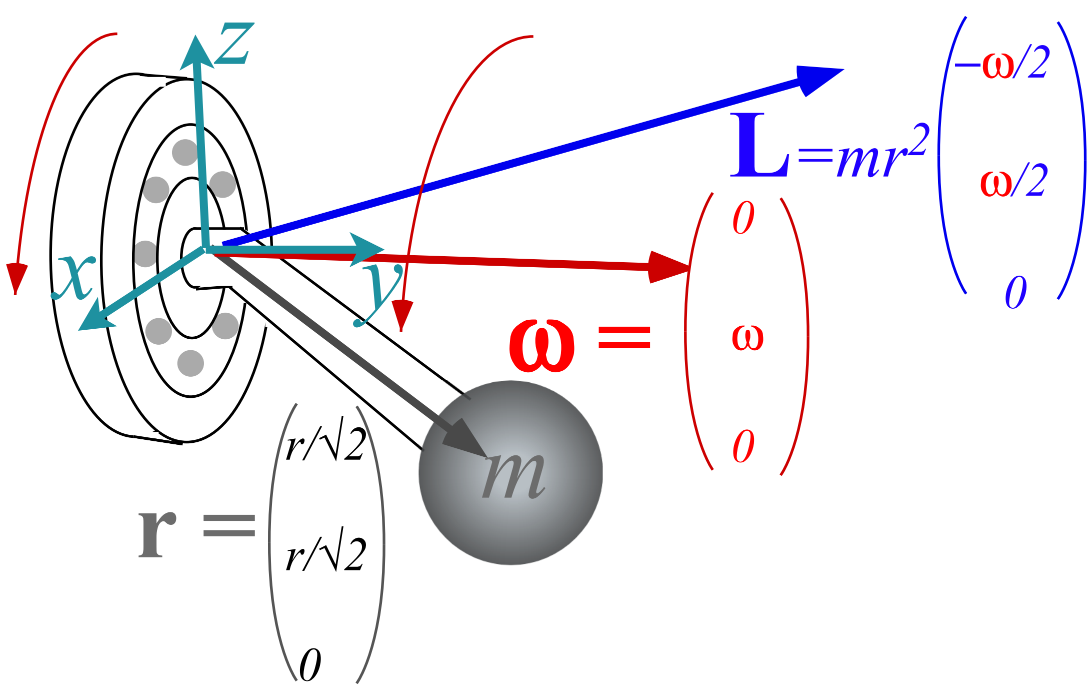

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in the ω-to-$\mathbf{L}$ relation:

$$\mathbf{L} = \sum_{j=1}^{N} m_j \left[(\mathbf{r}_j \bullet \mathbf{r}_j)\boldsymbol{\omega} - (\mathbf{r}_j \bullet \boldsymbol{\omega})\mathbf{r}_j \right] = \sum_{j=1}^{N} m_j \left[(\mathbf{r}_j \bullet \mathbf{r}_j)\mathbf{1} - \mathbf{r}_j\mathbf{r}_j \right] \bullet \boldsymbol{\omega} = \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega}$$

Matrix form of the ω-to-$\mathbf{L}$ relation using the *inertia matrix* $\langle \mathbf{I} \rangle$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \sum_{j=1}^{N} m_j \begin{pmatrix} y_j^2 + z_j^2 & -x_j y_j & -x_j z_j \\ -y_j x_j & x_j^2 + z_j^2 & -y_j z_j \\ -z_j x_j & -z_j y_j & x_j^2 + y_j^2 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$\langle \overleftrightarrow{\mathbf{I}} \rangle = \sum_{j=1}^{N} \langle \overleftrightarrow{\mathbf{I}}_j \rangle = \sum_{j=1}^{N} m_j \begin{pmatrix} y_j^2 + z_j^2 & -x_j y_j & -x_j z_j \\ -y_j x_j & x_j^2 + z_j^2 & -y_j z_j \\ -z_j x_j & -z_j y_j & x_j^2 + y_j^2 \end{pmatrix}$$

Consider mass m instantaneously at $\mathbf{r}_m = (x_m, y_m, z_m) = r(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ on a bent axle rotating in a fixed bearing:

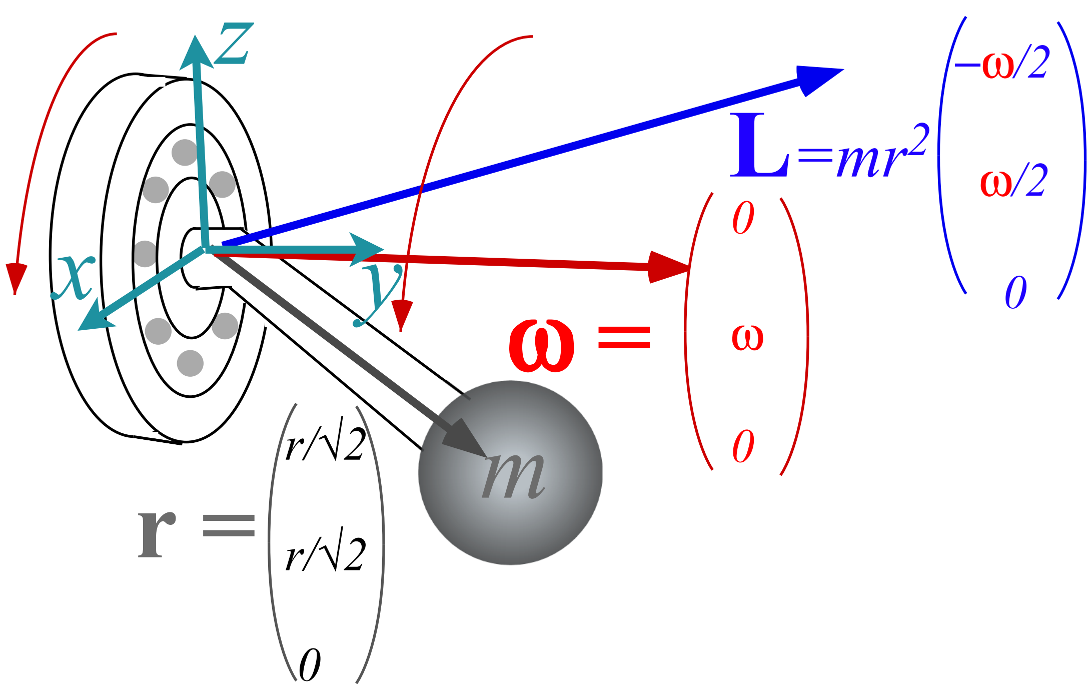

Fig. 6.5.1 Angular momentum for mass rotating on bent axle.

Inertia tensors

Consider N-body *angular velocity* $\boldsymbol{\omega}$ and *angular momentum* $\mathbf{L}$ relations with Levi-Civita analysis

$$\dot{\mathbf{r}}_j = \boldsymbol{\omega}\times\mathbf{r}_j \qquad \text{and} \qquad \mathbf{L} = \sum_{j=1}^{N} \mathbf{r}_j \times m_j\dot{\mathbf{r}}_j = \sum_{j=1}^{N} m_j\mathbf{r}_j \times\left(\boldsymbol{\omega}\times\mathbf{r}_j\right) \qquad \text{with} \qquad \mathbf{A}\times\left(\mathbf{B}\times\mathbf{C}\right)=\left(\mathbf{A}\bullet\mathbf{C}\right)\mathbf{B}-\left(\mathbf{A}\bullet\mathbf{B}\right)\mathbf{C}$$

This produces the *rotational inertia tensor* $\mathbf{I}$:

$$\overleftrightarrow{\mathbf{I}} = \sum_{j=1}^{N}\overleftrightarrow{\mathbf{I}}_j = \sum_{j=1}^{N} m_j\left[\left(\mathbf{r}_j\bullet\mathbf{r}_j\right)\mathbf{1}-\mathbf{r}_j\mathbf{r}_j\right]$$

in the $\boldsymbol{\omega}$-to-$\mathbf{L}$ relation:

$$\mathbf{L} = \sum_{j=1}^{N} m_j\left[\left(\mathbf{r}_j\bullet\mathbf{r}_j\right)\boldsymbol{\omega}-\left(\mathbf{r}_j\bullet\boldsymbol{\omega}\right)\mathbf{r}_j\right] = \sum_{j=1}^{N} m_j\left[\left(\mathbf{r}_j\bullet\mathbf{r}_j\right)\mathbf{1}-\mathbf{r}_j\mathbf{r}_j\right]\bullet\boldsymbol{\omega} = \overleftrightarrow{\mathbf{I}}\bullet\boldsymbol{\omega}$$

Matrix form of the $\boldsymbol{\omega}$-to-$\mathbf{L}$ relation using the *inertia matrix* $\langle\mathbf{I}\rangle$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \sum_{j=1}^{N} m_j \begin{pmatrix} y_j^2+z_j^2 & -x_jy_j & -x_jz_j \\ -y_jx_j & x_j^2+z_j^2 & -y_jz_j \\ -z_jx_j & -z_jy_j & x_j^2+y_j^2 \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \qquad \left\langle\overleftrightarrow{\mathbf{I}}\right\rangle = \sum_{j=1}^{N}\left\langle\overleftrightarrow{\mathbf{I}}_j\right\rangle = \sum_{j=1}^{N} m_j \begin{pmatrix} y_j^2+z_j^2 & -x_jy_j & -x_jz_j \\ -y_jx_j & x_j^2+z_j^2 & -y_jz_j \\ -z_jx_j & -z_jy_j & x_j^2+y_j^2 \end{pmatrix}$$

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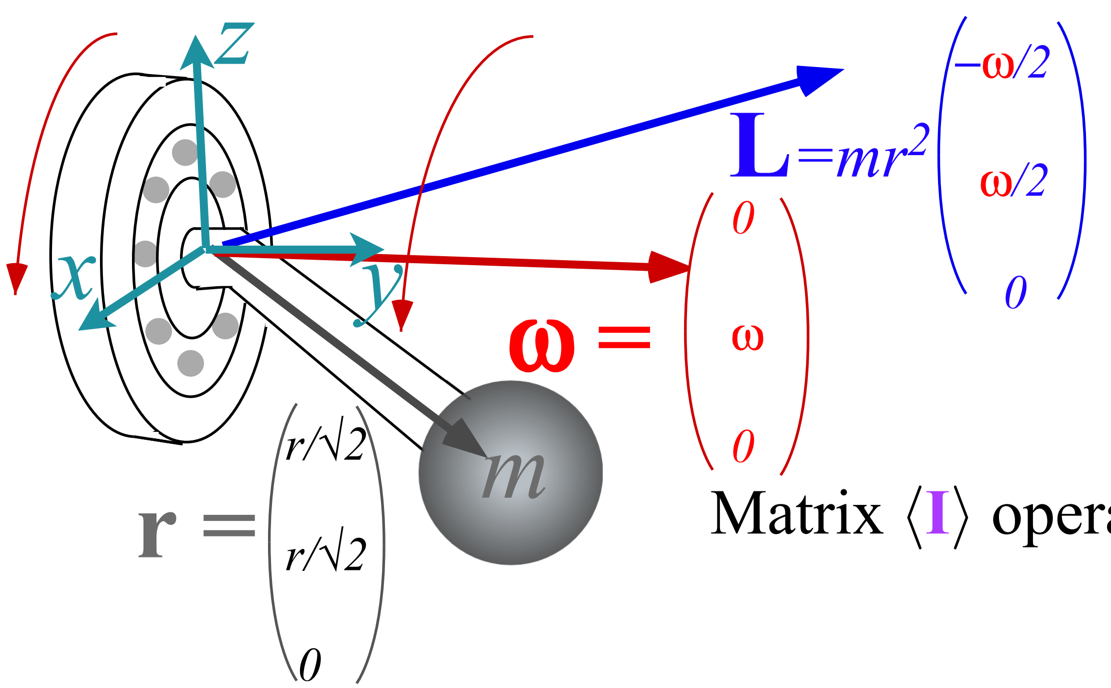

Instantaneous matrix $\langle\mathbf{I}\rangle$ of inertia is:

$$\left\langle\overleftrightarrow{\mathbf{I}}\right\rangle = mr^2\begin{pmatrix} \left(1/\sqrt{2}\right)^2+0 & -\left(1/\sqrt{2}\right)\left(1/\sqrt{2}\right) & -\left(1/\sqrt{2}\right)0 \\ -\left(1/\sqrt{2}\right)\left(1/\sqrt{2}\right) & \left(1/\sqrt{2}\right)^2+0 & -\left(1/\sqrt{2}\right)0 \\ -0\left(1/\sqrt{2}\right) & -0\left(1/\sqrt{2}\right) & \left(1/\sqrt{2}\right)^2+\left(1/\sqrt{2}\right)^2 \end{pmatrix} = mr^2\begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrix $\langle\mathbf{I}\rangle$ operates on angular velocity $\boldsymbol{\omega}$ to give angular momentum $\mathbf{L}$

Fig. 6.5.1 Angular momentum for mass rotating on bent axle.

Inertia tensors

Consider N-body *angular velocity* $\boldsymbol{\omega}$ and *angular momentum* $\mathbf{L}$ relations with Levi-Civita analysis

$$\dot{\mathbf{r}}_j = \boldsymbol{\omega}\times\mathbf{r}_j \qquad \text{and} \qquad \mathbf{L} = \sum_{j=1}^{N} \mathbf{r}_j \times m_j\dot{\mathbf{r}}_j = \sum_{j=1}^{N} m_j\mathbf{r}_j\times\left(\boldsymbol{\omega}\times\mathbf{r}_j\right) \qquad \text{with} \qquad \mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(\mathbf{A}\bullet\mathbf{C})\mathbf{B}-(\mathbf{A}\bullet\mathbf{B})\mathbf{C}$$

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$$\mathbf{L} = \sum_{j=1}^{N} m_j\left[\left(\mathbf{r}_j\bullet\mathbf{r}_j\right)\boldsymbol{\omega}-\left(\mathbf{r}_j\bullet\boldsymbol{\omega}\right)\mathbf{r}_j\right] = \sum_{j=1}^{N} m_j\left[\left(\mathbf{r}_j\bullet\mathbf{r}_j\right)\mathbf{1}-\mathbf{r}_j\mathbf{r}_j\right]\bullet\boldsymbol{\omega} = \overleftrightarrow{\mathbf{I}}\bullet\boldsymbol{\omega}$$

Matrix form of the $\boldsymbol{\omega}$-to-$\mathbf{L}$ relation using the *inertia matrix* $\langle\mathbf{I}\rangle$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \sum_{j=1}^{N} m_j \begin{pmatrix} y_j^2+z_j^2 & -x_jy_j & -x_jz_j \\ -y_jx_j & x_j^2+z_j^2 & -y_jz_j \\ -z_jx_j & -z_jy_j & x_j^2+y_j^2 \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$\langle\overleftrightarrow{\mathbf{I}}\rangle = \sum_{j=1}^{N}\langle\overleftrightarrow{\mathbf{I}}_j\rangle = \sum_{j=1}^{N} m_j \begin{pmatrix} y_j^2+z_j^2 & -x_jy_j & -x_jz_j \\ -y_jx_j & x_j^2+z_j^2 & -y_jz_j \\ -z_jx_j & -z_jy_j & x_j^2+y_j^2 \end{pmatrix}$$

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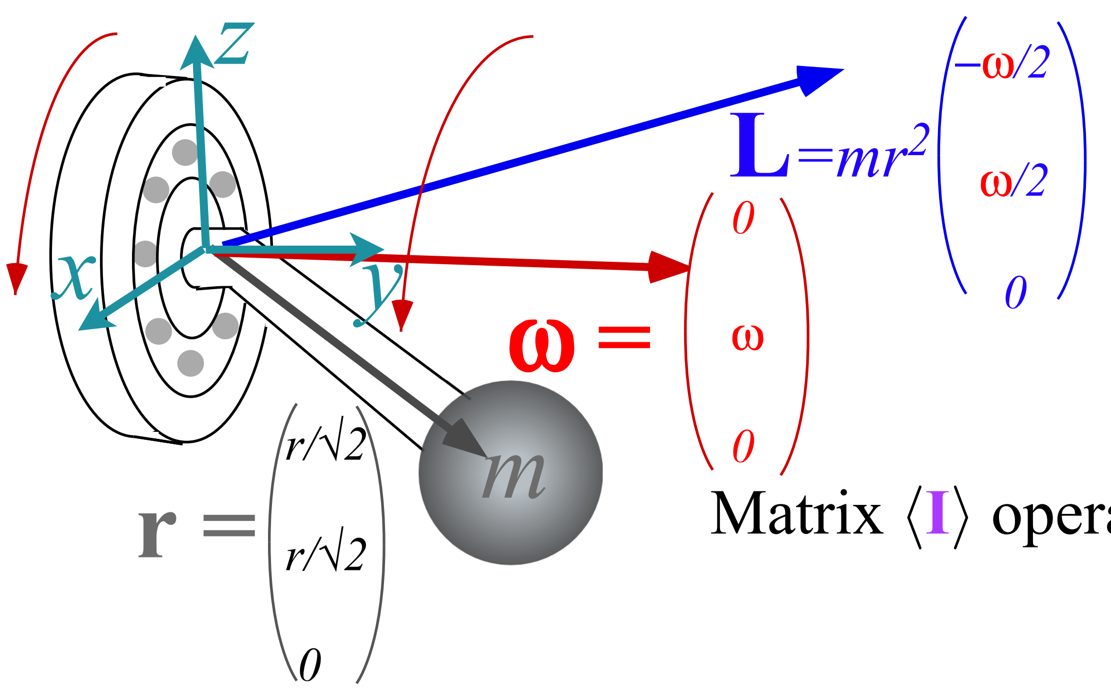

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Matrix $\langle\mathbf{I}\rangle$ operates on angular velocity $\boldsymbol{\omega}$ to give angular momentum $\mathbf{L}$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = mr^2\begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ \omega \\ 0 \end{pmatrix} = mr^2\begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix}\omega$$

Fig. 6.5.1 Angular momentum for mass rotating on bent axle.

Inertia tensors

Consider N-body *angular velocity* $\boldsymbol{\omega}$ and *angular momentum* $\mathbf{L}$ relations with Levi-Civita analysis

$$\dot{\mathbf{r}}_j = \boldsymbol{\omega}\times\mathbf{r}_j \qquad \text{and} \qquad \mathbf{L} = \sum_{j=1}^{N} \mathbf{r}_j \times m_j\dot{\mathbf{r}}_j = \sum_{j=1}^{N} m_j\mathbf{r}_j \times\left(\boldsymbol{\omega}\times\mathbf{r}_j\right) \qquad \text{with} \quad \mathbf{A}\times\left(\mathbf{B}\times\mathbf{C}\right)=\left(\mathbf{A}\bullet\mathbf{C}\right)\mathbf{B}-\left(\mathbf{A}\bullet\mathbf{B}\right)\mathbf{C}$$

This produces the *rotational inertia tensor* $\mathbf{I}$:

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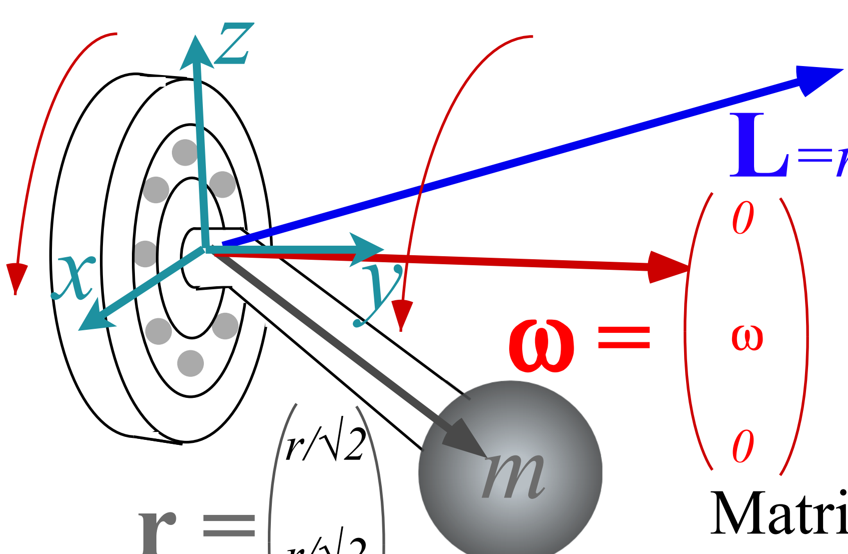
in the $\boldsymbol{\omega}$-to-$\mathbf{L}$ relation:

$$\mathbf{L} = \sum_{j=1}^{N} m_j\left[\left(\mathbf{r}_j\bullet\mathbf{r}_j\right)\boldsymbol{\omega}-\left(\mathbf{r}_j\bullet\boldsymbol{\omega}\right)\mathbf{r}_j\right] = \sum_{j=1}^{N} m_j\left[\left(\mathbf{r}_j\bullet\mathbf{r}_j\right)\mathbf{1}-\mathbf{r}_j\mathbf{r}_j\right]\bullet\boldsymbol{\omega} = \overleftrightarrow{\mathbf{I}}\bullet\boldsymbol{\omega}$$

Matrix form of the $\boldsymbol{\omega}$-to-$\mathbf{L}$ relation using the *inertia matrix* $\langle\mathbf{I}\rangle$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \sum_{j=1}^{N} m_j \begin{pmatrix} y_j^2+z_j^2 & -x_jy_j & -x_jz_j \\ -y_jx_j & x_j^2+z_j^2 & -y_jz_j \\ -z_jx_j & -z_jy_j & x_j^2+y_j^2 \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \qquad \left\langle\overleftrightarrow{\mathbf{I}}\right\rangle = \sum_{j=1}^{N}\left\langle\overleftrightarrow{\mathbf{I}}_j\right\rangle = \sum_{j=1}^{N} m_j \begin{pmatrix} y_j^2+z_j^2 & -x_jy_j & -x_jz_j \\ -y_jx_j & x_j^2+z_j^2 & -y_jz_j \\ -z_jx_j & -z_jy_j & x_j^2+y_j^2 \end{pmatrix}$$

Consider mass m instantaneously at $\mathbf{r}_m = (x_m, y_m, z_m) = r(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ on a bent axle rotating in a fixed bearing:



$$\mathbf{L} = mr^2\begin{pmatrix} -\omega/2 \\ \omega/2 \\ 0 \end{pmatrix} \qquad \boldsymbol{\omega} = \begin{pmatrix} 0 \\ \omega \\ 0 \end{pmatrix} \qquad \mathbf{r} = \begin{pmatrix} r/\sqrt{2} \\ r/\sqrt{2} \\ 0 \end{pmatrix}$$

Instantaneous matrix $\langle\mathbf{I}\rangle$ of inertia is:

$$\left\langle\overleftrightarrow{\mathbf{I}}\right\rangle = mr^2\begin{pmatrix} \left(1/\sqrt{2}\right)^2+0 & -\left(1/\sqrt{2}\right)\left(1/\sqrt{2}\right) & -\left(1/\sqrt{2}\right)0 \\ -\left(1/\sqrt{2}\right)\left(1/\sqrt{2}\right) & \left(1/\sqrt{2}\right)^2+0 & -\left(1/\sqrt{2}\right)0 \\ -0\left(1/\sqrt{2}\right) & -0\left(1/\sqrt{2}\right) & \left(1/\sqrt{2}\right)^2+\left(1/\sqrt{2}\right)^2 \end{pmatrix} = mr^2\begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrix $\langle\mathbf{I}\rangle$ operates on angular velocity $\boldsymbol{\omega}$ to give angular momentum $\mathbf{L}$

Bearing torque is: $\mathbf{N}=\dfrac{d\mathbf{L}}{dt}=\boldsymbol{\omega}\times\mathbf{L}$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = mr^2\begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ \omega \\ 0 \end{pmatrix} = mr^2\begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix}\omega$$

Fig. 6.5.1 Angular momentum for mass rotating on bent axle.

Kinetic energy in terms of velocity ω and rotational Lagrangian

Kinetic energy T of a rotating rigid body can be expressed in terms of the inertia matrix $\mathbf{I}$

$$T=\frac{1}{2}\sum_{j=1}^{3} m_j\dot{\mathbf{r}}_j\bullet\dot{\mathbf{r}}_j=\frac{1}{2}\sum_{j=1}^{3} m_j\left(\boldsymbol{\omega}\times\mathbf{r}_j\right)\bullet\left(\boldsymbol{\omega}\times\mathbf{r}_j\right)$$

Levi-Civita identity

$$\left(\mathbf{A}\times\mathbf{B}\right)\times\left(\mathbf{C}\times\mathbf{D}\right)=\left(\mathbf{A}\bullet\mathbf{C}\right)\left(\mathbf{B}\bullet\mathbf{D}\right)-\left(\mathbf{A}\bullet\mathbf{D}\right)\left(\mathbf{B}\bullet\mathbf{C}\right)$$

$$T=\frac{1}{2}\sum_{j=1}^{3} m_j\left[\left(\boldsymbol{\omega}\bullet\boldsymbol{\omega}\right)\left(\mathbf{r}_j\bullet\mathbf{r}_j\right)-\left(\boldsymbol{\omega}\bullet\mathbf{r}_j\right)\left(\mathbf{r}_j\bullet\boldsymbol{\omega}\right)\right]$$

$$=\frac{1}{2}\boldsymbol{\omega}\bullet\sum_{j=1}^{3} m_j\left[\left(\mathbf{r}_j\bullet\mathbf{r}_j\right)\mathbf{1}-\left(\mathbf{r}_j\right)\left(\mathbf{r}_j\right)\right]\bullet\boldsymbol{\omega}$$

$$=\frac{1}{2}\boldsymbol{\omega}\bullet\vec{\mathbf{I}}\bullet\boldsymbol{\omega}$$

Kinetic energy is a *quadratic form*

$$T=\frac{1}{2}\begin{pmatrix}\omega_x & \omega_y & \omega_y\end{pmatrix}\begin{pmatrix}I_{xx} & I_{xy} & I_{xz}\\ I_{yx} & I_{yy} & I_{yz}\\ I_{zx} & I_{zy} & I_{zz}\end{pmatrix}\begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix}\langle\omega|x\rangle & \langle\omega|y\rangle & \langle\omega|z\rangle\end{pmatrix}\begin{pmatrix}\langle x|\mathbf{I}|x\rangle & \langle x|\mathbf{I}|y\rangle & \langle x|\mathbf{I}|z\rangle\\ \langle y|\mathbf{I}|x\rangle & \langle y|\mathbf{I}|y\rangle & \langle y|\mathbf{I}|z\rangle\\ \langle z|\mathbf{I}|x\rangle & \langle z|\mathbf{I}|y\rangle & \langle z|\mathbf{I}|z\rangle\end{pmatrix}\begin{pmatrix}\langle x|\omega\rangle\\ \langle y|\omega\rangle\\ \langle z|\omega\rangle\end{pmatrix}\quad\text{(Dirac notation)}$$

$$=\frac{1}{2}\begin{pmatrix}\omega_x & \omega_y & \omega_y\end{pmatrix}\sum_{j=1}^{3} m_j\begin{pmatrix}y_j^2+z_j^2 & -x_jy_j & -x_jz_j\\ -y_jx_j & x_j^2+z_j^2 & -y_jz_j\\ -z_jx_j & -z_jy_j & x_j^2+y_j^2\end{pmatrix}\begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix}$$

Simplifies in *principle inertial axes* $\{X,Y,Z\}$ or *body eigen-axes*

$$T=\frac{1}{2}\begin{pmatrix}\omega_X & \omega_Y & \omega_Z\end{pmatrix}\begin{pmatrix}I_{XX} & I_{XY} & I_{XZ}\\ I_{YX} & I_{YY} & I_{YZ}\\ I_{ZX} & I_{ZY} & I_{ZZ}\end{pmatrix}\begin{pmatrix}\omega_X\\ \omega_Y\\ \omega_Z\end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix}\omega_X & \omega_Y & \omega_Z\end{pmatrix}\begin{pmatrix}I_{XX} & 0 & 0\\ 0 & I_{YY} & 0\\ 0 & 0 & I_{ZZ}\end{pmatrix}\begin{pmatrix}\omega_X\\ \omega_Y\\ \omega_Z\end{pmatrix}=\frac{I_{XX}\omega_X^2}{2}+\frac{I_{YY}\omega_Y^2}{2}+\frac{I_{ZZ}\omega_Z^2}{2}$$

Kinetic energy in terms of momentum L and rotational Hamiltonian

$$\mathbf{L} = \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega} , \quad \text{generally implies:} \quad \boldsymbol{\omega} = \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$$

Express kinetic energy T in terms of angular velocity ω , momentum $\mathbf{L}$, or both at once. once

$$T = \frac{1}{2}\boldsymbol{\omega} \bullet \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega} \bullet \mathbf{L} = \frac{1}{2}\mathbf{L} \bullet \boldsymbol{\omega} = \frac{1}{2}\mathbf{L} \bullet \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$$

$$T = \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix}\begin{pmatrix} I_{XX} & I_{XY} & I_{XZ} \\ I_{YX} & I_{YY} & I_{YZ} \\ I_{ZX} & I_{ZY} & I_{ZZ} \end{pmatrix}^{-1}\begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix}\begin{pmatrix} 1/I_{XX} & 0 & 0 \\ 0 & 1/I_{YY} & 0 \\ 0 & 0 & 1/I_{ZZ} \end{pmatrix}\begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix} = \frac{L_X^2}{2I_{XX}} + \frac{L_Y^2}{2I_{YY}} + \frac{L_Z^2}{2I_{ZZ}}$$

Kinetic energy in terms of momentum L and rotational Hamiltonian

$\mathbf{L} = \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega}$, generally implies: $\boldsymbol{\omega} = \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$

Express kinetic energy T in terms of angular velocity ω , momentum **L**, or both at once. once

$$T = \frac{1}{2}\boldsymbol{\omega} \bullet \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega} \bullet \mathbf{L} = \frac{1}{2}\mathbf{L} \bullet \boldsymbol{\omega} = \frac{1}{2}\mathbf{L} \bullet \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$$

$$T = \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix}\begin{pmatrix} I_{XX} & I_{XY} & I_{XZ} \\ I_{YX} & I_{YY} & I_{YZ} \\ I_{ZX} & I_{ZY} & I_{ZZ} \end{pmatrix}^{-1}\begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix}$$

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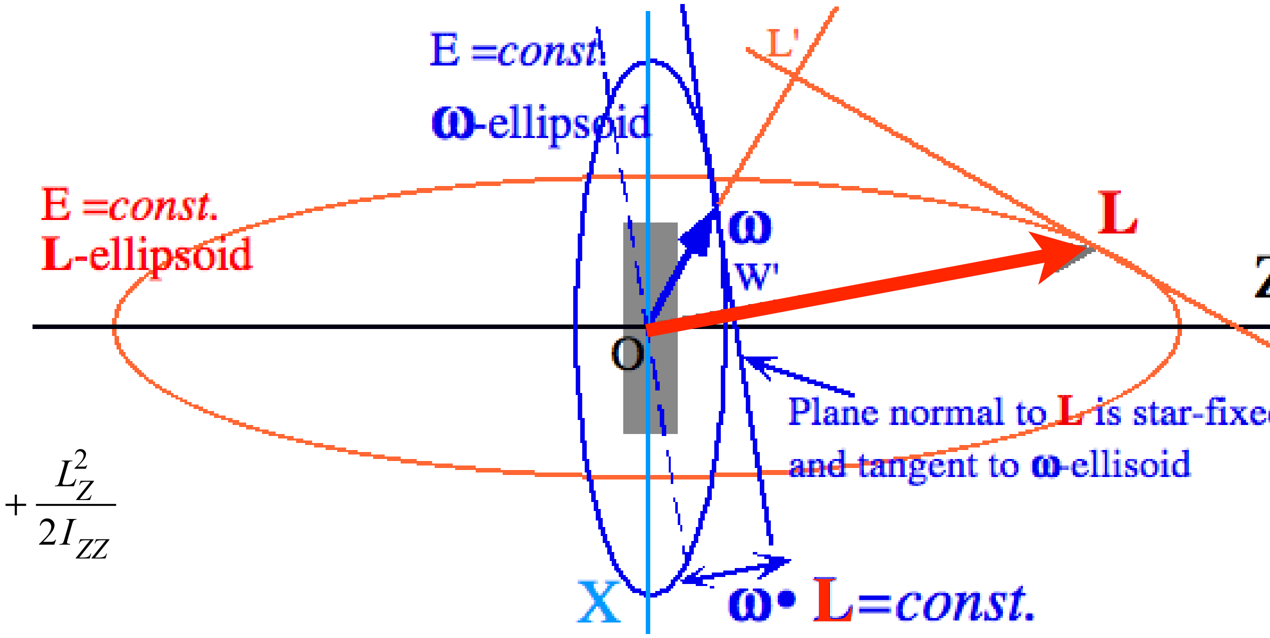


Hamiltonian form is the equation of the *angular momentum or* **L***-ellipsoid*

Lagrangian form is the equation of the *angular velocity or ω-ellipsoid*

Kinetic energy in terms of momentum L and rotational Hamiltonian

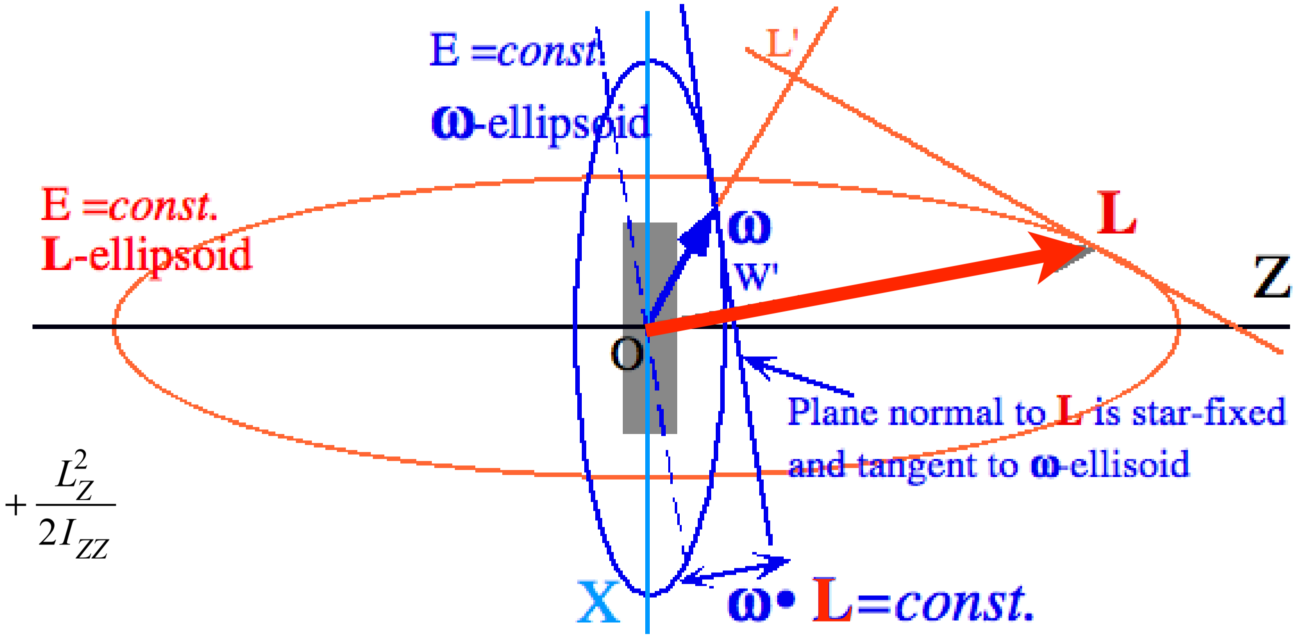
$\mathbf{L} = \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega}$, generally implies: $\boldsymbol{\omega} = \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$

Express kinetic energy T in terms of angular velocity ω , momentum **L**, or both at once. once

$$T = \frac{1}{2}\boldsymbol{\omega} \bullet \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega} \bullet \mathbf{L} = \frac{1}{2}\mathbf{L} \bullet \boldsymbol{\omega} = \frac{1}{2}\mathbf{L} \bullet \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$$

$$T = \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix}\begin{pmatrix} I_{XX} & I_{XY} & I_{XZ} \\ I_{YX} & I_{YY} & I_{YZ} \\ I_{ZX} & I_{ZY} & I_{ZZ} \end{pmatrix}^{-1}\begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} L_X & L_Y & L_Z \end{pmatrix}\begin{pmatrix} 1/I_{XX} & 0 & 0 \\ 0 & 1/I_{YY} & 0 \\ 0 & 0 & 1/I_{ZZ} \end{pmatrix}\begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix} = \frac{L_X^2}{2I_{XX}} + \frac{L_Y^2}{2I_{YY}} + \frac{L_Z^2}{2I_{ZZ}}$$

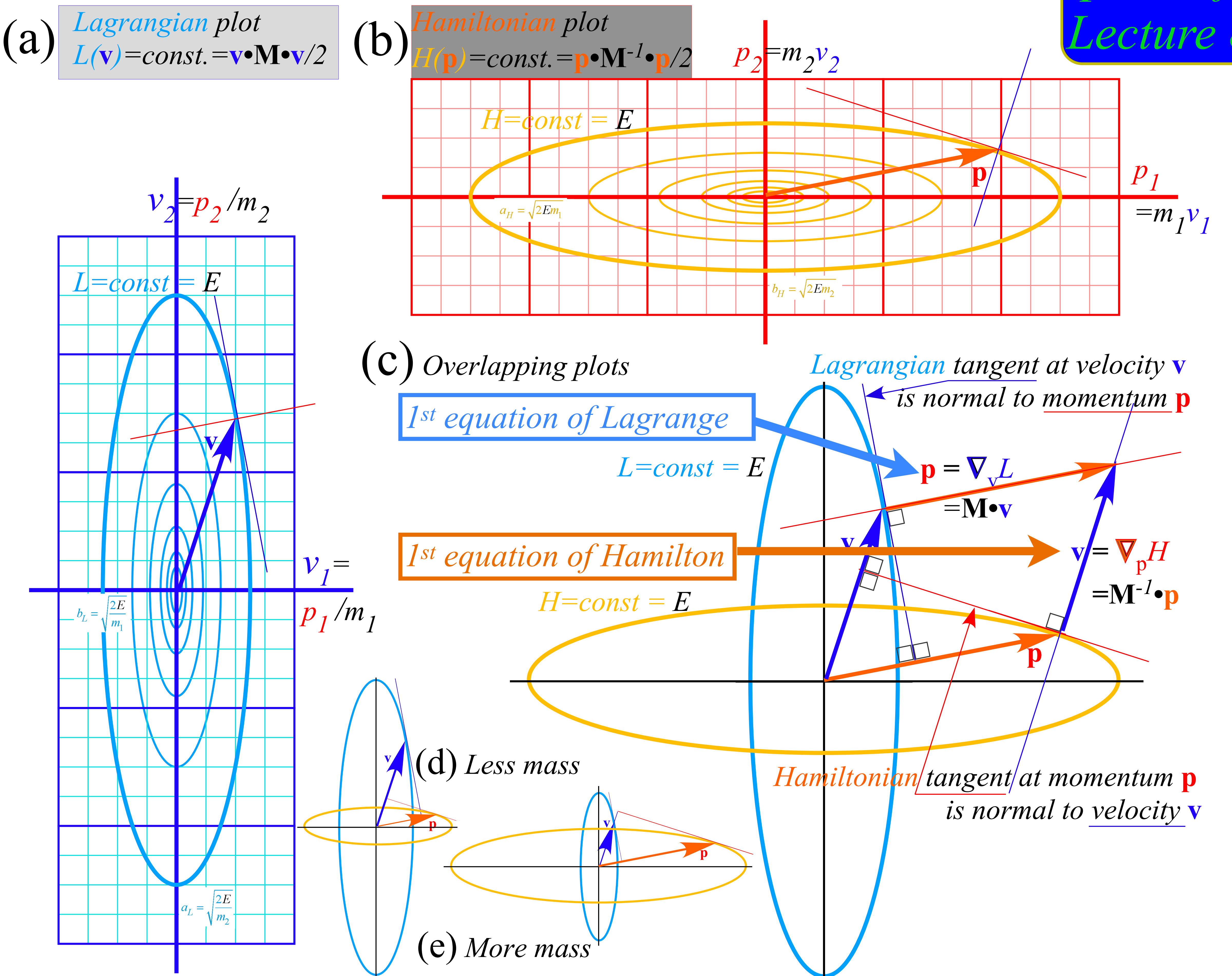


Hamiltonian form is the equation of the *angular momentum or* **L**-*ellipsoid*

Lagrangian form is the equation of the *angular velocity or ω-ellipsoid*

Recall quadratic forms for Lagrangian and Hamiltonian in Lecture 8 and CMwBANG! Unit 1 Ch.12

Unit 1
Fig. 12.2



Kinetic energy in terms of momentum L and rotational Hamiltonian

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$$T = \frac{1}{2}\boldsymbol{\omega} \bullet \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega} \bullet \mathbf{L} = \frac{1}{2}\mathbf{L} \bullet \boldsymbol{\omega} = \frac{1}{2}\mathbf{L} \bullet \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$$

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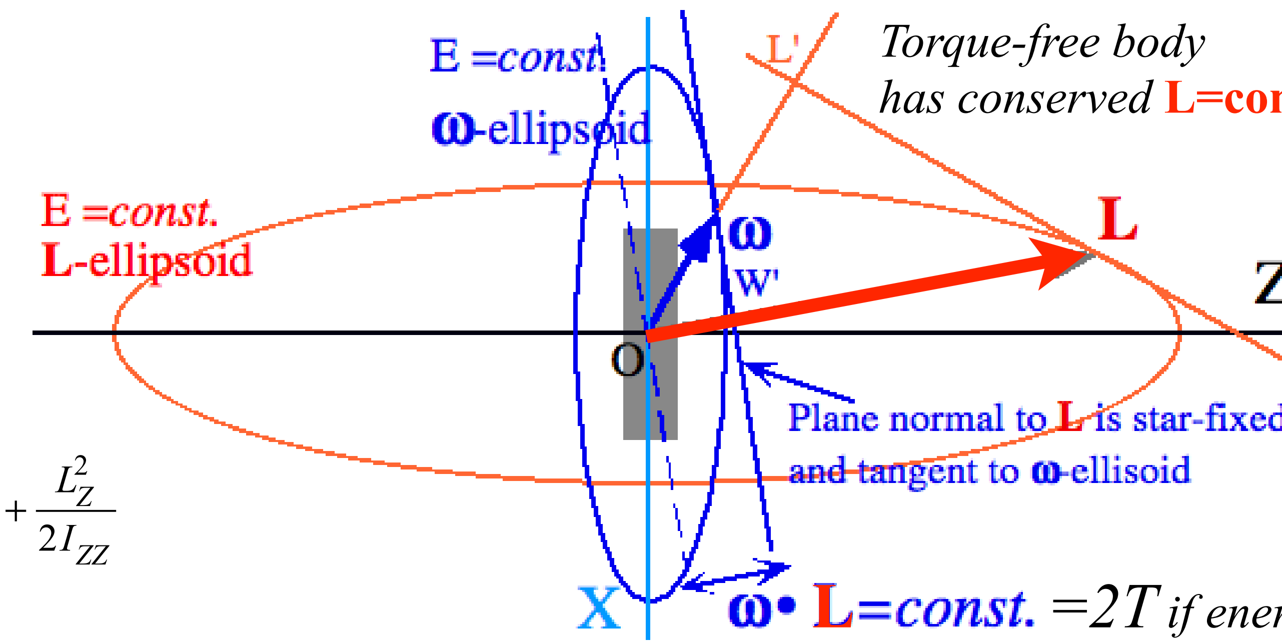

$\boldsymbol{\omega}\bullet$ **L**=*const.* $=2T$ *if energy is not dissipated internally*

Hamiltonian form is the equation of the *angular momentum or* **L***-ellipsoid*

Lagrangian form is the equation of the *angular velocity or ω-ellipsoid*

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Kinetic energy in terms of momentum L and rotational Hamiltonian

$\mathbf{L} = \overleftrightarrow{\mathbf{I}} \bullet \boldsymbol{\omega}$, generally implies: $\boldsymbol{\omega} = \overleftrightarrow{\mathbf{I}}^{-1} \bullet \mathbf{L}$

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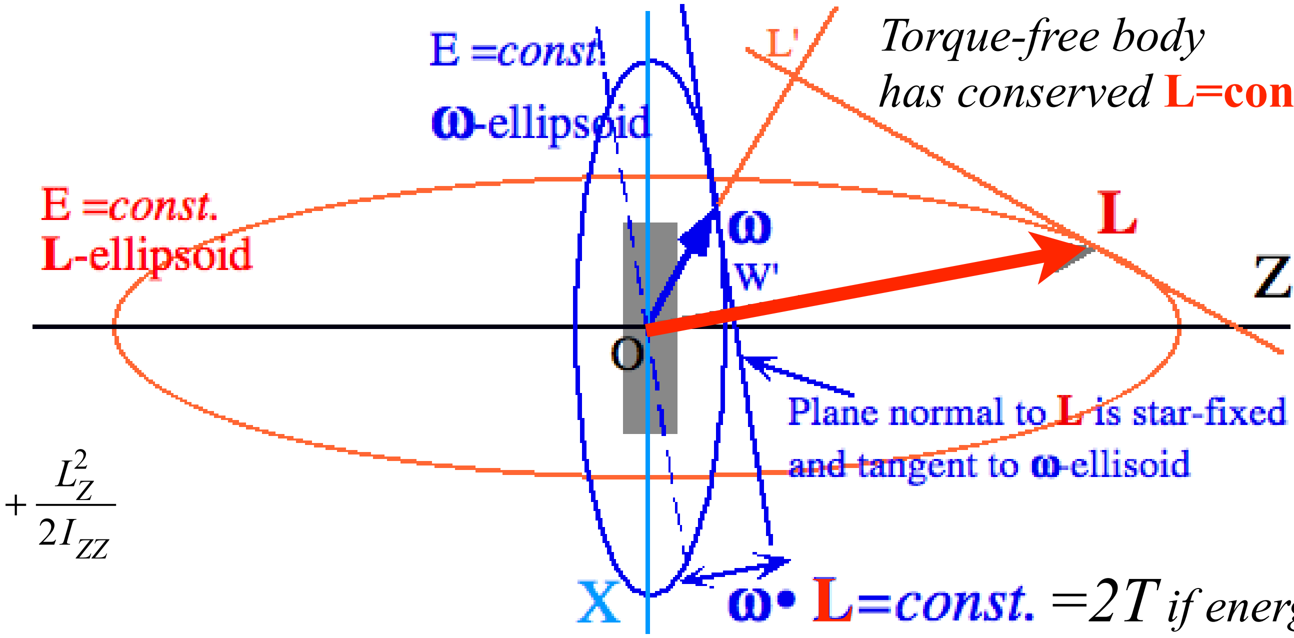

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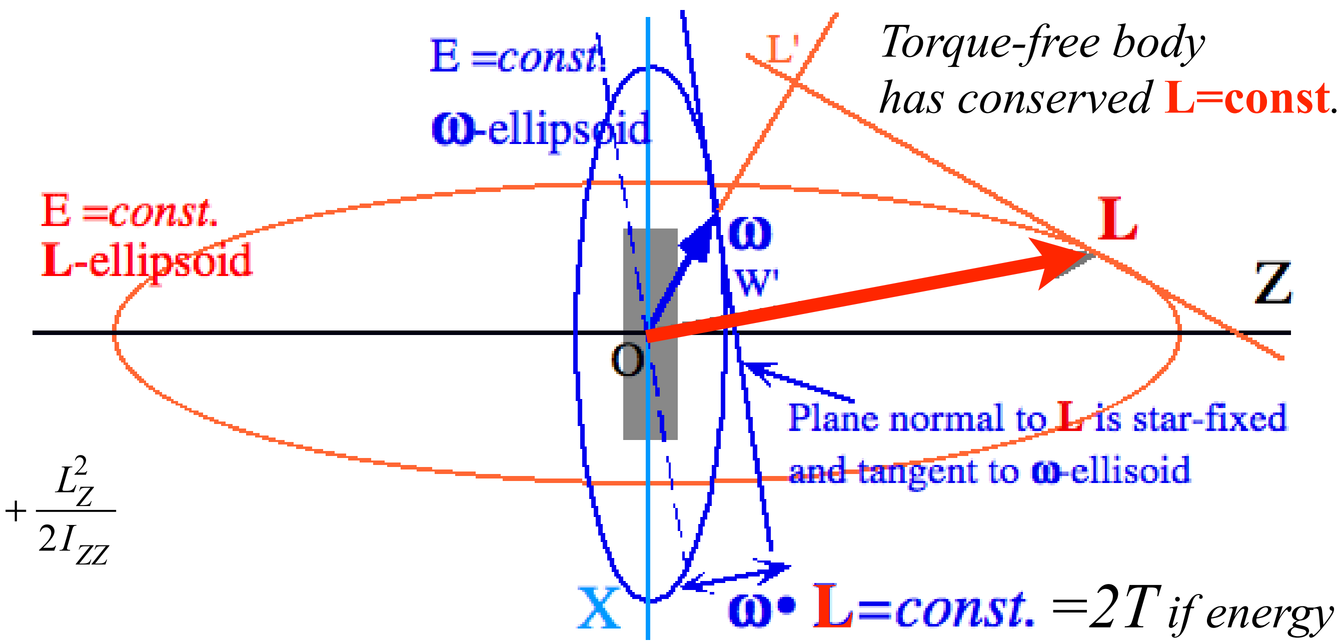

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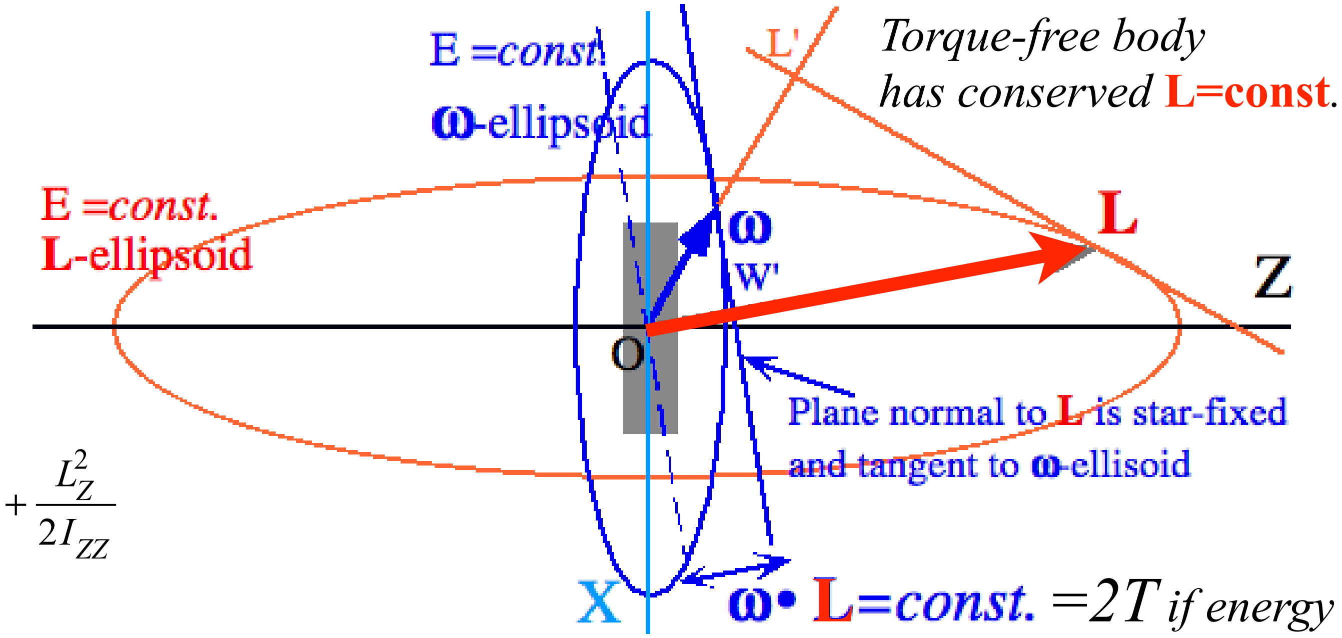

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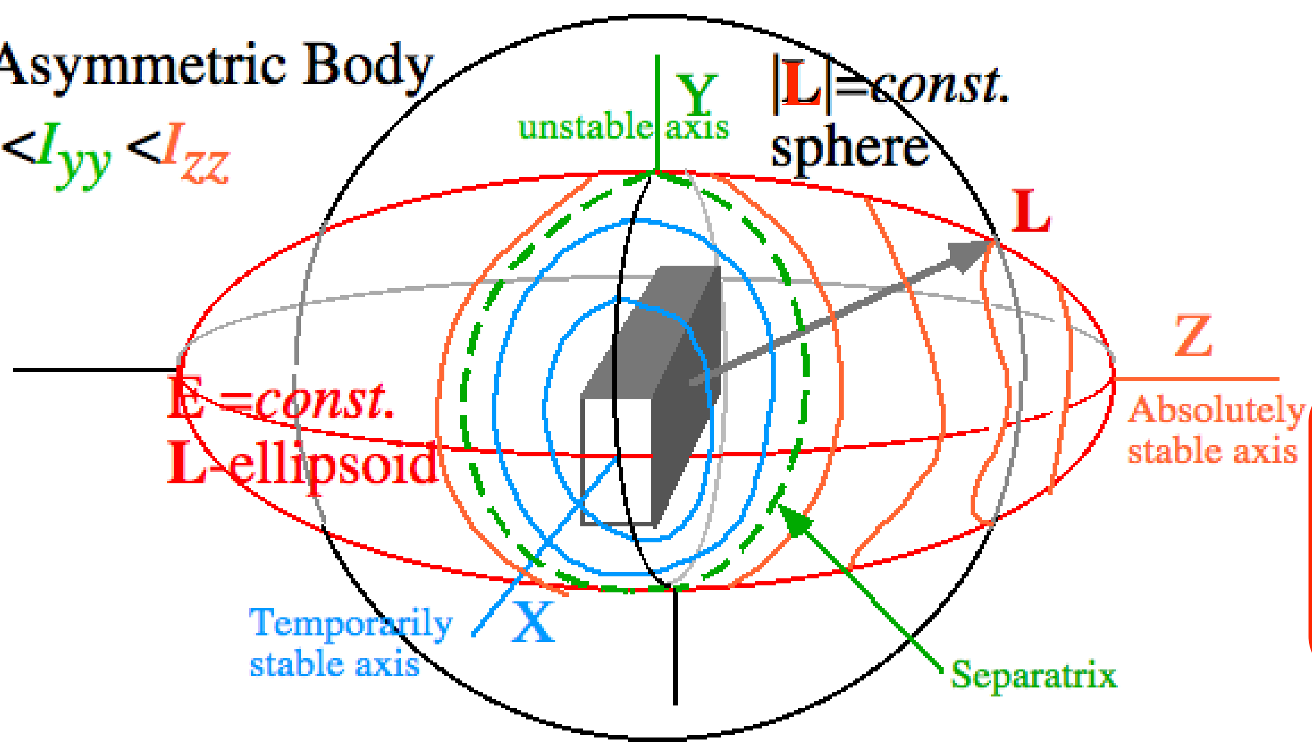
General Asymmetric Body

$I_{xx} < I_{yy} < I_{zz}$



$$\boxed{\begin{array}{l} \text{Canonical momentum:} \quad p_\mu = \dfrac{\partial L}{\partial \dot{q}^\mu} \quad (\text{where: } L = T) \\ \mathbf{L} = \dfrac{\partial T}{\partial \boldsymbol{\omega}} = \nabla_{\boldsymbol{\omega}} T = \dfrac{\partial}{\partial \boldsymbol{\omega}} \dfrac{\boldsymbol{\omega} \bullet \mathbf{I} \bullet \boldsymbol{\omega}}{2} = \mathbf{I} \bullet \boldsymbol{\omega} \end{array}}$$

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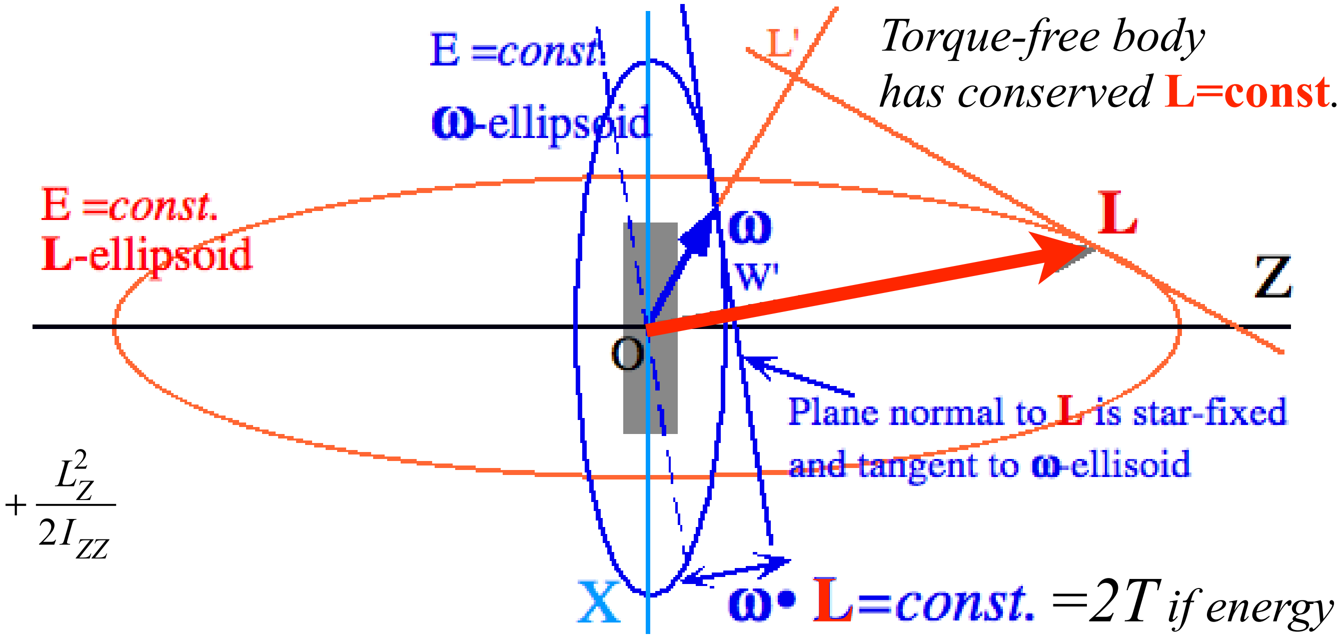

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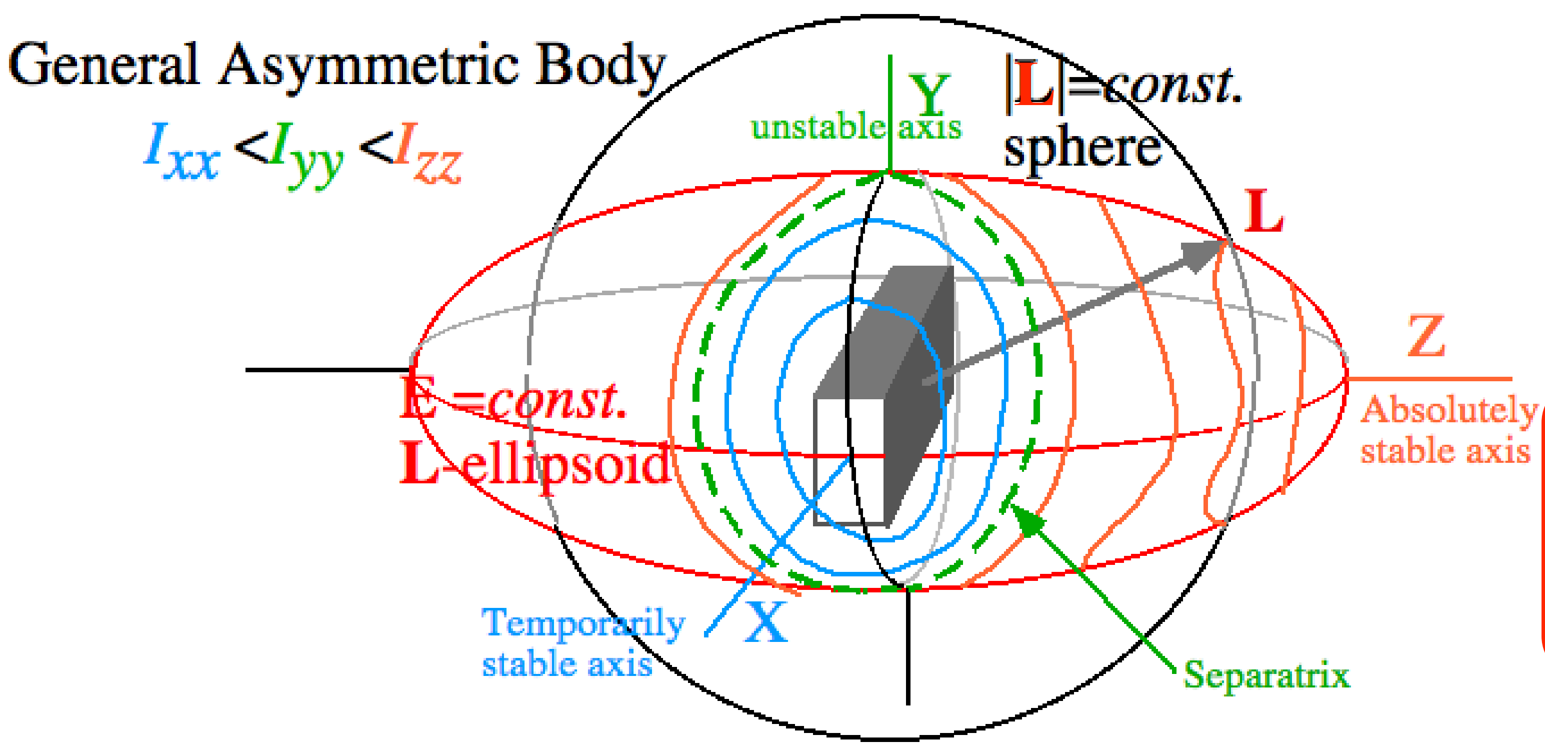

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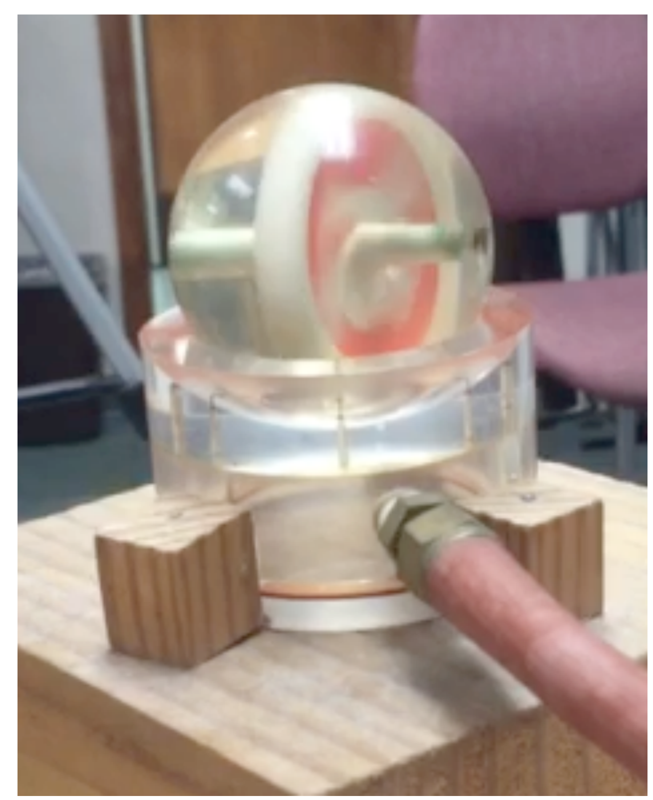
In body frame momentum **L** *moves along intersection of* ***L****-ellipsoid and* ***L****-sphere (Length |***L***| is constant in any classical frame.)*

Asymmetric Top
Demo video

Rotational Energy Surfaces (RES)

→ *Symmetric, asymmetric, and spherical-top dynamics (Constant* **L***)*

BOD-frame cone rolling on LAB frame cone



Singular Motion of
Asymetric Rotators
AJP 44, 11 p1080 1976

Asymmetric-top dynamics (Constant **L***)*

1. NASA Space station video

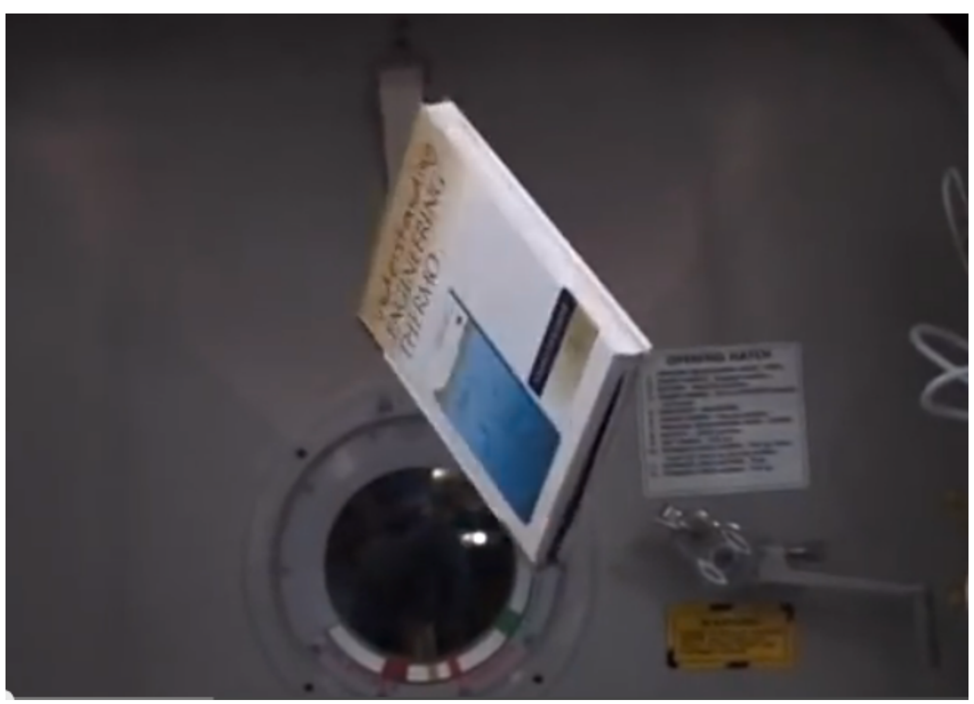
https://youtu.be/1n-HMSCDYtM

For those physist who are brave of heart, make note the video's comments

2. UAF lab air-supported asymmetric top video

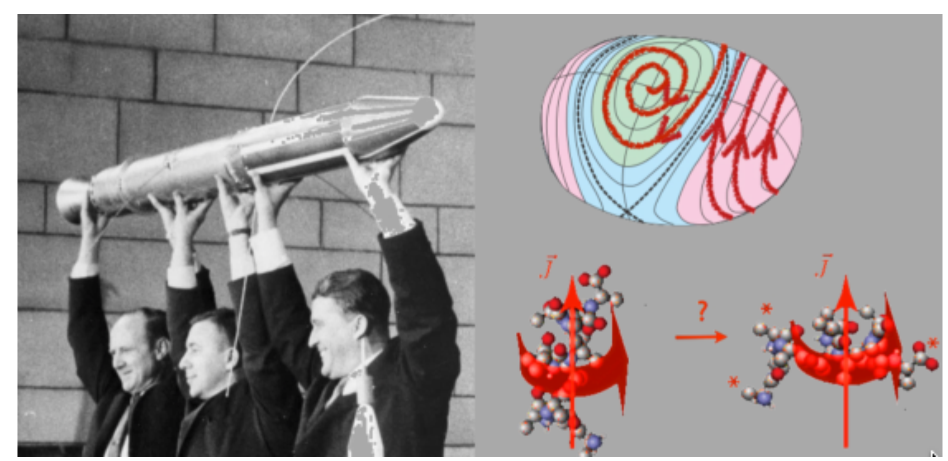
https://youtu.be/HWjGvCaqx5g

3. NASA-Rotating Solid Bodies in Microgravity (2008)



https://www.youtube.com/watch?v=BPMjcN-sBJ4

4. Early NASA-JPL satellite blunder (1958)



To be Continued ⇒several pages ahead

Singular Motion of Asymetric Rotators AJP 44, 11 p1080 1976

Comments following Space Lab video of asymmetric rotation show that it is not a widely understood phenomenon

Bagnon DuJour • 3 months ago

As the handles spins out it dips down a bit before becoming detached and that linear momentum travels through the angular momentum until the equilibrium requires the flip to maintain the path of least resistance. If they could spin it perfectly without the dip, it would not turn like that.

• Reply • Share ›

Bill Aldridge → Bagnon DuJour • 3 months ago

So you are saying, when they put their hands on the tip, i dip, you dip, we dip.

• Reply • Share ›

EVERYONE is born an atheist → Bagnon DuJour • 3 months ago

Exactly. Not sure why this was even posted. Maybe it was just going to b used as a basic physics example for schools.

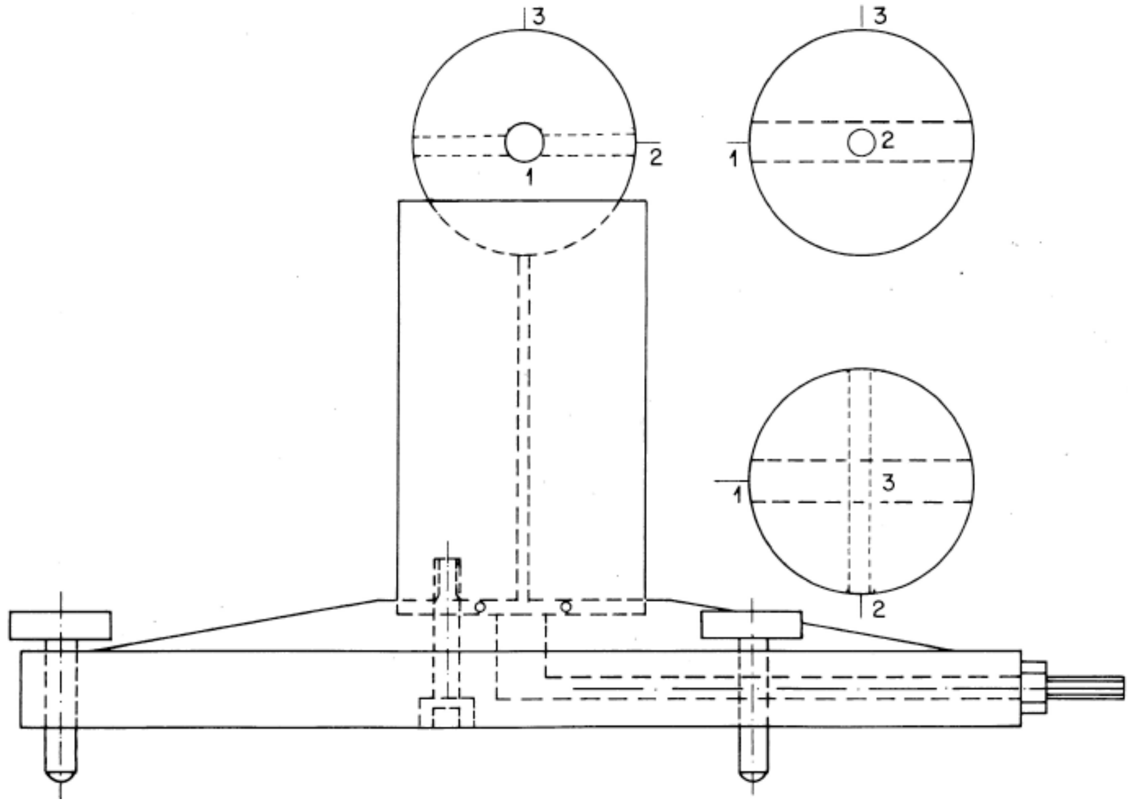
• Reply • Share ›

Tim Johnson → Bagnon DuJour • 3 months ago

It sounds like you have a handle on what’s going on here.

1 • Reply • Share ›

Bocce-Ball Asymmetric Top we built at USC *(donated to Cal. Museum of Science & Industry)*



$$I_1 = (2M/5 + m_2/3)R^2 + m_1 r_1^{\,2}/2 + m_2 r_2^{\,2}/4,$$
$$I_2 = (2M/5 + m_1/3)R^2 + m_2 r_2^{\,2}/2 + m_1 r_1^{\,2}/4, \quad (8)$$
$$I_3 = (2M/5 + m_1/3 + m_2/3)R^2 + m_1 r_1^{\,2}/4 + m_2 r_2^{\,2}/4,$$

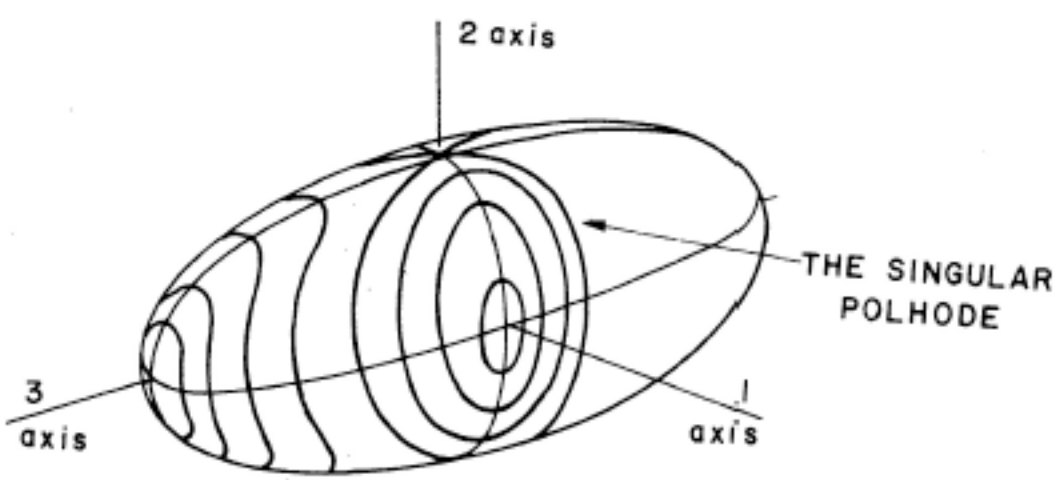


Fig. 3. Polhodes. A family of constraint curves for the vector ω in the body system, or "polhodes," are separated into two distinct groups by a curve called the singular polhode.

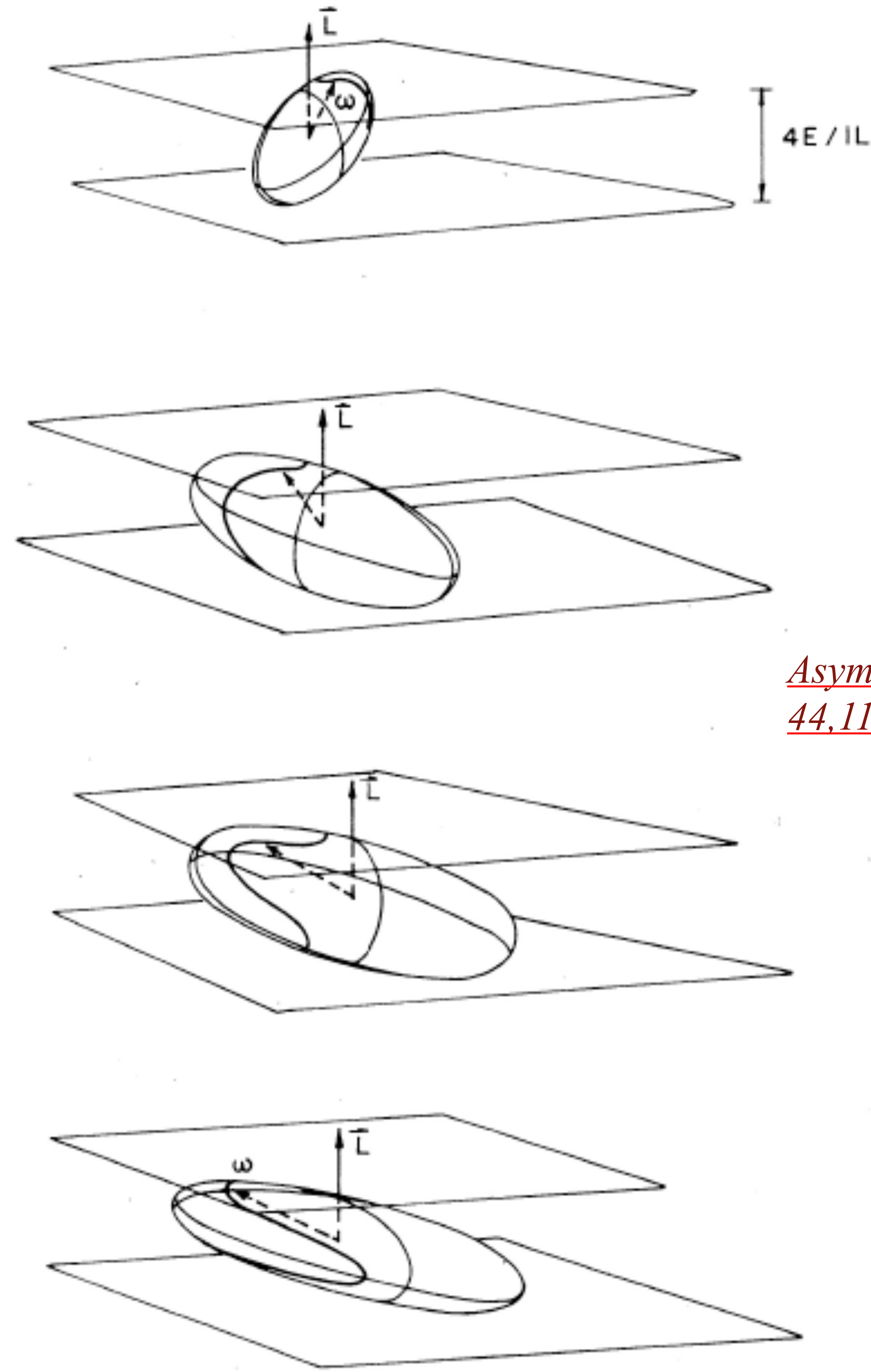


Fig. 4. Model of rotational motion near the singular polhode.

Asym. Rotor AJP 44,11 1976

Singular Motion of Asymetric Rotators AJP 44, 11 p1080 1976

Bocce-Ball Asymmetric Top Motion solved by Euler's equation and elliptic integrals

$$\dot{\mathbf{L}} = \omega \times \mathbf{L}, \tag{9}$$

which takes the following form for the 2 component:

$$\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3)/I_2 = 0. \tag{10}$$

Solving Eq. (10) for $\omega \equiv \omega_2$ using Eqs. (5) and (6), we obtain the following:

$$\dot{\omega} = (a - b\omega^2)^{1/2}(c - d\omega^2)^{1/2}/I_2(I_1I_3)^{1/2}, \tag{11}$$

where the constants a–d [Eq. (12)] depend on initial conditions and the inertial moments as follows:

$$\begin{gathered} a = 2EI_3 - L^2, \quad b = I_2(I_3 - I_2), \\ c = L^2 - 2EI_1, \quad d = I_2(I_2 - I_1), \\ a = I_2(I_3 - I_2)W^2 \cos^2 \epsilon, \\ c = [I_2(I_2 - I_1)\cos^2 \epsilon + I_3(I_3 - I_1)\sin^2 \epsilon)W^2, \end{gathered} \tag{12}$$

where we have assumed initial conditions

$$\omega_1(0) = 0, \quad \omega_2(0) = W\cos\epsilon, \quad \omega_3(0) = W\sin\epsilon. \tag{13}$$

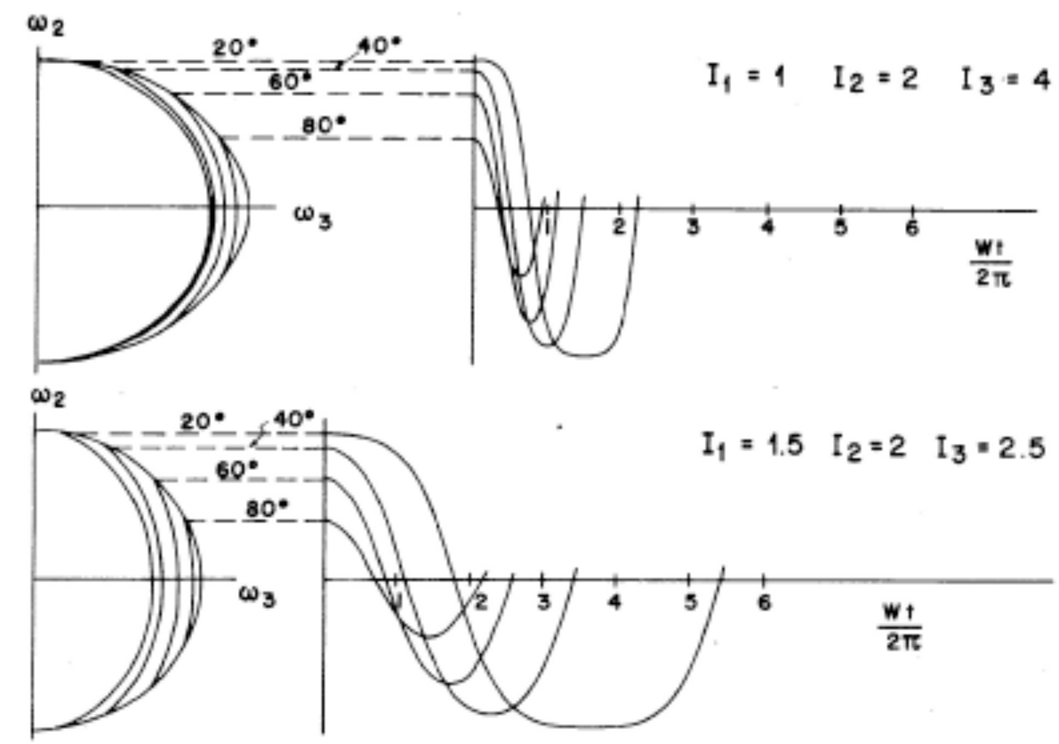


Fig. 6. Exact solutions. The motion of the ω vector for an asymmetric and a not-so-asymmetric body are compared. Various polhodes are shown on the left-hand side while the corresponding time behavior is plotted on the right-hand side.

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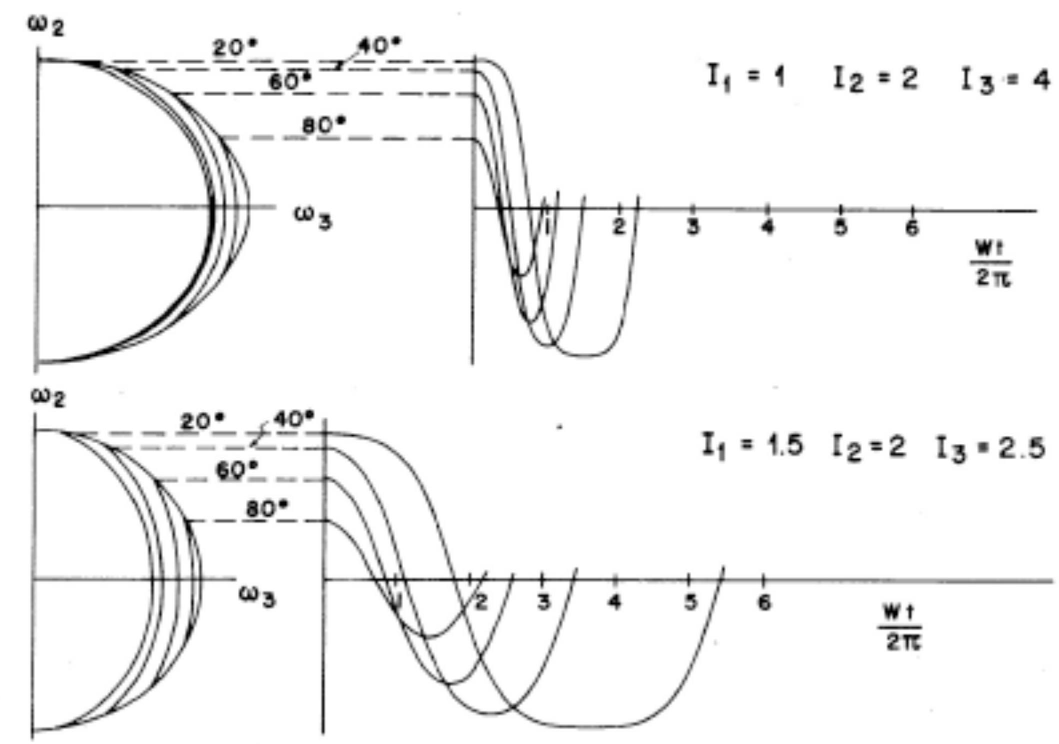
where the following substitutions were made:

$$k = (ad/bc)^{1/2}, \quad \omega = (a/b)^{1/2}\Omega = \Omega W\cos\epsilon. \tag{15}$$

A further substitution $\Omega = \sin\phi$ reduces the integral

$$\int_0^{\Omega'} \frac{d\Omega}{(1 - \Omega^2)^{1/2}(1 - k^2\Omega^2)^{1/2}} = \int_0^{\phi'} \frac{d\phi}{(1 - k^2\sin^2\phi)^{1/2}} \equiv \mathrm{sn}^{-1}(\phi', k). \tag{16}$$



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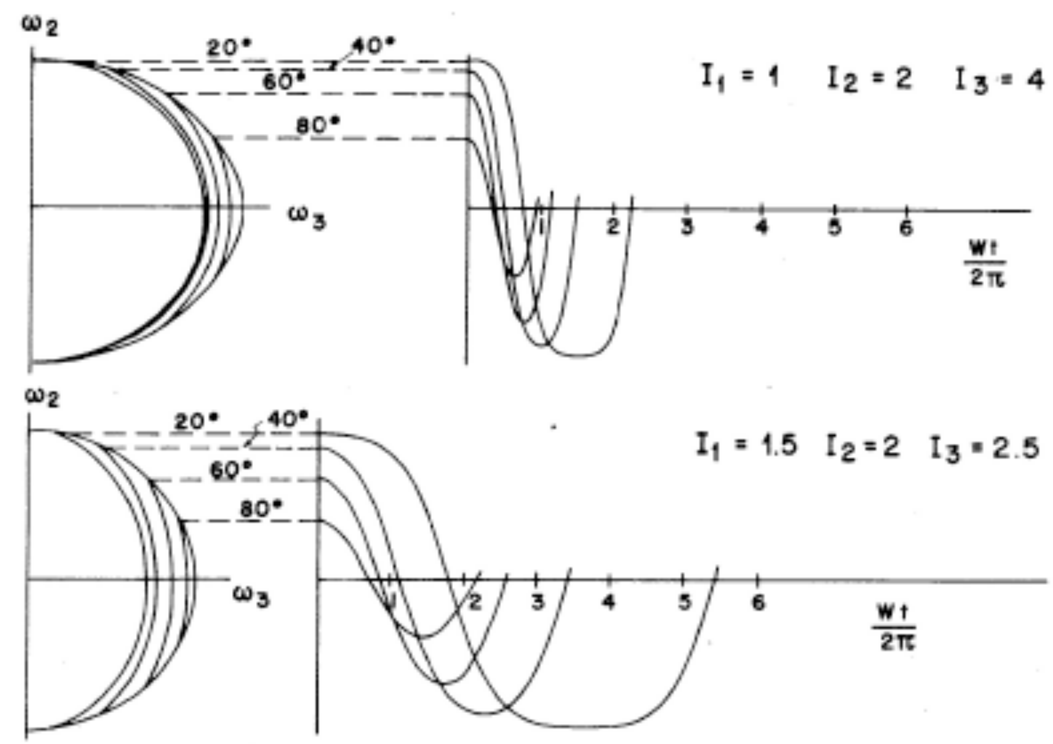

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$$t \to \frac{2}{W}\left(\frac{I_1I_2}{(I_3 - I_2)(I_2 - I_1)}\right)^{1/2} \mathrm{sn}^{-1}\left(\frac{\pi}{2}, k\right), \qquad (17b)$$

where

$$k = \left(\frac{I_2(I_2 - I_1)}{I_2(I_2 - I_1)\cos^2\epsilon + I_3(I_3 - I_1)\sin^2\epsilon}\right)^{1/2} \cos\epsilon, \qquad (18a)$$

$$k \to 1 - (I_3/I_2)[(I_3 - I_1)/(I_2 - I_1)](\epsilon^2/2). \qquad (18b)$$

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where the constants a–d [Eq. (12)] depend on initial conditions and the inertial moments as follows:

$$\begin{gathered} a = 2EI_3 - L^2, \quad b = I_2(I_3 - I_2), \\ c = L^2 - 2EI_1, \quad d = I_2(I_2 - I_1), \\ a = I_2(I_3 - I_2)W^2 \cos^2 \epsilon, \\ c = [I_2(I_2 - I_1)\cos^2 \epsilon + I_3(I_3 - I_1)\sin^2 \epsilon)W^2, \end{gathered} \qquad (12)$$

where we have assumed initial conditions

$$\omega_1(0) = 0, \quad \omega_2(0) = W\cos \epsilon, \quad \omega_3(0) = W \sin \epsilon. \qquad (13)$$

$$t = \left(\frac{I_1 I_2 I_3}{(I_3 - I_2)(L^2 - 2EI_1)}\right)^{1/2} \times \int_0^{\Omega'} \frac{d\Omega}{(1-\Omega^2)^{1/2}(1-k^2\Omega^2)^{1/2}}, \qquad (14)$$

where the following substitutions were made:

$$k = (ad/bc)^{1/2}, \quad \omega = (a/b)^{1/2}\Omega = \Omega W \cos \epsilon. \qquad (15)$$

A further substitution $\Omega = \sin\phi$ reduces the integral

$$\int_0^{\Omega'} \frac{d\Omega}{(1-\Omega^2)^{1/2}(1-k^2\Omega^2)^{1/2}} = \int_0^{\phi'} \frac{d\phi}{(1-k^2\sin^2\phi)^{1/2}} \equiv \mathrm{sn}^{-1}(\phi', k). \qquad (16)$$

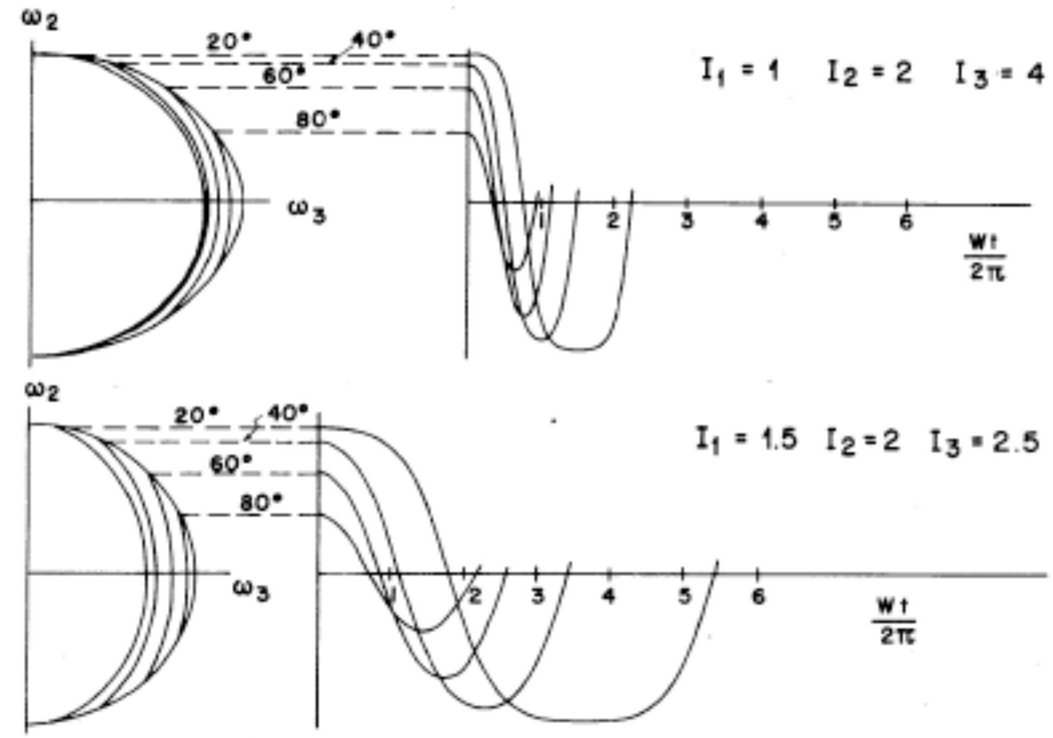
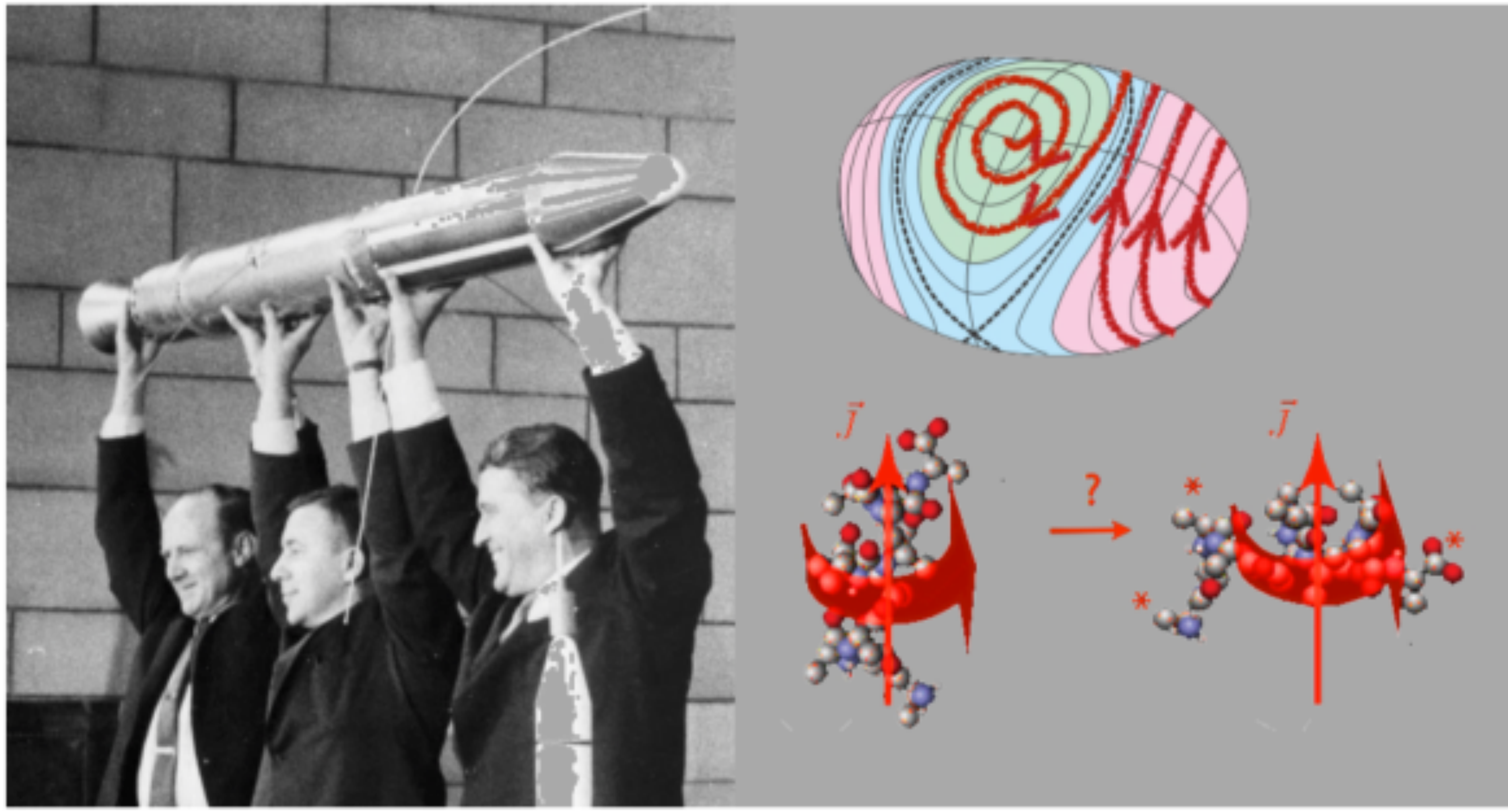


Fig. 6. Exact solutions. The motion of the ω vector for an asymmetric and a not-so-asymmetric body are compared. Various polhodes are shown on the left-hand side while the corresponding time behavior is plotted on the right-hand side.

The limiting forms [Eqs. (17) and (18b)] become good approximations for $\epsilon < 10°$. The approximate number of revolutions accomplished by a body before it overturns is given by the product of $W/2\pi$, the number of revolutions per second, and the right-hand side of Eq. (17b). Exact solutions for various I_j and ϵ are displayed in Fig. 6.

If one desires to increase the reversal time, it should be done through the first factor in Eq. (17b). The integral in the second factor is usually only as large as 7 or 8 in our experiments ($\epsilon = 10°$ gives 3.1, $\epsilon = 1°$ gives 5.4, and $\epsilon = 0°$ 1′ gives 9.5). This is a good demonstration of the behavior of an elliptic function near its singularity.

4. Early NASA-JPL satellite blunder (1958)



*From text*in preparation by Rick Heller on semi-classical dynamics of polyatomic molecules*

**Now available:*

Eric Heller,

The Semiclassical Way

Princeton University Press (2018)

Figure 10.3: NASA-JPL early blunder. Rockets are not rigid bodies, especially with floppy whip antennas attached. The Explorer 1 satellite was the first one launched successfully by the United States. Seen in the left panel are James van Allen (center), William Pickering (left), and Werner von Braun, with a full-size model of the satellite, just after it was successfully orbited in 1958. As this press conference took place, the satellite was busily tumbling out of control. Van Allen soon realized that the intermittent signal from the satellite was due to tumbling. Fortunately, enough antennas were bristling from the satellite that it still gave much useful data, resulting in discovery of the van Allen radiation belts. The tumbling took place because friction due to slight wobbling is converted to heat, lowering the rotational energy, but not changing the angular momentum. The only way for this to happen is for the satellite to start rotating around a lower energy axis, until it bottoms out in end and over and tumbling at the lowest possible rotational energy for the given angular momentum. The author thanks Prof. William Harter for pointing out the existence and the physics of this story.

Rotational Energy Surfaces (RES) and Constant Energy Surfaces (CES) replace Lagrange Poinsot $\frac{1}{2}\omega\bullet I\bullet\omega$

Rotational Energy Surface (RES) is* quadratic multipole function plotted radially

$$E=\frac{\mathbf{J}_x^2}{2I_x}+\frac{\mathbf{J}_y^2}{2I_y}+\frac{\mathbf{J}_z^2}{2I_z}\quad \text{with } J=const.$$

$$=J^2\left(\frac{\sin^2\theta\cos^2\phi}{2I_x}+\frac{\sin^2\theta\sin^2\phi}{2I_y}+\frac{\cos^2\theta}{2I_z}\right)$$

Constant Energy Surface (CES) is asymmetric ellipsoid of constant E

$$E=\frac{\mathbf{J}_x^2}{2I_x}+\frac{\mathbf{J}_y^2}{2I_y}+\frac{\mathbf{J}_z^2}{2I_z}=const.$$

$$or:\quad \frac{\mathbf{J}_x^2}{2EI_x}+\frac{\mathbf{J}_y^2}{2EI_y}+\frac{\mathbf{J}_z^2}{2EI_z}=1$$

Here notation L or $\mathbf{L}$ for angular momentum is replaced by J or $\mathbf{J}$

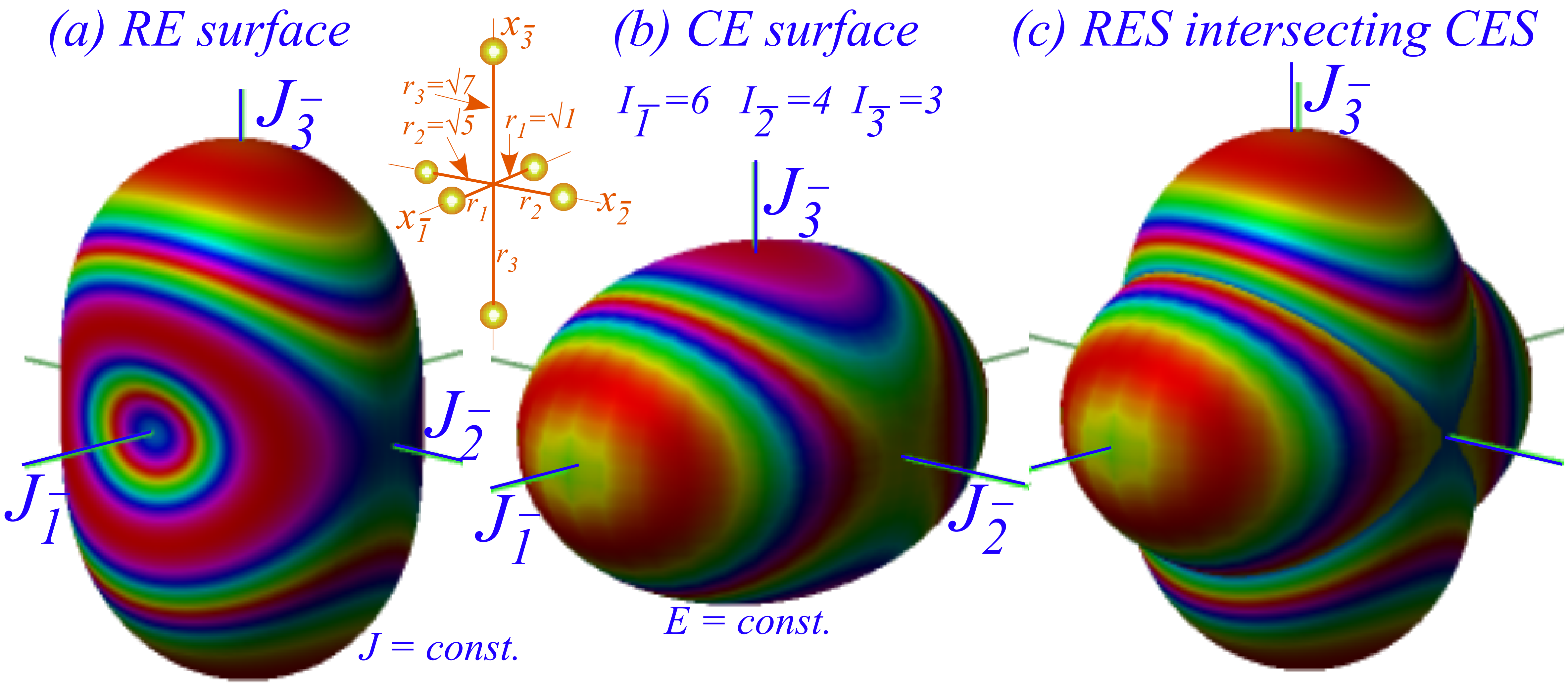


Fig. 6.8.1 Rigid rotor surfaces (a) RES polynomial, (b) CES ellipsoid, and (c) RES and CES intersected.

* First published in:
J. Chem. Phys. 80, 4241 (1984)
W. G. Harter and C. W. Patterson

RES and CES for nearly-symmetric prolate rotors and nearly-symmetric oblate rotors

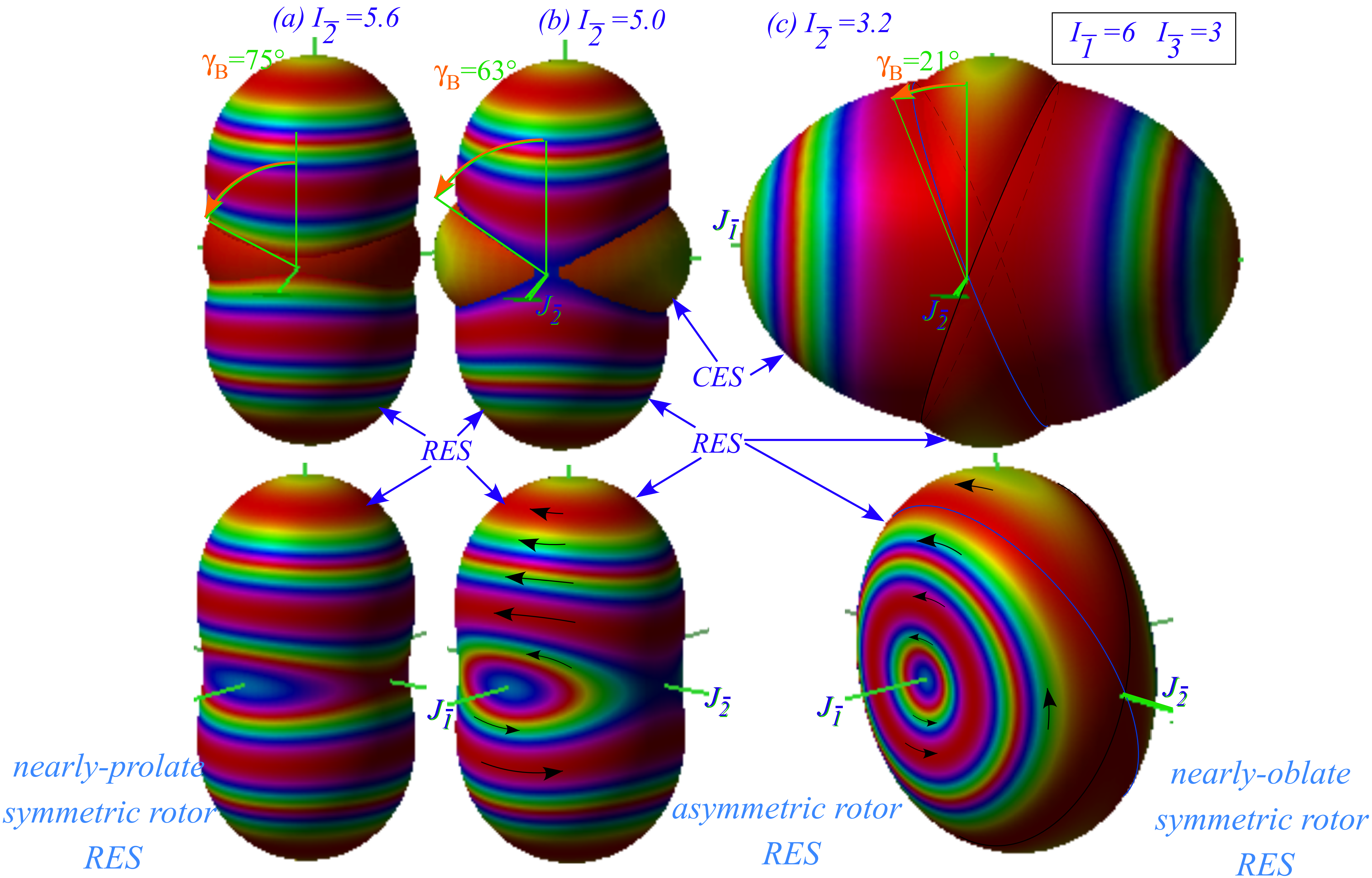
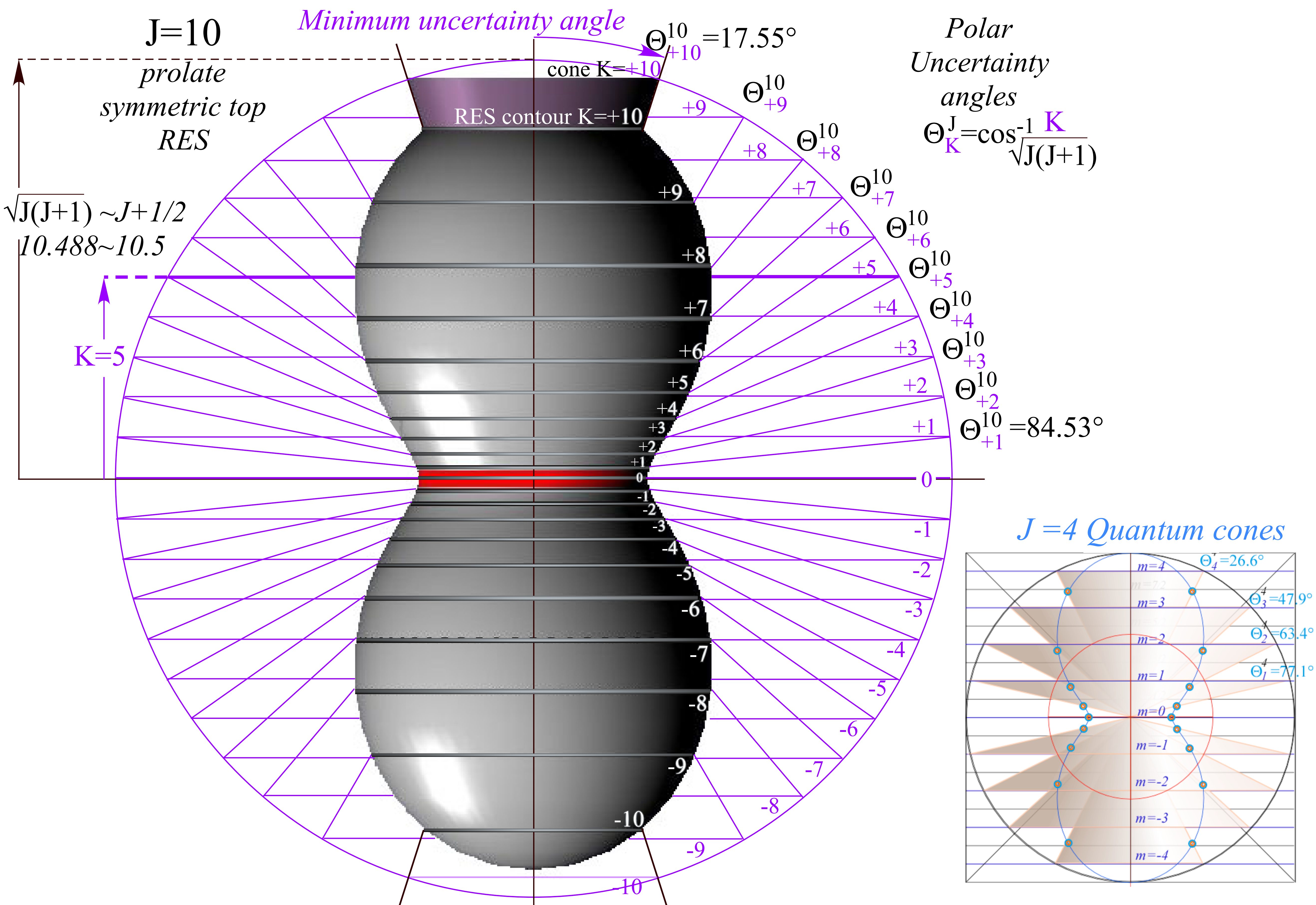


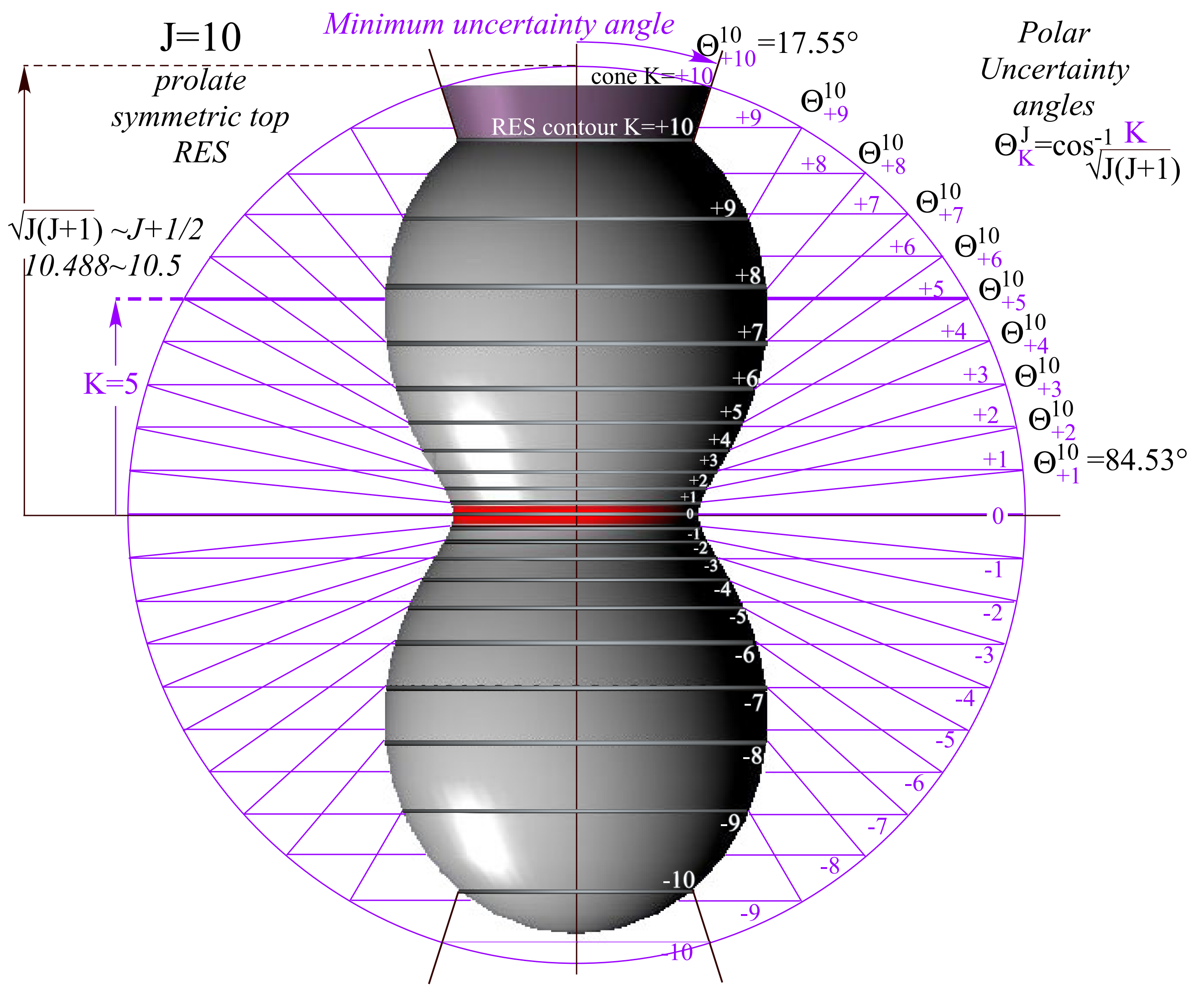
Fig. 6.8.2 Fixed-J- RES with CES at separatrix $E=J^2/2I_{\bar{2}}$ *as* $I_{\bar{2}}$ *varies. (a)* $I_{\bar{2}}$=5.6 and γ_B=75.5° *(Nearly prolate low-E CES), (b)* $I_{\bar{2}}$=5.0 and γ_B= 63.4°, *(c)* $I_{\bar{2}}$=3.2 and γ_B= 20.7° *(Nearly oblate high-E CES).*

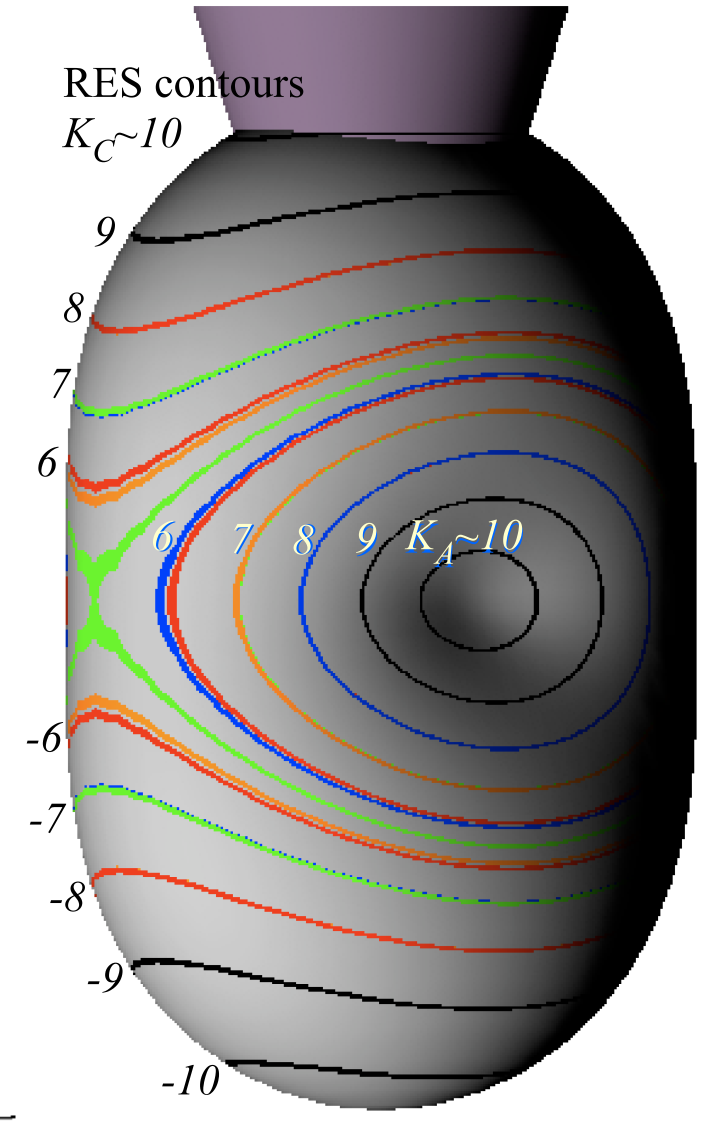
RES for symmetric prolate rotor locates J =10 quantum (-J<K<J) levels (at RES-quantum cone intersections)

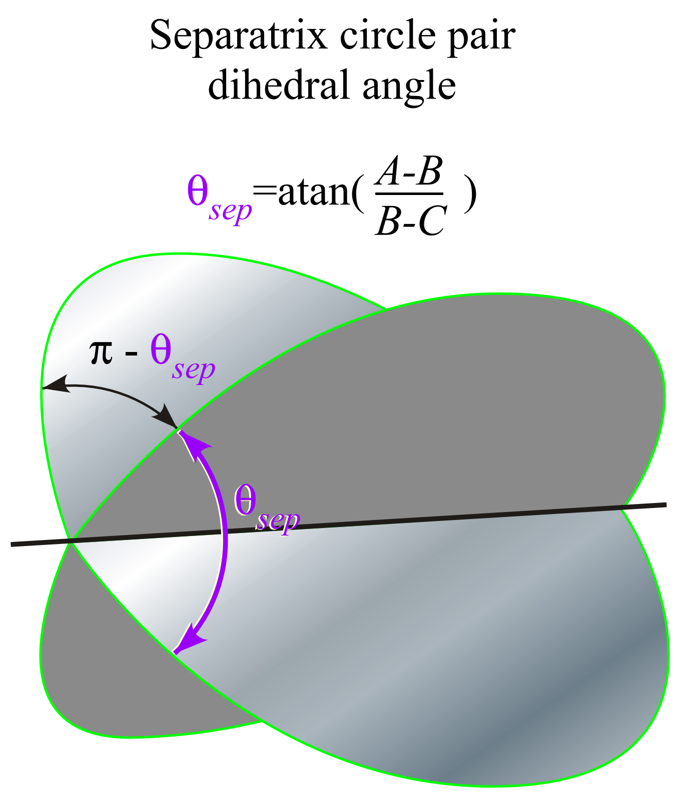


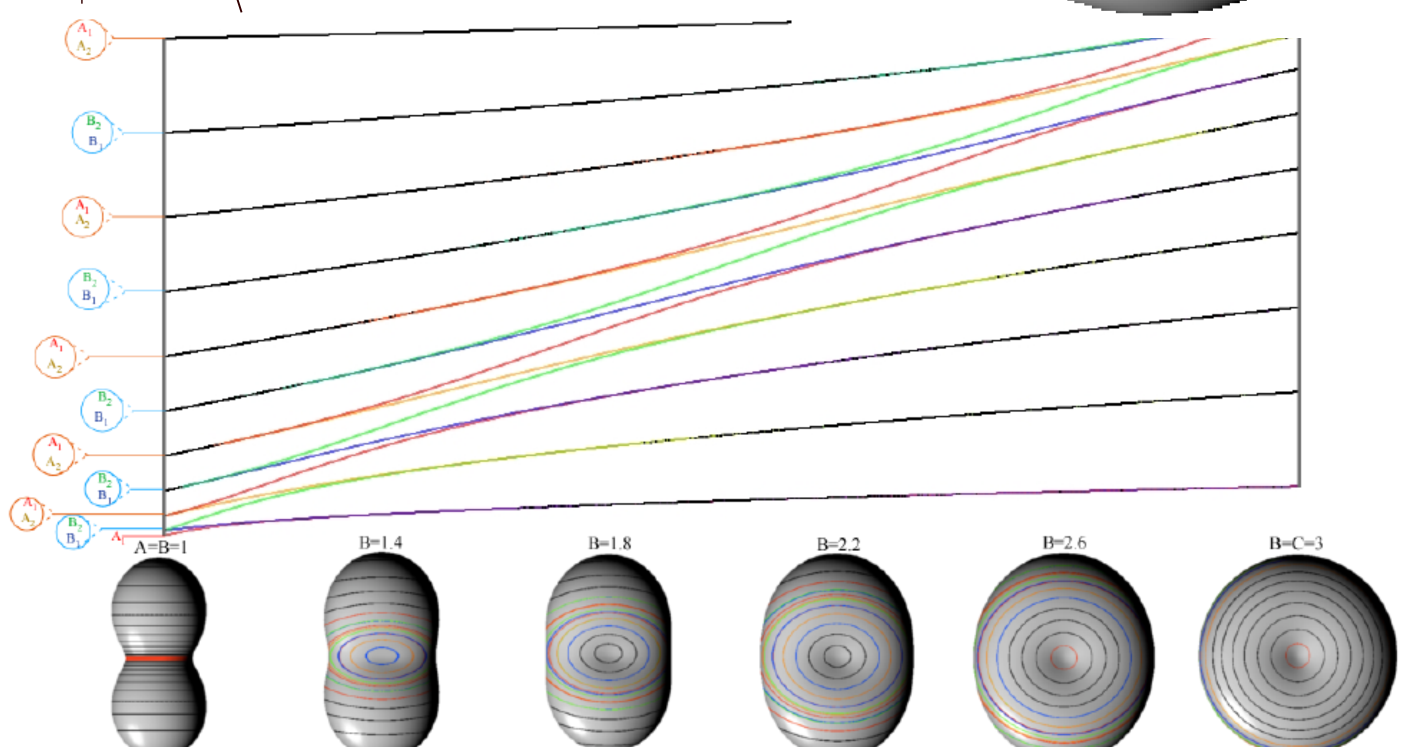
Link to pdf of: *W. G. Harter and J C. Mitchell ,International Journal of Molecular Science, 14, 714-806 (2013) Fig. 1-5 p.730*

RES for symmetric and asymmetric rotor approximates J =10 (-J<K<J) levels (near RES-quantum cone levels)







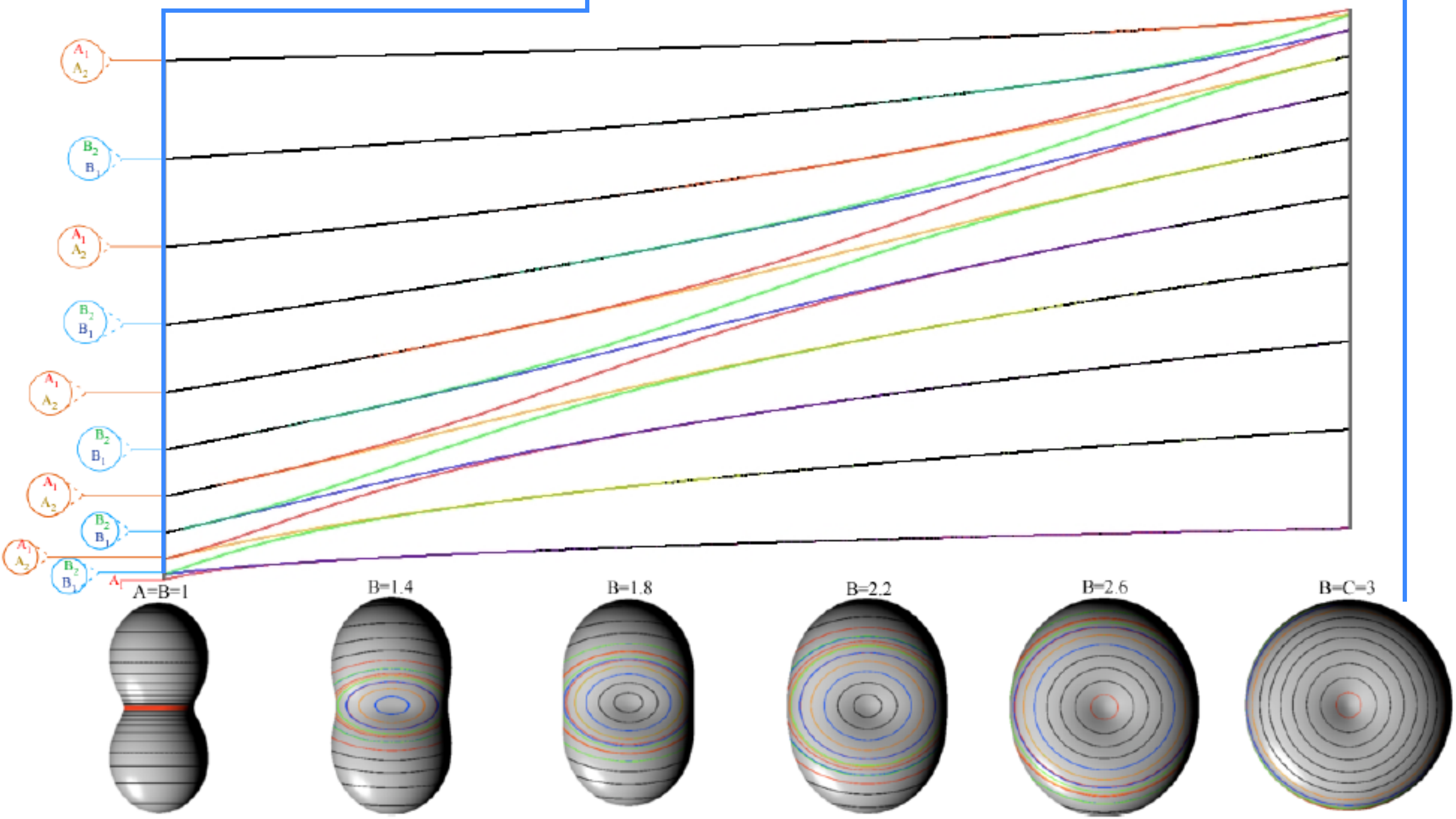


Link to pdf of: *W. G. Harter and J C. Mitchell ,International Journal of Molecular Science, 14, 714-806 (2013) Fig. 1-5 p.730*

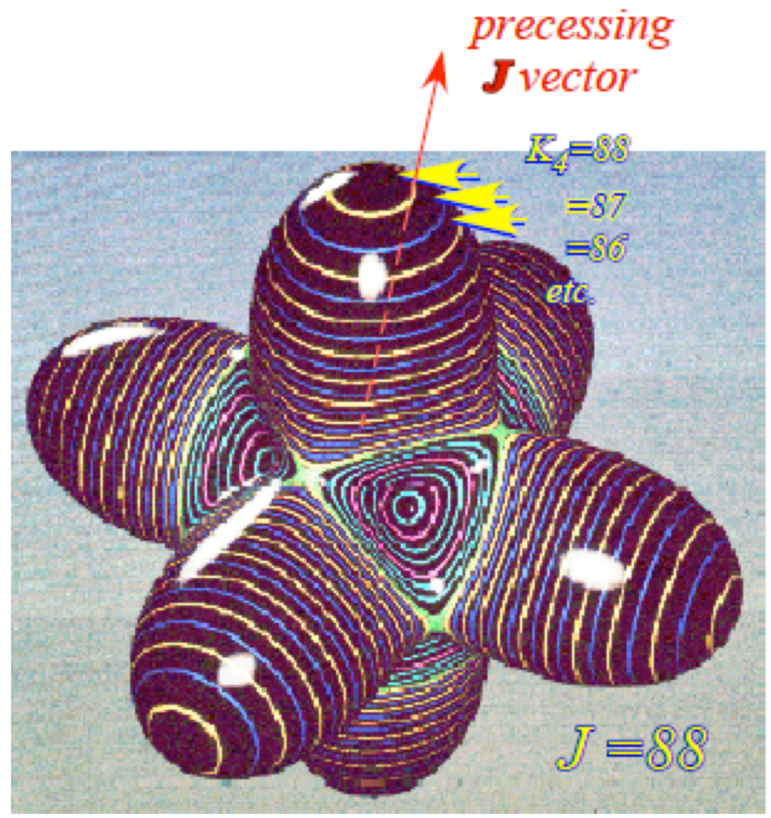
RES for symmetric prolate rotor locates $J=10$ quantum $(-J<K<J)$ levels (at RES-quantum cone intersections)

$$E = A\mathbf{J}_x^2 + B\mathbf{J}_y^2 + C\mathbf{J}_z^2 \quad \text{with } J = const.$$

Spectra varies as symmetric prolate RES changes through a range of asymmetric RES to oblate RES



Link to pdf of: *W. G. Harter and J C. Mitchell ,International Journal of Molecular Science, 14, 714-806 (2013) Fig. 1-5 p.730*



Rotational Energy Surfaces (RES)

→ *Symmetric, asymmetric, and spherical-top dynamics (Constant **J**)*

BOD-frame cone rolling on LAB frame cone

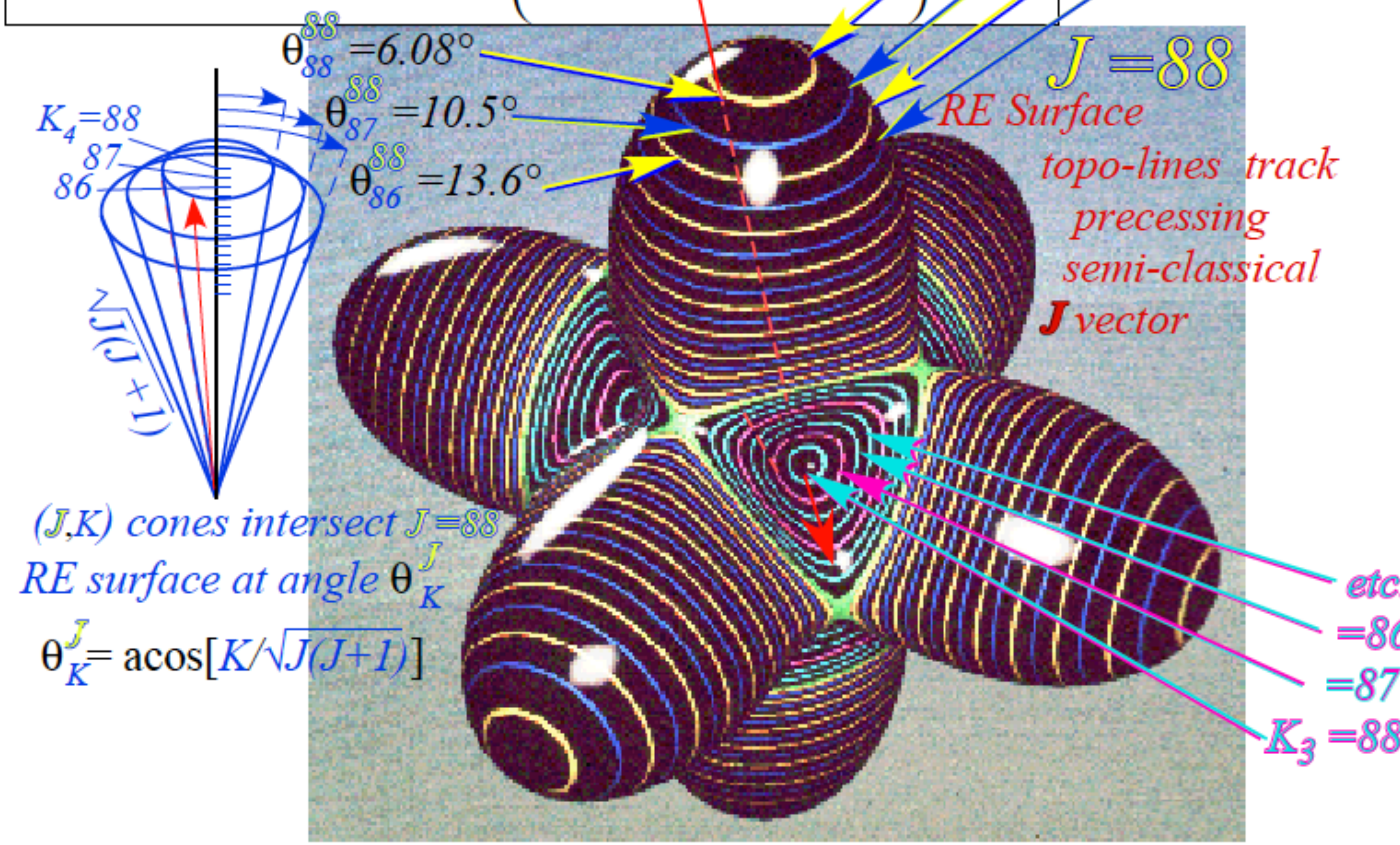
RES for spherical rotor approximates J =88 (-J<K<J) levels of SF_6

$\langle H\rangle \sim \nu_{vib} + BJ(J+1) + \langle H^{\text{Scalar Coriolis}}\rangle + \langle H^{\text{Tensor Centrifugal}}\rangle + \langle H^{\text{Tensor Coriolis}}\rangle + \langle H^{\text{Nuclear Spin}}\rangle + \ldots$

O_h or T_d Spherical Top: *(Hecht CH_4 Hamiltonian 1960)*

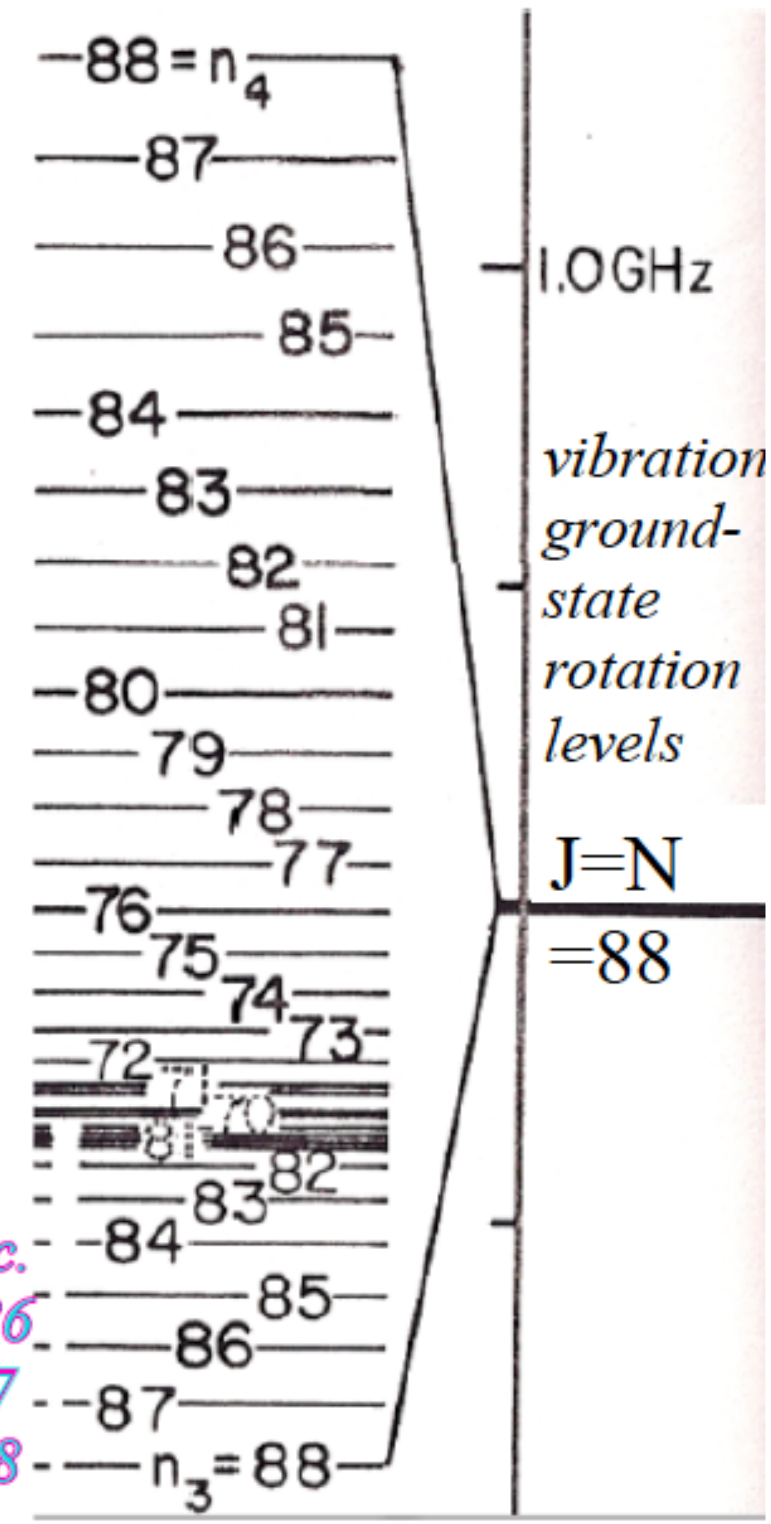
$$\mathbf{H} = B\left(\mathbf{J}_x^2 + \mathbf{J}_y^2 + \mathbf{J}_z^2\right) + t_{440}\left(\mathbf{J}_x^4 + \mathbf{J}_y^4 + \mathbf{J}_z^4 - \frac{3}{5}J^4\right) + \cdots$$

$$= BJ^2 + t_{440}\left(\mathbf{T}_0^4 + \sqrt{\frac{5}{14}}\left[\mathbf{T}_4^4 + \mathbf{T}_{-4}^4\right]\right) + \cdots$$



(J,K) cones intersect J =88
RE surface at angle θ^J_K

$\theta^J_K = \mathrm{acos}[K/\sqrt{J(J+1)}]$



(next page shows slice)

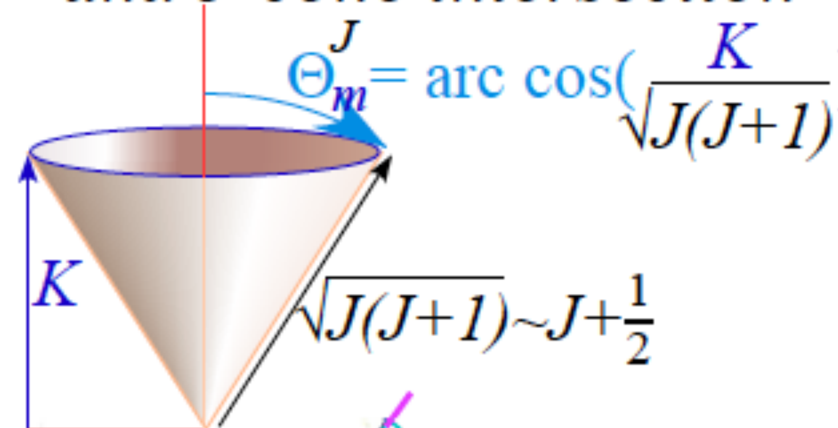
Link to pdf of: *W. G. Harter and J C. Mitchell ,International Symposium on Molecular Spectroscopy, OSU Columbus (2009)*

SF_6 Spectra of O_h Ro-vibronic Hamiltonian described by RE Tensor Topography

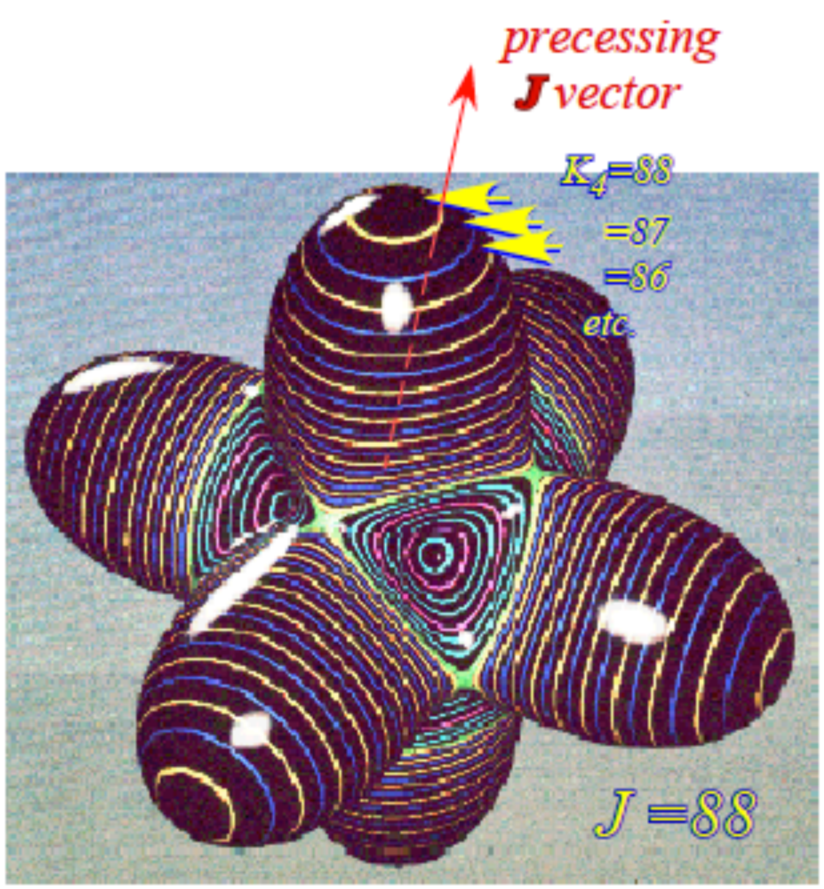
and J-cone intersection

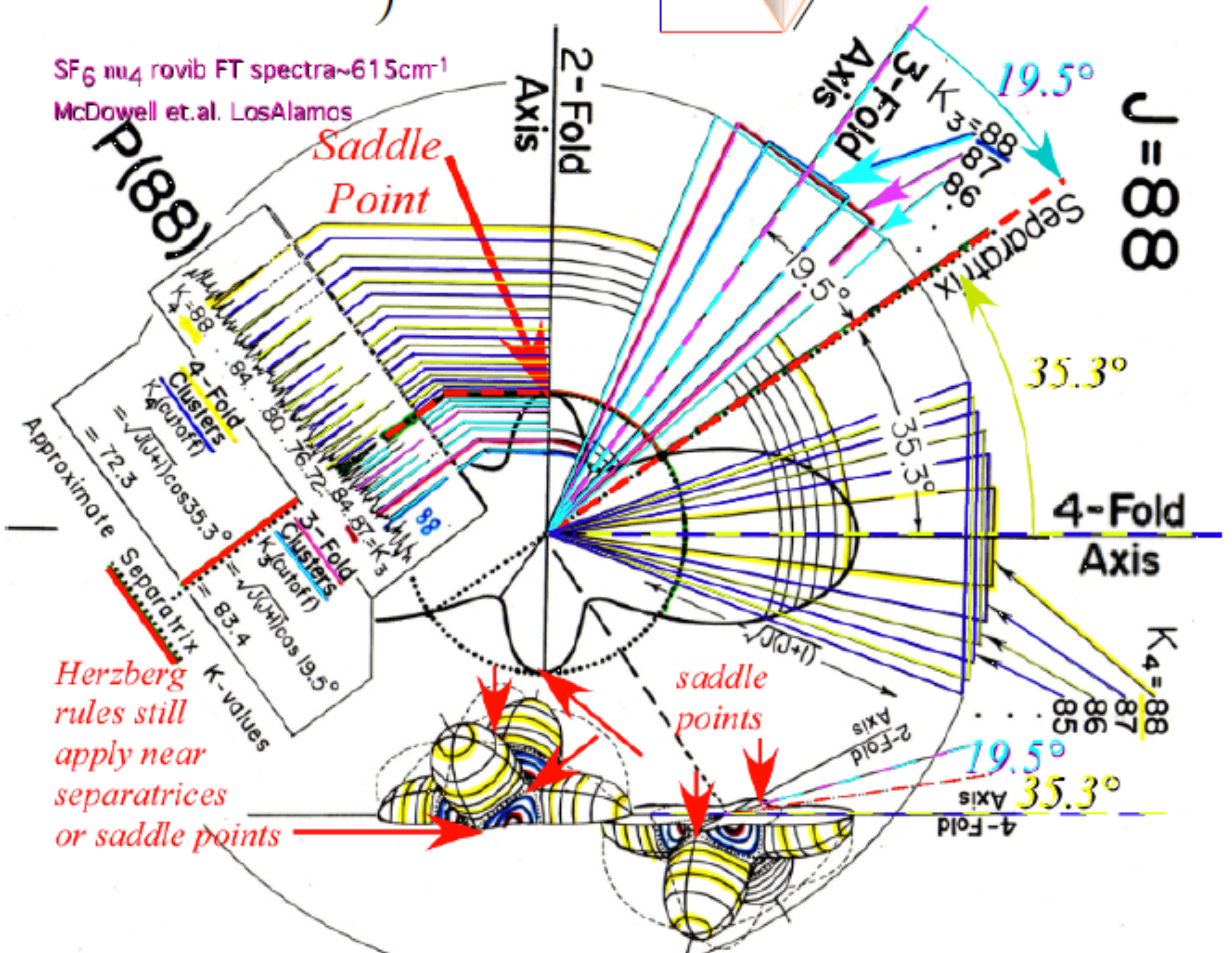
$$\mathbf{H} = B\left(\mathbf{J}_x^2 + \mathbf{J}_y^2 + \mathbf{J}_z^2\right) + t_{440}\left(\mathbf{J}_x^4 + \mathbf{J}_y^4 + \mathbf{J}_z^4 - \frac{3}{5}J^4\right) + \cdots$$

$$= \quad B\mathbf{J}^2 \quad + t_{440}\left(\mathbf{T}_0^4 + \sqrt{\frac{5}{14}}\left[\mathbf{T}_4^4 + \mathbf{T}_{-4}^4\right]\right) + \cdots$$



Rovibronic Energy (RE) Tensor Surface



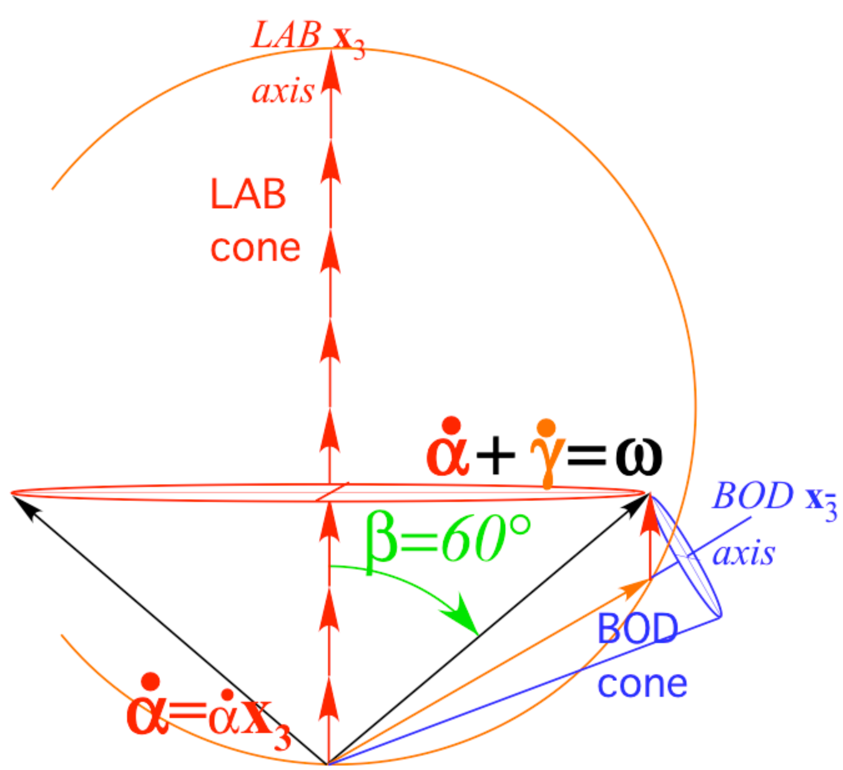


Link to pdf of: *W. G. Harter and J C. Mitchell ,International Symposium on Molecular Spectroscopy, OSU Columbus (2009)*

Rotational Energy Surfaces (RES)

Symmetric, asymmetric, and spherical-top dynamics (Constant **J***)*

→ *BOD-frame cone rolling on LAB frame cone*



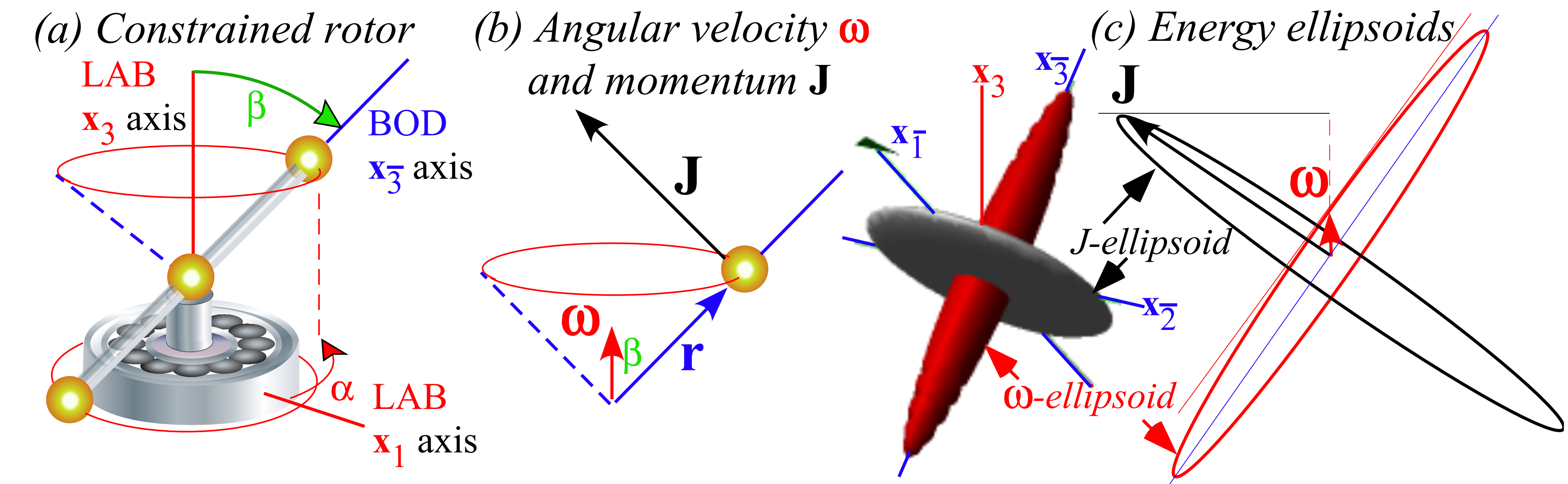


Fig. 6.7.1 Elementary ω-constrained rotor and angular velocity-momentum geometry.

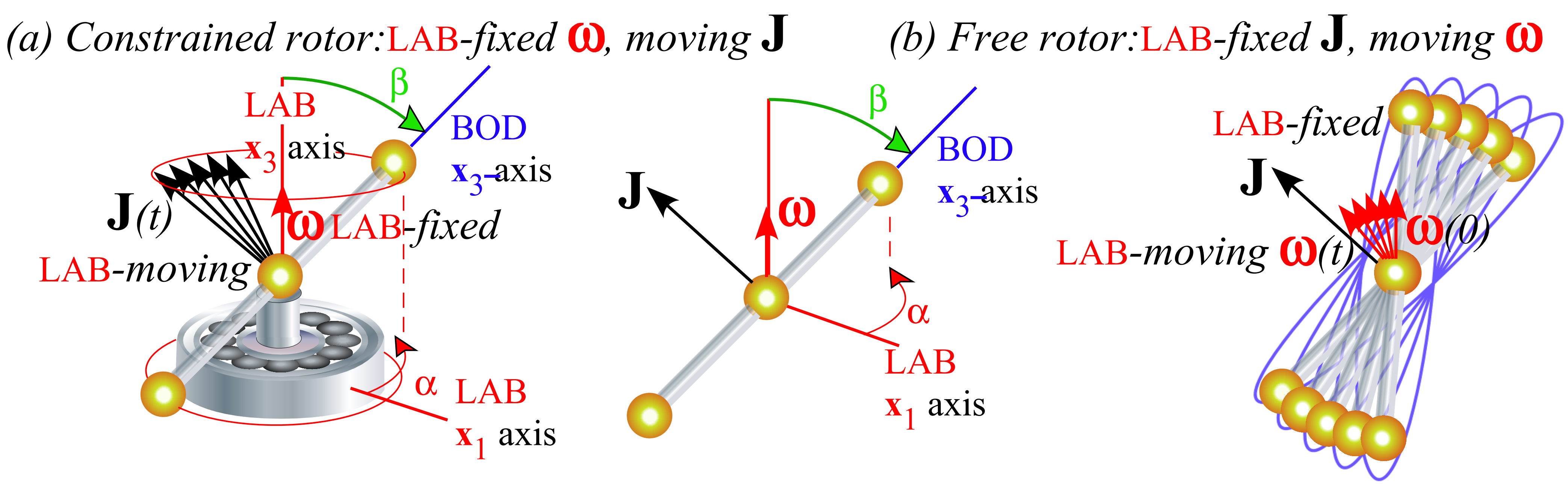
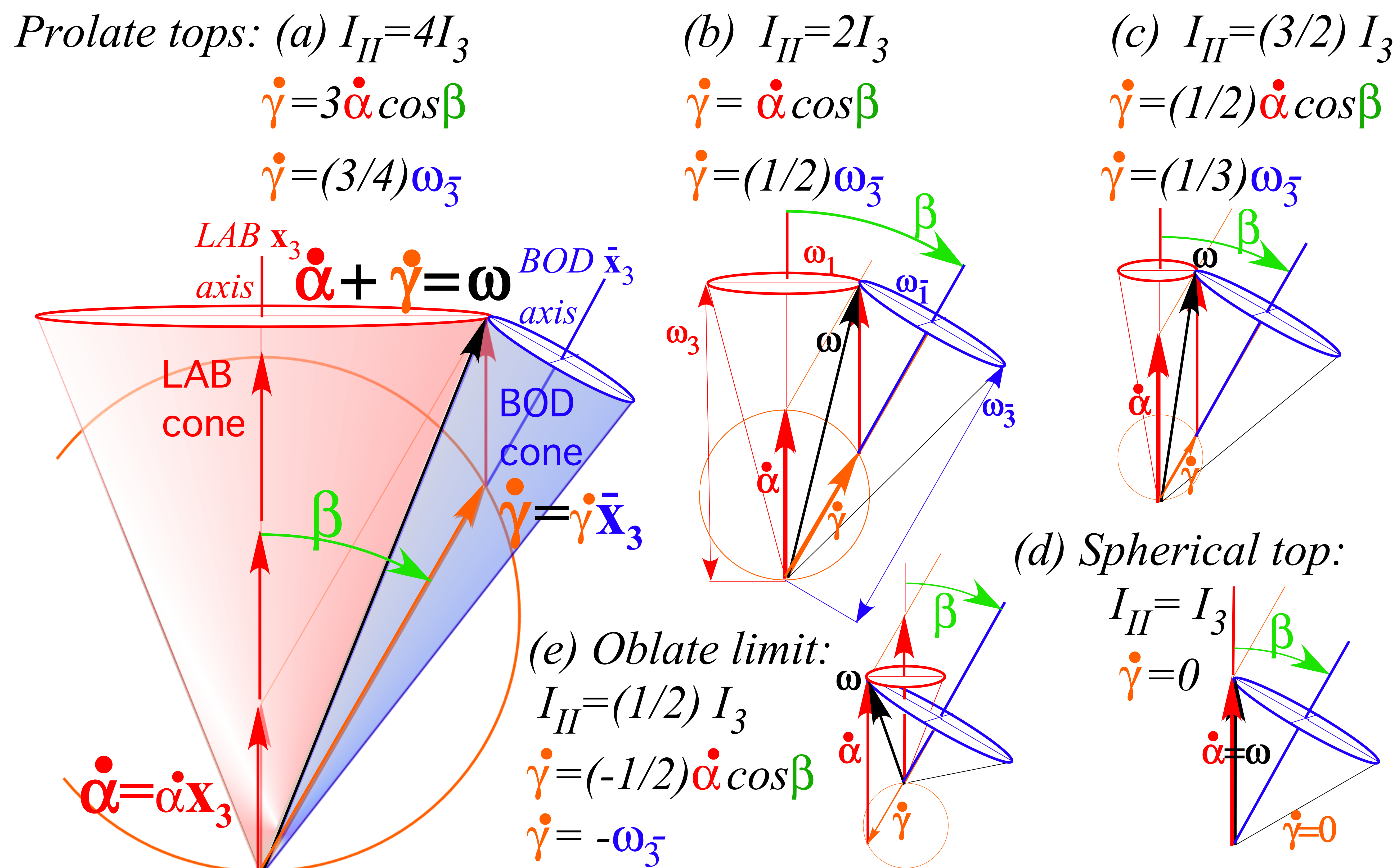


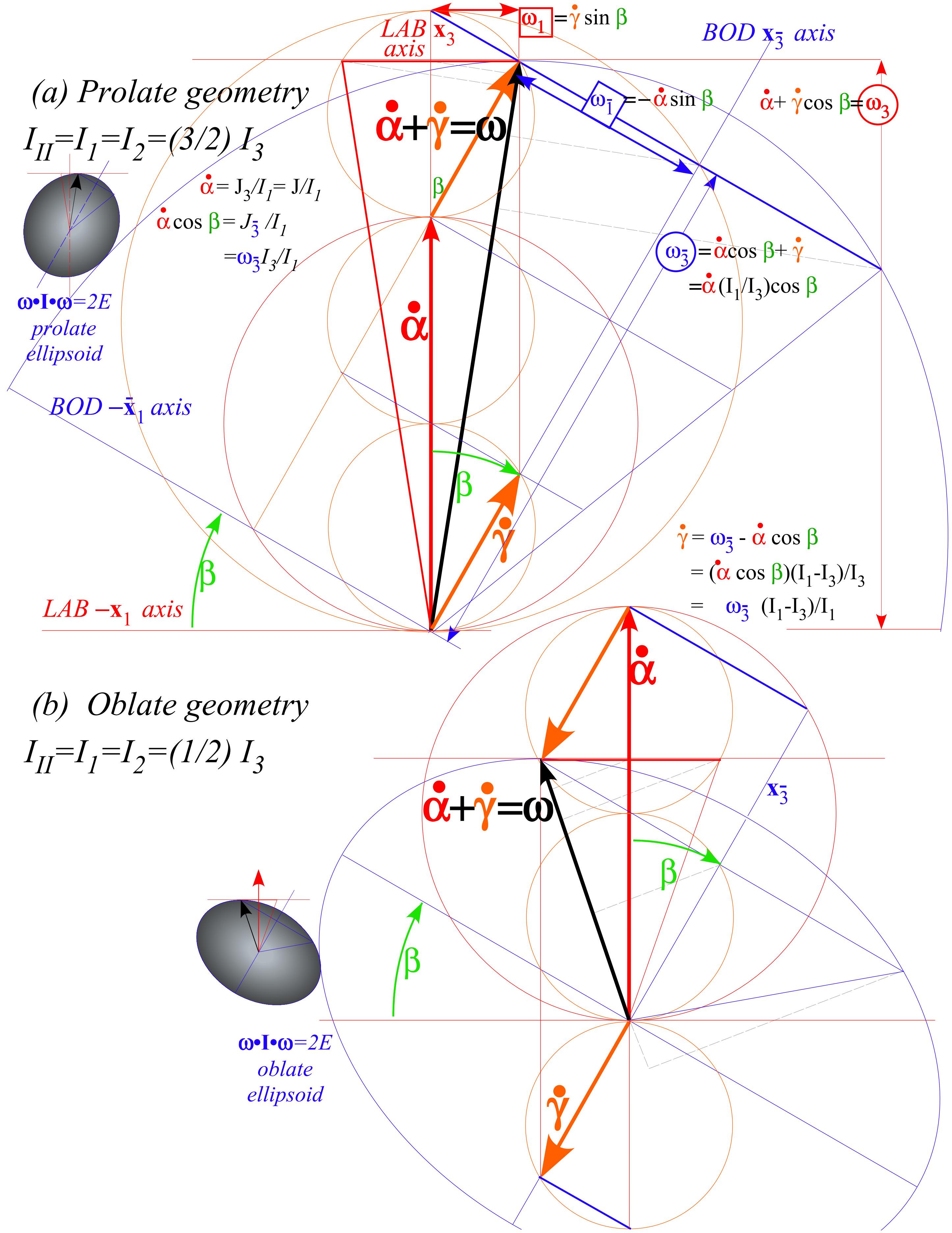
Fig. 6.7.2 Free rotor cut loose from LAB-constraining ω-axis changes dynamics accordingly.

..this was the kind of dynamics that started me dropping superballs...



Blue BOD-frame cones roll (around ω-sticking axis) without slipping on red LAB-frame cone

Fig. 6.7.3 Symmetric top ω-cones for β=30° and inertial ratios: (a) $\frac{I_{II}-I_3}{I_3}=3$, *(b) 1, (c)* $\frac{1}{2}$, *(d) 0, (e)* $-\frac{1}{2}$.



Blue BOD-frame cones roll without slipping on red LAB-frame cone

Fig. 6.7.4 Detailed geometry of symmetric top kinetics. (a) Prolate case. (b) Most-oblate case

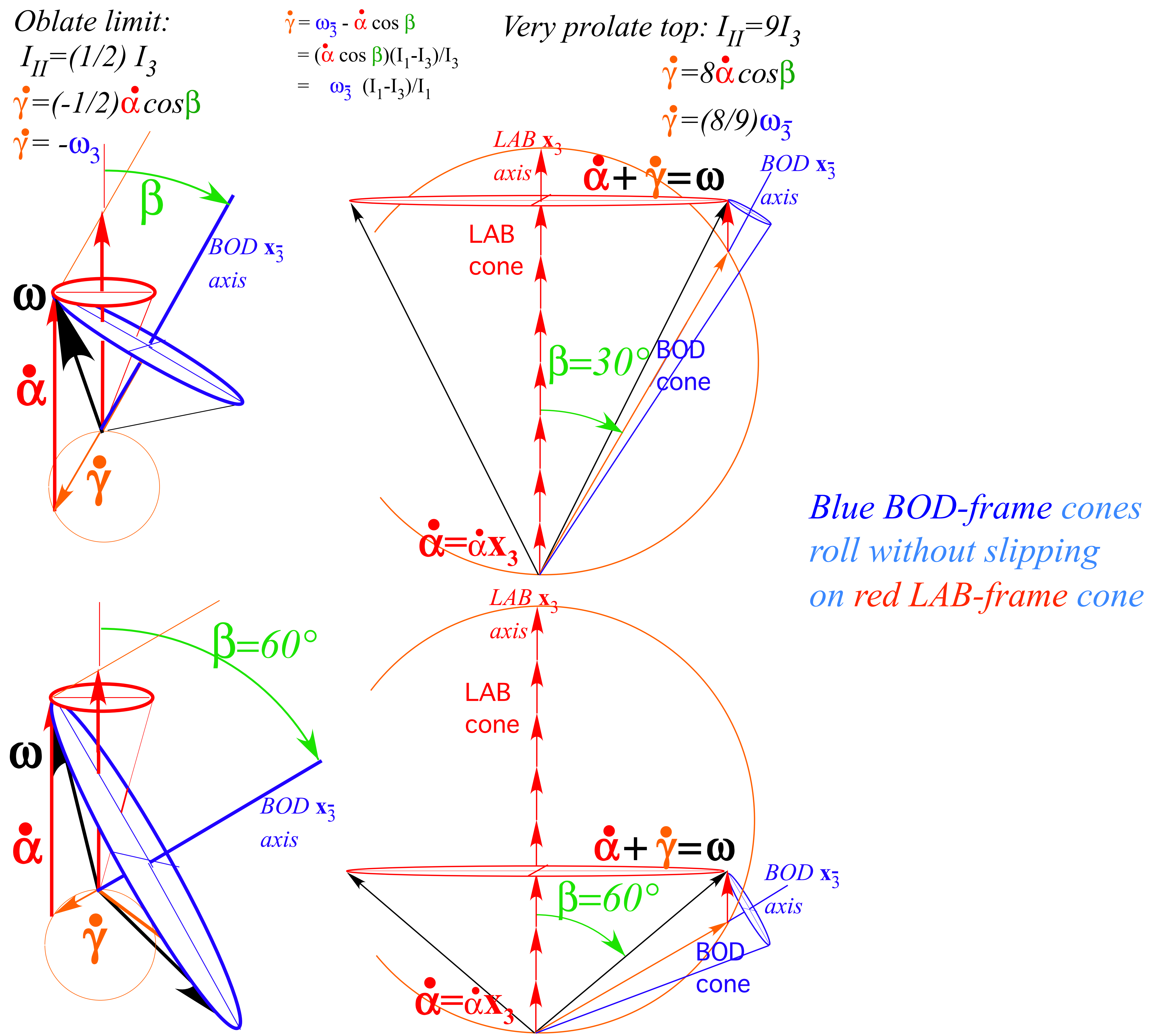
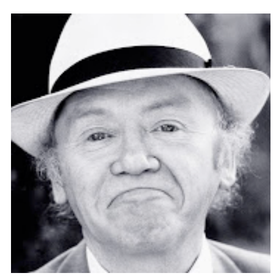


Fig. 6.7.5 Extreme cases (Oblate vs. Prolate) of symmetric-top geometry.

The following treatment of lever rotation was covered in Lecture 18 (2019)

Cycloidal geometry of flying levers
Practical poolhall application
(See above link)

Now enjoy this famous hit physics song…



Aluminum boomerang from p.43

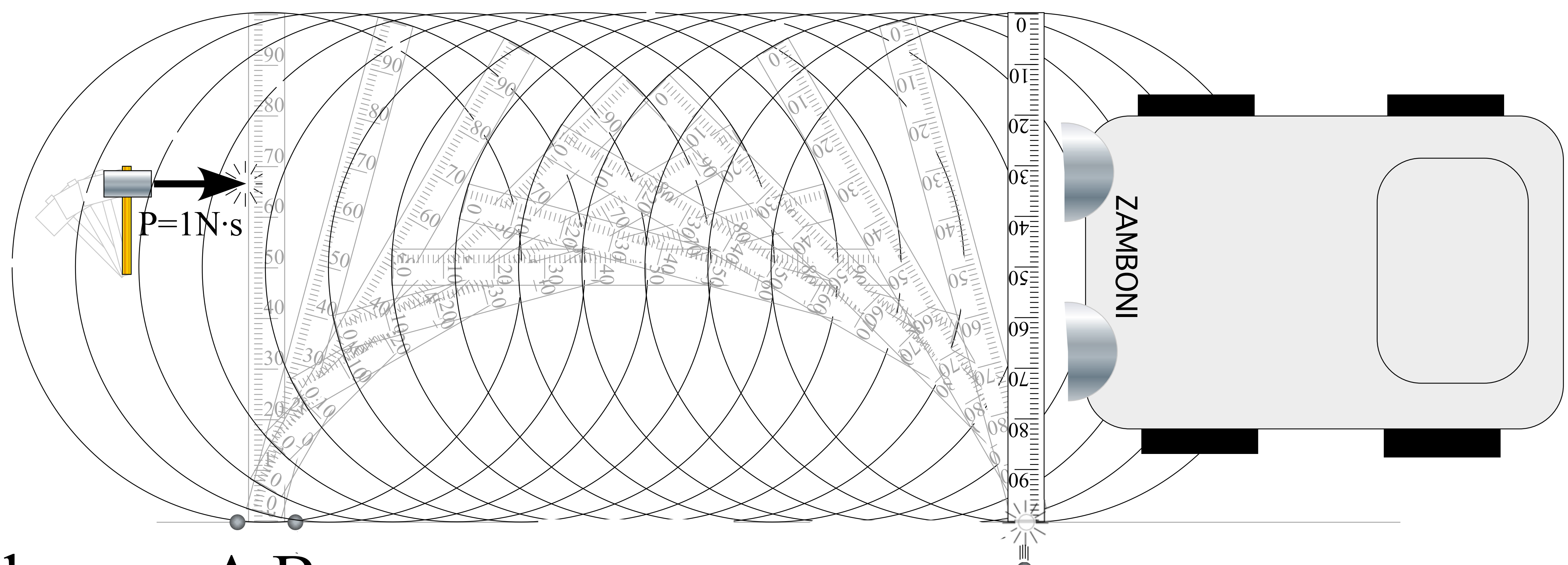
https://www.youtube.com/playlist?list=PLGwmGldCxzLxbPlFVG8Z89WZIBuT4m0Ii





The Zamboni-Ice-Shot problem

(Assumes frictionless ice rink)



Marbles: A B

C

Where on a meter-stick do you hit it so as to not disturb marbles A or B and...

...knock marble C down as shown.