Lect. 27 Self-study of (R,γ) parameters for ruler-compass graph-paper constructions Wed. 11.25.2019

Geometry and Symmetry of Coulomb Orbital Dynamics

Review of lecture 26 *(Ch. 2-4 of Unit 5 12.05.17)*

Eccentricity vector **ε** *and (ε,λ)-geometry of orbital mechanics*

Analytic geometry derivation of **ε***-construction*

Connection formulas for (a,b) and (ε,λ) with (γ,R)

Detailed ruler & compass construction of **ε***-vector and orbits*

(R=-0.375 elliptic orbit)

(R=+0.5 hyperbolic orbit)

Properties of Coulomb trajectory families and envelopes

Graphical **ε***-development of orbits*

Launch angle fixed-Varied launch energy

Launch energy fixed-Varied launch angle

Launch optimization and orbit family envelopes

Geom.of Coulomb Scatt.AJP 40 4 (1972)

Lenz Vector...analog computersAJP 44 4 (1974)

The Lenz vector and orbital analog computers*

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A single geometrical diagram based on the Lenz vector shows the qualitative and quantitative features of all three types of Coulomb orbits. A simple analog computer can be made for an overhead projector by using this theory, and a number of interesting effects can be efficiently demonstrated.

I. INTRODUCTION: THE ECCENTRICITY VECTOR

Occasionally, a geometrical construction and accompanying picture is worth a great deal more to the physicist or the physics student than pages of equations and solutions, especially now that computer graphics are so available. Since Newton's time the geometrical approach has come to be regarded as more clumsy than other methods of thought, and some very pretty pictures and proofs of physical phenomena have undoubtedly been lost. An example of such a construction involving Rutherford scattering was discussed in a recent article[1] by my students and myself, and the following is an improvement of this which describes general Coulombic orbit mechanics.

The generalization we shall describe below is based partly on a more recently discovered quantity called the Lenz–Runge vector[2,3] or the "eccentricity" vector $\boldsymbol{\epsilon}$ defined by Eq. (1). There $\mathbf{r}$ is the position vector of the orbiting particle, $\mathbf{L}$ is its angular momentum, $\mathbf{p}$ is its linear momentum, m is its mass, and k is the gravitational (or electrostatic) coefficient:

$$\boldsymbol{\epsilon} = \mathbf{r}/r - \mathbf{L}\times\mathbf{p}/km. \tag{1}$$

Lately this quantity has received a flurry of attention in group theoretical studies of the hydrogen atom[4]; however, we shall use only its geometrical and classical properties.

In particular, the main property of $\boldsymbol{\epsilon}$ is that it is a constant vector for any particle moving according to a Coulomb field. Vector $\boldsymbol{\epsilon}$ points along the major axis of ellipse, parabola, or hyperbola, whichever is the appropriate orbit of the particle. Furthermore, the magnitude ϵ of this vector is the eccentricity of the orbit.

To show that this is consistent with the usual formulation, we take the dot product of this vector $\boldsymbol{\epsilon}$ with the position $\mathbf{r}$ as in Eq. (2). This then reduces to the following equation (3) of a conic section in polar coordinates, which is the general orbit equation[5]:

$$\epsilon r\cos\theta = \boldsymbol{\epsilon}\cdot\mathbf{r} = r - \mathbf{L}\times\mathbf{p}\cdot\mathbf{r}/km \tag{2}$$

$$= r + \mathbf{L}\cdot\mathbf{L}/km,$$

$$r = -(L^2/km)(1-\epsilon\cos\theta)^{-1}. \tag{3}$$

In Sec. II a simple geometric construction using these properties is shown to describe qualitatively and quantitatively the Coulomb orbits for all three cases: namely, the attractive case ($k<0$) with positive energy, with negative energy, and the repulsive case ($k>0$).

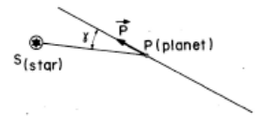

Fig. 1. Initial position and momentum must be given before construction of the resulting orbit is possible.

Finally, it is shown how this construction leads to an analog computer of orbits that can be made for a few dollars to fit onto an overhead projector. This device can be appreciated by elementary classes (even *large* elementary classes if you use the right projector) when they know only a little about conic sections, since the more tedious mathematics is built into the device.

II. COMPUTING ORBITS BY RULER AND COMPASS

We start by simply listing three steps of an orbit construction while demonstrating their application to a particular case of a satellite orbiting a star. Then a general proof of the steps will be provided along with further discussion and applications.

Suppose you are given the initial position and velocity of a satellite relative to some very massive star. If these quantities are given in a pictorial form which shows the angle γ between momentum vector $\mathbf{p}=m\mathbf{v}$ and the radius line PS in Fig. 1, and if the magnitude of $\mathbf{v}$ is given by the ratio $R=T/V$ of the kinetic energy ($T=mv^2/2$) to the potential energy ($V=k/r$), then the construction below proceeds immediately. Otherwise, these quantities must be calculated before proceeding. (In the potential energy of the star's gravity we have $k=-GMm$, where M is the star mass and G is the universal constant of gravitation.) Note that R is minus the squared ratio of initial velocity to the escape velocity in

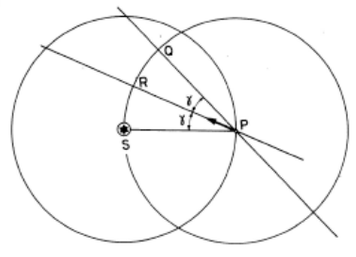

Fig. 2. Doubling the angle between the momentum and the position vectors gives a line QP which must contain the orbit focus.

Review of lecture 26

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

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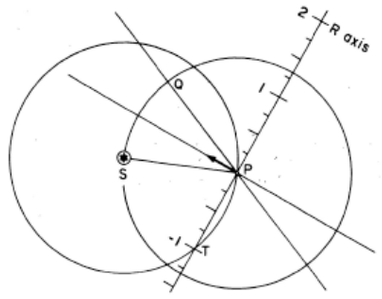


Fig. 3. A line perpendicular to the momentum will serve as an energy scale, linear in the initial R = KE/PE.

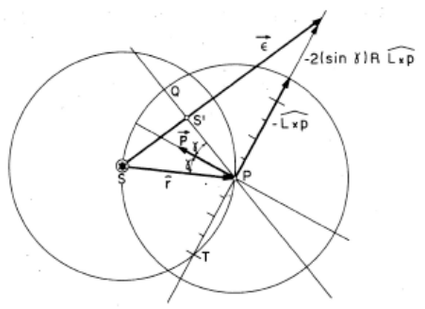


Fig. 5. The algebra of the Lenz vector is shown to correspond with the construction of Figs. 1–4. (This figure is drawn for positive R.)

the case of attractive Coulomb force. For our example below we will take R to be $-¼$. The construction is given now.

Step 1. Draw circles around and through the star S and satellite or planet P as shown in Fig. 2. Mark off arc QS so it is double the arc RS on the circle around the planet, and draw straight line through P and Q.

Step 2. Copy the arc SRQ onto the arc PT of the circle around the star. A line drawn through P and T should be perpendicular to $\mathbf{p}$ and will serve as the $R = T/V$ axis. Point P corresponds to $R = 0$ while T corresponds to $R = -1$, and the R scale is linear (see Fig. 3).

Step 3. Locate the given ratio on the R scale (for the example we take $R = -¼$) and extend a line from the star S through this point until it intersects the line QPS′ at S′. Point S′ is the second focus of the conic section orbit tangent to vector $\mathbf{p}$. In Fig. 4 the appropriate ellipse has been drawn by using the foci S and S′.

Had we picked $R < -1$, the intersection S′ on line QP would have been in the upper left-hand side of Fig. 4, whereupon S′ and S would be foci of a *hyperbola* tangent to $\mathbf{p}$.

The first thing to understand about the construction is step 1, where we claim that line QP must contain the other focus. This is easier to see for the ellipse where the physical definition of the foci involves the emission of light or sound at one focus followed by the convergence at the other. Then step 1 makes the angle of incidence at P equal to the angle of reflection. Similar properties hold for the hyperbola except that the reflecting surfaces must be *normal* to the curve.

So we are left with the problem of showing where on line QP the other focus resides. The solution to this involves the construction of the eccentricity vector ϵ of Eq. (1), which is done in steps 2 and 3. To prove this is correct, we rewrite Eq. (1) in the form below (the notation $\hat{}$ is used to denote unit vectors:

$$\begin{aligned}\epsilon &= \hat{r} - \mathbf{L}\times\mathbf{p}/km \\ &= r - [(mvr)(mv)\sin\gamma/km]\mathbf{L}\hat{\times}\mathbf{p}, \\ \epsilon &= \hat{r} - \frac{mv^2/2}{k/r}\,2\sin\gamma\mathbf{L}\hat{\times}\mathbf{p}, \\ \epsilon &= \hat{r} - 2R\sin\gamma\mathbf{L}\hat{\times}\mathbf{p}.\end{aligned} \tag{4}$$

This is labeled in Fig. 5 where, for convenience, the ratio R is taken to be positive in order to correspond to a repulsive ($k > 0$) Coulomb field, wherein the focus S′ falls between Q and P for all initial speeds. However, the generality of the construction for all cases can be seen.

With this construction in mind now, one may think of quite a number of orbit problems which can be solved geometrically. For example, returning to the attractive gravity of the star S in Fig. 1, suppose we desired to find in what direction (if any) a projectile at P could be launched so that it would pass through the point P′ in Fig. 6(a), if its initial speed is assumed to be such that we have the energy ratio $R = -¼$.

Now it is well known that the major axis $2a$ of an orbit is a function of energy of the orbiting particle only.[6] For the orbit drawn in Fig. 4 this distance $2a$ is equal to the sum of distances SP and PS′. Now we propose to make a projectile at P with the same energy (with $R = -¼$) go through P′ in Fig. 6(a). Such an orbit must have its second focus on a circle around P that includes point S′. In Fig. 6(b) it is seen that two points, labeled S″ and S‴, on this circle give orbits through P and P′. These are found by marking distance SM′ on line SP′ equal to the distance SM (this distance is the desired axis length $2a$) and then finding S″ and S‴ at the intersections of a circle of radius P′M′ around P′ with the $R = -¼$ circle about P. The resulting two orbits are drawn in Fig. 6(c).

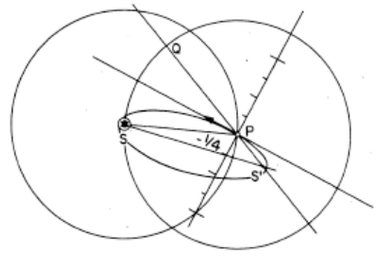


Fig. 4. The desired energy $R = -¼$ on the R scale points to the correct orbit focal point at S′ on the focus locus.

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ) *geometry of orbital mechanics*

Isotropic field $V=V(r)$ guarantees conservation *angular momentum vector* $\mathbf{L}$

(Review of Lect. 26)

$$\mathbf{L} = \mathbf{r}\times\mathbf{p} = m\,\mathbf{r}\times\dot{\mathbf{r}}$$

Coulomb $V=-k/r$ also conserves *eccentricity vector* $\boldsymbol{\varepsilon}$

$$\boldsymbol{\varepsilon} = \frac{\mathbf{p}\times\mathbf{L}}{km} - \frac{\mathbf{r}}{r} = \frac{\mathbf{p}\times(\mathbf{r}\times\mathbf{p})}{km} - \frac{\mathbf{r}}{r}$$

(...for sake of comparison...)

IHO $V=(k/2)r^2$ also conserves *Stokes vector* $\boldsymbol{S}$

$$S_A = \tfrac{1}{2}(x_1^2+p_1^2-x_2^2-p_2^2)$$
$$S_B = x_1p_1 + x_2p_2$$
$$S_C = x_1p_2 - x_2p_1$$

$\mathbf{A} = km\cdot\boldsymbol{\varepsilon}$ is known as the *Laplace-Hamilton-Gibbs-Runge-Lenz vector*. Generate symmetry groups: $U(2)\subset U(2)$ or: $R(3)\subset R(3)\times R(3)\subset O(4)$

Consider dot product of $\boldsymbol{\varepsilon}$ with a radial vector $\mathbf{r}$:

$$\boldsymbol{\varepsilon}\bullet\mathbf{r} = \frac{\mathbf{r}\bullet\mathbf{r}}{r} - \frac{\mathbf{r}\bullet\mathbf{p}\times\mathbf{L}}{km} = r - \frac{\mathbf{r}\times\mathbf{p}\bullet\mathbf{L}}{km} = r - \frac{\mathbf{L}\bullet\mathbf{L}}{km}$$

...or of $\boldsymbol{\varepsilon}$ with momentum vector $\mathbf{p}$:

$$\boldsymbol{\varepsilon}\bullet\mathbf{p} = \frac{\mathbf{p}\bullet\mathbf{r}}{r} - \frac{\mathbf{p}\bullet\mathbf{p}\times\mathbf{L}}{km} = \mathbf{p}\bullet\hat{\mathbf{r}} = p_r$$

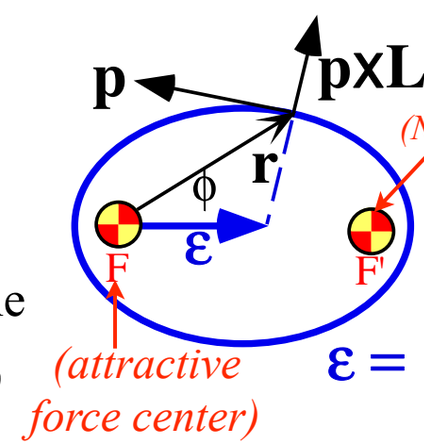
Let angle ϕ be angle between $\boldsymbol{\varepsilon}$ and radial vector $\mathbf{r}$

$$\varepsilon\, r\cos\phi = r - \frac{L^2}{km} \quad \text{or:} \quad r = \frac{L^2/km}{1-\varepsilon\cos\phi}$$

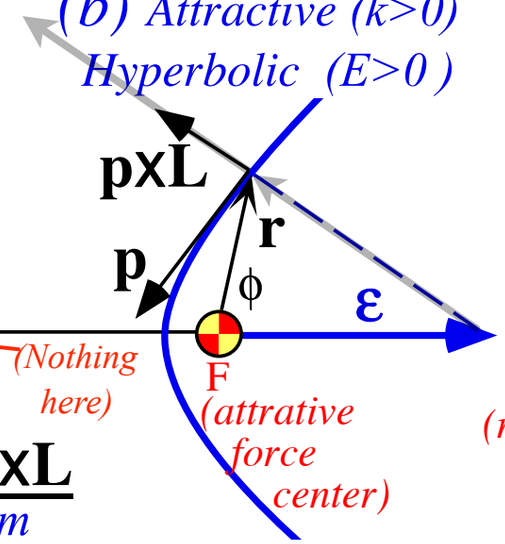
For $\lambda=L^2/km$ that matches: $r = \dfrac{\lambda}{1-\varepsilon\cos\phi} = \begin{cases} \dfrac{\lambda}{1-\varepsilon} & \text{if: } \phi=0 \quad \textit{apogee} \\ \lambda & \text{if: } \phi=\dfrac{\pi}{2} \quad \textit{zenith} \\ \dfrac{\lambda}{1+\varepsilon} & \text{if: } \phi=\pi \quad \textit{perigee} \end{cases}$

(Rotational momentum $\mathbf{L}=\mathbf{r}\times\mathbf{p}$ is normal to the orbit plane.)

(a) Attractive (k>0) Elliptic (E<0)

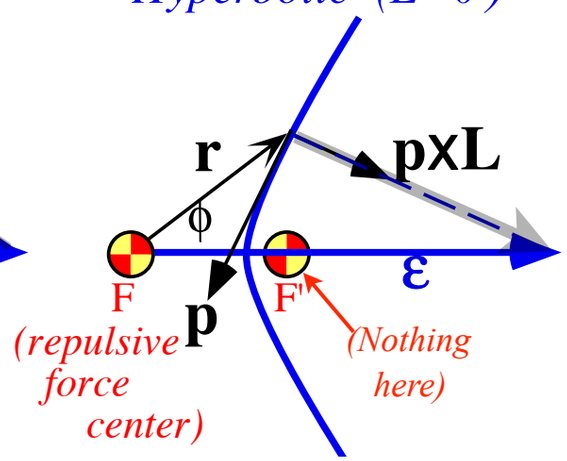


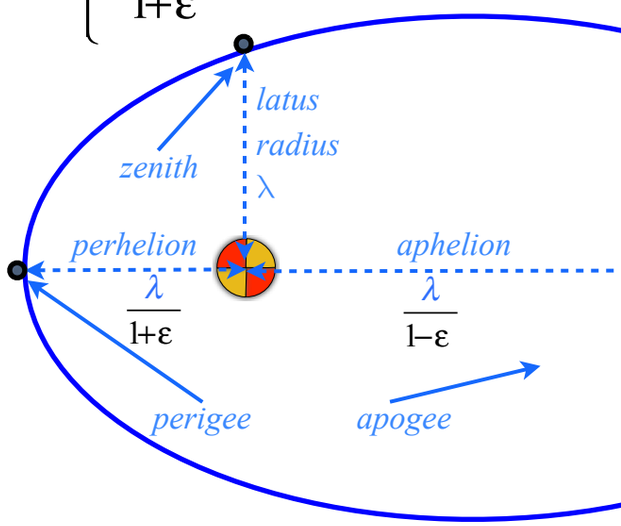
(b) Attractive (k>0) Hyperbolic (E>0)



$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{pxL}}{km}$$

(c) Repulsive (k<0) Hyperbolic (E>0)

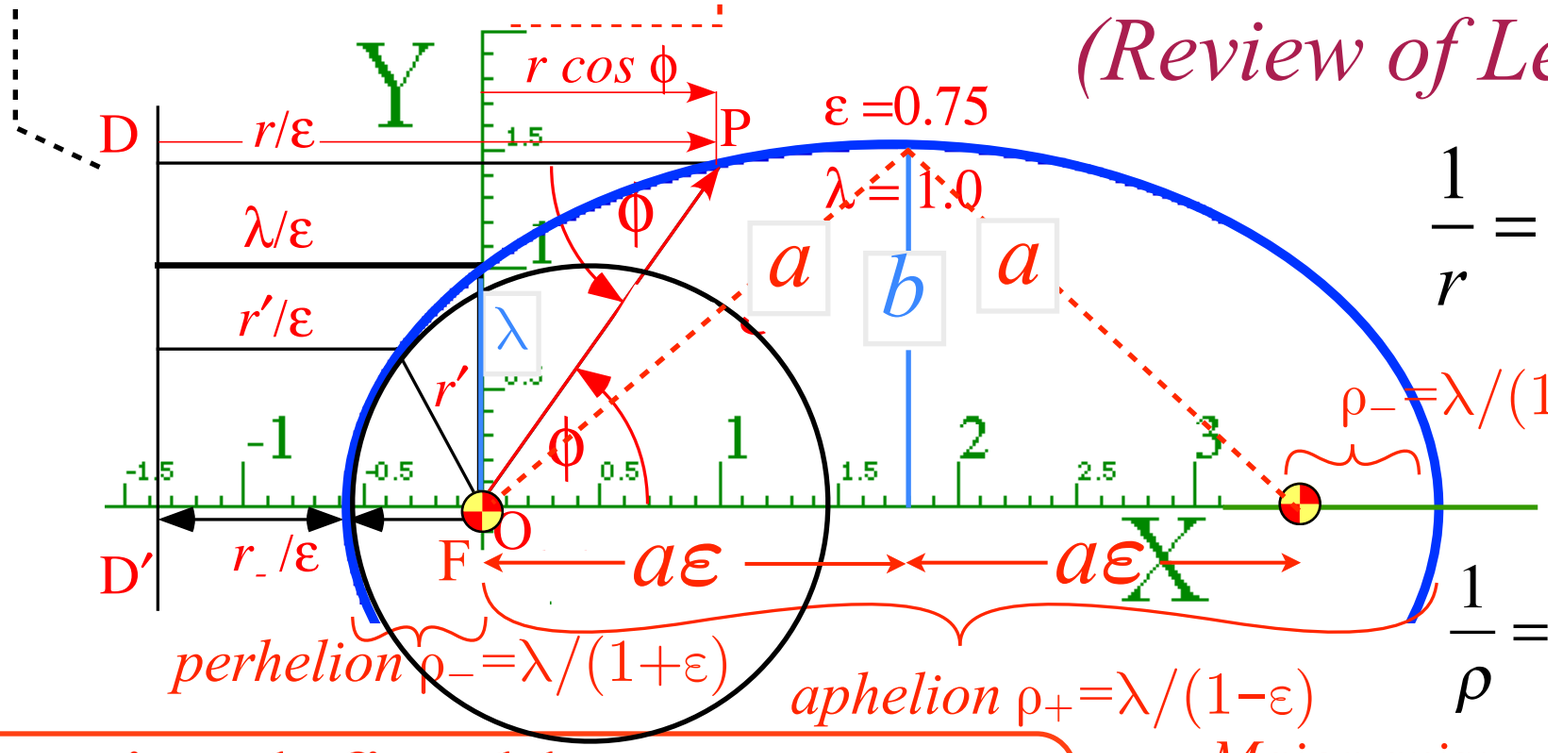




(From Lecture 26 p. 49) # *Geometry of Coulomb orbits (Let: $r = \rho$ here)*

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r = \frac{\lambda}{1-\varepsilon\cos\phi}$$

(Review of Lect. 25)



$$\frac{1}{r} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

$\rho_- = \lambda/(1+\varepsilon)$ *perhelion*

$$\frac{1}{\rho} = \frac{-k}{\mu^2/m} + \frac{\sqrt{k^2 + 2E\mu^2/m}}{\mu^2/m}\cos\phi$$

All conics defined by:

Defining ***eccentricity*** ε

Distance to Focal-point $= \varepsilon \cdot$ Distance to Directrix-line

Major axis: $\rho_+ + \rho_- = 2a$

$\rho_+ + \rho_- = [\lambda(1+\varepsilon) + \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda/|1-\varepsilon^2|$

Focal axis: $\rho_+ - \rho_- = 2a\varepsilon$

$\rho_+ - \rho_- = [\lambda(1+\varepsilon) - \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda\varepsilon/|1-\varepsilon^2|$

Minor radius: $b = \sqrt{(a^2 - a^2\varepsilon^2)} = \sqrt{(a\lambda)}$ *(ellipse:* $\varepsilon<1$*)*

Minor radius: $b = \sqrt{(a^2\varepsilon^2 - a^2)} = \sqrt{(\lambda a)}$ *(hyperb:* $\varepsilon>1$*)*

(x,y) *parameters*	*physical constants*	(r,ϕ) *parameters*
$a = \dfrac{k}{2E}$	$E = \dfrac{k}{2a}$	$\varepsilon = \sqrt{\dfrac{k^2 m + 2L^2 E}{k^2 m}} = \sqrt{1 \pm \dfrac{b^2}{a^2}}$
$b = \dfrac{L}{\sqrt{2m\lvert E\rvert}}$	$L = \sqrt{km\lambda}$	$\lambda = \dfrac{L^2}{km} = \dfrac{b^2}{a}$

$\varepsilon^2 = 1 - \dfrac{b^2}{a^2}$ (ellipse: $\varepsilon<1$) $\dfrac{b^2}{a^2} = \sqrt{1-\varepsilon^2}$

$\varepsilon^2 = 1 + \dfrac{b^2}{a^2}$ (hyperbola: $\varepsilon>1$) $\dfrac{b^2}{a^2} = \sqrt{\varepsilon^2 - 1}$

$\lambda = a(1-\varepsilon^2)$ *(ellipse:* $\varepsilon<1$*)*

$\lambda = a(\varepsilon^2-1)$ *(hyperb:* $\varepsilon>1$*)*

Review of lecture 26

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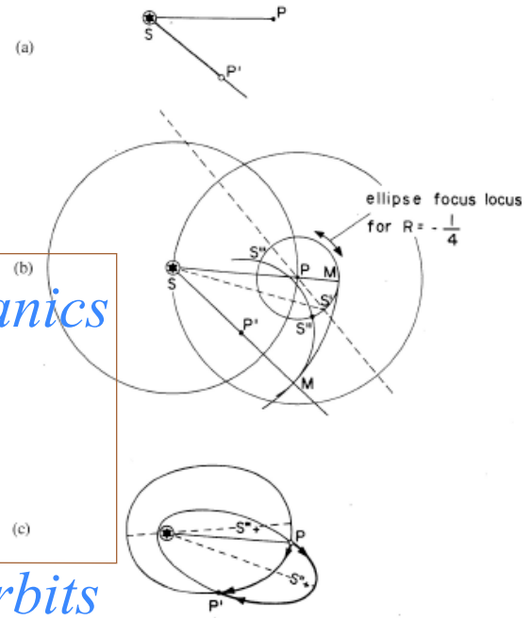


Fig. 6. Typical orbit problem solved geometrically. (a) *Problem:* Find orbits that carry a satellite with initial $R = -\frac{1}{4}$ from P to P′. (b) The foci of all $R = -\frac{1}{4}$ orbits reside on a circle around P. The focus S′ from Fig. 4 determines the circle, and then new foci S″ and S‴ are found as described in the text. (c) The two solutions to the problem are shown.

Clearly, there exist a set of points P′ that can be reached from P with the initial energy ratio of $R = -\frac{1}{4}$ along two orbits. Outside this set will be points that cannot be reached by *any* $R = -\frac{1}{4}$ orbit. Now it can be seen that the boundary of this set is composed of points that can be reached by just *one* orbit, and this envelope of orbits is an ellipse with foci at S and P.

It is possible to demonstrate this phenomena and other interesting things with a simple analog computer described in Sec. III.

III. AN ORBIT DEMONSTRATION COMPUTER

The computer–demonstrator described below is based upon some principles which were mentioned in Sec. II. The scale of the computer [Fig. 7(a)] gives the position of the movable second focus (the first focus is fixed at the center of the Coulomb field), which varies with the angle and magnitude of the initial velocity or "injection vector," as it is called in space jargon. The concentric circles labeled $-6/10$, $-5/10$, $-4/10$, . . . , each form the focus locus for an orbit with fixed $R = -6/10$, $-5/10$, $-4/10$, respectively, and variable injection angle. This injection angle is measured as an elevation above the "horizon" drawn in the figure, where one can imagine injecting a ballistic missile or satellite into orbit from a point near the earth's surface. In Fig. 7(a) the injection point is the center of the $-6/10$, $-5/10$, $-4/10$, . . . , circles and corresponds to point P in Fig. 6(b).

The scale pictured in Fig. 7(a), or one very similar to it, is to be printed on a transparency and glued to a

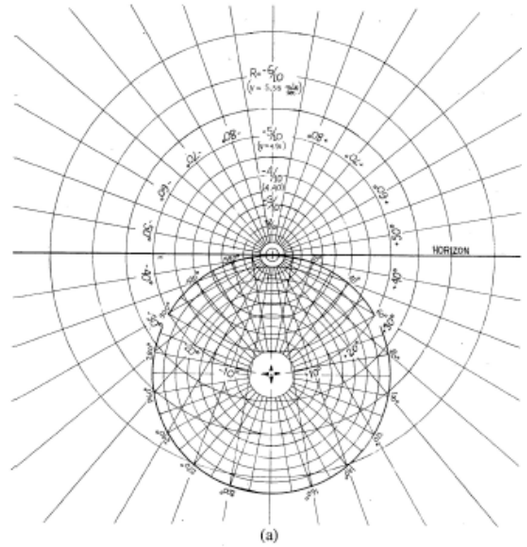
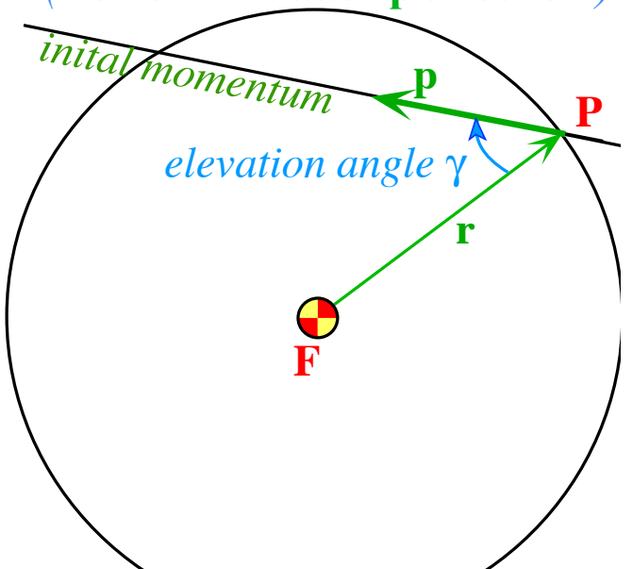


Fig. 7. Coordinate grids for orbital analog computers. (a) Elliptical orbit scale ($0 > R > -1$). This can be used with the apparatus in Figs. 8 or 9. Radial lines marked $\pm 10°$, $\pm 20°$, . . . , are each the focus locus for orbits with an initial velocity $\pm 10°$, $\pm 20°$, . . . , above the horizon line. The circle marked 20°, 40°, . . . , 340° can be taken as the Earth's surface, or any circle inside this one can be taken to be the surface of any celestial body. The R values apply correctly in either case, while the velocity values are marked for the former case only. (b) Hyperbolic orbit scale ($0 < R < \infty$) and ($-\infty < R < -1$). This can only be used with the apparatus shown in Fig. 9. Outer circles locate foci for orbits of particles attracted to the force center, while inner circles locate foci for orbits in a repulsive field. In either case a radial line marked $\pm 10°$, $\pm 20°$, . . . , is the focus locus for an orbit with the initial velocity an angle $\pm 10°$, $\pm 20°$, . . . , above the nadir line.

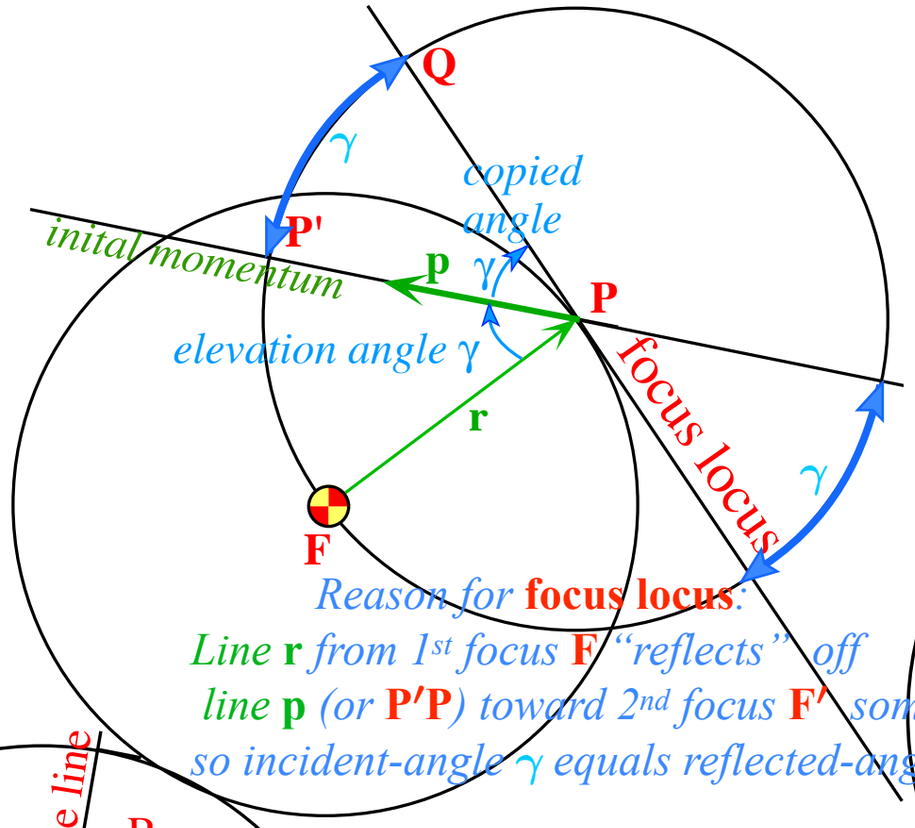
Plexiglas plotting board that will fit on an overhead projector. The plotting board and attachments shown in Fig.

ε-vector and Coulomb orbit construction steps

Pick launch point P
(radius vector **r** *)*
and elevation angle γ *from radius*
(momentum initial **p** *direction)*

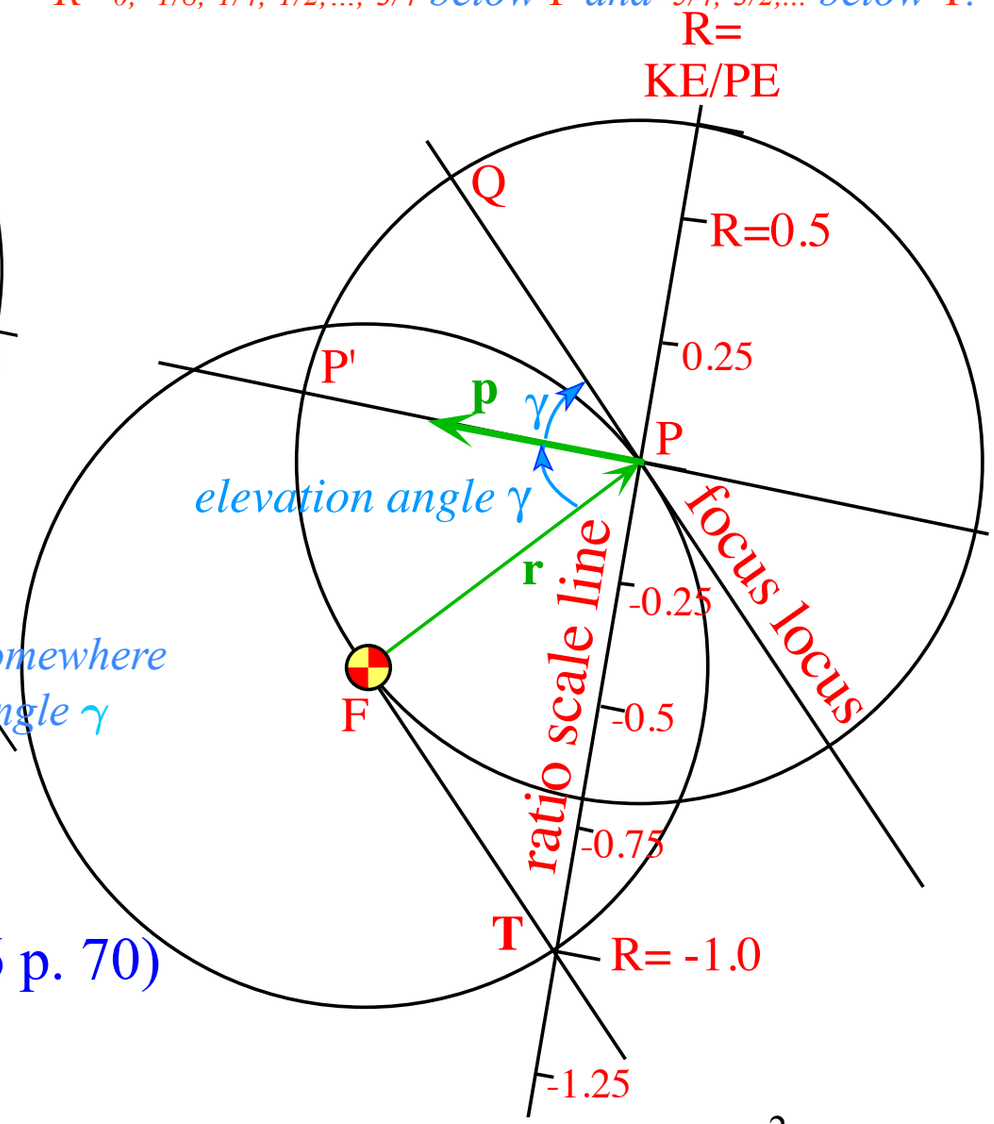


Copy F-center circle around launch point P
Copy elevation angle γ *(∠*FPP′*) onto ∠*P′PQ
Extend resulting line QPQ′ *to make* **focus locus**

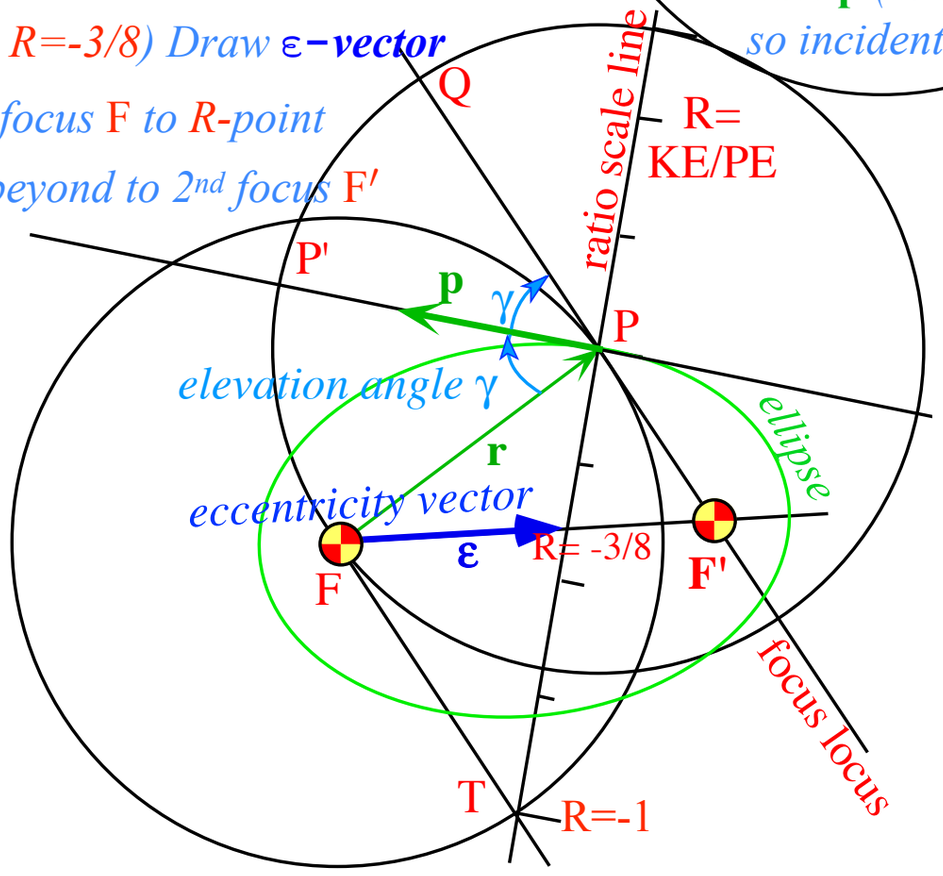


Reason for **focus locus**:
Line **r** *from 1st focus* **F** *"reflects" off line* **p** *(or* **P′P***) toward 2nd focus* **F′** *somewhere so incident-angle* γ *equals reflected-angle* γ

Copy double angle 2γ *(∠*FPQ*) onto ∠*PFT
*Extend ∠*PFT *chord* PT *to make R-***ratio scale line**
Label chord PT *with R=0 at* P *and R=-1.0 at* T.
*Mark R-***line** *fractions R=0, +1/4, +1/2,... above* P *and R=0, -1/8,-1/4,-1/2,...,-3/4 below* P *and -5/4,-3/2,... below* T.



Pick initial R=KE/PE value
(here R=-3/8) Draw **ε−vector**
from focus F *to R-point*
and beyond to 2nd focus F′



(From Lecture 26 p. 70)

$$R = \frac{\text{Initial KE}}{\text{Initial PE}} = \frac{m\,v^2(0)/2}{-k/r(0)}$$

$$= \pm\left(\frac{\text{Initial velocity}}{\text{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

focus F *and 2nd focus* F′ *allow final construction of orbital trajectory. Here it is an R=-3/8 ellipse.*

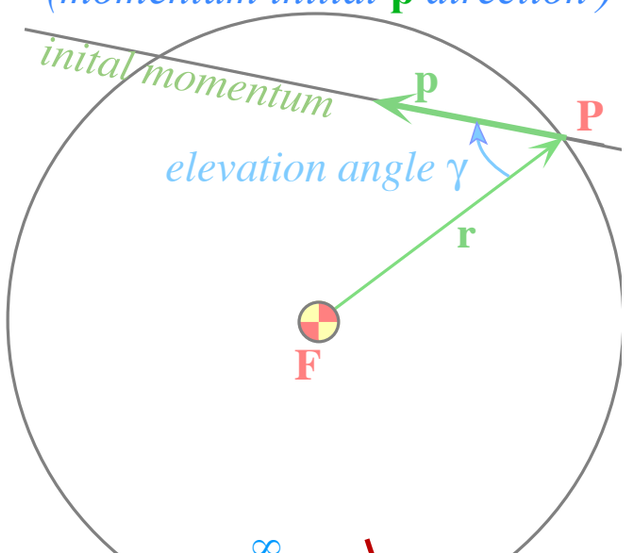
(Detailed Analytic geometry of ε*-vector follows.)*

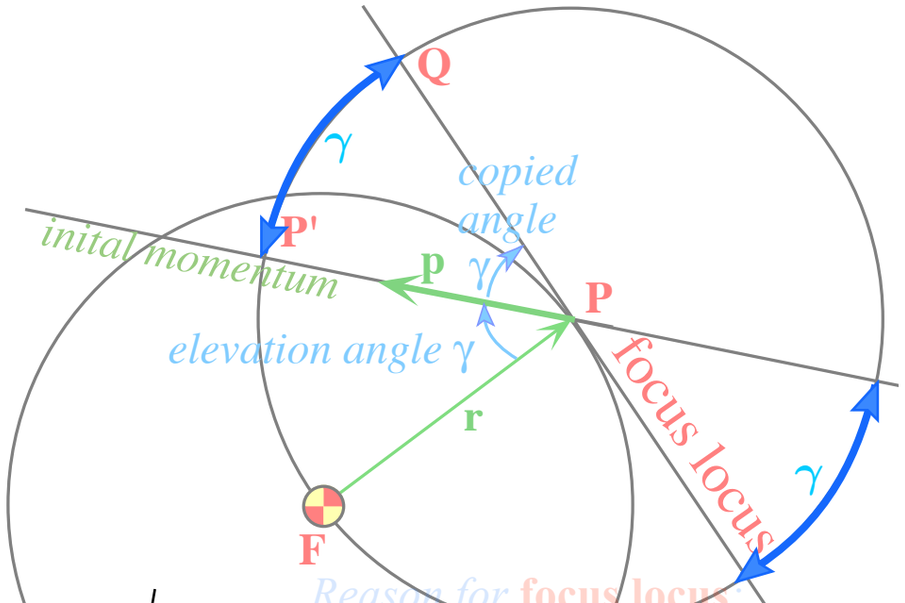
ε-vector and Coulomb orbit construction steps

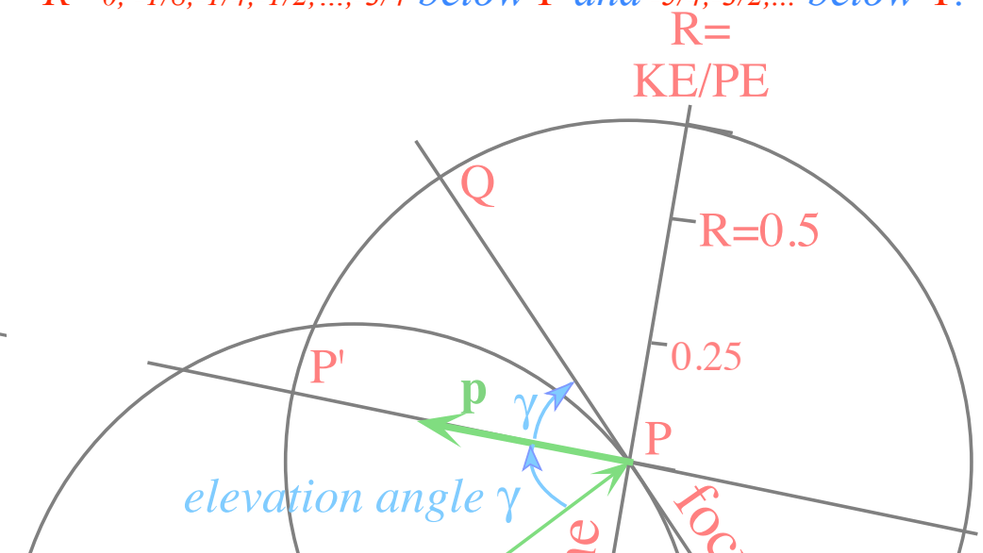
Pick launch point P
(radius vector **r** *)*
and elevation angle γ *from radius*
(momentum initial **p** *direction)*

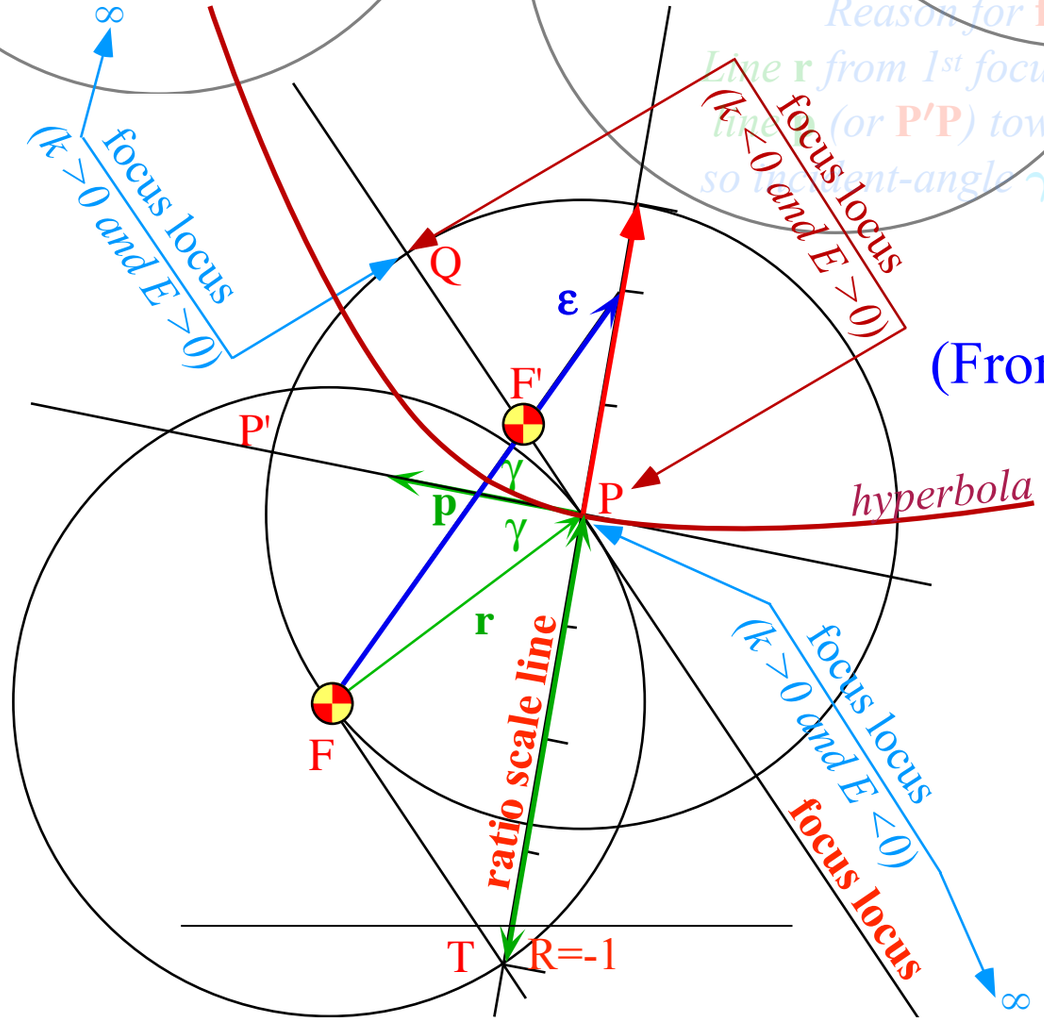
Copy F-center circle around launch point P
Copy elevation angle γ *(∠*FPP′*) onto ∠*P′PQ
Extend resulting line QPQ′ *to make* **focus locus**

Copy double angle 2γ *(∠*FPQ*) onto ∠*PFT
*Extend ∠*PFT *chord* PT *to make R-***ratio scale line**
Label chord PT *with R=0 at* P *and R=-1.0 at* T.
*Mark R-***line** *fractions R=0, +1/4, +1/2,... above* P *and R=0, -1/8,-1/4,-1/2,...,-3/4 below* P *and -5/4,-3/2,... below* T.









(From Lecture 26 p. 71)

Pick initial R=KE/PE value
(here R=+1/2) Draw **ε−*vector***
from focus F *to R-point*
(Here it intersects 2nd focus F′

focus F *and 2nd focus* F′ *allow final construction of orbital trajectory. Here it is an R=+1/2 hyperbola.*

(Detailed Analytic geometry of ε*-vector follows.)*

$$R = \frac{\text{Initial KE}}{\text{Initial PE}} = \frac{m\,v^2(0)/2}{-k/r(0)}$$

$$= \pm\left(\frac{\text{Initial velocity}}{\text{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

Analytic geometry derivation of ε-constructions

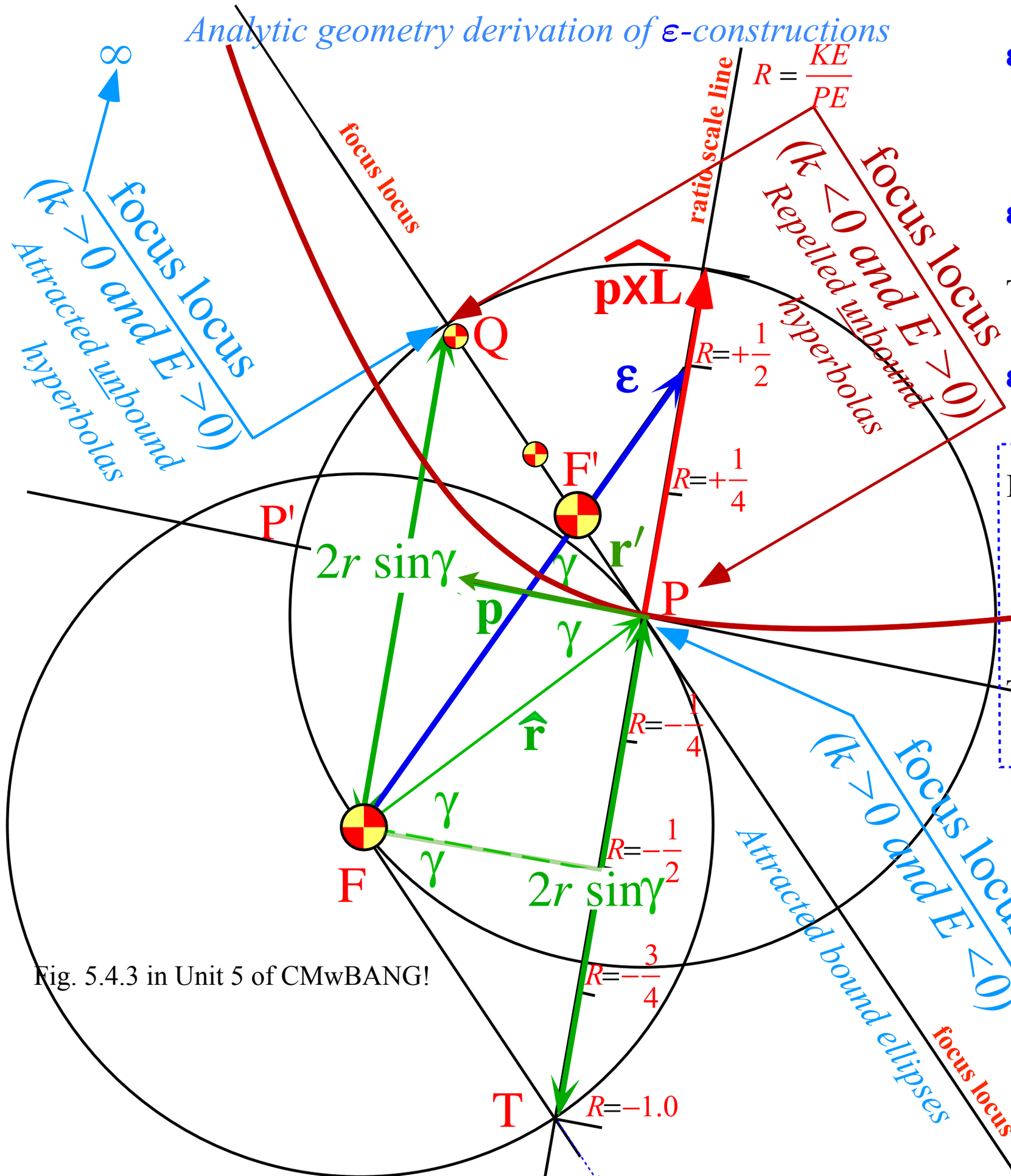


Fig. 5.4.3 in Unit 5 of CMwBANG!

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p}\times\mathbf{L}}{km} = \hat{\mathbf{r}} - \frac{(mv_0)(mv_0 r_0)\sin\gamma}{km}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

where: $\mathbf{L}_{\mathbf{p}\times} \equiv \mathbf{p}\times\mathbf{L}$

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} + 2\sin\gamma\,\frac{mv_0^{\,2}/2}{-k/r_0}\hat{\mathbf{L}}_{\mathbf{p}\times} = \hat{\mathbf{r}} + 2\sin\gamma\,\frac{KE}{PE}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

The *eccentricty* vector is:

$$\boldsymbol{\varepsilon} = \begin{pmatrix}\cos\gamma\\ \sin\gamma\end{pmatrix} + 2\sin\gamma\begin{pmatrix}0\\1\end{pmatrix} R = \begin{pmatrix}\cos\gamma\\ (2R+1)\sin\end{pmatrix}$$

For: $\gamma = 45°$ and: $R = +\frac{1}{2}$

$$\boldsymbol{\varepsilon} = \begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}(2R+1)\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\ 2/\sqrt{2}\end{pmatrix},$$

The *eccentricty* parameter defined by:

$$\varepsilon^2 = \cos^2\gamma + (2R+1)^2\sin^2\gamma = 1 \pm \frac{a^2}{b^2}$$

$$= 1 + 4R(R+1)\sin^2\gamma = \frac{5}{2}$$

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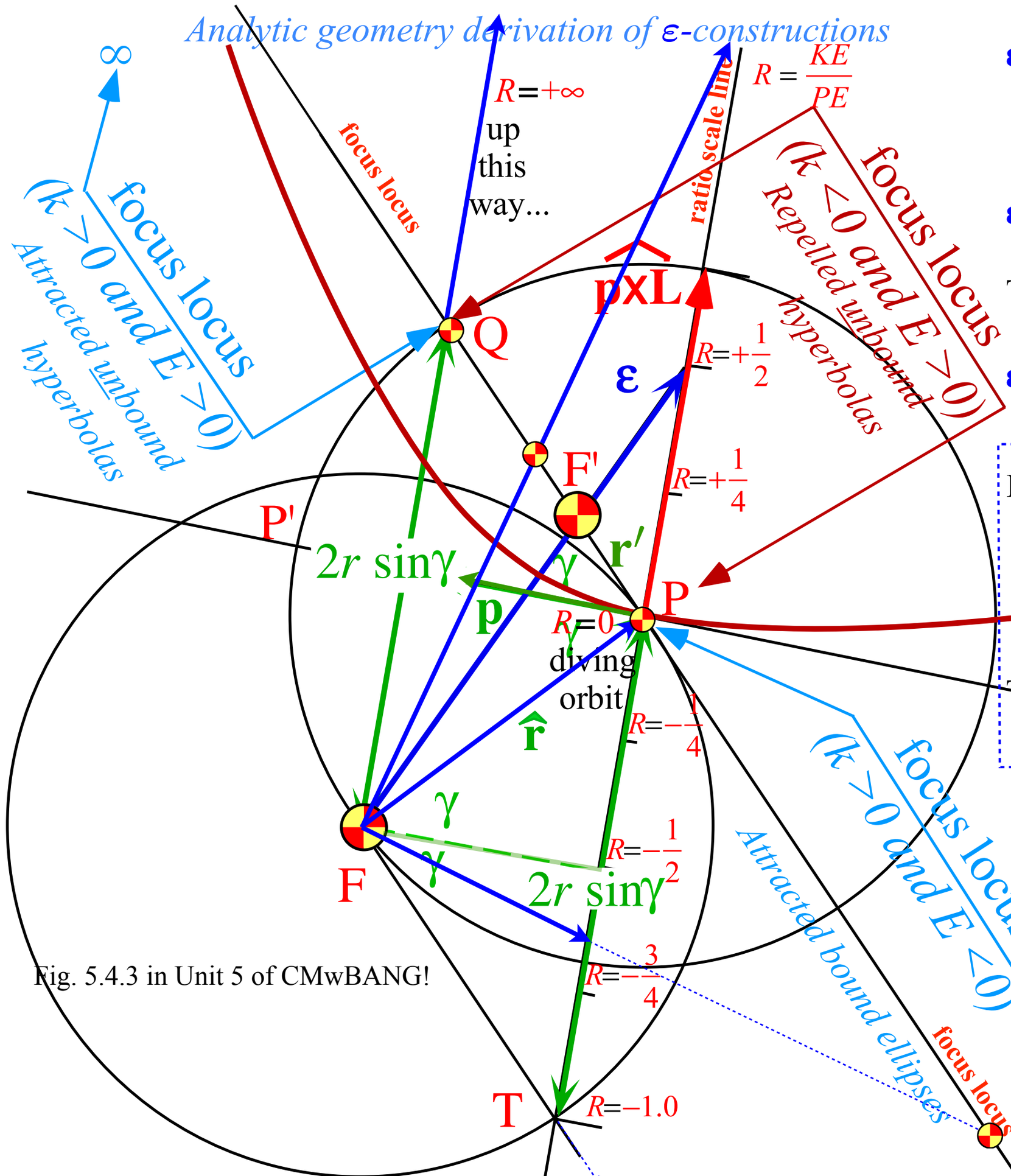

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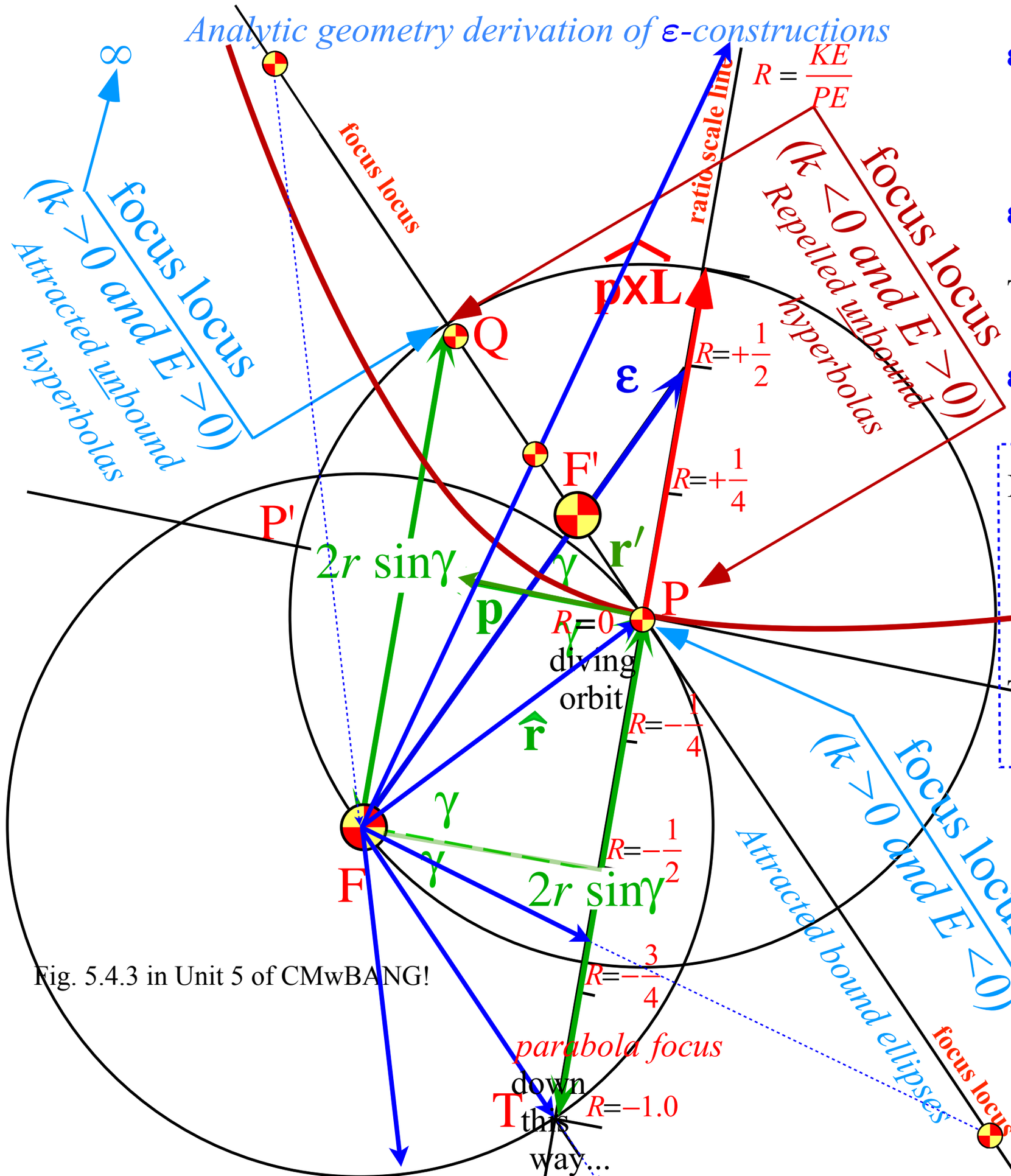
Fig. 5.4.3 in Unit 5 of CMwBANG!

Analytic geometry derivation of ε-constructions



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focus locus ($k>0$ and $E>0$) Attracted unbound hyperbolas

focus locus

$R=+\infty$ up this way...

ratio scale line

$R = \frac{KE}{PE}$

focus locus ($k<0$ and $E>0$) Repelled unbound hyperbolas

pxL

Q

ε

$R=+\frac{1}{2}$

F'

$R=+\frac{1}{4}$

P'

$2r\sin\gamma$

γ

r′

p

P

$R=0$ diving orbit

r̂

$R=-\frac{1}{4}$

γ

γ

$R=-\frac{1}{2}$

F

$2r\sin\gamma^2$

$R=-\frac{3}{4}$

parabola focus

$R=-\infty$ down this way...

T

down this way...

$R=-1.0$

focus locus ($k>0$ and $E<0$) Attracted bound ellipses

focus locus

Fig. 5.4.3 in Unit 5 of CMwBANG!

Review of lecture 26

Eccentricity vector ε and (ε,λ)-geometry of orbital mechanics

Analytic geometry derivation of ε-construction

➔ *Connection formulas for (a,b) and (ε,λ) with (γ,R)*

Detailed ruler & compass construction of ε-vector and orbits

(R=-0.375 elliptic orbit)

(R=+0.5 hyperbolic orbit)

Properties of Coulomb trajectory families and envelopes

Graphical ε-development of orbits

Launch angle fixed-Varied launch energy

Launch energy fixed-Varied launch angle

Launch optimization and orbit family envelopes

Lenz Vector...analog computersAJP 44 4 (1974)

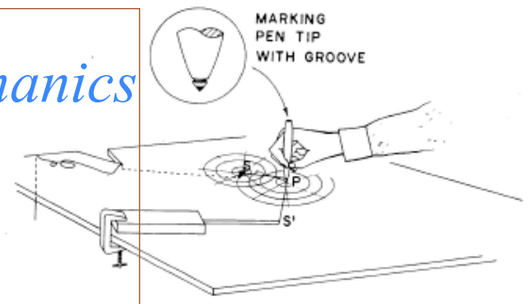

Fig. 8. Orbital computer design: cheaper model that computes elliptical orbits only using the scale of Fig. 7(a). A Plexiglas sheet that is about ⅛ in. thick has a string hole at the orbit's center. A transparency (Xerox, 3-M, etc.) of Fig. 7(a) is taped in position on the underside. (*Caution:* One should avoid marking pens that permanently mark plastic.)

8 go with Fig. 7(a) and can be assembled in a few minutes by using odds and ends. A more sophisticated apparatus is shown in Fig. 9. The simple apparatus of Fig. 8 produces the well-known elliptical orbits and trajectories of planets and satellites, but not the hyperbolic trajectories characteristic of higher-than-escape velocity meteors or of the repulsive Coulomb force problems. The second apparatus (Fig. 9) is designed to handle all cases, provided that the appropriate focal point scale is inserted.

The operation of either plotter begins with the positioning of second focal point S′ according to the scale on the plotting board. Then the marking pen is poked into a small identation at P and held while the strings to S and S′ are tightened. Finally, you slide the pen out along the board in such a way that the strings stay tight and the desired trajectory is drawn.

The apparatus in Fig. 8 will thus make an ellipse since the sum of distances SP and S′P is constant. The apparatus in Fig. 9 does the same when the spool brakes are

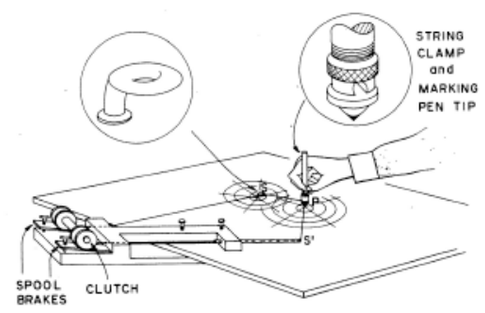

Fig. 9. Orbital computer design: a more elaborate model that computes general Coulomb orbits. The ellipse drawing mode is obtained when, first, the clamp is opened to allow the string to slide and then the spool brakes are tightened after the initial adjustment has been made with the use of the scale in Fig. 7(a). The hyperbola drawing mode is obtained when the string clamp and spool clutch are tightened but the brakes are released. The spools must turn together after the initial adjustment has been made with the use of the scale in Fig. 7(b). One hand can maintain the string tension while drawing the orbit, and the other hand can control the paying out of string. (Alternately, springs on the spool axis accomplish the same thing.)

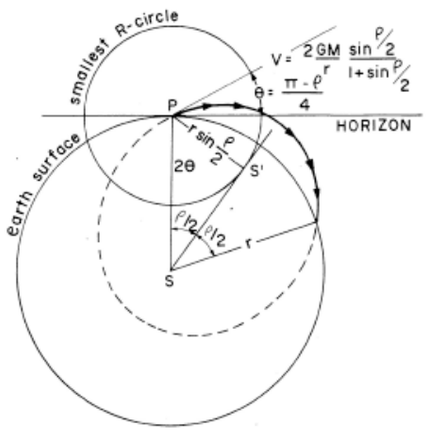

Fig. 10. Sample computer trajectory problem. One finds the minimum-energy trajectory for a given range ρ. Initial velocity v and θ follow easily from the geometry of the computer scale in Fig. 7(a).

tightened and the string clamp on the marking pen is loosened.

The apparatus in Fig. 9 will produce a hyperbola if the *difference* between distances SP and S′P is constant. This is accomplished by tightening the string clamp and the clutch so that two string spools reel equal amounts of string in or out and the constant difference in length is maintained.

IV. SOME USES FOR COMPUTERS

For the computer to be set up to draw elliptical orbits, there is one important question that can be answered immediately: To throw with *minimum* initial velocity a free falling spacecraft between two fixed points near the earth and a distance ρ apart, what initial angle θ and speed v are needed? (We imagine a fixed coordinate system here, not a rotating one.)

Measuring this range by a great circle angle ρ, we see that the focus of the orbit must be on a line through the Earth's center, making an angle $\rho/2$ with the launch point P. The smallest R circle [recall Fig. 7(a)] intersecting this line is the one tangent to it and represents the solution to the problem. Indeed, the algebraic solution to this problem follows from the diagram in Fig. 10 and is given there. Note that the angle θ approaches 45° as the range becomes small compared to the radius of the Earth.

Note that one may change the radius of the starting point P by simply reinterpreting the scale of the computer. For example, if the starting point is located at a height of, say, four times the Earth's radius, then the velocities marked on this scale are all divided by the square root of this factor, in this case, by 2.

With the computer set up to draw hyperbolic orbits and the appropriate scales available, there are a number of interesting problems to examine. For example, the attractive-field positive-energy scale allows one to exhibit the paths of meteorites. Given the impact direction and speed, one can extrapolate to find its origin.

The hyperbolic computer setup can be used to demonstrate Rutherford scattering for either the repulsive field (see Ref. 1) or the attractive field. At the same time

Algebra of ε-construction geometry

Three pairs of parameters for Coulomb orbits:
1.Cartesian (a,b), 2.Physics (E,L), 3.Polar (ε,λ)
Now we relate a 4th pair: 4.Initial (γ,R)

The *eccentricty* parameter relates ratios $R=\frac{KE}{PE}$ and $\frac{b^2}{a^2}$

$$\varepsilon^2 = 1+4R(R+1)\sin^2\gamma$$

$$= 1-\frac{b^2}{a^2} \quad \text{for ellipse} \quad (\varepsilon<1) \quad \text{where:} \quad 4R(R+1)\sin^2\gamma = -\frac{b^2}{a^2} = \varepsilon^2-1 \quad \text{implying:} \quad R(R+1)<0 \qquad \begin{matrix}(\text{or: } -R^2>R)\\(\text{or: } 0>R>-1)\end{matrix}$$

$$= 1+\frac{b^2}{a^2} \quad \text{for hyperbola } (\varepsilon>1) \quad \text{where:} \quad 4R(R+1)\sin^2\gamma = +\frac{b^2}{a^2} = \varepsilon^2-1 \quad \text{implying:} \quad R(R+1)>0 \qquad \begin{matrix}(\text{or: } -R^2<R)\\(\text{or: } 0<R<-1)\end{matrix}$$

Total $\frac{-k}{2a}=E=\textit{energy}=KE+PE$ relates ratio $R=\frac{KE}{PE}$ to individual radii a, b, and λ.

$$\frac{-k}{2a}=E=KE+PE=R\cdot PE+PE=(R+1)PE=(R+1)\frac{-k}{r} \quad \text{or:} \quad \frac{1}{2a}=(R+1)\frac{1}{r}=(R+1)$$

$$a=\frac{r}{2(R+1)}=\left(\frac{1}{2(R+1)} \text{ assuming } \textbf{unit} \text{ initial radius } (r\equiv 1).\right)$$

$$4R(R+1)\sin^2\gamma=\mp\frac{b^2}{a^2} \quad \text{implies:} \quad 2\sqrt{\mp R(R+1)}\sin\gamma=\frac{b}{a} \quad \text{or:} \quad b=2a\sqrt{\mp R(R+1)}\sin\gamma$$

$$b=r\sqrt{\frac{\mp R}{R+1}}\sin\gamma\left(=\sqrt{\frac{\mp R}{R+1}}\sin\gamma \text{ assuming } \textbf{unit} \text{ initial radius } (r\equiv 1)\right)$$

Latus radius is similarly related:

$$\lambda=\frac{b^2}{a}=\mp 2r\,R\sin^2\gamma$$

(Review of Lect. 26 p.107-108)

Algebra of ε-construction geometry

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$$\varepsilon^2 = 1+4R(R+1)\sin^2\gamma$$

$$= 1-\frac{b^2}{a^2}\ \text{ellipse}(\varepsilon<1) \qquad 4R(R+1)\sin^2\gamma = -\frac{b^2}{a^2}$$

$$= 1+\frac{b^2}{a^2}\ \text{hyperbola}\ (\varepsilon>1) \quad 4R(R+1)\sin^2\gamma = +\frac{b^2}{a^2}$$

$$a=\frac{r}{2(R+1)}=\left(\frac{1}{2(R+1)}\ \text{assuming } \textbf{unit}\ \text{initial radius } (r\equiv 1).\right)$$

$$b=r\sqrt{\frac{\mp R}{R+1}}\sin\gamma\left(=\sqrt{\frac{\mp R}{R+1}}\sin\gamma\ \ \text{assuming } \textbf{unit}\ \text{initial radius } (r\equiv 1)\right)$$

Latus radius is similarly related:

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From ε^2 result (at top):

$$\frac{b}{a}=2\sqrt{\mp R(R+1)}\sin\gamma=\sqrt{\pm(1-\varepsilon^2)}$$

(Review of Lect. 26 p.107-109)

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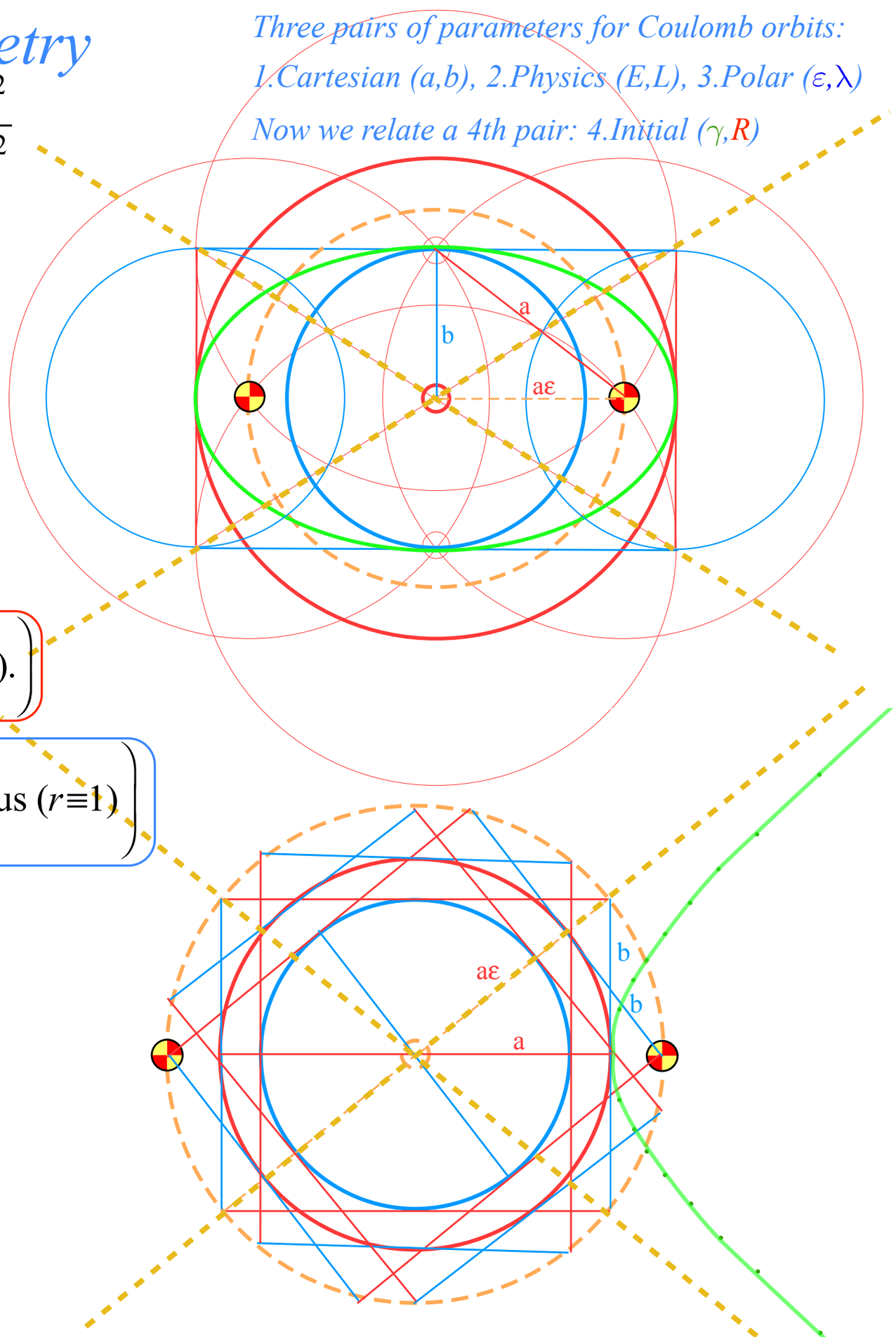

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Analytic geometry derivation of $\boldsymbol{\varepsilon}$*-construction*

Connection formulas for (a,b) *and* (ε,λ) *with* (γ,R)

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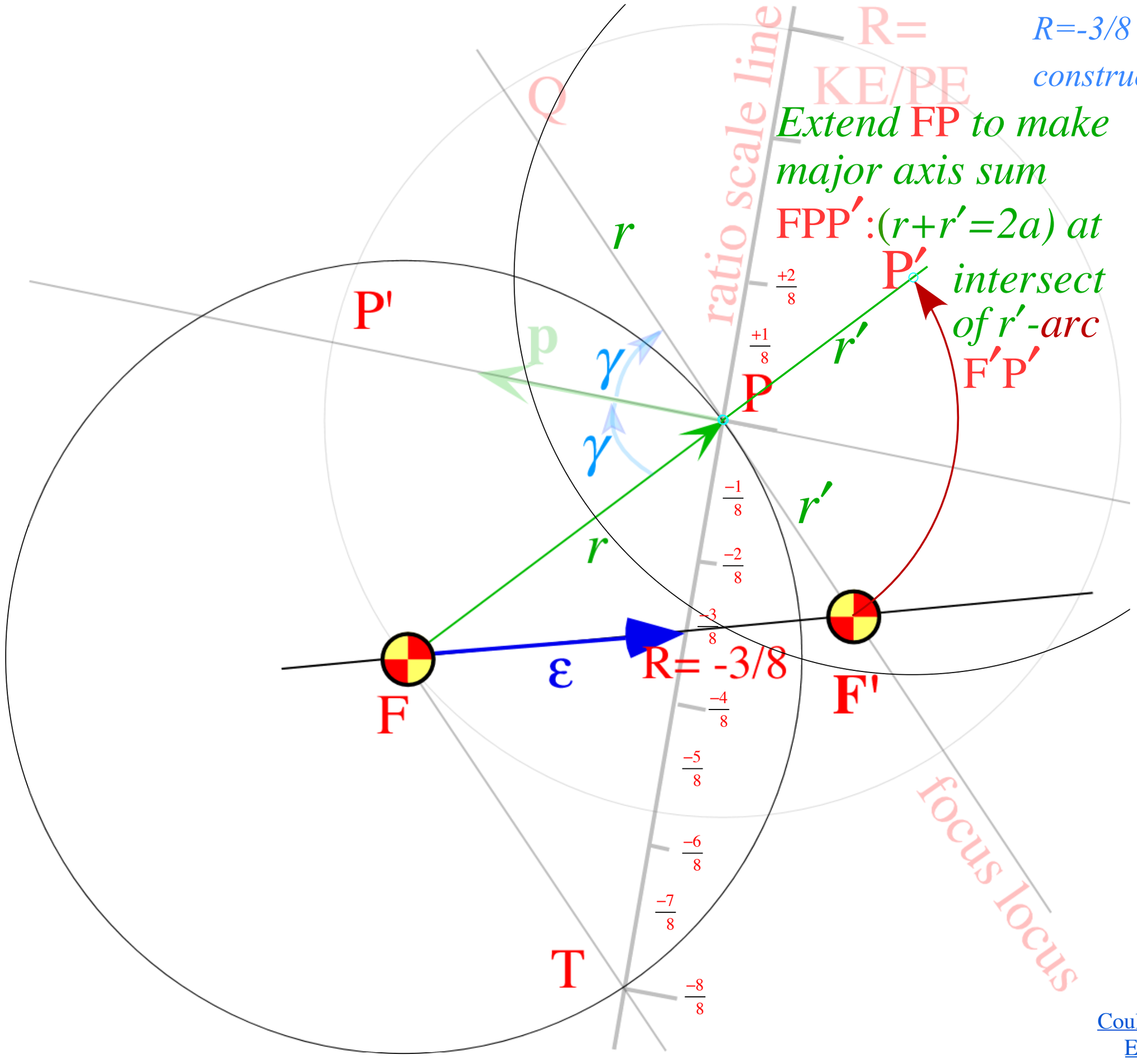
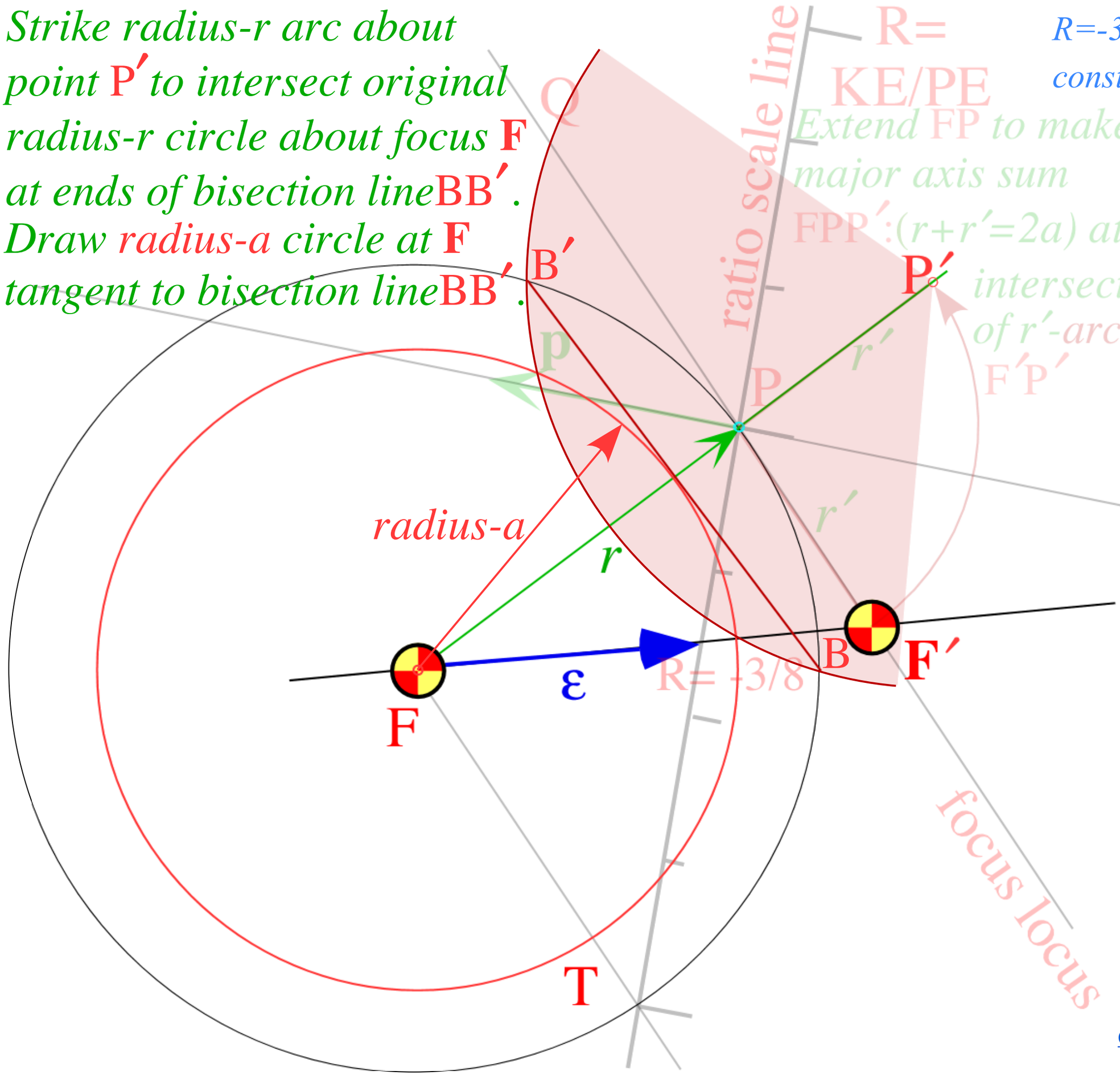


[CoulIt Web Simulation
Elliptical R=-3/8]

Strike radius-r arc about point P′ *to intersect original radius-r circle about focus* **F** *at ends of bisection line* BB′.
Draw radius-a circle at **F** *tangent to bisection line* BB′.



R=-3/8 elliptic orbit construction

R=-3/8

γ=45°

[CoulIt Web Simulation
Elliptical R=-3/8]

R=-3/8 elliptic orbit construction

R=-3/8

γ=45°

Strike radius-r arc about point P′ to intersect original radius-r circle about focus F at ends of bisection line BB′.

Draw radius-a circle at F tangent to bisection line BB′.

Q

R= KE/PE

ratio scale line

Extend FP to make major axis sum FPP′:(r+r′=2a) at P′ intersect of r′-arc F′P′

B′

P

P

r′

+b

radius-a

r′

r

B

F′

ε

O

R= -3/8

F

-b

focus locus

T

Draw radius-a circle at F′

Draw radius-a and radius-b circles at O (Center of bisection line (±b).

CoulIt Web Simulation
Elliptical R=-3/8

Strike radius-r arc about point P′ *to intersect original radius-r circle about focus* F *at ends of bisection line* BB′.
Draw radius-a circle at F *tangent to bisection line* BB′.

Extend FP *to make major axis sum* FPP′:(*r*+*r*′=2*a*) *at* P′ *intersect of r*′*-arc* F′P′

R=-3/8 elliptic orbit construction

R=-3/8

γ=45°

Q

R=
KE/PE

ratio scale line

B′

p

P

r′

P′

radius-a

r

r′

B

F′

ε

O

F

R= -3/8

-b

focus locus

T

Draw radius-a circle at F′
Draw radius-a and radius-b circles at O
(Center of bisection line (±b).

$$\varepsilon=\sqrt{1+4R(R+1)\sin^2\gamma}=\frac{\sqrt{34}}{8}=.73$$

$$a=\frac{1}{2(R+1)}=\frac{4}{5}$$

$$b=\sqrt{\frac{R}{R+1}}\sin\gamma=\sqrt{\frac{3}{10}}=.54$$

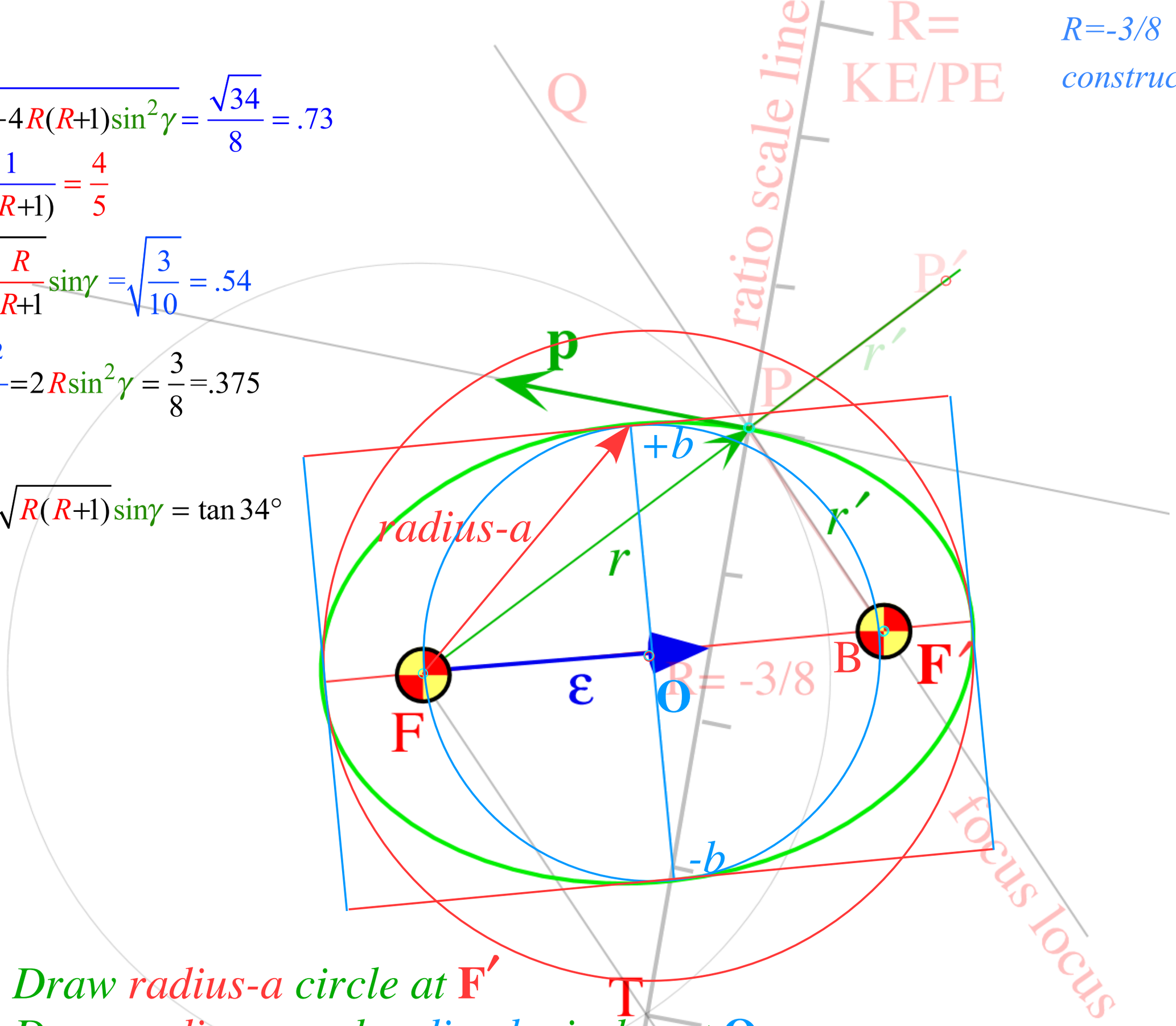
$$\lambda=\frac{b^2}{a}=2R\sin^2\gamma=\frac{3}{8}=.375$$

$$\frac{b}{a}=2\sqrt{R(R+1)}\sin\gamma=\tan 34°$$

R=-3/8 elliptic orbit construction

R=-3/8

γ=45°



Draw radius-a circle at **F′**

Draw radius-a and radius-b circles at **O**

(Center of bisection line (±b).) Do (a,b)-ellipse construction.

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Analytic geometry derivation of $\boldsymbol{\varepsilon}$*-construction*

Connection formulas for (a,b) *and* (ε,λ) *with* (γ,R)

Detailed ruler & compass construction of $\boldsymbol{\varepsilon}$*-vector and orbits*

(R=-0.375 elliptic orbit)

➔ *(R=+0.5 hyperbolic orbit)*

Properties of Coulomb trajectory families and envelopes

Graphical $\boldsymbol{\varepsilon}$*-development of orbits*

Launch angle fixed-Varied launch energy

Launch energy fixed-Varied launch angle

Launch optimization and orbit family envelopes

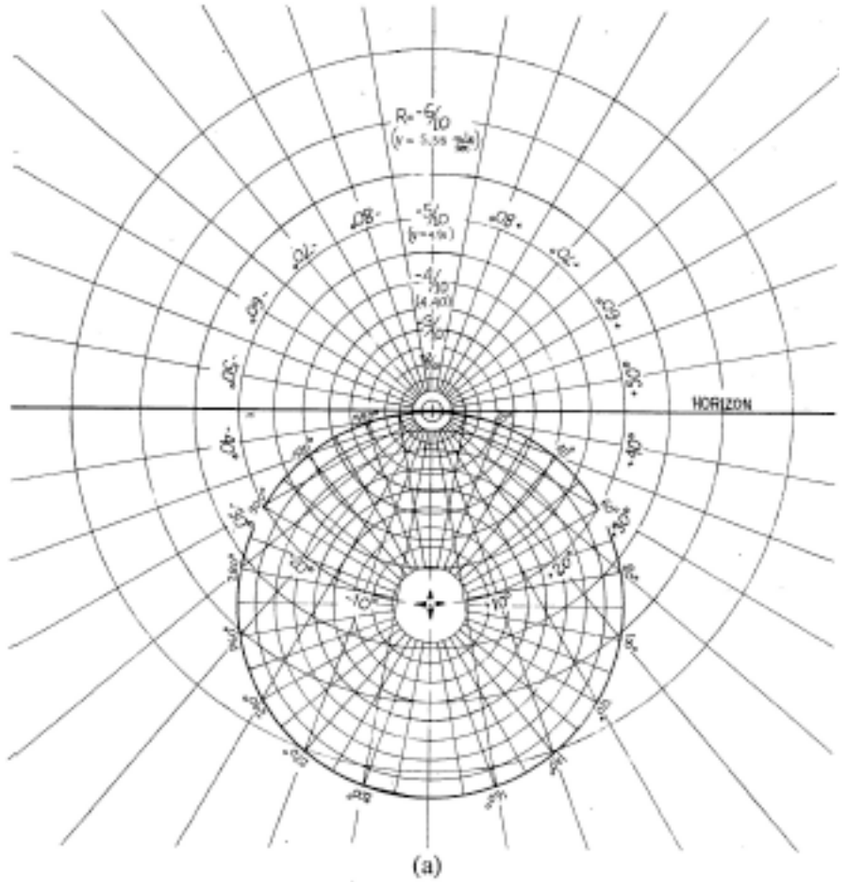


Fig. 7. Coordinate grids for orbital analog computers. (a) Elliptical orbit scale ($0 > R > -1$). This can be used with the apparatus in Figs. 8 or 9. Radial lines marked $\pm 10°$, $\pm 20°$, . . . , are each the focus locus for orbits with an initial velocity $\pm 10°$, $\pm 20°$, . . . , above the horizon line. The circle marked 20°, 40°, . . . , 340° can be taken as the Earth's surface, or any circle inside this one can be taken to be the surface of any celestial body. The R values apply correctly in either case, while the velocity values are marked for the former case only. (b) Hyperbolic orbit scale ($0 < R < \infty$) and ($-\infty < R < -1$). This can only be used with the apparatus shown in Fig. 9. Outer circles locate foci for orbits of particles attracted to the force center, while inner circles locate foci for orbits in a repulsive field. In either case a radial line marked $\pm 10°$, $\pm 20°$, . . . , is the focus locus for an orbit with the initial velocity an angle $\pm 10°$, $\pm 20°$, . . . , above the nadir line.

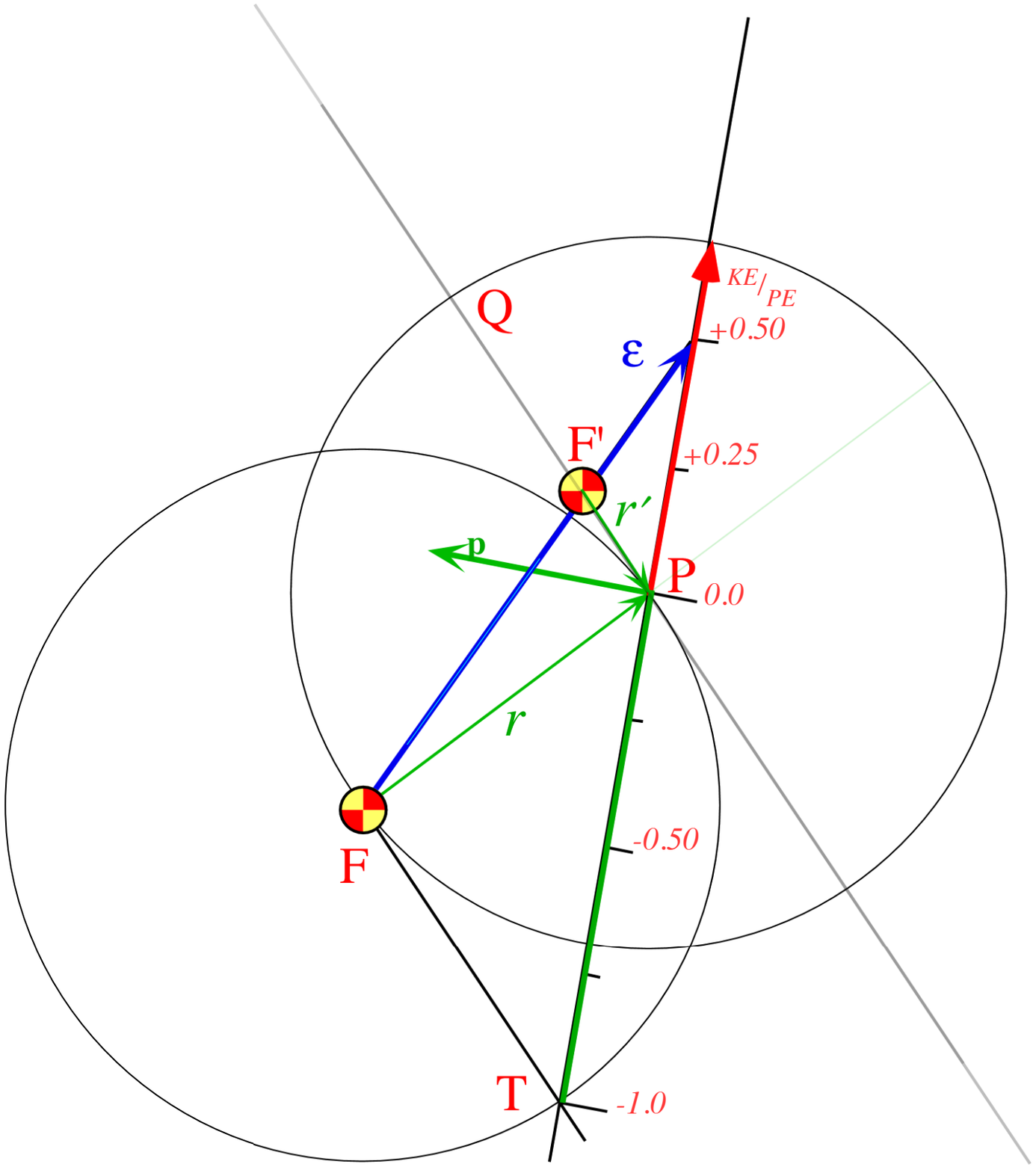
Lenz Vector...analog computersAJP 44 4 (1974)

Major diameter 2a is difference (r-r′=2a).
Major radius a is half of difference (r-r′)/2=a
Major diameter 2a needs to be centered on F-F′ focal axis

R=+1/2 hyperbolic orbit construction

R=+1/2

γ=45°



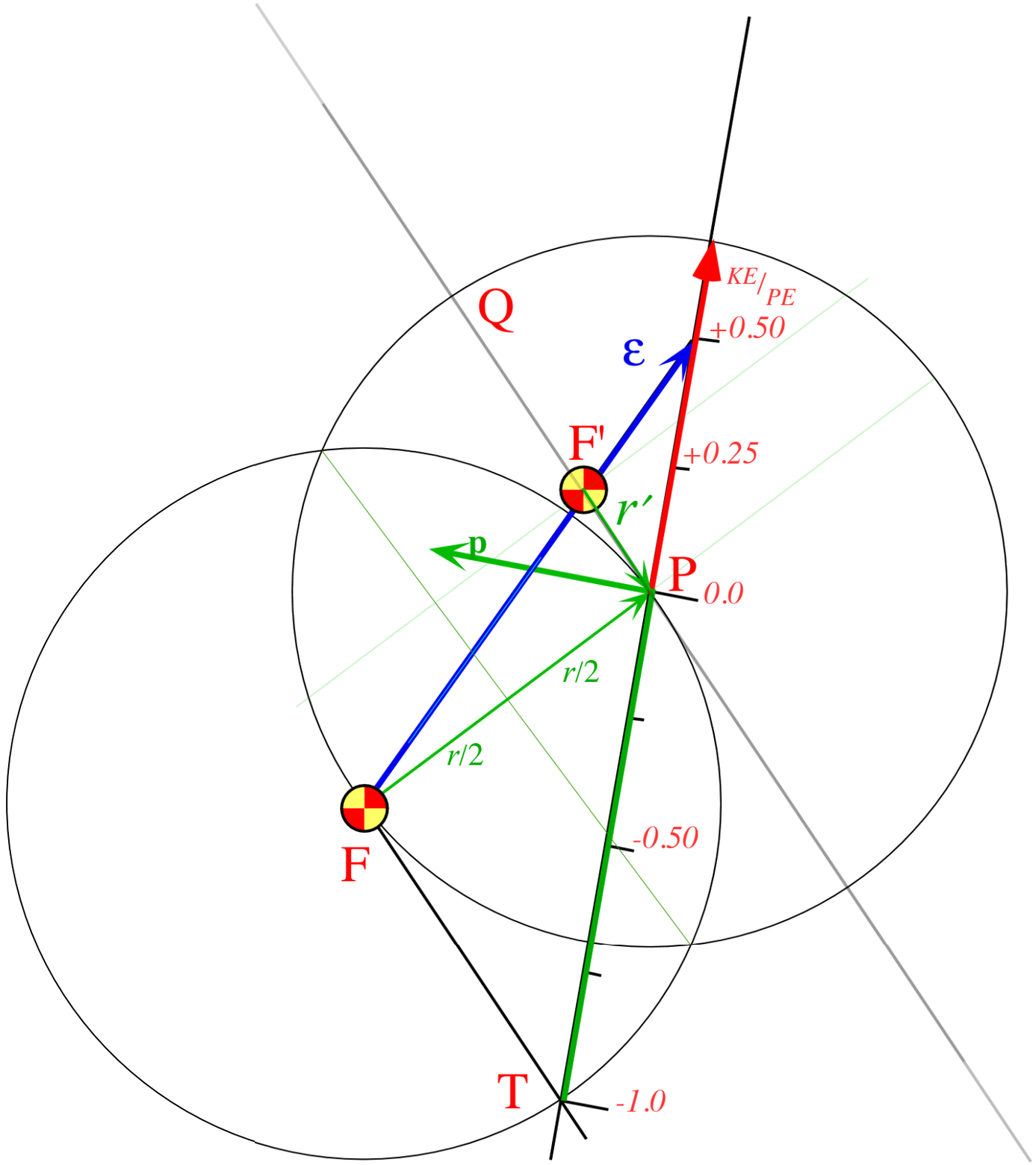
CoulIt Web Simulation
Hyperbolic R=1/2

Major diameter 2a is difference (r-r′=2a).
Major radius a is half of difference (r-r′)/2=a
Major diameter 2a needs to be centered on F-F′ focal axis
1. Bisect F-P radius r using F-P circle intersections to define r/2 sections.

R=+1/2 hyperbolic orbit construction

R=+1/2

γ=45°



CoulIt Web Simulation
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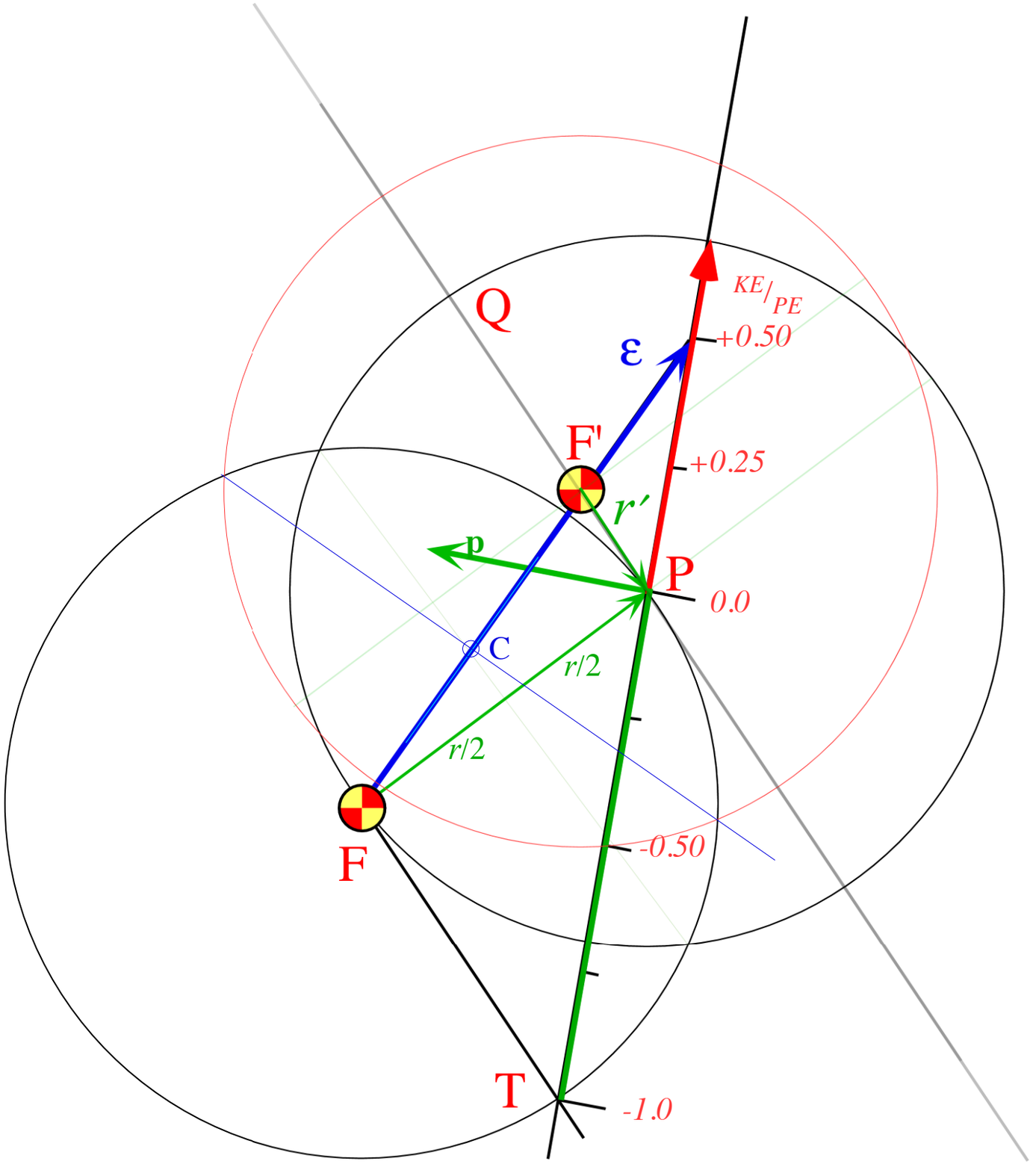
1. Bisect F-P radius r using F-P circle intersections to define r/2 sections.
2. Bisect F-F′ focal axis using F-F′ circle intersections to locate orbit center C.

R=+1/2 hyperbolic orbit construction

R=+1/2

γ=45°



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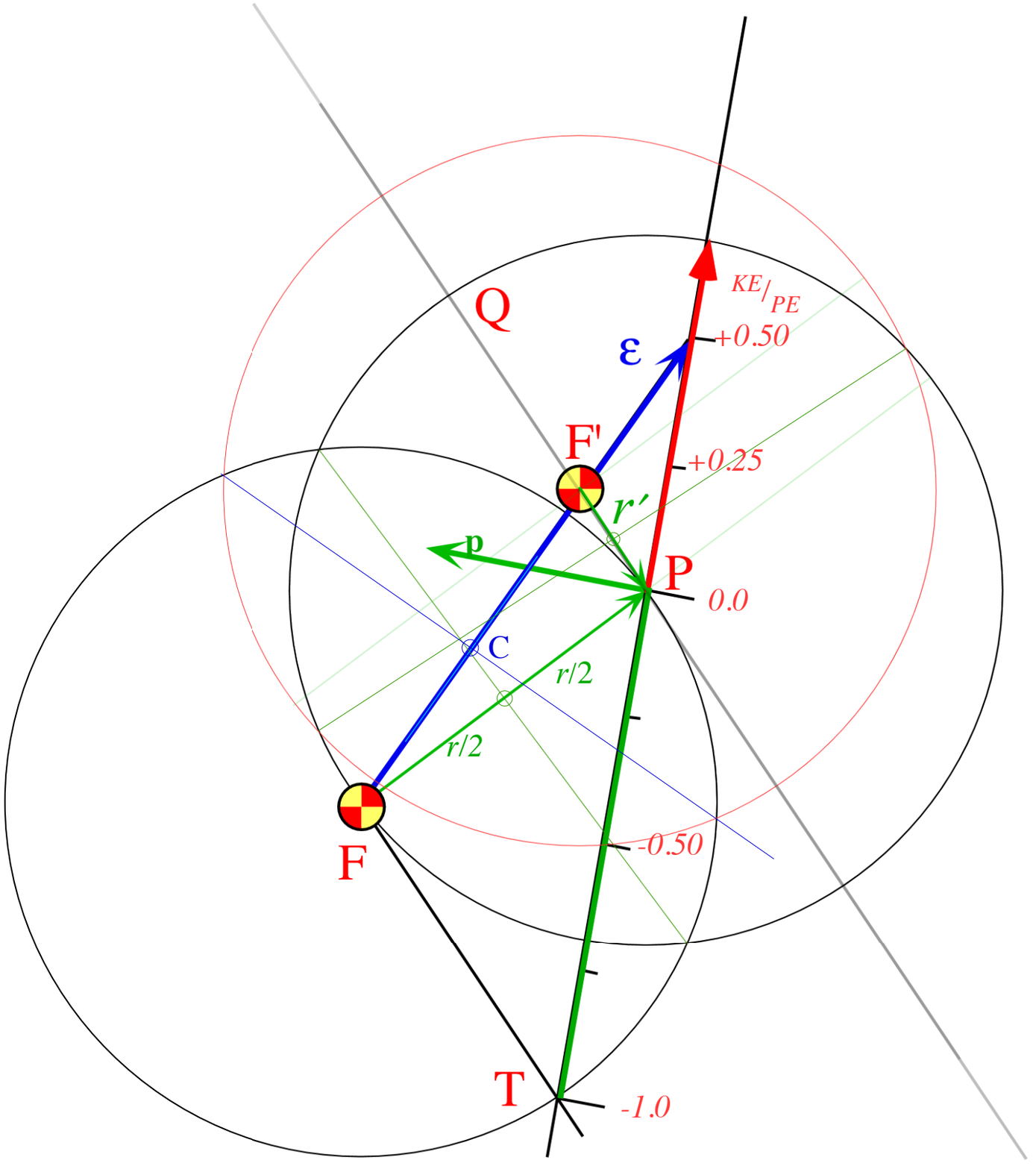
3. Bisect F′-P radius r′ using F′-P circle intersections.

R=+1/2 hyperbolic orbit construction

R=+1/2

γ=45°

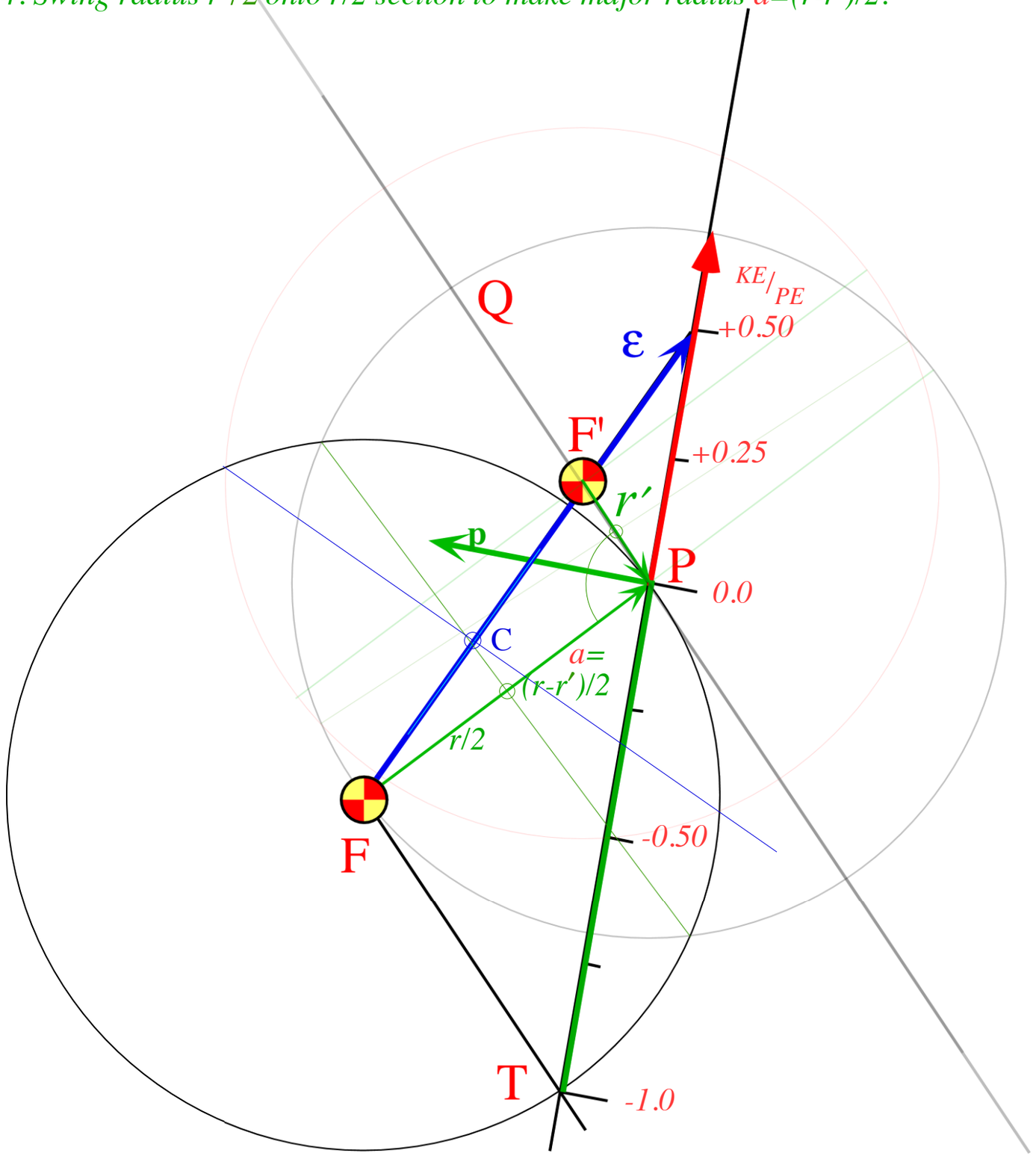


Major diameter 2a is difference (r-r′=2a).
Major radius a is half of difference (r-r′)/2=a
Major diameter 2a needs to be centered on F-F′ focal axis
1. Bisect F-P radius r using F-P circle intersections to define r/2 sections.
2. Bisect F-F′ focal axis using F-F′ circle intersections to locate orbit center C.
3. Bisect F′-P radius r′ using F′-P circle intersections.
4. Swing radius r′/2 onto r/2 section to make major radius a=(r-r′)/2.

R=+1/2 hyperbolic orbit construction

R=+1/2

γ=45°



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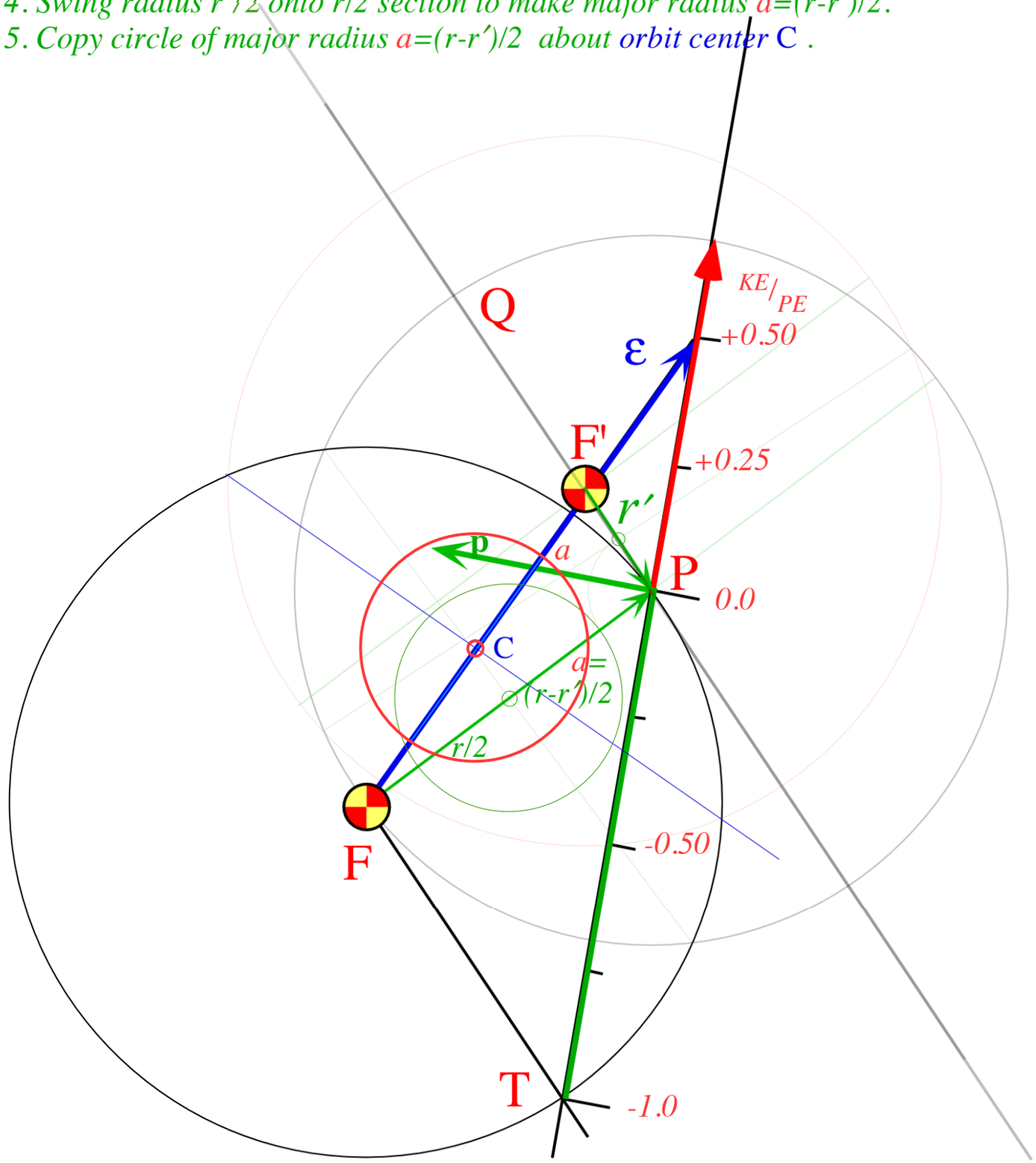
Major diameter 2a needs to be centered on F-F′ focal axis
1. Bisect F-P radius r using F-P circle intersections to define r/2 sections.
2. Bisect F-F′ focal axis using F-F′ circle intersections to locate orbit center C.
3. Bisect F′-P radius r′ using F′-P circle intersections.
4. Swing radius r′/2 onto r/2 section to make major radius a=(r-r′)/2.
5. Copy circle of major radius a=(r-r′)/2 about orbit center C .

R=+1/2 hyperbolic orbit construction

R=+1/2

γ=45°



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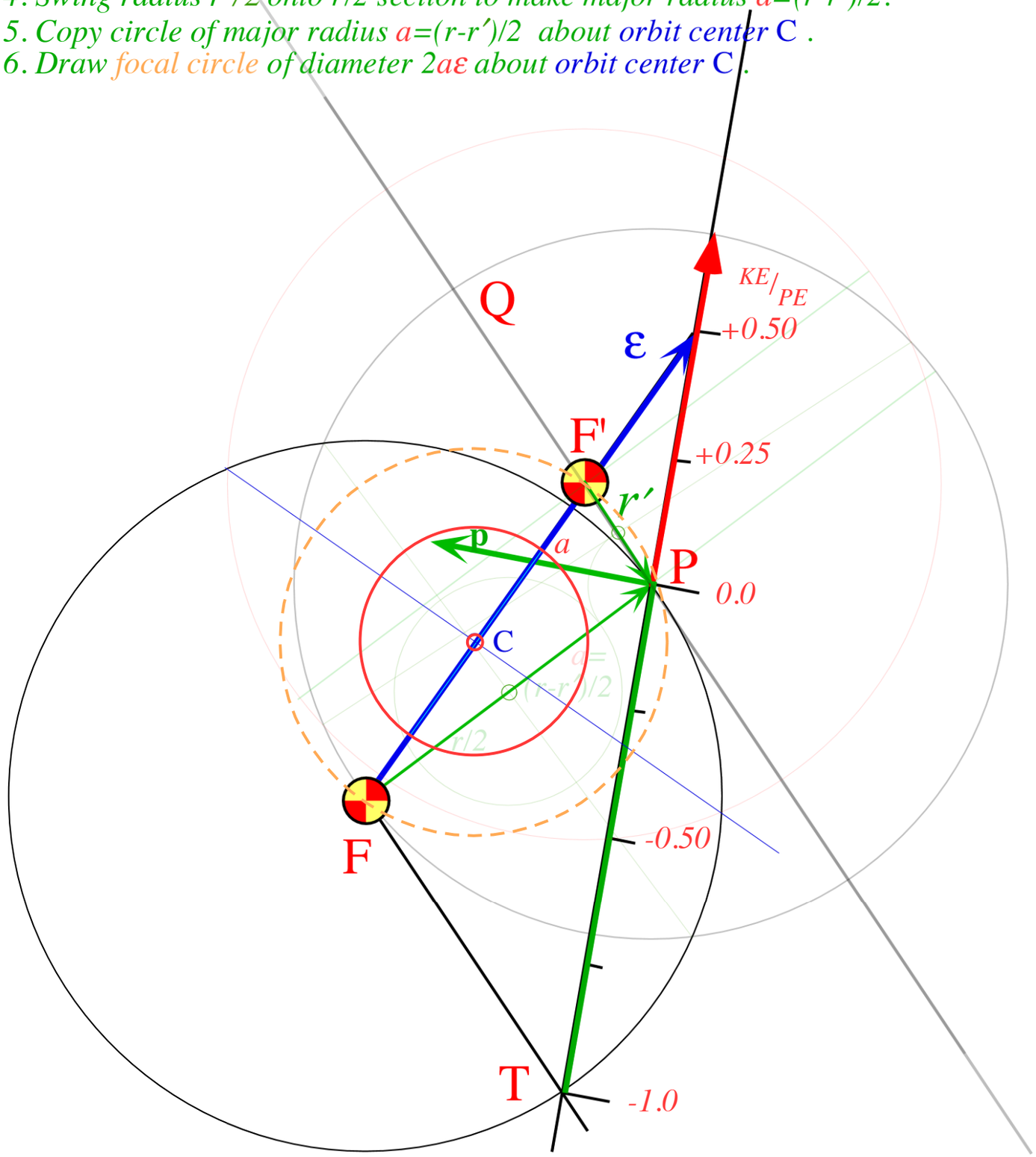
Major diameter 2a needs to be centered on F-F′ focal axis
1. Bisect F-P radius r using F-P circle intersections to define r/2 sections.
2. Bisect F-F′ focal axis using F-F′ circle intersections to locate orbit center C.
3. Bisect F′-P radius r′ using F′-P circle intersections.
4. Swing radius r′/2 onto r/2 section to make major radius a=(r-r′)/2.
5. Copy circle of major radius a=(r-r′)/2 about orbit center C .
6. Draw focal circle of diameter 2aε about orbit center C.

R=+1/2 hyperbolic orbit construction

R=+1/2

γ=45°



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Major diameter 2a needs to be centered on F-F′ focal axis
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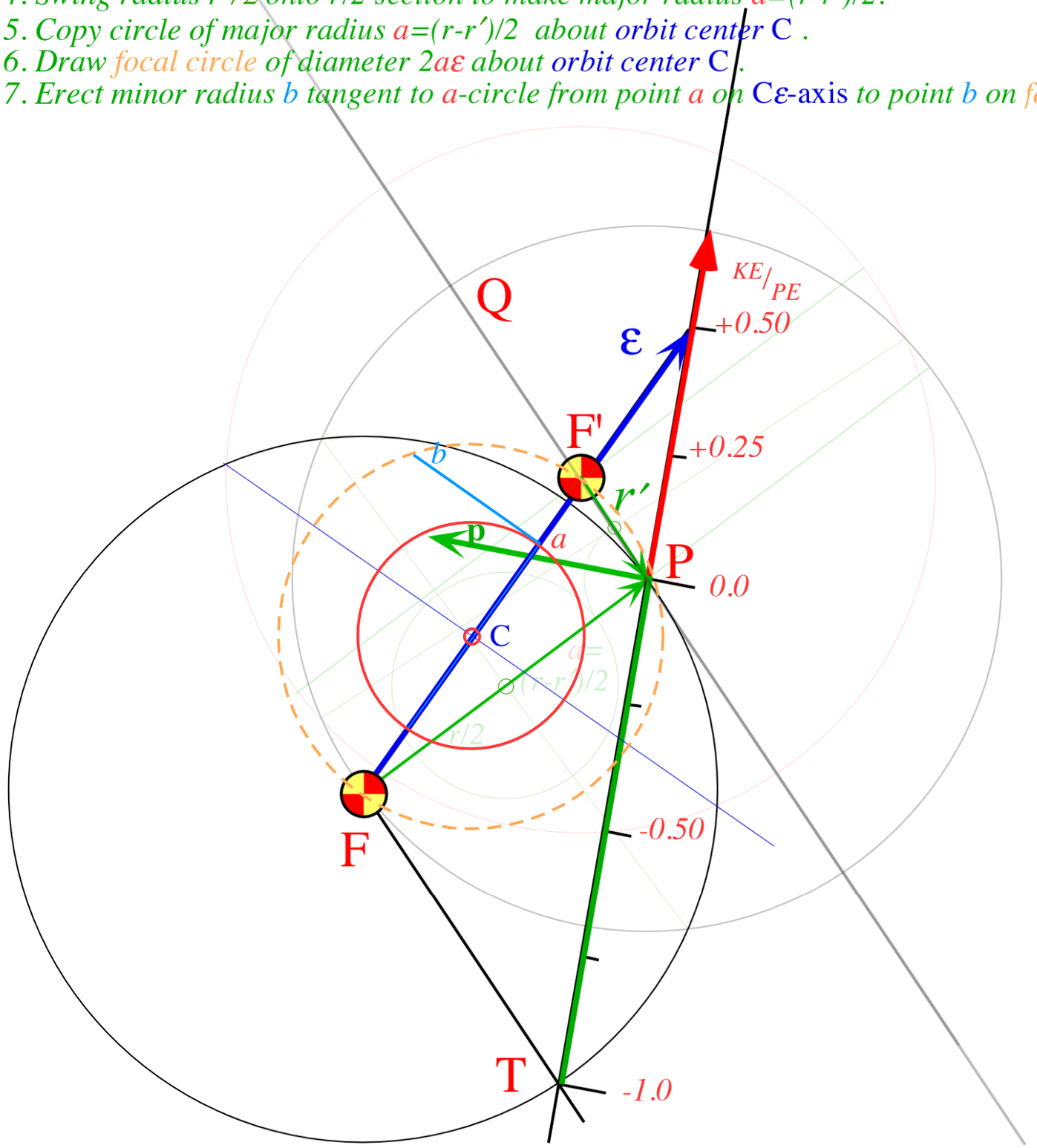
2. Bisect F-F′ focal axis using F-F′ circle intersections to locate orbit center C.
3. Bisect F′-P radius r′ using F′-P circle intersections.
4. Swing radius r′/2 onto r/2 section to make major radius a=(r-r′)/2.
5. Copy circle of major radius a=(r-r′)/2 about orbit center C .
6. Draw focal circle of diameter 2aε about orbit center C.
7. Erect minor radius b tangent to a-circle from point a on Cε-axis to point b on focal circle.

R=+1/2 hyperbolic orbit construction

R=+1/2

γ=45°



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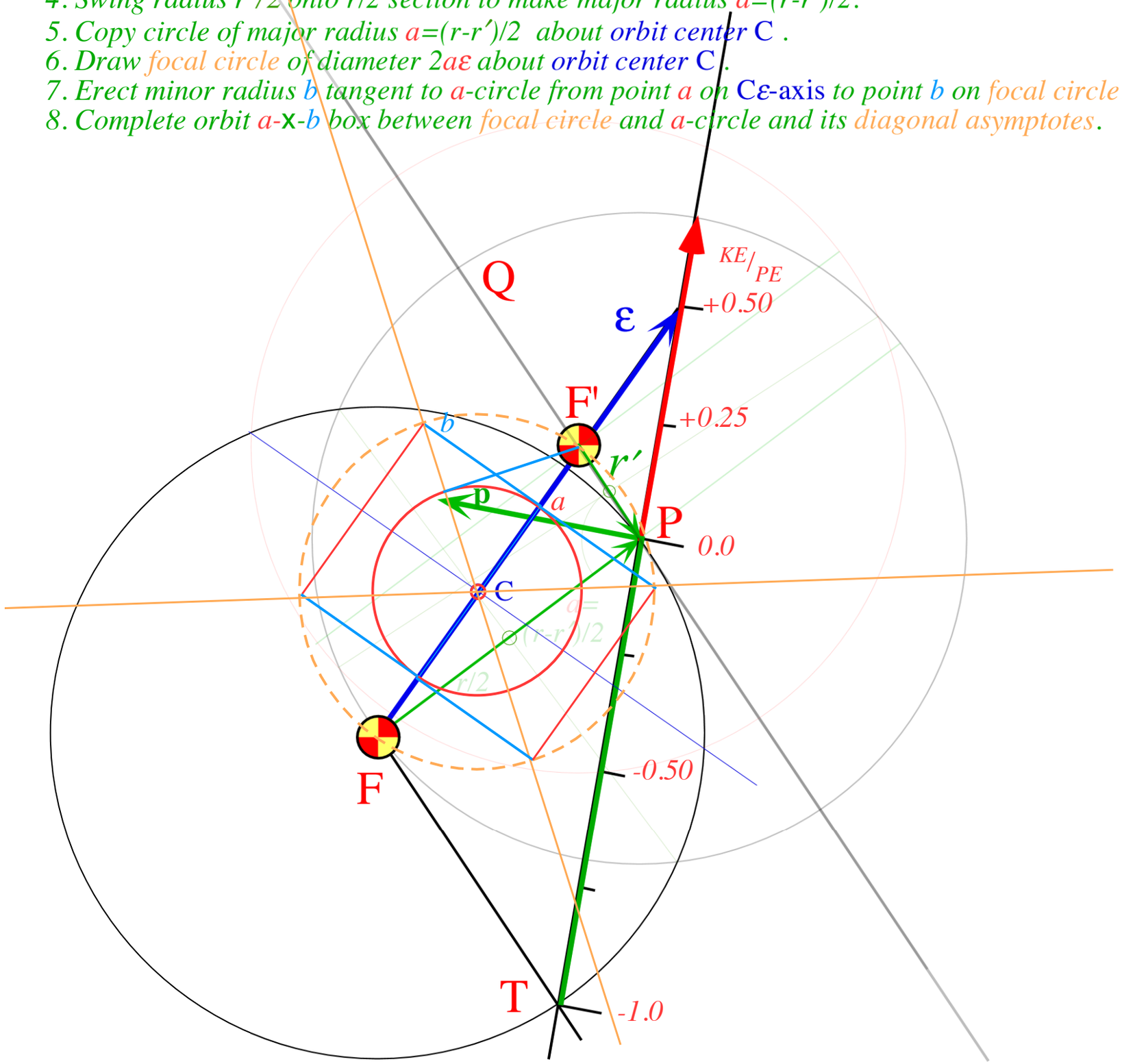
4. Swing radius r′/2 onto r/2 section to make major radius a=(r-r′)/2.
5. Copy circle of major radius a=(r-r′)/2 about orbit center C .
6. Draw focal circle of diameter 2aε about orbit center C.
7. Erect minor radius b tangent to a-circle from point a on Cε-axis to point b on focal circle.
8. Complete orbit a-x-b box between focal circle and a-circle and its diagonal asymptotes.

R=+1/2 hyperbolic orbit construction

R=+1/2

γ=45°

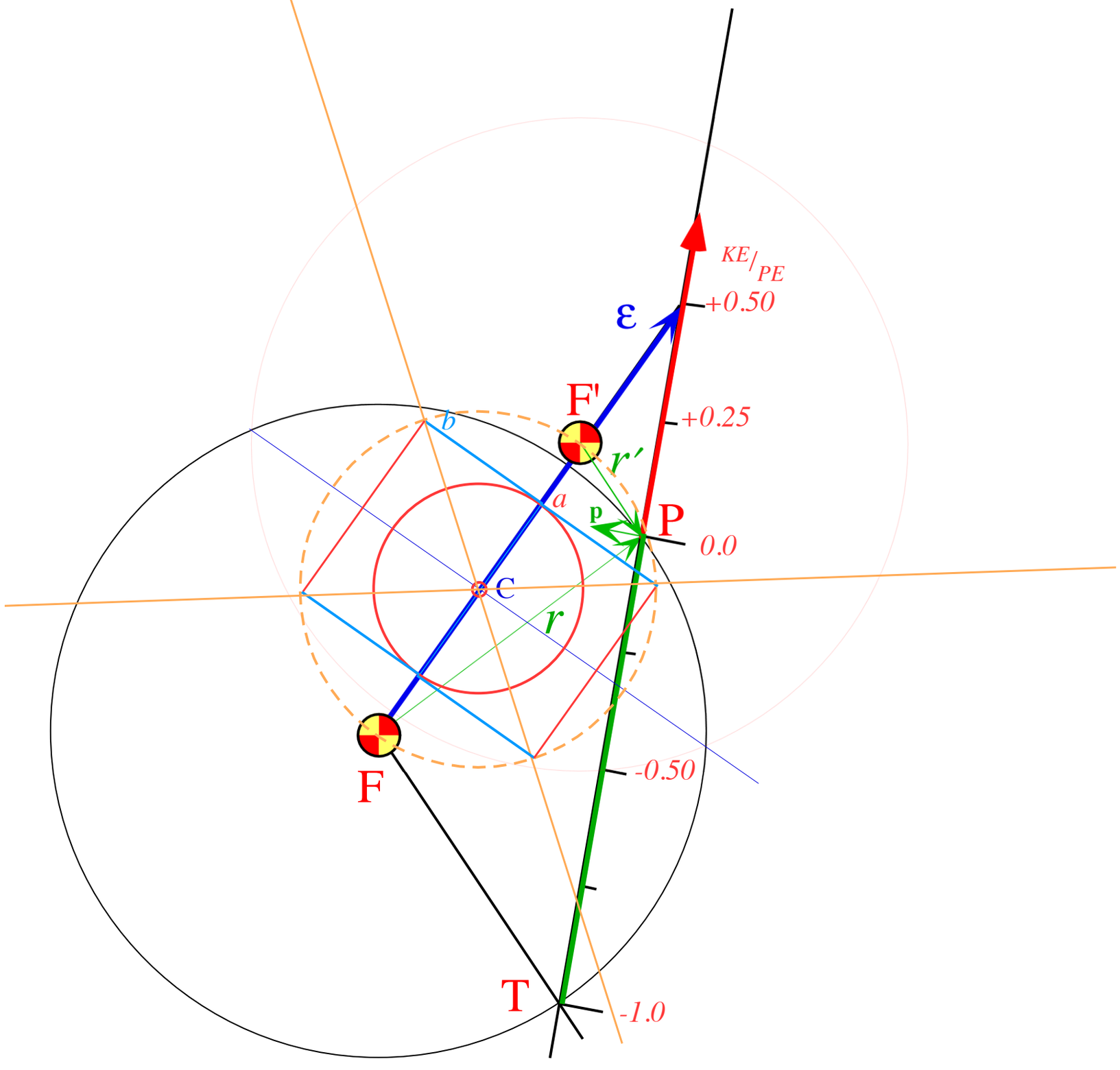


9. Draw section of hyperbolic orbit.

R=+1/2 hyperbolic orbit construction

R=+1/2

γ=45°

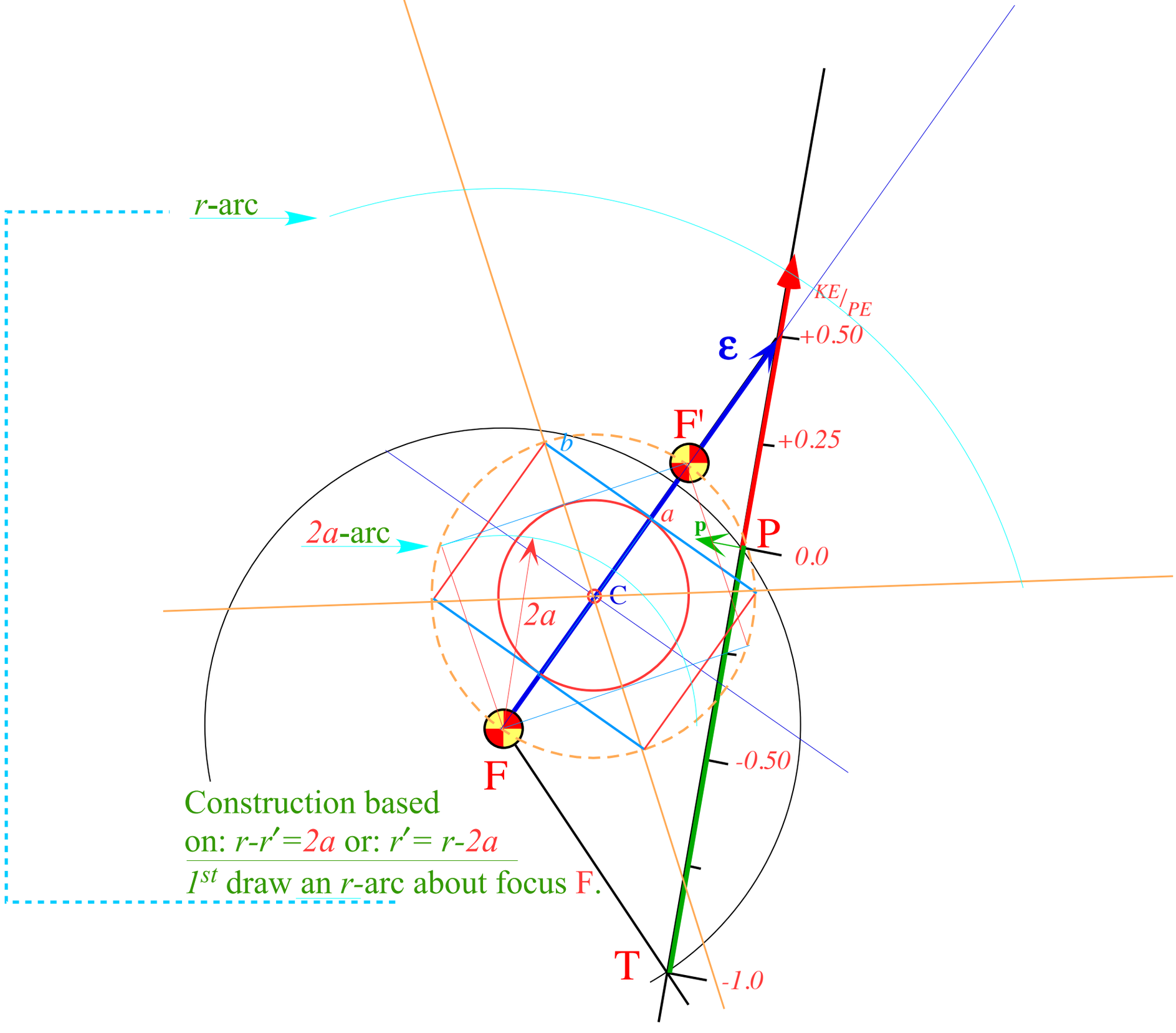


R=+1/2 hyperbolic orbit construction

R=+1/2

γ=45°

9. Draw section of hyperbolic orbit.



Construction based
on: $r\text{-}r'=2a$ or: $r'=r\text{-}2a$

1st draw an *r*-arc about focus F.

R=+1/2 hyperbolic orbit construction

R=+1/2

γ=45°

9. Draw section of hyperbolic orbit.

r-arc

r-arc-minus-*2a* distance (also on Cε-*line*)

2a-arc

KE/*PE*

+0.50

+0.25

0.0

-0.50

-1.0

ε

F'

F

P

T

C

a

b

2a

p

Construction based on: *r*-*r′*=*2a* or: *r′*= *r*-*2a*

1st draw an *r*-arc about focus F.

2st set compass to *(r-2a)* using *r*-arc-minus-*2a* on Cε-*line*.

R=+1/2 hyperbolic orbit construction

R=+1/2

γ=45°

9. Draw section of hyperbolic orbit.

r-arc

r-arc-minus-*2a* distance (also on Cε-*line*)

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+0.50

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ε

F'

F

T

P

C

a

b

2a

p

Construction based on: *r*-*r′*=*2a* or: *r′*= *r*-*2a*

1st draw an *r*-arc about focus F.

2st set compass to *(r-2a)* using *r*-arc-minus-*2a* on Cε-*line*.

3rd draw *(r-2a)*-arc about focus F′.

R=+1/2 hyperbolic orbit construction

R=+1/2

γ=45°

9. Draw section of hyperbolic orbit.

r-arc

r-arc-minus-*2a* distance (also on Cε-*line*)

r

r′= r-2a

ε

KE/PE

+0.50

F'

+0.25

b

a

p

P

0.0

2a-arc

C

2a

F

-0.50

T

-1.0

Construction based on: *r-r′=2a* or: *r′= r-2a*

1st draw an *r*-arc about focus F.

2st set compass to *(r-2a)* using *r*-arc-minus-*2a* on Cε-*line*.

3rd draw *(r-2a)*-arc about focus F′.

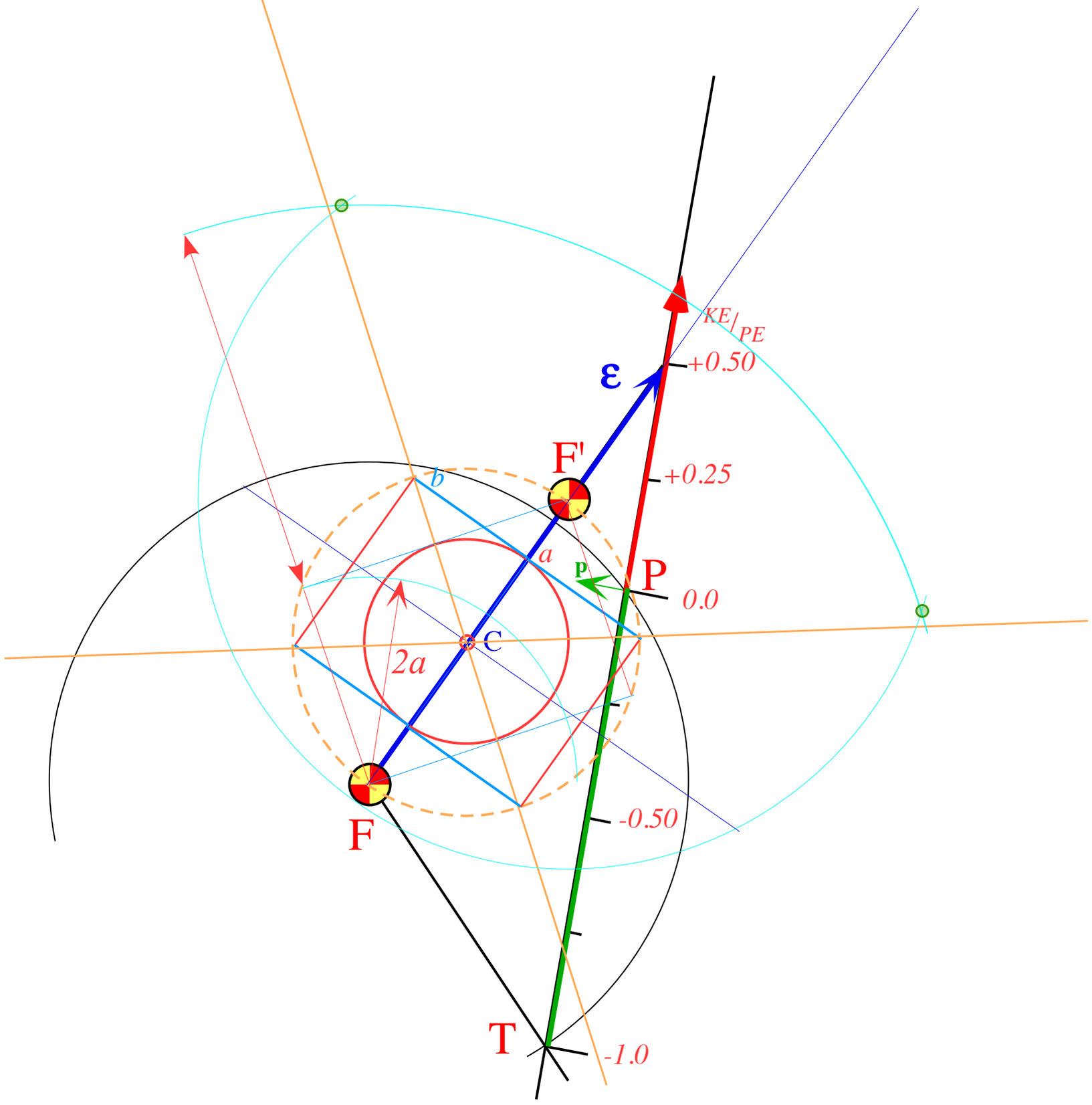
Orbit points at intersections.

9. Draw section of hyperbolic orbit.

R=+1/2 hyperbolic orbit construction

R=+1/2

γ=45°

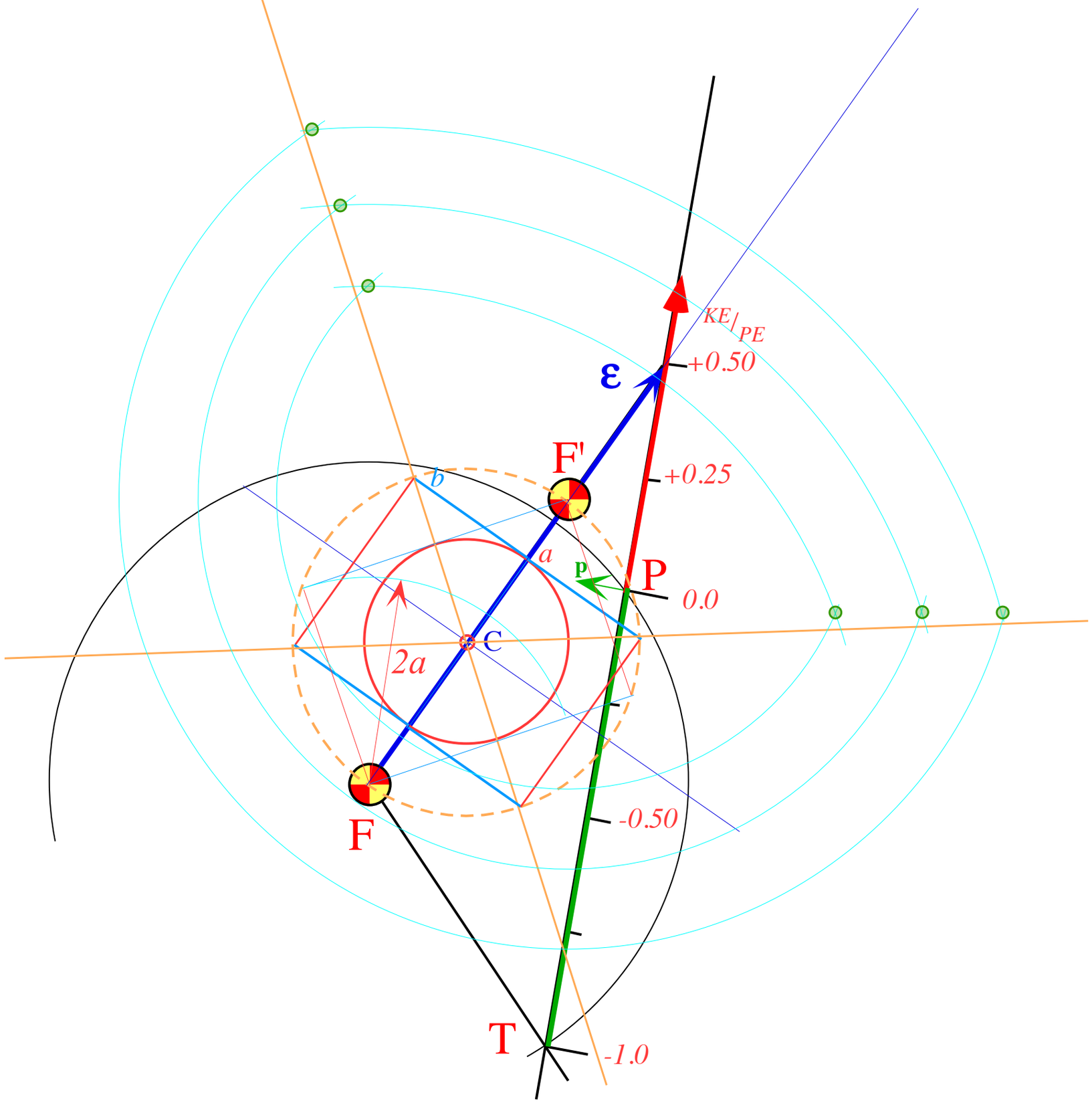


9. Draw section of hyperbolic orbit.

R=+1/2 hyperbolic orbit construction

R=+1/2

γ=45°



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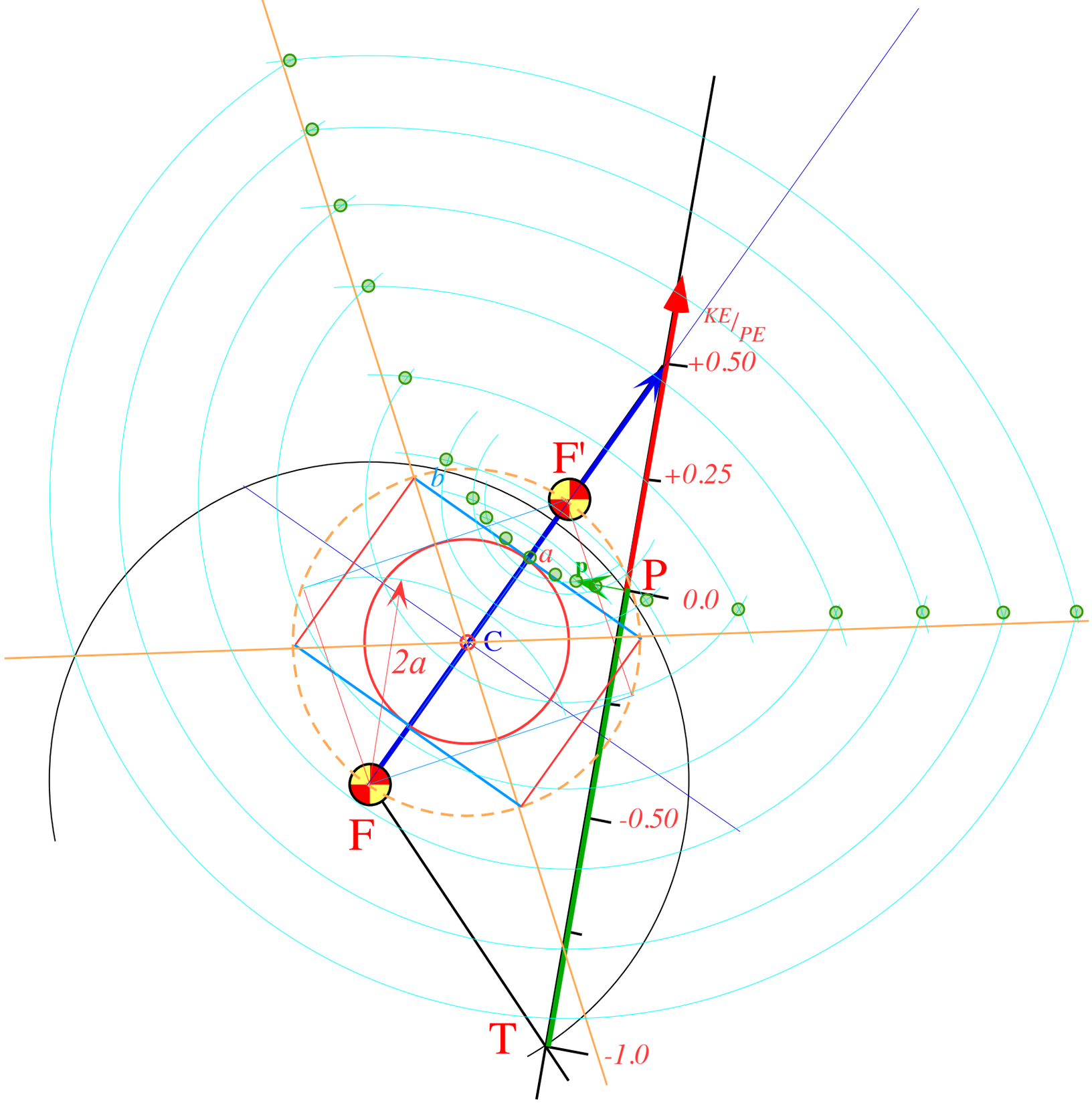
R=+1/2 hyperbolic orbit construction

R=+1/2

γ=45°



9. Draw section of hyperbolic orbit.

R=+1/2 hyperbolic orbit construction

R=+1/2

γ=45°

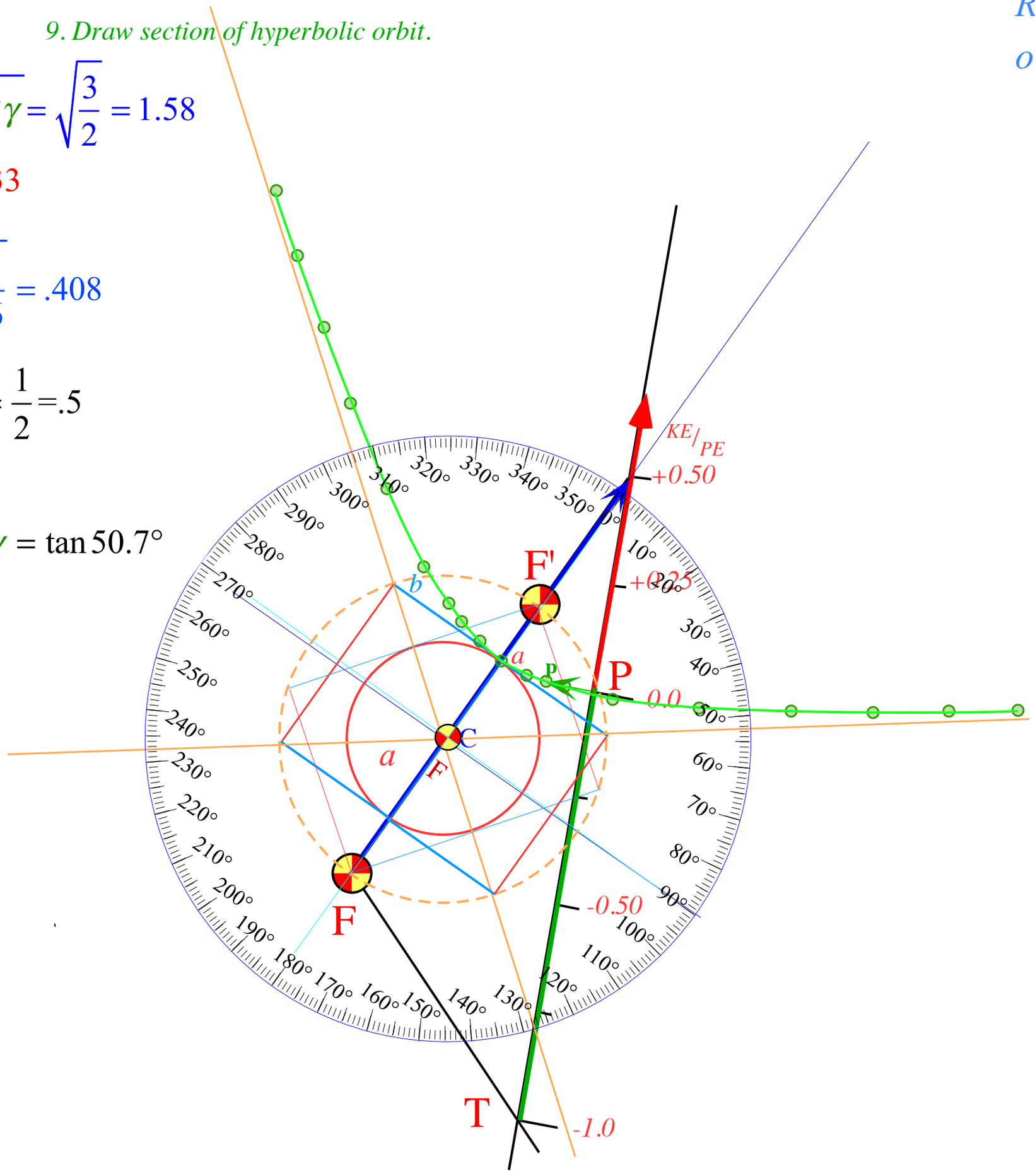
$$\varepsilon=\sqrt{1+4R(R+1)\sin^2\gamma}=\sqrt{\frac{3}{2}}=1.58$$

$$a=\frac{1}{2(R+1)}=\frac{1}{3}=.33$$

$$b=\sqrt{\frac{R}{R+1}}\sin\gamma=\sqrt{\frac{1}{6}}=.408$$

$$\lambda=\frac{b^2}{a}=2R\sin^2\gamma=\frac{1}{2}=.5$$

$$\frac{b}{a}=2\sqrt{R(R+1)}\sin\gamma=\tan 50.7°$$



Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Analytic geometry derivation of $\boldsymbol{\varepsilon}$*-construction*

Connection formulas for (a,b) *and* (ε,λ) *with* (γ,R)

Detailed ruler & compass construction of $\boldsymbol{\varepsilon}$*-vector and orbits*

(R=-0.375 elliptic orbit)

(R=+0.5 hyperbolic orbit)

Properties of Coulomb trajectory families and envelc

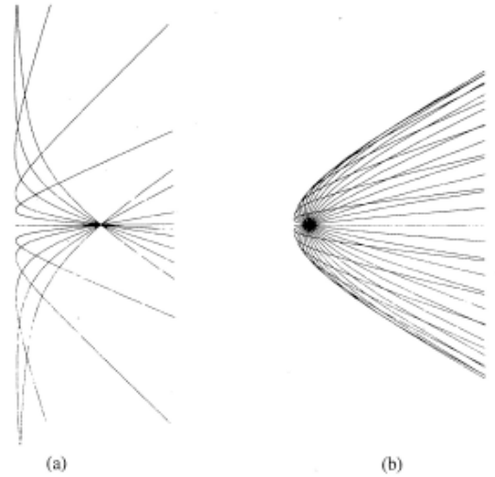
➔ *Graphical* $\boldsymbol{\varepsilon}$*-development of orbits*

➔ *Launch angle fixed-Varied launch energy*

Launch energy fixed-Varied launch angle

Launch optimization and orbit family envelopes

Lenz Vector...analog computersAJP 44 4 (1974)



(a) (b)

Fig. 11. Sample computer trajectories. (a) Family of hyperbolic orbits with $R = 1$. (b) Family of hyperbolic orbits with $R < 1$.

one can show approximately the effects of the solar wind, if we imagine that the force field for each floating dust particle is proportional to the local solar flux (i.e., $1/r^2$) minus the solar gravity. The latter is proportional to the mass of the particle, while the former is proportional to the particle cross section, so we expect the repulsive solar wind to become larger for smaller particles. In any case if some fixed or nearly fixed object is emitting particles in all directions with about the same speed, we can use the computer to track them and show the envelopes of the trajectories that are analogous to the ones for the elliptical orbits. Indeed, the envelope changes from a concave hyperpola (see Ref. 1, Fig. 2) to a plane [Fig. 11(a)] to a convex hyperbola [Fig. 11(b)] as the field becomes more repulsive (i.e., as R becomes less). The latter reminds one of the shape of a comet.

V. CONCLUSION

So far, I have only used the simple model of the computer in class, and I am waiting to try out the more complicated instrument. However, I have found that most students are quite fascinated by this business and many have been stimulated to learn much more about it.

Undoubtedly, many alternative schemes are possible. For example, a large plotting board in front of the class might be better than the overhead projector. (One could still use it in an unruly class where dimming the lights becomes a hazard!) Another suggestion involves replacing the scales in Fig. 7 with a simple piece of polar graph paper made into a movable transparency. Now for each "shot" you must compute the R scale. [An R circle has a radius equal to $\pm R/(R+1)$ times the distance to the force center, where the sign taken is that of the total energy $T+V$.] However, in this way you may vary your initial position and make midcourse corrections.

It is hoped that the ideas for problems and solutions suitable for this type of analog computer, some of which were listed in Sec. III and at the end of Sec. II, can lead to interesting projects for many students.

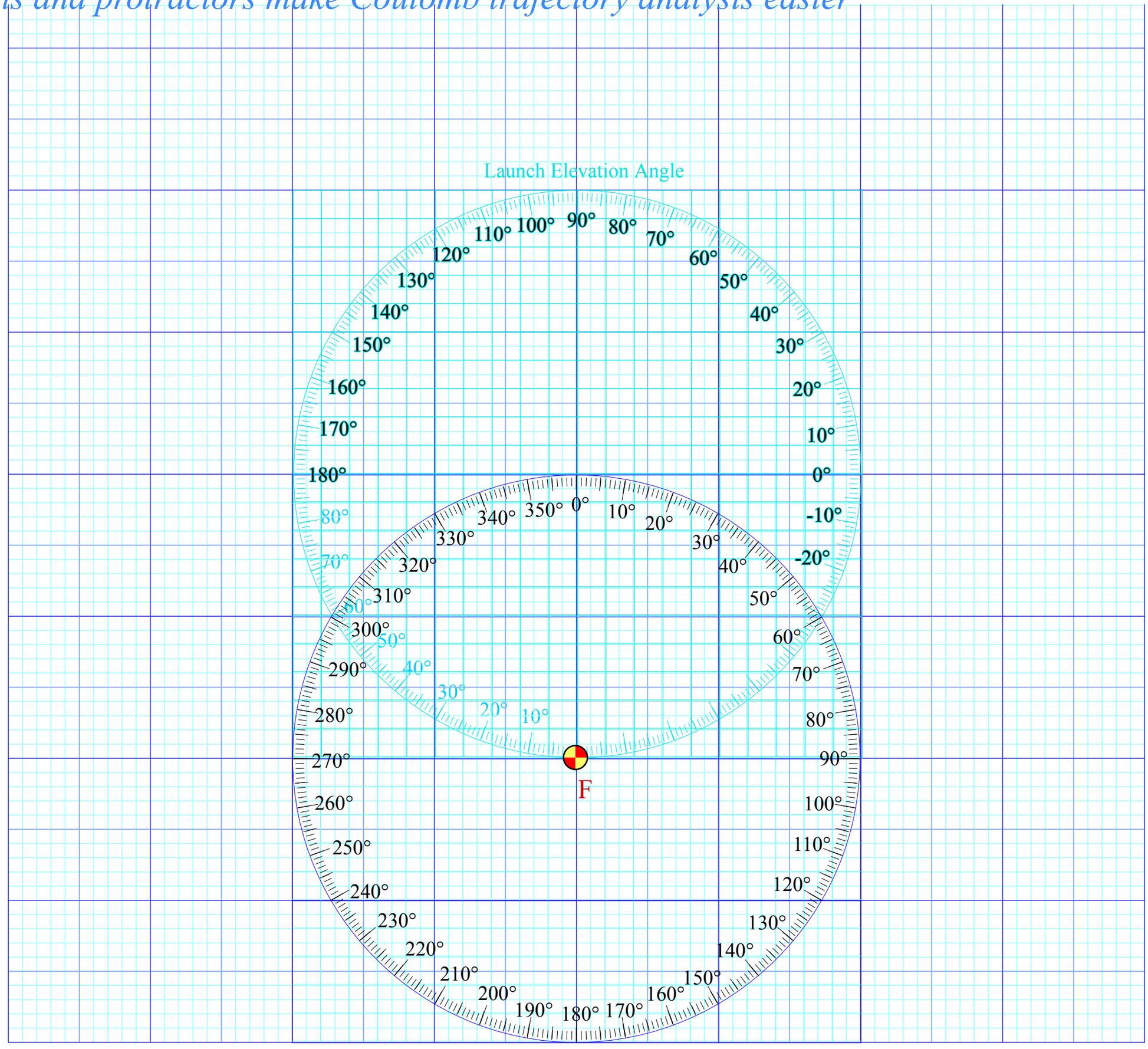
ACKNOWLEDGMENTS

The author would like to thank Professor N. Jannuzzi and E. Lujan for making possible the construction of these devices.

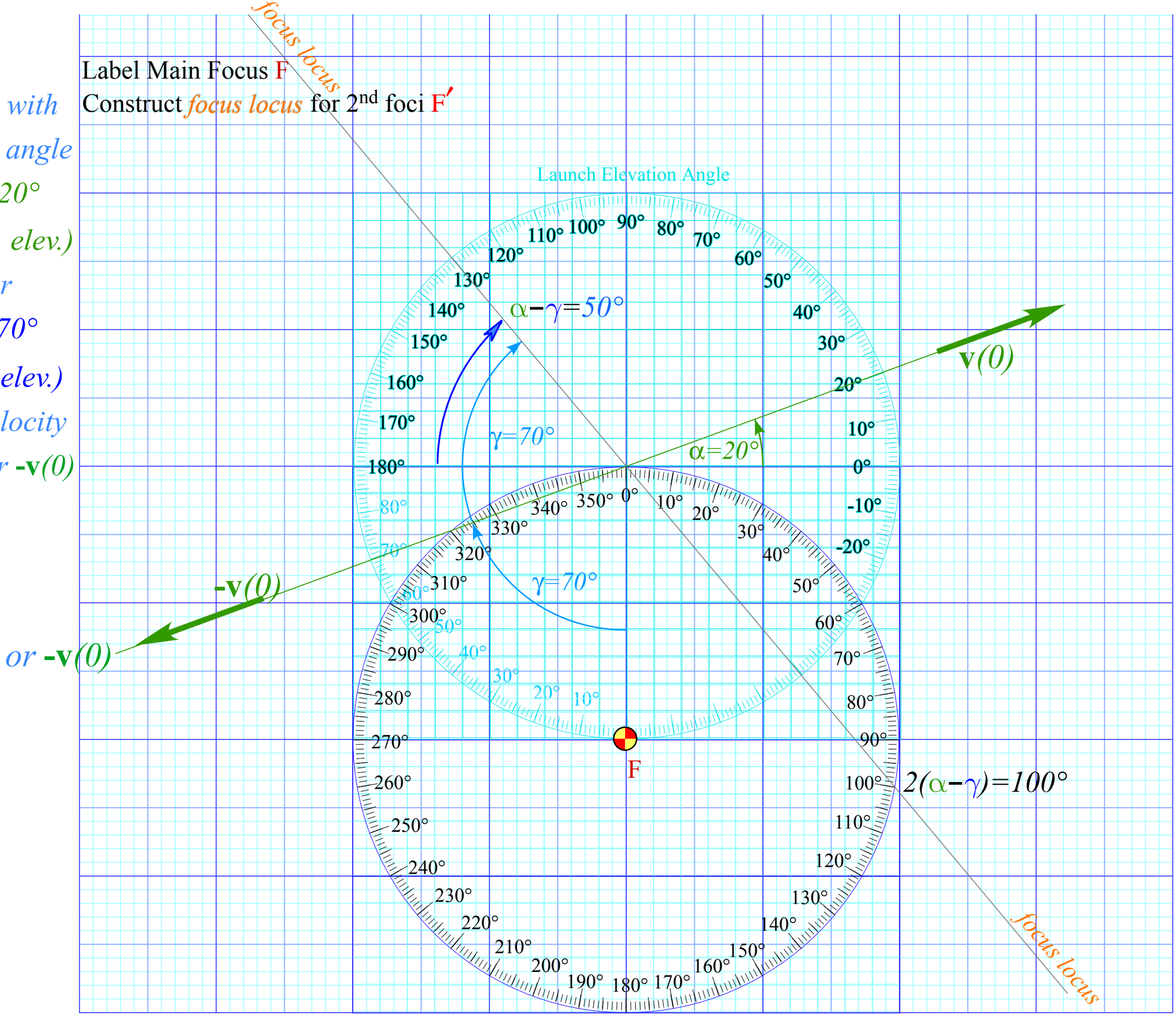
*Research supported in part by Centro Tecnico Aerospacial, S.P., Brasil.

[1]J. W. Adolph, A. L. Garcia, W. G. Harter, G. C. McLanghlin, R. R. Shiffman, and V. G. Surkus, Am. J. Phys. **40,** 1852 (1972).
[2]W. Pauli, Z. Phys. **36,** 336 (1926).
[3]W. Lenz, Z. Phys. **24,** 197 (1924).
[4]B. G. Wyebourne, *New Directions in Atomic Physics,* edited by E. U. Condon (Yale U.P., New Haven, CT, 1972), p. 96.
[5]H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, MA, 1950).
[6]Reference 5, p. 79.

Graphs and protractors make Coulomb trajectory analysis easier



Start with initial angle $\alpha=20°$ *(horiz. elev.) or* $\gamma=70°$ *(rad. elev.) for velocity* $\mathbf{v}(0)$ *or* $-\mathbf{v}(0)$

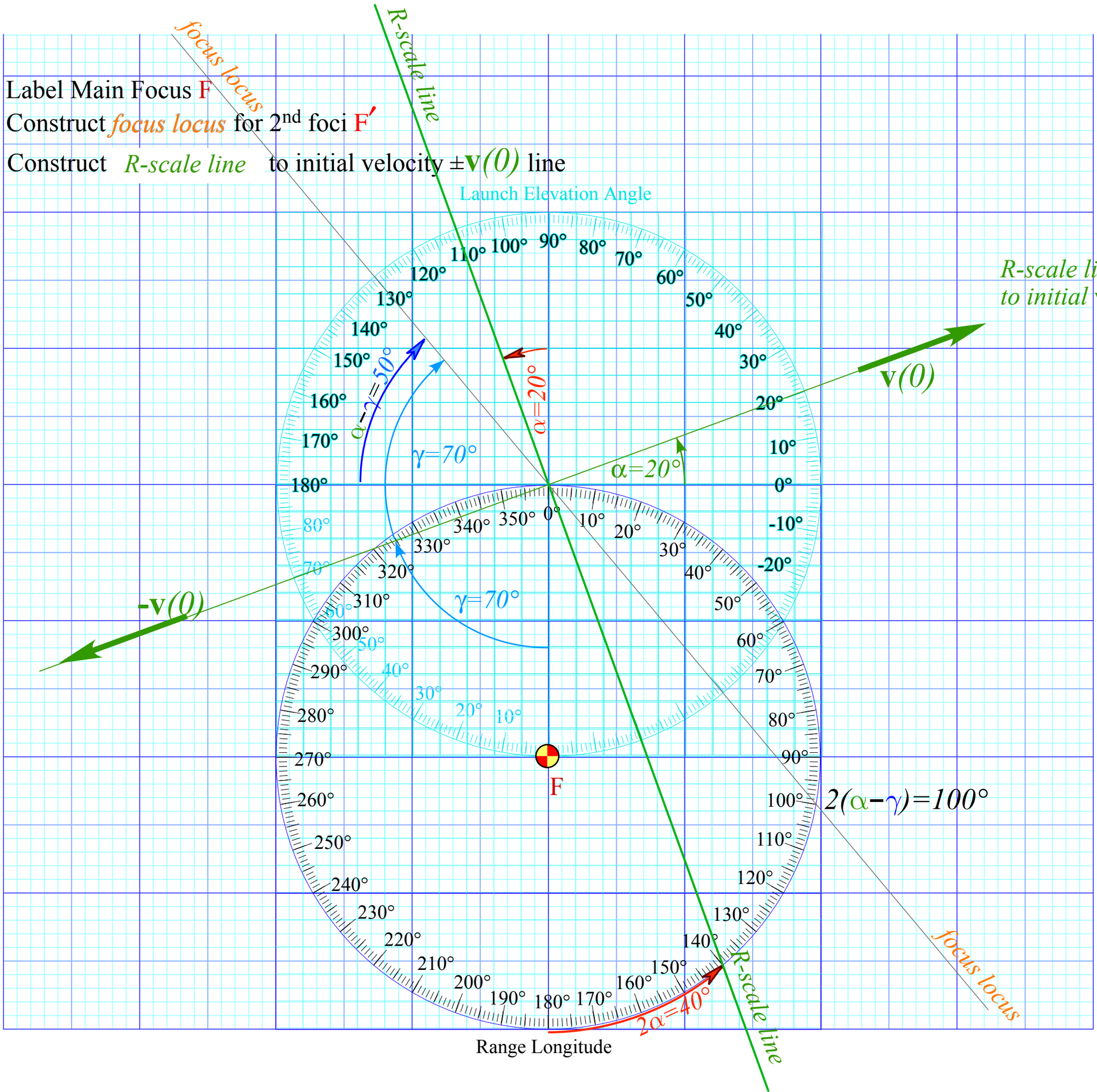


Start with initial angle $\alpha=20°$ *(horiz. elev.) or* $\gamma=70°$ *(rad. elev.) for velocity* $\mathbf{v}(0)$ *or* $-\mathbf{v}(0)$

Label Main Focus F

Construct *focus locus* for 2nd foci F′

Construct *R-scale line* to initial velocity $\pm\mathbf{v}(0)$ line



Start with initial angle $\alpha=20°$ *(horiz. elev.) or* $\gamma=70°$ *(rad. elev.) for velocity* $\mathbf{v}(0)$ *or* $-\mathbf{v}(0)$

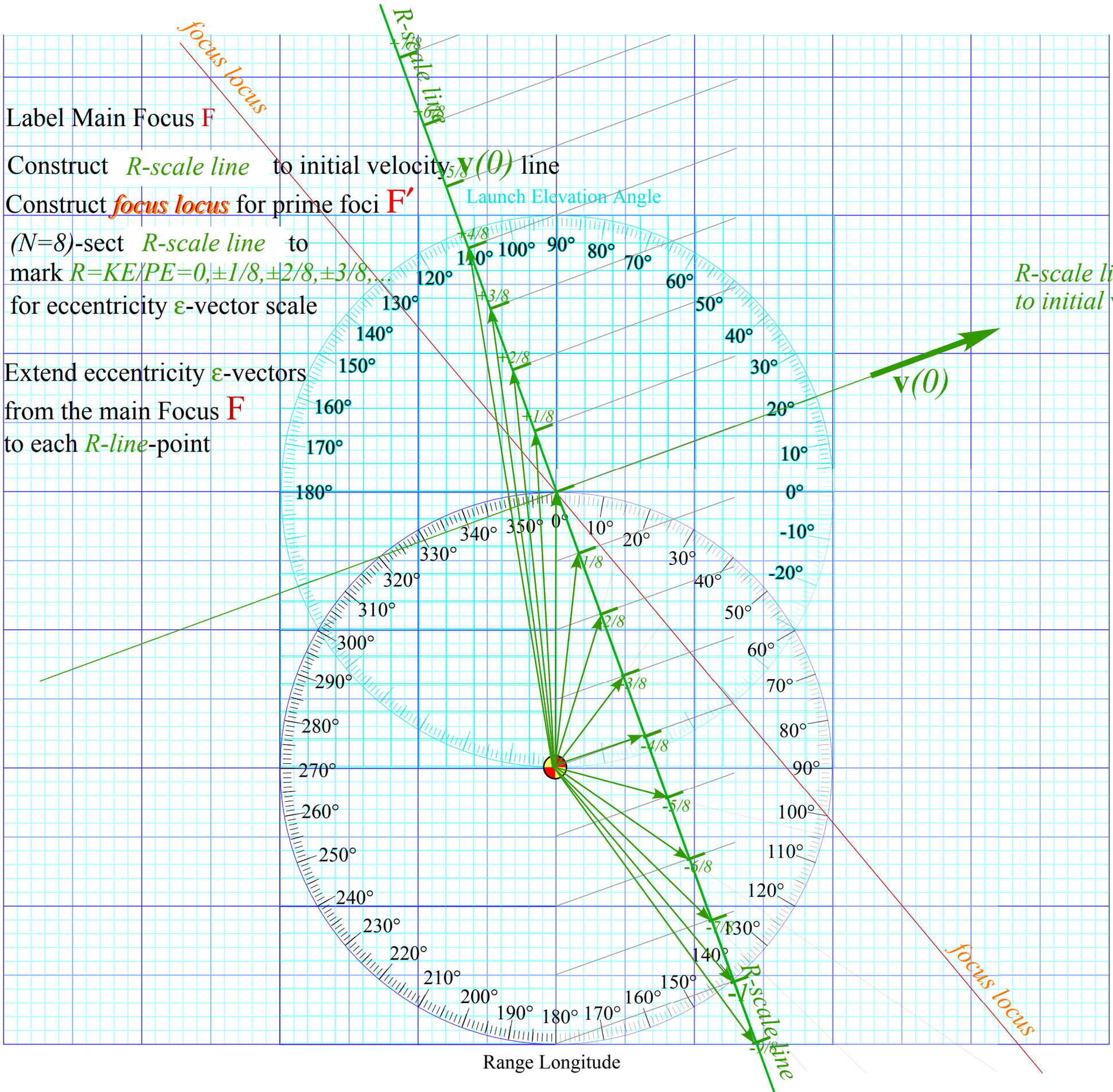
Label Main Focus F

Construct *R-scale line* to initial velocity $\mathbf{v}(0)$ line

Construct ***focus locus*** for prime foci F′

(N=8)-sect *R-scale line* to mark $R=KE/PE=0,\pm1/8,\pm2/8,\pm3/8,\ldots$ for eccentricity ε-vector scale

Extend eccentricity ε-vectors from the main Focus F to each *R-line*-point



or $-\mathbf{v}(0)$

Start with initial angle $\alpha=20°$ *(horiz. elev.) or* $\gamma=70°$ *(rad. elev.) for velocity* $\mathbf{v}(0)$ *or* $-\mathbf{v}(0)$

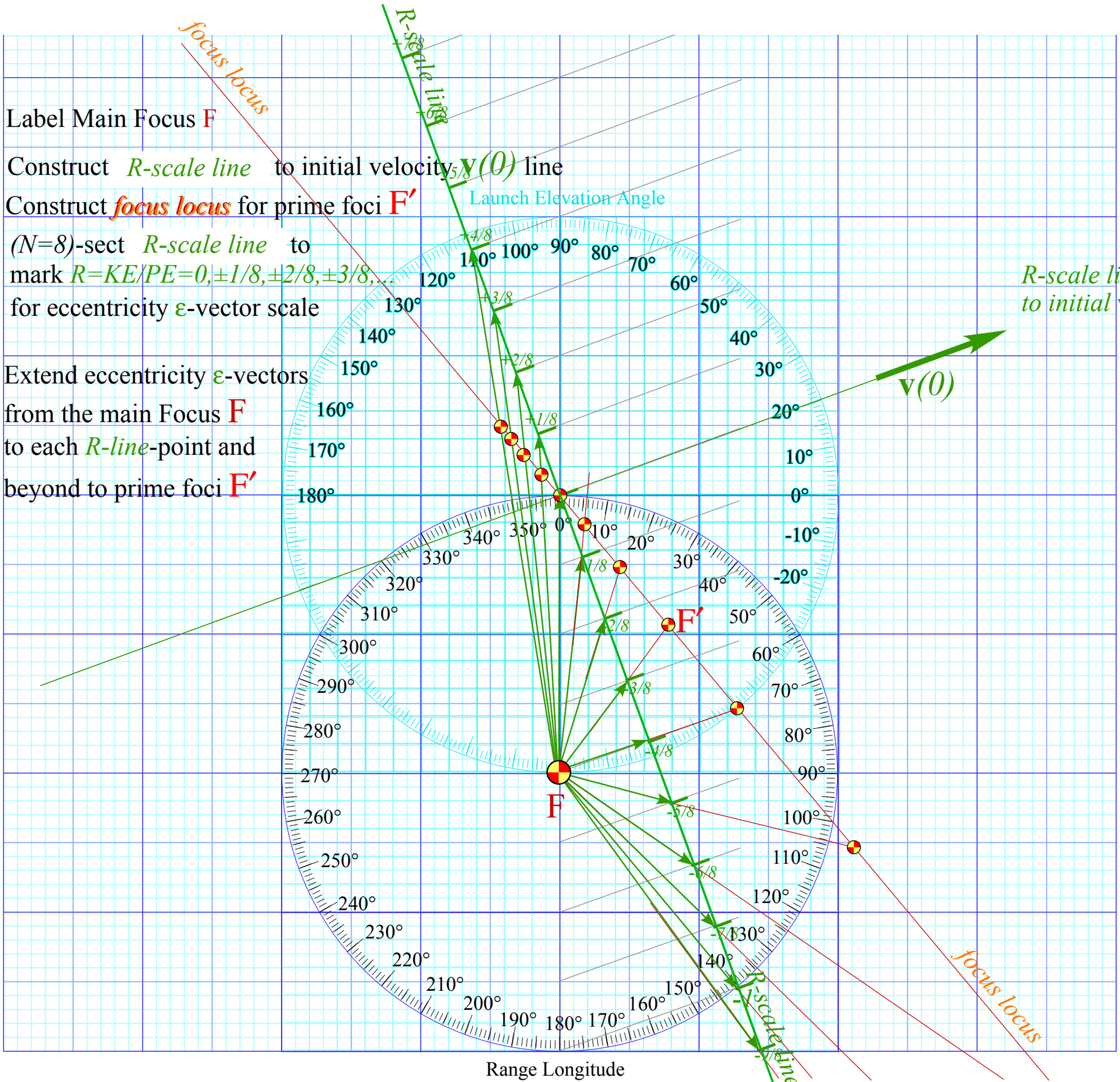
Label Main Focus F

Construct *R-scale line* to initial velocity $\mathbf{v}(0)$ line

Construct ***focus locus*** for prime foci F′

(N=8)-sect *R-scale line* to mark $R=KE/PE=0,\pm1/8,\pm2/8,\pm3/8,\ldots$ for eccentricity ε-vector scale

Extend eccentricity ε-vectors from the main Focus F to each *R-line*-point and beyond to prime foci F′



Start with initial angle $\alpha=20°$ *(horiz. elev.) or* $\gamma=70°$ *(rad. elev.) for velocity* $\mathbf{v}(0)$ *or* $-\mathbf{v}(0)$

initial velocity $\pm\mathbf{v}(0)$ line

Launch Elevation Angle

R-scale line

focus-locus

[R=1 ε-line parallel to focus-locus]

R-scale line is normal to initial $\mathbf{v}(0)$*-line*

$\mathbf{v}(0)$

$-\mathbf{v}(0)$

$\alpha=20°$

$\gamma=70°$

$\alpha=50°$

$\gamma=70°$

$2(\alpha-\gamma)=100°$

$2\alpha=40°$

F

R=1 *focal ε-axis for parabola*

focus locus

R-scale line

Range Longitude

Start with initial angle $\alpha=20°$ *(horiz. elev.) or* $\gamma=70°$ *(rad. elev.) for velocity* $\mathbf{v}(0)$ *or* $-\mathbf{v}(0)$

[(R=±∞) ε-line is parallel to R-scale line.]

R-scale line

focus-locus

(R=±∞) ε-line intersects focus-locus on unit circle.

Con… …ocity ±**v**(*0*) line

Launch Elevation Angle

[R=1 ε-line parallel to focus-locus]

±∞ point

R-scale line is normal to initial **v**(**0**)*-line*

v(*0*)

-**v**(*0*)

$\alpha=20°$

$\gamma=70°$

$\alpha=50°$

$\alpha=-40°$

$2(\alpha-\gamma)=100°$

F

R=1 focal ε-axis for parabola

focus locus

R-scale line

Range Longitude

This (R=-1) c

Start with initial angle $\alpha=20°$ *(horiz. elev.) or* $\gamma=70°$ *(rad. elev.) for velocity* **v**(0) *or* **-v**(0)

L

Cons ocity **v**(0) line

Constru Launch Elevation Angle

(N= 8)-sec

mark R= KEPLER= 1/8 1/4 3/8 1/2 5/8 3/4 7/8

for eccentricity -vector scale

Extend eccentricity ε-vectors from the main Focus F to each *R-line*-point and beyond to prime foci F′

This (R=-9/8) ε-line hits focus-locus way up to left.

focus-locus for R<-1 (attracted hyperbolae)

(R=±∞) ε-line intersects focus-locus on unit circle.

[(R=±∞) ε-line is parallel to R-scale line.]

±∞ point

focus-locus for ∞>R>0 (repelled hyperbolae)

R-scale line is normal to initial **v**(0)-line

v(0)

F

F′

focus locus

Range Longitude

This (R=-1) c

Start with initial angle

α=20°

(horiz. elev.)

or

γ=70°

(rad. elev.)

for velocity

v(*0*) *or* -**v**(*0*)

L

Cons

Constru

(N= 8)-sec

mark R= KE/rL= 1/8,1/4,3/8,1/2,5/8,3/4,7/8

for eccentricity -vector scale

Extend eccentricity ε-vectors

from the main Focus F

to each *R-line*-point and

beyond to prime foci F′

This (R=-9/8) ε-line hits focus-locus way up to left.

focus-locus for R<-1 (attracted hyperbolae)

(R=±∞) ε-line intersects focus-locus on unit circle.

[(R=±∞) ε-line is parallel to R-scale line.]

ocity **v**(*0*) line

Launch Elevation Angle

±∞ *point*

focus-locus for ∞>R>0 (repelled hyperbolae)

R-scale line is normal to initial **v**(*0*)*-line*

v(*0*)

F

F′

focus locus

Range Longitude

This (R=-1) c

Start with initial angle α=20° (horiz. elev.) or γ=70° (rad. elev.) for velocity **v**(0) or -**v**(0)

L

Cons ocity **v**(0) line

Constru

(N=8)-sec

mark R= KE/rL= 1/8 1/4 3/8

for eccentricity -vector scale

Extend eccentricity ε-vectors from the main Focus F to each *R-line*-point and beyond to prime foci F'

This (R=-9/8) ε-line hits focus-locus way up to left.

focus-locus for R<-1 (attracted hyperbolae)

(R=±∞) ε-line intersects focus-locus on unit circle.

[(R=±∞) ε-line is parallel to R-scale line.]

±∞ point

focus-locus for ∞>R>0 (repelled hyperbolae)

Launch Elevation Angle

R-scale line is normal to initial **v**(0)-line

v(0)

focus-locus for -∞<R<0 (attracted ellipses)

F

Range Longitude

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Analytic geometry derivation of $\boldsymbol{\varepsilon}$*-construction*

Connection formulas for (a,b) *and* (ε,λ) *with* (γ,R)

Detailed ruler & compass construction of $\boldsymbol{\varepsilon}$*-vector and orbits*

(R=-0.375 elliptic orbit)

(R=+0.5 hyperbolic orbit)

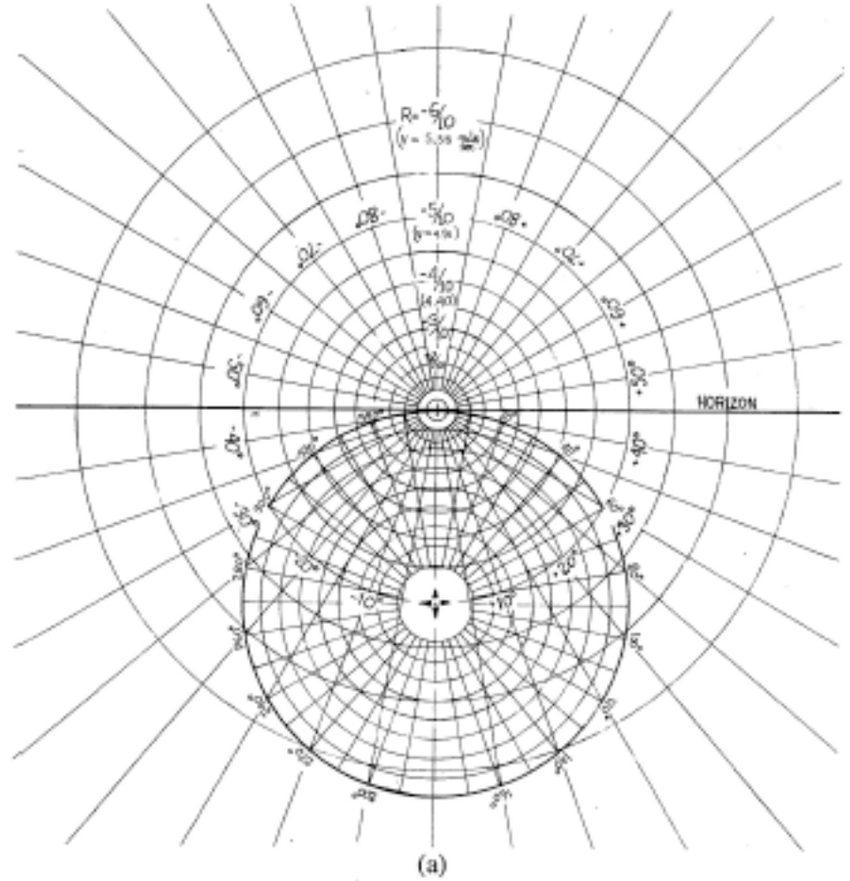
Properties of Coulomb trajectory families and envelopes

➔ *Graphical* $\boldsymbol{\varepsilon}$*-development of orbits*

➔ *Launch angle fixed-Varied launch energy*

Launch energy fixed-Varied launch angle

Launch optimization and orbit family envelopes



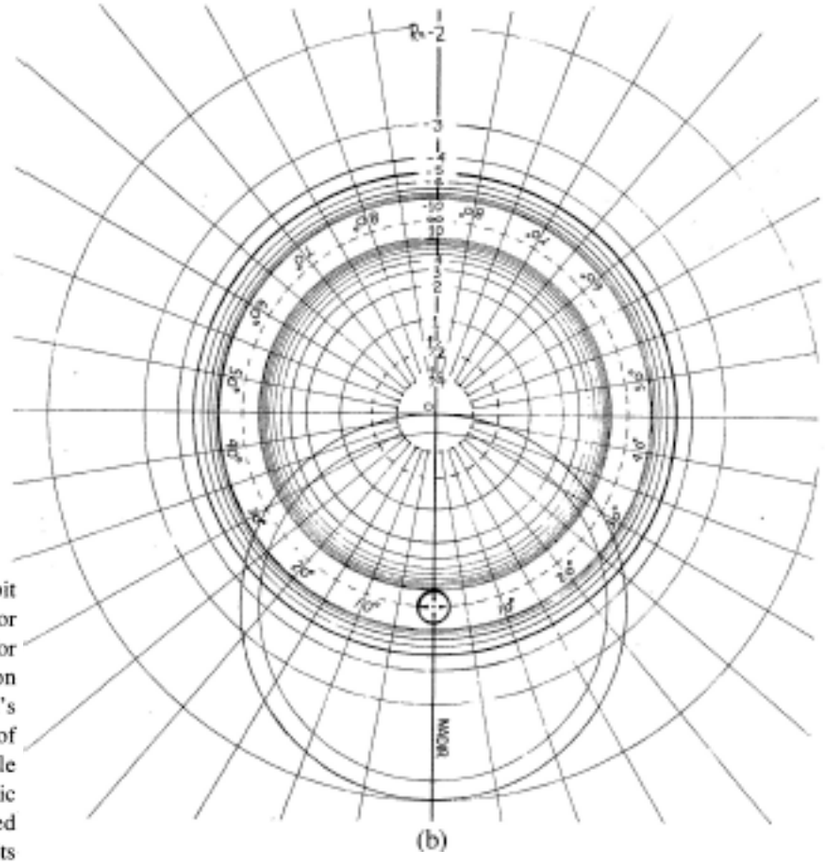


Fig. 7. Coordinate grids for orbital analog computers. (a) Elliptical orbit scale ($0 > R > -1$). This can be used with the apparatus in Figs. 8 or 9. Radial lines marked $\pm 10°$, $\pm 20°$, . . . , are each the focus locus for orbits with an initial velocity $\pm 10°$, $\pm 20°$, . . . , above the horizon line. The circle marked 20°, 40°, . . . , 340° can be taken as the Earth's surface, or any circle inside this one can be taken to be the surface of any celestial body. The R values apply correctly in either case, while the velocity values are marked for the former case only. (b) Hyperbolic orbit scale ($0 < R < \infty$) and ($-\infty < R < -1$). This can only be used with the apparatus shown in Fig. 9. Outer circles locate foci for orbits of particles attracted to the force center, while inner circles locate foci for orbits in a repulsive field. In either case a radial line marked $\pm 10°$, $\pm 20°$, . . . , is the focus locus for an orbit with the initial velocity an angle $\pm 10°$, $\pm 20°$, . . . , above the nadir line.

Lenz Vector...analog computersAJP 44 4 (1974)

Start with initial velocity **v***(0)*

or **-v***(0)*

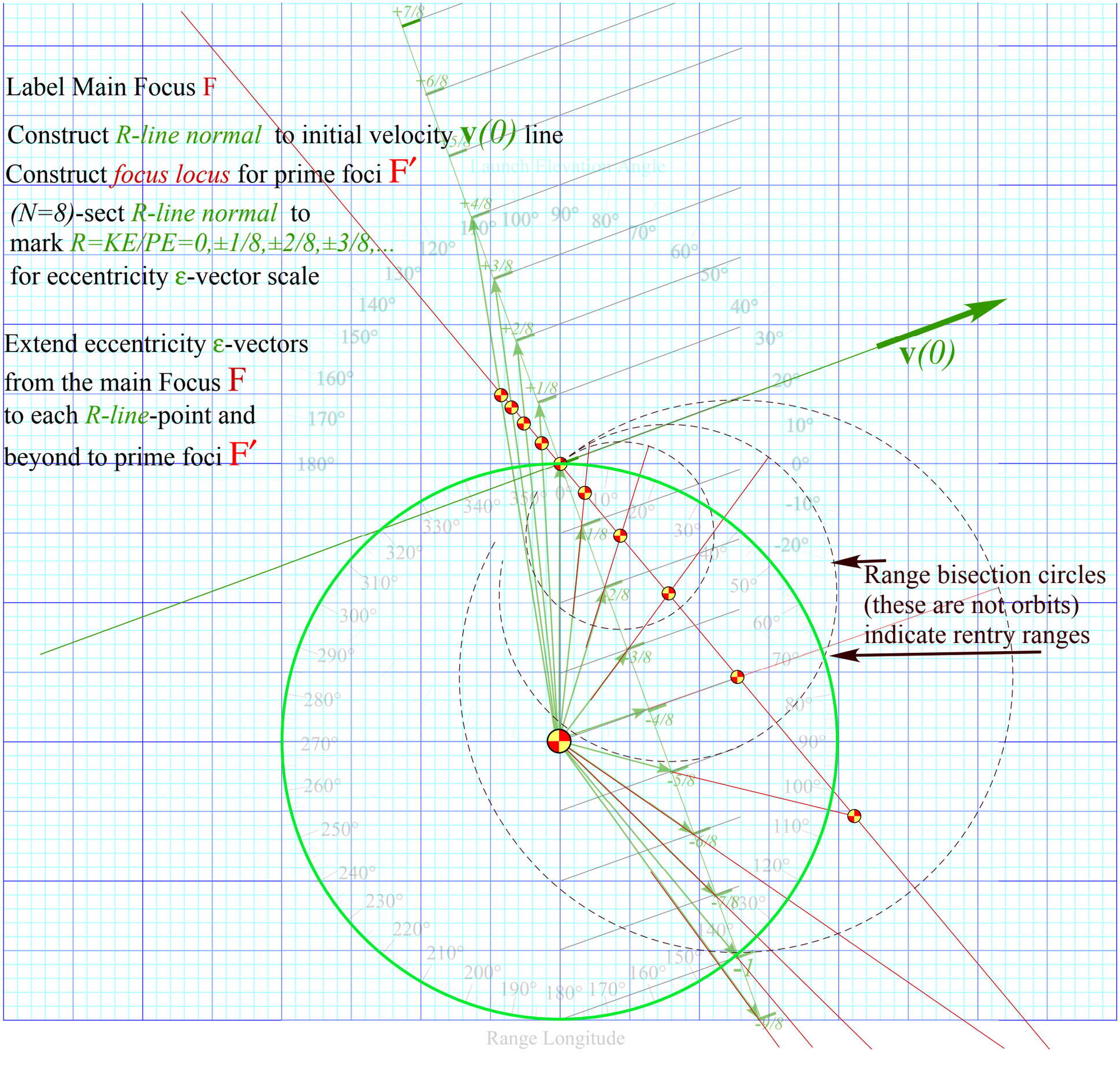
Label Main Focus F

Construct *R-line normal* to initial velocity **v***(0)* line

Construct *focus locus* for prime foci F′

(N=8)-sect *R-line normal* to mark *R=KE/PE=0,±1/8,±2/8,±3/8,...* for eccentricity ε-vector scale

Extend eccentricity ε-vectors from the main Focus F to each *R-line*-point and beyond to prime foci F′



or **-v***(0)*

Start with initial velocity **v***(0)*

or **-v***(0)*

Label Main Focus F

Construct *R-line normal* to initial velocity **v***(0)* line

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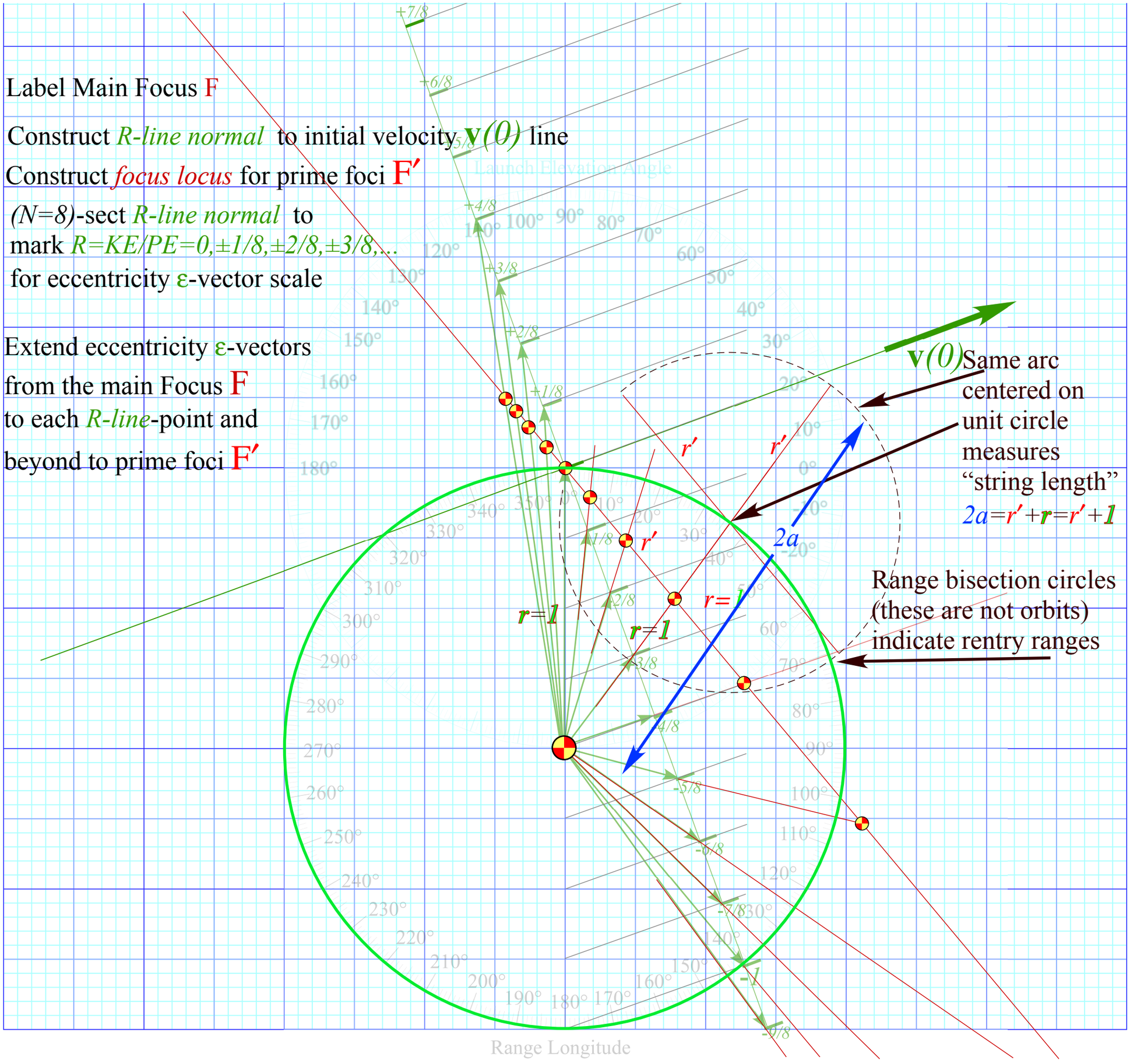

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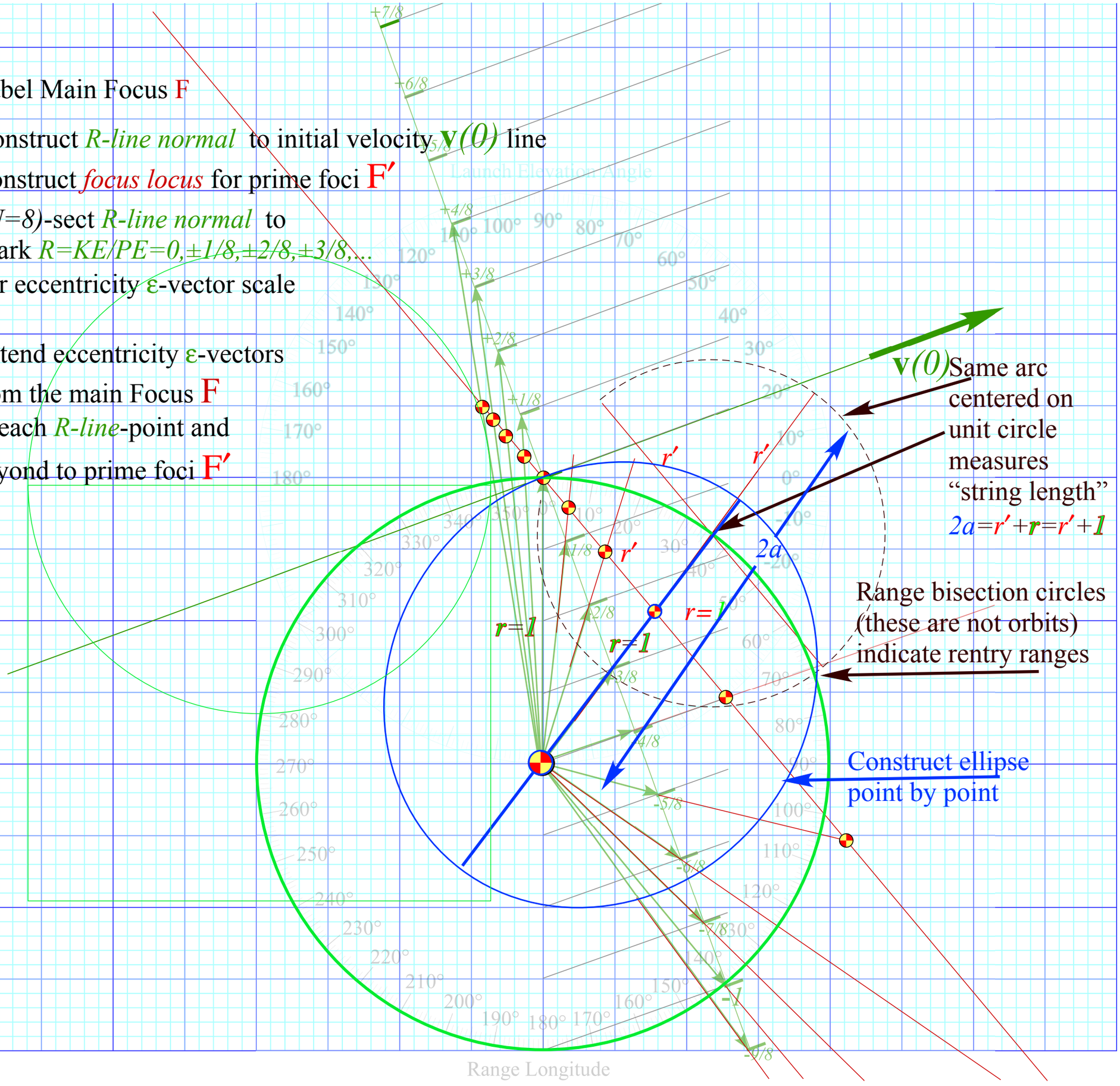
Extend eccentricity ε-vectors from the main Focus F to each *R-line*-point and beyond to prime foci F′

v*(0)*

Same arc centered on unit circle measures "string length" *2a=r′+r=r′+1*

Range bisection circles (these are not orbits) indicate rentry ranges

Construct ellipse point by point



Label Main Focus F

Construct *R-line normal* to initial velocity **v**(*0*) line

Construct *focus locus* for prime foci F′

(N=8)-sect *R-line normal* to
mark *R=KE/PE=0,±1/8,±2/8,±3/8,...*
for eccentricity ε-vector scale

Extend eccentricity ε-vectors
from the main Focus F
to each *R-line*-point and
beyond to prime foci F′

Launch Elevation Angle

v(*0*)

Range bisection
indicates re-entry ranges

Maximum range limit
for this elevation angle
α=20° has *R= -1*
and ε=1 (*parabola*)

r′

r=1

r=1

r=1

ε

[*(R=-1) ε=1-line parallel to focus-locus*]

Range Longitude

This puts 2nd focus at ∞.

Label Main Focus F

Construct *R-line normal* to initial velocity **v**(*0*) line

Construct *focus locus* for prime foci F′

(N=8)-sect *R-line normal* to
mark *R=KE/PE=0,±1/8,±2/8,±3/8,...*
for eccentricity ε-vector scale

Extend eccentricity ε-vectors
from the main Focus F
to each *R-line*-point and
beyond to prime foci F′

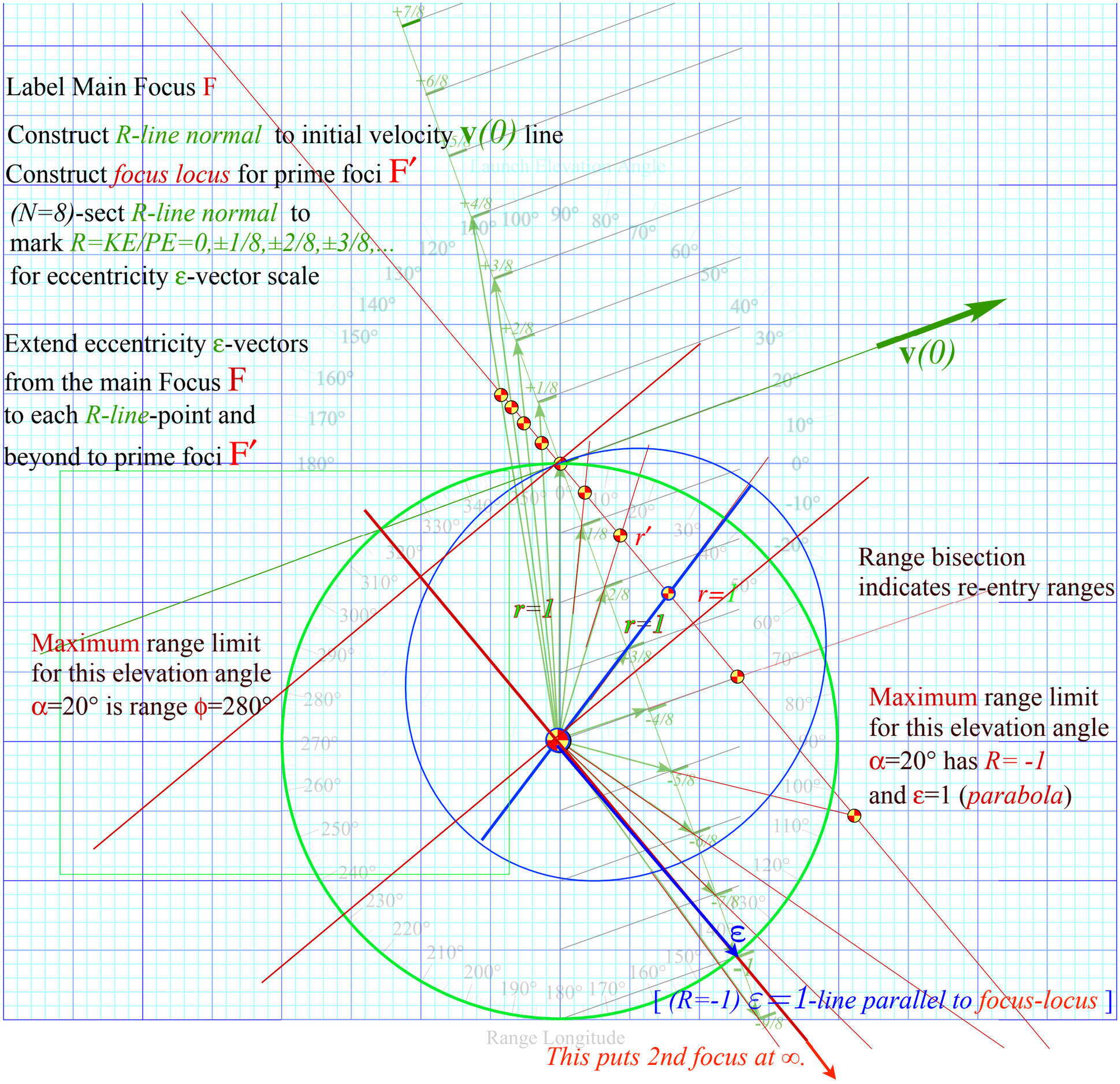
Maximum range limit
for this elevation angle
α=20° is range ϕ=280°

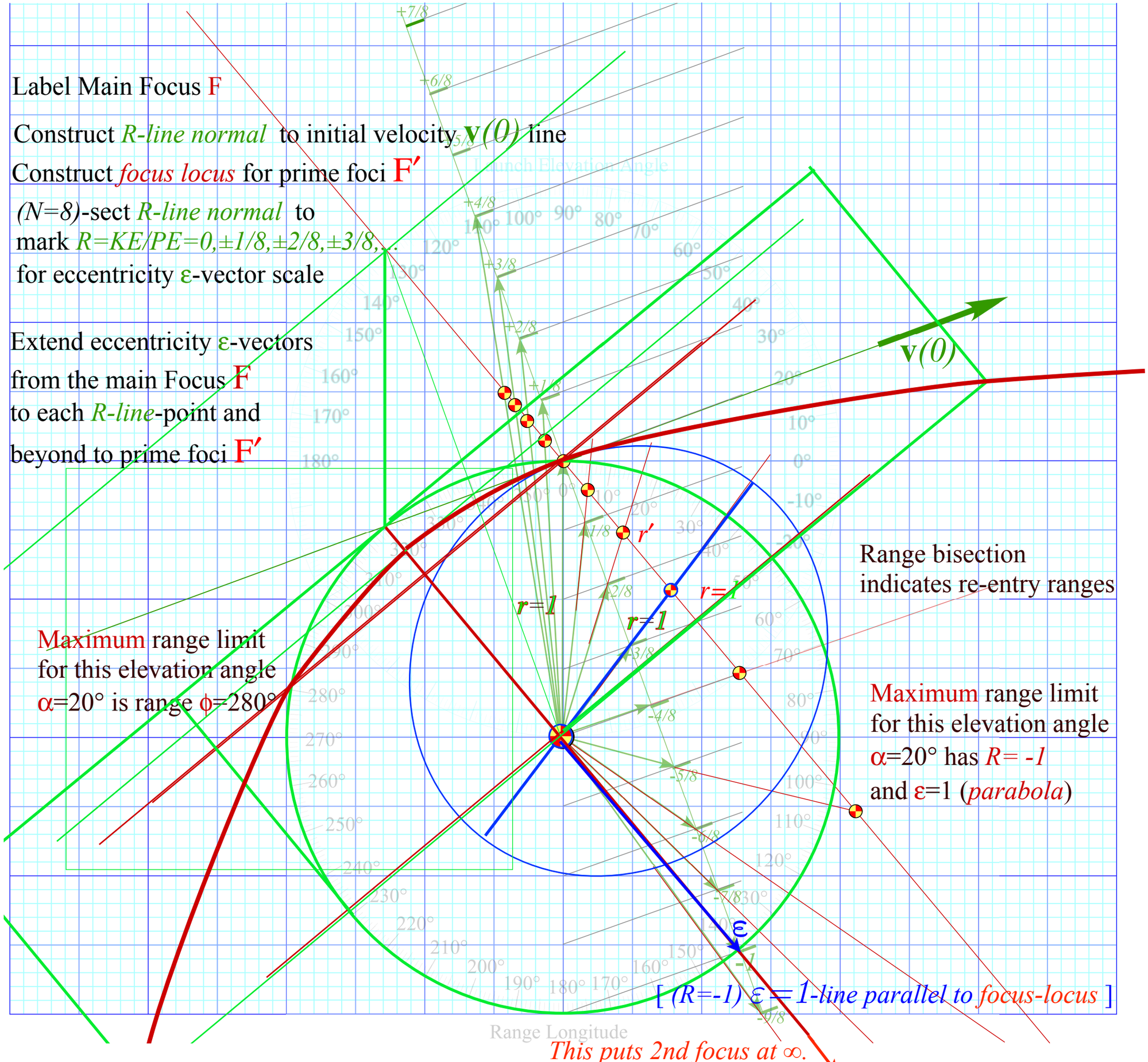
Range bisection
indicates re-entry ranges

Maximum range limit
for this elevation angle
α=20° has *R= -1*
and ε=1 (*parabola*)

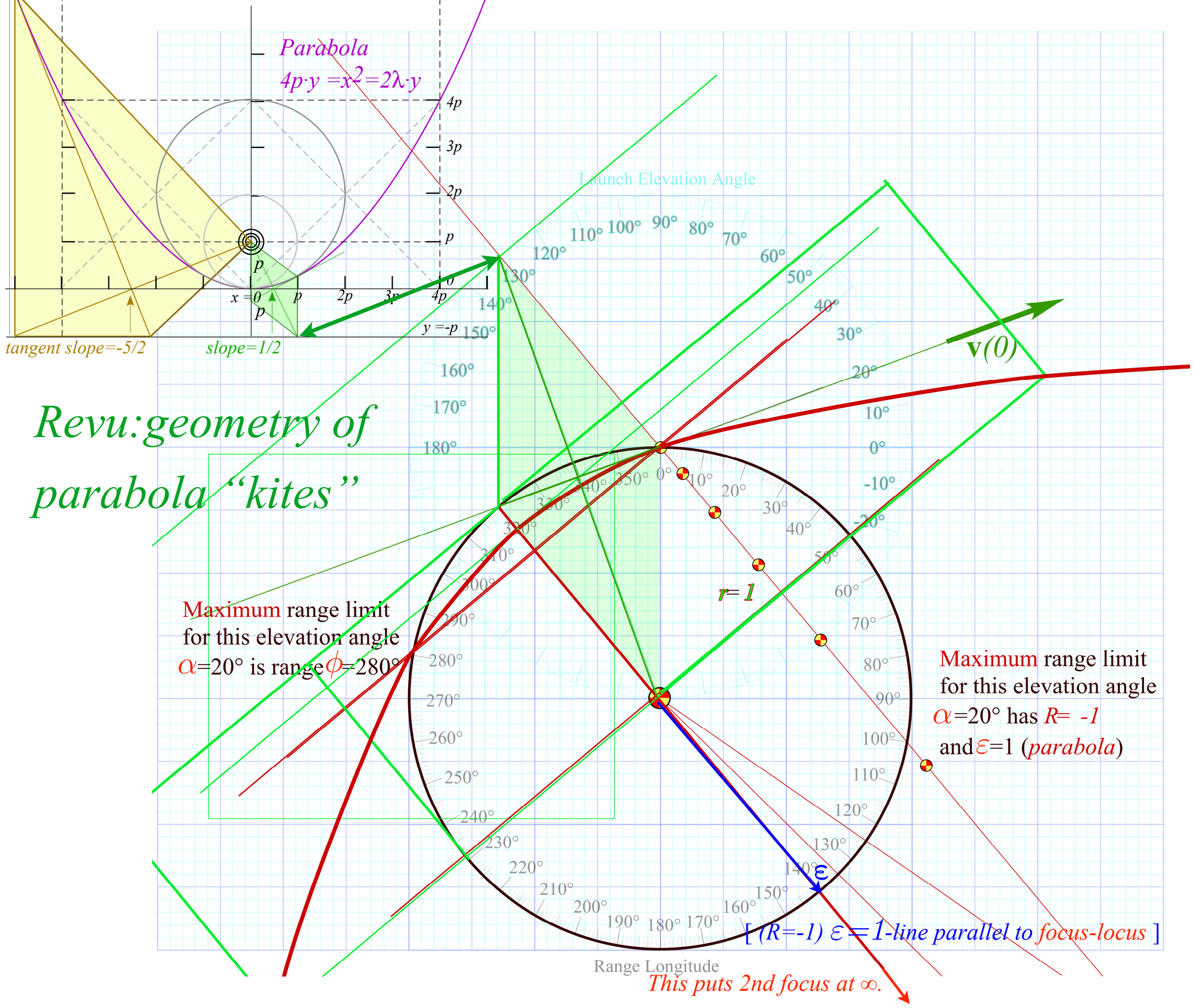
[*(R=-1) ε=1-line parallel to focus-locus*]

This puts 2nd focus at ∞.




Label Main Focus F
Construct R-line normal to initial velocity v(0) line
Construct focus locus for prime foci F′
(N=8)-sect R-line normal to mark R=KE/PE=0,±1/8,±2/8,±3/8,...
for eccentricity ε-vector scale
Extend eccentricity ε-vectors from the main Focus F to each R-line-point and beyond to prime foci F′
Maximum range limit for this elevation angle α=20° is range ϕ=280°
Range bisection indicates re-entry ranges
Maximum range limit for this elevation angle α=20° has R= -1 and ε=1 (parabola)
v(0)
ε
r′
r=1
r=1
r=1
[(R=-1) ε = 1-line parallel to focus-locus]
This puts 2nd focus at ∞.
Launch Elevation Angle
Range Longitude

Revu:geometry of parabola "kites"



Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Analytic geometry derivation of $\boldsymbol{\varepsilon}$*-construction*

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Detailed ruler & compass construction of $\boldsymbol{\varepsilon}$*-vector and orbits*

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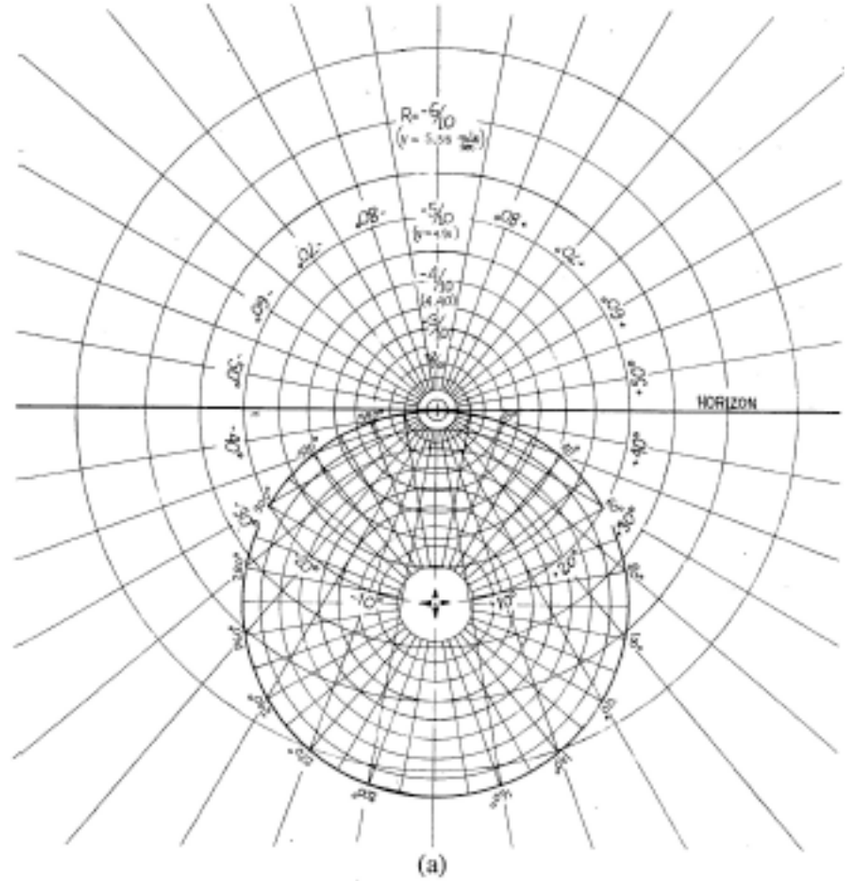

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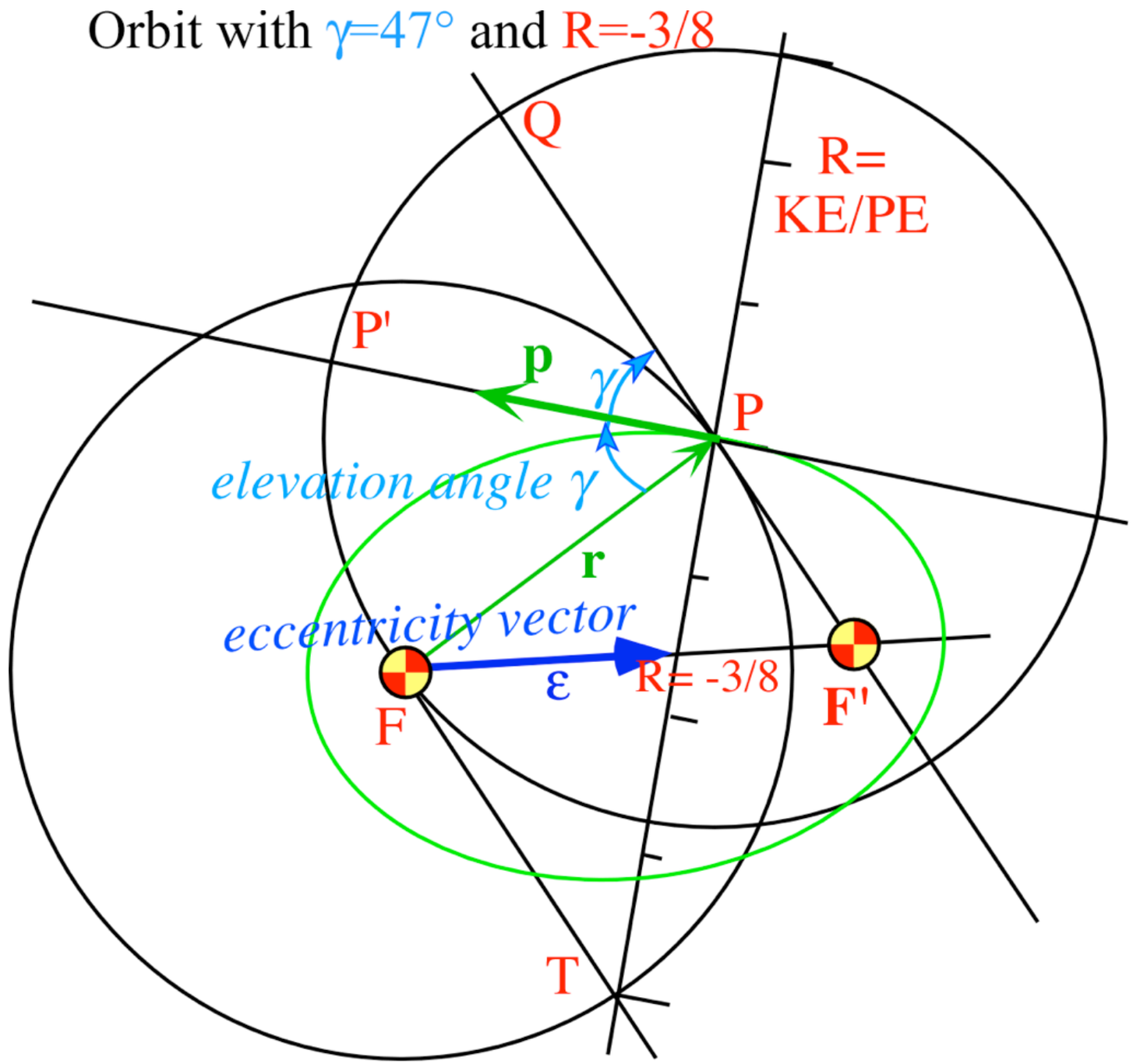
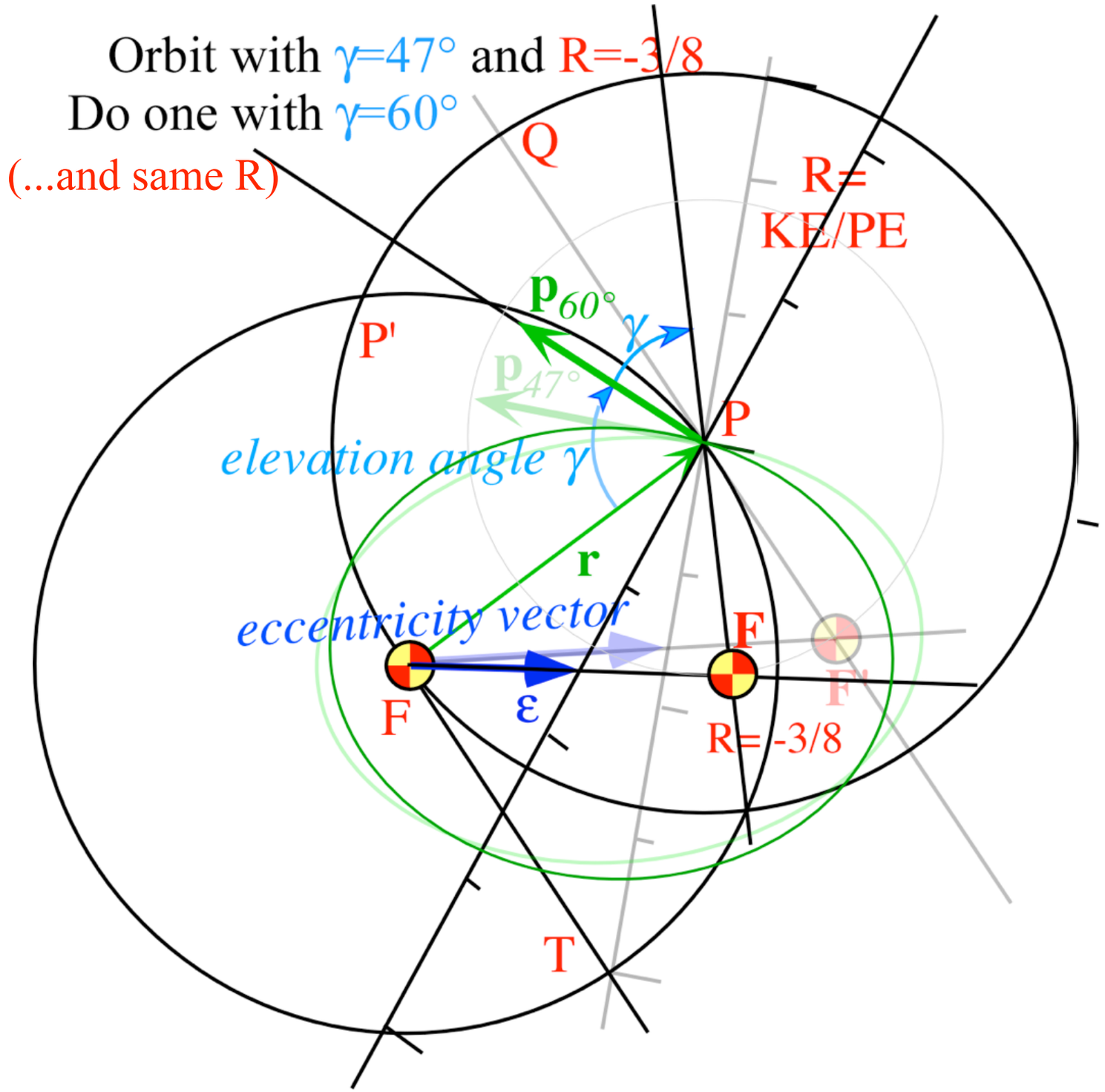
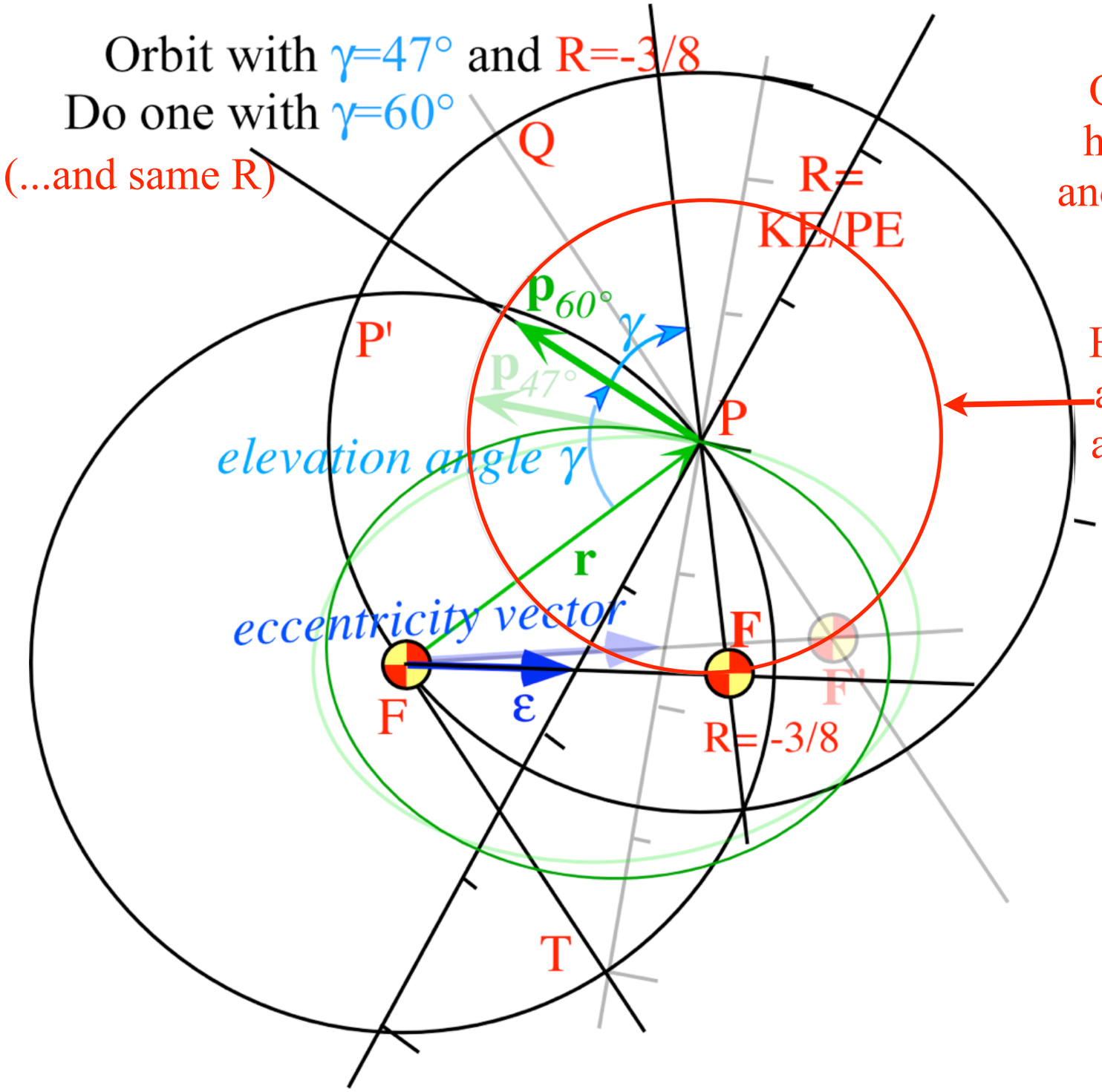
Lenz Vector...analog computersAJP 44 4 (1974)


Orbit with γ=47° and R=-3/8
Q
R=
KE/PE
P'
p
γ
P
elevation angle γ
r
eccentricity vector
ε
R= -3/8
F
F'
T


Orbit with γ=47° and R=-3/8
Do one with γ=60°
(...and same R)
Q
R=
KE/PE
P'
p60°
γ
p47°
P
elevation angle γ
r
eccentricity vector
F
F
ε
F'
R=-3/8
T



Orbits with the same R
have the same energy E
and the same major radii *a*

Hence their foci lie on
a circle of radius *2a-r*
around launch point P

Orbit with γ=47° and R=-3/8
Do one with γ=60°
(...and same R)

Orbits with the same R
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and the same major radii *a*

Hence their foci lie on
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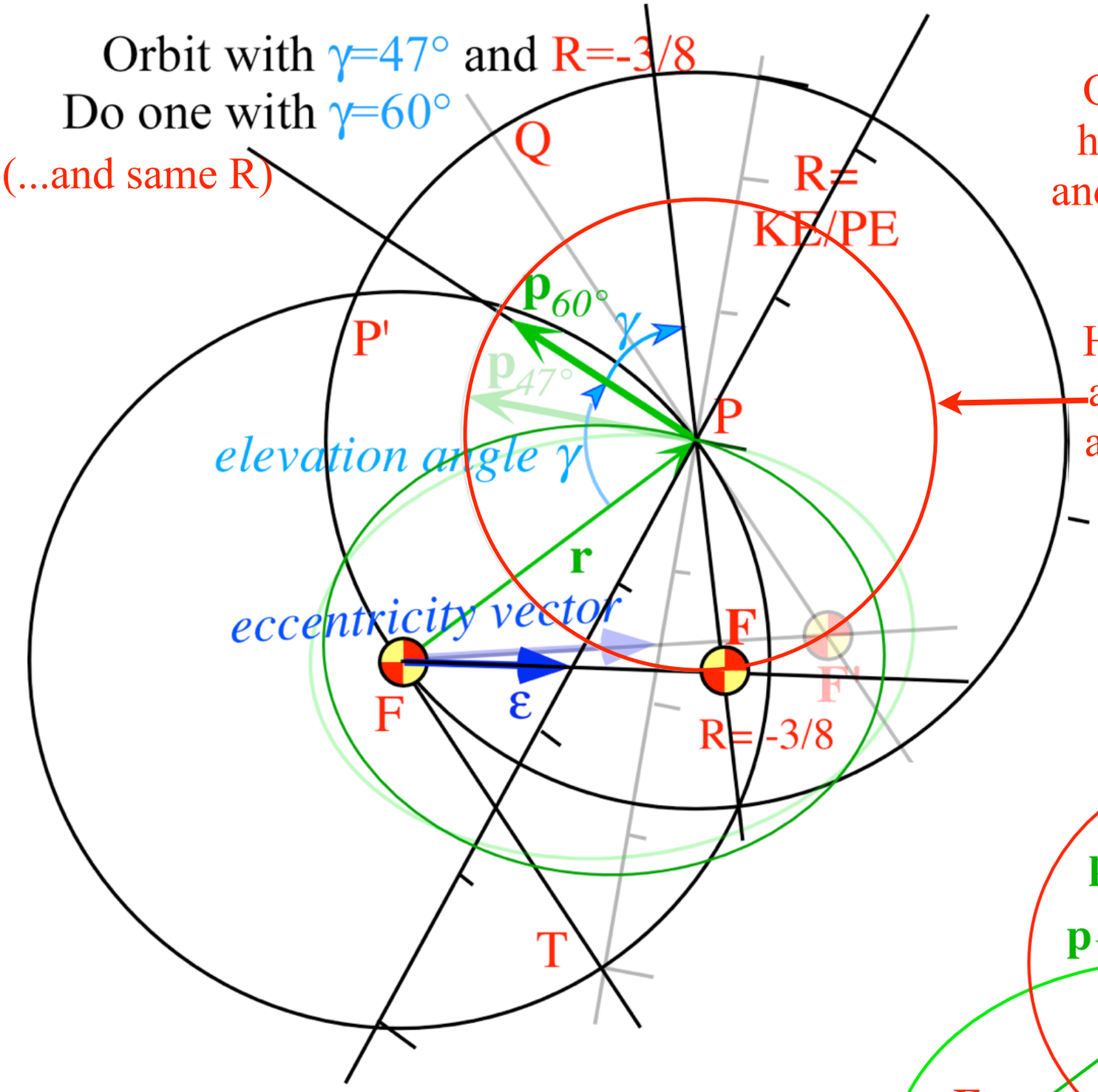
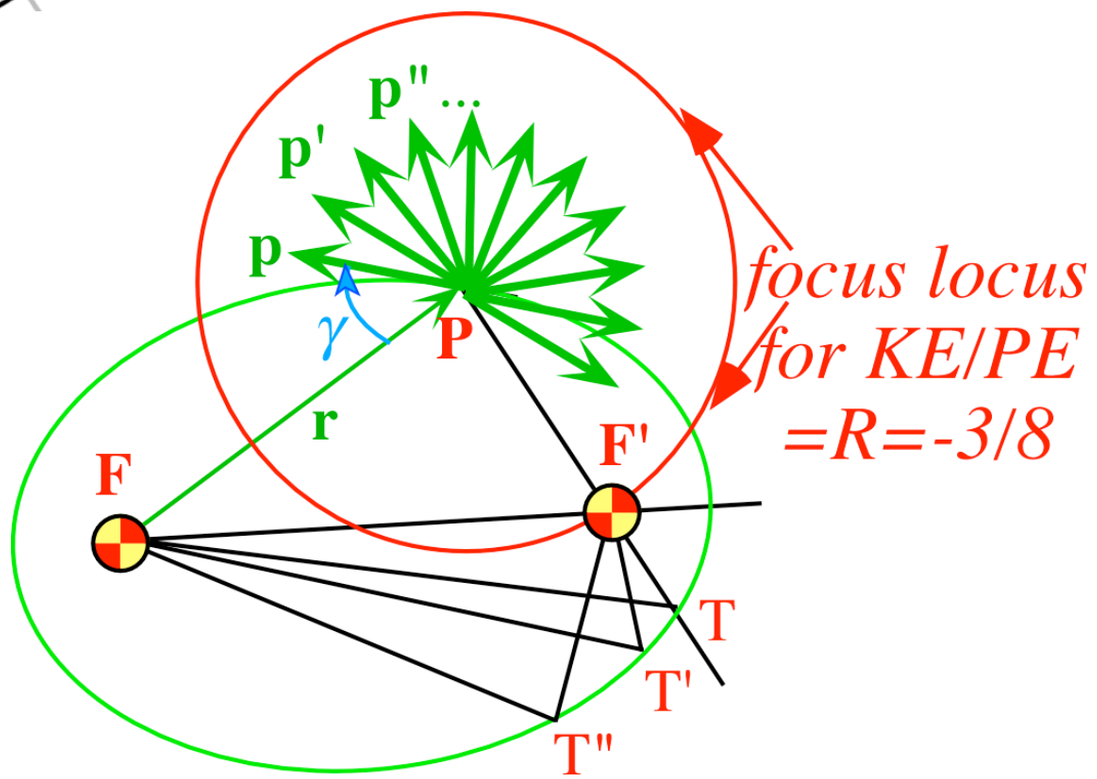
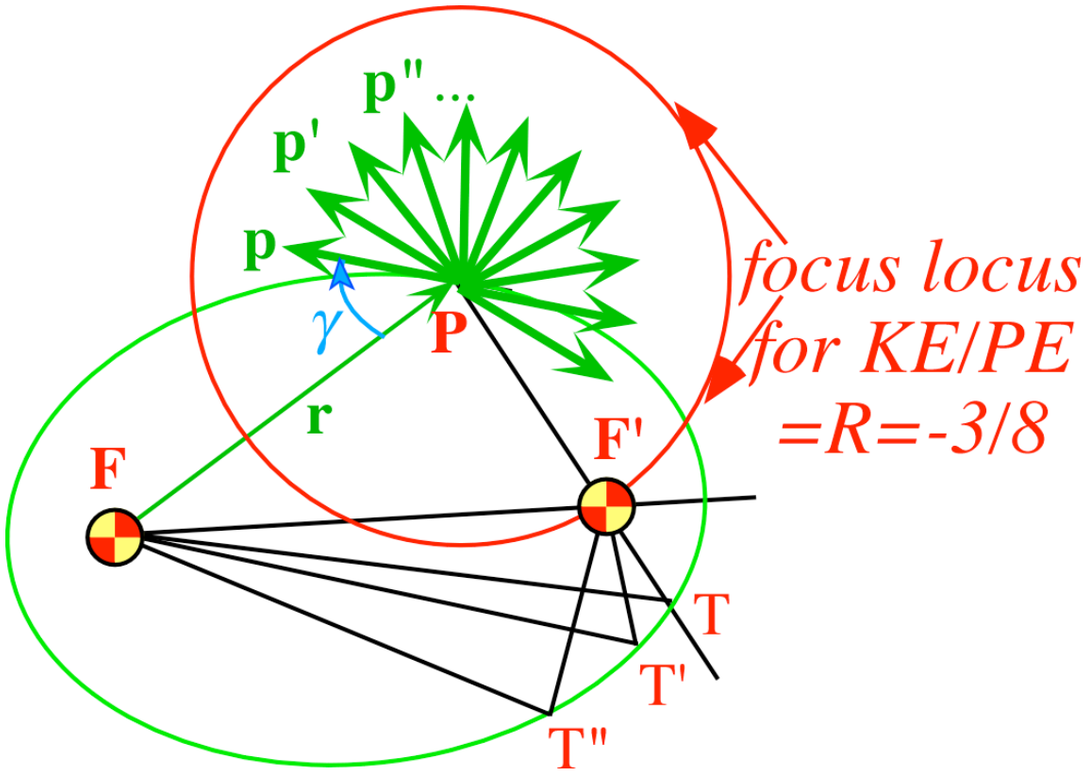
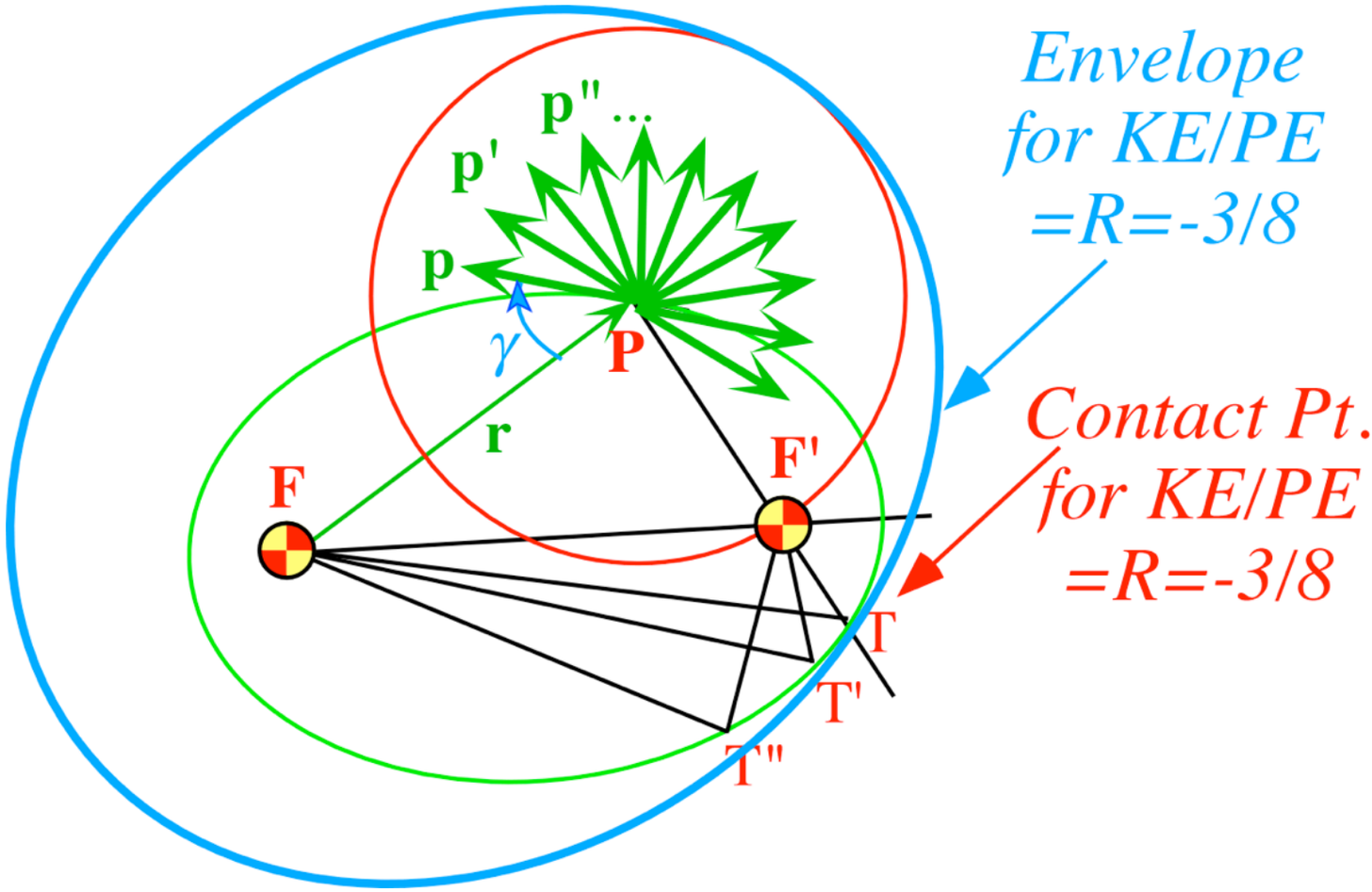





p"...
p'
p
γ
P
r
F
F'
T
T'
T"
focus locus
for KE/PE
=R=-3/8


p"...
p'
p
γ
P
r
F
F'
T
T'
T"
Envelope
for KE/PE
=R=-3/8
Contact Pt.
for KE/PE
=R=-3/8

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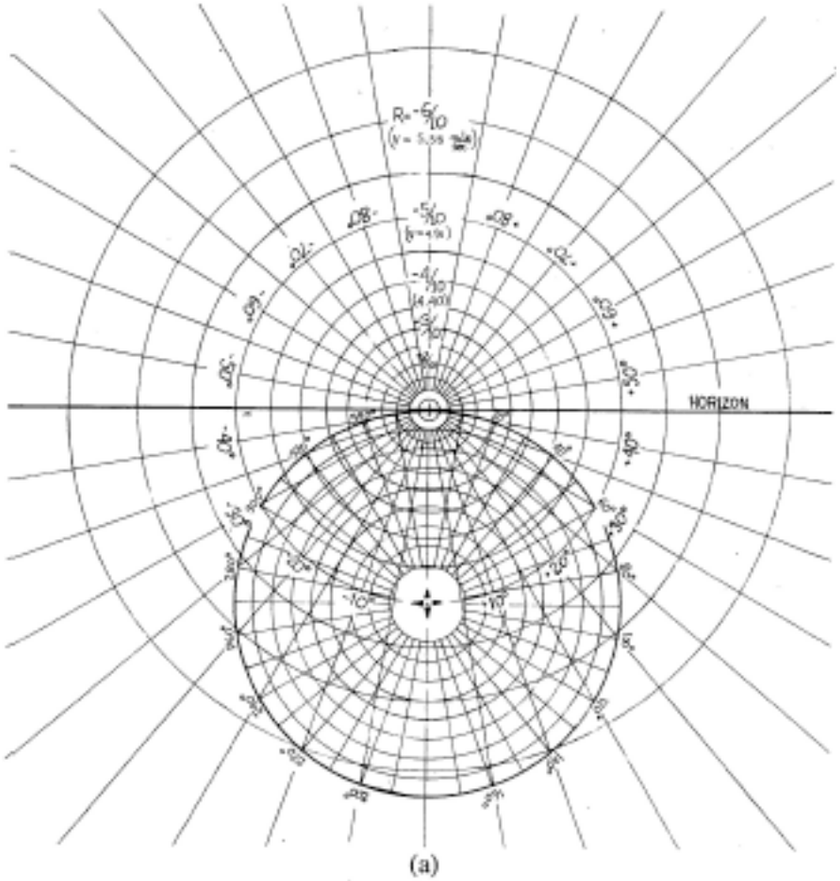
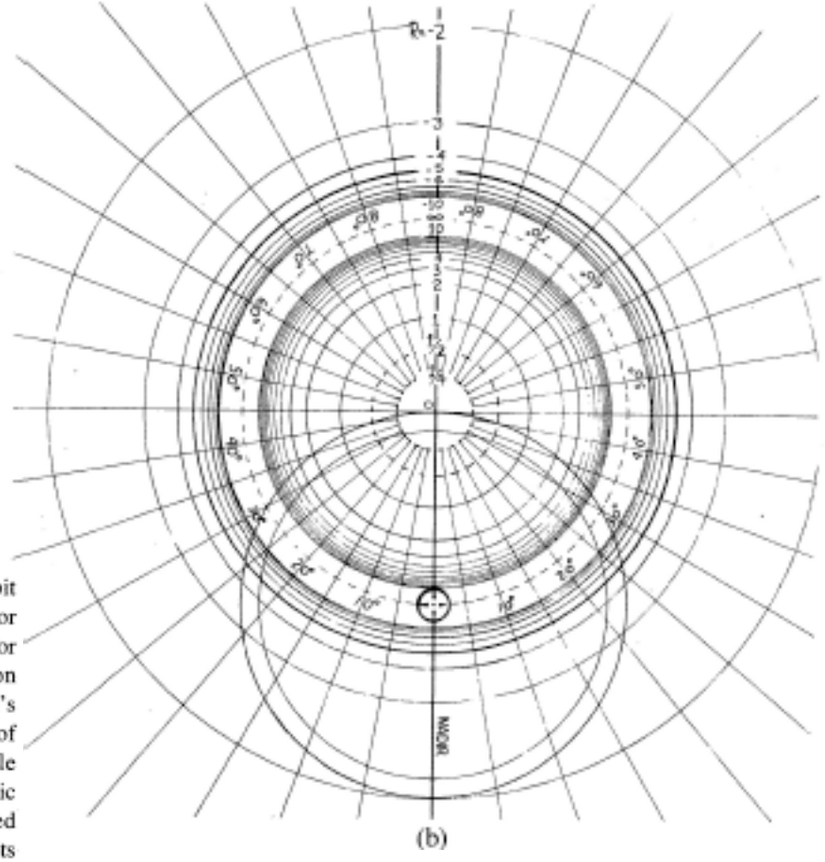
➔ *Launch energy fixed-Varied launch angle*

➔ *Launch optimization and orbit family envelopes*





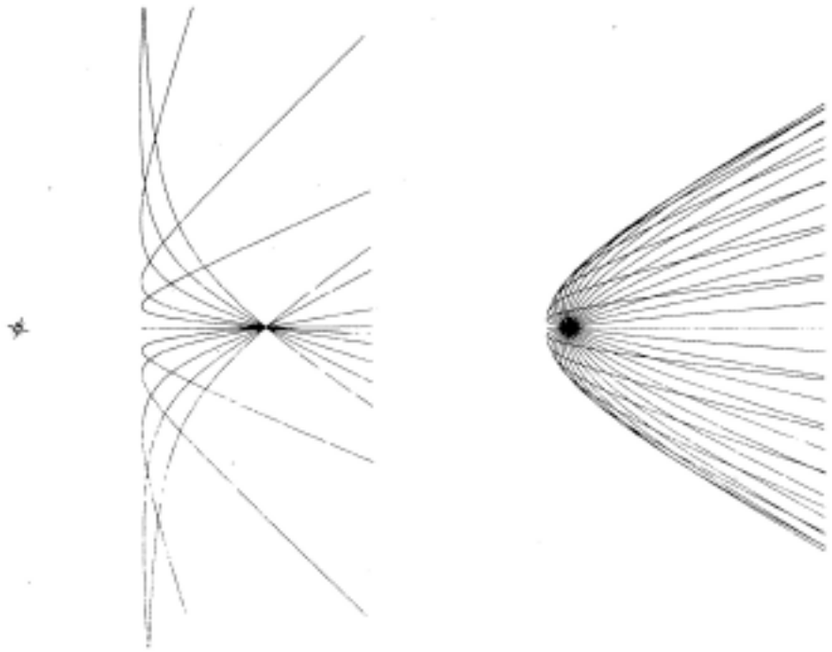
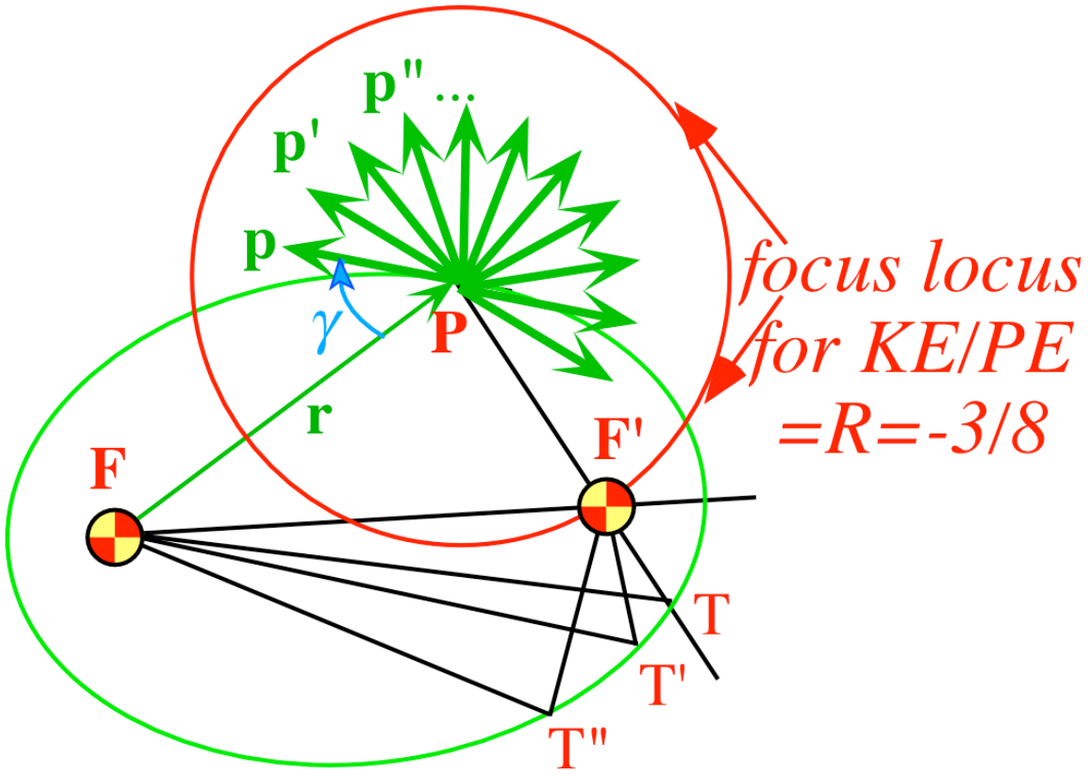
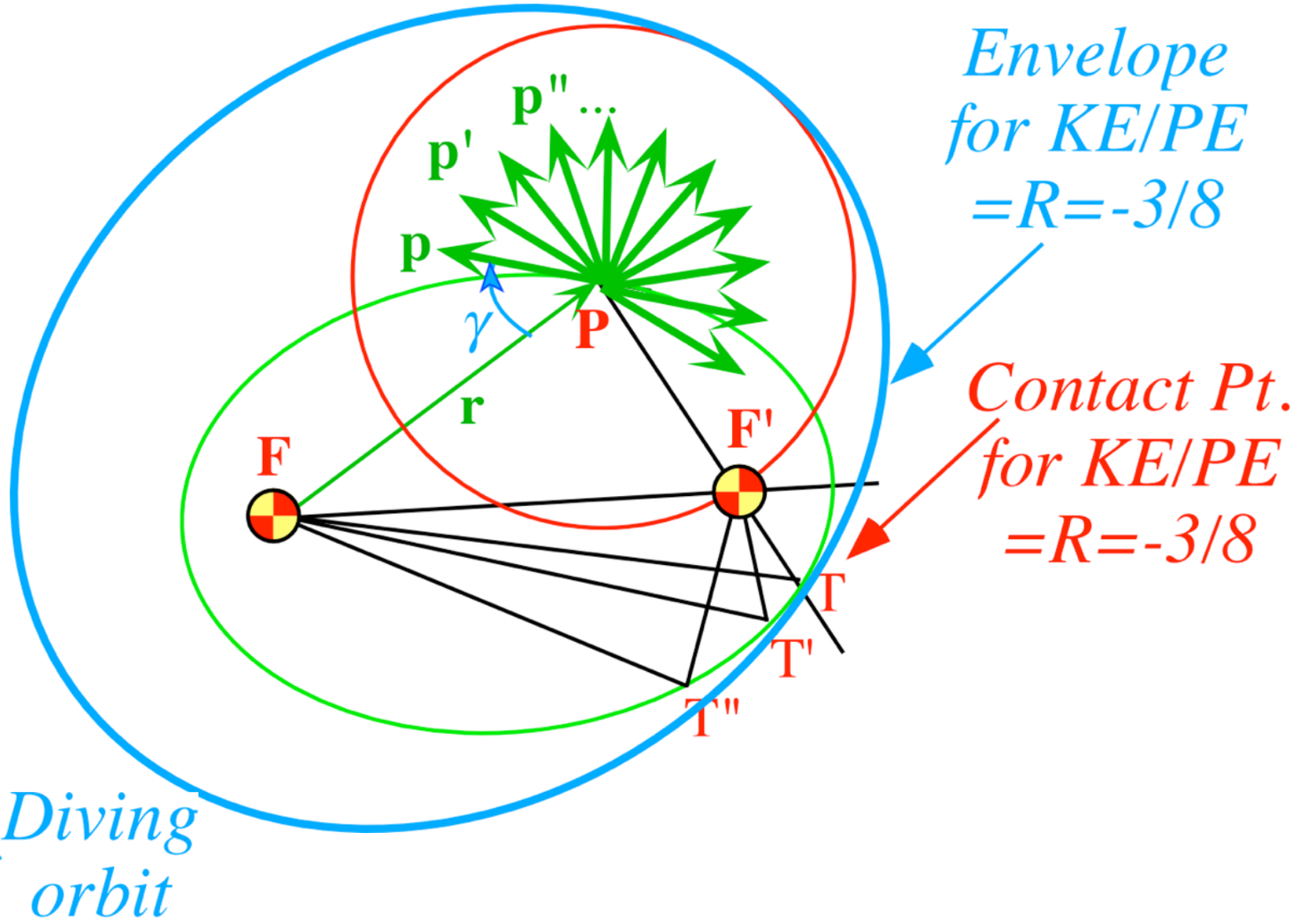


Fig. 11. Sample computer trajectories. (a) Family of hyperbolic orbits with $R = 1$. (b) Family of hyperbolic orbits with $R < 1$.

Fig. 7. Coordinate grids for orbital analog computers. (a) Elliptical orbit scale $(0 > R > -1)$. This can be used with the apparatus in Figs. 8 or 9. Radial lines marked $\pm 10°$, $\pm 20°$, . . . , are each the focus locus for orbits with an initial velocity $\pm 10°$, $\pm 20°$, . . . , above the horizon line. The circle marked 20°, 40°, . . . , 340° can be taken as the Earth's surface, or any circle inside this one can be taken to be the surface of any celestial body. The R values apply correctly in either case, while the velocity values are marked for the former case only. (b) Hyperbolic orbit scale $(0 < R < \infty)$ and $(-\infty < R < -1)$. This can only be used with the apparatus shown in Fig. 9. Outer circles locate foci for orbits of particles attracted to the force center, while inner circles locate foci for orbits in a repulsive field. In either case a radial line marked $\pm 10°$, $\pm 20°$, . . . , is the focus locus for an orbit with the initial velocity an angle $\pm 10°$, $\pm 20°$, . . . , above the nadir line.



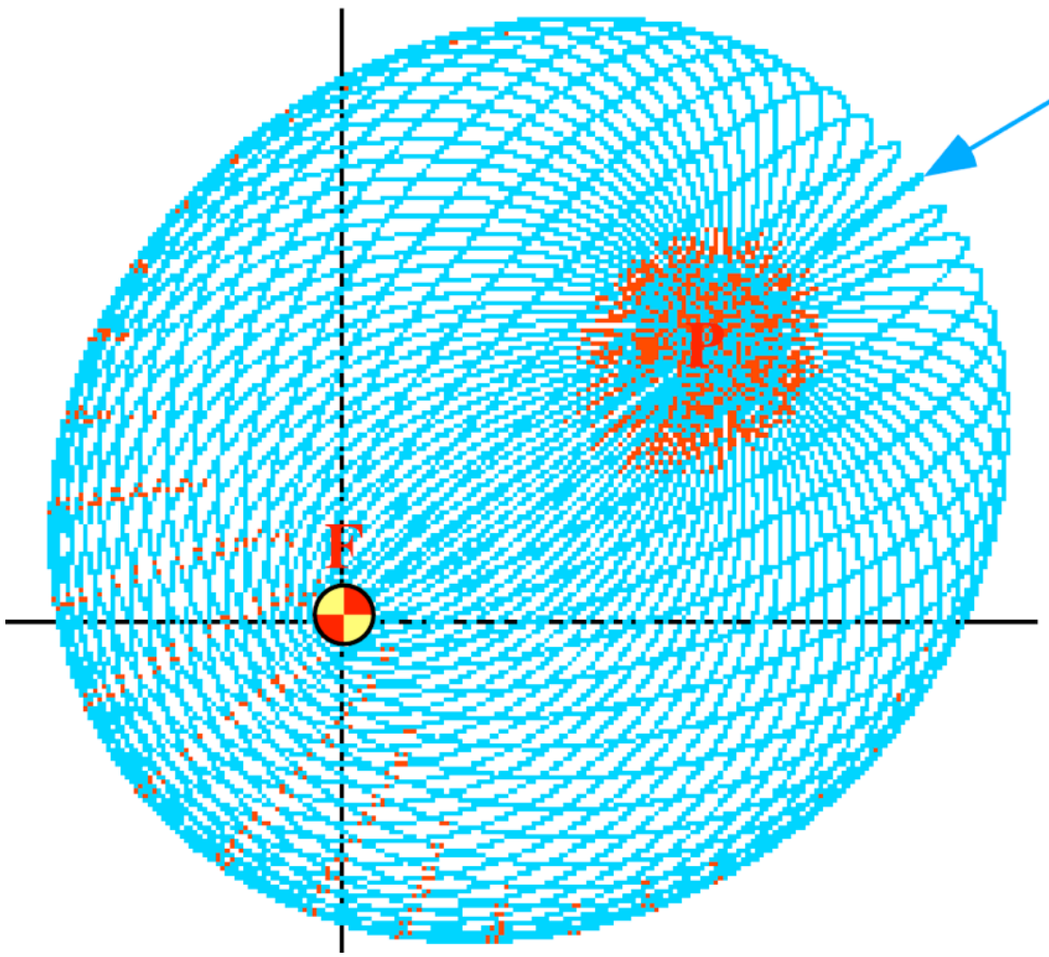
focus locus for KE/PE =R=-3/8



Envelope for KE/PE =R=-3/8

Contact Pt. for KE/PE =R=-3/8

Diving orbit



Coulomb envelope geometry

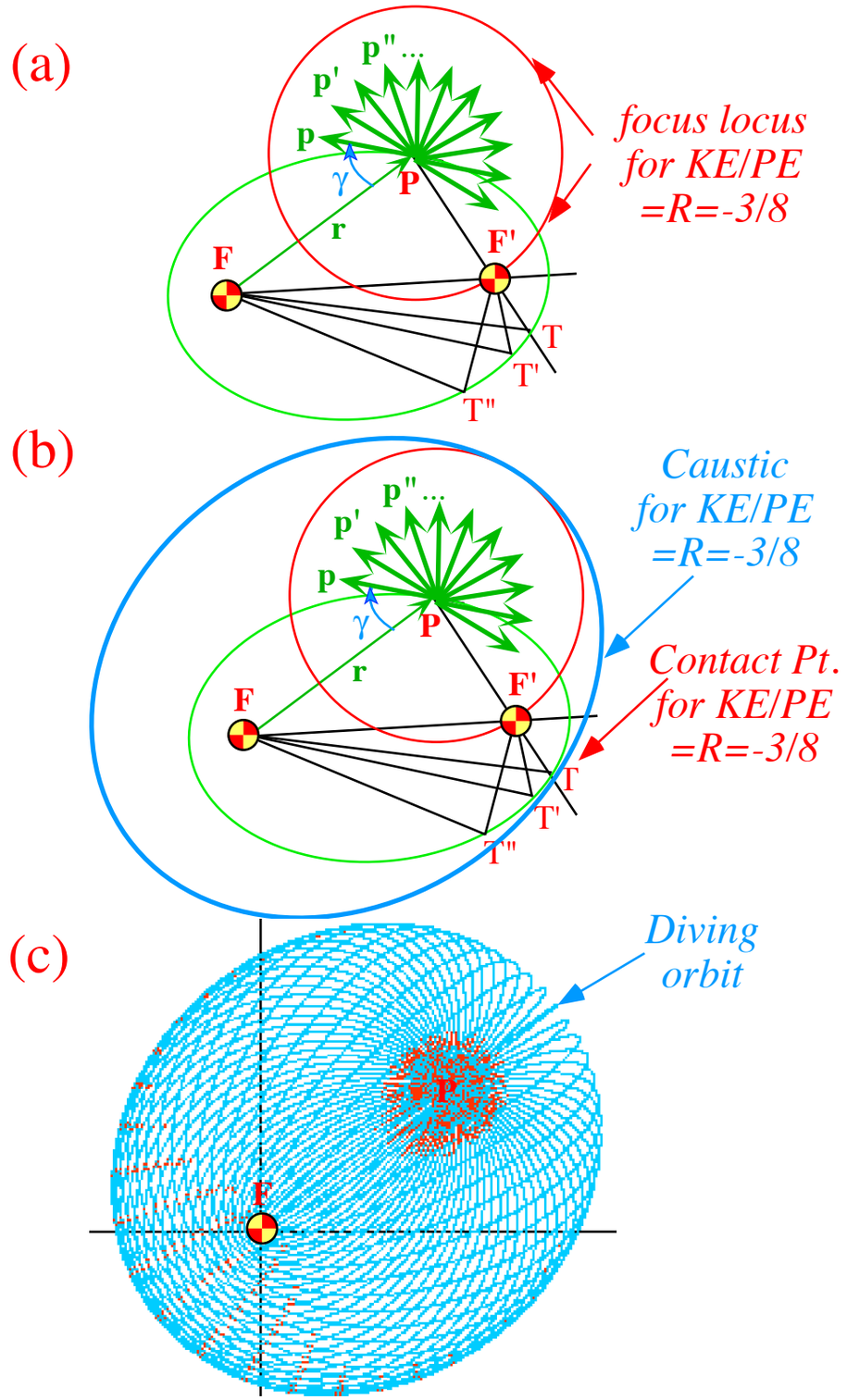
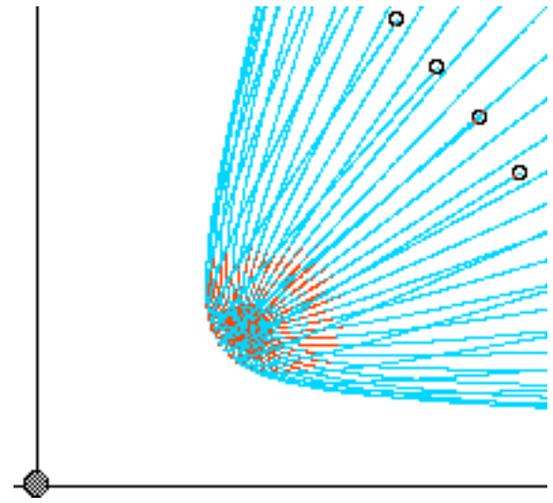


Fig. 5.4.4 in Unit 5 of CMwBANG!



Ideal comet "heads" or "tails" in solar wind

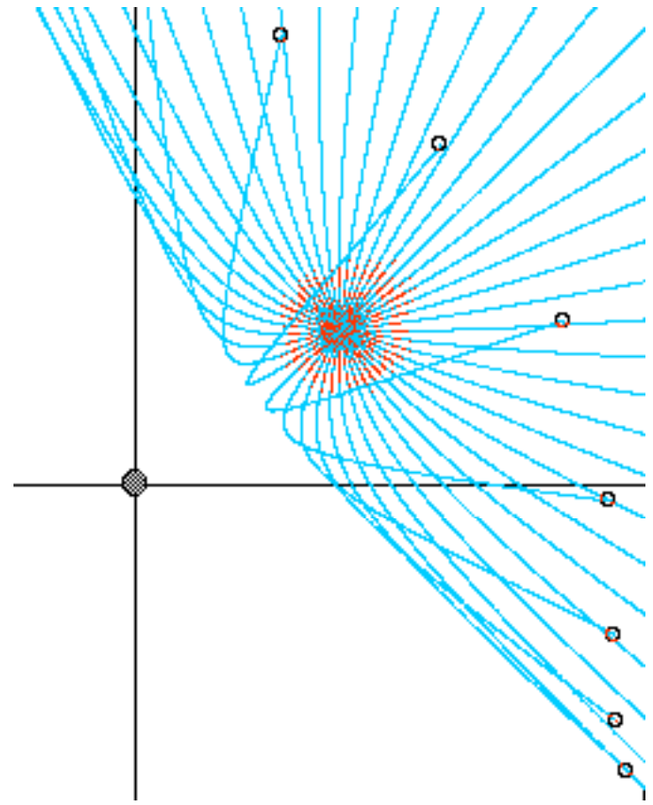


Fig. 5.4.5 in Unit 5 of CMwBANG!

Start with initial velocity **v***(0)* *or* **-v***(0)*

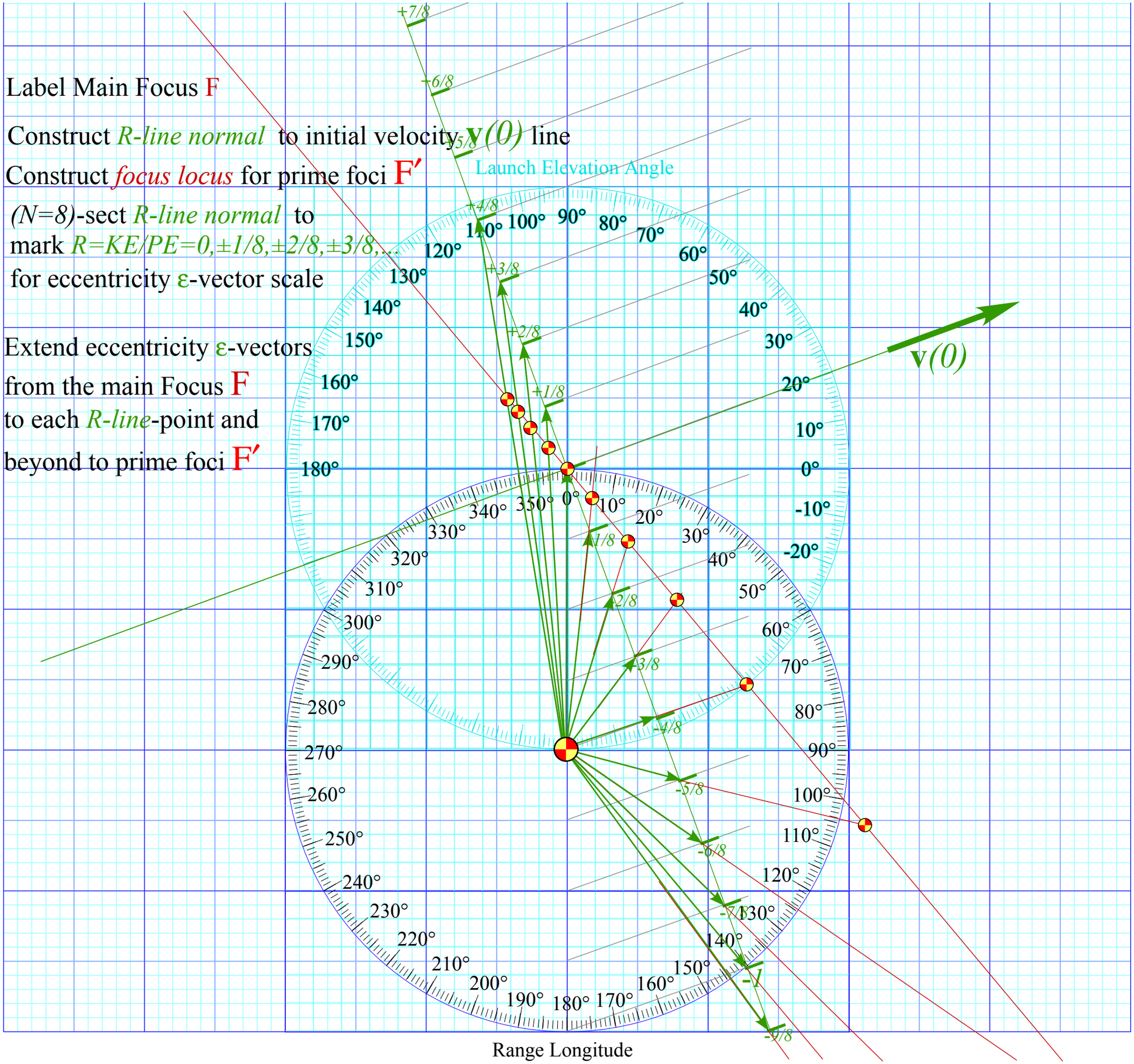
Label Main Focus F

Construct *R-line normal* to initial velocity **v***(0)* line

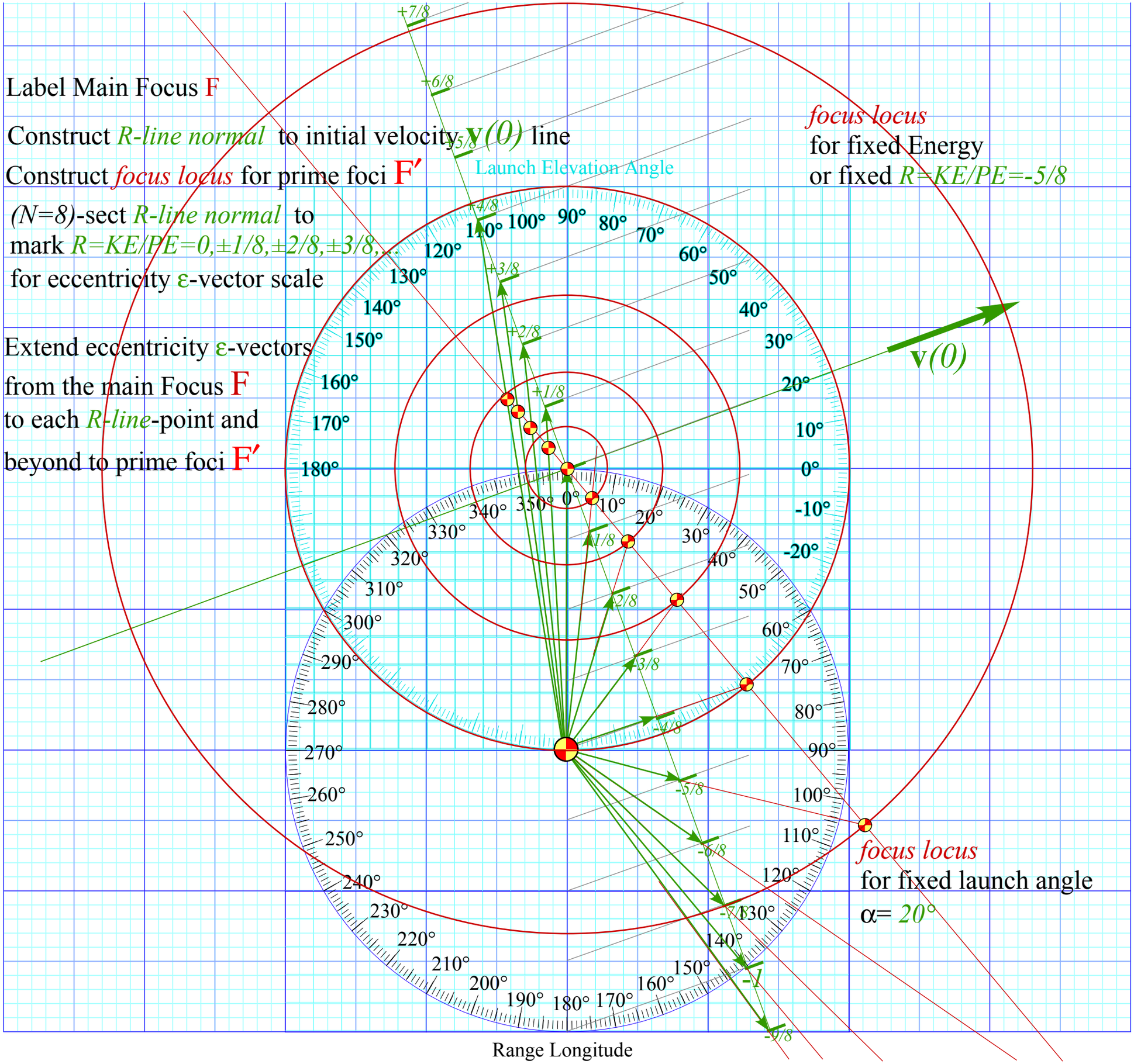
Construct *focus locus* for prime foci F′

(N=8)-sect *R-line normal* to mark *R=KE/PE=0,±1/8,±2/8,±3/8,…* for eccentricity ε-vector scale

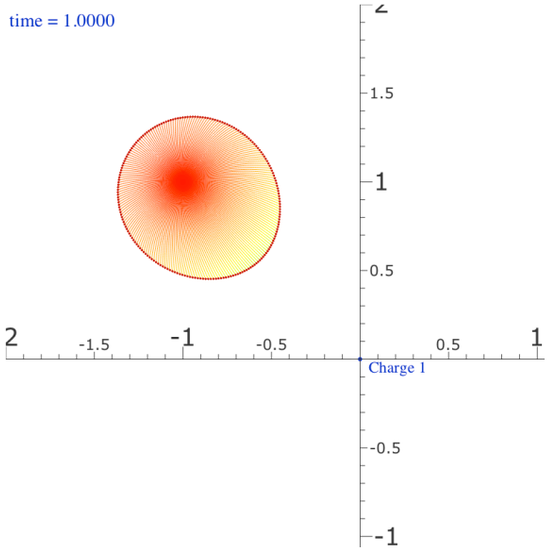
Extend eccentricity ε-vectors from the main Focus F to each *R-line*-point and beyond to prime foci F′

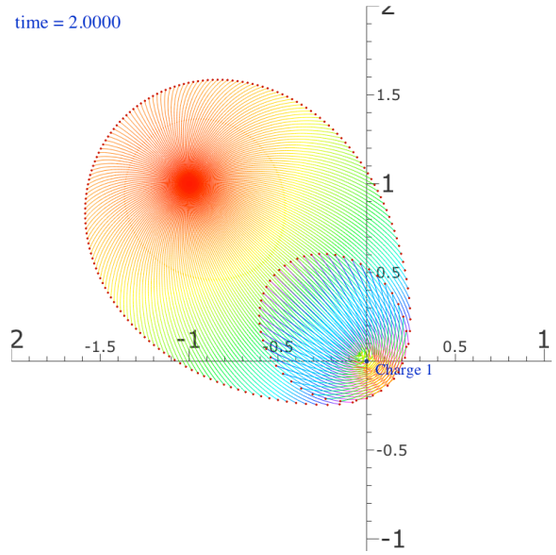


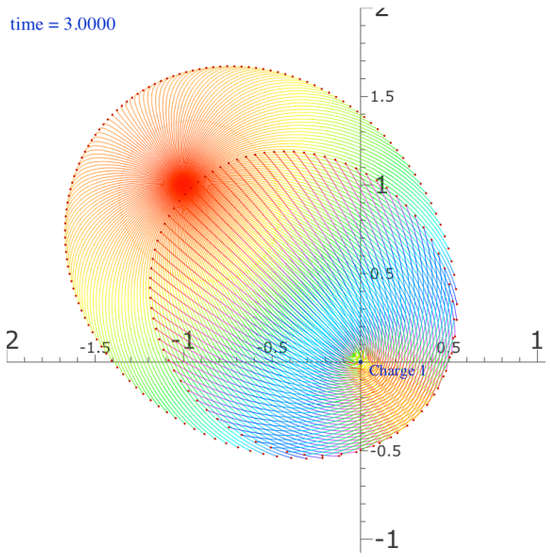
or **-v***(0)*

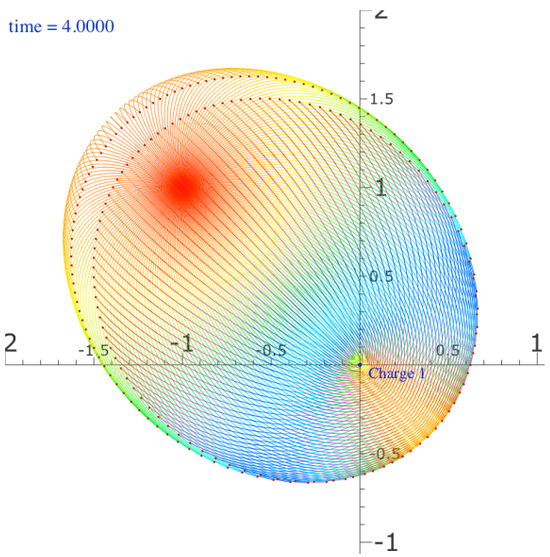

Label Main Focus F
Construct R-line normal to initial velocity v(0) line
Construct focus locus for prime foci F′
(N=8)-sect R-line normal to mark R=KE/PE=0,±1/8,±2/8,±3/8,...
for eccentricity ε-vector scale
Extend eccentricity ε-vectors from the main Focus F to each R-line-point and beyond to prime foci F′
Launch Elevation Angle
focus locus
for fixed Energy
or fixed R=KE/PE=-5/8
v(0)
focus locus
for fixed launch angle
α= 20°
Range Longitude

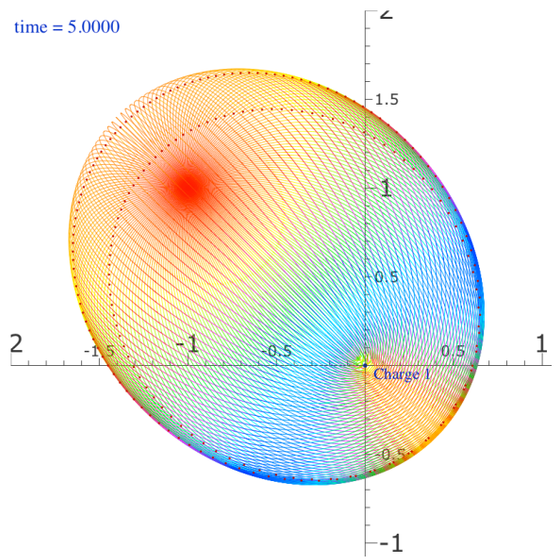
[CoulIt Web Simulation](#)
[Attractive Coulomb Burst](#)

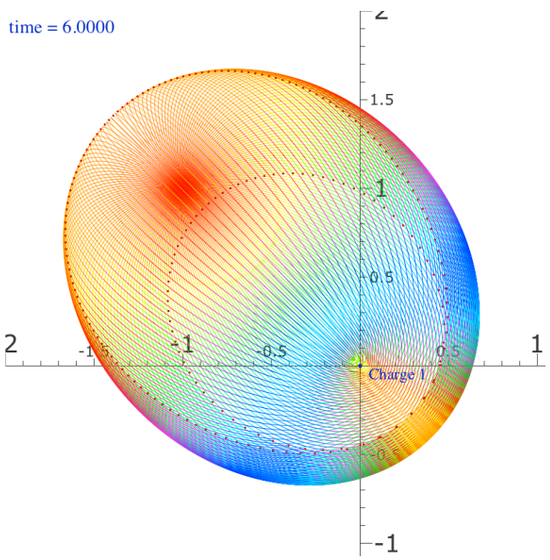


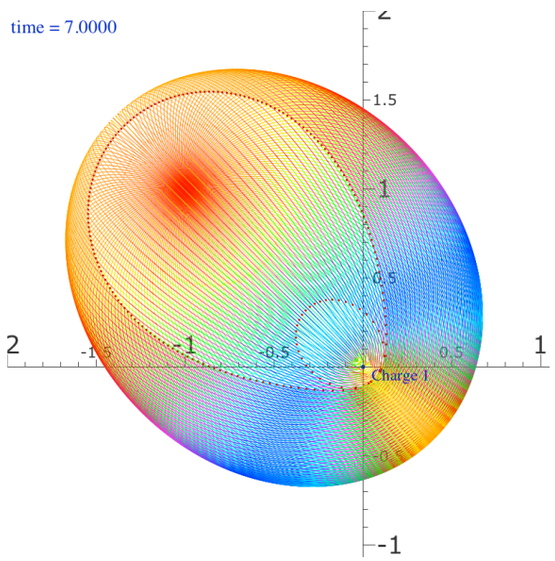


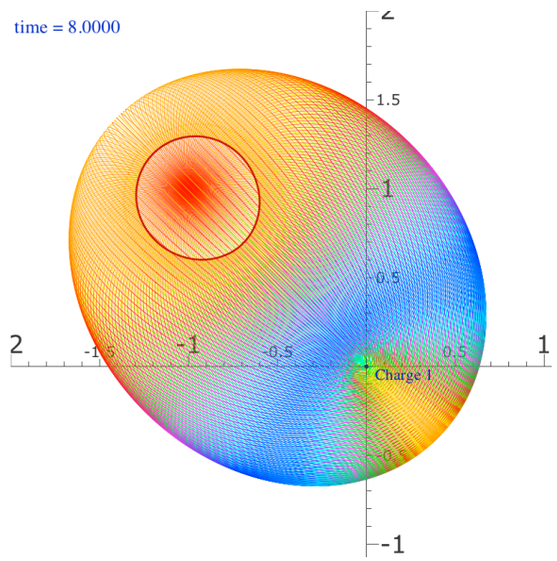


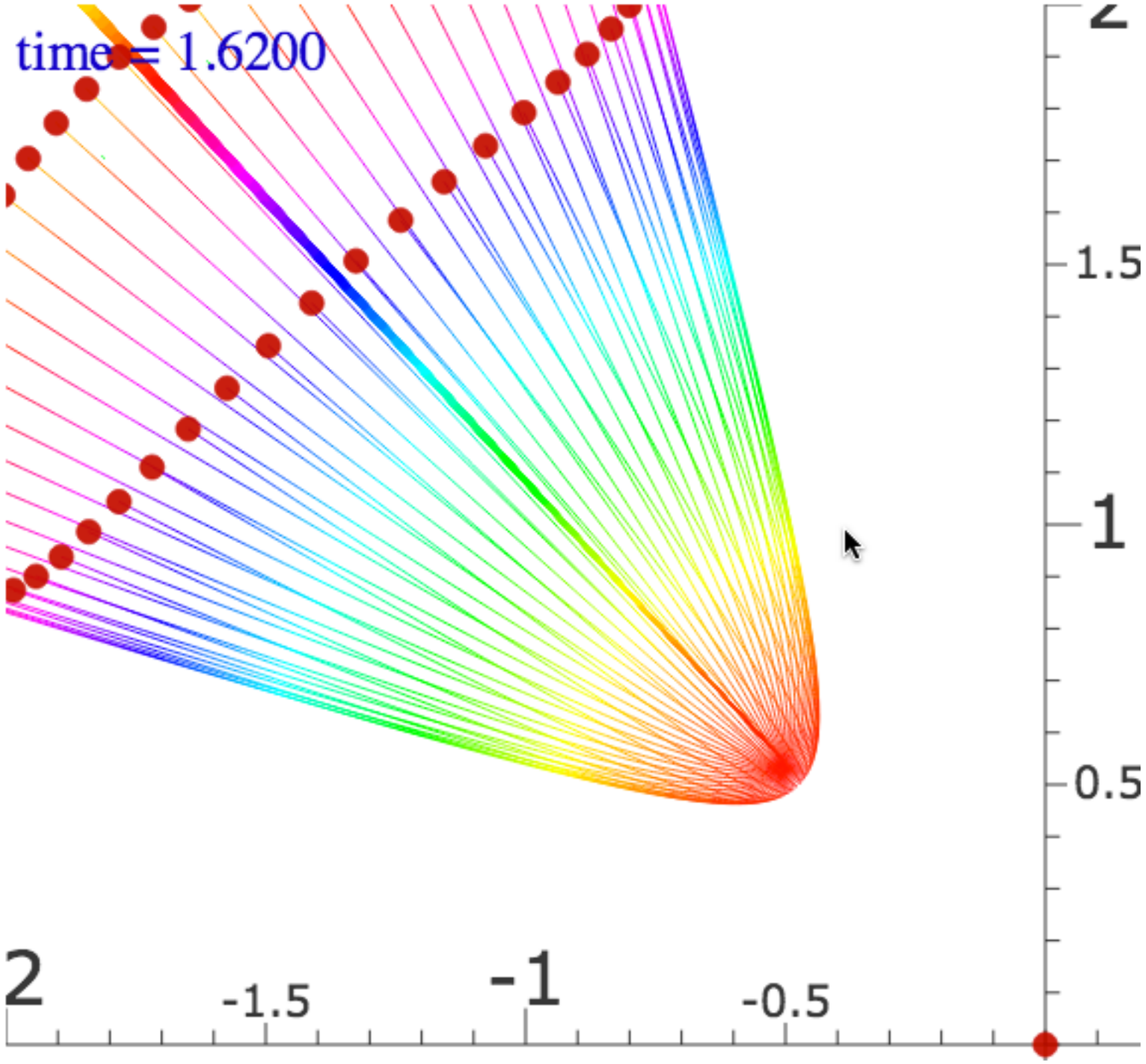












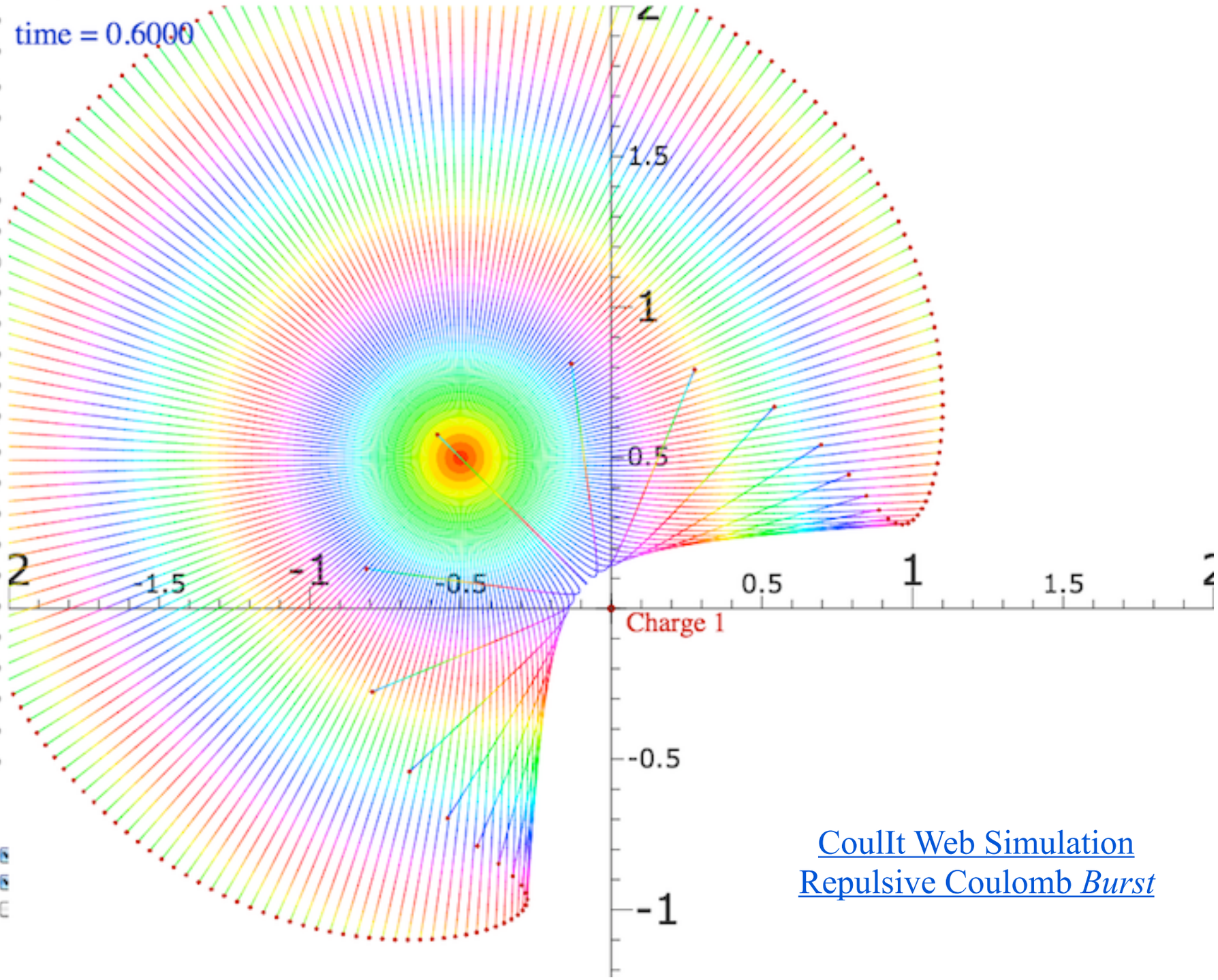
[CoulIt Web Simulation Repulsive Coulomb Burst - Tight]

Main Control | Toggle Local | Pause | Reset T=0 | Erase Paths

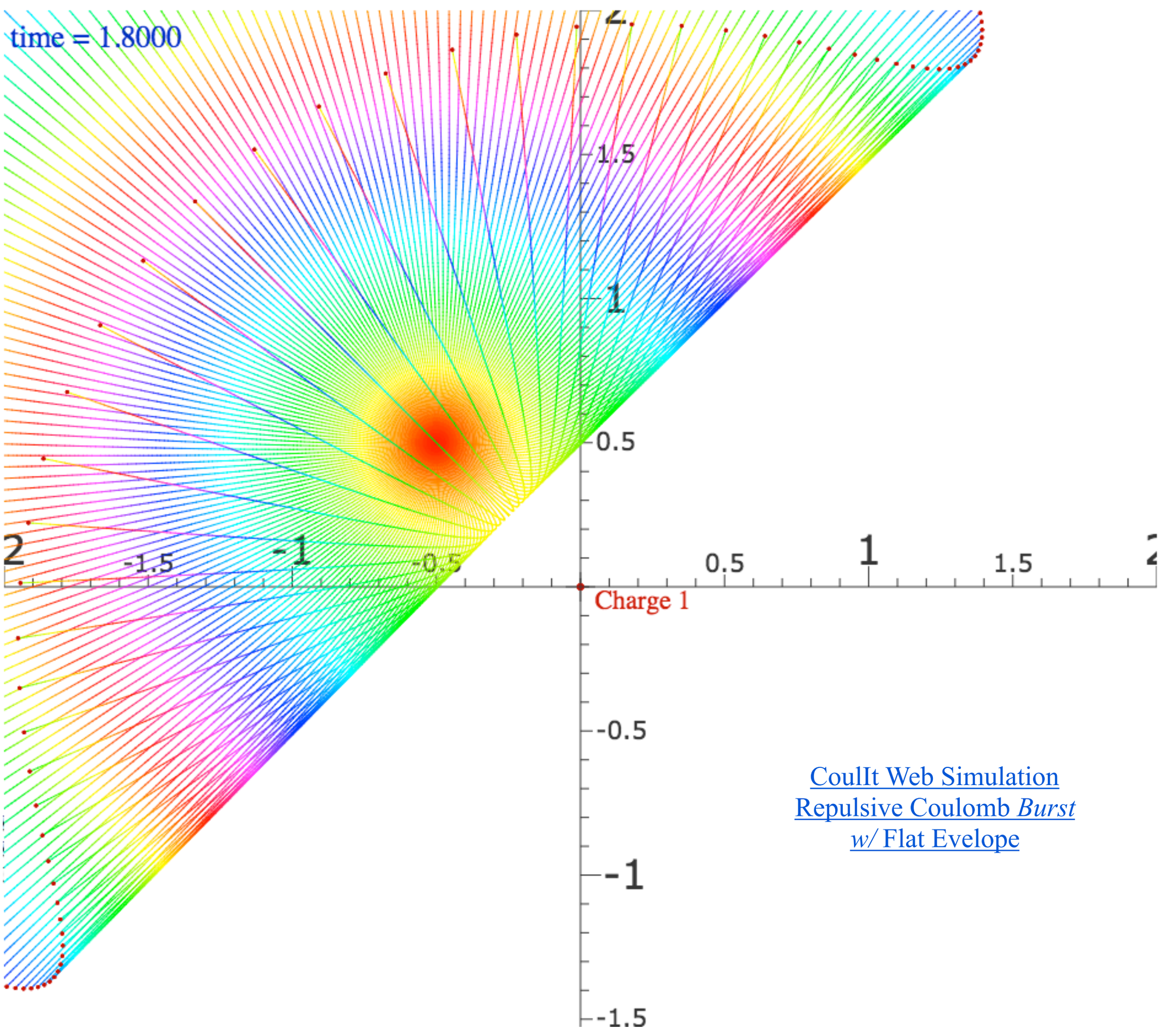
Initial position x(0) = -0.5
Initial position y(0) = 0.5
Initial momentum p(0) = 2.7
Initial momentum ϕ(0) = 90

Terminal time t(off) = 0.6
Maximum step size dt = 0.01
Start launch angle phi1 = -180
Start launch angle phi2 = 180
Number of burst paths = 200
Charge of Nucleus 1 = 0.5
x-Position of Nucleus 1 = 0
y-Position of Nucleus 1 = 0
Charge of Nucleus 2 = 0
Coulomb (k12) = -1
Core thickness r = 1e-32
x-Stark field Ex = 0
y-Stark field Ey = 0
Zeeman field Bz = 0
Diamagnetic strength k = 0
Plank constant h-bar = 2
Color quantization hues = 256
Color quantization bands = 2
Fractional Error (e^{-x}), x = 8
Particle Size = 1

Fix r(0) ☐ Fix p(0) ☐ Do swarm ☑ Beam ☐
Plot r(t) ☑ Plot p(t) ☐
Color action ☑ No stops ☐ Field vectors ☑ Info ☑
Draw masses ☑ Axes ☑ Coordinates ☐ Lenz ☑
Set p by ϕ ☑ Elastic ☑ 2 Free ☐
Save to GIF ☐



[CoulIt Web Simulation
Repulsive Coulomb *Burst*]



CoulIt Web Simulation
Repulsive Coulomb *Burst*
w/ Flat Evelope

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Analytic geometry derivation of $\boldsymbol{\varepsilon}$*-construction*

Connection formulas for (a,b) *and* (ε,λ) *with* (γ,R)

Detailed ruler & compass construction of $\boldsymbol{\varepsilon}$*-vector and orbits*

(R=-0.375 elliptic orbit)

(R=+0.5 hyperbolic orbit)

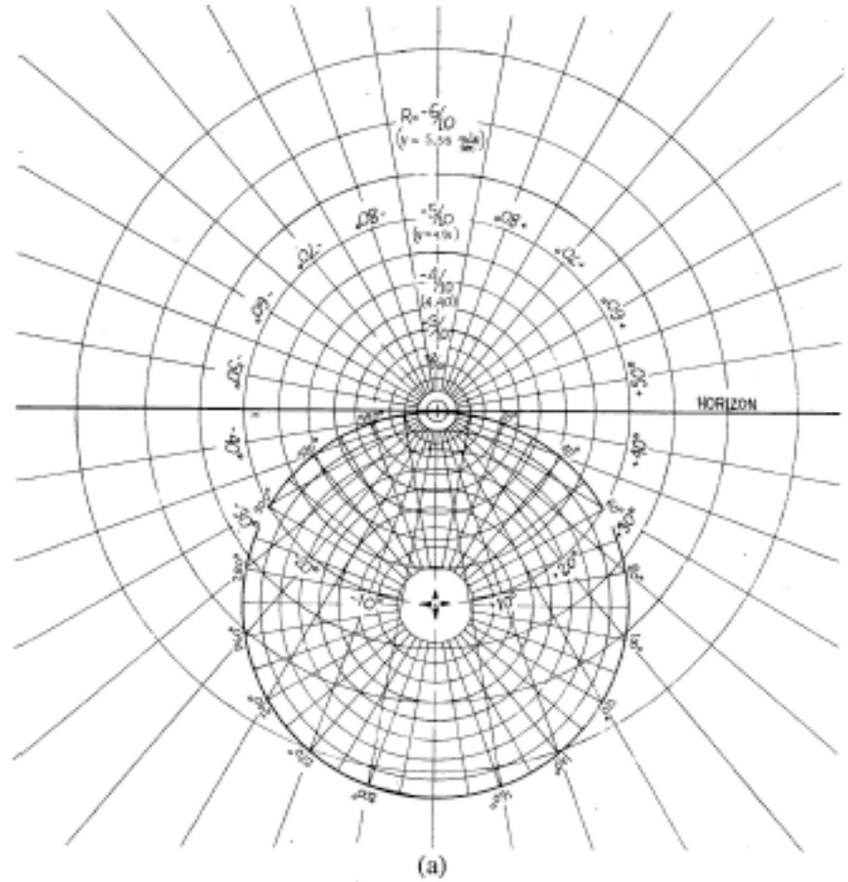
Properties of Coulomb trajectory families and envelopes

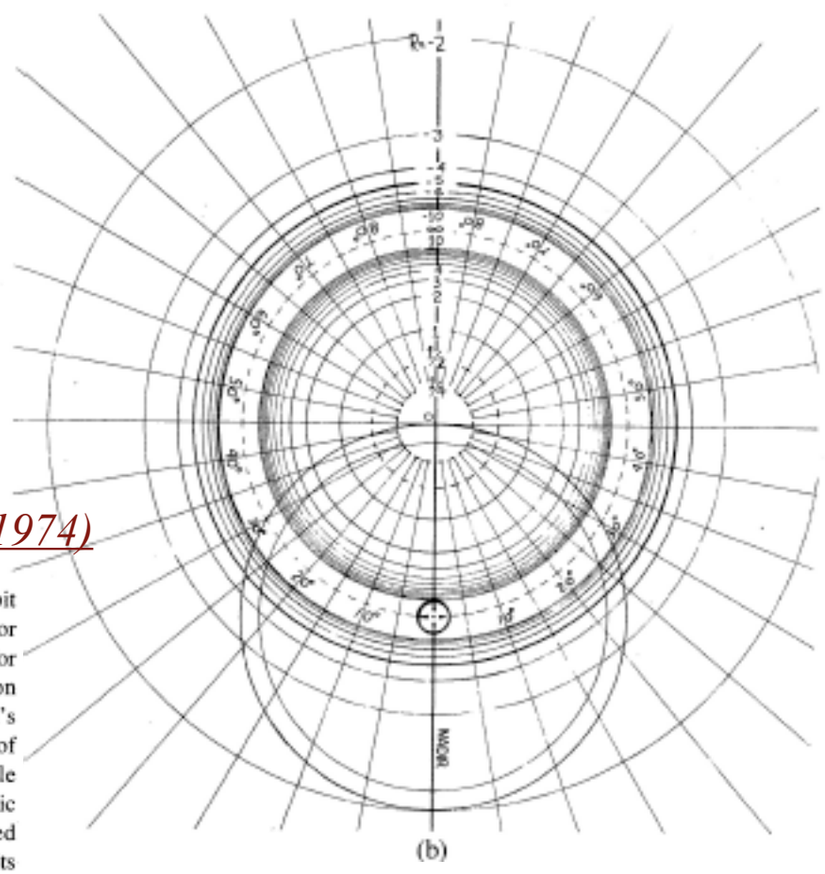
➤ *Graphical* $\boldsymbol{\varepsilon}$*-development of orbits*

Launch angle fixed-Varied launch energy

Launch energy fixed-Varied launch angle

➤ *Launch optimization and orbit family envelopes*





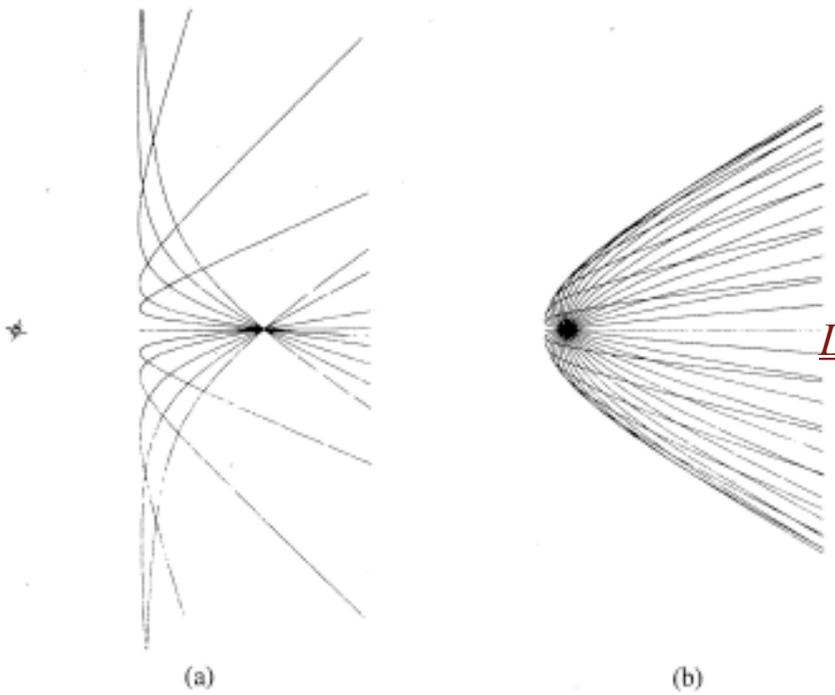


Fig. 11. Sample computer trajectories. (a) Family of hyperbolic orbits with $R = 1$. (b) Family of hyperbolic orbits with $R < 1$.

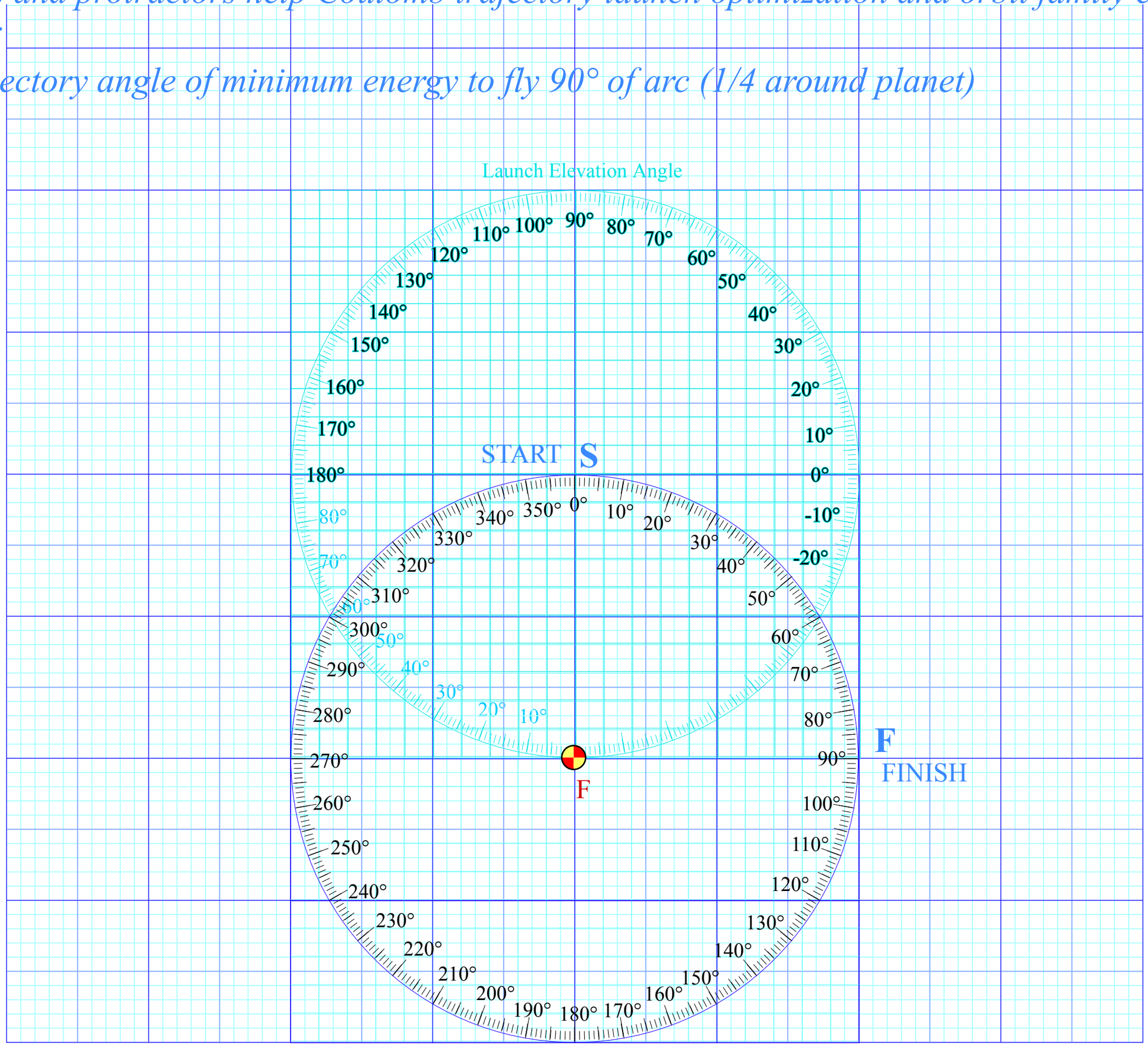
Lenz Vector...analog computersAJP 44 4 (1974)

Fig. 7. Coordinate grids for orbital analog computers. (a) Elliptical orbit scale ($0 > R > -1$). This can be used with the apparatus in Figs. 8 or 9. Radial lines marked $\pm 10°$, $\pm 20°$, . . . , are each the focus locus for orbits with an initial velocity $\pm 10°$, $\pm 20°$, . . . , above the horizon line. The circle marked 20°, 40°, . . . , 340° can be taken as the Earth's surface, or any circle inside this one can be taken to be the surface of any celestial body. The R values apply correctly in either case, while the velocity values are marked for the former case only. (b) Hyperbolic orbit scale ($0 < R < \infty$) and ($-\infty < R < -1$). This can only be used with the apparatus shown in Fig. 9. Outer circles locate foci for orbits of particles attracted to the force center, while inner circles locate foci for orbits in a repulsive field. In either case a radial line marked $\pm 10°$, $\pm 20°$, . . . , is the focus locus for an orbit with the initial velocity an angle $\pm 10°$, $\pm 20°$, . . . , above the nadir line.

Graphs and protractors help Coulomb trajectory launch optimization and orbit family envelopes

Problem:

Find trajectory angle of minimum energy to fly 90° of arc (1/4 around planet)

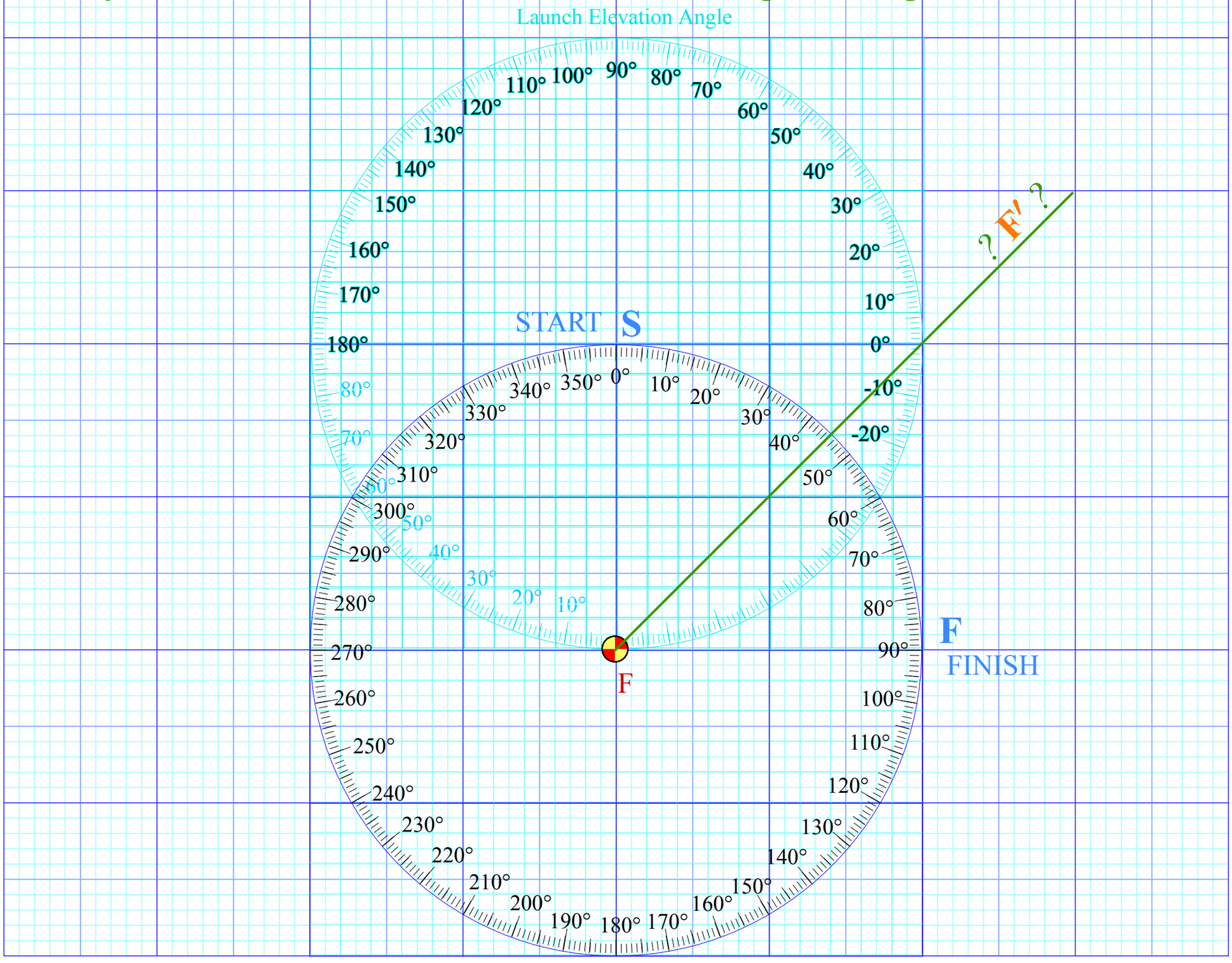


Graphs and protractors help Coulomb trajectory launch optimization and orbit family envelopes

Problem:

Find trajectory angle of minimum energy to fly 90° of longitude (1/4 around planet)

Solution: Prime focus **F′** *lies on radial line that bisects longitude angle*

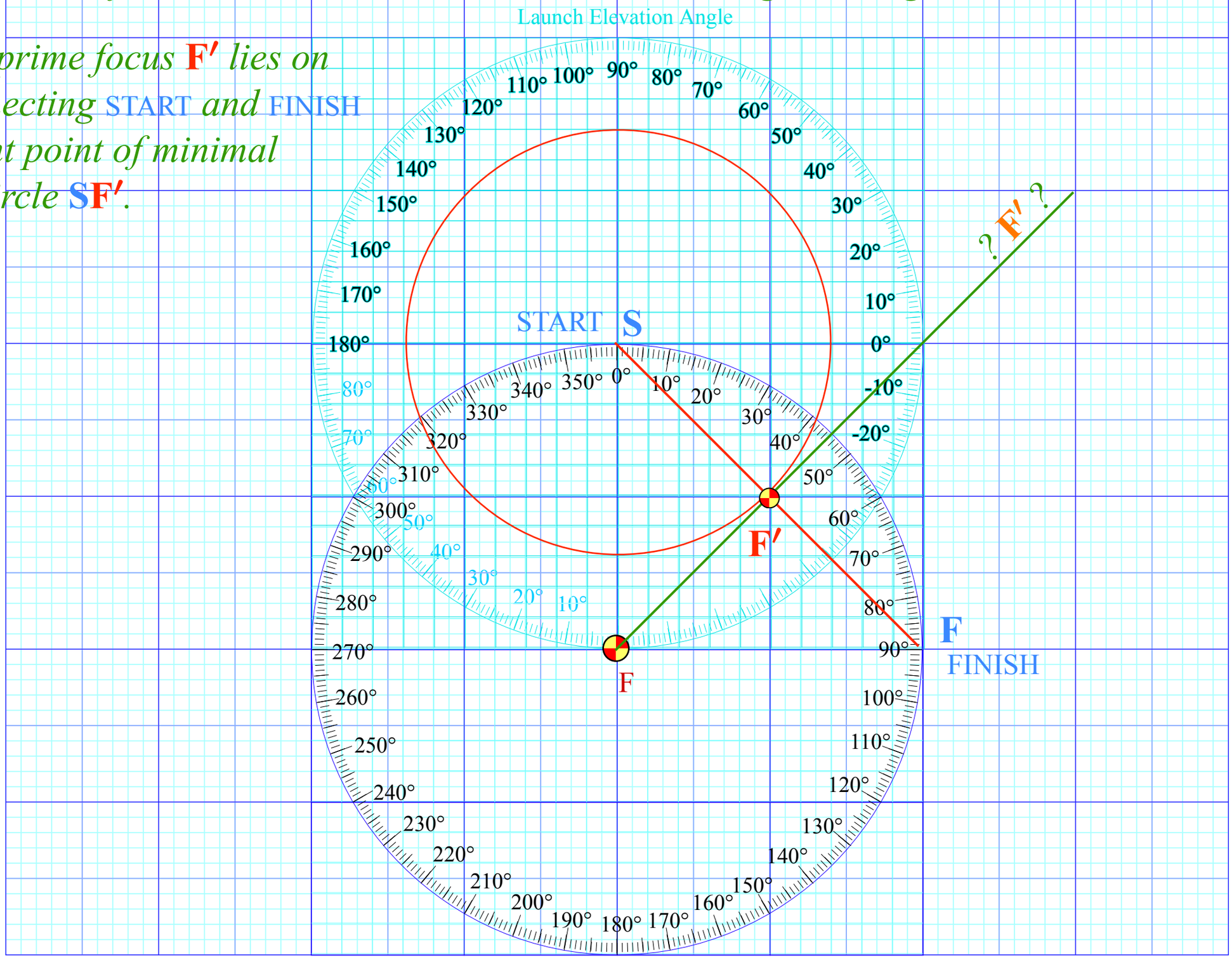


Graphs and protractors help Coulomb trajectory launch optimization and orbit family envelopes

Problem:
With launch angle $\alpha = 22.5°$ *find maximum range of trajectory.*

Solution: Prime focus **F′** *lies on radial line that bisects longitude angle*

Optimal prime focus **F′** *lies on line connecting* START *and* FINISH *at tangent point of minimal energy circle* **SF′**.



Graphs and protractors help Coulomb trajectory launch optimization and orbit family envelopes

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With launch angle $\alpha = 22.5°$ *find maximum range of trajectory.*

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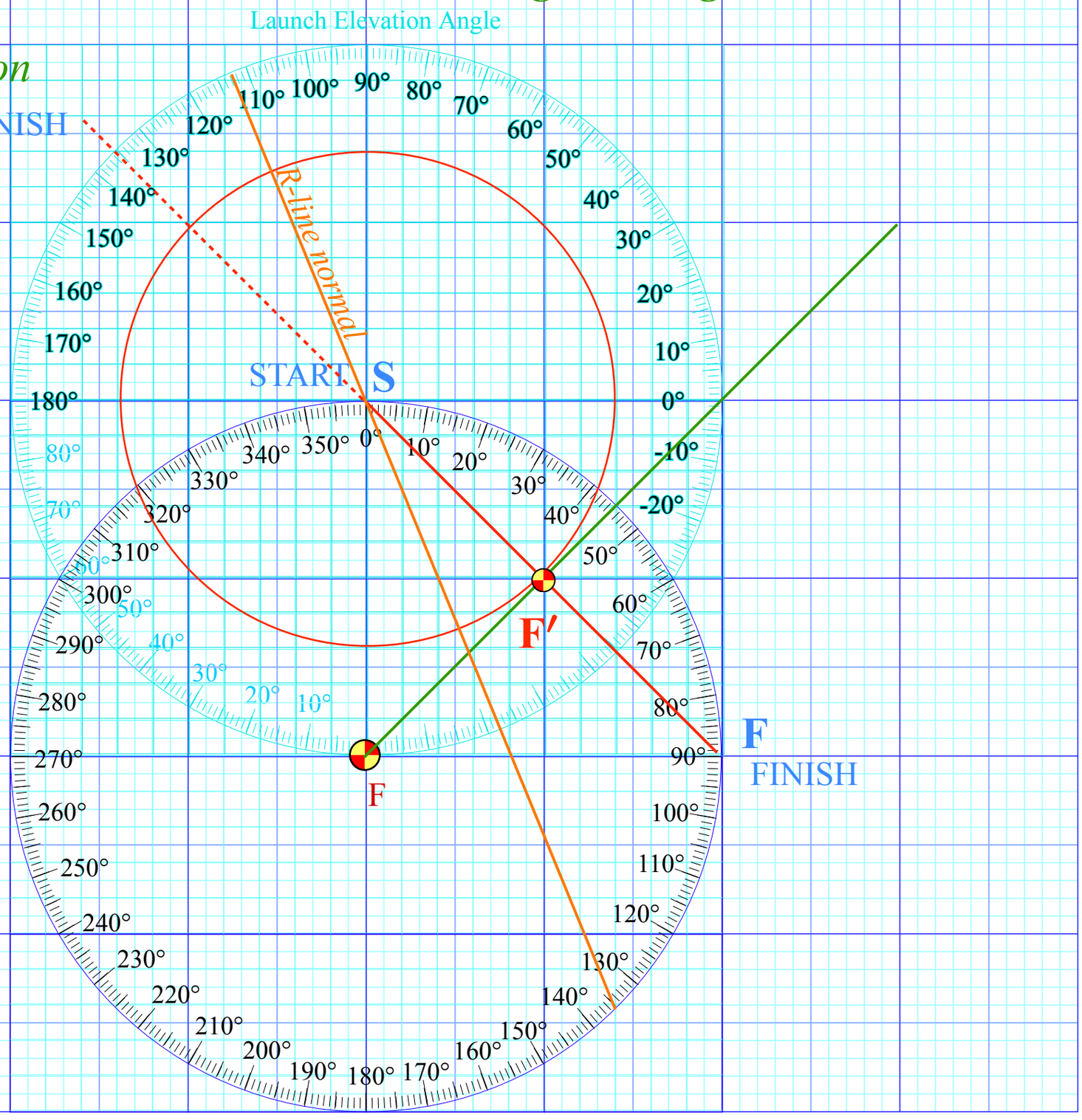
R-line normal must bisect angle **FSF′** *connecting foci* **F** *and* **F′** *and is normal to initial launch vector* $\mathbf{v}_0$



Graphs and protractors help Coulomb trajectory launch optimization and orbit family envelopes

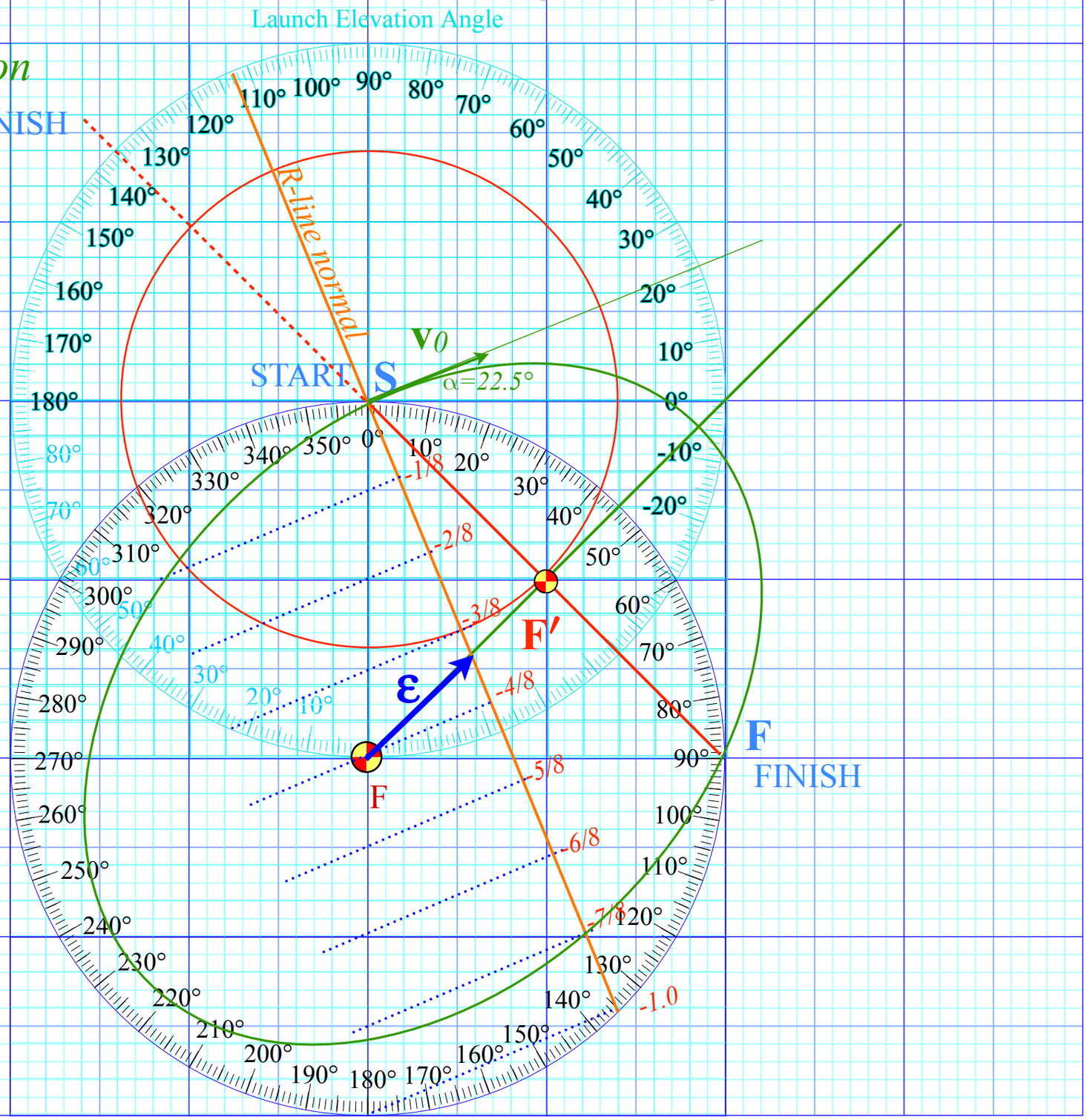
Problem:
With launch angle α=22.5° find maximum range of trajectory.

Solution: Prime focus **F′** *lies on radial line that bisects longitude angle*

Optimal prime focus **F′** *lies on line connecting* START *and* FINISH *at tangent point of minimal energy circle* **SF′**.

R-line normal must bisect angle **FSF′** *connecting foci* **F** *and* **F′** *and is normal to initial launch vector* $\mathbf{v}_0$ *with launch angle α=22.5°*

The ε*-vector and R-value: slightly below R=-3/8...*

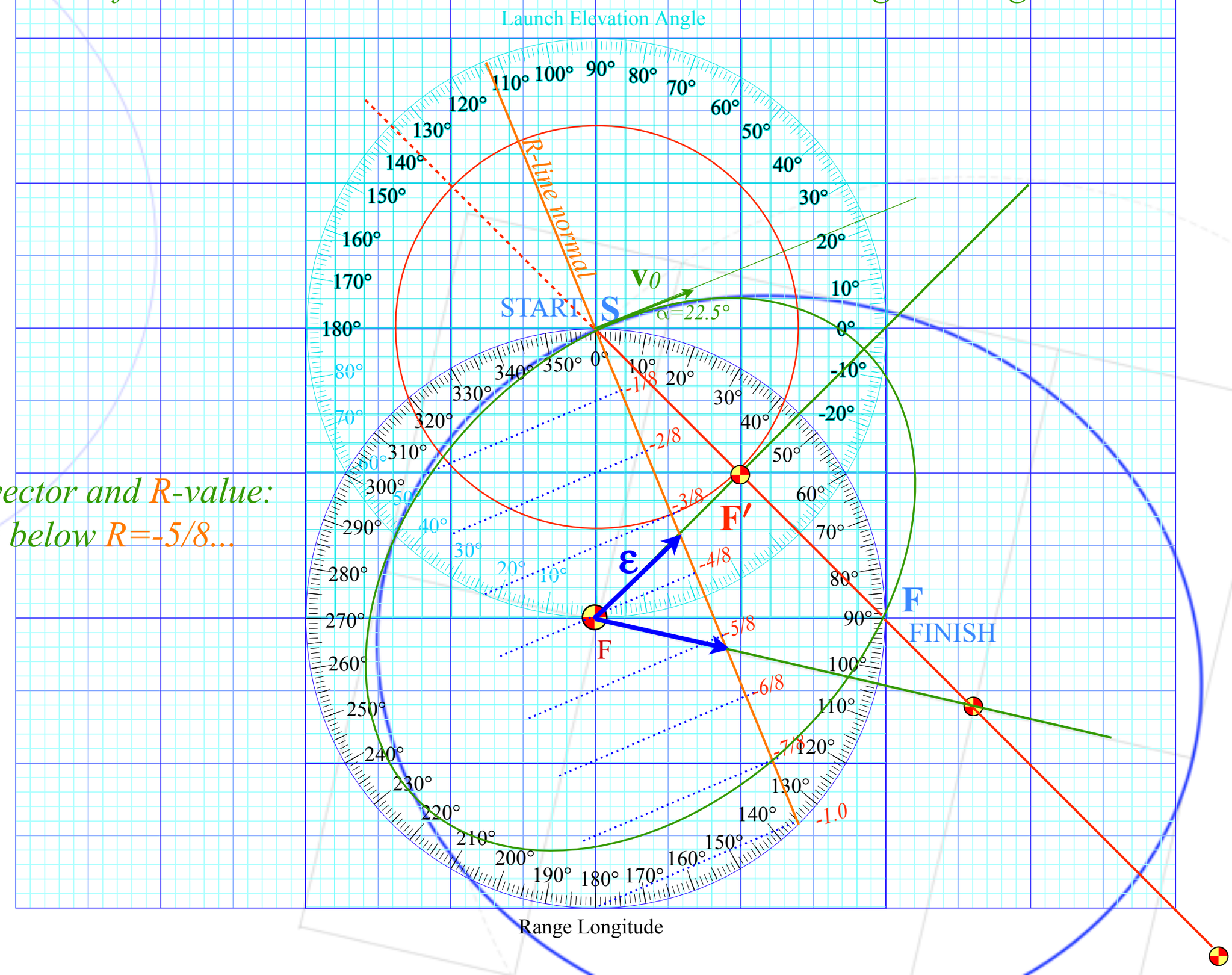


Graphs and protractors help Coulomb trajectory launch optimization and orbit family envelopes

Problem:
With launch angle α=22.5° find maximum range of trajectory.

Solution: Prime focus **F′** *lies on radial line at 103.5° that bisects longitude angle 207°*

The ε-vector and R-value: slightly below R=-5/8...



Graphs and protractors help Coulomb trajectory launch optimization and orbit family envelopes

Problem:
With launch angle $\alpha=22.5°$ find maximum range of trajectory.

Solution: Prime focus **F′** *lies at infinity and gives parabola ($\varepsilon_\infty=1, R=-1$) trajectory.*

Trajectory axis is at 135°.

Trajectory would hit Earth at 270°

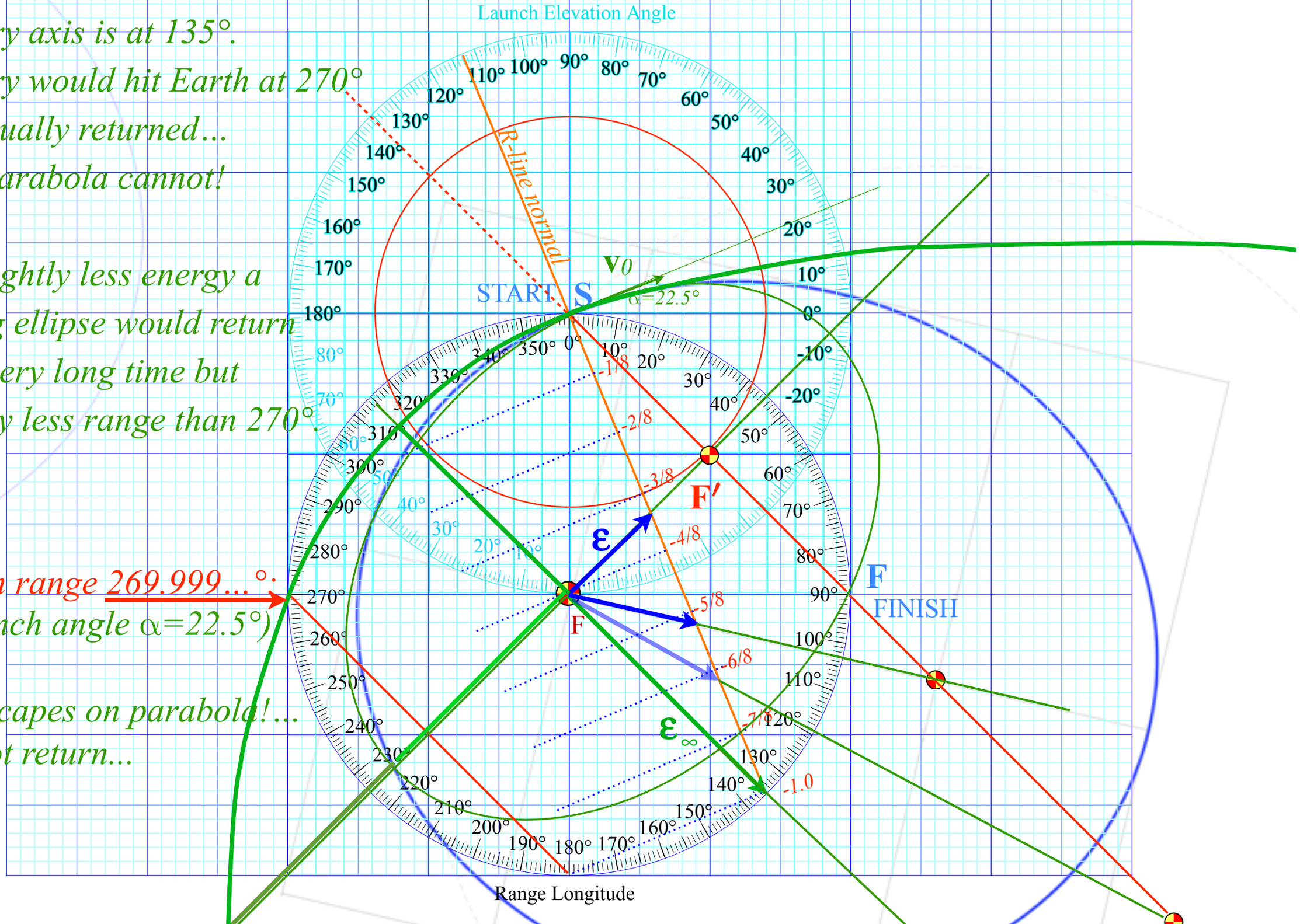
...if it actually returned...

...but a parabola cannot!

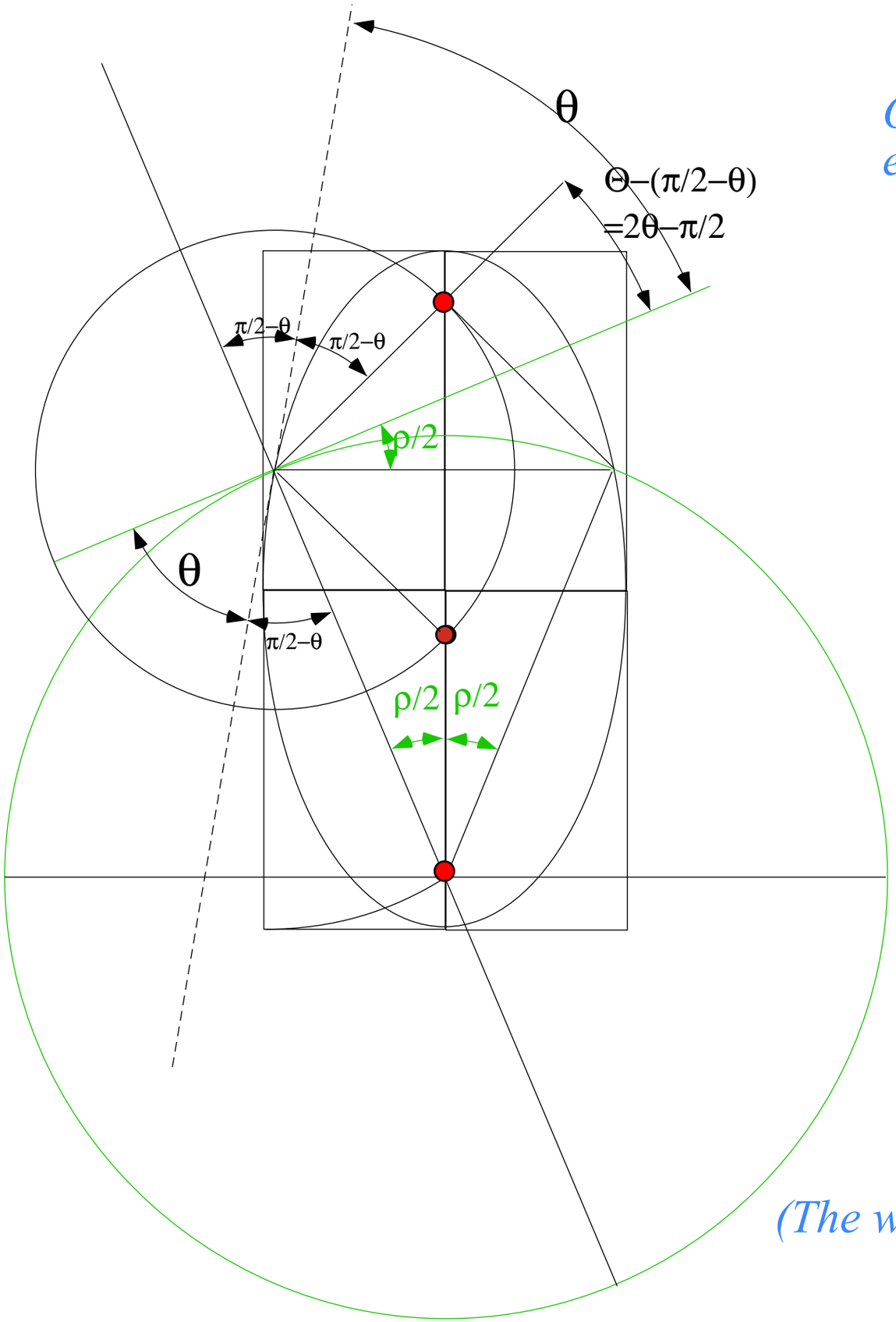
But at slightly less energy a very long ellipse would return after a very long time but at slightly less range than 270°.

Maximum range 269.999...°.
(with launch angle $\alpha=22.5°$)

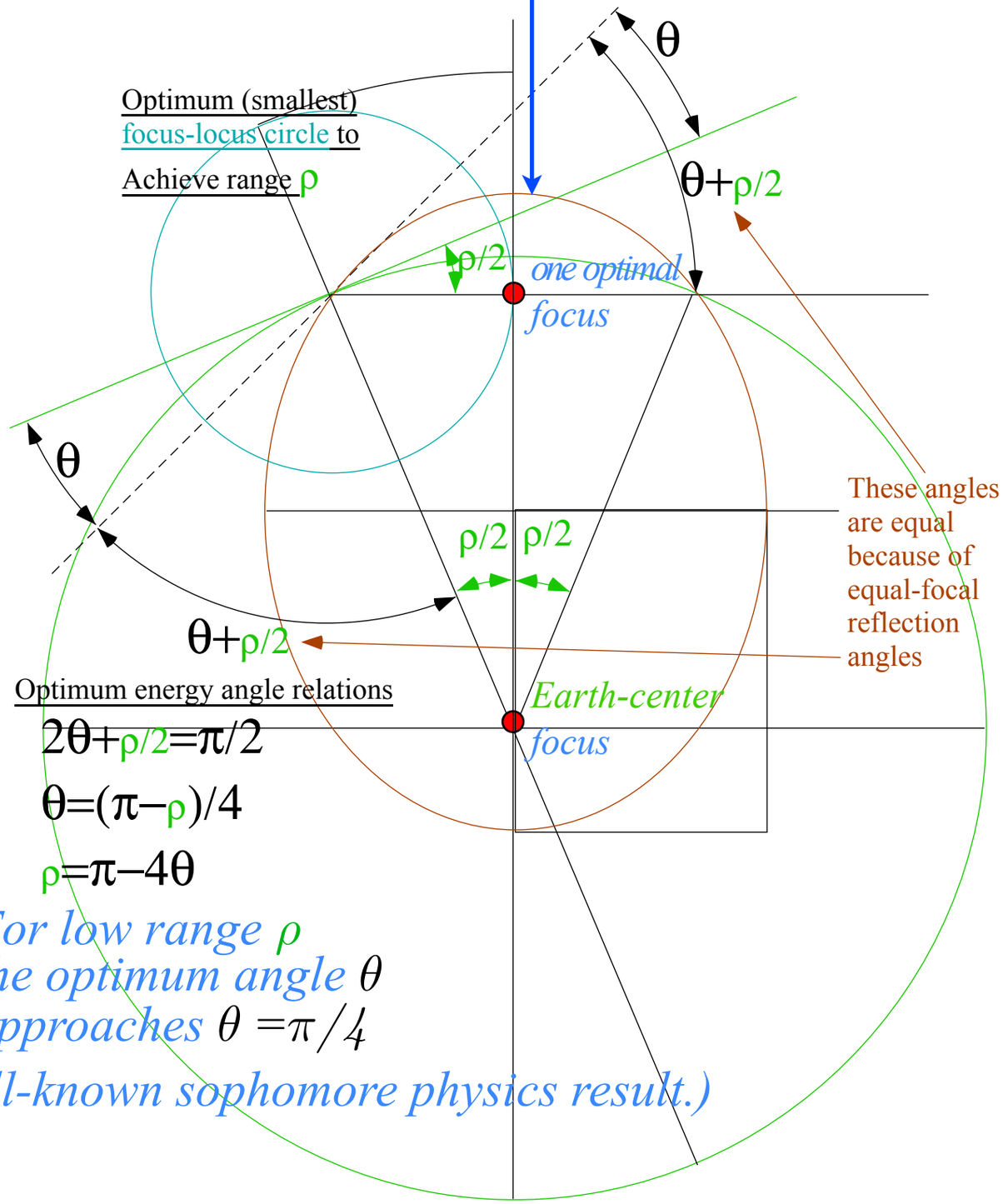
$v_0+\delta$ escapes on parabola!...
...cannot return...



Launch optimization



Optimal v_0 has only one elliptic path to target



Optimum energy angle relations

$2\theta+\rho/2=\pi/2$

$\theta=(\pi-\rho)/4$

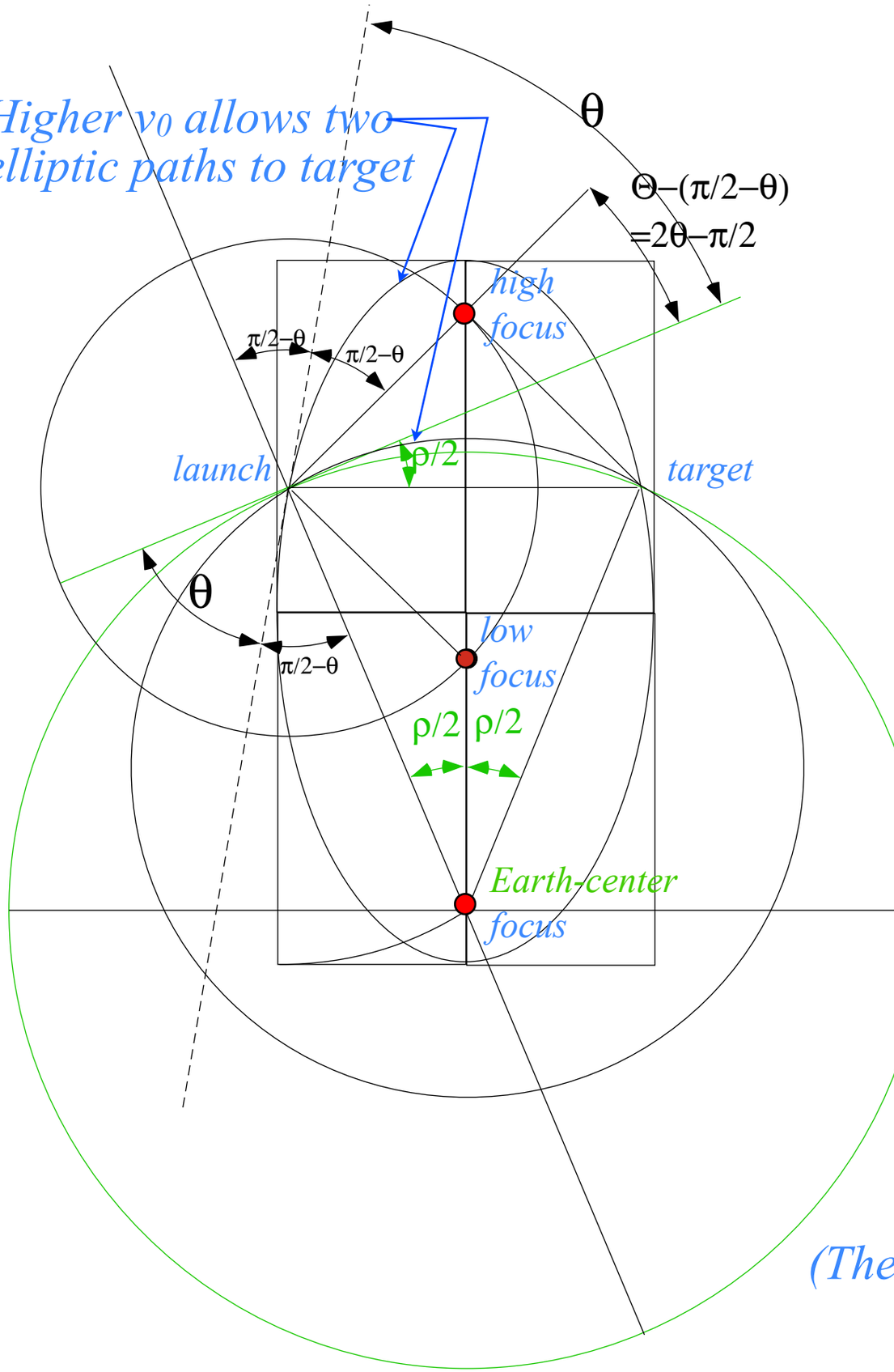
$\rho=\pi-4\theta$

For low range ρ the optimum angle θ approaches $\theta=\pi/4$

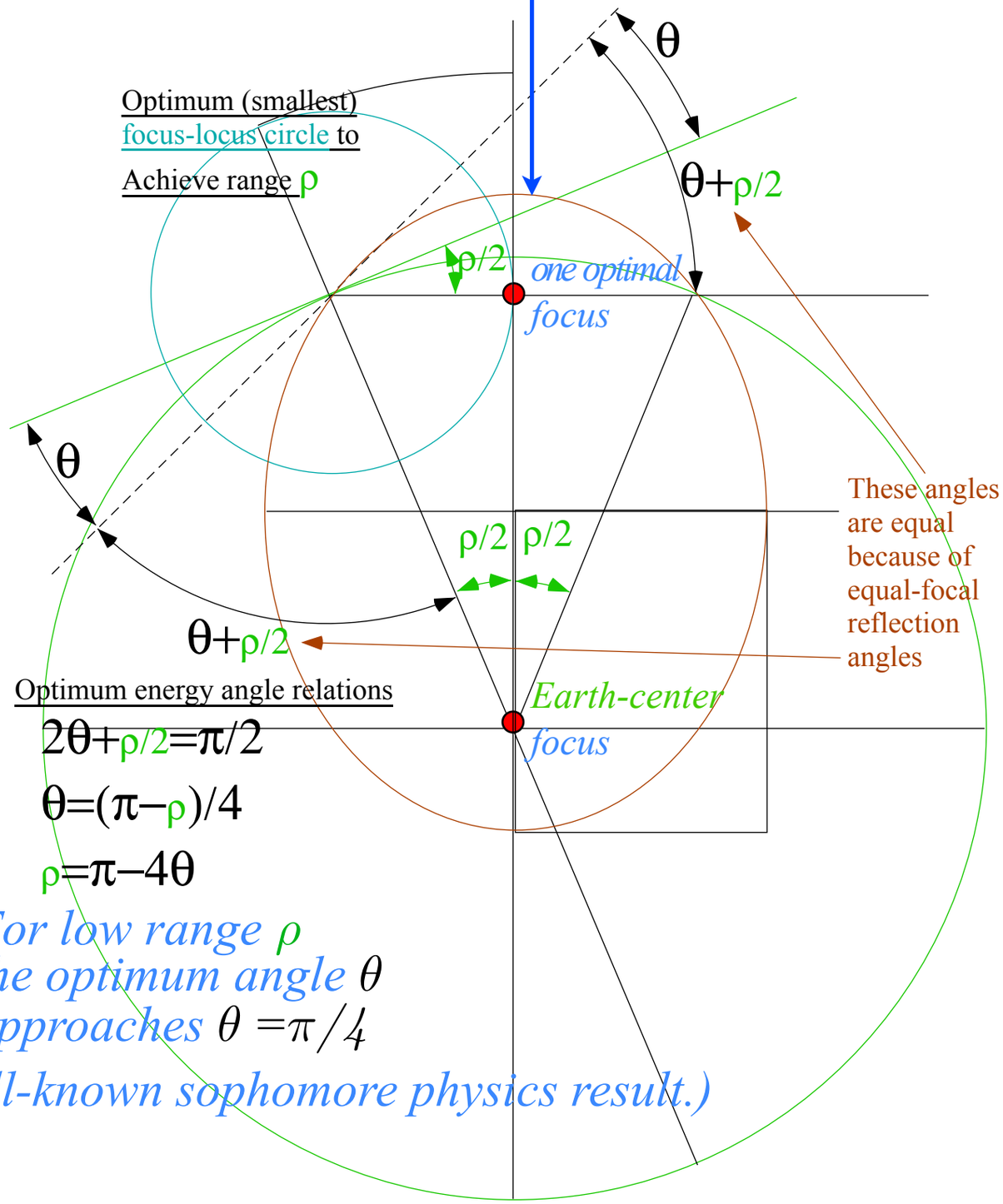
(The well-known sophomore physics result.)

Launch optimization

Higher v_0 allows two elliptic paths to target



Optimal v_0 has only one elliptic path to target



Optimum energy angle relations

$2\theta+\rho/2=\pi/2$

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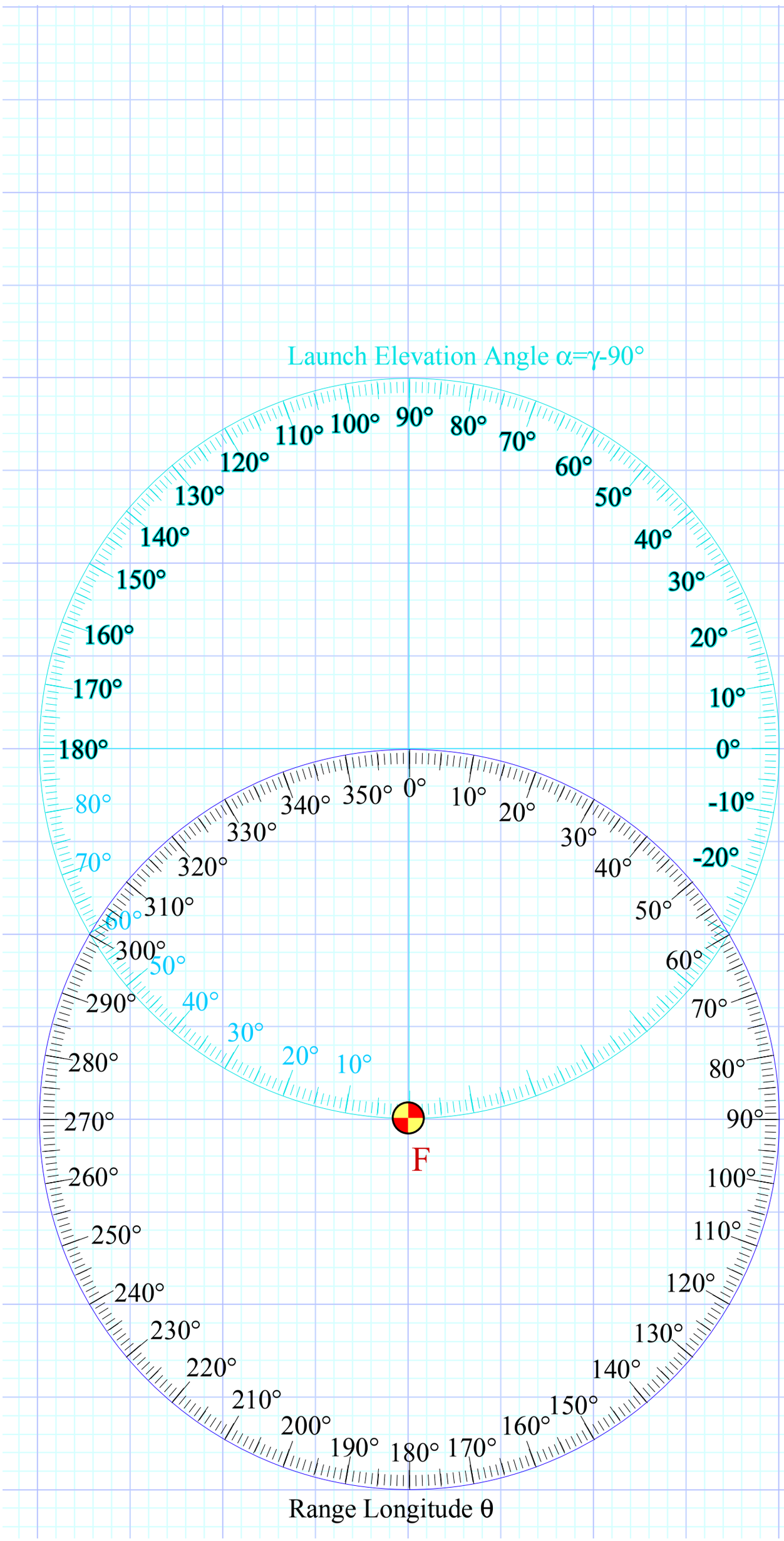
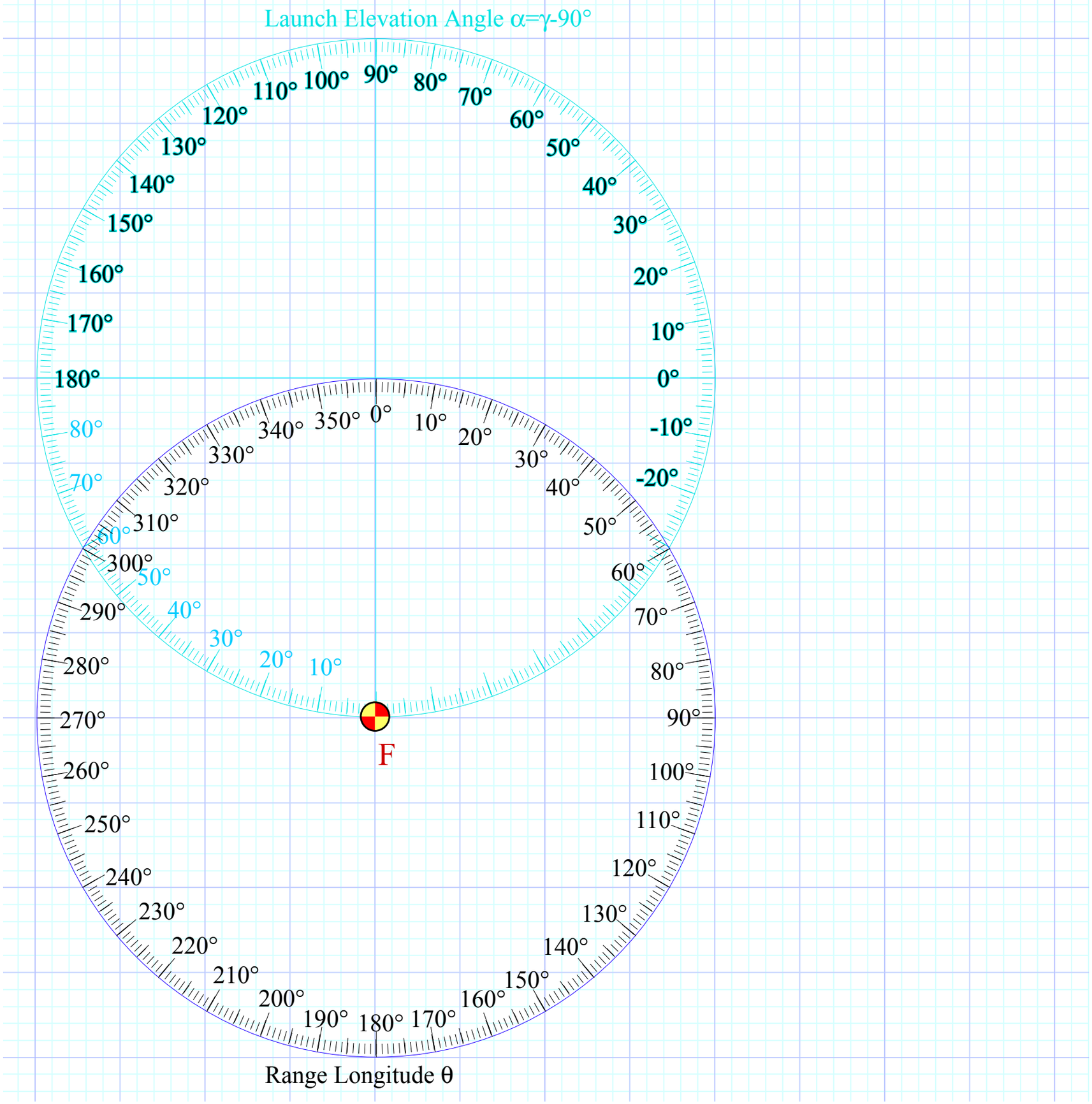
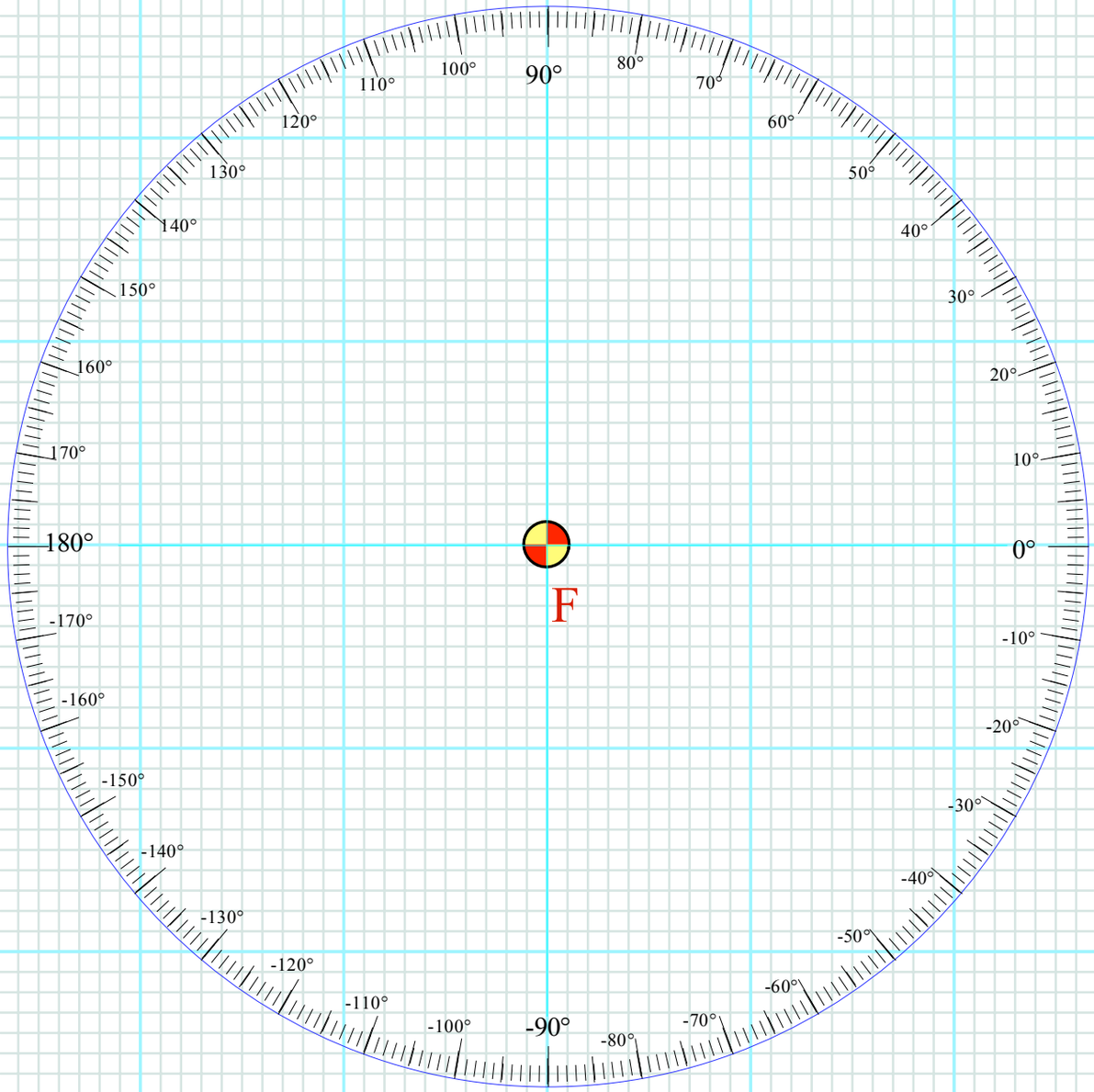
(The well-known sophomore physics result.)


Launch Elevation Angle α=γ-90°
90°
80°
70°
60°
50°
40°
30°
20°
10°
0°
-10°
-20°
100°
110°
120°
130°
140°
150°
160°
170°
180°
350°
340°
330°
320°
310°
300°
290°
280°
270°
260°
250°
240°
230°
220°
210°
200°
190°
F
Range Longitude θ


Launch Elevation Angle α=γ-90°
90° 80° 70° 60° 50° 40° 30° 20° 10° 0° -10° -20°
110° 100°
120° 130° 140° 150° 160° 170° 180°
80° 70° 60° 50° 40° 30° 20° 10°
0° 10° 20° 30° 40° 50° 60° 70° 80° 90° 100° 110° 120° 130° 140° 150° 160° 170° 180° 190° 200° 210° 220° 230° 240° 250° 260° 270° 280° 290° 300° 310° 320° 330° 340° 350°
F
Range Longitude θ


90°
80°
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-80°
-90°
-100°
-110°
-120°
-130°
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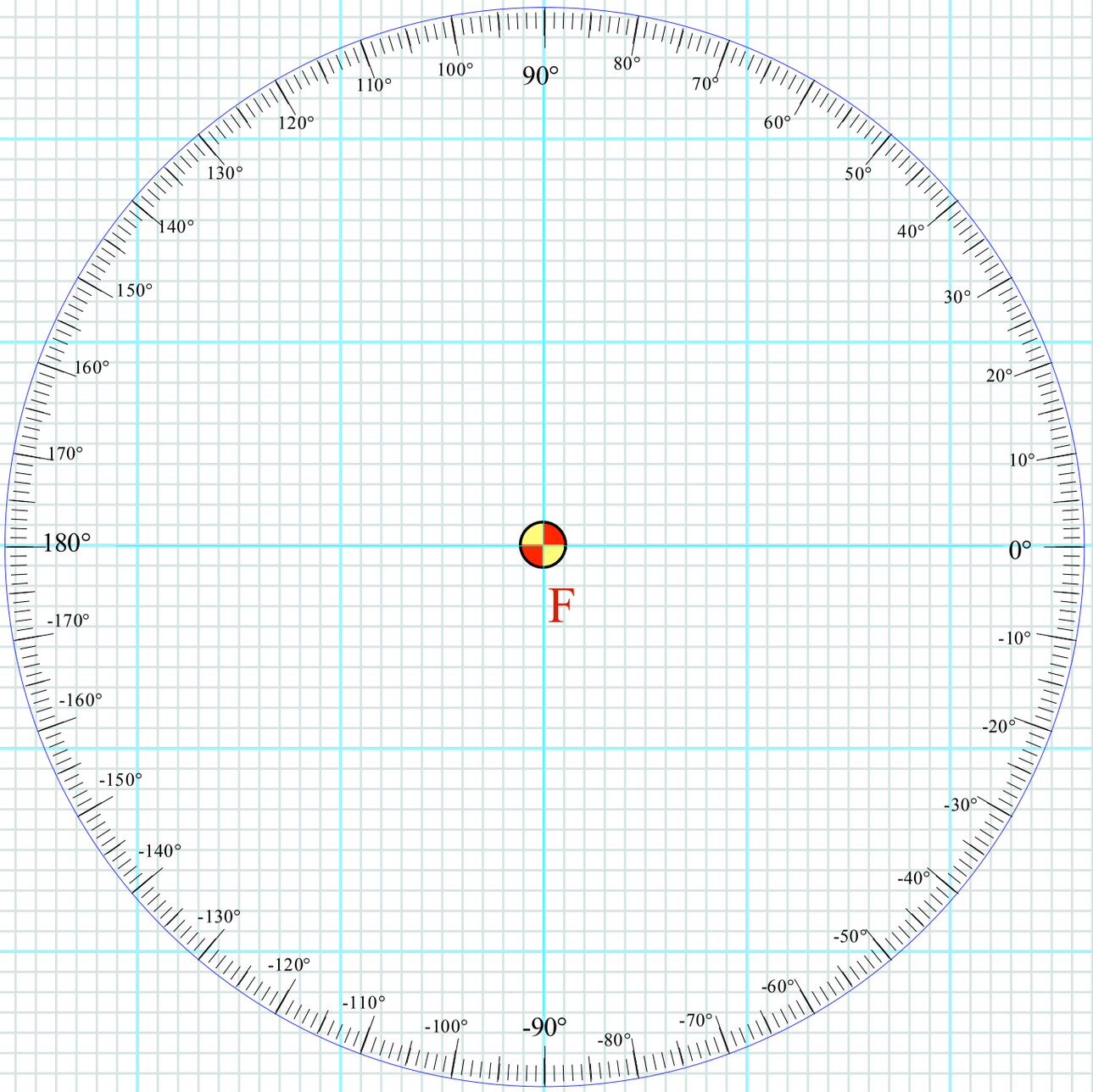
120°
130°
140°
150°
160°
170°
F


90°
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-160°
-170°
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F